ΟΙΚΟΝΟΜΙΚΟ ΠΑΝΕΠΙΣΤΗΜΙΟ ΑΘΗΝΩΝ

ATHENS UNIVERSITY OF ECONOMICS AND BUSINESS

Athens University of Economics and Business
Department of International and European Economic Studies

"Options Pricing with discrete distributions"

Maria Ioanna Tzortzaki
Student ID: 4180125
Subject: Diplomatic Thesis
Supervisor: Nikolaos Topaloglou

“I, Maria Ioanna Tzortzaki, certify that the work prepared and presented in the submitted thesis is exclusively mine. Any information and material contained has been drawn from other sources, has been properly referenced in this thesis. In addition, I am aware that in the event that it is found that what is certified by me is not the case, my work will be nullified."

Summary

This thesis presents an overview of the literature of the valuation models that there are for the pricing of the European option, focusing mainly on the Black-Scholes model developed in 1973 and two alternative option pricing formulas that were developed according to the discrete distributions in order to produce more accurate and realistic results than the initial Black-Scholes model. The first alternative method is basically the Black and Scholes model augmented with two higher order moments, skewness and excess kurtosis of the empirical distribution of the underlying asset log-returns. The second alternative model is the Risk-Neutral Probabilities model that is assuming a no-arbitrage environment, in which the European option is being computes as the payoff of its expected value discounted by the risk-free interest rate. The performance and numerical results of both the alternative option pricing models are compared with the performance and numerical results of the Black- Scholes model, using actual prices of the S&P 500 Index. The conclusions we came to derive from our analysis were that the Black and Scholes model is overpricing deep out-of-the-money call options (OTM) and underpricing deep out-of-the-money put options (OTM). The reason for that is because of the asymmetric and leptokurtic characteristics of the distribution of the underlying assets prices. Another conclusion we made was that the two alternative pricing methods, which capture the asymmetric and leptokurtic characteristics of the underlying assets distributions in contrast with the B-S model, form feasible alternative option pricing methods for pricing accurately and realistic the European put and call options.

Keywords: Options pricing, European options, call and put options, Black & Scholes model, Higher order moments, Risk – Neutral Probabilities , Alternative option pricing methods

Acknowledgments

I would like to express my gratitude and many thanks to my Professor Mr. Nikolas Topaloglou, for his help and guidance through the course of this thesis on Options Pricing on Discrete Distribution and for providing continuous support and help to my many questions and answering all my emails without time to waste. He played a very big role in helping me properly prepare my thesis.

Table of Contents

1. Introduction

In recent years and as economies have evolved more and more over time, derivatives have gradually acquired a key role in the Global Economy. Futures and options are now actively traded on several exchanges around the world. Many different types of derivatives such as Futures, Forwards, Swaps, Option and many others are traded by financial institutions, end managers and investors in mainly Over -the-counter market transactions (OTC) or in Exchanges such as the Chicago Options Exchange established in 1973 in Chicago. In the era we live in, it is now a necessary condition for anyone working in the financial sector to understand how derivatives work, how they can be used in practice and how they are priced.

Derivatives got their name because of their use: they are financial instruments whose value is dependent on or derived from the value of other more basic underlying assets, such as assets, indices, interest rates, etc. Also, quite often we encounter the phenomenon that the variables underlying derivatives are prices of traded assets. They are widely used in finance for hedging (i.e., protection against sharp movements of the prices), speculation (i.e., positioning on the future direction of the financial market) and arbitrage (i.e. sure profits).

From the beginning of their establishment, the main objective of these products is to neutralize the risks arising from price fluctuations or to ensure the purchase of assets at a pre-agreed price.
Options trading has a long and illustrious history but it also underwent a revolutionary change in 1973. At that time Fischer Black and Myron Scholes presented the first complete equilibrium satisfying option pricing model. At the same time, Robert Merton extended their model in several important ways. These articles pioneered Options pricing and have formed the basis for a plethora of forthcoming academic research studies that have followed. Since then, financial analysts have been able to calculate the value of a stock option (stock option) with a fairly high degree of accuracy, whether it is a put or call option, respectively, with the invention of the Black-Scholes model.

Options are financial instruments that give their owners the right but not the obligation to buy or sell an asset at a pre-agreed price in a

specific time period. They also provide the investors who own them with a mechanism to offset risk, which allows them to offer their services at a much better price than they would otherwise.

Options trading goes back many years but due to their complicated costs, their transactions were not so easy to arrange until the Chicago Board Options Exchange established them in April 1973. Their origins go back centuries and more in particular the first examples are found in Ancient Greece where options were used by farmers for predictions about olives and harvest in order to be able to make profit through selling them.

In applied finance, the pricing of options is a quite complicated process because the values of options depend on many different variables apart from the underlying asset. Basically, the option pricing theory is an approach that used probabilities to estimate the price of an option, also widely known as premium. Some of the most known and used models for pricing options are the Black and Scholes model, the Binomial Trees and many more.

The Black-Scholes model for options pricing made a very important breakthrough by deriving a differential equation that must be satisfied by the price of any security-dependent derivative on a non-dividend-paying stock. It is a well-known pricing method, which was originally created to assign value to a European Option. The key principle of the model is to hedge the option by buying (or taking a long position in the option) and selling (or taking a short position) the underlying asset in ways to help eliminate the risk, otherwise known as delts hedging. It was first published in the Journal of Political Economy under the title "The Pricing of Options and Corporate Liabilities" in 1973.

The basic assumptions on which this model is based are the following 6:

- markets are always open,
- there is no cost of sure profit (no arbitrage),
- the risk-free interest rate is constant over time
- the volatility of the price of the underlying asset is constant

- the price movements of the underlying asset follow a logarithmic distribution which implies a normal distribution of capitalized returns.

In other words, the price of an option is explicitly determined by the evolution of the price of the underlying asset. More specifically, in order to calculate it, it is based on the current share price, the exercise price of the options, the period until expiration, the risk-free interest rate, dividends and volatility. The last three must be calculated in order to find the price of an option in a rational way, while the first three are given.

The Black & Scholes formula, assuming that the stock value follows a log-Brownian process, constructs a strategy in which banks can very accurately create a true copy of the buyer's Portfolio. In such a way that for the Bank this has absolutely no cost.

It is quite obvious at first sight that the simplified characteristics of Black-Scholes, which are derived from the assumptions mentioned above, show that economic data shows a tendency for non-continuous leap forward. It was created under certain strict assumptions that quite often do not correspond to reality. For this very reason, a fairly large number of models have been created, which try to improve the original model, remove its disadvantages and expand some of its conditions in order to reflect and contain the real development of the market with more rational way.

Before deriving the model, it was observed that the non-existence of sure profit (arbitrage) is a critical condition that leads to the differential equation solved by the Black-Scholes formula. But in practice, as stated by Ambrož (2002), this is violated and leads to anomalies. Another important condition for the derivation of the equation of the Black-Scholes model is the perfect replication of the derivative from the stock and the risk-free instrument, which, however, cannot be achieved without any transaction costs.

Grossman and Zhou (1996) found that volatility is correlated not only with stock price but also with trading volume and cost. This results in creating new models, which reflect transaction costs as for example Davis et al. (1993) or Taksar et al. (1998) as well as models that propose a modification of the original model with variability that is not constant such as the model of Hull and White (1987). Stock prices are affected by specific past events that occurred before trading began. In addition, the original model does not take into account the

payment of dividends for the underlying stock, yet the majority of corporate stocks pay dividends.

During their research, Black-Scholes (1973) produced clear pricing formulas for both call and put options based on the assumption that stock prices follow a Brownian geometric distribution. Despite its big popularity and wide use across the world, the Black and Scholes model is structured upon very specific assumptions, that are not applicable in the real-world conditions. That is why because its crucial underlying factors assumed either to be known or to remain constant through the whole duration of the option till its expiration, facts that unfortunately are not true and are out of step with the real world.

The main deficiencies of the model that make it unrealistic and proven to incorrectly price out-of-the-money options are the following assumption it makes; a) the stock price follows a random walk , or either known as the Geometric Brownian Motion , b) the volatility is constant overtime, c) the assumption that the interest rates are known and constant , d) the log-normally distributed underlying asset price are normally distributed, e) the underlying assets do not pay dividends during the time till expiration of an option contract.

For the above reasons, several attempts have been made to find new models that will solve the problems presented by the Black-Scholes model. Harrison and Kreps (1979) developed an alternative method to price options, assuming a non-arbitrage environment where the price of the European option is being computed as the discounted payoff of its expected value by the risk free interest rate. Dempster and Hutton (1999) used linear programming to price American and European options and to compare these results with prices observed in the market. Bakshi (2003) developed a method to relate the physical probability measures with the risk neutral probability measures using a pricing kernel and created from it a transformation between the physical probabilities and the risk – neutral probabilities, in the case of a discrete distribution. Rubinstein (1994) and Jackwerth with Rubinstein (1996) found a solution to the constant volatility problem using binomial trees, which are based on real option values. Jarrow and Rudd (1982) and Corrado and Su (1996) simply added to the existing Black–Scholes model new features, which take into account the influences of higher order moments on options pricing (skewness and kurtosis).

The main objective of this thesis is the implementation and validation of rational and certified methods for pricing options that are consistent with discrete asset price scenarios. These pricing models also take into account statistical characteristics such as skewness and kurtosis that have been observed empirically. More specifically, two methodologies have been used to price options, both of which price options on underlying assets with discrete allocations.

The first methodology to follow is an extension of the original Black and Scholes model implemented by Corrado and Su (1996), who added two features that take into account higher order moments (skewness and kurtosis) of the empirical distribution of the underlying asset log returns.

The second method is based on the transformation between the physical probability measure and the risk-neutral probabilities in the case of the discrete distribution that is reflected on the subset of nodes on a scenario tree. Pricing becomes very easy and fast once these risk-free probabilities are calculated.

We compare the Black and Scholes model and the performance of the alternative option pricing methods with the real market prices of the options. For the execution of the alternative options pricing methods, we generated scenarios of the asset prices that match the statistical features of the underlying asset as they are observed in practice. We assume that the log- normal assumptions are not working and can't produce results that agree with the real market prices. Through the various tests that we made, the conclusion is one; the European options pricing methods that are based on binomial trees that take into accounts asymmetries that are presented in the in-practice distribution of the underlying asset price , yield results that are obviously better that the ones the Black and Scholes model yields and therefore is a pretty accurate and trusting method for pricing options.

The following thesis is structured as follows: in Chapter we start with summarizing the key points of the thesis, or in other words the introduction. In Chapter 2, we provide a detailed analysis and description of the o the options and especially the European put and call options, their characteristics, their payoffs and their upper and lower bounds. In Chapter 3, we focus on the Black and Scholes model, its characteristics, its deficiencies and we present the one and two step Binomial Trees. In Chapter 4, we describe the two alternative pricing option methods, the augmented Black and Scholes model with higher order moments (skewness and kurtosis) and the risk-neutral probability model (Corrado and Su). Last but not least, in chapter 6 we present the conclusions and summarize our findings.

2.Options

Options are the second largest class of derivatives, after Futures, and are widely used worldwide either for risk hedging or for speculation. An important advantage they have over owning the underlying asset is the possibility of leverage (i.e., borrowing). By the term hedging we refer to an investment that reduces the risk of risk occurring in an already existing position, such as another investment.

Apart from the fact that they were discovered several years ago, their development was limited due to various technical issues related to the calculation of their value, as well as because there was low liquidity in the unorganized Stock Exchanges, where they were traded. At the same time, they significantly strengthened the options market and the Black and Scholes model and its establishment by the Chicago Stock Exchange (Chicago Board Options Exchange) and in almost three years, options they covered all underlying assets and products, which were offered for trading. (Mylonas 2005).

Options are defined as the right to buy or sell a specified security or commodity at a specified price during the period specified by the contract. This specific price is referred to as the exercise price of the right (exercised price-striking price-contract price). The last day the contract is valid is called the termination date-maturity date. Options, with which securities are bought are called call options (right to buy), while options with which securities are sold are called put options (right to sell). If the option can be exercised before the expiration date, then it is an American option, while on the contrary, if it can only be exercised on the expiration date then it is a European option.

Options granted by companies are defined as rights or warrants. Rights are usually exercisable for a very short period of time, unlike options, which can be exercisable for long periods. Contracts that are used more for the purchase of goods than for the purchase of financial instruments are called Commodity options.

An important distinction to be made from the side of buyers (owners) and sellers (creators). Buyers (long position) have the right but not the obligation to exercise any option they wish. Accordingly, sellers (short position) have the right but not the obligation to buy or sell any underlying asset corresponding to the option, at any time when it is exercised by the buyers. Those who take a short position – sellers or creators of the option – receive some cash in advance (call/put premium – essentially like a risk premium), but then have other potential responsibilities in case the options are exercised by those who they took a long position (buyers).

The four types of position an investor can take on an option are as follows:

I. long position in call option
II. short position in call option
III. long position in put option
IV. short position in put option

They are shown in the figure below:

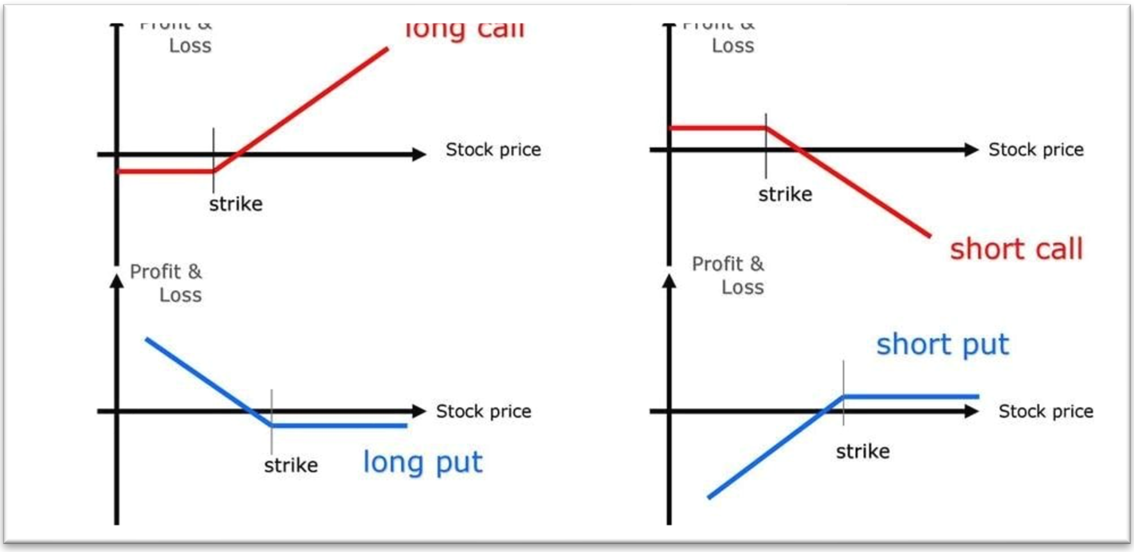


Figure 1: Payoffs of the positions of the call and put options

Where

Strike is the exercise price of the option and
Stock price (S0) the underlying asset price.

2.1 European call option

A call option gives the buyer the right, but not the obligation, to buy an underlying asset at a predetermined price (strike price). There are two different scenarios regarding the strike price of options: the first is that the strike price is greater than the price at a certain moment (spot price) (K>So) and the second scenario is that the spot price is greater than the strike price. (K<So). It is also important to mention the fact that if at the time when the option expires the exercise price is above the price at the expiration of the underlying asset (e.g., share price) (K>St) then the buyer will not exercise under any circumstances the option. This results in the option not being exercised and the buyer losing the premium he paid by buying this option.

In the scenario in which the price at expiration is greater than the exercise price (St>K) we have that the buyer will definitely exercise the option and make a profit from it. This will happen because the buyer has agreed to buy this underlying asset at a price lower than the price of the underlying asset at that time (spot price), so the difference between them will be his profit. In addition to this we have that the profits that can be obtained from a call option increase linearly and depend on how much the underlying asset at that moment in time exceeds the strike price.

In the event that the spot price exceeds the exercise price strike price but does not exceed it when the premium paid by the buyer is added to it, then anyone would exercise this option in order to minimize the losses they will have, managing to "catch" a part of the premium of what was originally paid.

Payoffs of the European Call Options

long position

The payment of the buyer at the expiration of the purchase option, who has taken a purchase position in it, is as follows:

$$Payoff\ Long\ Call = \max\ [ST - K,\ 0] \qquad (1)$$

This shows us that the option, the only way to be exercised will be the price at the expiration of the underlying asset is greater than the exercise price ($St-K>0 \rightarrow St>K$), if the opposite is true the option does not will practice ($ST<K$).
The net benefit will be profit, after calculating the premium of the purchase option and is:

$$Netprofit=\max\ [ST\ -K, 0]\ -c \qquad (2)$$

Where c is the price of the call option.

short position

An investor, who has taken a sell position on an option (European call option), expects the spot price of the underlying asset to remain either constant, or to be lower than the exercise price, so that the call option premium he paid to buy it is not worth The Black-Scholes model is constantly mispricing deep OTM call option, because of the observed negative skewness premium on that particular selected date. exercising and holding on to.
Therefore, the sellers of options are faced with huge levels of risk because they cannot put limits on how much damage and loss of money they will have once they decide to exercise the option, they will have to pay the buyers.

The payment of the investor who has taken a buy position on the European call option is:

$$Payoff\ Short\ Call = \min\ [K - ST,\ 0] \qquad (3)$$

Or

$$= \ -\max\ [ST\ -K, 0] \qquad (4)$$

In the diagram below, the net profits of both different types of positions that an investor can take on an option are clearly seen.

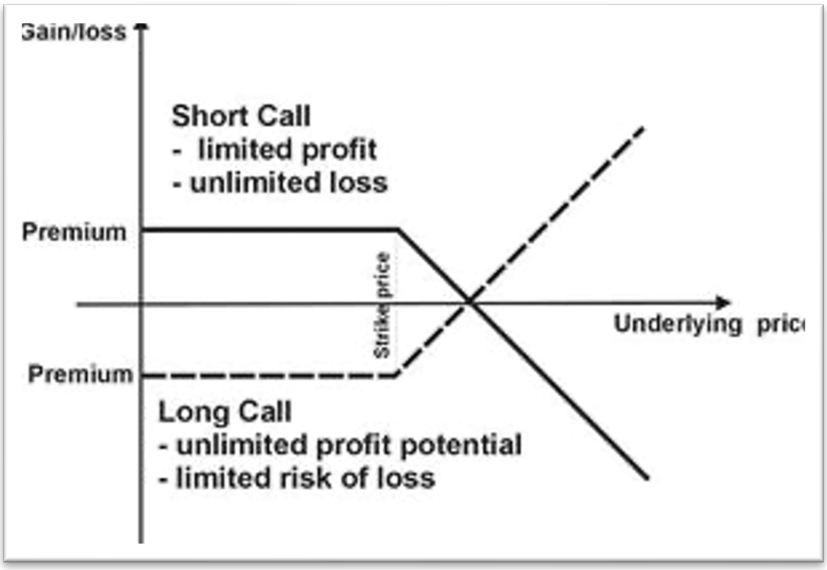


Figure 2: Net profits of the positions on call option

It is evident from the above diagram that the profits that arise for the buyers if the options are exercised are the corresponding money losses for the sellers and vice versa if the option is not exercised the call premium is transferred from the buyer to the seller. Therefore, the long position in a call option emphasizes the positive possibility in the event that there is a positive movement in the market.

2.2 European put option

A put option gives the buyer the right, but not the obligation, to sell an underlying asset at a specified time at a predetermined price. The buyer of the put option, in order to make a profit from this market, expects the price of the underlying asset to move below the exercise price of the option, unlike the one who owns a call option who expects the exact opposite. In conclusion, this option will not be exercised if the exercise price is below the price at the specific time (spot price), i.e., valid ($S_t > K$). This is because it is not profitable for someone to sell something very cheaply and buy it right after at a more expensive price.

A necessary condition for the holder to exercise the option is that the exercise price is at a higher level than the price in the expiration period ($St < K$), thus obtaining an immediate profit. The reason this is done is because he sells something at a high price and at the same time buys it back at a price lower than the prevailing market price.

Worth mentioning is the fact that in order to obtain a profit by having a long position in a put option, the exercise price must exceed the sum of the price at that particular moment in time with the premium paid initially. In the event that the exercise price exceeds the instantaneous price without the premium being added to it, then the one who has the option will definitely exercise it in order to minimize the losses that may be caused by sequestering a part of the premium paid at the beginning.

It is generally true that profits increase as the current price of the underlying asset decreases and overall, a put option provides protection in the event that the price of the underlying asset falls sharply.

Payoff of European Put Options

Long position

The payoff for an investor who has taken a long position in a put option is:

$$Payoff \textbf{ Long Put} = \max[K - S_T\ ,\ 0] \qquad (5)$$

Therefore, the option will only be exercised if $K - St > 0 \rightarrow K > St$ and will not be exercised if $K < St$.

The net profit is equal to:

$$Net\ profit = \max[K - S_T, 0] - p \qquad (6)$$

where p is the price of the put (the so-called put premium).

The writer of a put option expects the option price to remain constant or exceed the strike price so that the option is not exercised and retains the put premium. For this very reason, the sellers of options are faced with very big risks because there are no limits at all to how much loss they will have, since they have the obligation to pay the buyers if they decide to exercise the option.

