“International Portfolio Management”

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1 Introduction

Risk is something infinitely dimensional and depends on all of the possible events in the world. Financial advisors, organizations and investors aim to structure their investment strategies combining assets that will offer a balance between risk and return. Since 1952, when Markowitz published his paper about portfolio selection in which he tries to maximize portfolio expected return for a given amount of risk, researchers have created various risk measures for optimal portfolio selection. However, it is not attainable to determine risk precisely. The only accurate answer we can give to the question "What is the maximal loss we can suffer over some time horizon?" is, "We can lose everything!" However, the probability of this event is very small.

The current thesis is separated in a theoretical and an empirical part. As far as the theoretical part concerns, we analyze the various kinds of risks that an investor faces while investing in financial assets. Secondly, we cite the risk measures and their mathematical formulation that have been published in literature and are used commonly in modern economies for the portfolio selection. These measures are applied to financial, engineering and insurance sector by the managers in order to make optimal decisions. In the last section of the theoretical part, we investigate the benefits of investing internationally for investors who come from developed and development countries. The impact of international diversification depends on a variety of factors that we analyze in that section from a financial and an econometrical point of view, according to the international literature.

As far as the empirical part concerns, we implement a portfolio optimization problem using CVaR model as risk measurement. Particularly, we apply the optimization problem on an investor from Germany who has the option to invest in a domestic and an international portfolio with assets from Germany, U.S.A and Japan. After receiving the historical data for the returns of the assets and the exchange rates for a selected period of time, we run the optimization problem on GAMS model, and we get the optimal weights for each asset. By determining the efficient frontiers and conducting a backing test for domestic and international portfolios, we reach to a conclusion about the profitability of the investments. In the end, we interpret the outputs, by analyzing the historical returns of the assets in tandem with the effect of exchange rates.

2 Kinds of risk

The term "risk" describes the uncertainty that the realized return of an investment will not be equal to the expected return. The financial instruments contain various risks, according to their specific characteristics and the transactions among them comprise high risks of reducing or losing the initial (or multiple) investing capital. Thereby, the transaction with financial instruments is appropriate to investors who understand and interpret both the movements of these instruments and the undertaken risks. There are several forms of risk that affect the performance of a portfolio and the investors are trying to mitigate them.

2.1 Market risk:

is the risk which can cause fluctuations both to the fair value and to the future cash flows of the financial instrument. It is also common known as "Systematic risk" and is the probability of a loss as a result of negative fluctuations in the market. For instance, a decrease in the price of equities, interest rates, currencies, commodities, valuable metals and generally values that are traded in the market, can reduce the invested capital. Market risk can affect the whole market and the investors can mitigate it by investing in a variety of asset classes to delegate the possibilities.

2.2 Exchange risk:

is the risk of the impact on evaluating an investment in foreign currency, due to a fluctuation of exchange rates. Furthermore, we can define the Exchange risk as the risk that can be caused from the changes in exchange rates, affecting the fair value or the future cash flows of a financial instrument. Once an investor has to deal with assets that are exposed to foreign currencies, he is prone to unpredictable fluctuations of the currency's price.

2.3 Interest rate risk:

contains the possibility the value of an investment to be undermined due to a change in interest rates. As interest rate rise equity's and bond's prices fall and vice versa. . The relationship between the interest rate and bond prices can be interpreted by opportunity risk. When an investor buys bonds, assumes that if the interest rate increases, he will miss out the opportunity of purchasing the bonds with better returns. In that case, the demand for the bonds with lower returns declines because of the new opportunities that appear. The choice of new bonds with higher return rates that are issued, are now more attractive for the investors, than the previous bonds that they hold. The interest rate risk is called "the source" of market risk as the fluctuations in interest rates affect the market in total. In conclusion, it is a risk through fair value and future cash flows can either be affected negatively or have intense fluctuations.

2.4 Credit risk:

is the risk of one of the two parts to abrogate its obligation, causing loss to the other part. It is known that a bond is a kind of loan, issued by a government or an organization. Although the risk of issuer's default is the greatest factor of setting the appropriate borrowing rate, usually the market can predict it, especially in periods of growth. When an investor buys a bond or a kind of loan has expectations for a return. The return is a summary of interest rate risk and credit risk. The interest rate risk is reflected in reference rate and the credit risk in credit spread. The credit risk wasn't able to be isolated until the appearance of credit options. Through these instruments, in a market, credit risk was attainable to be transferred to the investors who had the willing to take it. (Through the composition of a bond and an asset swap, the credit risk could be isolated).

The euro system has adapted rules and institutions for the assessment of credit risk among the countries. The rating agencies (Moody's, Standard & Poor's, Fitch e.t.c.) have the duty to assess credit risk of the countries and big companies, providing useful information to the investors. If a bond has a low rating (B or C), the issuer has a high risk of default. Conversely, if it has a high rating (AAA, AA, or A), it's considered to be a safe investment. Also, Central Banks like Deutsche Bundesbank, Banque de France, Banco de Espana, Oesterreichische National bank provide detailed reports for the credit assessment.

2.5 Liquidity risk:

stems from the lack of marketability of an investment that happens when a financial product cannot be sold, as a result of not being attractive to the investors. In that case, the investor might be unable to convert an asset into cash without giving up capital and income due to a lack of buyers or an inefficient market. Usually, it is reflected either in bid-ask-spreads (the amount by which the ask price exceeds the bid price for an asset in the market) or in big price movements. These facts depict uncertainty and insecure based on the fact that a borrower cannot meet its short-term debt obligations. The liquidity risk is used from the investors as a measurement of the level of risk within a company. In the level of financial institutions, their sustainability depends on the ability to meet their debt obligations without realizing great losses.

2.6 Inflation risk:

is the risk that inflation (the increase in the prices of a basket of selected goods and services over a period of time) will erode the value and expected profit of an investment. When we interpret the performance of an investment without taking into consideration the inflation, the outcome that turns out is the nominal return. However, an investor should take into account the real return of its investment which is linked to the purchase power. Bonds are the financial instruments that are most vulnerable to inflationary risk. Most of the times, bonds obtain a fixed coupon rate that does not increase. Thus, if an investor buys a bond that pays a fixed interest rate, but inflation rises more than the interest rate, then the investor faces losses. In case this situation will go on until the maturity of the bond, the holder will lose more purchasing power, regardless of how safe investment it is.

2.7 Settlement risk:

is the risk that one party will fail to deliver the terms of a contract with another party at the time of settlement. It can happen when one side fails to fulfill its obligations either by not delivering the underlying asset or by not paying the cash value of the contract. The organized markets try to confine this kind of risk by adopting the appropriate procedures. In not structured markets (out of money market) the investor faces the credit risk of the counterparty.

2.8 Sovereign risk:

contains the possibility a central bank will apply exchange rules that will affect negatively the worth of the FOREX (Foreign Exchange) contracts that have been done by foreign investors. A FOREX contract is a kind of currency transaction, where the two parties make an agreement in order to exchange two designated currencies at a specific future time. It is a common way for the byer to protect from currency price's fluctuations. Foreign exchange traders have to deal with the danger that a central bank will alter its monetary policy and as a result will affect currency trades. Another factor that causes sovereign risk is the political instability that takes place when a government refuses to adhere to the payments agreement of its sovereign debt.

3 Risk measures and mathematical formulations

Portfolio optimization problem is an important topic since the pioneering Markowitz work on optimal portfolio selection. Specifically, an investor is interested in determining the optimal combination of n risky assets in such a way that the obtained portfolio has minimal risk and maximal expected gain. Although the idea of risk seems to be intuitively clear, it is difficult to formalize it. There is an efficient way to measure and quantify risk in almost every single market. However, each method is deeply associated with its specific market and cannot be used directly to other markets. For instance, the most useful way to measure risk on an equity portfolio is the volatility of returns, whereas for a government bond is the interest rate risk. Furthermore, for portfolios of options, the risk can be measured by the rate of the change in price when only one of the parameters changes (like the underlying asset, volatility, time to maturity). These indices are suited to measure the very specific types of risk related to a specific financial instrument.

Based on investors preferences and risk tolerance several alternative formulations are available that will be analyzed in the current thesis. Prior to presenting the various risk measures and the mathematical formulation of them, it is necessary to set the "scenario generation" which is the base of the analysis.

Scenario generation:

Scenarios are used in order to include and measure all the disparate sources of risk that affect the performance of portfolio. Each scenario depicts a range of different parameters and values that react on the portfolio. Thus, is a way to represent uncertainty that stems from the variations of the included parameters.

Probabilities	Scenarios	r_1	r_2	…..	r_n
$Prob_1$	S_1	r_{11}	r_{12}	…..	r_{1n}
$Prob_2$	S_2	r_{21}	r_{22}	…..	r_{2n}
$Prob_3$	S_3	r_{31}	r_{32}	…..	r_{3n}
…...	…..	…..	…..	…..	…..
$Prob_T$	S_T	r_{T1}	r_{T2}	…..	r_{Tn}

- Let us assume that we have an universe of n instruments (i = 1,2,…, n)

- $\tilde{r} = (\widetilde{r_1}, \widetilde{r_2}, \widetilde{r_3}, \dots, \widetilde{r_n})^T$, where $\tilde{r}$ is the vector of the random instrument's return, which are unknown at the time of portfolio selection and are treated as random variables.
- $E(\tilde{r}) = \bar{r} = (\overline{r_1}, \overline{r_2}, \overline{r_3}, \dots, \overline{r_n})^T$, where E($\tilde{r}$) is the Expected return or the mean of a random instrument.
- $x = (x_1, x_2, \dots, x_n)^T$, where x is the allocation of an investor's budget in the assets.
- $\sum_{i=1}^{n} x_i = 1$ (budget constraint)
- $X = \{x : x^T 1 = 1 , x \geq 0\}$, which is the basic portfolio constraints in vector notation by using the conformable vector $1 = (1,1, \dots ,1)^T$of ones.
- $R(x, \tilde{r}) = x^T \cdot \tilde{r} = \sum_{i=1}^{n} x_i \cdot \widetilde{r_i}$, where $R(x, \tilde{r})$ is the portfolio's uncertain performance at the end of the holding period.
- $\Omega = \{s | s = 1,2, \dots , T\}$ is the finite set of discrete scenarios.
- $Prob_s > 0$ is the probability under a particular scenario (s) take the returns $\tilde{r}$
- $E[R(x, \tilde{r})] = x^T \cdot \bar{r} = \sum_{i=1}^{n} x_i \cdot \overline{r_i}$, where $E[R(x, \tilde{r})]$ is the Expected performance of portfolio.
- $F(x, n) = Prob(R(x, \tilde{r}) \leq n)$, where F(x,n) is the market portfolio.

According to the above table:

- $R(x, r^1) = \sum_{i=1}^{n} x_i \cdot r_i^1$
- $E[R(x, r)] = \sum_{s=1}^{S_T} Prob(s) \cdot R(x, r^s)$

We use the function $\varphi(x, r)$ as a risk measure. The choice of the risk measure is related to the investor's profile and his preferences. Then for a certain minimal expected portfolio return μ, the φ-efficient portfolio is the result of the below problem solution:

$$\min \varphi(x, r)$$

$$\text{s.t. } E[R(x, r)] \geq \mu$$

Axiomatic investigation of risk measures was suggested by Artzner et al. [1]. Rockafellar [2]defined a function as a coherent risk measure in the extended sense if it meets the following axioms:

1. Sub-additivity: $\varphi(\tilde{x} + \tilde{y}) \leq \varphi(\tilde{x}) + \varphi(\tilde{y})$
2. Homogeneity: $\varphi(\lambda \cdot \tilde{x}) = \lambda\varphi(\tilde{x})$
3. Monotonicity: $\varphi(\tilde{x}) \leq \varphi(\tilde{y}) \Rightarrow \tilde{x} \ll \tilde{y}$ (we prefer $\tilde{y}$ than $\tilde{x}$)
4. Risk-free condition:$\left(\tilde{x} - n \cdot r_f\right) = \varphi(\tilde{x}) - n$, where r_f is the riskless rate.

A function is called a coherent risk measure if it satisfies axioms 1,2,3,4.

It's vital for the risk measure to satisfy the first axiom (sub-additivity) because it allows the diversification. Diversification is a risk management strategy that mixes a wide variety of investments within a portfolio. Thus, a diversified portfolio has a significant lower risk because of the limited exposure to any single asset or risk. Last but not least, sub-additivity and homogeneity ensure that the risk measure function is convex, which is consistent with the theory of risk aversion.

