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ΑΘΗΝΩΝ

ATHENS UNIVERSITY
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“Pricing Options with discrete distributions”

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Abstract

This thesis develops an overview literature of the famous Black & Scholes model and the advancements made by the years. Our aim is to provide suitable and alternative methods for pricing European options which produce more consistent results than the Black & Scholes model and are in accordance with the discrete scenario sets of assets prices. We develop a scenario tree with subset of nodes in order to display the discrete distribution of the asset. Both methods take into account the statistical characteristics of asset returns as they are observed in their empirical distributions. The first method is an expansion of the Black & Scholes model augmented with two additional segments, skewness and kurtosis. In the second method we derive the risk-neutral probabilities that are linked with the postulated price outcomes represented on the scenario tree. Through the empirical tests we applied using real market prices of S&P 500 index and options addressed to the index, we make appropriate comparisons between the proposed approaches and the Black & Scholes model and deduced that the above procedures perform better than the Black & Scholes model especially in out of the money options and are closer to the real market prices.

Keywords: Black & Scholes model, Pricing under discrete distributions, European Options, Higher order moments

Table of Contents

Introduction

In modern economies, market of derivatives composes an integrant component of money markets and capital, giving investors and traders the ability to conduct their portfolios more efficiently in the markets where the negotiation of the underlying assets takes place.

From the beginning of the establishment of those products, their main use was to ease or even better neutralize the risks that arise from the fluctuation of the prices or ensure the purchase of some assets in a predetermined price.

Options trading have existed for centuries, but they remained relatively imprecise financial tools until the introduction of a listed options exchange in 1973 by the Chicago Board Options Exchange. Since then, trading of options has been massively expanded in American underlying asset markets.

The option pricing is considered to be one of the most complex fields of all applied areas of Finance. Researchers have attended to establish a precise formula in order to achieve consistent results in accordance with the market prices but without any success. In 1973 this field of research underwent a remarkable revolution with the invention of the first complete equilibrium option pricing model by the seminal work of Fisher Black and Myron Scholes. In the same year Robert Merton expanded their model making crucial additions and adjustments. Those articles have been the pioneers of option pricing and created the basis for various upcoming academic research studies that followed. Since then, financial analysts have been able to calculate with tremendous accuracy the value of a stock option either a put or call option with the invention of the Black-Scholes model.

Options belong in the category of derivatives such as forwards, futures and swaps and they are directly associated with transferring risk from one entity in the economy to another. Derivatives took their name due to their use; they are securities that derive their value from an underlying asset. Options are financial tools that give the owners the right but not the obligation to buy or sell an asset at a predetermined price over a specific period of time.

The presence of options trading tracks its roots many years ago but because of their complex and significant costs, their transactions were not easy to arrange until the Chicago Board Options Exchange established them in April 1973. These instruments were not originated from Wall Street or some other financial institution, but their roots are traced back hundreds of years ago. The first options were thought to be used in ancient Greece in order to speculate on crops.

The Black & Scholes formula for pricing European options depends on six factors in order to be computed: the current stock price, the exercise price, the time to maturity, the risk- free interest rate, the dividends and the volatility. From all of these, the first three factors are known and the other three must be calculated. Black & Scholes made the assumption of an ideal world in their analysis and considered the last three as constants. But this could not be the case because the only way to evaluate these parameters accurately is to compute them when the option expires, and so future values of these arguments need to be calculated in order to price an option correctly.

The Black & Scholes model is known to be biased [Rubinstein 1985] and that's because it assumes that stock returns are log-normally distributed with constant and known mean and variance. Volatility is the most crucial of these three undetermined factors and has the biggest impact on the pricing result of an option. There are other models assuming that volatility follows a stochastic process, and this allow any individual to make a more precise pricing of derivatives.

It must be noted that every option pricing formula has to make three basic assumptions: the process that the price of the underlying security follows (distributional assumption), the election of the interest rate process and the market price of factor risks.There are plenty of techniques in order to obtain correct and consistent results about option pricing; many of them have Black & Scholes formula as the basic part and make changes when are needed.

A number of empirical studies showed that Black & Scholes model tend to overprice deep out of the money options and misprice deep in the money options. Several attempts were made in order to find a solution to these problems (constant volatility, mispricing, skewness and kurtosis). Ostensive Dempster and Hutton [1999] solved part of these problems using linear programming in order to calculate American and European options and compared these results to the observed market prices. Rubinstein [1994], Jackwerth and Rubinstein [1996] solved the problem of the constant volatility using binomial trees based on real prices of options. Jarrow and Rudd [1982], Corrado and Su [1996] added to the Black & Scholes two new features that take into consideration the effects of higher order moments of the asset return distribution (skewness and kurtosis) regarding the option pricing.