Short position

The payoff for an investor who has a short position in a put option is:

$$Payoff\ Short\ Put = \min[S_T - K,\ 0] \quad (7)$$

Or:

$$= -\max[K - ST\ ,\ 0] \quad (8)$$

T are the net profits and long short positions in put options are simply described with the following:

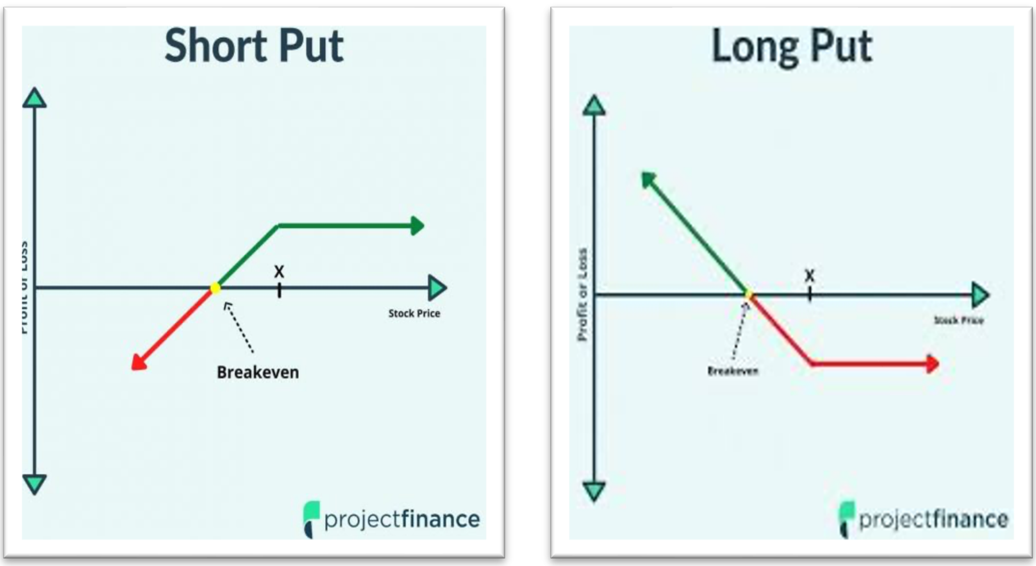


Figure 3: Net profits of positions on put option

What we can conclude from the above two charts with the positions one can take in a put option is that the profits or losses of an investor who has taken a long position (buyer of the option) are the exact opposite of someone who has taken a sell position / short position (option writings).

2.3 Factors that affect option pricing

Options can be affected by some factors, which are as follows:

- The price of the underlying asset at the specific moment we are talking about, $\mathbf{S_0}$
- The strike price, **K**
- The expiration of the option, **T**
- The volatility of the asset, **σ**
- The riskless rate, **r**
- And the dividends that are expected to be, **D**

Below will be presented how each factor affects the price of the option as well as some more details about each one. In the following chart we have concluded the variables and the effect, positive or negative, they have on the put and call options. We derived from that chart the below two figures, where we used indicators 1, -1 and 0 to describe the positive, negative and the unknown effect respectively

VARIABLE	EUROPEAN CALL	EUROPEAN PUT	EFFECT ON CALL	EFFECT ON PUT
current stock price	+	-	1	-1
exercise price	-	+	-1	1
time to expiration	?	?	0	0
volatility	+	+	1	1
dividends	+	-	1	-1
risk free interest rate	-	+	-1	1

Table 1: How factors affect the price of the options

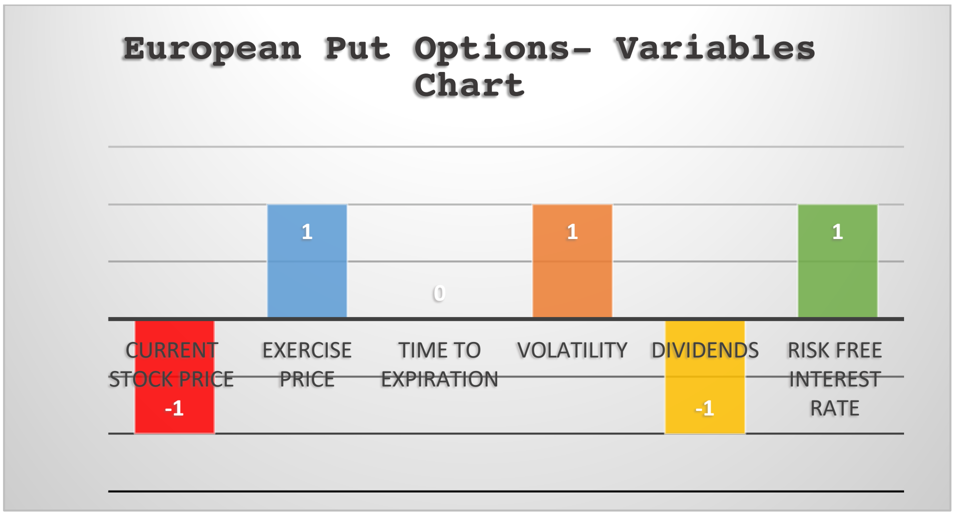


Figure 4: Effect of the variables on the European Call options

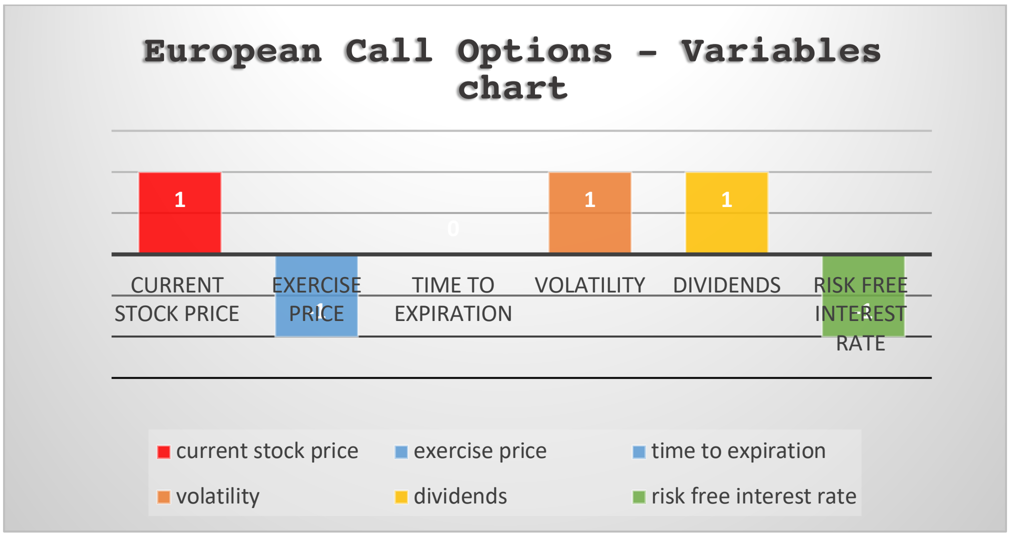


Figure 5: Effect of the variables on the European Put options

Where

- **1** is the representation of the positive effect the variable has on both the European call and put options

- **-1** is the representation of the negative effect the variable has on the European call and put option

- **0** represents the fact that we don't know the exact effect the variable has on the European put or call option, it depends

Before we go on about the effect of the above parameters on the price of the put/call option, we need to provide more information about the symbols of the above diagram. When using the + we mean that the value of the option follows the same direction as the value of the variable does, while the – means that the value of the option follows the exact opposite direction. Moreover, the symbol, shows that the relation between the value of the price and the variable does not have a clear relationship, there is uncertainty. That is also one difference the effect the variables have on the European and American put / call options: the time to expiration has clearly a positive relationship with the value of both the American put option and the American call option, while all the others have the exact same effect they have on the European call option, on the American call option and accordingly to the European put and American put.

The above-mentioned variables have the same result on both the European call option and American call option as do on the European put option and American put option.

Overall, from the previous diagram what we can conclude is the following: some of those variables can have a positive effect on the price of the option while some others don't. The variables that have a positive effect on the price of a put option have the exact opposite result on the call option and vice versa. As we can see above, the current stock price of the underlying asset, the volatility and dividends have a positive result on the call option and a negative on the put option. What that means is that the value of the European call option increases as the value of those variables increases. With that same perspective in mind, the value of the call option decreases as the exercise price or the dividends increases. On the contrary, the value of the put option increases as the value of the exercise price, the volatility or the dividends increases. Basically, the variables that make the value of the call option increase, at the same time make the put option decrease. While, the current stock price and the risk-free rate make the price of the put option decrease.

Now let's talk more about each one separately.

Exercise price & current price of the underlying asset

As it was said above, when a call option is exercised the payoff, it gives to the investor is the difference between the current price of the underlying asset and the exercise price. In other words, is the amount by which the stock price exceeds the exercised price. That is why the value of the call options has a positive relationship with the price of the underlying asset price and a negative with the exercise price, meaning that when the stock price increases it causes the value of the call option to increase too. While in contrast when the exercise price increases, the value of the call to decreases.

On the other side of the spectrum, for the put option when it is exercised, the payoff will be amount by which the exercise price goes above the stock price. For the out, we have the exact opposite effect, that is to say that when the exercise price increases that causes the value of the put to increase and decrease when the price of the underlying asset increases.

Time to expiration

It is very important for the put and call options because especially for the American put and call options it can affect how valuable they are as the time to expiration increases. It is easier for the owner of the option that has an increased time to expiration, to explore and have more opportunities to exercise the option than the owner who owns an option with a shortened time to expiration. But a necessary condition for that to be able to happen is if the option with the longest time has as much value as the one with the short time, something that it is not always possible due to the dividends that are given from the stock. To be more precise, even if we have an option with a long time till expiration, that does not mean that it has more value if the dividends that are expected to be given are quite large because that will have as a result for the stock price to decrease and that will give the option with the short time more value than the one with the long time.

Volatility

The volatility of the underlying asset can be described as a measure of uncertainty because we don't know and cannot predict how the future price is going to move.

Dividends

They don't really have an immediate result on the price of the stock option and when they are given that means that the stock price decreases and as a result the value of the call option decreases too and the value of the put increases. As a general rule, the value of the call option has a negative relation with the number of dividends, while the put option has a positive relation with the number of dividends.

Risk free rate

A very important issue is to find what rate is more suitable for traders to use as a risk-free rate. A common thought that used to occur was that the traders would use the rates implied by the Treasury bill and bonds as the risk-free rate in their exchanges. But this is not the case because the Treasury rates are very low due to reasons addressed with tax issues. For that reason, the rate used as the risk-free rate in option trading is the LIBOR2 rate.

2.4 Strategies involving options

There are certain strategies that concern specific combinations of options in hedging applications. That is why we consider the following option trading strategies: Straddle, Strip, Strap and Strangle, and Bearspread. These strategies involve combinations of positions in call and put options. Each of them generates a different payoff profile. The decision of the best choice for the traders is based on their view regarding potential movements in the value of the underlying and his preferences for protection in case of such movements happening.

Straddle strategy

This strategy suits investors who really like the idea of being able to exit the market as soon as possible in the event of an unexpected movement of the stock price either in a more positive or negative way. It is designed to provide protection in the event of increased volatility and yields payoffs if there is a substantial movement in the price of the underlying security, either that is downside or upside. A long straddle consists of long position in one call and one put on the same underlying asset, same strike price and with the same time to expire. An investor enters a long straddle position when he anticipates an increase in volatility but is not very sure about which direction the movement will take, yet he wants to be covered and safe in the unlikely event of sharp changes in the stock price in either direction, either that is positive or negative. Moreover, for a long straddle position to yield a profit, the underlying asset must move sufficiently low or high in order to cover the total cost of the option premiums in the case an unpredictable movement happens. Once of the cost of the option premium is cleared there is a linear profit against the movement of the stock index in either direction. Generally, if the stock price is relatively the same amount as the strike price of the option, then the straddle price results in a loss; the maximum that loss can reach is the sum of the call and put premiums (purchase prices) that constitute the straddle. But on the other hand, when the stock index happens to experience a large movement in either direction, then the straddle makes a substantial profit. Along these lines, the loss from a potential large decrease in the underlying asset index can be recovered from a long position in the straddle.

Strip Strategy

A strip strategy and especially a long strip strategy consists of a long position in one call and two options with the exact same expiration date and strike price. An investor takes a long strip position when he expects a large movement in the underlying asset index and believes that a decrease in the index is much more likely to happen than an increase. So again, here the investor basically buys protection against large unpredictable fluctuating in the stock price but gives more weight on the coverage against downward movements. As we said on the straddle strategy again, we have that for limited range of changes around the stock price, the strip strategy results in a loss. While for large movements it yields a positive payoff no matter what direction they are, the payoff for downward movements is much steeper than for upwards movements.

Strap strategy

From a specific point of view, some may say that a strap strategy is exactly the same as the strip strategy. Along strap consists of a long position in two calls and one put with, again, the same strike price and time to expiration. An investor takes a long strap position when he is waiting for large movements in the stock price but believes that an increase is more likely to happen than a decrease. Once again, the investor yields a profit in the case of sharp movements of the stock price index in either direction, increase or decrease, but gives more importance to gains from likely upward movements that downwards movements.

The options that dominate these three categories of strategy are typically the at-the-money options; where as we said before, their strike price is equal to the current stock price. The closer the strike price f a call or a put option is to the current stock price, the more expensive is the option, therefore it acquires more a bigger premium. That is why these strategies are rather costly especially the strip and strap that the straddle, because they acquire an additional option in order to be taken into action.

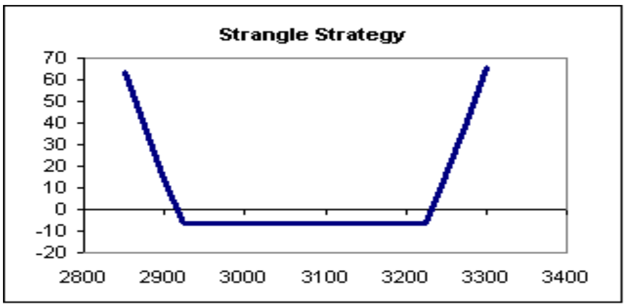
Strangle strategy

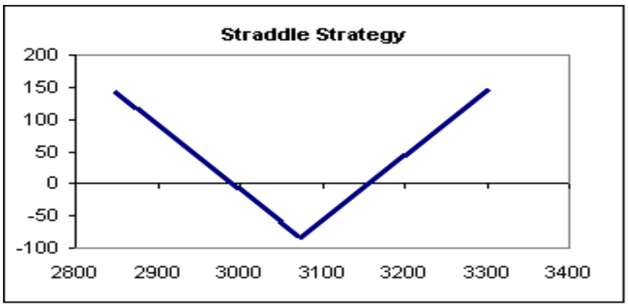
This one is very similar to the straddle because a long straddle consists of a long call and a long put on the same stock price with the same expiration date, while in strangle strategy the two options have different strike prices. The exercise price of a call option is higher than the current price of the underlying asset, while that of the put is lower than the current stock price. A long straddle yields a profit when there is a large movement of the stock index in either direction. For the strangle to yield a profit a necessary condition is for the index in the strangle to move further that in a straddle. The downside risk if there is a limited change in the value of the stock price is less with the strangle strategy than with the straddle because the strangle is way cheaper than the straddle as the prices of the options are lower than those in the straddle. When an investor has a long strangle, buys coverage against large movements of the stock index in either direction, in other words he covers against volatility. The payoff here depends on how close the strike price of the call and put options are; if both of these prices approach the current stock price then the payoffs of the strangle resemble those of the straddle. The further apart the strike prices are from one another the lower the cost of the strategy, but in order for the strategy to yield a profit the stock price must move a lot farther.

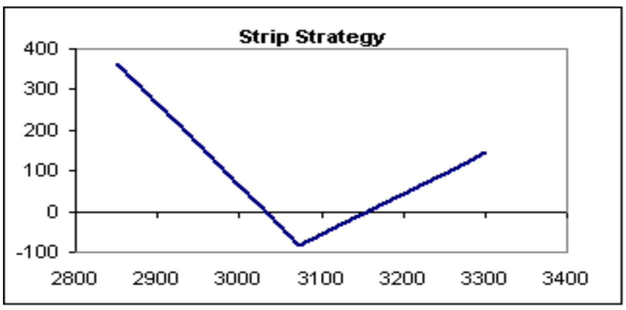
BearSpread strategy

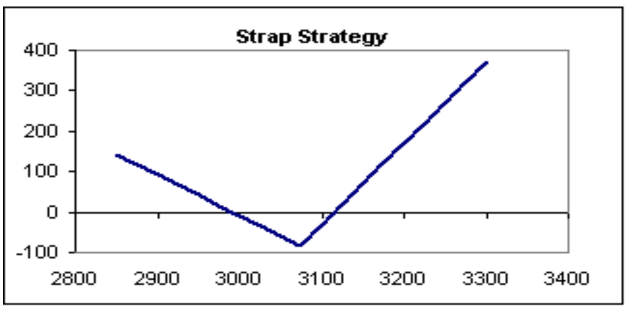
This strategy consists of two put options with the same expiration date. It involves a long in-the-money put and a short -out-of-the-money put. The maximum payoff from this strategy is the difference between the two strike prices, without the net premium for the two put option. The break- even point is the difference between the higher strike price of the in-the-money option and the net premium. Again, the maximum loss is the net premium for the positions in the tow options.

Below we can see the payoffs of the positions as they are presented in diagrams:









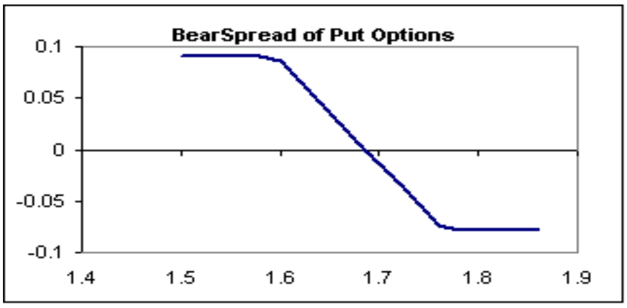


Figure 6: The strategies of the Options

2.5 Lower and upper bounds of European call and put options

It is time to finally talk about the upper and lower bounds of the options prices in this section. If the option price is above the upper or below the lower bound, then arbitrage opportunities may arise for investors.

Before diving into it, let's indicate the notations we are going to use:

S_0: spot price

K: strike price

T: time to expiration

S_t: price of the underlying asset at expiration time

R: continuously compounded risk-free rate

c: value of European call option (call premium)

p: value of European put option (put premium)

Upper bounds of European call options

A call option gives the holder the right but not the obligation, to buy one share of a stock or an asset for a certain pre-determined price. Whatever happens, the option can never be worth more than the stock price. That is why the stock price is an upper bound to the option price

$$C \le S0 \qquad (9)$$

If this relationship is not valid, an arbitrageur can very easily make a riskless profit by selling the call option and buying the stock.

Upper bounds of European put options

A put option gives the holder the right but not the obligation to sell one share or an underlying asset for K. It doesn't matter how low the price of the stock can become; the option can never be worth more than that value K. That is why the upper bound of the put option is:

$$p <= K \tag{10}$$

It is a known fact at time T the option will not be worth more that K, therefore it follows that its value today cannot be more than the present value of K:

$$p<=e^{-T}K \tag{11}$$

If the below is not valid then an arbitrageur could make a riskless rate profit by selling the option and investing the proceeds from it by selling the risk-free rate.

Lower bounds for call options

Generally, the price of the option reflects the difference between the future expected value of the stock and the strike price, therefore the future expected value if the underlying asset can never be less than S0eRT. This is happening due to the fact that if the future value is less than S0eRT,then something that is very likely to happen is short selling the underlying asset , invest the amount of money gained at risk free rate and make a profit at T time. Hence, the future expected price of the stock cannot be less than the spot price compounded at risk free rate and so the value of the option cannot be less than the strike price discounted at risk free interest rate. If this doesn't happen, the investor would buy the call option, sell the stock , invest the money at risk free rate and then he will be left with a riskless profit no matter the future movements of the stock price. So the lower bound for a European call option is:

$$c \geq S-Ke^{-RT} \text{ or } c \geq \max(S-Ke^{-RT},0) \tag{12}$$

Lower bound for European put option

As we said before the current price of the underlying asset cannot be less than the discounted value of future expected price or the discounted value of the strike price. In that event, that has to be portrayed in the price of the put option , therefore the lower bound for a European put option is :

$$p \geq Ke^{-RT} - S_0 \ \text{or} \ p \geq \max(Ke^{-RT} - S_0, 0) \qquad (13)$$

2.6 Put- Call Parity

We now can derive an important relationship between the Put price p and the Call price c. We consider the following two portfolios:

- Portfolio A: One call option plus the amount of cash equal to $e^{-rT}X$
- Portfolio B: one European put option plus one share

Both are worth:

$$\mathbf{Max(S_T, X)} \qquad (14)$$

At expiration of the option: Because both options are European, they can't be exercised prior to the expiration date, therefore the portfolios must have identical values today.
This means that:

$$\mathbf{C + Xe^{rT} = P + S_T} \qquad (15)$$

This relationship is known as put-call parity. It shows that the value of the European call with a certain strike price and exercise date can be deduced from the value of the European put with the same strike price and exercise price and vice versa.
If the above equation does not work, there are arbitrage opportunities for the investors.

2.7 Moneyness

The first category when it comes to moneyness is called in-the-money (ITM) and is about options that the price of the underlying asset is greater than the exercise price and are profitable if exercised. For a call option to be described as ITM essential condition is for the strike price to be below the spot price, while for a put option is for the strike price to be above spot price.

A Call option is I In-the-money if $\mathbf{K<St}$
A put option is In-the-money if $\mathbf{K> S_t}$

The second category is called out-of-the-money and is about options that don't have intrinsic value (very close to zero or equal to zero), have only time value and generally they are unprofitable to be exercised. For a call option to be described as OTM essential condition is for the strike price to be above the stock price and for put options the strike price to be below the stock price.