After setting the scenario generation, the next section defines the risk measures and their basic properties through their mathematical formulation:

3.1 Variance:

The Mean-Variance model, which was introduced by Markowitz in 1952 and he published the article *Portfolio selection* in Journal of Finance 1952 [3], laid the foundations of the contemporary risk management. The model presents two vital parameters with regard to portfolio optimization problems: Expect return and risk. Portfolios have to tackle the problem of how to allocate wealth among several assets. However, an investor always prefers to maximize the returns of the portfolio and concurrently to make the risk as small as possible. Nevertheless, a high return always accompanied with a higher risk. That problem is one of the important research fields in modern risk management. Markowitz applied variance (or standard deviation) as a measure of risk and the mathematical formulation of it is:

$$\sigma^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij}$$

In that case, in order to make the notation simpler we introduce *weights*, where w_i is the proportion of the portfolio held in assets i $(i = 1, \dots, N)$ and w_j is the proportion of the portfolio held in assets j $(j = 1, \dots, N)$. Also, σ_{ij} is the covariance between assets between i and j.

The demise of the risk in a portfolio implies a low variance(σ^2). Hence, the fundamental problem of a portfolio requires to minimize the variance with respect to a fixed expected return $\bar{r}$. This is known as the *Markowitz model* [4]

$$\min \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij}$$

$$\text{s.t.} \quad \textstyle\sum_{i=1}^{N} w_i \bar{r}_i = \bar{r}$$

$$\sum_{i=1}^{N} w_i = 1$$

Alternatively, we want to maximize the portfolio return given constraints and given the portfolio volatility:

$$max \sum_{i,j=1}^{N} w_i \bar{r}_i$$

$$\text{s.t.} \quad \textstyle\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij} = \sigma^2$$

$$\sum_{i=1}^{N} w_i = 1$$

The volatility of the portfolio is a highly non-linear function of the weights, whereas most of the other constraints are linear. Also, the portfolio return is linear in the weights. The volatility of the portfolio is a highly non-linear function of the weights, while the majority of the rest constraints are linear, included the portfolio return in the weights. By considering the mathematics of optimization we are able to select the formulation of the problem. Thus, we prefer to minimize risk (the standard deviation or the variance of the portfolio), instead of maximizing return given non-linear constraints (for the level of risk). We can explain that choice since it is easier to minimize a non-linear function with linear constraints than the alternative. Also, it is important to mention the one-to-one relation between the standard deviation and the variance of the assets. Because of that property we can minimize the variance and get the same results for the portfolio allocation as would get if we minimized the standard deviation. Furthermore, another reason we prefer to minimize the risk of the portfolio is that variance is a quadratic function and several convenient mathematical methods exist in order to optimize them. One of them is called the method of the *Langrangean multipliers*.

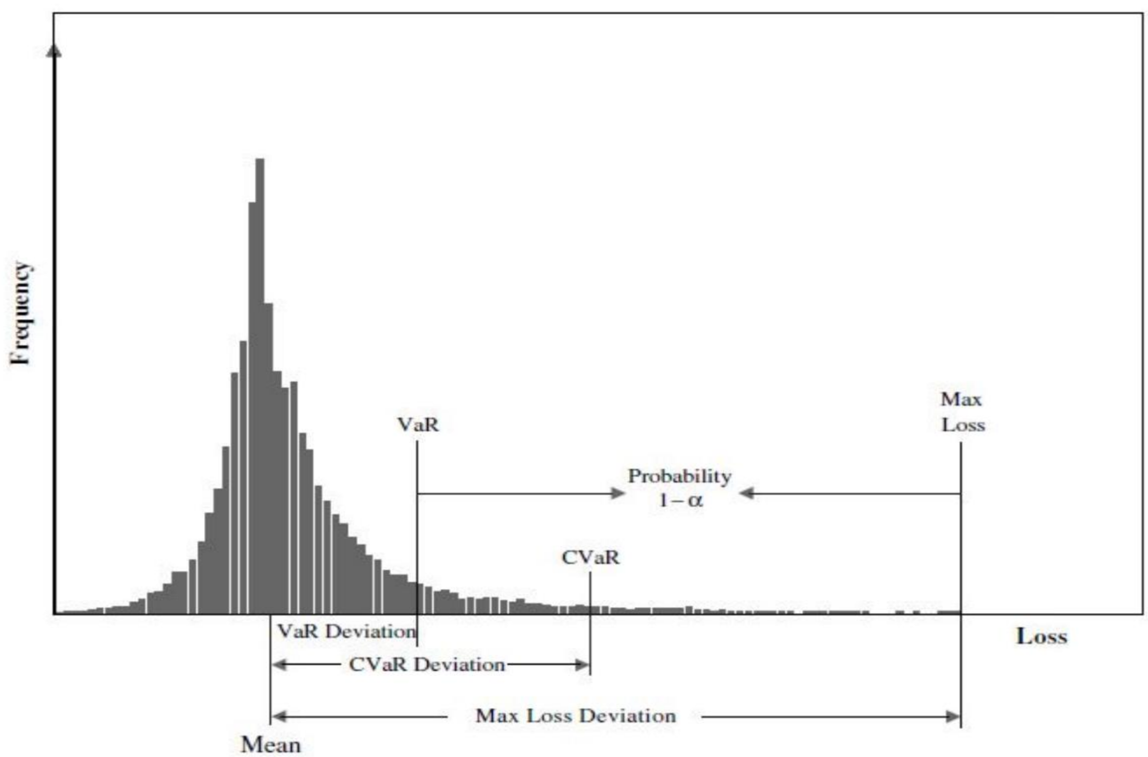
After presenting the portfolio optimization problem, using the variance of a portfolio as a risk measure, it is important to mention the theoretical problems of Markowitz's approach. Many researchers do not blindly depend on the Mean-Variance model of Markowitz as it is based on the assumptions that (for a risk averse investor):

- the distribution of the rate of return is multivariate normal.
- the utility function of is a quadratic function of the rate of return.

However, none of these assumptions applies in practice, leading the researchers and investors to develop a plethora of models in order to tackle the Mean-Variance model's defects. Thus, a lot of improved models from computational and theoretical view have been made. In the current thesis a plethora of these models are presented below.

3.2 Value at Risk (VaR):

Information about future return of a portfolio is unknown at the time of portfolio allocation, so every decision brings some risk. The Value at Risk is an important measure of the risk exposure of a given portfolio. It is now the main tool in financial industry and part of regulatory mechanism.



Let X be a random variable with the cumulative distribution function: $F_X(z) = P\{X \leq z\}$.X may have meaning of loss or gain, but in the particular situation X has meaning of loss.

The VaR of X with confidence level α ∈[0, 1] is:

$$VaR(x, a) = \min\{u|F(x, z) \geq 1 - a\} = \min\{u|P[R(x, \tilde{r}) \leq u] \geq 1 - a\}, \text{ where } a\text{=95\%(or 99\%)}$$

The VaR of a portfolio is defined as the maximum loss that will occur over a given period at a given probability level. Since we are interested in the most unfavorable outcomes, we would like to measure the level of losses which will not be exceeded with some fixed probability. The critical level typically used for this fixed probability is either 5% or 1%.

Historically, VaR was introduced by financial companies in the late 80's. Since then this approach has become popular among practitioners, the academic community and – most significantly – among central banks.

The greatest advantage of VaR is the recognition and the acceptance that has as a widely accepted standard. Also, it is a good basis for comparison and measurement of risk between portfolios and investments, since the industry and the regulators agreed to use it as the primary risk management measure. The two valuable factors of the success of the current model is its simplicity in use and the stability and the consistency of the results.

However, VaR risk measure has a grave disadvantage since it does not control scenarios exceeding VaR. More specifically, VaR does not take into account the extreme tails of the distribution that contain large losses with very low probability (less than$(1 - a) \cdot 100\%$. Thus, the indifference of VaR to extreme negative returns may be an undesirable property, allowing the investor to take extraordinary and undesirable risks with a very small chance. Although this indifference of VaR hides a risk for the investors, it may have a positive result in statistics as its estimators are robust and are not affected by excessive losses. Particularly, these huge losses may "confuse" the statistical estimation procedure and produce misleading results.

Furthermore, VaR model has another important drawback that affects seriously the estimated risk. Specifically, the model does not consider the property of sub-additivity which is a vital condition for a risk measure to be coherent. The lack of sub-additivity depicts that VaR of two different investment portfolios may be greater than the sum of the individual VaRs. As a result, the property of "diversification" is not satisfied, and that fact limits the use of VaR.

Last, VaR is relatively difficult to optimize due to be a non-convex discontinuous function and exhibits multiple local extrema. Although significant progress has been made in this direction, efficient algorithms for solving such functions are lacking.

In conclusion, VaR is not a coherent measure of risk due to the facts below:

- It does not take into consideration the extreme tails of the distribution that entail big losses.
- It does not urge and, indeed, sometimes prohibit diversification.
- It is difficult to optimize it.

In order to deal with these disadvantages and to avoid large errors in the final decision, an alternative risk measure was introduced.

3.3 Conditional Value at Risk (CVaR):

CVaR is a coherent risk measure that estimates the mean of the beyond-VaR tail region, which may be widely used in the optimization of the portfolio. VaR answers the question: what is the maximum loss with

the confidence level $\alpha \cdot 100\%$ over a given time horizon? Thus, its calculation reveals that the loss will exceed VaR with likelihood$(1-\alpha) \cdot 100\%$, but no information is provided on the amount of the excess loss, which may be significantly greater. By contrast, CVaR is considered a more consistent measure of risk than VaR. CVaR supplements the information provided by VaR and calculates the quantity of the excess loss.
Below, CVaR is defined as the conditional expectation of portfolio returns below VaR for continuous distributions, which measures the expected value of the $(1-a) \cdot 100\%$ lowest portfolio's returns:

$$CVaR = E[R(x, \tilde{r}) | R(x, \tilde{r}) \leq VaR]$$

However, the formula does not provide a coherent risk measure for discrete distributions as it gives a non-convex function in portfolio positions. Furthermore, another drawback in case there is a discrete distribution is the lack of a vital property that must be applied, sub-additive. In order to face these problems, we define CVaR model for general distributions with the formula:

$$CVaR = \left(1 - \frac{\sum_{\{s \in \Omega | R(x, r_s) \leq z\}} Prob_s}{1-a}\right) \cdot z + \frac{1}{1-a} \cdot \sum_{\{s \in \Omega | R(x, r_s) \leq z\}} Prob_s R(x, r_s),$$ where $z = VaR$

The priority of each investor is to minimize the losses of his investments. In order to achieve that, as VaR and CVaR tends more to the right part of the distribution than to the other direction where the bigger losses exist, the investment is more profitable and appealing to the investors. To solve this risk management problem:

$$\max z - \frac{1}{1-a} \sum_{i=1}^{s} Prob_s \cdot y_s^+$$

$$s.t. \quad R(x, r_s) = \sum_{i=1}^{n} x_i \cdot r_i^s$$

$$E[R(x, r_s)] = \sum_{s=1}^{S} Prob_s \cdot R(x, r_s) \geq \mu$$

$$y_s^+ \geq z - R(x, r_s)$$

$$y_s^+ \geq 0$$

$$\sum_{i=1}^{n} x_i = 1, x_i \geq 0 \ \forall \ i=(1, 2, \ldots n)$$

Where $z = VaR$ and $y_s^+$is an instrumental variable. Also, we define the expected portfolio return with μ.

At this point, it is crucial to define the instrumental variable which is a tool for the above optimization problem. Hence, for every scenario $s \in \Omega$:

$$y_s^+ = \max[0, z - R(x, r_s)]$$

This variable turns 0 when the return of the portfolio exceeds VaR for a particular scenario. In contrast, when the portfolio return is below VaR the variable equals to the difference between VaR and the realized performance.

Through the above maximization problem, the "CVaR-efficient frontier" is generated, which is the result of the CVaR's optimization. It applies on each expected return μ, related to each discrete scenario. Concurrently, apart from the "CVaR-efficient frontier" the VaR is calculated, providing a completed view from the whole tail of the returns distribution which depicts all the chances, even the lowest ones, to realize losses. At the end, the investor is informed about the allocation of their budget and the structure of their portfolio since the optimized portfolio is produced.