If we take all the parameters of the Black & Scholes model as granted and we solve the equation for the only unknown factor; the volatility, we can observe that volatility is far from constant, presenting different values across different strike prices and maturities forming ''smile'' patterns and as a direct consequence the assumption of the constant volatility is transgressed.

The main purpose of this thesis is the implementation and validation of appropriate methods for pricing options that are consistent with discrete scenario sets of assets prices. These pricing methods also account for the statistical characteristics (kurtosis and skewness) that have been observed in their empirical distributions Two approaches are used for pricing options. Both approaches price options on underlying assets with discrete distributions.

The first approach is an extension of the Black & Scholes model which has been applied by Jarrow and Rudd [1982] and Corrado and Su [1996] adding up two arguments which account for the effects of higher moments: skewness and kurtosis.

In the second approach we define a risk-neutral probability measure from the physical probability measure that is associated with the forecasted price outcomes on the scenario tree that has been constructed (scenario generation). The risk neutral probability is computed under the assumption that the future expected value of all financial assets is equivalent to the future pay off discounted with the risk-free interest rate. After the risk neutral probabilities are calculated, pricing the options is very simple.

We use as our reference point for comparisons the Black & Scholes model and test the performance of the alternative methods with the real market prices of the options. For the implementation of these tests we generated scenarios of asset prices that matches the statistical characteristics of the underlying asset as they are observed in their empirical distribution. We deduced that the log-normal assumption is violated and cannot produce results that are consistent with the real market prices. Through the empirical tests that were made, we came to the conclusion that pricing of European options with methods based on scenario tree that take into account the asymmetries and heavy tails as they appear in the empirical distribution of the underlying asset produce results that are significant better than the Black & Scholes method and thus, can be accounted for viable valuation methods.

The following thesis is organized as follow: in Chapter 1 the literature of the Black & Scholes model is reviewed with the improvements and critics made by the years and also Stochastic Programming is displayed with its advantages; in Chapter 2 there is a descriptive analysis about the options and their characteristics; in Chapter 3 the two seminal methods for pricing options are displayed with alternative derivations and deficiencies; in Chapter 4 the proposed methods for pricing options are analytically presented and the empirical results from the comparisons are depicted; in Chapter 5 conclusions are drawn and summarized.

Chapter 1

1.1 Black & Scholes Literature and Advancements by the years

During the 1970s, Fisher Black, Myron Scholes and Robert Merton were the pioneers of pricing European stock options. More specific Fisher Black and Myron Scholes developed the famous Black & Scholes model through their paper << The Pricing of Options and Corporate Liabilities>> which was published on May 1973 to the journal of Political Economy.

This major achievement gave Merton and Scholes the Nobel Prize for economics (1997); Fisher Black died in 1995 otherwise he would be undoubtedly nominated for the price. The invention of this famous model has had a huge impact on the way that traders and analysts hedge and price derivatives.

In their paper they created a model in order to compute the prices of European Options during their lifetime. This model suggests that the price of the underlying assets follow a geometric Brownian motion with constant mean and variance. When this model is applied it takes into account the continuous volatility of the prices of the stocks, the time value of money, the exercise price and the time to maturity of the option (see Black and Scholes [1973]).

The trigger for the equation was given by Fisher Black, when he started working on a valuation formula regarding stock warrants. According to this formula, while using derivatives he could calculate how much the discount rate of a mortgage was different depending on the time and the price of the stock. Soon after he collaborated with Myron Scholes, they derived an exact option pricing template. Hence, the result that they came up with was based on a model that was made by A. James Boness in his dissertation to the University of Chicago. Black & Scholes demonstrated that the risk-free interest rate is the suitable discount rate, based on the assumption of the absence of risk preferences of the investors.

The biggest part of the previous researches on the valuation of options have been expressed in terms of warrants (see Ayres [1963], Samuelson [1965]) and nearly all of them produced valuation formulas of the same form. Thus, their formulas were not complete because it involved one or more arbitrary parameters.

Black & Scholes model is considered among the most important discoveries of the modern financial theory despite the deficiencies addressed to this model.

Charles Castelli put the basis for options with his book << The Theory of Options in Stocks and Shares>> in 1877, inducting for first time the definition of speculation and hedging on options. On 29 March 1900 Louis Bachelier displayed the first analytical review of options on his dissertation <<Theorie de la Speculation>> at the Sorbonne University and many considered this day to be the day the mathematic finance was born.

Soon after in 1962 Boness with his Ph. D. dissertation on University of Chicago, which was primarily affected by the work of Bachelier and showed high resemblance, developed an option pricing model which is considered to be the precursor of the evolution for the Black & Scholes model. Boness research was focused in discrete time framework in contrast to Black & Scholes which was on continuous time framework (see Galai [1978]).