A Call option is Out-of-the-money if $\mathbf{K>St}$
A put option is Out-of-the-money if $\mathbf{K<St}$

The third and final category is called at-the-money and is about options that their exercise price is equal to the price of the underlying asset (this goes for both put and call options) In other words, we are talking about options that does not have intrinsic value (is equal to zero), the options have only the time value and the investors who hold the options are indifferent between exercising them or not.

A call option is At-the-money If $\mathbf{K=St}$
A put option is At-the-money if $\mathbf{K=St}$

It is really important to underline the fact that even if an option is in-the-money, that doesn't particularly mean that the holder of the option will yield a profit from its exercise. This problem is related to the fact that the initial cost (premium) will possibly cover the return that is going to occur from the exercise of the option. In this case, the investor will still wish to exercise the option because the positive return that is going to come out of it will help cover the total loss.

At his ending point in this theoretical section and after we talked about options and their characteristics in more depth, it is very important to underline the reasons why investors use options and why they are important.

One of the first and most important reasons is hedging. Options due to their asymmetric and nonlinear profits payoffs, can provide the means to protect the value of the holdings in an asset in the unfortune event of substantial variations in the market price of the underlying security. They can be used from either an individual investor or a financial institution in order to provide protection for their portfolios or ensure that their investments are protected from fluctuations of the price of the underlying securities.

They can also be used for speculation, which helps investors generate profits in the event of large changes in the value of the underlying asset in either direction. More specifically, when the value of the underlying asset follows a downward movement, profit is acquired by taking a long position in a put option, while when the underlying asset follows a upward movement, to make a profit the investor needs to take a long position in a call option. A very positive characteristic that the options have is that there is absolutely no limit in making profits, just by paying a premium (the price of the call or put option), which considering the possible profits that could arise from investing in a option, is not so expensive.

The third and last reason is about arbitrage. Arbitrageurs take long positions in options to lock in a profit. That means that no matter if we are talking about a put or call option, they are going to make sure they yield a profit from them. The strategic they follow is, buy the cheap and sell the expensive, even that is the underlying asset or the overall option.

2.8 Put-Call Parity

The Put- Call parity is a relationship between the European put and call options, with the same strike price (K) and same time to expiration (T) as described below:

Hull (2015) derives the put-call parity assuming two portfolios;

- portfolio A that consists of a European call option and a zero-coupon bond that has a payoff K at time T and a
- portfolio B consisting of a European put option and a share of the stock.

This stock pays no dividends, while the put and call options have the same strike price K and time to expiration T as said above.

In the scenario that the terminal stock price S_T at time T is greater than K the strike price, then the European call option of the Portfolio A will be exercised and the value of the portfolio will be

ST=(ST–K)+K at time T (16)

In the scenario that the terminal stock price ST is less than K , the European call option of the Portfolio A will expire without having any significant value and the value of the Portfolio will be equal to K at time T.

Regarding the Portfolio B, the share of the stock will have a value of ST at time T. If the ST is less than K, then the European put option is exercised and the value of the Portfolio will be K=(K-ST)+ST at time T. In the other side, if the ST is greater than K, the European call option of the Portfolio B will expire without having any significant value and the value of the portfolio will be ST at time T.

PORTFOLIO A		
Constituent Parts	**S_T<K**	**S_T>K**
European call option	S_T-K	0
zero coupon bond	K	K
total value	S_T	K

PORTFOLIO B		
Constituent Parts	**S_T<K**	**S_T>K**
European put option	0	K-S_T
share of stock	S_T	S_T
total value	S_T	K

Table 2: Scenarios regarding the option value at time T

In the case of ST>K, as described above, both the Portfolios A and B have a value of St at time T while in the opposite case that the ST<K both Portfolios have a value of K at time T and therefore both of them have a value equal to (ST, K) at time T , as it is impossible for both the put and call option to be exercised prior to maturity time T. As they have the same value at time T, they should also have the same value at time 0.

Therefore, we have the following:

$$c = K^* e^{-r_f T} = p + S_0 \quad (17)$$

The above equation is the Put -Call parity and proves that the value of a European call option (c) with a particular strike price (K) and time to expiration (T) can be deducted from the value of a European put option with the same strike price (K) and time to expiration (T), and the opposite.

3. The Black – Scholes pricing model and how it evolves through the years

3.1 The Black and Scholes model

The most important feature when pricing derivatives is to find the pricing model/method that suits them the most in order to get the most accurate results that are closer to the market prices. Therefore, in this segment are presented and analyzed two pricing formulas; the Black and Scholes pricing method with the differential equation as presented by Merton and the one step and two steps Binomial Trees. Moreover, some deficiencies and problems that make the models not so accurate pricing methods are presented.

First, we are going to talk about the Black & Scholes pricing formula.

The Black and Scholes Model (BSM) is a renowned pricing method originally created for the valuation of the European options and is one of the most crucial concepts is the modern financial theory both in terms of approach and applicability. It was developed by Fisher Black, Myron Scholes and later on Robert Merton in the early 1970s. It is considered to be a model of price variation overtime of financial instruments such as stocks, assets that can among very other things, be used as a help in order to determine the price of a European call or put option. The model was first derived and published in the JOURNAL OF POLITICAL ECONOMY under the title The Pricing of Options and Corporate Liabilities in 1973. Black and Sholes and later on Merton, created the model based on a crucial assumption that an option can be very easily replicated through the purchase and sale of the basic financial instrument and a risk free asset which eliminates risk. In more simpler words the value of an option is determined by the development of the price of the underlying asset. The Black and Scholes model is based on the following five key variables:

- ⇒ The current market price of the underlying asset
- ⇒ The strike price of the option
- ⇒ The cost of having a position (long/short) on the option
- ⇒ The volatility of the underlying asset
- ⇒ The time to expiration

One of the biggest strengths of the model is the possibility of estimating market volatility od=f an underlying asset generally as a function of price and time without, for instance, direct reference to the expected yield or utility functions. The second strength is the act that an investor can buy and sell derivatives by the strategy and never experience loss. In other words, hedging; explicit trading strategy in underlying assets and risk less bonds whose terminal payoff equals to the payoff of a derivative security at maturity. Which means that it is very possible to create a hedged position constituted by a long position on a stock and short position to an option the value of which will not depend on the price of the stock value (Black-Scholes, 1973). In order to better understand the model and how it works, an important requirement is to take into account the assumptions that the model is based on.

The authors of the model consider the ideal market conditions such as:

I. Constant volatility
II. The stock price follows a random walk
III. Stock that moves normally distributed
IV. Interest rates(risk free rate)are constant and known through time
V. The stock pays zero dividends
VI. There are no commissions and transactions costs in buying or selling the stock option
VII. Markets are perfectly liquid
VIII. The option is European, therefore it can only be exercised at maturity

The formula is derived under the simple assumption that the time interval between the observations is very small and the log prices follow a random walk with normally distributed innovations, which is not affected by any linear drift in the random walk. The model for the prices of the underlying asset is called geometric Brownian motion. A very crucial input to the BSM formula is the **σ** which is called standard deviation of the asset's continuously compounded rate of return. As said before, on the condition that the time interval between observations is significantly small, **σ** is just the standard deviation of the innovations of the random walk, which in other words means that σ can be considered as a measure of the volatility of the stock price. According to the random walk model, **σ** is constant over time and its value is not known but it can be estimated from the data that is available. In essence, the BSM model believes that the price of the underlying assets follows a geometric Brownian motion with fixed drift and volatility.

3.2 The geometric Brownian motion

A Geometric Brownian motion is referred as a Wiener process and it is a specific type of Markov stochastic process, that has a mean variance equal to zero and a variance rate of 1 (as said by Hull 8th ed.). Brownian motion is very important because it provides a framework to capture fluctuations in stock prices, which have upwards and downwards movements due to unpredictable circumstances (Morters, P., Peres , Y, Schramm, Werner, W.). It implies that the series of first differences of the log prices must be uncorrelated. The BS theorem assumes that the stocks move in a way that is considered as random walk; random walk means that at any given time the price of the underlying asset can move in either direction, with the same probability. However, this assumption is not very realistic as stock prices are determined by many different factors that cannot be assigned with the same exact probability in the way they are going to affect the movement of the asset price.

Brownian motion is closely linked to normal distribution.

If a variable x is following the Brownian motion, then the following properties should be satisfied:

Property 1

The Δz during a period of time Δt is

$$\Delta z = \epsilon\sqrt{\Delta t} \qquad (18)$$

where ϵ is following a normal distribution φ (0,1)

A direct consequence is that Δz also follows a normal distribution with the mean being 0 ($\Delta z=0$) and the variance being $\Delta z=\Delta t$

Property 2

The values the Δz gets for two different time intervals Δt are independent, implying in that way that z follows a Markov process. With a Markov process we mean a distinct stochastic process, by which only the current value of a variable is appropriate about predicting the future expected value, while the past movements are uncorrelated. It is common for stock prices to follow a Markov process.

An also very important condition in order for the Black Scholes model to exist is that the asset price follows a log normal distribution. For that reason, the below essential conditions need to exist:

- ⇒ The volatility of the asset to be constant
- ⇒ The price of the underlying asset experiences smooth changes, without intense fluctuations.

But in reality, neither of the below conditions can exist and be satisfied. The reason for that is, that in practice the volatility of the exchange rate cannot be constant overtime and exchange rates experience quite intense changes and movements in either direction.

A Markov process is a distinct stochastic process, by which only the current value of a variable is appropriate about predicting the future expected value, while the past movements are uncorrelated. It is common for stock prices to follow a Markov process.

In the Δt the mean of the return is μΔt and the standard deviation is $\sigma\sqrt{T}$ and it follows:

$$\frac{\Delta s}{S} \sim \varphi(\mu\Delta t, \sigma 2\Delta t) \quad (19)$$

Where,

μ: is the expected return of the underlying asset

σ: is the volatility of the underlying asset

Δs: is the change of the stock price in time Δt

φ (x,σ) implies a normal distribution with mean x and variance σ

The model implies that:

$$\ln ST - \ln S_0 \sim \varphi[(\mu - \sigma^2/2)T, \sigma^2 T] \quad (20)$$

$$\ln ST \sim \varphi[(\mu - \sigma^2/2)\ T, \sigma^2 T] \quad (21)$$

and

$$\ln ST \sim \varphi[\ln S_0 + (\mu - \sigma 2\ /2)T, \sigma\ 2T] \quad (22)$$

where S_t is stock price at time t and S_0 is the stock price at time t=0.

The last equation we got implies that the $\ln St$ is normally distributed with mean $\ln S_0 + (\mu - \sigma^2/2)T$ and variance $\sigma^2 T$, therefore the S_T is lognormal distributed.

Therefore, for risk neutral investors with dividends equal to zero the Black and Scholes pricing formula

- For a call option is:

$$c = S_0 \times N(d_1) - K \times e^{-i_1 T} \times N(d_2) \quad (23)$$

- For a put option

$$p = K \times e^{rT} \times N(d_2) \times S_0 \times N(d_1) \quad (24)$$

Where

p: is the European put option price

c: is the European call option price

S_0: is the stock price at time t = T

K: is the strike price

R: is the continuous compounded risk-free rate

σ: is the volatility of the stock price

T: is the time to maturity

d_1,d_2 are calculated as follows :

$$d_1 = \frac{\ln\left(\frac{S0}{X}\right)+\left(i1+\frac{\sigma 2}{2}\right)T}{\sigma\sqrt{T}} \quad (25)$$

$$d_2 = d - \sigma\sqrt{T} \quad (26)$$

3.3 Black and sholes model with dividend policy

The Merton model in 1973 was published not long after the Black and Scholes model. A very crucial difference between the two models is the fact that the Merton model also takes into account the dividends, which are not included in the B-S model. In other words, the Merton model is an expanded version of the Black and Scholes. Nevertheless, this model did not get as much popularity as the former one due to the fact that is operates in with a continuous dividend paid in the same account. The dividend assumptions are very restrictive and make the model a not so realistic one, however later on it turned out that regardless of the restriction, the model was (and still is) suitable for valuating futures and currency options and so on. Dividends have a very immediate result on the stock prices, meaning that when dividends are given, the stock prices usually decrease by the amount of the dividend paid. The payment of a dividend yield at ate q results to the reduction of the growth rate of the stock price by amount q. More specifically, in the case of a non-dividend payments a stock price will grow from S_0, that is todays value, to S_t in time t = T. Therefore, the valuation of the

European option on a stock that pays dividends equal to q results from the reduction of the current stock price of S_0 to $S_T \times e^{-qT}$

The expanded Merton formula to dividend paying stocks is the following:

$$c = S_0 \times e^{-\delta T} \times N(d1) - X \times e^{-i_1 T} \times N(d2) \qquad (27)$$

$$p = K \times e^{-rT} \times N(-d2) - S_0 \times e^{-dT} \times N(-d1) \qquad (28)$$

Where

$$d1 = \frac{\ln\left(\frac{S0}{X}\right)+\left(i1-\delta+\frac{\sigma 2}{2}\right)T}{\sigma\sqrt{T}} \qquad (29)$$

$$d2 = d1 - \sigma\sqrt{T} \qquad (30)$$

where δ are the dividends paid out during the lifetime of the option.

3.4 The Black and Scholes differential equation

The Black -Scholes and Merton differential equation is an equation that can be used to help find the value of any derivative that depends on an underlying asset and pays no dividends. Something really important to note is that the B-S-Merton portfolio which is used in order to derive the differential equation is risk free for only a very short period of time. Therefore, we need to clarify the below:

t: is the price of the derivative at time t

T: is the time to expiration of the option

f: is the price of the call option or any other derivative S

First, we assume that the stock price follows a Geometric Brownian motion|:

$$dS = \mu \times S \times dt + \sigma \times S \times dz \qquad (31)$$

Then we have

$$df = \left(\frac{\theta f}{\theta s} \times \mu \times S + \frac{\theta f}{\theta t} + \frac{1}{2} \times \frac{\theta^2 f}{\theta S^2} \times \sigma^2 \times S^2\right) \times dt + \frac{\theta f}{\theta S} \times \sigma \times S \times dz \qquad (32)$$

The discrete versions of the above equations are the following

$$\Delta S = \mu \times S \times \Delta t + \sigma \times \Delta z \quad (33)$$

and

$$\Delta f = \left(\frac{\theta f}{\theta s} \times \mu \times S + \frac{\theta f}{\theta t} + \frac{1}{2} \times \frac{\theta^2 f}{\theta S^2} \times \sigma^2 \times S^2\right) \times \Delta t + \frac{\theta f}{\theta S} \times \sigma \times S \times \Delta z \quad (34)$$

where

Δf and ΔS are the changes in f and S in a small interval Δt.

Π is the value of the portfolio so we have by definition that the holder of the portfolio is short one derivative and long an amount θf / θS of shares as shown below:

$$\Pi = -f + \frac{\theta f}{\theta S} S \quad (40)$$

Therefore the change $\Delta\Pi$ in the value of the portfolio in a time interval Δt is:

$$\Delta\Pi = r\,\Pi\,\Delta t \quad (41)$$

Where

r is the risk-free interest rate.

If we put the above two equations in the Black -Scholes and Merton differential equation we get the following result:

$$\left(\frac{\theta f}{\theta t} + \frac{1}{2}\frac{\theta^2 f}{\theta S^2}\sigma^2 S^2\right)\Delta t = r\left(f - \frac{\theta f}{\theta S} S\right)\Delta t \quad (42)$$

So we have

$$\frac{\theta f}{\theta t} + rS\frac{\theta f}{\theta S} + \frac{1}{2}\sigma^2 S^2 \frac{\theta^2 f}{\theta S^2} = rf \quad (43)$$

Where we can clearly see the deviations of the spot price, time ,risk free rate and volatility of the stock price.

The above equation is considered to be the Black – Scholes and Merton differential equation which can get many different solutions according to the many different derivatives that can be describes with the S as the underlying asset. The outstanding positive feature that this model has is the fact that the results of the differential equation are not affected by the preferences the investors may have regarding risk.

3.5 Deficiencies of the Black - Scholes Model

In order to continue our analysis about the Black and Scholes model, a necessary step is to investigate the discrepancies between the assumptions the model makes with the real-world assumptions. The Black and Scholes model is based on simple and ideal market assumptions that are often violated when compared to the real market conditions. It is obvious at first sight that the simplifying features of the model implied by these assumptions, such as normal distribution or continuous process, very often fail when compared to the real market data. Embrechts et. Al. (1999) state that the financial data we have available tend to show non- continuous jump like progress.

Overall, the theory and examination of the differential equation, has become a popular subject for studying potential price problems after the Black and Scholes model was designed. Over the last two decades, the B-S formula has been a very popular subject for researchers, as it is a very effective simple and accurate instrument for option pricing (as the model includes all the key factors affecting the price of the option like the current stock price of the underlying asset, the strike price of the option, the interest rate, the volatility and the time to expiration. Also, it is very quick regarding its computation speed because it can price options contracts very quickly making investors take imminent decisions, especially in the case that the underlying asset is a stock, whose prices change very quickly and fast. Another important thing is that it is flexible as it can be used to price options on underlying asset not only stocks but also bonds, commodities and a variety of other assets.

This does not change the fact that the original version was designed under very strict assumptions that are not in sync with reality. For that reason a large number of models have been created , attempting to improve the initial model and remove it's drawback and improve and expand some of the conditions in order for them to reflect the real market development more precisely. The main deficiencies of the Black and Scholes model are described below:

<u>The geometric Brownian motion</u>

A main assumption of the Black -Scholes model is that the price of the underlying asset follows a random walk, which is known as the geometric Brownian Motion. Basically, the meaning of the random walk is that the price of the underlying asset (in this case stock) can move either upwards either backwards, having the same probability at any given time. That is not realistic, because of the fact that the price of the underlying stock is determined by many factors that is impossible to assign the same probability, in a way that they affect the movement of the price.

<u>The assets returns are not normally distributed</u>

Another main assumption of the Black and Scholes model, is that the log normally distributed stock prices are normally distributed. Hull (2015) describes that the asset returns have a finite variance and semi heavy tails , something that are in contrast to the stable distributions like log-normal with infine variance and heavy tails.

<u>Volatility</u>

The hypothesis that the volatility is constant is a very unrealistic assumption. Although the volatility is proven that in can be relatively constant over a short period of time, it is observed that it is not constant in long period of time. It is statically proven that in periods with high volatility follow immediately a large change downwards, something that is known as volatility clustering. Hull (2015) describes that the investors usually work with implied volatilities that are the volatilities implied by the stock prices observed in the market. Implied volatilities are known to be a very efficient way of measuring the option market's opinion regarding the volatility of a particular underlying asset – stock. Arriojas et al. (2007) and Kazmerchuk et al. (2007) a) assume that the modeling of the real development can be improved by volatility that is dependent on the history of stock prices, since investors have a tendency to monitor the historical development of prices of the instruments before they decide to invest. Generally, the stock prices are influenced by certain previous events that took place prior to the start of trading. Therefore, Traders usually use implied volatilities in order to make a hypothesis about the implied volatilities of other options.

That is why the volatility parameter is the most imperative input for pricing an option and has a very significant role on option pricing. For investors who use options often, volatility is a measure of the rate return they acquire over the holding period and it's forecasting is essential because it helps investors understand the extent of the risk, they face by trading options. Therefore, it is understandable that the volatility parameter is the most crucial one of the Black and Scholes model. The model, acquires for volatility to be constant but this is not the case in practice. The higher the value of the volatility variable, the more fluctuation the price will have and as a result an issue that occurs is the high uncertainty regarding the accomplishment of the expected return. Volatility estimations plays a essential role in order to price accurately derivatives and get precise results. In the Black and Scholes model the volatility is estimated through historical data and that has taken a lot of criticism. James D. Macbeth and Larry J. Merville (1979) stated that the biggest issue emerges from the B-S model that occurs from the variance rates. They got similar tests like Blattberg and Gonedes [1974], Latane and Rendlman [1976], Chiras and Manaster [1977]; all these testes were focused on the calculation of the standard deviation which is derived by substituting the observed market prices of the options on the B-S model while having all the other parameters constant. Macbeth and Merville deduced from their empirical results that the variance rate is not constant and changes over time and that these differences are also related to the differences in the stock prices the exercise prices and the time to expiration.

Interest rates

The assumption that the interest rates are known and constant overtime is a not so accurate one in the real world. In practice, the risk-free rates used in the B-S model, such as the U.S. Government Treasury Bill 30-day rate change over time of high volatility.

Dividends

The initial model does not account for the payment of dividends for the underlying asset, yet the biggest majority of the stock corporations pay dividends. Thus, Pavlat (1994) said that the option holders are at a significant disadvantage compared to shareholders. This issue has been solved by the Merton model that really gained popularity among the investor public.

No Transactions costs, perfect liquidity, trading

Another very important and necessary condition in order to derive the Black and Scholes differential equation, is that there are no fees for taking a long position or a short position in options and stocks and there are also no barriers to trading, something that is unrealistic. Moreover, it assumes that the markets have perfect liquidity and gives the investors the chance to buy or sell any amount of stocks or options at any given time, which are also unrealistic in the real world especially during the financial crisis in 2007-2008 and 2020 during the COVID-19 pandemic as traders had limited money to invest therefore buying and selling options wasn’t possible.

3.6 Binomial trees

The Black and Scholes formula caused a big shock amongst the economist at the time of its introduction. The economic ideas underlying the B-S model, such as the principles of risk – neutrality and riskless portfolios, shook the theory of option pricing to it's core. However, it's involved mathematical background based on diffusion models might have been complicated or too academic so that motivated various economists to search for simpler modeling framework that preserves the economically relevant properties of the B-S model but at the same time is more accessible. The binomial approach to option pricing grew out of a discussion between M. Rubinstein and W.F. Sharpe at a conference in Ein Borek, Israel (see Rubinstein 1992 for the historical background). They realized that the economic idea behind the BS formula can be reduced to the following principle; If an economy incorporating three securities can only attain two future states, one such security will be, that is to say that each single security can be replicated by the other two , a fact later on referred to as market completeness. It is understandable that with the above in mind, that one should introduce such a two-state model and verify that the economic properties of the Black and Scholes approach are preserved. This was the creation and birth of the binomial model for option pricing.