The use of CVaR, as risk measure, is common in a lot of industries due to its properties that turn it into a consistent and a coherent measure of risk. CVaR is a convex and continuous function, allowing the optimization to be done quite efficiently. Also, its popularity is based on its property to take into account and control scenarios exceeding VaR, on the extreme tails of the distribution. It allows the investors to protect from the undesired risk which can hide substantial losses.

The problem of choice between the two risk measures (VaR and CVaR), especially in financial risk management, has been quite popular in academic literature. The differences in mathematical properties, simplicity of optimization procedures and the accuracy in calculation and control the risk are the main factors that affect an investor's decision. VaR has achieved popularity as an appropriate risk measure in a plethora of engineering applications, including the financial sector. On the other hand, CVaR is heavily used in the insurance industry since the achievement of profits is related to the effective minimization of risk. However, there is no doubt that both are widely popular for risk measure so in the academic research as in the economy.

3.4 Mean Absolute Deviation (MAD)

The MAD model is a risk measure which is widely used to solve large-scale portfolio optimization problems. It was introduced by Konno and Yamazaki (1991) due to the standard Mean-Variance model (MV) of Markowitz couldn't deal with huge models. The innovative model uses the absolute-deviation of the rate of return of a portfolio instead of the variance to measure the risk. Although these two measures are almost the same from mathematical view, they have significant differences from computational view. Specifically, MAD model can be optimized by a linear program, whereas MV leads to a convex quadratic programming problem. The former can be solved distinctively faster than the latter and this is one of the main reasons we prefer MAD model.

The definition of MAD model describes risk as the mean absolute deviation of portfolio return from its expected value. As it is conspicuous from the mathematical formulation:

$$MAD(x) = E\{\, |E[R(x,\tilde{r})] - R(x,\tilde{r})|\}$$

When we refer to a discrete scenario set instead of a random asset's return:

$$MAD(x) = \sum_{s=1}^{S} Prob_s \cdot |E(R) - R_s|$$

We could define the MAD model as the average of the positive distances of each point from the mean. The larger the MAD, the greater variability there is in the data. In order to optimize MAD, we need to solve the following linear problem:

$$\min \sum_{s=1}^{S} Prob_s \cdot y_s$$

s.t. $\quad R(x,r_s) = \sum_{s=1}^{N} x_s \cdot r_s$

$$E[R(x,r_s)] = \sum_{s=1}^{N} Prob_s \cdot R(x,r_s) \geq \mu$$

$$\sum_{i=1}^{n} x_i = 1,\ x_i \geq 0 \forall\ \text{i=(1, 2,…,n)}$$

$$y_s \geq E[R(x,r)] - R(x,r_s)\ \forall\ s = 1,2,\dots,S$$

$$y_s \geq R(x,r_s) - E[R(x,r)]\ \forall\ s = 1,2,\dots,S$$

$$y_s \geq 0$$

Comparable to CVaR model, we introduce an instrumental variable to linearize the absolute value expression. By setting the above conditions, we ensure that y_s gathers the positive difference of the mean, either a bit is higher or lower than this. In case $y_s < 0$ then it equals 0. The notion we follow is the same as it was in CVaR model. By solving the linear problem, the MAD- efficient frontier is generated according to the different values of the expected return μ. For each expected return μ the investor manages the risk, since the output of MAD model is the allocation of an investor's budget. Thereby, we have achieved to solve large- scale portfolio optimization problems with a wider computational simplicity than using the MV model and to build the efficient portfolio for each μ, based on the MAD-efficient frontier. These models have been used in a plethora of portfolio optimization problems because of the groundbreaking method that was introduced in the 90's.

3.5 Tracking error models

In the tracking error models, we introduce a different concept for allocation of the capital in comparison to the previous ones. In this case, the professional money manager (institutional investor) has to structure the portfolio according to a pre-specified benchmark. In other terms, the goal of a tracking error portfolio model is to replicate, as close as possible, a given benchmark. Therefore, the portfolio returns must relate to the performance of the imitated target. The allocation decision problem is based on the difference between the manager's return and the benchmark return, the so-called tracking error. The tracking-error optimization problems could be solved by finding a portfolio with maximum expected performance relative to the benchmark for a given tracking error volatility. For the implementation, it is necessary to control the deviations between the portfolio returns and the target. To achieve it, we impose an infinite penalty for any deviations that are more than a user specified parameter $\epsilon R(x,\bar{r})$ below the target. The condition contains both the cases where deviations being within $\epsilon R(x,\bar{r})$ below or above the benchmark and greater than it. Thus, we do not take into consideration returns that do not comply with that condition. The next step is to set the mathematical formulation:

$$\max \sum_{s=1}^{S} p_s R(x,r_s)$$

$$\text{s.t.}\quad x \in X$$

$$R(x,r_s) \geq g_s - \epsilon R(x,\bar{r}),\quad s = 1,2,\dots,S$$

where $g_s$is the return of the benchmark for each scenario $s \in \Omega$.

It is conspicuous that the condition affects the downside risk which is defined with respect to the target. The target may be set equal to the return of an alternative investment opportunity or to a fixed value and thereby the target is the desired return we want the portfolio to achieve. Parameter ϵ is determined by the user and it is utilized to estimate the weights of each asset. In other words, the parameter is used to produce the optimal portfolio. For that reason, it is essential to set an appropriate ϵ that is suitable with the data of the problem. The goal is to find the parameter with which we will produce the optimal portfolio for each scenario. In order to structure a portfolio close to the target, it is conducive to set the parameter equal to a small number because as we can observe from the above condition a low ϵleads to a portfolio return closer to the selected return we want to achieve. In conclusion, the parameterϵ depicts that the smaller it gets, the closer our portfolio approaches the benchmark.

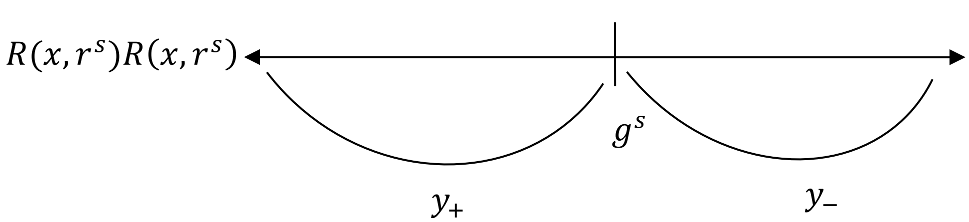
3.6 Put/Call Efficient Frontiers Model

The main notion of the model is to build up a portfolio that trades off the risk (downside) against the reward (upside) given a benchmark (target). It is known that the portfolio's reward is calculated through the expected return of the portfolio. The main goal of that model is to create a portfolio that performs better than the target for each scenario. In order to achieve that, it is crucial to measure both the portfolio's outperformance from the target and the underperformance from the target. The tools that will provide all the needed information are two auxiliary variables counting the positive and the negative deviations of the portfolio return from the benchmark. Thus, we introduce y_+ in order to measure the portfolio returns and y_- in order to captivate the risk:

$$y_+ = \max[\, R(x, \tilde{r}) - \tilde{g}, 0]$$

$$y_- = \max[\, \tilde{g} - R(x, \tilde{r}), 0]$$

These auxiliary variables have a crucial role as they distinguish the model from the other risk measures. As it is conspicuous from the name of the model, the payoffs of the upside potential on the future portfolio return are similar to a call option, having as a strike price the price of the target. In contrast, the payoffs of the downside payoffs on the future portfolio return are similar to those of a short position in a put option. In other terms, when the return of the portfolio is above the target, the gains are identical to the payoffs of an exercised call option. The investor gains the difference between the performance of the portfolio and the price of the benchmark. Obviously in that case, there is a zero- downside potential and the put option is not exercised. On the contrary, when the portfolio return is below the target, the downside payoffs are identical to those of a short position in a put option. Then, the investor gains the difference between the price of the benchmark and the low portfolio performance. Again, in that case, the call option cannot be exercised as the upside potential is zero. The following figure depicts the Put/Call Efficient Frontier Model:



Also, y_+ is used to measure the upside potential of the portfolio to outperform the benchmark, while $y_-$is used as a measurement of the downside risk of the portfolio. In discrete scenario setting we can express the definitions of the auxiliary variables as systems of inequalities:

$$y_+^s \geq R(x, r_s) - g_s, \quad y_+^s \geq 0, \quad \forall s \in \Omega$$

$$y_{-}^{s} \geq g_s - R(x, r_s), \quad y_{-}^{s} \geq 0, \quad \forall s \in \Omega$$

The goal of the model is to produce portfolios that achieve the higher call for a given put. Under this occasion, the investment is considered to have the greatest performance and the portfolio is called put/call efficient. By definition, a portfolio is efficient when it offers the highest expected return for a defined level of risk or the lowest risk for a given level of expected return. The mathematical formulation of the Put/Call Efficient Frontier Model in discrete scenario setting is represented with a linear program for tracing the efficient frontier.

$$\max \sum_{s=1}^{S} p_s y_{+}^{s}$$

s.t $\quad \sum_{s=1}^{S} Prob_s y_{-}^{s} \geq w$, w: risk target

$$y_{+}^{s} \geq 0 \ \ \forall s$$

$$y_{-}^{s} \geq 0 \ \ \forall s$$

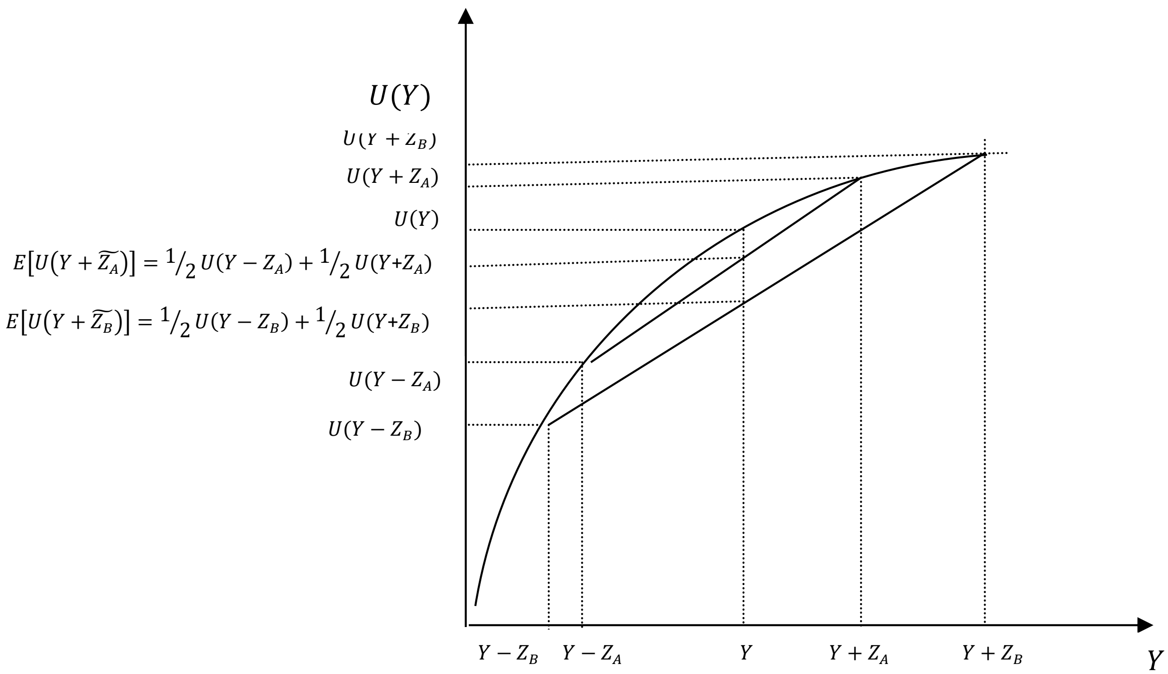
$$y_{+}^{s} - y_{-}^{s} - [R(x, r^s)] - g_s] = 0, \ \ \forall s \in \Omega$$

The output of the maximization of the model is the portfolio of which the returns are above the target, limiting at the same time the downside losses.

3.7 Expected Utility Maximization Model

The previous models optimized a pre-specified measure of risk against a pre-specified measure of reward, such as the expected return of the portfolio. However, the investor's choices and behavior have a vital role in portfolio selection and in setting a target return. In order to take into account such a need in portfolio optimization, we use the expected utility maximization model which allows the users to optimize according to their own criteria when trading of risk and rewards. The current model bases on the Expected Utility criterion which takes into consideration the risk preferences of an investor. It derives from the ability of a concave curve to depict the preferences of a risk-averse investor. According to that criterion, we can classify two mutually excluded investment plans because of their expected utility, since the criterion choose the one with the highest expected utility.