Nowadays analysts, traders and most of the techniques that are used are based primarily upon Black & Scholes model which allowed them to compute with tremendous accuracy the value of an option. Fisher Black and Myron Scholes within their paper introduced the first general equilibrium formulation on the problem of options pricing, based on the assumption that the stocks follow the Brownian model which produce a log-normal distribution for stock price between any two points in time.

In this model (Black & Scholes) the only input that is used is the volatility of the stocks; something that is not easily observable and because it is not computed with the most appropriate method it tends to misprice deep in the money and overprice out of the money options; the volatility estimates in the Black-Scholes formula, implied by market prices of options and their underlying securities, differ across exercise prices and maturities forming “smile” patterns, violating the constant volatility assumption; a phenomenon known as volatility skew or volatility smile (see Topaloglou et al. [2008]).

The same year Merton evidenced that if a share does not give any dividend, it would not be worth exercised before its maturity date; he extended the Black & Scholes formula and deduced that the basic formation of the model would stand even if payout restriction was lifted, the option can be exercised prior to the expiration date and the interest rate is stochastic (see Merton [1973]). Moreover, he added on the Black & Scholes model dividend policy and showed that the only reasons for premature exercising are lack of protection against dividends or sufficiently unfavorable exercise price changes.

On 1978 William F. Sharpe displayed the binomial option pricing model through his textbook “Investors and Markets” presenting a more efficient method in order to determine numerical values for many complicated options having various exercise dates as well as other characteristics which do not permit closed form solutions (see Litzenberger [1991]). This model is commonly used in computing American options on a variety of financial instruments which include call options on dividend paying common equities, mortgage backed securities subject to prepayment risk and options on interest rates and currency swaps.

One year later on 1979 Cox, Ross and Rubinstein studied and analyzed in depth the binomial pricing model and through their paper ‘’Option Pricing: A Simplified Approach’’ displayed a pricing formula concerning call and put options on stocks where dividends are non-zero. Due to the model flexibility it is said to be more accurate in its results compared to Black & Scholes after observing market prices (see Cox et. al [1979]).

Subsequently Rubinstein [1983] developed a new option pricing formula; The Displaced diffusion model, which assumes that a firm hold only two assets one of which pertains risk and the other is riskless. The value of the riskless asset follows a lognormal diffusion process with an annualized instantaneous volatility of σ and the riskless asset grows at the discrete annualized compounded rate $r-1$. He suggested that the distribution of the stock depends on the distribution of the real assets of which the company holds.

This model entails stochastic volatility and variables concerning dividend policy which makes it in some way more consistent to the real world. From empirical results compared with the Black & Scholes outputs, he derived that out of the money call options valued more (than B-S), in the money call options valued less (than B-S) and these discrepancies become larger as the maturity date is getting close (see Rubinstein [1983].

Another solution about the issue of stochastic volatility was given by Hull and White. They made empirical tests with the assumption that stochastic volatility is independent to the stock price and also when volatility is related to the stock price. While volatility assumed stochastic, they deducted that Black and Scholes model tends to overvalue at the money options and undervalue deep in and out of the money options. In the latter case where volatility is correlated to the stock price they tested about positive and negative correlation extrapolating that when positive correlation occurs the Black & Scholes model tend to overvalue in the money options and undervalue out of the money options (see Hull and White [1987]).

Rubinstein also made nonparametric tests such as the Pure jump model (see Rubinstein [1985], the Mixed Diffusion-Jump model (see Merton [1976]), the Displaced diffusion model (see Rubinstein [1983]), the Compound option diffusion model (see Geske [1979]) and the Constant elasticity of variance diffusion model (see Cox and Ross [1976]) on the 30 most active option classes from the Chicago Board of Exchange and compared the results with the Black & Scholes model. He deduced that the short maturity out of the money calls are priced significantly higher unlike to other calls that the Black & Scholes would forecast. After observing the market prices with the results of these models including Black & Scholes in most of the cases Black & Scholes tend to have bigger errors estimations from all the other models.

On 1982 R. Jarrow and A. Rudd showed how a granted possibility distribution could be replaced by an arbitrary distribution in terms of series expansion taking into account effects of higher moments (skewness and kurtosis). They concentrated on the distribution of the underlying security and resulted in an equation concerning the Black & Scholes formula with adding up three new features; the skewness, kurtosis and the differences that occur in variance. The main scheme of their dissertation was to adapt the inconsistencies that occur between the log normal distribution Black & Scholes model was based and the real stock price distribution (see Jarrow and Rudd [1982]).

Similar to Jarrow and Rudd, Corrado and Su [1996] developed and derived an option pricing model which extends the Black & Scholes model to account for skewness and kurtosis. Corrado and Su used a Gram-Charlier series in contrast to Jarrow and Rudd who used an Edgeworth expansion of the lognormal probability density function in order to model the distribution of the stock prices. They observed that the distribution of stock prices displays significant skewness and kurtosis unlike the log-normal distribution the Black & Scholes model suggests and also found that modification of skewness and kurtosis are useful for reducing systematic strike price biases occurring from the Black & Scholes (see Corrado and Su [1996]).Skewness is the result of fat-tailed physical index risk aversion distributions addressed to investors for the risks of price jumps that have to compete with, and kurtosis is referring to the result of jumps and is implicitly connected to the risk premium on deep in and out of the money options.