Binomial trees are referring to the development of a diagram that represents different possible paths that the stock price might follow during the duration of an option. An essential assumption in this option pricing models that the price of the underlying asset follows a random walk. In other words , at each step of the binomial tree , there is a certain probability that the price will follow an upward movement and a certain probability that it is going to follow a downward movement. This model is used for pricing mainly American options.

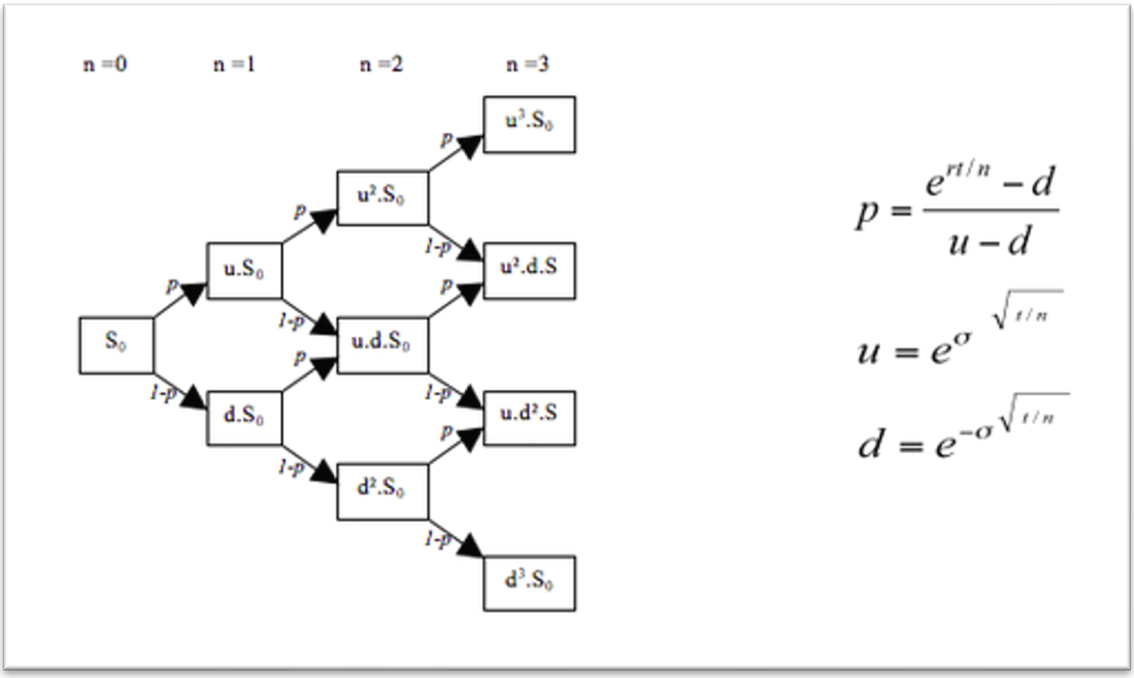


Figure 7: Binomial Tree options pricing model

In order to apply the Binomial Tree method, the following assumptions need to me made:

- No arbitrage opportunities
- The interest rate for borrowing is equal to the interest rate for investing and it is constant
- Time is discrete
- No taxes or transaction costs
- Markets are competitive meaning that any investor can buy (or sell) as much of any underlying asset as he wants without having to worry about the price of the asset.

The binomial tree option pricing model (is also known as the lattice approach) was presented by John Cox, Stephen Ross and Mark Rubinstein in 1979 in the paper that is titled "Options pricing: A simplified approach", published in the Journal Of Finance.

A very interesting feature of the Binomial Tree model, is that as the time period (steps) s getting smaller, the model converges to the Black-Scholes model. The Binomial model allows for the calculation of the prices of the underlying asset and the option for various periods of time along with the range of the possible outcomes over each period time. The most significant advantage of the model is that it provides a visualization of the changes of the prices of the underlying asset and the option from period of period. Another advantage is that it helps the investors decide when the option should be exercised and when it should be held and exercised at a future period, by providing them insight.

3.7 One step Binomial tree

We are going to start out with a one-step binomial tree, assuming that there are no arbitrage opportunities in the market. We consider the price of the underlying asset to be S_0 and the option that is addressed to the asset to be f. We also consider that the option's time to expiration is T and that during its life the price of the option can either move upwards from S_0 to S_0u (where u>1) or move downwards from S_0 to S_0d (where d<1). When the stock price goes up the percentage increase is u-1, while when the stock price goes down the percentage decreases d-1. In the event that the stock price goes upwards towards S_0u the option payoff is fu and if the stock price goes downwards to S_0d the payoff of the option is fd. All the above are described in the below diagram:

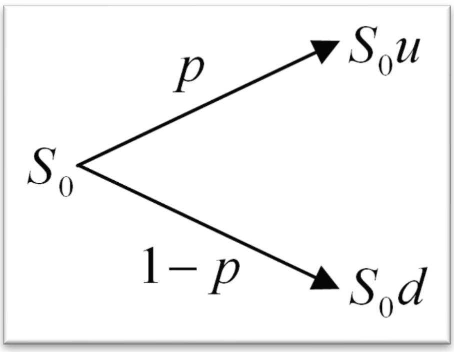


Figure 8: One step binomial tree

In order to continue, we consider a portfolio that consists of a long position in Δ shares and a short position is an option. We will find the value of the Δ in order to set the portfolio to be risk free.

In the first scenario we have that the stock price follows an upwards motion, then the value of the portfolio at the time to expiration T of the option is:

$$S_0 \times u \times \Delta - fu \quad (44)$$

In the second scenario we have that the stock price follows a downwards motion, then the value of the portfolio will be:

$$S_0 \times d \times \Delta - fd \quad (45)$$

The above two equations need to be equal in order to get the portfolio to be risk free:

$$S_0 \times u \times \Delta - fu = S_0 \times d \times \Delta - fd \quad (46)$$

That has a result:

$$\Delta = \frac{fu - fd}{S0 \times u - S0 \times d} \quad (47)$$

The above equation shows that Δ is the ratio of the change of the option price to the current asset price at time T. Now we have that the portfolio is riskless, there are no arbitrage opportunities and the investors get the risk-free interest rate. But is important to underline that in order to have no arbitrage opportunities we require $u > r > d$, if this relationship does not exist then arbitrage opportunities arise involving the stock price and the riskless rate. (COX et al. 1985)

If we denote the risk-free rate by r, the present value of the portfolio will be:

$$(S_0 \times u \times \Delta - fu)e^{-rT} \quad (48)$$

The cost in order to create the portfolio is:

$$S_0 \times \Delta - f \quad (49)$$

And we have that:

$$S_0 \times \Delta - f = (S_0 \times u \times \Delta - fu)e^{-rT} \quad (50)$$

$$f = e^{-rT} \{p \times fu + (1-p) \times fd\} \quad (51)$$

$$p = \frac{e^{-rT}-d}{u-d} \quad (52)$$

The above two equations are created through the one step binomial tree with the only assumption that there are no arbitrage opportunities in the market. But this model has a very crucial problem in it; the probabilities of the fluctuations of the underlying asset price are not included in it, therefore it is very easy when the price of the asset moves upwards it affects the price of the option because the value of a call option will increase and the value of the put option will decrease.

We have two very important comments to make regarding the above one step binomial formula:

⇒ The preferences of an investor regarding risk are not affected by the value of the call option, therefore the formula will be the same even if investors are risk lovers, risk averse or risk neutral.

⇒ The value of the call option depends only on the stock price which is the only stochastic value it depends on.

3.8 Two step binomial tree

The two-step binomial tree formula is basically an expanded version of the above simple one. For that reason, we will assume a call option with two periods before it's maturity time. Now the stock can take three possible values after two periods. The assumptions we made before are the same on every stage of the binomial tree. The stock price is S_0 and during each step it either moves up to S_0u or it either moves down to S_0d as we saw before. The duration of each step is Δt and r is the risk-free rate. An obvious difference from the one step binomial tree is the fact that now the value of the duration has changed from T to Δt so we have the below equations:

Figure 11.6 Stock and option prices in general two-step tree

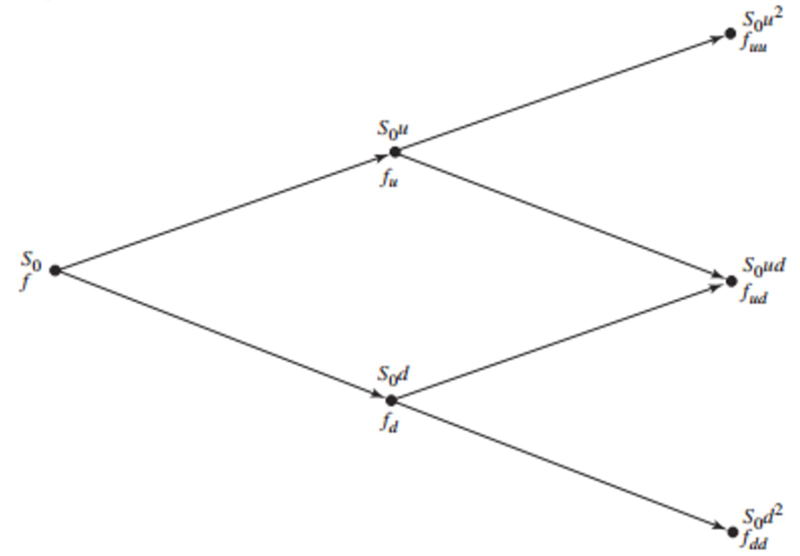


Figure 9: Two step binomial tree

$$f = e^{-r\Delta t}[\ pfu + (1-\ p)fd] \quad (53)$$

where

$$p = \frac{e^{-r\Delta t} - d}{u - d} \quad (54)$$

the results we have from the above two equations (53) and (54) are:

$$fu = e^{-r\Delta t}[pfuu + (1-\ p)fud] \quad (55)$$

$$fd = e^{-r\Delta t}[\ pfud + (1-\ p)fdd] \quad (56)$$

$$f = e^{-r\Delta t}[\ pfu + (1-\ p)fd] \quad (57)$$

If we substitute the first two equations (55) and (56) into the last one (57) we get the following:

$$f = e^{-2r\Delta t}\ [p^2fuu + 2p\ (1-p)\ fud + (1-p)^2fdd] \quad (58)$$

where

p^2 is the upper probability of the node, 2p(1-p) is the middle and $(1-p)^2$ is the lowest.

This equation gives the price of the call option when we have two periods. This equation can be expanded to more time steps.

3.9 Volatility estimation

The parameter of volatility is very crucial when developing a binomial tree model. The volatility is the same for both the real world and the risk neutral world. We assume that the expected return of the stock price is μ and the volatility of the stock price is σ. We also assume that p* is the probability of an upwards move in the real world and p is the probability of an upward move in the risk neutral world. In real world the expected stock price will be $S_0 e^{\mu \Delta t}$.

The expected stock price is

$$p^* \times S_0 \times u + (1-p^*) \times S_0 \times d \qquad (59)$$

In order for the expected stock return to be in accordance to the trees parameters we need the above two equations to be equal, therefore:

$$p^* \times S0 \times u + (1-p^*) \times S0 \times d = S0 \times e^{\mu \Delta t} \qquad (60)$$

then we have

$$p^* = \frac{e^{\mu \Delta t} - d}{u-d} \qquad (61)$$

Cox, Ross and Rubinstein in 1979 proposed the values of the u and d in accordance with the volatility:

$$u = e^{\sigma\sqrt{\Delta t}} \qquad (62)$$

and

$$d = e^{-\sigma\sqrt{\Delta t}} \quad (63)$$

Binomial trees can be applied also on options that are connected to stocks paying continuous dividends, currencies and futures .

One crucial principle of this formula is the fact that we have the opportunity to assume the world is risk neutral when we are pricing an option and therefore the results, we derive are equal to the same options prices that are in force in the real world.

This model can value more accurately the price of the American options because it takes into consideration other factors like dividends when compared to the Black and Scholes model. As every model has disadvantages so has the binomial tree formula compared to the B-S model because binomial tree option pricing model is much more complicated, difficult, slower and not so useful if we want to calculate thousands of options in a short time unlike the Black and Scholes option pricing formula which is easier and quicker.

4. Alternative pricing methods

The B-S (1973) option pricing model is widely accepted and used worldwide in order to be able to value option contracts. However, its big acceptance the model has known to be a not so accurate pricing option model because it prices wrongly and deep in-the-money and out-of-the-money options. Specialist in the option pricing field, refer to this problem as a volatility skew or smile (as the Corrado and Su method said). With the phrase volatility skew we refer to the pattern that comes from calculating implied volatilities through the range of of exercise prices. Generally, it is most common that the the skew pattern is related to the extent the options are in-the-money or out-of-the-money. That is something that the Black and Scholes formula cannot incorporate into their model and therefore cannot predict it because of the reason that volatility is a property of the underlying asset and the same implied volatility value should be observed through all options on the same asset.
For this reason, new more appropriate models have been used that cover the deficiencies of the Black & Scholes model, but we for the purpose of this thesis, we are going to analyze two of those alternative methods and those are; the Corrado and Su (1996)model and the second model is about determing risk neutral probabilities.
First, we are going to talk about the Corrado and Su model which in more simple words, is basically the B-S formula but augmented with higher order moments; the skewness and kurtosis.

4.1 First model: Black-Scholes model augmented with Skewness and Kurtosis (Corrado Su method)

As it was said above, the first method that is going to be shown is the Corrado and Su (1996) model, that is basically an expanded version of the Black and Scholes model; it is augmented with two higher order moments which are the skewness and kurtosis. We use that specific model in order to account for nonnormal skewness and kurtosis in stock return distributions. This method is based on fitting the first four of a distribution to a pattern of observed option prices. [Corrado and Su -1996]. The mean of this distribution is being calculated by the option pricing theory, but it is estimated through the estimation of implied values for variance, skewness and kurtosis of the distribution of the stock returns. The basis of this method was first presented by Jarrow and Rudd (1982), while Brown and Robinson (2002) made some corrections on the formula.

The derivation of this model is based on the following notations:

τ: the term of the European option

S_0: the market price of the underlying asset at the time of the expiration of the option

$\tilde{S}_\tau$: the random price of the underlying asset at the options expiration date

r_f: the risk-free rate

K: the strike price of the option

The log- return of the underlying asset during the period of the holding is computed as follows:

$$r_f = \ln\tilde{S}_\tau - \ln S_0 = \ln(\tilde{S}_\tau / S_0) \quad (64)$$

Where the conditional distribution of $\tilde{S}_\tau$ depends on that of the log-return $\tilde{r}_t$, as follows :

$$\tilde{S}\tau = S0 \times e^{\tilde{r}\tau} \quad (65)$$

Therefore, the price of c_0 of a call option in the risk – neutral environment is calculated as follows:

$$c_0 = S_0 \times (e - r_f)^\tau \times E[(\tilde{S}\tau - K)^+] = (e - r_f)^\tau \times \int_{\ln(\frac{K}{S0})}^{\infty} (S_0 \times e^x - K) \times f(x)\, dx \quad (66)$$

where

$f(\cdot)$ is the conditional density of $\tilde{r}_\tau$

Corrado and Su (1996) adapt in their method, a Gram – Charlier series expansion of the standard normal density function to describe the empirical probability density of asset log returns in order to extract an option pricing formula, that sums the B-S option pricing formula plus adjustments terms for nonnormal skewness and kurtosis. Through this expansion, they have managed to approximate the underlying distribution with the alternative and more tractable log-normal distribution. The coefficient in this series expansion are functions of the higher moments of the original and the approximating distributions.

In this formula, the first two terms of this series expansion, reflect the initial Black and Scholes pricing model, while the third and fourth terms of this series reflect the effect or higher order of the assets returns distribution on the option price, such as skewness and kurtosis of the asset returns, respectively.

Applying a four – term Gram Charlier series expansion of the conditional density $f(\cdot)$, Corrado and Su solved equation (66) from above and derived the following expression for the prices of a call option:

$$\mathbf{c_0 = c_{BS} + \gamma_1 \times Q_3 + (\gamma_2 - 3) \times Q_4} \qquad \textbf{(67)}$$

where

- $\mathbf{c_{BS}}$**:** is the price of the call option calculated by the Black-Scholes model
- $\mathbf{Q_3}$**:** reflects the adjustments of the initial Black – Scholes model for non-zero Skewness
- $\mathbf{Q_4}$**:** reflects the adjustments of the initial Black- Scholes model for non-zero Kurtosis.

Generally, the nonnormal skewness and kurtosis give rise to the implied volatility skews as it can be shown in the below figure, that shows exactly that rise.

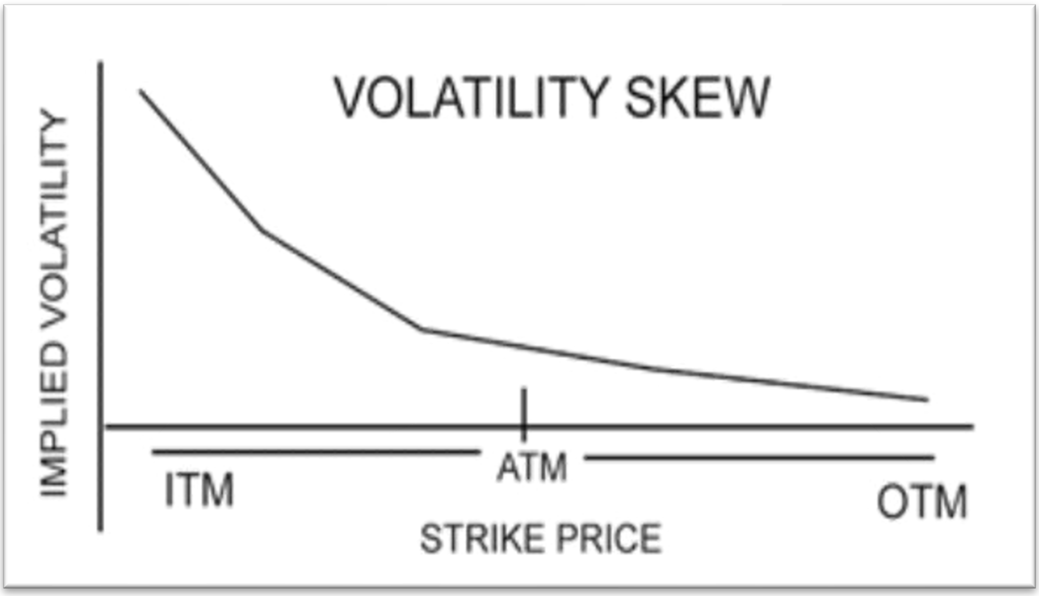


Figure 10: Presentation of the implied Volatility Skew caused by Q_3 and Q_4

The terms of the equation (67) are computed as shown below:

$$c_{BS} = S_0 \times N(d) - K \times e^{-rf \times \tau} \times N(d - \sigma_\tau) \quad (68)$$

$$Q_3 = \frac{1}{3!} \times S_0 \times \sigma_\tau \times [(2 \times \sigma_\tau - d) \times \varphi(d) + \sigma_\tau^2 \times N(d)] \quad (69)$$

$$Q_4 = \frac{1}{4!} \times S_0 \times \sigma_\tau \times [(d^2 - 3 \times d \times \sigma_\tau + 3 \times \sigma_\tau^2 - 1) \times \varphi(d) + \sigma_\tau^3 \times N(d)] \quad (70)$$

$$d = \frac{\ln\left(\frac{S_0}{K}\right) + r_f \times \tau + \sigma_\tau^2/2}{\sigma_\tau} \quad (71)$$

where

$\varphi(\cdot)$: is the standard normal density

N $(\cdot)$: is the cumulative normal distribution

γ_1 , γ_2 : are the fisher parameters for the skewness and kurtosis respectively

σ_τ : is the standard deviation of the underlying asset returns

Corrado and Su (1996) described in their thesis that Q_3 and Q_4 of the above equations (69) and (70) respectively, represent the marginal effect of non-normal skewness and excess kurtosis on the price c_{BS} of the option. In other words, if the underlying asset returns are normally distributed, then that means that $\gamma_1 = 0$ and $\gamma_2 = 3$ and the equation (68) from above is the same as the basic Black – Scholes model. All in all, the equation (67) is basically the option price calculated by the basic Black – Scholes model plus the calculated adjusted higher order moments, non-normal skewness and kurtosis.

Mailard (2018) calculated the first, second, third and fourth order moments, as shown below:

$$\mu_1 = 0 \tag{72}$$

$$\mu_2 = 1 + \frac{1}{96} \times K^2 + \frac{25}{1296} \times S^4 - \frac{1}{36} \times K \times S^2 \tag{73}$$

$$\mu_3 = S + \frac{1}{4} \times S \times K - \frac{76}{216} \times S^3 + \frac{1}{32} \times S \times K^2 - \frac{13}{144} \times K \times S^3 + \frac{85}{1296} \times S^5 \tag{74}$$

$$\mu_4 = 3 + K + \frac{7}{16} \times K^2 + \frac{3}{32} \times K^3 + \frac{31}{3072} \times K^4 - \frac{7}{216} \times S^4 - \frac{25}{486} \times S^6 + \frac{21665}{486559872} \times S^8 - \frac{7}{12} \times K \times S^2 + \frac{113}{432} \times K \times S^4 - \frac{5155}{46656} \times K \times S^6 - \frac{7}{24} \times K^2 \times S^2 + \frac{2455}{20736} \times K^2 \times S^4 + \frac{65}{1152} \times K^3 \times S^2 \tag{75}$$

where

μ_i is the *i*th central moment of underlying asset returns.