Furthermore, the expected utility criterion can classify investment plans that have different risk, in spite of having the same expected return. Although the same level of expected returns implies same utility for the risk-averse investor, it would be wrong if we were indifferent between these two choices. Expected utility gives a solution on that selection problem, as it considers the risk preferences of the investor.



The above figure illustrates two investment plans having the same expected returns:

$$E\left(Y+\widetilde{Z_A}\right) = \frac{1}{2}(Y-Z_A) + \frac{1}{2}(Y+Z_A) = E\left(Y+\widetilde{Z_B}\right) = \frac{1}{2}(Y-Z_B) + \frac{1}{2}(Y+Z_B) = Y$$

but also, different variance as plan B shows higher deviations than A does. However, both of them depict the same utility for the expected returns:

$$U\left[E\left(Y+\widetilde{Z_A}\right)\right] = U\left[E\left(Y+\widetilde{Z_B}\right)\right] = U(Y)$$

The conclusion is that the investor can distinguish which of the two investment plans appeals to his profile only by using the expected utility criterion.

From a mathematical view, the ability of the criterion to classify the investment plans with different risks and the same returns is attributed to the concavity of the utility function. Consequently, the expected utility has different prices for two plans that have the same expected performance. For a concave utility function, the higher expected utility of the plan with the lower deviation is consistent to the investing behavior of a risk-averse investor. Such a behavior is caused because of that plan includes a risk of lower loss in comparison to the alternative plan, since the returns of the latter have higher variance.

From the view of a portfolio, the expected utility criterion can be generalized for each scenario s, where $s \in \Omega$. Thus, the expected utility is defined as the expected average price of the utility levels for each scenario:

$$V[S_1, S_2, \ldots, S_N;\ Prob_1, Prob_2, \ldots, Prob_N] = \sum_{s=1}^{N} Prob_s \cdot U[R(x, r_s)]$$

While the models of the previous sections are specialized to produce frontiers of efficient portfolios –from which the user has to select one- the expected utility maximization model allows the user to choose a unique portfolio that maximizes the user's utility function. The maximization problem is presented as follows:

$$\max \sum_{s=1}^{S} Prob_s \cdot U[R(x, r_s)]$$

$$\text{s.t} \qquad x \in X$$

$$R(x, r_s) = \sum_{i=1}^{n} x_i r_i^s$$

$$\sum_{i=1}^{n} x_i = 1$$

4 International Diversification – Does it pay to go internationally?

In this part, we utilize the available literature and models to determine the benefits from investing overseas. Specifically, we investigate how the benefits of international portfolio diversification differ across the countries, taking into consideration a plethora of factors that could affect the result. Also, the analysis includes the short selling constraints in investments that are imposed in many developing countries to control capital flows. The outputs appear to be statistically significant and provide strong evidence about the advantages of international diversification.

The first research we are going to analyze comes from Joost Driessen and Luc Laeven [5] who carried out an investigation about how the benefits of international portfolio diversification differ across countries from the perspective of a local investor. Particularly, they investigate for a large cross-section of countries with both large and small stock markets, what benefits occur by adding international stock investment opportunities for a domestic investor that invests in local stocks only. They also measure the economic size of these benefits. Although the literature on international portfolio diversification takes a U.S. perspective, this paper takes the viewpoint of domestic investors in many different countries, since the benefits of investing abroad for investors in countries with less well-diversified and developed stock markets as those in the United States are likely to benefit more from investing abroad than U.S investors do. Also, the writers consider both the case of frictionless markets, where the investors don't face any costs to short assets, and the case where short selling constraints prevail especially in developing countries. Furthermore, the results are calculated both in local currencies since we consider a local investor that uses it, and in U.S. dollar as in practice, investors in many counties use it as their base currency.

The analysis bases on two types of diversification opportunities appeared for an investor that currently invests in the local equity index:

A. Regional diversification: A regional equity index for the country's region
B. Global diversification: MSCI indices for the US, Europe and Far East.

The reason for distinguishing between these two scenarios is that it is interesting to see how much of the international diversification benefits can be achieved by investing regionally and globally.

To measure the benefits of diversification for domestic investors, we use Markowitz's notion about the standard mean-variance, assuming either a mean-variance utility function for the investor, or normally distributed asset returns. The concept is to examine if an addition in portfolio's assets will shift the mean-variance frontier.

In order to measure the economic size of diversification benefits, we use the increase in Sharpe ratios and in expected return that is obtained when adding assets to the investment set. An alternative way to measure it is through the expected return. If the expected return is increased when adding to the portfolio the new N assets given the same (or lower) variance as for the optimal portfolio of K benchmark assets, then the diversification benefits are economic significant.

To calculate the statistical significance of the diversification possibilities, they use the regression tests for mean-variance spanning developed by Huberman, Gur and Kandel (1987) [6] , DeRoon, Nijman, and Werker (2001) [7], and DeRoon, Frans and Nijman (2001) [8]. In that test, it is examined whether adding N new assets to a given set of K benchmark assets leads to a significant shift in the mean-variance frontier. In the research, the domestic index is the K benchmark assets, so that K equals to 1, and the N additional assets are given by asset sets A or B (either regional or global diversification is implemented). In case of frictionless markets, the test can be performed using the regression:

$$r_{t+1} = a + \beta R_{t+1} + \varepsilon_{t+1}$$

Where r_{t+1} is a N-dimensional column vector with the N returns on the additional assets, R_{t+1} is a K-dimensional return vector for the K benchmark assets, a is a N-dimensional constant term, $\beta$is a N by K matrix with coefficients and $\varepsilon_{t+1}$is a N-dimensional vector with zero-expectation error terms.

The null hypothesis that the K benchmark assets span the entire market of all $K + N$ assets is equivalent to the restrictions:

$$a = 0 \; and \; \beta\iota_K = \iota_N$$

where ι_N is a N-dimensional vector with ones.

If the equations hold, the return on each asset that we add to the portfolio can be decomposed into the return on a portfolio of benchmark assets and a zero-expectation error term that is uncorrelated with the benchmark portfolio return. Hereby, from the view of mean-variance framework, such an additional asset can affect the variance of the portfolio return and not the expected return and the investor will not include the additional asset in his strategy. In other terms, if the spanning hypothesis holds, the optimal mean-variance portfolio consists of the K benchmark assets.

Also, it is vital to analyze how global diversification benefits change over the time. They use the country risk measure to interpret the correlations of local returns with global indices and the variances of local indices.

Finally, as far as the data concerns, for developed countries the writers use the stock market index developed by Morgan Stanley Capital International (MSCI) and for developing countries they use the S&P/IFC Global Index from Standard & Poor', on a monthly base for a period 1985 to 2002. The research includes 23 developed markets and 29 developing markets. Also, they download country risk ratings for each country s reported by the International Country Risk Guide maintained by the Political Risk Services group and they use the ICRG composite index as a general measure of country risk. The rating ranges from 0 to 100, with a higher rating implying less country risk. Furthermore, they download monthly currency data from Datastream in order to returns in U.S.D. terms and in terms of the local currency.

4.1 Empirical Results

First of all, they estimate both increases in Sharpe ratios and in expected returns given variance, when a regional index or three global MSCI indices are added to a local stock index portfolio, in order to measure the diversification benefits for the local investors of the 7 regions (Latin America, Asia, Eastern Europe, Middle East & Africa, Europe 16, Pacific, North America. The results for each region are divided into two types of diversification opportunities: Regional Diversification and Global Diversification. Also, the Sharpe ratios are calculated for each country both with and without short-selling constraints and in terms of local currency and U.S.D. returns. Short-selling constraints are either applied to all countries or only to developing countries. The following table reports the results summarized by averaging country-by-country results within each region.

	Latin America	Asia	Eastern Europe	Middle East & Africa	Europe 16	Pacific	North America
			Regional Diversification (Local Currency)				
SR: no frictions	0.039	0.014	0.095	0.047	0.035	0.015	0.050
SR: frictions, developing	0.038	0.010	0.084	0.025	0.035	0.001	0.050
SR: frictions, all	0.038	0.010	0.070	0.017	0.033	0.001	0.025
ER: no frictions	0.133	0.000	0.224	0.089	0.084	0.017	0.198
ER: frictions, developing	0.030	0.000	0.224	0.000	0.084	0.000	0.198
ER: frictions, all	0.030	0.000	0.224	0.000	0.000	0.000	0.000
			Regional Diversification (U.S. dollars)				
SR: no frictions	0.042	0.019	0.096	0.082	0.036	0.017	0.057
SR: frictions, developing	0.021	0.003	0.069	0.005	0.036	0.003	0.057
SR: frictions, all	0.014	0.003	0.041	0.001	0.034	0.003	0.023
ER: no frictions	0.223	0.000	0.291	0.099	0.092	0.027	0.259
ER: frictions, developing	0.000	0.000	0.291	0.000	0.092	0.000	0.259
ER: frictions, all	0.000	0.000	0.291	0.000	0.000	0.000	0.000
			Global Diversification (Local Currency)				
SR: no frictions	0.182	0.137	0.220	0.139	0.057	0.101	0.079
SR: frictions, developing	0.181	0.137	0.217	0.135	0.057	0.100	0.079
SR: frictions, all	0.141	0.115	0.195	0.107	0.037	0.087	0.059
ER: no frictions	1.116	0.981	1.703	0.749	0.340	0.609	0.349
ER: frictions, developing	1.037	0.852	1.623	0.715	0.340	0.503	0.349
ER: frictions, all	0.274	0.204	0.366	0.262	0.076	0.307	0.225
			Global Diversification (U.S. dollars)				
SR: no frictions	0.129	0.129	0.129	0.137	0.070	0.124	0.086
SR: frictions, developing	0.127	0.126	0.122	0.134	0.070	0.122	0.086
SR: frictions, all	0.089	0.106	0.071	0.091	0.054	0.105	0.059
ER: no frictions	1.476	1.190	1.855	0.991	0.423	0.780	0.423
ER: frictions, developing	1.442	1.178	1.840	0.964	0.423	0.765	0.423
ER: frictions, all	0.274	0.204	0.366	0.271	0.076	0.307	0.226

Table 1: Increase in Sharpe ratio and Expected return by region

It is conspicuous from the *Table 1* that the regional diversification benefits appear largest on average for countries in Eastern Europe, even when allowing for short selling constraints. For instance, the expected return for Eastern European countries of investing in the region is 0.291% per month (3.492% per year) higher than the return of the investment in the local market only, in U.S.D. terms, even when short selling constraints prevail.

For the majority of the investors of the seven regions, the global diversification benefits are significantly higher than the regional diversification benefits, according to both of Sharpe ratio and Expected returns, with the largest benefits appear for the countries in Eastern Europe. These figures suggest that the economic size of global diversification benefits is large for most countries, since the increases in Sharpe ratio and in expected returns implies economic significance of the diversification benefits.

In a plethora of developing countries, short-selling constraints are imposed due to the government restricts the investors to invest in their home country, which is potentially costly for these investors, since these restrictions reduce the diversification possibilities. Therefore, the writers use short-selling constraints in order to have more realistic results. However, it is obvious from the table that the results of global diversification benefits are not affected significantly when assuming short sales constraints in developing countries, due to for almost all countries it is optimal to invest a part of the total asset portfolio to the local stock market index. But when we assume short sails constraints for all countries, which is a less realistic assumption than the imposed constraints in developing countries, there is a substantial decrease in the benefits of global diversification for the majority of the countries, especially when diversification benefits are measured in terms of increasing in expected returns. Nevertheless, the global diversification benefits for most of the regions are substantial and statistically significant as they were without short sales constraints.

Finally, as far as the currency concerns, there isn't a significant difference in the results of diversification benefits when we interpret them either in U.S.D. terms or in the local currency terms.

The next step is to calculate the statistical significance of the results by using the regression tests for mean-variance spanning. *Table 2* depicts the results of the spanning tests on a regional basis. Specifically, it includes the percentages of cases in which the spanning hypothesis is rejected.

	S&P Global Latin America	S&P Global Asia	S&P Global Eastern Europe	S&P Global Middle East & Africa	MSCI Europe 16	MSCI Pacific	MSCI North America
Regional Diversification (Local Currency)							
Total number of countries	7	10	6	6	16	5	2
No frictions	100%	100%	83%	100%	100%	100%	50%
Frictions, developing	29%	0%	50%	50%	100%	0%	50%
Frictions, all	29%	0%	33%	17%	19%	0%	50%
Regional Diversification (U.S. $)							
Total number of countries	7	10	6	6	16	5	2
No frictions	100%	100%	83%	100%	100%	100%	50%
Frictions, developing	14%	0%	50%	0%	100%	0%	50%
Frictions, all	0%	0%	17%	0%	19%	0%	50%
Global Diversification (Local Currency)							
Total number of countries	7	10	6	6	16	5	2
No frictions	100%	100%	100%	100%	100%	100%	100%
Frictions, developing	100%	100%	100%	100%	100%	100%	100%
Frictions, all	100%	90%	100%	83%	31%	80%	50%
Global Diversification (U.S. $)							
Total number of countries	7	10	6	6	16	5	2
No frictions	100%	100%	100%	100%	100%	100%	100%
Frictions, developing	100%	100%	100%	100%	100%	100%	100%
Frictions, all	57%	90%	33%	67%	63%	80%	50%

Table 2: Rejection of Spanning Tests by Region

When the global diversification takes place, the spanning hypothesis is rejected for all countries at 1% confidence level, both when the short sales constraints are imposed to the investors in developing countries and when the investors do not face any constraints at all. That means the global diversification benefits are statistically significant for domestic investors around the world, due to tiny p-values that force us to reject the spanning hypothesis for all confidence levels.

However, when we introduce short-selling constraints for all countries, which is not a realistic scenario, the spanning hypothesis is not rejected for several countries in the regions, proving that the constraints disappear both the regional and global diversification benefits for a plethora of countries. Despite that fact, there are still many countries that gain benefits from diversification.

In the next section, the writers aim to interpret which factors affect the benefits of international diversification in the various countries. Particularly, they explore whether the country risk and other country characteristics have an effect on the gains from international diversification. In order to test it, they use a cross-country regression as a mean to recognize the factor that affects the benefits. They use as dependent variable the difference between the Sharpe ratio of the global portfolio (including the local index) and the Sharpe ratio of the local index so as to measure the economic benefits for a mean-variance investor who invests not only in the local but also in the global market. The higher figure this variable indicates the larger benefits the investor gains. The regression is the following:

$$ISR_{i,t} = a + bICRG_{i,t} + cSMC_{i,t} + dTrGDP_{i,y} + ePCGDP_{i,t} + \varepsilon_{i,t}$$

Where ISR is the Increase in Sharpe Ratio, a is a constant, $ICRG$ is the country risk rating, SMC is the Stock Market Capitalization in U.S.D, $TrGDP$ is the trade to GDP, $PCGDP$ is the private credit to GDP and ε is the error. The dependent variables are calculated using stock market return and interest rate data for the period 1985-2002.Generally, these variables proxy for the level of institutional development in the country. It is expected that more development countries benefit less from international portfolio diversification. The results of the regression are presented in *Table 3*:

	Panel A: Local currency returns		**Panel B: U.S. dollar returns**	
	Increase in Sharpe ratios	Increase in Sharpe ratios	Increase in Sharpe ratios	Increase in Sharpe ratios
Country risk rating	-0.468*** (0.100)	-0.390*** (0.115)	-0.196*** (0.060)	-0.191** (0.083)
Private credit to GDP	—	-3.007 (2.578)	—	0.145 (2.865)
Trade to GDP	—	-0.001 (1.446)	—	0.153 (1.427)
Market cap	—	-0.124 (0.333)	—	-0.392 (0.333)
R-squared	0.42	0.43	0.16	0.17
Number of observations	52	51	52	51

Table 3: Cross-country differences in the benefits of foreign diversification

In comparison to all dependent variables used in the estimation, they found only the Country risk rating to be statistically significant for the increase in the Sharpe ratio variable. Also, a vital output is that the international diversification benefits are larger for countries with higher country risk. Nothing special changes when we interpret the results in local currency terms than in U.S.D. terms and the short selling constraints do not affect the results. To control robustness, they excluded the region of East Asia and Russian due to the sample period covers both the Asian and Russian debt crisis of 1997 and the results did not change significantly.

Except for the paper of Driessen and Laeven, another research carried out by Wan-Jiun Paul Chiou [10] analyzes the benefits of international diversification by using two different measures in order to estimate them. Specifically, the writer utilizes two straightforward measures: the increase in risk-adjusted premium by investing in the maximum risk-adjusted return (MRR) portfolio and the reduction in the volatility by investing the minimum variance portfolio (MVP) on the international efficient frontiers. The mathematical formulation of the increment in risk-adjusted premium requires setting the maximum risk-adjusted return (MRR). Thus, if there is no investment restriction neither for the developing nor the developed markets:

$$MRR = \max\left\{\left(\frac{w_p^T\mu}{w_p^T V w_p}\right)^{1/2} \middle| w_p^T \in S\right\}$$

Where S is the set of all real vectors w that define the weights such that $w^T 1 = 1$. The mean and variance-covariance of asset returns can be expressed as a vector μ and a positive definite matrix V, respectively. From the above maximization problem, we expect to take the weights of the various assets in the portfolio. However, the investor takes into consideration various constraints before investing in a foreign market and the mathematical formulation can be obtained by the following:

$$MRR = \max\left\{\left(\frac{w_p^T \mu}{w_p^T V w_p}\right)^{1/2} \middle| w_p^T \in P_J\right\}$$

Where $P_J = \{w_p \in S : 0 \leq w_i \leq Jw(Cap)_i \leq 1, i = 1,2, \ldots, N\}$, J is any number greater than one and $w(Cap)_i$ is the weight of the market value of each country i. By setting the MRR, the greatest increment of unit-risk performance brought by international diversification, for the domestic investor, is:

$$\delta_i = 1 - \frac{RR_i}{MRR}$$

Where RR_i is the risk-adjusted performance of domestic portfolio in country i. The second measure used for the estimation of the international diversification benefits is the reduction in volatility. Elton and Gruber (1995) support that the estimation of the expected returns may hide a difficulty for the investors and for that reason they seek to minimize the variance of a portfolio. A common way to minimize the variance of a portfolio is by obtaining the vector of weights of the minimum-variance portfolio (MVP), w_{MVP}. According to the research of Li et al. (2003) [12], the maximum decline in volatility of domestic portfolio in country i, by international diversification is:

$$\varepsilon_i = 1 - \left[\frac{w_{MVP}^T V w_{MVP}}{V_i}\right]^2$$

Where $V_i$is the variance of return of the domestic portfolio. The data used in this study are the U.S. Dollar-denominated monthly returns of the MSCI indices for 21 developed countries and 13 developing countries from January 1988 to December 2004. The differences of the benefits of global diversification for the developed and developing countries with various investment constraints are reported below:

Country	RR	S.D.	w_i(MRRP)	w_i(MVP)	δ (%)	ε (%)
Panel A: Developed countries						
Australia	0.3809	0.1865	0.0000	0.1130	54.28	32.17
Austria	0.3471	0.2304	0.0000	0.0927	58.34	45.09
Belgium	0.4578	0.1813	0.0000	0.0883	45.05	30.22
Canada	0.4206	0.1764	0.0000	0.0000	49.52	28.35
Switzerland	0.6055	0.1735	0.2585	0.0580	27.32	27.13
Germany	0.3370	0.2266	0.0000	0.0000	59.55	44.19
Denmark	0.6147	0.1862	0.2410	0.0524	26.22	32.10
Spain	0.3396	0.2214	0.0000	0.0000	59.24	42.88
Finland	0.2696	0.3369	0.0000	0.0000	67.64	62.49
France	0.4701	0.1972	0.0000	0.0000	43.57	35.89
U.K.	0.3930	0.1588	0.0000	0.0236	52.82	20.43
Hong Kong	0.3311	0.2734	0.0000	0.0000	60.25	53.74
Ireland	0.4155	0.2006	0.0000	0.0000	50.13	37.00
Italy	0.2277	0.2356	0.0000	0.0000	72.66	46.32
Japan	−0.0529	0.2338	0.0000	0.0610	106.35	45.96
Netherlands	0.4708	0.1726	0.0000	0.0000	43.49	26.83
Norway	0.3731	0.2350	0.0000	0.0000	55.22	46.19
New Zealand	0.0836	0.2332	0.0000	0.0075	89.97	45.77
Singapore	0.2546	0.2511	0.0000	0.0000	69.44	49.67
Sweden	0.4514	0.2578	0.0000	0.0000	45.81	50.99
U.S.A.	0.6547	0.1438	0.1996	0.4287	21.41	12.12
Panel B: Emerging markets						
Argentina	0.2703	0.5340	0.0091	0.0000	67.56	76.33
Brazil	0.2314	0.5970	0.0000	0.0000	72.23	78.82
Chile	0.5335	0.2535	0.1592	0.0575	35.96	50.15
Greece	0.2876	0.3453	0.0000	0.0000	65.48	63.39
Indonesia	0.0979	0.5144	0.0000	0.0000	88.25	75.43
Korea, S.	0.1012	0.3917	0.0000	0.0000	87.85	67.73
Malaysia	0.1444	0.3216	0.0000	0.0000	82.67	60.70
Mexico	0.5660	0.3432	0.1325	0.0000	32.06	63.16
Philippines	0.0602	0.3372	0.0000	0.0000	92.77	62.51
Portugal	0.0889	0.2281	0.0000	0.0175	89.33	44.55
Thailand	0.0739	0.4197	0.0000	0.0000	91.13	69.89
Turkey	0.1124	0.6100	0.0000	0.0000	86.50	79.28
Taiwan	0.1241	0.3989	0.0000	0.0000	85.10	68.32

Table 4: Portfolio weights and benefit from global diversification

The significant cross-market differences in risk-adjusted premiums, implies that domestic investors in the countries of low performance can improve the mean-variance efficiency of their portfolios by optimally allocating funds to overseas assets. According to *Table 4*, the countries where the investors can gain the highest mean-variance efficiency by investing internationally are Japan, Philippines, Thailand, New Zealand, Portugal, Indonesia and Korea. Also, local investors can eliminate their portfolio's volatility by investing in strategies of MVP. The general conclusion is that investments from developing markets have substantial margins of benefits by investing internationally, in contrast to investors that come from developed countries.

According to the MRRP index, 6 countries only are proposed to invest in, whereas the MVP suggests 11 countries. Furthermore, the weights in small capital markets are excessive, while in the major financial markets are significantly low. Based on these two indices, an investor should invest 80% of fund in Argentina, Chile, Denmark, Mexico and Switzerland, which represent only 4% of global market value and 20% in the US markets. The results comply with the outcomes of Driessen and Laeven, since both of them depict that there are substantial benefits for the investors of developing countries by investing globally.

Finally, the writer sites in *Table 5* the unconstrained efficient frontiers composed by global portfolios, countries of different developmental stages and various regions.