The Fisher parameters γ_1 and γ_2 for skewness and kurtosis respectively are calculated as following :

$$\gamma_1 = \frac{\mu_3}{\mu_2^{(3/2)}} \tag{76}$$

$$\gamma_2 = \frac{\mu_4}{\mu_2^2} \tag{77}$$

To further complete the model, based on the Put-Call Parity Formula, the price p_0 of a put option with the same exercise price is calculated as follows:

$$p_0 = c_0 + K \times e^{-r_f \tau} - S_0 \tag{78}$$

The biggest advantages this model has are the implementation simplicity and the computational efficiency. In comparison with the initial Black-Scholes model, this model does not require complex calculations that were developed through optimization programs. The only thing that it requires is the estimates of the higher order moments of the underlying asset's returns, which are determined in the context of the discrete outcomes of the asset prices at the option's maturity.

4.2 Second model: Determing risk neutral probabilities

In the environment of the no arbitrage hypothesis the price of the European options is being computed as the expectation of its discounted by the riskless rate payoff (always by paying respect to the risk-neutral measure). [Topaloglou, Vladimirou, Zenios 2007]. This principle was first implemented by Harrison and Kreps [1979], who applied this method and introduced this pricing model, where the price of the European option is computed as the discounted payoff of its expected value by the risk-free interest rate r_f. The critical input in order to price correctly a European option with τ term maturity periods at a non-terminal node $n_0 \in N \setminus N_T$ of a scenario tree is the distribution of the underlying assets price at the time to expiration of the option, conditional on node n_0.

The leaf nodes L_T of a subtree of a scenario tree represent the desired conditional distribution, such as the possible pieces of the underlying asse on the options maturity, conditional on the initial price an node n_0 at the date of the valuation. The discrete support of this distribution of the underlying price is
$\Omega = \{\{\omega^n = S^n: n \in L_T\}$, while the corresponding conditional possibilities of the physical distribution are $P = \{\{p_n = \pi_n/\pi_{n0}: n \in L_T\}$, where π_n w s defied a the unconditional probability mass of node n.

In order to derive this alternative option pricing method on a scenario tree we need the following notations:

τ : the term of the European option priced at the root node n_0
S_0: the market price of the underlying asset at root node n_0
$\widetilde{S}_\tau$: the random price of the underlying asset at the options maturity date
S^n: the price of underlying asset at node $n \in L_T$ of the subtree
r_f : the risk-free rate
K: the exercise price of the option
P: the physical probability measure for the discrete conditional distribution of the underlying asset price at the options maturity date
$\bar{P}$: an equivalent risk-neutral probability measure for the same discrete distribution

A main theorem in risk neutral probability valuation [Jacob and Shiryaev 1998] describes that a model that has asset prices is free of any arbitrage opportunities if and only if there is a probability $\bar{P}$ (risk neutral measure), under which we have that the discounted process of the asset prices is a martingale. When we are talking about a martingale condition , in probability theory, we mean that over a particular time interval , under the risk-neutral measure $\bar{P}$, the return of the underlying asset is equal to the riskless return over the exact same time interval (Neftci , 1996, Chapter 15). In other words is:

$$E_{\bar{p}} \times \left[e^{-r_f \tau} \times \widetilde{S_\tau} | S_0 | \right] = S_0 \quad (79)$$

Following the above, an equivalent martingale measure over the discrete support, of the asset prices, must follow the below:

$$\sum_{n \in L_\tau} \overline{p_n} \times S_i^n = S_{0i} \times (e^{rf})^\tau \quad , \ i=1,\dots,I, \quad (80)$$

$$\sum_{n \in L_\tau} \overline{p_n} = 1 \quad (81)$$

$$\overline{p_n} > 0, \forall n \in L_\tau \quad (82)$$

Equation (81) makes sure that the $\bar{P} = \{\overline{p_n} : n \in L_\tau\}$ is the right measure we need and the equation (82) makes sure that the required equivalence between the P and the risk-neutral, $\bar{P}$, is a proper probability measure.

The equivalence between P and $\bar{P}$, makes a necessary condition that the both of them need to associate nonnegative probability to the same domains. More specifically:

$P(Z) > 0 \leftrightarrow \bar{P}(Z) > 0$, $P(Z) = 0 \leftrightarrow \bar{P}(Z) = 0 \ \forall Z \subset \Omega$

In our analysis, both the P and the $\bar{P}$, are being defined over the exact same discrete set, Ω and equation (IV) requires that the risk-neutral measure need to match each positive mass, $\overline{p_n}$, to each possible outcome of Ω (more specifically, each conditional outcome of asset prices over the nodes, L_τ, of the subtree which we use to price the options).

In their analysis, Harrison and Kreps (1979) indicated that all the asset price models that have a no "Free lunches" policy, have an equivalent martingale measure (in our analysis we also use the no "free lunches" policy). As a result, that bring us to the point of existence of an equivalent risk-neutral (in other words martingale) measure on the set of the asset prices scenarios.

Usually in most problem cases, the essential martingale conditions that are equations (80) to (82) are not very sufficient and effective in order to determine the risk -neutral probabilities. The reason for this is that this linear equation system is in most cases underdetermined. Therefore, for this system to be completely determined, in order to result in a unique martingale result, it is very crucial for the number of the linear independent securities, I, to be equal to the amount of the possible outcomes $|L_\tau|-1$.

That is something that does not typically happen because the number of the price outcomes of the scenario tree to approximate the distribution of random prices, is much larger than the number of assets. Hence, we turn to developments in market equilibrium approach to price options.

A very common assumption in these models is that the market participants can be united into a representative agent. The utility function of that representative agent is a choice among various classes and the more appropriate choice is the one that represents the aggregate market as well.

In equilibrium, the relationship between P and $\bar{P}$ is shown through the following expression for the current asset price whose random value at τ is $\widetilde{S_\tau}$:

$$\mathbf{S_0} = (e^{-r_f})^\tau \times \mathbf{E_P}\,[\xi\widetilde{S_\tau}] \qquad (83)$$

Where

ξ: is the stochastic discount factor (pricing kernel)

$E_P[\cdot]$: shows the expectation operator with respect to the physical probability measure P.

We are going to follow the method of Bakshi et. al. (2003), which relates the physical and risk-neutral probability measures through a pricing kernel. We adapt this method to the case of a discrete distribution of asset returns. In a continuous setting, the equilibrium equation (V) relates the physical, P, and the equivalent risk-neutral measure $\bar{P}$ as shown below:

$$\mathbf{S_0} = (e^{-r_f})^\tau\ \mathbf{E_P}\,[\xi\widetilde{S_\tau}] = (e^{-r_f})^\tau \int_\Omega \widetilde{S_\tau}(\omega)\,d\bar{P}(\omega) = (e^{-r_f})^\tau \int_\Omega \widetilde{S_\tau}(\omega)\,d\bar{P}(\omega) \qquad (84)$$

Also

$$\xi(\omega)\,dP(\omega) = d\bar{P}(\omega) \Rightarrow \frac{d\bar{P}(\omega)}{dP(\omega)} = \xi(\omega) \qquad (85)$$

The P and $\bar{P}$ respectively, are defined on a measurable space (Ω, F).

This result is very essential, when the support of the asset price distribution is a finite discrete set, as is the case with the discrete outcomes $\Omega = \{\omega^n = S^n: n \in L_\tau\}$ that we use in order to present the distribution of the asset price $\tilde{S}_\tau$. In our situation, F includes all the subsets of Ω. In discrete distribution setting, the equation (VII) is:

$$\xi\ (\omega^n) = \frac{\bar{P}(\omega^n)}{P(\omega^n)} \Rightarrow \bar{P}(\omega^n) = \xi\ (\omega^n)\ P(\omega^n),\ \forall \in \Omega \tag{86}$$

Bakshi et. al. (2003) derived a transformation between the physical, P, and the risk-neutral probabilities, $\bar{P}$. This transformation in the case of a discrete support is the following:

$$\tilde{p}_n = \frac{E_p[\xi|S^n]p_n}{\sum_{n\in L_\tau} E_p[\xi|S^n]p_n}, \quad n \in L\tau \tag{87}$$

Where

ξ: is the general change of measure pricing kernel

To continue, we will make the basic assumption that the hypothesis of a power utility function of a representative market agent, the pricing kernel can be the following:

$$E_P[\xi \mid S^n] = (S^n)^{-\gamma} = e^{-\gamma \ln(S^n)}, \quad n \in L\tau \tag{88}$$

Where

γ: is the coefficient of relative risk aversion

Putting equation (88) in (87) and deriving both the denominator and the numerator by the $S_0^{-\gamma}$ we get:

$$\tilde{p}n = \frac{e^{-\gamma \ln(S^n/S_0)} p_n}{\sum_{n\in L_\tau} e^{-\gamma \ln(S^n/S_0)} p_n} = \frac{e^{-\gamma R^n} p_n}{\sum_{n\in L_\tau} e^{-\gamma R^n} p_n}, \quad n\in L\tau, \tag{89}$$

where we have that

$R^n = \ln(S^n/S_0)$ is the return of the underlying asset at the leaf node $n \in L_\tau$

The equation (89) is supposed to introduce a risk-neutral measure for the first time, by converting the physical measure, always in sync with the principles regarding equilibrium. In other words, we are talking about a transformation of the physical measure into a risk-neutral one and it depends on the parameter γ, which basically is the risk-neutral parameter. The value of this parameter, is estimated mainly empirically through the observed market prices of options and the underlying asset and can be affected by errors that may come through the estimation and wrong measurements. More specifically, we estimate the value of parameter γ by using the observed option prices at a relatively recent date (before the date we do our analysis of course) and then we use the same exact value to price options at stages of the scenario tree that correspond to later period.

$$\gamma = argmin_{\gamma}\sum_{i=1}^{m}((CP_i(\gamma') - MP_{i)}/MP_i)^2 \quad (90)$$

It is very important to determine the necessary martingale conditions in the estimation of risk – neutral probabilities before we continue on to complete the pricing of options at a node $n_0 \in N/N\tau$ of the scenario tree. We obtain the risk-neutral probabilities of the price at the expiration date of the option from the solution of the below program:

$$\textbf{Minimize } \overline{p_n} \in L_\tau \quad \sum_{n \in L_\tau}(\overline{p_m - \widehat{p_n}})2$$

$$\text{s.t } \sum_{n \in L_\tau}\overline{p_n} \times Sn = S0(e^{r_f})^T \quad (91)$$

$$\sum_{n \in L_\tau}\overline{p_n} = 1$$

$$\overline{p_n} > 0 \ \forall n \in L_\tau$$

The values $\hat{p}_n$ are reflecting the conditional probabilities for the discrete states $n \in L_\tau$ implied by the principles there are in equilibrium environment.

Once we estimate the $\bar{p}_n$ for discrete outcomes, S^n, $n \in L_\tau$ of the underlying asset's price the pricing of option is the easy part and very straightforward.

The fair price of a European call option at node n_0, with K as the strike price, is the expected value of its payoff over the risk-neutral measure, discounted by the riskless rate as seen below:

$$c_0\ (S_0\ ,K) = (e^{-rf})^T \times \sum_{n\in L_\tau} \overline{p_n} \times [\max(S^n - K, 0)] \quad (92)$$

Whiles the fair price of a European put option is the below:

$$p_0\ (S_0,K) = (e-rf)^\tau \times \sum_{n\in L_\tau} \overline{p_n} \times [\max(K - S^n, 0)] \quad (93)$$

Basically, the equation (90) that is the transformation and the (92) and the (93) are very easy to compute, while the equation (91), the quadratic program, must be solved at each node of the scenario tree.

King (2002) develops a mathematical basis for the analysis of the claims in discrete time. Basically, what he does is the fact that he models the hedging problem at a stochastic program. He makes a very essential point that the absence of arbitrage in the hedging problem translates in the dual to the existence of a valuation operator that makes a discrete discounted price into a martingale. The dual problem establishes very essential asset pricing theorems and determing the unique martingale measure. It also provides multiple solutions that give rise to the range of prices for the claim.

In complete market environment, both King and the method shown above can determine a very unique result as a martingale measure over the discrete set of price outcomes. That is the reason why , we don't have to solve the equation (XI) because there is a sufficient number of linear independent securities so that the system that corresponds to the martingale condition shown before is completely determined the martingale measure it is needed .However in the case of not so complete markets (incomplete) we tend to seek solution in the principles of the equilibrium environment in order to determine an approximation of the options price rather than having a price range within a bid ask interval.

5. Empirical Application – Pricing on S&P 500

In this 5th section of our analysis, we empirically test and implement with numerical tests the pricing methods for the European options as they are described above. More specifically, we are going to be concentrated on the comparison of the three pricing models, the Black and Scholes model, the Corrado and Su method that is the B-S formula but augmented with the skewness and kurtosis and the Risk neutral probabilities method. As said before the last two methods take into account higher order moments as they show on the empirical distribution of the returns of the underlying asset.

We will run numerical tests in order to check the correctness and the performance of the models. For that reason, we assume that the Black and Scholes is our main comparison base and therefore the estimations we get from it are our basis in order to review the other two methods. The main purpose is to compare the estimations we get from the Black and Scholes model and the other two piecing methods with the real market data and decide which methods produce the most accurate and consistent with the market prices results. A very important disclaimer that needs to be made here, is the fact that it is near impossible that any of these methods can produce results that can keep up exactly with the observed market route.

5.1 Data description

We mainly focus on European options, call and put, addressed to the S&P 500 index with expiration date T= 30 days and the exercise (strike) prices K that vary from 0.9*S0 (in-the-money) to 1.10*S0(out-of-the-money). The historical monthly prices on the S&P 500 index are from 01/01/2018 until 31/12/2022. The fluctuation of the monthly data prices of the S&P 500 can be seen in the below graph:

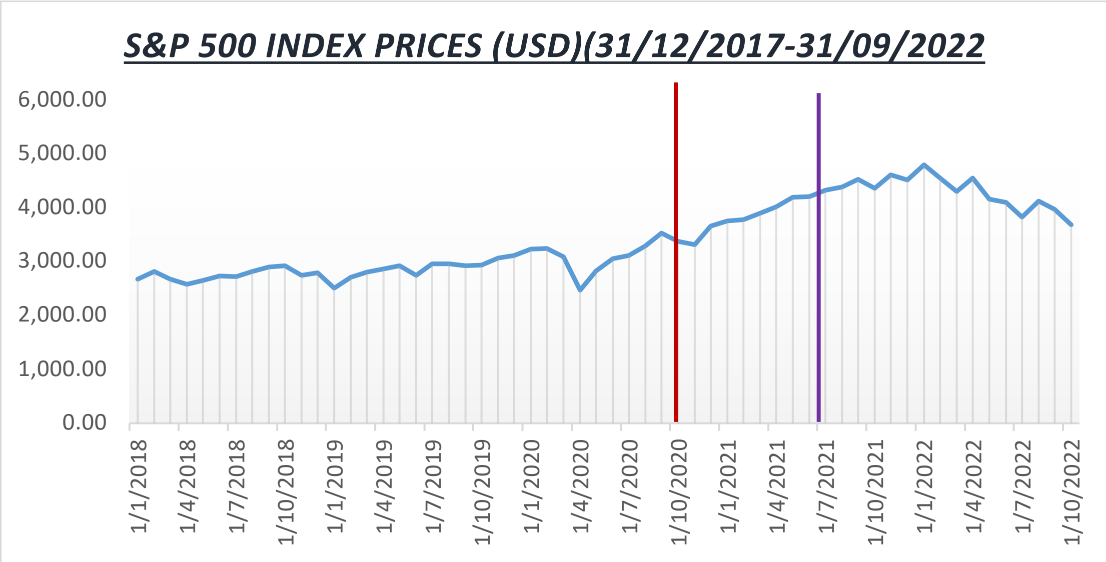


Figure 11: Monthly Historic prices on the S&P500 index (31/12/2017-30/09/2022)

In order to continue with our testing, we are going to use two different dates; the first date is going to be the one where the stock index is high, which means that the investors assume that the market will follow an upward movement. As far as the second date goes, it is going to be the one where the index is low, which means that the investors assume the market is going to follow an downward movement. More specifically, we have tested the two alternative methods, shown in chapter 4, on ***30Th of September 2020*** when the S&P 500 index is expected to move downwards and on the ***30th of June 2021*** when the S&P 500 is expected to move upwards as we can see in the red and purple lines above.

The prices on these dates are; on 30st of September 2020 the price of the S&P500 Index is S0= 3,380.80 and on 30th of June 2021 is S0 = 4,319.94. For the calculations that we need, we are going to use the 3-month US Treasure Bill Secondary Market Rate, as our risk-free rate. We got the data for the rate from the Thomson Reuters DataStream and because of the fact that DataStream provides the values of the US T-bill in quarterly rates, we are going to use the following equation in order to find the annualized rates:

$$(1 = r_{usquart})^4 = 1 + r_{usquart} \Rightarrow r_{usquart} = (1 = r_{usquart})^4 - 1$$

5.2 Numerical test results of the Black and Scholes Model

We priced the European put and call option with 30 days to maturity and for exercise prices (strike) from 0.90*S0 to 1.10*S0 on the two dates we have selected below; on the 30st of September 2020 where the S0= 3,380.80 and on the 30th of June 2021 where the price S0=4,319.94. In order to calculate and price the option with the Black & Scholes formula we are going to use the equations (27), (28), (29), (30) described above in section 3. We have estimated the volatility based on the returns of the S&P 500 index for the recent 10 years. After our calculations we have extracted the results, using the data in Tables 3 and 4 below, as shown in the Appendix; in Table 5 and 6 we have the estimation of European call and put option prices for 28th of February 2020 (01/03/2020) and in Tables 7 and 8 we have the estimation of European call and put option prices for 30th of June 2021 01/07/2021

DATA INPUT FOR 28th FEBRUARY 2020 (01/03/2020)	
S&P 500 PRICE S0	**3,090.23**
Time to expiration T	0.08333
Volatility - monthly σ_m	0.039763173
Volatility - Annually σ_a	0.137421507
Risk free interest rate - 3 months $r_{usquart}$	0.0125
Risk free interest rate - annually r_{usann}	0.050945337

Table 3: Data input for 01/03/2020

DATA INPUT FOR 30th JUNE 2021 (01/07/2021)	
S&P 500 PRICE S0	**4,319.94**
Time to expiration T	0.08333
Volatility - monthly σ_m	0.044895437
Volatility - Annually σ_a	0.155522355

Risk free interest rate - 3 months $r_{usquart}$	0.00005
Risk free interest rate - annually r_{usann}	0.002001501

Table 4: Data input for 01/07/2021

5.3 Numerical test results of the Black and Scholes model augmented with skewness and kurtosis (Corrado and Su [1996])

Again, for this numerical test we priced the European put and call option with time to maturity 30 days, for strike price we used number range from 0.90*S0 to 1.10*S0 on the selected dates we said before, i.e. the 28th of February (01.03.2020) and on the 30th June 2021(i.e. 01/07/2021), where the stock price is S0= 3,380.80 and S0=4,319.94 respectively , using equations (68),(69),(70),(71) as described above in section 4. . We have estimated the volatility and the higher order moments based on the actual returns of the S&P 500 index for the recent 10 years (i.e. 120 months) in order to derive the Skewness and Kurtosis for the adjusted Black and Scholes model by Corrado and Su using equations (72),(73),(74),(75) and the Corner – Fisher parameters using equations (76),(77). After our calculations we have extracted the following inputs and outputs of the price estimations of the Black and Scholes model augmented with the Skewness and Kurtosis, using the data from Tables 9 and 10 below, as they are presented in the tables in Appendix; in Tables 10, 11 we have the estimated prices for European call and put options respectively options using the Skewness and Kurtosis adjusted Black-Scholes model for 28th February 2020(01/03/2020) and in Tables 12, 13 we have the same Estimations but for the 30TH of June 2021(01/07/2021):

Data input for Skewness and Kurtosis augmented B-S model (28/02/2020 ie 01/03/2020)	
S&P500 Index price	**3,090.23**
Time to maturity	0.08333
Volatility-monthly	0.039763173
Volatility-annually	0.137421507

Risk free interest rate – 3 months	0.0125
Risk free interest rate -annually	0.050945337
Mean	0.009322084
Variance	0.001592515
Skewness	-0.587208131
Kurtosis	0.880673072
Fisrt moment μ1	0
Second moment μ2	1.001937316
Third moment μ3	-0.647963968
Fourth moment μ4	4.050569265
Cornish – Fisher γ1	-0.646085551
Cornish – Fisher γ2	4.034920288

Table 9: Data input for Skewness and Kurtosis augmented Black & Scholes model-01/03/2020

Data input for Skewness and Kurtosis augmented B-S model (30/06/2021 ie 01/07/2021)	
S&P500 Index price	**4,319.94**
Time to maturity	0.08333
Volatility-monthly	0.044895437
Volatility-annually	0.155522355
Risk free interest rate - 3 months	0.00005
Risk free interest rate -annually	0.002001501
Mean	0.010834537
Variance	0.002031967
Skewness	-0.900486884
Kurtosis	4.055200353
Fisrt moment μ1	0
Second moment μ2	1.092641341
Third moment μ3	-1.790756791
Fourth moment μ4	16.07872136
Cornish - Fisher γ1	-1.567906514
Cornish - Fisher γ2	13.46778742

Table 10: Data input for Skewness and Kurtosis augmented Black & Scholes model-01/07/2021

The equation (69), (70) represent the marginal effect of the non-normal skewness and kurtosis of the option as calculated by the Black-Scholes model.