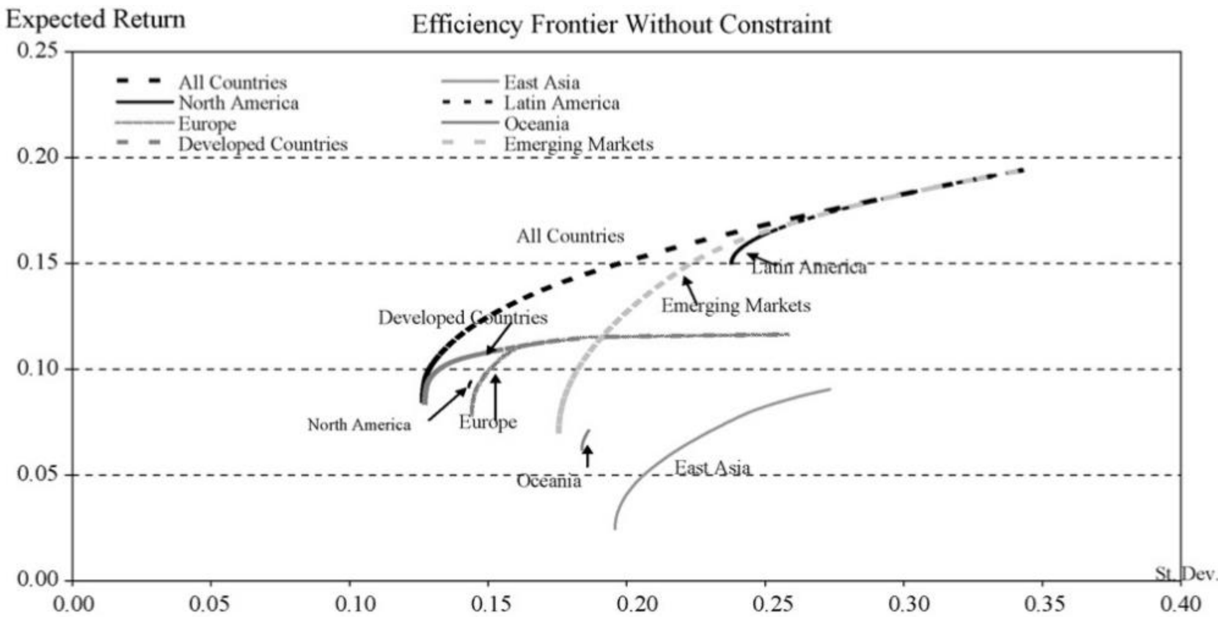


Table 5: Efficient Frontiers without constraints

The different positions of international portfolios composed by various groups of countries depict different impacts on mean-variance efficiency. Specifically, the efficient frontiers that derive from developed countries, Europe and North America are of low risk and low return, while the efficient frontiers of investments in emerging markets and Latin America are of high risk with high return.

In conclusion, the two researches provide useful information about the differed benefits of international portfolio diversification across countries from the perspective of a local investor. The first finding is that there are substantial benefits from investing both regional and global for domestic investors in both developed and developing countries, even under the realistic assumption that investors cannot short sell stocks in developing markets. However, the benefits of international portfolio diversification are larger for developing countries than for developed ones. Concerning the factors that play a crucial role in determination of the benefits, country risk is vital, since countries with higher country risk have a greater potential benefit of global diversification. The results are statistically significant. Another output of the investigation is the decreasing trend of diversification benefits across the countries, due to the improvements in country risk over time.

5 Empirical Application

In this section we implement a portfolio optimization problem of an investor who invests firstly in domestic market and secondly in international capital markets. The problem is applied on an investor based in Germany which means that the first part of the empirical application refers to the German market only, while in the second one two foreign markets are added. We picked the United States of America (U.S.A) and Japan as the two countries where the investor can invest in, except for his country, because of the different level of development of the their financial markets. The assets which the investor can select for his portfolio are 1 stock index and 2 bond indices from each country. Thus, in the case of the domestic portfolio optimization problem, the portfolio can be consisted of up to 3 assets, whereas the international portfolio up to 9 assets.

In order to calculate and evaluate the performance of both domestic and international portfolio during the selected past period, we implement a method called back testing. Back testing is a methodology for assessing the viability of a strategy by discovering how it would perform using historical data. Particularly, it displays the level in which a portfolio would have performed ex-post and as a result it provides strong evidence about the outcome of the investment when implemented in reality. The notion is that a portfolio that would have performed poorly in the past will probably perform poorly in the future, and vice versa. Thus, in our sample we implement back testing in order to assess the performance of the optimal portfolio during a long period of time and to come to a conclusion about the future results of that specific investment. However, before implementing the back testing, it is necessary to calculate the optimal weights of each asset that an investor should invest in. In order to achieve it, we run a CVaR model on GAMS system which solves the portfolio optimization problem.[1] Hence, by holding the optimal weights of the assets we can export the real returns of the portfolio that were performed in the past.

Apart from the back testing we construct the efficient frontier for the domestic and the international portfolio since it is a way to understand how a portfolio's expected returns vary considering the amount of risk taken. Specifically, the efficient frontier is the locus between the returns and the CVaR (risk) of all the combinations of the portfolios that offer the highest expected return for a defined level of risk. Those portfolios that lie below the efficient frontier are sub-optimal because they do not provide enough return for risk assumed. The efficient frontier is a commonly known path for the investors to select investments since it provides information for all the kinds of investors. According to efficient frontier, a risk-seeing investor would select portfolios that lie on the right end of the efficient frontier, since there are located portfolios that are expected to have a high degree of risk coupled with high potential returns. Contrarily, risk-averse investors will select portfolios that lie on the left end of the efficient frontier.

Last, it is vital to interpret the factors that affect the asset's optimal weighs coming from the CVaR's model solution, so as to understand the results from a financial point of view. The two main reasons in international portfolio which contribute to the optimal level of each asset are the descriptive statistics of the indices' returns and the exchange risk since the investor based in Germany has to deal with assets that are exposed to U.S. Dollar and Yen. Apparently, in the case of domestic portfolio the exchange risk does not exist, due to the fact that the investments share joint currency.

[1] The GAMS (General Algebraic Modeling System) is a high-level modeling system for mathematical programming. It can formulate complex and large-scale optimization models in a notation similar to their algebraic notation. Gams is designed to solve linear, non-linear and mixed-integer optimization problems and thus it can use many different solvers.

5.1 Data Description

As referred, we are considering 3 assets from each country (1 stock index and 2 bond indices). Particularly, as far as Germany concerns, we focus on the DAX 30 index, on FTSE German Government Bond Index 1-3 year sector and on FTSE German Government Bond Index 7-10 year sector. Furthermore, regarding to U.S.'s financial markets, we examine S&P 500 index, the US FTSE World Government Bond Index (WGBI) US, 1-3Y and the US FTSE WGBI US, 7-10Y. Last, in Japan, we take into consideration the NIKKEI 225 Stock index, the FTSE Japanese Government Bond Index 1-3 year sector and the FTSE Japanese Government Bond Index 7-10 year sector. We have collected historical prices for all the indices starting from 31/12/2004 until 30/4/2020, on a monthly basis. Thus, we were able to compute the returns for each asset through all the months of our period. Also, due to the investments in foreign currencies, we have collected data regarding to exchange rates, specifically the exchange rate for U.S. Dollar (U.S.D.) to Euro and the rate for Yen to Euro. Again, the exchange rates are available on a monthly basis, the same periods as the 9 assets were.

5.2 Methodology

In this part is made an analytical description of the steps that were followed using the GAMS model. First of all, we selected the CVaR mathematical model in order to solve the optimization problem. CVaR is a very popular coherent risk measure for optimal portfolio selection, because it goes beyond the information revealed by VaR and thus it takes into account the distribution of the tail, where the losses can be huge there.

5.2.1 Domestic Portfolio

Efficient Frontier: As referred previously, we need to find the locus between the returns and the standard deviation (risk) of all the combinations of the portfolios that offer the highest expected return for a defined level of risk. The data which we import in the GAMS model are returns of the 3 German assets for the last 10 years (120 observations). In the first step, we find the bottom end of the efficient frontier's curve, where the risk is at its lowest level as it relates to return. In order to calculate that point in GAMS model, we maximize CVaR model. Thus, we get the expected return of the portfolio which lies on the left end of the efficient frontier. In the second step, we export the upper limit of the efficient frontier which lies on the right tail, by solving the problem of maximization of total returns on GAMS model. After these steps, we hold the two extreme points on the efficient frontier's curve which display the minimum and the maximum expected returns. In order to determine the rest points, we separate the gap between the extreme points in 8 equal spaces, getting 8 different expected returns. Then, we run the CVaR model 8 times, by assuming every time one of the 8 expected returns that we calculated previously, as the expected portfolio's target return. Therefore, we locate the middle points and the efficient frontier is completed.

Back testing: In order to implement this dynamic test, for each specific month we run the model, we have to adjust the historic data to the optimization problem. The period of implementation starts from April 2017 until April 2020, which means that we solve the optimization problem 37 times (37 months). For every month we run the model, we insert a fixed number of historical data which is 147, since we got data for the returns of the assets for 147 months before April 2017 (the beginning of the test). Thus, the sample of historical data is determined as the last 147 observations prior to the month we examine. For example, when solving the optimization problem for June 2018, the inserted returns in the model are from March 2006 until May 2018. After solving the portfolio optimization problem through GAMS model, we get the optimal weights for the 3 assets that an investor should invest in, monthly. In order to evaluate the performance of the investment for all the months of the test, we multiply the optimal weights with the realized returns in the specific month we examine. We repeat that procedure for all the tested months and in the end, we have created the monthly gains or losses of the portfolio and by summarizing them we get the final performance of the portfolio. Therefore, we are able to evaluate the return of a portfolio consisted of 3 German assets the last 3 years, utilizing the CVaR model as a mean of selecting the optimal portfolio.

5.2.2 International Portfolio

Efficient frontier: We implement the same methodology as we did in domestic portfolio, including however the exchange rate of the foreign currencies. In order an investor from Germany to invest in American and Japanese capital markets, it is necessary to convert euro in the currency that each asset can be traded in. Thus, we need to find a mechanism for changing one currency into another, to be able to invest in foreign assets. Our aim is to detect the return of each asset for all the months in the sample. The notion is the following:

In order an investor from Germany to invest 1€ in a foreign country (USD or Yen), he has to convert it in the specific currency. If the exchange rate is USD to Euro (or Yen), the type to convert 1€ in USD (or in Yen) is:

$$1 * e_c$$

e_c: is the exchange rate in country c

Also, the performance of the asset i in scenario n in 1 month is:

$$1 * e_c * (1 + R_{i,c}^n)$$

where $R_{i,c}^n$ is the return of asset i in scenario n.

In order to convert the performance in € (domestic currency):

$$\frac{1 * e_c * (1 + R_{i,c}^n)}{e_c^n}$$

where e_c^n is the exchange rate in scenario n.

In conclusion, the performance of the asset i is:

$$\frac{e_c * (1 + R_{i,c}^n)}{e_c^n} - 1$$

However, this type is applied on our test since the exchange rates are USD to Euro and Yen to Euro. In case the rates were expressed vice versa (Euro to Yen and Euro to USD), the performance of the asset i is:

$$\frac{e_c^n * \left(1 + R_{i,c}^n\right)}{e_c} - 1$$

By utilizing this type, we are able to create the returns for all the foreign assets on a monthly basis. Then, we follow the same procedure as we did in domestic portfolio. The only different point is that the portfolio is consisted of 9 assets, instead of 3 as it was in domestic portfolio. Hence, through the steps we implemented in domestic portfolio related to maximization of CVaR model in GAMS, we draw the efficient frontier of the international portfolio.

Back Testing: We repeat the methodology as we did in domestic portfolio by adding the 6 foreign assets. In order to take into account the exchange rates that affect the returns of the investment in foreign assets, we have to implement the steps that we did previously, so as to export the returns gained by an investor from Germany. Hence, the notion of the back testing remains the same and we examine the performance of the international portfolio, according to the returns performed in the past.

5.3 Empirical results

We present below the results of the efficient frontier and back testing for the domestic and the international portfolio.

5.3.1 Efficient Frontier

Domestic Efficient Frontier		
	CVaR	Expected Returns
Bottom end	-0,00814	0,01897
m=0,001105	-0,00815	0,01899
m=0,00214	-0,00816	0,01901
m=0,003175	-0,00819	0,01903
m=0,00421	-0,00822	0,01905
m=0,005245	-0,00824	0,01907
m=0,00628	-0,00827	0,01909
m=0,007315	-0,0083	0,01911
Top end	-0,03	0,01913

Table 6: Domestic Efficient Frontier

International Efficient Frontier		
	CVaR	Expected Returns
Bottom end	-0,00425	0,00007
m=0,001105	-0,00599	0,00111
m=0,00214	-0,00922	0,00214
m=0,003175	-0,01317	0,00318
m=0,00421	-0,01725	0,00421
m=0,005245	-0,02532	0,00525
m=0,00628	-0,04367	0,00628
m=0,007315	-0,0648	0,00732
Top end	-0,125	0,00835

Table 7: International Efficient Frontier

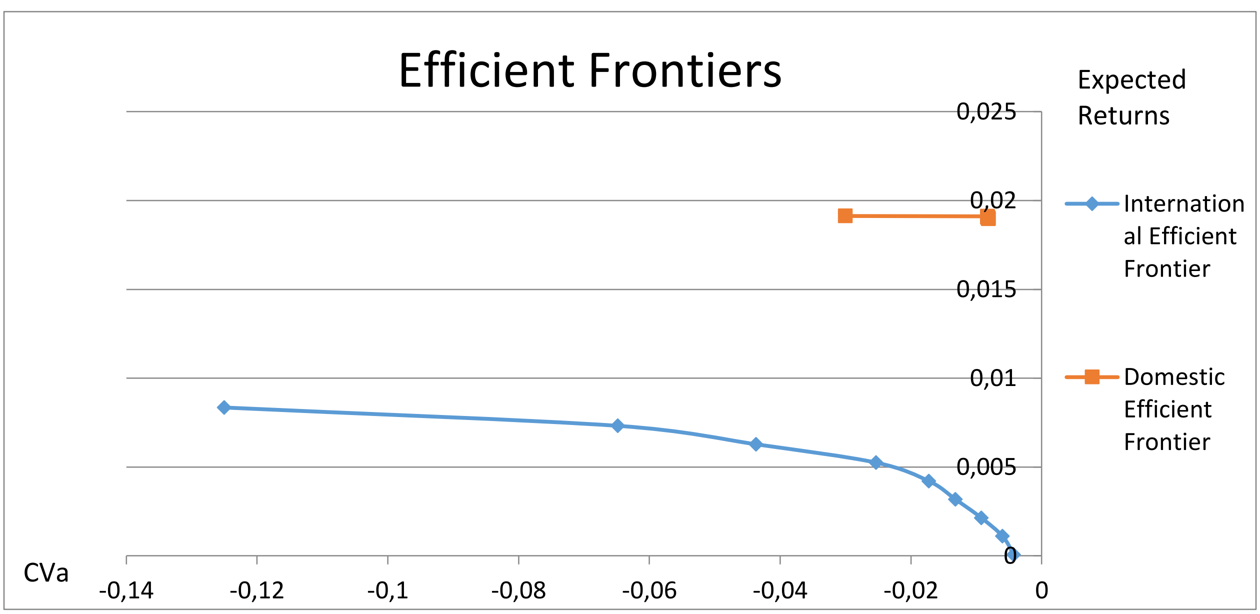


Figure 1: Efficient Frontiers

In *Table 6* and *Table 7* are the results that we received from the GAMS model, after solving the optimization problem using CVaR as risk measure. These results are represented in *Figure 1* and by connecting all the points we get the efficient frontier for each portfolio.

It is obvious from the graph that Domestic portfolio does not have significant volatility in expected returns, since in the bottom end of the efficient frontier it is 0,01897 while in the top end it is 0,01913. The risk is computed -0,00814 and -0,03 respectively, which shows that the German investor does not face a substantial risk by investing in domestic portfolio. Furthermore, the efficient frontier of the international portfolio displays that the maximum return an investor from Germany can gain is 0,00835 whereas the minimum return is 0,00007. Also, the minimum price of CVaR (risk) is -0.125 and the maximum is -0.00425.

Comparing the expected returns and the CVaR of the portfolios, we can observe that there is a positive correlation between expected returns and CVaR. Specifically, in domestic portfolio there is higher CVaR than in international one, due to the higher expected returns. The interpretation is attributed to the definition of CVaR. Since CVaR quantifies the expected losses that occur beyond the VaR breakpoint, it is more likely to be higher in a portfolio with larger expected returns. On the contrary, in international portfolio the potential returns are low for a German investor and therefore they are associated with low levels of extreme losses.

5.3.2 Back Testing

Domestic Portfolio					
	Portfolio's returns	Cumulative portfolio's returns	Optimal Weight S1	Optimal Weight S2	Optimal Weight S3
Months	-0,000293672	1	0,01342	0,98658	0
April '17	-0,000381804	0,999706328	0,01342	0,98658	0
May '17	-0,003693683	0,999324635	0,01342	0,98658	0
June '17	0,00096082	0,995633447	0,01342	0,98658	0
July '17	0,000610809	0,996590071	0,01342	0,98658	0
Aug. '17	-0,00022372	0,997198797	0,01342	0,98658	0
Sept. '17	0,000868697	0,996975704	0,01342	0,98658	0
Oct. '17	-0,001730693	0,997841774	0,01342	0,98658	0
Nov. '17	-0,001634322	0,996114816	0,01342	0,98658	0
Dec. '17	-0,002161012	0,994486843	0,01342	0,98658	0
Jan. '18	-3,90672E-05	0,992337745	0,01342	0,98658	0
Feb. '18	-0,000111754	0,992298977	0,01342	0,98658	0
Mar. '18	-0,000309253	0,992188084	0,01342	0,98658	0
April '18	0,001752117	0,991881247	0,01342	0,98658	0
May '18	-0,001120883	0,993619139	0,01342	0,98658	0
June '18	-0,001354472	0,992505408	0,01342	0,98658	0
July '18	-0,000283257	0,991161087	0,01342	0,98658	0
Aug. '18	-0,00176035	0,990880334	0,01342	0,98658	0
Sept. '18	0,000521172	0,989136038	0,01342	0,98658	0
Oct. '18	-0,000827312	0,989651548	0,01342	0,98658	0
Nov. '18	-0,001137356	0,988832798	0,01449	0,98551	0
Dec. '18	-0,000723619	0,987708143	0,01449	0,98551	0
Jan. '19	-0,00065075	0,986993419	0,01449	0,98551	0
Feb. '19	0,00104477	0,986351133	0,01449	0,98551	0
Mar. '19	0,000114	0,987381643	0,01449	0,98551	0
April '19	0,000441616	0,987494204	0,01449	0,98551	0
May '19	0,001436525	0,987930297	0,01449	0,98551	0
June '19	-0,000212691	0,989349484	0,01449	0,98551	0
July '19	0,001543428	0,989139059	0,01449	0,98551	0
Aug. '19	-0,00304921	0,990665724	0,01449	0,98551	0
Sept. '19	-0,001696144	0,987644976	0,01449	0,98551	0
Oct. '19	-0,000892558	0,985969788	0,01449	0,98551	0
Nov. '19	-0,001240872	0,985089753	0,01449	0,98551	0
Dec. '19	0,000417119	0,983867382	0,01449	0,98551	0
Jan. '20	-7,67644E-05	0,984277772	0,01449	0,98551	0
Feb. '20	-0,004440918	0,984202214	0,01449	0,98551	0
Mar. '20	0,001023008	0,979831453	0,01351	0,98649	0
April '20	-0,000293672	0,980833828	0,01342	0,98658	0

Table 8: Domestic portfolio - back testing

International Portfolio											
Months	Portfolio's returns	Cumulative portfolio's returns	Opt. W. S1	Opt. W. S2	Opt. W. S3	Opt. W. S4	Opt. W. S5	Opt. W. S6	Opt. W. S7	Opt. W. S8	Opt. W. S9
Apr. '17	-0,00029	0,99970633	0,01342	0,98658	0	0	0	0	0	0	0
May '17	-0,00038	0,99932464	0,01342	0,98658	0	0	0	0	0	0	0
June '17	-0,00369	0,99563345	0,01342	0,98658	0	0	0	0	0	0	0
July '17	0,00096	0,99659378	0,01305	0,9868	0	0	0	0	0,00015	0	0
Aug. '17	0,00064	0,99723358	0,01128	0,98769	0	0	0	0	0,00103	0	0
Sep. '17	-0,00002	0,99721284	0,00799	0,98569	0	0	0	0	0,00632	0	0
Oct. '17	0,00083	0,99803731	0,01127	0,9877	0	0	0	0	0,00102	0	0
Nov. '17	-0,00172	0,99632185	0,00794	0,98578	0	0	0	0	0,00628	0	0
Dec. '17	-0,00153	0,99480088	0,01021	0,98185	0	0	0	0	0,00794	0	0
Jan. '18	-0,00223	0,99258248	0,01024	0,98188	0	0	0	0	0,00788		
Feb. '18	-0,00002	0,99256195	0,01301	0,98683	0	0	0	0	0,00016	0	0
Mar '18	-0,00003	0,99253311	0,01143	0,98779	0	0	0	0	0,00078	0	0
Apr. '18	-0,00037	0,99216487	0,01128	0,98771	0	0	0	0	0,00101	0	0
May '18	0,00175	0,99390305	0,01351	0,98649	0	0	0	0	0	0	0
June '18	-0,00112	0,99278695	0,01351	0,98649	0	0	0	0	0	0	0
July '18	-0,00141	0,99138377	0,01128	0,98769	0	0	0	0	0,00103	0	0
Aug. '18	-0,00029	0,99109986	0,01351	0,98649	0	0	0	0	0	0	0
Sep. '18	-0,00181	0,98931035	0,01129	0,9877	0	0	0	0	0,00102	0	0
Oct. '18	0,00052	0,98982001	0,01351	0,98649	0	0	0	0	0	0	0
Nov. '18	-0,00083	0,98899969	0,01351	0,98649	0	0	0	0	0	0	0
Dec. '18	-0,00108	0,98793474	0,01351	0,98649	0	0	0	0	0	0	0
Jan. '19	-0,00078	0,98716200	0,01351	0,98649	0	0	0	0	0	0	0
Feb. '19	-0,00068	0,98648887	0,01351	0,98649	0	0	0	0	0	0	0
Mar '19	0,00104	0,98751966	0,01351	0,98649	0	0	0	0	0	0	0
Apr. '19	0,00004	0,98756265	0,01351	0,98649	0	0	0	0	0	0	0
May '19	0,00049	0,98804832	0,01351	0,98649	0	0	0	0	0	0	0
June '19	0,00138	0,98941278	0,01351	0,98649	0	0	0	0	0	0	0
July '19	-0,00020	0,98921878	0,01351	0,98649	0	0	0	0	0	0	0
Aug. '19	0,00157	0,99076724	0,01351	0,98649	0	0	0	0	0	0	0
Sep. '19	-0,00309	0,98770285	0,01351	0,98649	0	0	0	0	0	0	0
Oct. '19	-0,00170	0,98602385	0,01439	0,98561	0	0	0	0	0	0	0
Nov. '19	-0,00090	0,98514081	0,01439	0,98561	0	0	0	0	0	0	0
Dec. '19	-0,00124	0,98391815	0,01439	0,98561	0	0	0	0	0	0	0
Jan. '20	0,00042	0,98433062	0,01439	0,98561	0	0	0	0	0	0	0
Feb. '20	-0,00007	0,98426345	0,01439	0,98561	0	0	0	0	0	0	0
Mar '20	-0,00442	0,97990839	0,01439	0,98561	0	0	0	0	0	0	0
Apr. '20	0,00087	0,98076346	0,0119	0,9881	0	0	0	0	0	0	0

Table 9: International portfolio – back testing

In *Table 8* and *Table 9*, we present the optimal weights of all the assets where the investor should invest in, after solving the optimization problem in GAMS model for each month of our sample. As we referred in methodology section, we maximize CVaR model in each month by taking into account the historical returns of 147 months prior to the examined month. Thus, the optimal weights that we hold for each month, are based on the unique sample of historical data that we entry in GAMS model in order to solve the optimization problem and indicate the optimal asset allocation for the months we examine. Since 37 months are contained in our sample, we will repeat the procedure of the optimization 37 times. For the sake of brevity, we illustrate the DAX 30 index, the FTSE German Government Bond Index 1-3 year sector and the FTSE German Government Bond Index 7-10 year sector as S1, S2, S3 respectively, the S&P 500 index, the US FTSE World Government Bond Index (WGBI) US 1-3Y and the US FTSE WGBI US 7-10Y as S4, S5, S6 and finally, the NIKKEI 225 Stock index, the FTSE Japanese Government Bond Index 1-3 year sector and the FTSE Japanese Government Bond Index 7-10 year sector as S7, S8, S9.

As far as the domestic portfolio concerns, generally, the outputs of GAMS model display that it is optimal for an investor from Germany to invest the majority of his funds in German Government Bond Index 1-3 year sector (almost 98%), while only a small amount in DAX 30 index (almost 2%). We remind that the weights illustrate the best investment solution for a German investor who holds a domestic portfolio during a specific month of the sample. According to the produced weights, we have computed the portfolio's returns by multiplying the optimal weights with the historical returns of the assets. After that, we exposure the cumulative returns that are represented graphically in *Figure 2 (blue line).*

As far as the international portfolio concerns, according to the solution of the optimization problem, it is optimal for the investor to place the largest part of the portfolio in German Government Bond Index 1-3 year sector (almost 98%). Also, a tiny part of the portfolio (1%) should include DAX 30 index, while for specific months of our sample it is optimal to invest in NIKKEI 225 Stock index a meager part of the portfolio. In *Figure 2* (red line), we illustrate the cumulative returns of the international portfolio.