In order to further complete our analysis we compute the moneyness using the equation below:

$$Moneyness\ (\%) = \frac{K \times e^{-rf \times T} - S_0}{\times e^{-rf \times T}} \times 100$$

From the tables 10,11,12 and 13 in Appendix we get that the negative(positive) skewness causes the Black-Scholes model to overprice (or underprice) out-of-the-money options and underprice (or overprice) in-the-money options.

5.3 Numerical test results of the Risk Neutral Probabilities Model

Once again, we have calculated the prices of the European put and call option with time to expiration 30 days and for strike prices from 0.90*S0 to 1.10*S0 on the two dates we have selected, i.e. on the 28th of February 2020(01/03/2020) and on the 30th of June 2021(01/07/2021), using the equations described in section 4, of the model. We have generated 120 scenarios that are based on the actual prices of the returns of the S&P500 for the recent 10 years (i.e., 120 months). For this purpose, we used the following equation $S_i = S_0\ (1+r_i)$ in order to calculate the estimated prices S_i, using the actual price S_0 and the actual return of the index r_i as observed for the recent 10 years or I other words 120 months. Based on the estimated prices for those 120 months, we estimated the physical probabilities for the discrete conditional distribution and then we estimated the equivalent risk neutral probability measure for that same discrete distribution as it is shown below:

⇒ *Estimation of the physical and the equivalent risk-neutral probabilities for 28/02/2020 i.e., 01/03/2020*

In Table 14 we have estimated the Physical Probabilities and the equivalent Risk-Neutral Probabilities, using the 120 scenarios we got from the date of the 10 previous years (i.e., 120 months) from 01/03/2020 and in tables 15,16 we have estimated the European call and put prices respectively using the Risk-Neutral Probability model for the 28th of February 2020(01/03/2020). For the below estimations we used the data as shown below:

- $\mathbf{S_0}$ = 3,090.23
- $\mathbf{T}$ = 0.08333
- $\mathbf{r_i}$ = 0.0509453
- $\mathbf{P_n}$ = 0.0083333

⇒ *Estimation of the physical and the equivalent risk-neutral probabilities for 30/06/2021 i.e 01/07/2021*

In Table 17 we have estimated the Physical Probabilities and the equivalent Risk-Neutral Probabilities, using the 120 scenarios we got from the date of the 10 previous years (i.e., 120 months) from 01/07/2021 and in tables 18,19 we have estimated the European call and put prices respectively using the Risk-Neutral Probability model for the 30th of June 2021(01/07/2021). For the below estimations we used the data as shown below:

- $\mathbf{S_0}$ = 4,319.94
- $\mathbf{T}$ = 0.08333
- $\mathbf{r_i}$ = 0.002002
- $\mathbf{P_n}$ = 0.0083333

5.5 Comparison of the Black & Scholes model and the Alternative option pricing methods

The Comparison of the numerical results of the Black-Scholes model and the two Alternative option pricing models, the adjusted with the skewness and excess kurtosis Black-Scholes mode and the Risk-Neutral Probabilities model, regarding the estimation of European call and put options for a maturity of 30 days on the two selected dates, i.e., ***February 28th 2020 (i.e. 01/03/2020)*** and ***June 30th 2021 (i.e. 01/07/2021)*** are presented in the tables below:

Comparison of the numerical results of the prices of the European call option [28/02/2020] (i.e01/03/2020)			
Strike price	**Black-Scholes**	**Skewness-Kurtosis**	**Risk-Neutral**
ITM 2,781	320.9053868	321.5105818	320.80
2,803	299.0895786	299.996771	298.90
2,825	277.3428219	278.6332907	276.99
2,847	255.7066503	257.4475854	255.08
2,869	234.2403466	236.4650038	233.18
2,891	213.0245466	215.7114458	211.27
2,913	192.1637199	195.2200595	189.36
2,935	171.786658	175.0416789	167.46
2,957	152.0441942	155.2570449	145.55
2,979	133.1036809	135.9873688	123.64
3,001	115.1402488	117.3991544	101.74
3,023	98.32546664	99.69985941	79.83
3,045	82.81460215	83.12298446	57.92
3,067	68.73408262	67.90399368	36.02
ATM 3,090	**55.624464**	53.65729077	13.09
3,112	44.69143956	41.80614795	0.000000
3,134	35.30520336	31.75183287	0.000000
3,156	27.40685451	23.47950415	0.000000
3,178	20.89636089	16.89051655	0.000000
3,200	15.64198228	11.81721197	0.000000
3,222	11.49141567	8.045438435	0.000000
3,244	8.283181086	5.339938465	0.000000
3,266	5.856956218	3.467962126	0.000000
3,288	4.061935199	2.217698399	0.000000

3,310	2.762729908	1.409851594	0.000000
3,332	1.842754107	0.902369909	0.000000
3,354	1.205362126	0.589567076	0.000000
3,376	0.773219862	0.397498442	0.000000
OTM 3,399	0.47847167	0.274194259	0.000000

Table 20: Comparison of the numerical results of the prices of the European call option [28/02/2020] (i.e. 01/03/2020)

Based on the above results, we conclude that the Skewness and Kurtosis adjusted Black - Scholes model is performing better that the Black-Scholes model and the Risk Neutral Probability model. The Black-Scholes model also performs slightly better that the Risk Neutral Probability model.

As far as the deep ITM call options are concerned, the Risk Neutral Probabilities model appears to be the most efficient, because it estimates the call option prices at a more realistic level than the other two models. To the contrary, as far as the OTM call option are concerned, the Skewness and Kurtosis adjusted Black – Scholes model appears to be the most efficient and more accurate, because it estimates the call option prices in a more realistic level as the other two models' price higher.

As we can clearly see, the risk neutral probability model, when the price of the option is negative it considers it as zero as it doesn't take into consideration negative prices, only positive.

The Black-Scholes model is mispricing deep OTM call options because of the observed negative skewness premium on this selected date.

Comparison of the numerical results of the prices of the European put option [28/02/2020] (i.e01/03/2020)			
Strike price	**Black-Scholes**	**Skewness-Kurtosis**	**Risk-Neutral**
ITM 2,781	0.099954632	0.705619944	0.000000

2,803	0.190944716	1.098611052	0.000000
2,825	0.350986128	1.641932713	0.000000
2,847	0.621612765	2.36302935	0.000000
2,869	1.062107226	3.287249656	0.000000
2,891	1.753105435	4.44049359	0.000000
2,913	2.799076949	5.855909203	0.000000
2,935	4.328813293	7.584330511	0.000000
2,957	6.493147599	9.706498389	0.000000
2,979	9.459432589	12.34362427	0.000000
3,001	13.40279867	15.66221171	0.000000
3,023	18.49481468	19.86971866	0.000000
3,045	24.89074838	25.19964564	0.000000
3,067	32.71702706	31.88745677	0.000000
ATM 3,090	42.53287274	40.56622207	0.000000
3,112	53.50664651	50.62188117	8.8157332
3,134	66.0272085	62.47436801	30.722535
3,156	80.03565785	76.1088412	52.629337
3,178	95.43196242	91.42665552	74.536139
3,200	112.084382	108.2601529	96.442941
3,222	129.8406136	126.3951812	118.34974
3,244	148.5391772	145.5964832	140.25654
3,266	168.0197505	165.6313088	162.16335
3,288	188.1315277	186.287847	184.07015
3,310	208.7391206	207.3868021	205.97695
3,332	229.725943	228.7861223	227.88375
3,354	250.9953492	250.3801214	249.79055
3,376	272.4700052	272.0948547	271.69736
OTM 3,399	294.8487931	294.6450905	294.3709

Table 21: Comparison of the numerical results of the prices of the European put option [28/02/2020] (i.e. 01/03/2020)

Based on the above results, we conclude that the Skewness and Kurtosis adjusted Black-Scholes model is performing better that the Black-Scholes model and the Risk Neutral Probability model. The Black-Scholes model also performs slightly better that the Risk Neutral Probability model.

As far as the ITM put options are concerned, the Skewness and Kurtosis adjusted Black - Scholes model appears to be the most efficient than the other two models, because it estimates put option prices in a more realistic level than the other two. As far as deep OTM are concerned, the Risk Neutral Probabilities model appears the most efficient as it estimates put option prices in a more realistic way that the other two which price more expensively.

The Black-Scholes model in constantly mispricing deep in-the-money (ITM) put options.

Comparison of the numerical results of the prices of the European call option [30/06/2021] (i.e01/07/2021)			
Strike price	**Black-Scholes**	**Skewness-Kurtosis**	**Risk-Neutral**
ITM 3,887	434.060064	448.4359288	433.48826
3,918	403.600264	421.6313529	402.6434
3,949	373.3530951	394.8210854	371.79855
3,980	343.4082138	367.5616457	340.95369
4,011	313.8792552	339.3690658	310.10884
4,041	284.9045646	309.8130303	279.26398
4,072	256.6457689	278.6283454	248.41913
4,103	229.2837825	245.8205707	217.57427
4,134	203.0121394	211.7390377	186.72942
4,165	178.0279213	177.0940352	155.88456
4,196	154.5209461	142.9056513	125.0397
4,226	132.6622143	110.3872346	94.19485
4,257	112.5928178	80.78246781	63.349995
4,288	94.41453409	55.1870187	32.50514
ATM 4,319	78.18316286	34.39011311	1.6602845
4,350	63.90532182	18.76692888	0.000000
4,381	51.53897874	8.240838745	0.000000
4,412	40.99752631	2.318725341	0.000000
4,442	32.15679312	0.187212536	0.000000
4,473	24.8640886	0.846542113	0.000000
4,504	18.94823962	3.254215851	0.000000
4,535	14.2295937	6.452633297	0.000000
4,566	10.52911546	9.662221978	0.000000

4,597	7.675942324	12.3313801	0.000000
4,628	5.51304319	14.14420025	0.000000
4,658	3.900890736	14.99430577	0.000000
4,689	2.719278643	14.93712935	0.000000
4,720	1.867569349	14.13357054	0.000000
OTM 4,751	1.263740652	12.79595009	0.000000

Table 22: Comparison of the numerical results of the prices of the European call option [30/06/2021] (i.e. 01/07/2021)

Based on the above table, we conclude that the Risk Neutral Probabilities model is performing better overall in comparison to the other two and also the Black and Scholes is performing better that the Skewness and Kurtosis Adjusted Black-Scholes model.

As far as the ITM call options are concerned, the Risk Neutral Probabilities model appears to be the most efficient, as it estimates the call option prices more realistic. As far as the OTM call option are concerned, the Risk-Neutral Probabilities model is the more efficient as it estimates the call option in a more realistic level that the other two, which price more expensively.

The Black-Scholes model is constantly mispricing deep out-of-the-money (OTM) call option, because of the observed negative skewness premium on that particular selected date.

Comparison of the numerical results of the prices of the European put option [30/06/2021] (i.e01/07/2021)			
Strike price	**Black-Scholes**	**Skewness-Kurtosis**	**Risk-Neutral**
ITM 3,887	0.57180799	14.9476727	0.000000
3,918	0.95686305	18.9879519	0.000000
3,949	1.55454927	23.0225396	0.000000
3,980	2.45452309	26.607955	0.000000
4,011	3.7704196	29.2602302	0.000000
4,041	5.6405841	30.5490497	0.000000
4,072	8.22664351	30.20922	0.000000

4,103	11.7095122	28.2463004	0.000000
4,134	16.2827242	25.0096226	0.000000
4,165	22.1433613	21.2094752	0.000000
4,196	29.4812412	17.8659464	0.000000
4,226	38.4673645	16.1923848	0.000000
4,257	49.242823	17.4324731	0.000000
4,288	61.9093945	22.6818791	0.000000
ATM 4,319	76.5228784	32.7298286	0.000000
4,350	93.0898924	47.9514995	29.1845706
4,381	111.568404	68.2702645	60.0294257
4,412	131.871807	93.1930062	90.8742808
4,442	153.875929	121.906348	121.719136
4,473	177.42808	153.410533	152.563991
4,504	202.357086	186.663062	183.408846
4,535	228.483295	220.706335	214.253701
4,566	255.627672	254.760778	245.098556
4,597	283.619354	288.274792	275.943411
4,628	312.30131	320.932467	306.788267
4,658	341.534012	352.627427	337.633122
4,689	371.197255	383.415106	368.477977
4,720	401.190401	413.456402	399.322832
OTM 4,751	431.431428	442.963637	430.167687

Table 23: Comparison of the numerical results of the prices of the European put option [30/06/2021] (i.e. 01/07/2021)

Based on the table above, we conclude that the Skewness and Kurtosis Adjusted Black-Scholes model is performing slightly better that the Black – Scholes model and the Risk-Neutral Probabilities model. Black-Scholes model is also performing better that the Risk Neutral Probabilities model.

As far as the in-the-money (ITM) put options are concerned, the Risk Neutral Probabilities model appears to be the most efficient of the three as it estimates in a more realistic level. As far as the out-the-money (OTM) put options are concerned the Skewness and Kurtosis Adjusted Black-Scholes model is the most efficient, as it estimates the put option prices more realistically, in contrast to the other two that price more expensively options

6. Conclusions

In conclusion, the present thesis presents an overview of the Options Theory, their meaning and their characteristics, and the most crucial and effective models for options pricing, with our main focus being the Black & Scholes model.

More specifically, derivatives are financial instruments whose values are dependent on the values of other, more simple and basic underlying variables, such as assets, interest rates etc. They are widely used across the world in the Finance sector for hedging, speculation and arbitrage. We use them for hedging in order to provide protection to the investors from the steep movements of the stock prices, for speculation in order to position the investors in the right direction of the financial market in the Future and for arbitrage in order to make sure investors make sure profits.

Options are the most common and most used derivative in the world. Options are contracts that provide their owner the right, but not the obligation to buy (call option) or sell (put option) an underlying asset by a specific date for a specific predetermined price. They are very useful in the hedging of portfolios because the allow the investors to reduce their risks from their payoffs and trade volatility movements instead of stock movements.

Generally, the pricing of option is a very difficult process as the values of the options depend on many different variables apart from the underlying asset. The most widely used model and the one our analysis focus mainly on in pricing options, is the Black-Scholes model and its advancements made by the years.

The main objectives of this thesis were the introduction of the main pricing model, the Black-Scholes model and the presentation of two alternative European option pricing methods, as well as evaluating their performance, through empirical numerical tests on the S&P 500 Index and its options.

We implemented the two alternative pricing methods where the underlying asset (in this thesis being the S&P 500 INDEX) follows a discrete distribution, which was constructed by a scenario tree. Also, we estimated the numerical results of the two Alternative methods, Skewness and Kurtosis Adjusted Black- Scholes model and the Risk Neutral Probabilities model and the main Black-Scholes Model in two different dates where the market prices were expected to move upwards or downwards and we compared the results by those computed by the Black- Scholes Model.

From the empirical results we concluded that the two Alternative models, which can capture the asymmetric and leptokurtic features of the distributions of the underlying assets prices, from accurate alternative methods for pricing European call and put options. Both these alternative methods produce mostly better results than the initial Black-Scholes Model.

The last conclusion we made from our analysis was that the Black – Scholes Model misprices OTM and ITM options. In other words, it misprices deep in-the-money (ITM) options and overprices deep out-of-the-money (OTM) options. This is why the Black & Scholes model due to its unrealistic assumptions does not produce accurate and realistic results for pricing options that are in agreements with the market conditions.

Last but not least, options form appropriate financial tools for maintaining risk management, all due to their asymmetric payoffs. That is why the efficiency in pricing of options is a crucial matter and has a direct result in increasing the safety net the investors have and help them for hedging.

7. Appendix

ESTIMATED EUROPEAN CALL OPTION PRICES - BLACK &SCHOLES FORMULA ->28/02/2020 IE 01/03/2020					
strike price	estimates price_bs formula	d1	d2	N(d1)	N(d2)
2,781	320.9054	2.782766	2.743096	0.997305120	0.996957
2,803	299.0896	2.584151	2.544481	0.995119046	0.994528
2,825	277.3428	2.387088	2.347418	0.991508787	0.990548
2,847	255.7067	2.191554	2.151884	0.985794127	0.984297
2,869	234.2403	1.997525	1.957854	0.977115885	0.974876
2,891	213.0245	1.804977	1.765307	0.964460894	0.961244
2,913	192.1637	1.61389	1.57422	0.946724350	0.942282
2,935	171.7867	1.42424	1.38457	0.922811507	0.916908
2,957	152.0442	1.236006	1.196336	0.891771905	0.884217
2,979	133.1037	1.049168	1.009498	0.852949569	0.843632
3,001	115.1402	0.863704	0.824034	0.806124745	0.79504
3,023	98.32547	0.679595	0.639925	0.751619442	0.738889
3,045	82.8146	0.49682	0.45715	0.690342128	0.676218
3,067	68.73408	0.315362	0.275692	0.623756473	0.608607
3,090	**55.62446**	**0.126854**	**0.087183**	**0.550471883**	**0.534737**
3,112	44.69144	-0.05197	-0.09164	0.479276054	0.463492
3,134	35.3052	-0.22954	-0.26921	0.409226471	0.393886
3,156	27.40685	-0.40586	-0.44553	0.342423536	0.327969
3,178	20.89636	-0.58096	-0.62063	0.280635175	0.267423
3,200	15.64198	-0.75485	-0.79452	0.225170818	0.213448
3,222	11.49142	-0.92754	-0.96721	0.176822023	0.166718
3,244	8.283181	-1.09907	-1.13874	0.135869261	0.127406
3,266	5.856956	-1.26943	-1.3091	0.102143510	0.09525
3,288	4.061935	-1.43865	-1.47832	0.075124480	0.069661
3,310	2.76273	-1.60674	-1.64642	0.054055173	0.049839
3,332	1.842754	-1.77372	-1.81339	0.038054438	0.034886
3,354	1.205362	-1.9396	-1.97927	0.026213942	0.023893
3,376	0.77322	-2.1044	-2.14407	0.017671824	0.016014
3,399	0.478472	-2.27384	-2.31351	0.011487941	0.010347

Table 5: Estimation of European call options for 01/03/2020 - Black & Scholes model

ESTIMATED EUROPEAN PUT OPTION PRICES BLACK &SCHOLES FORMULA ->28/02/2020(IE 01/03/2020)			
strike price	estimated price	N(-d1)	N(-d2)
2,781	0.099955	0.002695	0.003043
2,803	0.190945	0.004881	0.005472
2,825	0.350986	0.008491	0.009452
2,847	0.621613	0.014206	0.015703
2,869	1.062107	0.022884	0.025124
2,891	1.753105	0.035539	0.038756
2,913	2.799077	0.053276	0.057718
2,935	4.328813	0.077188	0.083092
2,957	6.493148	0.108228	0.115783
2,979	9.459433	0.14705	0.156368
3,001	13.4028	0.193875	0.20496
3,023	18.49481	0.248381	0.261111
3,045	24.89075	0.309658	0.323782
3,067	32.71703	0.376244	0.391393
3,090	42.53287	0.449528	0.465263
3,112	53.50665	0.520724	0.536508
3,134	66.02721	0.590774	0.606114
3,156	80.03566	0.657576	0.672031
3,178	95.43196	0.719365	0.732577
3,200	112.0844	0.774829	0.786552
3,222	129.8406	0.823178	0.833282
3,244	148.5392	0.864131	0.872594
3,266	168.0198	0.897856	0.90475
3,288	188.1315	0.924876	0.930339
3,310	208.7391	0.945945	0.950161
3,332	229.7259	0.961946	0.965114
3,354	250.9953	0.973786	0.976107
3,376	272.47	0.982328	0.983986
3,399	294.8488	0.988512	0.989653

Table 6: Estimation of European put options for 01/03/2020 – Black & Scholes model

ESTIMATED EUROPEAN CALL OPTION PRICES – BLACK &SCHOLES FORMULA ->30/06/2021 IE 01/07/2021					
strike price	estimated price_bs formula	d1	d2	N(d1)	N(d2)
3,887	434.06	2.3779	2.3330	0.9913	0.9902
3,918	403.60	2.2018	2.1569	0.9862	0.9845
3,949	373.35	2.0271	1.9822	0.9787	0.9763
3,980	343.41	1.8537	1.8088	0.9681	0.9648
4,011	313.88	1.6817	1.6368	0.9537	0.9492
4,041	284.90	1.5110	1.4661	0.9346	0.9287
4,072	256.65	1.3416	1.2968	0.9101	0.9026
4,103	229.28	1.1735	1.1286	0.8797	0.8705
4,134	203.01	1.0067	0.9618	0.8430	0.8319
4,165	178.03	0.8411	0.7962	0.7998	0.7870
4,196	154.52	0.6767	0.6318	0.7507	0.7362
4,226	132.66	0.5135	0.4686	0.6962	0.6803
4,257	112.59	0.3515	0.3066	0.6374	0.6204
4,288	94.41	0.1907	0.1458	0.5756	0.5580
4,319	78.18	0.0310	-0.0139	0.5124	0.4945
4,350	63.91	-0.1275	-0.1724	0.4493	0.4316
4,381	51.54	-0.2849	-0.3298	0.3878	0.3708
4,412	41.00	-0.4413	-0.4862	0.3295	0.3134
4,442	32.16	-0.5965	-0.6414	0.2754	0.2606
4,473	24.86	-0.7506	-0.7955	0.2264	0.2132
4,504	18.95	-0.9037	-0.9486	0.1831	0.1714
4,535	14.23	-1.0558	-1.1007	0.1455	0.1355
4,566	10.53	-1.2068	-1.2517	0.1138	0.1053
4,597	7.68	-1.3568	-1.4017	0.0874	0.0805
4,628	5.51	-1.5058	-1.5507	0.0661	0.0605
4,658	3.90	-1.6538	-1.6987	0.0491	0.0447
4,689	2.72	-1.8008	-1.8457	0.0359	0.0325
4,720	1.87	-1.9469	-1.9918	0.0258	0.0232
4,751	1.26	-2.0920	-2.1369	0.0182	0.0163

Table7: Estimation of European call options for 01/07/2021 - Black & Scholes model

ESTIMATED EUROPEAN PUT OPTION PRICES BLACK &SCHOLES FORMULA ->30/06/2021(IE 01/07/2021)			
strike price	estimated price	N(-d1)	N(-d2)
3,887	0.57	0.0087	0.0098
3,918	0.96	0.0138	0.0155
3,949	1.55	0.0213	0.0237
3,980	2.45	0.0319	0.0352
4,011	3.77	0.0463	0.0508

4,041	5.64	0.0654	0.0713
4,072	8.23	0.0899	0.0974
4,103	11.71	0.1203	0.1295
4,134	16.28	0.1570	0.1681
4,165	22.14	0.2002	0.2130
4,196	29.48	0.2493	0.2638
4,226	38.47	0.3038	0.3197
4,257	49.24	0.3626	0.3796
4,288	61.91	0.4244	0.4420
4,319	76.52	0.4876	0.5055
4,350	93.09	0.5507	0.5684
4,381	111.57	0.6122	0.6292
4,412	131.87	0.6705	0.6866
4,442	153.88	0.7246	0.7394
4,473	177.43	0.7736	0.7868
4,504	202.36	0.8169	0.8286
4,535	228.48	0.8545	0.8645
4,566	255.63	0.8862	0.8947
4,597	283.62	0.9126	0.9195
4,628	312.30	0.9339	0.9395
4,658	341.53	0.9509	0.9553
4,689	371.20	0.9641	0.9675
4,720	401.19	0.9742	0.9768
4,751	431.43	0.9818	0.9837

Table 8: Estimation of European put options for 01/07/2021 – Black & Scholes model

ESTIMATED EUROPEAN CALL OPTION PRICES -SKEWNESS AND KURTOSIS AUGMENTED B-S MODEL ->28/02/2020 IE 01/03/2020								
strike price	moneyness percentage	estimates price_SK-KU-AUGMENTED BS MODEL	CBS	d	Q3	Q4	φ(d)	N(d)
2,781	-11.58	321.51	320.91	2.76	-0.48	0.28	0.01	1.00
2,803	-10.71	300.00	299.09	2.57	-0.75	0.40	0.01	0.99
2,825	-9.85	278.63	277.34	2.37	-1.13	0.53	0.02	0.99
2,847	-9.00	257.45	255.71	2.17	-1.61	0.67	0.04	0.99
2,869	-8.16	236.47	234.24	1.98	-2.18	0.77	0.06	0.98
2,891	-7.34	215.71	213.02	1.79	-2.82	0.82	0.08	0.96
2,913	-6.53	195.22	192.16	1.60	-3.46	0.78	0.11	0.94
2,935	-5.73	175.04	171.79	1.41	-4.02	0.62	0.15	0.92
2,957	-4.94	155.26	152.04	1.22	-4.42	0.33	0.19	0.89
2,979	-4.17	135.99	133.10	1.03	-4.56	-0.07	0.23	0.85
3,001	-3.40	117.40	115.14	0.85	-4.37	-0.54	0.28	0.80
3,023	-2.65	99.70	98.33	0.66	-3.81	-1.04	0.32	0.75
3,045	-1.91	83.12	82.81	0.48	-2.90	-1.50	0.36	0.68

3,067	-1.18	67.90	68.73	0.30	-1.70	-1.84	0.38	0.62
3,090	-0.43	53.66	55.62	0.11	-0.24	-2.02	0.40	0.54
3,112	0.28	41.81	44.69	-0.07	1.22	-2.00	0.40	0.47
3,134	0.98	31.75	35.31	-0.25	2.59	-1.79	0.39	0.40
3,156	1.67	23.48	27.41	-0.42	3.75	-1.42	0.36	0.34
3,178	2.36	16.89	20.90	-0.60	4.62	-0.96	0.33	0.27
3,200	3.03	11.82	15.64	-0.77	5.15	-0.46	0.30	0.22
3,222	3.69	8.05	11.49	-0.95	5.34	0.02	0.26	0.17
3,244	4.34	5.34	8.28	-1.12	5.23	0.42	0.21	0.13
3,266	4.99	3.47	5.86	-1.29	4.86	0.73	0.17	0.10
3,288	5.62	2.22	4.06	-1.46	4.33	0.92	0.14	0.07
3,310	6.25	1.41	2.76	-1.62	3.71	1.00	0.11	0.05
3,332	6.87	0.90	1.84	-1.79	3.06	0.99	0.08	0.04
3,354	7.48	0.59	1.21	-1.96	2.44	0.92	0.06	0.03
3,376	8.08	0.40	0.77	-2.12	1.89	0.81	0.04	0.02
3,399	8.70	0.27	0.48	-2.29	1.40	0.67	0.03	0.01

Table 10: Estimation of European call options for Skewness and Kurtosis augmented Black & Scholes model-01/03/2020

ESTIMATED EUROPEAN PUT OPTION PRICES -SKEWNESS AND KURTOSIS AUGMENTED B-S MODEL ->28/02/2020 IE 01/03/2020		
strike price	moneyness percentage	estimates price_SK-KU-AUGMENTED BS MODEL
2,781	-11.58	0.705619944
2,803	-10.71	1.098611052
2,825	-9.85	1.641932713
2,847	-9.00	2.36302935
2,869	-8.16	3.287249656
2,891	-7.34	4.44049359
2,913	-6.53	5.855909203
2,935	-5.73	7.584330511
2,957	-4.94	9.706498389
2,979	-4.17	12.34362427
3,001	-3.40	15.66221171
3,023	-2.65	19.86971866
3,045	-1.91	25.19964564
3,067	-1.18	31.88745677
3,090	-0.43	40.56622207
3,112	0.28	50.62188117
3,134	0.98	62.47436801

3,156	1.67	76.1088412
3,178	2.36	91.42665552
3,200	3.03	108.2601529
3,222	3.69	126.3951812
3,244	4.34	145.5964832
3,266	4.99	165.6313088
3,288	5.62	186.287847
3,310	6.25	207.3868021
3,332	6.87	228.7861223
3,354	7.48	250.3801214
3,376	8.08	272.0948547
3,399	8.70	294.6450905

Table 11: Estimation of European put options for Skewness and Kurtosis augmented Black & Scholes model-01/03/2020

ESTIMATED EUROPEAN CALL OPTION PRICES -SKEWNESS AND KURTOSIS AUGMENTED B-S MODEL ->30/06/2021 IE 01/07/2021								
strike price	moneyness percentage	estimates price_SK-KU-AUGMENTED BS MODEL	CBS	d	Q3	Q4	φ(d)	N(d)
3887.1	-11.1538	448.4359	434.06	2.36	-1.811	0.851	0.025	0.991
3917.95	-10.2786	421.6314	403.6	2.18	-2.494	1.037	0.037	0.985
3948.8	-9.41705	394.8211	373.35	2.01	-3.296	1.19	0.053	0.978
3979.65	-8.56886	367.5616	343.41	1.83	-4.184	1.274	0.074	0.967
4010.5	-7.73371	339.3691	313.88	1.66	-5.094	1.251	0.1	0.952
4041.35	-6.91132	309.813	284.9	1.49	-5.943	1.091	0.131	0.932
4072.2	-6.10138	278.6283	256.65	1.32	-6.63	0.773	0.167	0.907
4103.05	-5.30363	245.8206	229.28	1.15	-7.048	0.299	0.205	0.876
4133.9	-4.51778	211.739	203.01	0.99	-7.103	-0.31	0.245	0.838
4164.75	-3.74358	177.094	178.03	0.82	-6.725	-0.99	0.285	0.794
4195.6	-2.98076	142.9057	154.52	0.66	-5.885	-1.69	0.322	0.744
4226.45	-2.22907	110.3872	132.66	0.49	-4.6	-2.33	0.353	0.689
4257.3	-1.48828	80.78247	112.59	0.33	-2.941	-2.83	0.378	0.63
4288.15	-0.75815	55.18702	94.415	0.17	-1.018	-3.14	0.393	0.568
4319	-0.03845	34.39011	78.183	0.01	1.026	-3.21	0.399	0.504
4349.85	0.671045	18.76693	63.905	-0.15	3.036	-3.04	0.395	0.441
4380.7	1.370544	8.240839	51.539	-0.31	4.867	-2.64	0.381	0.38
4411.55	2.060261	2.318725	40.998	-0.46	6.395	-2.08	0.359	0.322
4442.4	2.740398	0.187213	32.157	-0.62	7.536	-1.41	0.33	0.269
4473.25	3.411154	0.846542	24.864	-0.77	8.248	-0.71	0.296	0.22
4504.1	4.072721	3.254216	18.948	-0.92	8.532	-0.03	0.26	0.178
4534.95	4.725288	6.452633	14.23	-1.08	8.427	0.559	0.224	0.141

4565.8	5.369036	9.662222	10.529	-1.23	7.998	1.029	0.188	0.11
4596.65	6.004143	12.33138	7.6759	-1.38	7.328	1.359	0.155	0.084
4627.5	6.630782	14.1442	5.513	-1.53	6.502	1.55	0.124	0.063
4658.35	7.249121	14.99431	3.9009	-1.67	5.601	1.615	0.098	0.047
4,689	7.859324	14.93713	2.7193	-1.82	4.692	1.577	0.076	0.034
4,720	8.461551	14.13357	1.8676	-1.97	3.83	1.462	0.058	0.025
4,751	9.055957	12.79595	1.2637	-2.11	3.05	1.299	0.043	0.017

Table 12: Estimation of European call options for Skewness and Kurtosis augmented Black & Scholes model-01/07/2021

ESTIMATED EUROPEAN PUT OPTION PRICES -SKEWNESS AND KURTOSIS AUGMENTED B-S MODEL ->30/06/2021 IE 01/07/2021		
strike price	moneyness percentage	estimates price_SK-KU-AUGMENTED BS MODEL
3,887	-11.15383091	14.94767272
3,918	-10.2786039	18.98795192
3,949	-9.417052305	23.0225396
3,980	-8.568858102	26.60795502
4,011	-7.733713039	29.26023015
4,041	-6.911318283	30.54904975
4,072	-6.101384054	30.20922005
4,103	-5.303629286	28.24630042
4,134	-4.517781307	25.00962256
4,165	-3.743575519	21.20947515
4,196	-2.980755111	17.86594638
4,226	-2.229070767	16.19238478
4,257	-1.488280399	17.43247309
4,288	-0.758148886	22.68187908
4,319	-0.038447822	32.72982861
4,350	0.671044716	47.95149949
4,381	1.370544401	68.27026446
4,412	2.060260873	93.19300617
4,442	2.740397951	121.9063485
4,473	3.411153827	153.4105332
4,504	4.072721266	186.663062
4,535	4.725287788	220.7063346
4,566	5.369035844	254.7607784
4,597	6.004142986	288.2747916

4,628	6.630782033	320.9324669
4,658	7.249121224	352.6274275
4,689	7.859324374	383.4151062
4,720	8.461551012	413.4564025
4,751	9.055956525	442.9636371

Table 13: Estimation of European put options for Skewness and Kurtosis augmented Black & Scholes model-01/07/2021

ESTIMATION OF PHYSICAL AND RISK-NEUTRAL PROBABILITIES FOR 28/02/2020(01/03/2020)			
SCENARIOS	**PRICES S(n)**	**PROBABILITIES P(n)**	**RISK NEUTRAL PROBABILITIES P'(n)**
Scenario 1	3094.2473	0.008407155	0.009297
Scenario 2	3095.1744	0.008402119	0.009234
Scenario 3	3095.1744	0.008402119	0.009234
Scenario 4	3095.1744	0.008402119	0.009234
Scenario 5	3095.7924	0.008398765	0.009191
Scenario 6	3094.8653	0.008403797	0.009254
Scenario 7	3094.5563	0.008405476	0.009276
Scenario 8	3095.1744	0.008402119	0.009234
Scenario 9	3093.9383	0.008408834	0.009319
Scenario 10	3095.4834	0.008400442	0.009212
Scenario 11	3093.9383	0.008408834	0.009319
Scenario 12	3094.8653	0.008403797	0.009254
Scenario 13	3094.8653	0.008403797	0.009254
Scenario 14	3093.0112	0.008413876	0.009383
Scenario 15	3091.4661	0.008422289	0.00949
Scenario 16	3092.0841	0.008418922	0.009447
Scenario 17	3091.1571	0.008423973	0.009511
Scenario 18	3093.3202	0.008412195	0.009362
Scenario 19	3090.848	0.008425657	0.009533
Scenario 20	3090.848	0.008425657	0.009533
Scenario 21	3090.539	0.008427342	0.009554
Scenario 22	3090.539	0.008427342	0.009554
Scenario 23	3090.848	0.008425657	0.009533
Scenario 24	3092.0841	0.008418922	0.009447

Scenario 25	3092.7022	0.008415558	0.009405
Scenario 26	3092.3932	0.00841724	0.009426
Scenario 27	3093.3202	0.008412195	0.009362
Scenario 28	3092.3932	0.00841724	0.009426
Scenario 29	3093.0112	0.008413876	0.009383
Scenario 30	3093.6293	0.008410514	0.00934
Scenario 31	3093.0112	0.008413876	0.009383
Scenario 32	3093.3202	0.008412195	0.009362
Scenario 33	3093.6293	0.008410514	0.00934
Scenario 34	3092.7022	0.008415558	0.009405
Scenario 35	3091.7751	0.008420605	0.009468
Scenario 36	3092.3932	0.00841724	0.009426
Scenario 37	3093.6293	0.008410514	0.00934
Scenario 38	3092.3932	0.00841724	0.009426
Scenario 39	3091.7751	0.008420605	0.009468
Scenario 40	3091.4661	0.008422289	0.00949
Scenario 41	3091.4661	0.008422289	0.00949
Scenario 42	3091.4661	0.008422289	0.00949
Scenario 43	3091.1571	0.008423973	0.009511
Scenario 44	3090.848	0.008425657	0.009533
Scenario 45	3091.4661	0.008422289	0.00949
Scenario 46	3092.0841	0.008418922	0.009447
Scenario 47	3092.3932	0.00841724	0.009426
Scenario 48	3090.848	0.008425657	0.009533
Scenario 49	3091.7751	0.008420605	0.009468
Scenario 50	3091.7751	0.008420605	0.009468
Scenario 51	3091.1571	0.008423973	0.009511
Scenario 52	3091.4661	0.008422289	0.00949
Scenario 53	3091.4661	0.008422289	0.00949
Scenario 54	3091.1571	0.008423973	0.009511
Scenario 55	3091.1571	0.008423973	0.009511
Scenario 56	3090.848	0.008425657	0.009533
Scenario 57	3090.539	0.008427342	0.009554
Scenario 58	3090.848	0.008425657	0.009533
Scenario 59	3091.4661	0.008422289	0.00949
Scenario 60	3090.848	0.008425657	0.009533
Scenario 61	3090.848	0.008425657	0.009533
Scenario 62	3091.1571	0.008423973	0.009511
Scenario 63	3090.539	0.008427342	0.009554
Scenario 64	3090.539	0.008427342	0.009554
Scenario 65	3090.539	0.008427342	0.009554
Scenario 66	3092.7022	0.008415558	0.009405
Scenario 67	3092.7022	0.008415558	0.009405
Scenario 68	3089.921	0.008430714	0.009597

Scenario 69	3092.7022	0.008415558	0.009405
Scenario 70	3097.0285	0.008392062	0.009105
Scenario 71	3095.1744	0.008402119	0.009234
Scenario 72	3100.1187	0.00837534	0.008891
Scenario 73	3100.4278	0.00837367	0.008869
Scenario 74	3096.7195	0.008393737	0.009126
Scenario 75	3097.0285	0.008392062	0.009105
Scenario 76	3100.7368	0.008372001	0.008848
Scenario 77	3098.2646	0.008385367	0.00902
Scenario 78	3098.5736	0.008383695	0.008998
Scenario 79	3100.4278	0.00837367	0.008869
Scenario 80	3098.8826	0.008382023	0.008977
Scenario 81	3100.7368	0.008372001	0.008848
Scenario 82	3105.0631	0.008348688	0.008549
Scenario 83	3105.6812	0.008345365	0.008506
Scenario 84	3106.2992	0.008342045	0.008463
Scenario 85	3106.6082	0.008340385	0.008442
Scenario 86	3113.4067	0.008304001	0.007972
Scenario 87	3114.6428	0.008297411	0.007887
Scenario 88	3119.8962	0.008269491	0.007523
Scenario 89	3121.4413	0.008261307	0.007417
Scenario 90	3122.6774	0.008254768	0.007331
Scenario 91	3120.8233	0.008264579	0.007459
Scenario 92	3122.3684	0.008256402	0.007352
Scenario 93	3125.1496	0.008241713	0.00716
Scenario 94	3128.8579	0.008222188	0.006904
Scenario 95	3132.5662	0.008202733	0.006648
Scenario 96	3134.7293	0.008191416	0.006499
Scenario 97	3140.6007	0.008160817	0.006094
Scenario 98	3142.7639	0.008149587	0.005945
Scenario 99	3147.0902	0.008127195	0.005646
Scenario 100	3148.6353	0.008119221	0.005539
Scenario 101	3148.6353	0.008119221	0.005539
Scenario 102	3151.7256	0.008103307	0.005326
Scenario 103	3154.1978	0.00809061	0.005156
Scenario 104	3156.6699	0.008077942	0.004985
Scenario 105	3160.9963	0.008055845	0.004687
Scenario 106	3161.9233	0.008051122	0.004624
Scenario 107	3164.3955	0.008038547	0.004453
Scenario 108	3163.1594	0.008044831	0.004538
Scenario 109	3164.3955	0.008038547	0.004453
Scenario 110	3162.8504	0.008046403	0.00456
Scenario 111	3163.7775	0.008041688	0.004496
Scenario 112	3161.3053	0.008054271	0.004666

Scenario 113	3154.5068	0.008089025	0.005134
Scenario Scenario 114	3153.2707	0.008095368	0.00522
Scenario 115	3150.4895	0.008109667	0.005411
Scenario 116	3147.0902	0.008127195	0.005646
Scenario 117	3136.8925	0.008180123	0.00635
Scenario 118	3138.4376	0.00817207	0.006243
Scenario 119	3137.2015	0.008178511	0.006328
Scenario 120	3137.2015	0.008178511	0.006328

Table 14: Estimation of Physical and Risk Neutral Probabilities for 28/02/2020(01/03/2020)

ESTIMATION OF EUROPEAN CALL OPTIONS PRICES FOR 28/02/2020(01/03/2020)	
STRIKE PRICE	ESTIMATED PRICE
2,781	320.80
2,803	298.90
2,825	276.99
2,847	255.08
2,869	233.18
2,891	211.27
2,913	189.36
2,935	167.46
2,957	145.55
2,979	123.64
3,001	101.74
3,023	79.83
3,045	57.92
3,067	36.02
3,090	13.09
3,112	-8.82
3,134	-30.72
3,156	-52.63
3,178	-74.54
3,200	-96.44

3,222	-118.35
3,244	-140.26
3,266	-162.16
3,288	-184.07
3,310	-205.98
3,332	-227.88
3,354	-249.79
3,376	-271.70
3,399	-294.37

Table 15: Estimation of European call option prices for 01/03/2020 using the Risk Neutral Probabilities model

ESTIMATION OF EUROPEAN PUT OPTIONS PRICES FOR 28/02/2020(01/03/2020)	
STRIKE PRICE	ESTIMATED PRICE
2,781	-320.8049618
2,803	-298.8981599
2,825	-276.991358
2,847	-255.0845561
2,869	-233.1777542
2,891	-211.2709522
2,913	-189.3641503
2,935	-167.4573484
2,957	-145.5505465
2,979	-123.6437446
3,001	-101.7369427
3,023	-79.83014074
3,045	-57.92333882
3,067	-36.01653691
3,090	-13.0910687
3,112	8.815733217
3,134	30.72253513
3,156	52.62933705
3,178	74.53613897
3,200	96.44294089
3,222	118.3497428

3,244	140.2565447
3,266	162.1633466
3,288	184.0701486
3,310	205.9769505
3,332	227.8837524
3,354	249.7905543
3,376	271.6973562
3,399	294.3708962

Table 16: Estimation of European put option prices for 01/03/2020 using the Risk Neutral Probabilities model

ESTIMATION OF PHYSICAL AND RISK-NEUTRAL PROBABILITIES FOR 30/06/2021(01/07/2021)			
SCENARIOS	**PRICES S(n)**	**PROBABILITIES P(n)**	**RISK NEUTRAL PROBABILITIES P'(n)**
Scenario 1	4449.5382	0.012394774	0.021652
Scenario 2	4751.934	0.010867451	0.000000
Scenario 3	4406.3388	0.012639001	0.03529
Scenario 4	4406.3388	0.012639001	0.03529
Scenario 5	4363.1394	0.012890517	0.048619
Scenario 6	4363.1394	0.012890517	0.048619
Scenario 7	4406.3388	0.012639001	0.03529
Scenario 8	4579.1364	0.011703112	0.000000
Scenario 9	4665.5352	0.011273676	0.000000
Scenario 10	4622.3358	0.011485384	0.000000
Scenario 11	4751.934	0.010867451	0.000000
Scenario 12	4622.3358	0.011485384	0.000000
Scenario 13	4708.7346	0.011067769	0.000000
Scenario 14	4795.1334	0.010672523	0.000000
Scenario 15	4708.7346	0.011067769	0.000000
Scenario 16	4751.934	0.010867451	0.000000
Scenario 17	4795.1334	0.010672523	0.000000
Scenario 18	4665.5352	0.011273676	0.000000
Scenario 19	4535.937	0.01192709	0.000000
Scenario 20	4622.3358	0.011485384	0.000000
Scenario 21	4795.1334	0.010672523	0.000000