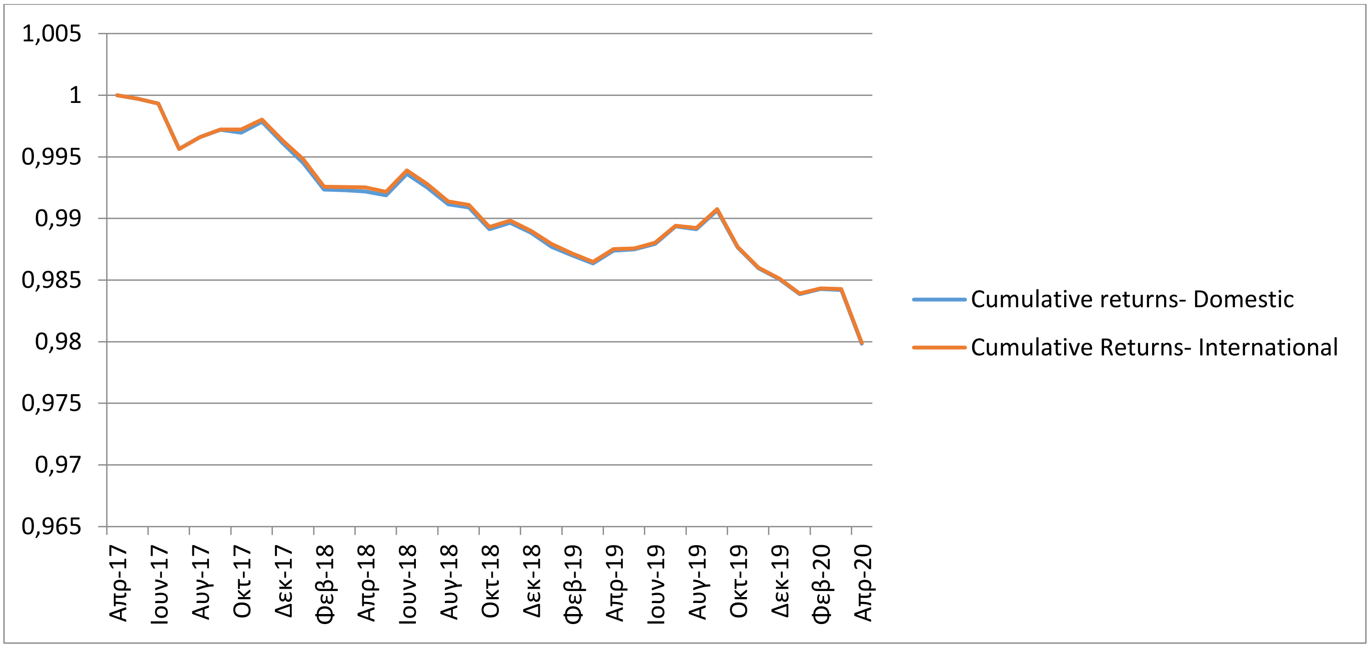


Figure 2: Domestic and International cumulative returns

Generally, according to the period and the assets of our sample, we observe that it is not profitable for an investor from Germany to allocate his funds in international assets (except for a tiny percentage in NIKKEI 225 Stock index). Due to that, domestic and international portfolios contain similar percentages of the same assets. Thus, the two curves (cumulative returns) depict that the performance of domestic and international portfolios will be in the same way.

Indeed, the performances of domestic and international portfolios are moving close, as we can observe from *Figure 2,* where the curves of the cumulative returns are almost coincided. Particularly, there is a significant negative performance for the portfolios which means that an investor from Germany would face losses if he had invested in the specific portfolios during April ’17 – April ’20. The size of the loss for the domestic and international portfolio would be 0.0192 (1.92%) and 0.0193 (1.93%) respectively. Comparing the performances, we conclude that is preferable for the German investor to invest in domestic portfolio, since it offers slightly fewer losses than the international.

5.4 Analysis

In this section, we will interpret the empirical results from a financial and statistical point of view. Our aim is to identify the factors that affect an investment decision, determining the asset allocation and therefore the portfolio’s performance. The analysis will be based on two pillars:

1. Descriptive statistics of returns
2. Exchange rates

5.4.1 Descriptive statistics of returns

	S1	S2	S3	S4	S5	S6	S7	S8	S9
Mean	0,00649	0,00119	0,00394	0,00567	0,00188	0,00449	0,00462	0,00024	0,00179
Standard Deviation	0,05217	0,00325	0,01378	0,04210	0,00360	0,01769	0,05500	0,00109	0,00726
Kurtosis	1,73523	4,76238	0,02216	2,01667	3,40740	1,73732	1,48384	4,47047	1,99475
Skewness	-0,63496	1,53482	0,33669	-0,72796	1,14670	0,46193	-0,70291	0,72307	-0,46245

Table 10: Descriptive statistics of monthly returns

In *Table 10* the statistical characteristics of monthly returns of the 9 assets are displayed. Focused on German Government Bond Index 1-3 year sector (S2), which is the most preferable asset for an investor from Germany in both domestic and international portfolios, we obtain several information about the features of the specific asset. More precisely, a vital factor that determines the strategy is the investment risk. One of the most common ways of measuring the risk an investment poses is standard deviation. By definition, standard deviation displays the spread of asset prices from their average price, helping us to project the future performance of each asset. The standard deviation of S2 asset is one of the second lowest among all the selected assets (and the lowest between the German assets) which implies that it is a credible and safe investment choice. Thus, an investment based on S2 makes sense because of the extremely low levels of risk.

Another tool that we utilize in order to assess an investment in a specific asset is the analysis of the return’s distribution. This analysis is performed using two statistical measures: skewness and kurtosis. Skewness measures the lack of symmetry in the data and it is a commonly used tool for the investors because it considers the extremes of the data set rather than focusing solely on the average. Skewness for a normal distribution and any symmetric data is 0. Negative values for the skewness indicate data that are skewed left (the left tail is longer or fatter), while positive values indicate data that are skewed right (the right tail is longer or fatter). Short-term and medium-term investors look at the extreme tails of the distribution, since they are less likely to hold a position long enough to be confident that the average return will prevail. Among the 9 assets, S2 is the one that it has the largest skewness which means that it is the asset with the largest concentration of the return distribution on the positive side. Positive distribution is desirable by the investors since there is some probability to gain huge profits that can offset all the frequent small losses. The high positive skewness explains in a big scale the attractiveness of German Government Bond Index 1-3 year sector (S2) for a German investor.

Apart from skewness, we use kurtosis to describe the distribution of returns. Whereas skewness differentiates extreme values in one versus the other tail, kurtosis measures extreme values in either tale. Kurtosis for a normal distribution is 3. Distributions with low kurtosis exhibit returns that are generally less extreme than the tails of the normal distribution, while in the case of high kurtosis the opposite occurs. According to *Table 5*, kurtosis is appeared to be higher than 3, which means that the investor will experience occasional extreme returns (either positive or negative).

As far as DAX 30 index (S1) concerns, we analyze the 4 moments so as to determine the reason of selecting to invest in that asset a tiny part of the whole portfolio (in both domestic and international portfolio). The fact that it has the highest mean among the assets reveals the high returns that the investor could obtain. Apparently, high returns are combined by high risk and therefore, a significant high standard deviation takes place. Hence, by investing a small percentage in a risky asset, the portfolio does not have a risk of losing significantly its value.

5.4.2 Exchange rates

The fluctuations of the exchange rates have a crucial role in the selection of the international portfolio. Many times, a change in exchange rates can encourage or discourage an investor to invest in foreign asset. In methodology section we analyzed the mechanism we utilize to convert the returns in € which is the domestic currency of the investor. The notion of how exchange rates affect the performance of the investment is simple: when the foreign currency is devaluated, the domestic investor has a benefit to invest, since 1 unit of the domestic currency corresponds to more units of the foreign currency. Thus, the domestic currency is becoming stronger and with 1 unit of it, the investor can buy more foreign assets. The opposite occurs when the foreign currency is revaluated, where 1 unit of domestic currency corresponds to less units of foreign currency. Then, the investor does not have the motivation to invest in that country, because he will lose a part of the investment's value, due to the weakness of his currency.

The revaluation of U.S.D against Euro is a crucial factor in the decision of a German investor not to invest in the 3 American assets. The revaluation reached the level of almost 20% from the beginning until the end of the examined period. Particularly, in December '04 the exchange rate (U.S.D. to Euro) was 1.359, while in April '20 was 1.095. The revaluation is computed about 20%, having an upward trend during the period, which is a daunting sign for a European investor. In conclusion, the constant revaluation of the U.S.D. against Euro has discouraged the investor from placing his funds in American assets.

Similar situation prevails in Japanese assets, because of the revaluation of Yen against Euro. In that case, an impressing steep revaluation of 30% took place from June '08 until April '20. That fact caused excess loses for those investors who had already invested in Japanese assets and discouraged potential investments from Europe. Also, we should take into account the high volatility of Yen against Euro that causes uncertainty to a foreign investor who intends to invest in Japanese market. Because of the extremely high standard deviation (16,19), investing in Japanese assets has a large degree of risk , especially currency risk, and therefore it is less likely a German investor to include them in his international portfolio.

5.5 Home Asset Bias

The preference of an investor to invest in domestic assets is a usual phenomenon that occurs in many developed countries. Although there are benefits from the international diversification, investors most of the times include in their portfolios mainly domestic assets. According to Cooper, I. and E. Kaplanis (1994) [13], we cite below *Table 11* that shows the investor's bias το invest in domestic stock.

	Rate of domestic market capitalization in global market	Percentage of domestic stock per country
France	2.6%	64.4%
Germany	**3.2%**	**75.4%**
U.S.A	36.4%	98.0%
Japan	43.7%	86.7%
Spain	1.1%	94.2%
Italy	1.9%	91.0%
Great Britain	10.3%	78.5%
Sweden	0.8%	100.0%

Table 11: Home Asset Bias

For instance, U.S. investors are less likely than the investors from any other country to invest in international stock, since they invest only the 2% of its portfolio value in foreign assets. The rest part is invested in domestic stock, despite the small size of the domestic market capitalization in global market. Similar situation takes place in the case of German investor, since more than 75% of the portfolio includes domestic stock.

The writers refer two reasons that explain the above phenomenon. First, there are obstacles in capital movements from country to country and various taxations of the foreign investments. Some examples are controls while entering or leaving a foreign market, the double taxes imposed on dividends or bond's coupons and the high transaction fees. However, the impact of these obstacles cannot be analyzed further since it is not a part of financial analysis. The second explanation has to do with the exchange risk that we analyzed above, and we concluded that the exchange rates of the certain period would cause losses for the investors in case they selected an international portfolio where American and Japanese assets are included.

6 Conclusion

In the current thesis, we analyzed from a theoretical point of view the risk measures and their mathematical formulations that are used in optimal portfolio selection. Each model optimizes a different measure of risk against a measure of reward, applied on a lot of investments done by various sectors of the economy. Also, by examining the diversification benefits of the international and regional portfolio investments from the view of an investor either from developed or developing country, we reached to the conclusion that it is more beneficial for the investors coming from developing countries to invest in international portfolios. Although, there are benefits for all the sides, the analyzed researches agree that these investors can earn significant large benefits. The most important factor that shapes the benefits is country risk, since there is a positive correlation between the risk of the investor's country and the potential benefits from investing international. However, there is a downward trend of diversification benefits through the years, because of the decreasing country risk that is caused by the constant improved financial environment in a lot of states.

Concerning the empirical part, we applied a maximization problem for a German investor who intends to invest in a domestic and an international portfolio. After taking into account the historical data of 3 German assets (used in domestic portfolio), 3 U.S. assets and 3 Japanese assets, and the exchange rates for a selected period of time, we solve the maximization problem in GAMS model in order to produce the optimal domestic and international portfolio for each month of our sample. The outputs of all the maximization problems showed that it is not optimal for a German investor to include the specific foreign assets in his portfolio. Subsequently, we constructed the efficient frontier of the two portfolios, and we conducted a back test which depicted a significant negative cumulative performance that would have been occurred in case the German investor had invested in the specific period of time in these portfolios. Finally, we interpreted the structure of the portfolios using the descriptive statistics of the asset's historical returns and the fluctuations of the exchange rates that affect significantly the foreign financial investments.

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