Scenario 22	4622.3358	0.011485384	0.000000
Scenario 23	4535.937	0.01192709	0.000000
Scenario 24	4492.7376	0.012157559	0.008324
Scenario 25	4492.7376	0.012157559	0.008324
Scenario 26	4492.7376	0.012157559	0.008324
Scenario 27	4449.5382	0.012394774	0.021652
Scenario 28	4406.3388	0.012639001	0.03529
Scenario 29	4492.7376	0.012157559	0.008324
Scenario 30	4579.1364	0.011703112	0.000000
Scenario 31	4622.3358	0.011485384	0.000000
Scenario 32	4406.3388	0.012639001	0.03529
Scenario 33	4535.937	0.01192709	0.000000
Scenario 34	4535.937	0.01192709	0.000000
Scenario 35	4449.5382	0.012394774	0.021652
Scenario 36	4492.7376	0.012157559	0.008324
Scenario 37	4492.7376	0.012157559	0.008324
Scenario 38	4449.5382	0.012394774	0.021652
Scenario 39	4449.5382	0.012394774	0.021652
Scenario 40	4406.3388	0.012639001	0.03529
Scenario 41	4363.1394	0.012890517	0.048619
Scenario 42	4406.3388	0.012639001	0.03529
Scenario 43	4492.7376	0.012157559	0.008324
Scenario 44	4406.3388	0.012639001	0.03529
Scenario 45	4406.3388	0.012639001	0.03529
Scenario 46	4449.5382	0.012394774	0.021652
Scenario 47	4363.1394	0.012890517	0.048619
Scenario 48	4363.1394	0.012890517	0.048619
Scenario 49	4363.1394	0.012890517	0.048619
Scenario 50	4665.5352	0.011273676	0.000000
Scenario 51	4665.5352	0.011273676	0.000000
Scenario 52	4276.7406	0.013416607	0.075275
Scenario 53	4665.5352	0.011273676	0.000000
Scenario 54	5270.3268	0.008834733	0.000000
Scenario 55	5011.1304	0.009772307	0.000000
Scenario 56	5702.3208	0.007546841	0.000000
Scenario 57	5745.5202	0.007433782	0.000000
Scenario 58	5227.1274	0.008981365	0.000000
Scenario 59	5270.3268	0.008834733	0.000000
Scenario 60	5788.7196	0.007323244	0.000000
Scenario 61	5443.1244	0.008282701	0.000000

Scenario 62	5486.3238	0.008152778	0.000000
Scenario 63	5745.5202	0.007433782	0.000000
Scenario 64	5529.5232	0.008025889	0.000000
Scenario 65	5788.7196	0.007323244	0.000000
Scenario 66	6393.5112	0.006003294	0.000000
Scenario 67	6479.91	0.005844274	0.000000
Scenario 68	6566.3088	0.005691489	0.000000
Scenario 69	6609.5082	0.005617334	0.000000
Scenario 70	7559.895	0.004293752	0.000000
Scenario 71	7732.6926	0.004103997	0.000000
Scenario 72	8467.0824	0.003422953	0.000000
Scenario 73	8683.0794	0.003254775	0.000000
Scenario 74	8855.877	0.003128999	0.000000
Scenario 75	8596.6806	0.003320526	0.000000
Scenario 76	8812.6776	0.00315975	0.000000
Scenario 77	9201.4722	0.00289837	0.000000
Scenario 78	9719.865	0.002597455	0.000000
Scenario 79	10238.2578	0.002341081	0.000000
Scenario 80	10540.6536	0.002208683	0.000000
Scenario 81	11361.4422	0.001901085	0.000000
Scenario 82	11663.838	0.001803788	0.000000
Scenario 83	12268.6296	0.001630333	0.000000
Scenario 84	12484.6266	0.001574408	0.000000
Scenario 85	12484.6266	0.001574408	0.000000
Scenario 86	12916.6206	0.001470858	0.000000
Scenario 87	13262.2158	0.0013952	0.000000
Scenario 88	13607.811	0.001325232	0.000000
Scenario 89	14212.6026	0.001214846	0.000000
Scenario 90	14342.2008	0.00119299	0.000000
Scenario 91	14687.796	0.00113751	0.000000
Scenario 92	14514.9984	0.001164755	0.000000
Scenario 93	14687.796	0.00113751	0.000000
Scenario 94	14471.799	0.001171719	0.000000
Scenario 95	14601.3972	0.001151012	0.000000
Scenario 96	14255.802	0.001207495	0.000000
Scenario 97	13305.4152	0.001386155	0.000000
Scenario 98	13132.6176	0.001422872	0.000000
Scenario 99	12743.823	0.001511016	0.000000
Scenario 100	12268.6296	0.001630333	0.000000
Scenario 101	10843.0494	0.002087208	0.000000

Scenario 102	11059.0464	0.002006472	0.000000
Scenario 103	10886.2488	0.002070675	0.000000
Scenario 104	10886.2488	0.002070675	0.000000
Scenario 105	9719.865	0.002597455	0.000000
Scenario 106	4795.1334	0.010672523	0.000000
Scenario 107	4708.7346	0.011067769	0.000000
Scenario 108	4924.7316	0.010118203	0.000000
Scenario 109	5011.1304	0.009772307	0.000000
Scenario 110	4708.7346	0.011067769	0.000000
Scenario 111	4795.1334	0.010672523	0.000000
Scenario 112	4751.934	0.010867451	0.000000
Scenario 113	4708.7346	0.011067769	0.000000
Scenario 114	4665.5352	0.011273676	0.000000
Scenario 115	4708.7346	0.011067769	0.000000
Scenario 116	4579.1364	0.011703112	0.000000
Scenario 117	4492.7376	0.012157559	0.008324
Scenario 118	4449.5382	0.012394774	0.021652
Scenario 119	4363.1394	0.012890517	0.048619
Scenario 120	4363.1394	0.012890517	0.048619

Table 17: Estimation of Physical and Risk-Neutral Probabilities for 30/06/2021(01/07/2021)

ESTIMATION OF EUROPEAN CALL OPTIONS PRICES FOR 28/02/2020(01/03/2020)	
STRIKE PRICE	ESTIMATED PRICE
3,887	433.4882561
3,918	402.6434009
3,949	371.7985458
3,980	340.9536907
4,011	310.1088356
4,041	279.2639805
4,072	248.4191254
4,103	217.5742703
4,134	186.7294152
4,165	155.8845601
4,196	125.0397049
4,226	94.19484984
4,257	63.34999473

4,288	32.50513962
4,319	1.660284505
4,350	0.000000
4,381	0.000000
4,412	0.000000
4,442	0.000000
4,473	0.000000
4,504	0.000000
4,535	0.000000
4,566	0.000000
4,597	0.000000
4,628	0.000000
4,658	0.000000
4,689	0.000000
4,720	0.000000
4,751	0.000000

Table 18: Estimation of European call prices for 01/07/2021 using the Risk-Neutral Probabilities model

ESTIMATION OF EUROPEAN PUT OPTIONS PRICES FOR 28/02/2020(01/03/2020)	
STRIKE PRICE	ESTIMATED PRICE
3,887	0.00000000
3,918	0.00000000
3,949	0.00000000
3,980	0.00000000
4,011	0.00000000
4,041	0.00000000
4,072	0.00000000
4,103	0.00000000
4,134	0.00000000
4,165	0.00000000
4,196	0.00000000
4,226	0.00000000
4,257	0.00000000
4,288	0.00000000
4,319	0.00000000

4,350	29.18457061
4,381	60.02942572
4,412	90.87428083
4,442	121.7191359
4,473	152.563991
4,504	183.4088462
4,535	214.2537013
4,566	245.0985564
4,597	275.9434115
4,628	306.7882666
4,658	337.6331217
4,689	368.4779768
4,720	399.3228319
4,751	430.167687

Table 19: Estimation of European put prices for 01/07/2021 using the Risk-Neutral Probabilities model

8. References

Black, F., & Scholes, M. S. (1973a). The Pricing of Options and Corporate Liabilities. Journal of Political Economy, 81(3), 637–654. https://doi.org/10.1086/260062

Black, F., & Scholes, M. S. (1973b). The Pricing of Options and Corporate Liabilities. Journal of Political Economy, 81(3), 637–654. https://doi.org/10.1086/260062

Cohen, J. D., Black, F., & Scholes, M. S. (1972). THE VALUATION OF OPTION CONTRACTS AND A TEST OF MARKET EFFICIENCY. Journal of Finance, 27(2), 399–417. https://doi.org/10.1111/j.1540-6261.1972.tb00969.x

Collan, M., Carlsson, C., & Majlender, P. (2003). Fuzzy Black and Scholes Real Options Pricing. Journal of Decision Systems, 12(3–4), 391–416. https://doi.org/10.3166/jds.12.391-416

Cox, J., Ross, S. L., & Rubinstein, M. P. (1979). Option pricing: A simplified approach. Journal of Financial Economics, 7(3), 229–263. https://doi.org/10.1016/0304-405x(79)90015-1

Del Giudice, M., Evangelista, F., & Palmaccio, M. (2015). Defining the Black and Scholes approach: a first systematic literature review. Journal of Innovation and Entrepreneurship. https://doi.org/10.1186/s13731-015-0030-8

Gay, G. D., & Hull, J. (1990). Options, Futures, and Other Derivative Securities. Journal of Finance, 45(1), 312. https://doi.org/10.2307/2328826

Jackwerth, J. C. (1999). Option-Implied Risk-Neutral Distributions and Implied Binomial Trees. Journal of Derivatives, 7(2), 66–82. https://doi.org/10.3905/jod.1999.319143

Manaster, S., & Koehler, G. J. (1982). The Calculation of Implied Variances from the Black-Scholes Model: A Note. Journal of Finance, 37(1), 227–230. https://doi.org/10.1111/j.1540-6261.1982.tb01105.x

Merton, R. C. (2022). Theory of Rational Option Pricing. Legare Street Press.

Orlando, G., & Taglialatela, G. (2021). On the approximation of the Black and Scholes call function. Journal of Computational and Applied Mathematics, 384, 113154. https://doi.org/10.1016/j.cam.2020.113154

Roman, S. (2004). Introduction to the Mathematics of Finance: From Risk Management to Options Pricing. http://ci.nii.ac.jp/ncid/BA68697901

Rubinstein, M. (1994). Implied Binomial Trees. Journal of Finance, 49(3), 771. https://doi.org/10.2307/2329207

Rubinstein, M. P. (1994). Implied Binomial Trees. Jou rnal of Finance, 49(3), 771–818. https://doi.org/10.1111/j.1540-6261.1994.tb00079.x

Smith, C. W., & Management, U. O. R. G. S. O. (197 C.E.). Option Pricing - a Review.

Wiggins, J. W. (1987). Option values under stochastic volatility: Theory and empirical estimates. Journal of Financial Economics, 19(2), 351–372. https://doi.org/10.1016/0304-405x(87)90009-2

Yalincak, O. H. (2005). Criticism of the Black-Scholes Model: But Why is it Still Used?: (The Answer is Simpler than the Formula). Social Science Research Network. https://doi.org/10.2139/ssrn.2115141

Ait-Sahalia, Y., & Lo, A. W. (1998a). Non-Parametric Risk Management and Implied Risk Aversion. Social Science Research Network. https://doi.org/10.2139/ssrn.94133

Ait-Sahalia, Y., & Lo, A. W. (1998b). Nonparametric Estimation of State-Price Densities Implicit in Financial Asset Prices. Journal of Finance, 53(2), 499–547. https://doi.org/10.1111/0022-1082.215228

Ayres, H. F. (1963). Risk aversion in the warrant markets. http://dspace.mit.edu/handle/1721.1/72779

Bahra, B. (1997). Implied Risk-neutral Probability Density Functions From Option Prices: Theory and Application. Social Science Research Network. https://doi.org/10.2139/ssrn.77429

Bali, T. G., Hu, J., & Scott, M. (2019). Option Implied Volatility, Skewness, and Kurtosis and the Cross-Section of Expected Stock

Returns. Option Implied Volatility, Skewness, and Kurtosis and the Cross-section of Expected Stock Returns. https://doi.org/10.2139/ssrn.2322945

Ball, C. A., & Roma, A. (1994). Stochastic Volatility Option Pricing. Journal of Financial and Quantitative Analysis, 29(4), 589. https://doi.org/10.2307/2331111

Ball, C. A., & Torous, W. N. (1984). The Maximum Likelihood Estimation of Security Price Volatility: Theory, Evidence, and Application to Option Pricing. The Journal of Business, 57(1), 97. https://doi.org/10.1086/296226

Black, F., & Scholes, M. S. (1973a). The Pricing of Options and Corporate Liabilities. Journal of Political Economy, 81(3), 637–654. https://doi.org/10.1086/260062

Black, F., & Scholes, M. S. (1973b). The Pricing of Options and Corporate Liabilities. Journal of Political Economy, 81(3), 637–654. https://doi.org/10.1086/260062

Bouchouev, I., & Isakov, V. (1997). The inverse problem of option pricing. Inverse Problems, 13(5), L11–L17. https://doi.org/10.1088/0266-5611/13/5/001

Britten-Jones, M., & Neuberger, A. (2000a). Option Prices, Implied Price Processes, and Stochastic Volatility. Journal of Finance, 55(2), 839–866. https://doi.org/10.1111/0022-1082.00228

Britten-Jones, M., & Neuberger, A. (2000b). Option Prices, Implied Price Processes, and Stochastic Volatility. Journal of Finance, 55(2), 839–866. https://doi.org/10.1111/0022-1082.00228

Brown, C. E., & Robinson, D. (2002). Skewness and Kurtosis Implied by Option Prices: A Correction. Journal of Financial Research, 25(2), 279–282. https://doi.org/10.1111/1475-6803.t01-1-00008

Cohen, J. D., Black, F., & Scholes, M. S. (1972). THE VALUATION OF OPTION CONTRACTS AND A TEST OF MARKET EFFICIENCY. Journal of Finance, 27(2), 399–417. https://doi.org/10.1111/j.1540-6261.1972.tb00969.x

Collan, M., Carlsson, C., & Majlender, P. (2003). Fuzzy Black and Scholes Real Options Pricing. Journal of Decision Systems, 12(3–4), 391–416. https://doi.org/10.3166/jds.12.391-416

Corrado, C. J., & Su, T. (1996). SKEWNESS AND KURTOSIS IN S&P 500 INDEX RETURNS IMPLIED BY OPTION PRICES. Journal of Financial Research, 19(2), 175–192. https://doi.org/10.1111/j.1475-6803.1996.tb00592.x

Corrado, C. J., & Su, T. (1997a). Implied volatility skews and stock return skewness and kurtosis implied by stock option prices. European Journal of Finance, 3(1), 73–85. https://doi.org/10.1080/135184797337543

Corrado, C. J., & Su, T. (1997b). Implied Volatility Skews and Stock Index Skewness and Kurtosis Implied by S&P 500 Index Option Prices. Journal of Derivatives, 4(4), 8–19. https://doi.org/10.3905/jod.1997.407978

Cox, J., Ross, S. L., & Rubinstein, M. P. (1979). Option pricing: A simplified approach. Journal of Financial Economics, 7(3), 229–263. https://doi.org/10.1016/0304-405x(79)90015-1

De Lange, P. E., Fleten, S., & Gaivoronski, A. A. (2004). Modeling financial reinsurance in the casualty insurance business via stochastic programming. Journal of Economic Dynamics and Control, 28(5), 991–1012. https://doi.org/10.1016/s0165-1889(03)00055-1

Del Giudice, M., Evangelista, F., & Palmaccio, M. (2015). Defining the Black and Scholes approach: a first systematic literature review. Journal of Innovation and Entrepreneurship. https://doi.org/10.1186/s13731-015-0030-8

Dempster, M. a. H., & Richards, D. (2000). Pricing American Options Fitting the Smile. Mathematical Finance, 10(2), 157–177. https://doi.org/10.1111/1467-9965.00087

Dumas, B., Fleming, J., & Whaley, R. E. (1998). Implied Volatility Functions: Empirical Tests. Journal of Finance, 53(6), 2059–2106. https://doi.org/10.1111/0022-1082.00083

Dupačová, J., Hurt, J., & Štěpán, J. (2003). Stochastic Modeling in Economics and Finance. In Kluwer Academic Publishers eBooks. https://doi.org/10.1007/b101992

Figlewski, S. (2010). Estimating the Implied Risk-Neutral Density for the US Market Portfolio*. In Oxford University Press eBooks (pp. 323–353). https://doi.org/10.1093/acprof:oso/9780199549498.003.0015

Gay, G. D., & Hull, J. (1990). Options, Futures, and Other Derivative Securities. Journal of Finance, 45(1), 312. https://doi.org/10.2307/2328826

Gonçalves, S., & Guidolin, M. (2004). Predictable Dynamics in the S&P 500 Index Options Implied Volatility Surface. The Dynamics of the S&P 500 Implied Volatility Surface. https://doi.org/10.2139/ssrn.406697

Jackwerth, J. C. (1999). Option-Implied Risk-Neutral Distributions and Implied Binomial Trees. Journal of Derivatives, 7(2), 66–82. https://doi.org/10.3905/jod.1999.319143

Jackwerth, J. C. (2000a). Recovering Risk Aversion from Option Prices and Realized Returns. Review of Financial Studies, 13(2), 433–451. https://doi.org/10.1093/rfs/13.2.433

Jackwerth, J. C. (2000b). Recovering Risk Aversion from Option Prices and Realized Returns. Review of Financial Studies, 13(2), 433–451. https://doi.org/10.1093/rfs/13.2.433

Janková, Z. (2018). Drawbacks and Limitations of Black-Scholes Model for Options Pricing. Drawbacks and Limitations of Black-Scholes Model for Options Pricing, 2018, 1–7. https://doi.org/10.5171/2018.79814

Kaut, M., & Wallace, S. W. (2007). Evaluation of scenario-generation methods for stochastic programming. Evaluation of Scenario-generation Methods for Stochastic Programming, 3(2), 257–271. https://doi.org/10.18452/8296

Manaster, S., & Koehler, G. J. (1982). The Calculation of Implied Variances from the Black-Scholes Model: A Note. Journal of Finance, 37(1), 227–230. https://doi.org/10.1111/j.1540-6261.1982.tb01105.x

Merton, R. C. (1973a). Theory of Rational Option Pricing. The Bell Journal of Economics and Management Science, 4(1), 141. https://doi.org/10.2307/3003143

Merton, R. C. (1973b). The Relationship between Put and Call Option Prices: Comment. Journal of Finance, 28(1), 183. https://doi.org/10.2307/2978180

Merton, R. C. (1976). Option pricing when underlying stock returns are discontinuous. Journal of Financial Economics, 3(1–2), 125–144. https://doi.org/10.1016/0304-405x(76)90022-2

Merton, R. C. (2022). Theory of Rational Option Pricing. Legare Street Press.

Mulvey, J. M., & Vladimirou, H. (1992). Stochastic Network Programming for Financial Planning Problems. Management Science, 38(11), 1642–1664. https://doi.org/10.1287/mnsc.38.11.1642

Orlando, G., & Taglialatela, G. (2021). On the approximation of the Black and Scholes call function. Journal of Computational and Applied Mathematics, 384, 113154. https://doi.org/10.1016/j.cam.2020.113154

Roman, S. (2004). Introduction to the Mathematics of Finance: From Risk Management to Options Pricing. http://ci.nii.ac.jp/ncid/BA68697901

Rubinstein, M. (1994). Implied Binomial Trees. Journal of Finance, 49(3), 771. https://doi.org/10.2307/2329207

Smith, C. W., & Management, U. O. R. G. S. O. (197 C.E.). Option Pricing - a Review.

Topaloglou, N., Vladimirou, H., & Zenios, S. A. (2008b). Pricing options on scenario trees. Pricing Options on Scenario Trees, 32(2), 283–298. https://doi.org/10.1016/j.jbankfin.2007.03.010

Wiggins, J. W. (1987). Option values under stochastic volatility: Theory and empirical estimates. Journal of Financial Economics, 19(2), 351–372. https://doi.org/10.1016/0304-405x(87)90009-2

Yalincak, O. H. (2005). Criticism of the Black-Scholes Model: But Why is it Still Used?: (The Answer is Simpler than the Formula). Social Science Research Network. https://doi.org/10.2139/ssrn.2115141

Zhao, Y., & Ziemba, W. T. (2008). Calculating risk neutral probabilities and optimal portfolio policies in a dynamic investment model with downside risk control. European Journal of Operational Research. https://doi.org/10.1016/j.ejor.2006.09.002

Ziemba, W. T., & Vickson, R. G. (2006). Stochastic Optimization Models in Finance. In World Scientific handbook in financial economic series. World Scientific. https://doi.org/10.1142/6101

Figure's References

https://quantra.quantinsti.com/glossary/Option-Payoff
https://ceopedia.org/index.php/Options
https://www.projectfinance.com/long-call-simplified/
https://en.wikipedia.org/wiki/Binomial_options_pricing_model
http://sfb649.wiwi.hu-berlin.de/fedc_homepage/xplore/tutorials/xlghtmlnode63.html
http://www.goddardconsulting.ca/option-pricing-binomial-index.html
https://quantra.quantinsti.com/glossary/Volatility-Skew