Because of the importance and the significant impact that kurtosis and skewness have on security distributions, many researches have been made in order to extract a formula which takes into account the role and the effect of these two higher moments regarding the deviances that create to the distribution prices on a risk neutral world. A risk-neutral word refers to an ‘ideal’ world where the expected return on a stock or any other underlying asset is the risk-free interest rate and the discount rate which is used to compute the expected payoff of an investment is the risk-free interest rate.

For instance, A. Kraus and R. H. Litzenberger [1976] extended the Capital Asset Pricing Model adding up three moments in order to take into consideration the impact of skewness in valuation formulas and made empirical tests. They deduced that systematic skewness is crucial to market valuation instead of total skewness and investors found to be favored by positive skewness.

Bakshi, Cao and Chen [1997] developed a valuation formula which encompasses stochastic volatility, stochastic interest’s rates and random jumps. They combined such features in order to take account skewness and kurtosis and reached to the conclusion that such a model produces not only better results than the Black & Scholes model but also are more realistic and applicable to market conditions (see Bakshi et al. [1997]).

Y. Ait-Sahalia and Andrew W. Lo [1998] constructed a nonparametric model for computing state price densities which are depended on the correlation of state-prices densities and option-prices densities and derived that the Black & Scholes model tend to be inaccurate due to the weakness of not accounting skewness and kurtosis (see Ait- Sahalia et al. [1998]).

Similarly, Madan, Carr and Chang [1998] developed a new valuation formula that has Black & Scholes model as a parametric special occasion and entails two added parameters which takes into account the skewness and kurtosis in the risk-neutral density. From their empirical test they extrapolated that there is negative skew and excess kurtosis on the underlying index and the result was also statistically significant compared to the Black & Scholes results (see Madan, Carr and Chang [1998]).

Bakshi, Kapadia and Madan [2000] studied and analyzed the largest 30 individual equity components which were addressed to the S&P 100 index. Through their empirical analysis they found that the slope of the individual underlying asset is far more negative than the index's and they also observed that volatility smile is getting more and more precipitous as the return distribution of the asset is getting negatively skewed (see Bakshi et al. [2000]).

Another alternative derivation for the problem of non-constant volatility was developed by Rubinstein [1994] and Jackwerth and Rubinstein [1996]. Rubinstein [1994] deployed a new method about deriving risk-neutral probabilities produced by the market prices of European options. These risk-neutral probabilities are employed in order to construct a binomial tree that is in accordance with the real market prices (see Rubinstein [1994] and Jackwerth and Rubinstein [1996]). A crucial factor here is the usage of Backtesting. Backtesting imposes the validity of a pricing model or a trading strategy on how it will react using historical data; if back testing results are consistent and accurate, then investors have the assertiveness to employ them in real terms.

It should be inferred that one of the most important principles in pricing of derivatives is known as the risk-neutral valuation method. This means that while valuing a derivative we can make the assumptions that all the investors are risk-neutral and so the world where we make the evaluation is called risk-neutral world. Of course, this is a theoretical approach because in no case is this world a risk neutral. Therefore, the higher risks the investors are willing to take, the higher the expected returns are waiting.

Miraculously, it can be seen that assuming a risk-neutral world gives us right option pricing results for both the world we live in and also for a risk-neutral world. Its major principle is that we get consistent prices for derivatives in all worlds. Options contain a high risk as referred to investments and so there is a question about how the risk preferences of investors can affect the way that the options are priced. Risk preferences does not play an important role when we are pricing options in terms of the price of the underlying asset. In the risk-neutral world there are two main features that clarify the pricing of derivatives: the expected return on the stock or any other underlying asset is the risk-free interest rate and the discount rate which is used to compute the expected payoff of an option or any other investment is the risk-free interest rate.

Afterwards, Jackwerth and Rubinstein [1996] in their article ''Recovering Probability Distributions from Option Prices'' generalized the proposed procedure by Rubinstein [1994] and came with a new optimization technique for computing probability distributions. They proposed a nonparametric method which not only takes into account the importance and presence of extreme events occurring in market but also produces a considerably larger amount of information regarding the construction of probability distribution. From their empirical application they extrapolated that the implied probability distributions in the pre-cash period they tested are leftskewed and platykurtic (see Jackwerth et al. [1996]).

1.2 Stochastic Programming

Most of the modern techniques for pricing options are based on stochastic programming which is thought to be one of the most complex mathematical applications in the applied field of Finance. The applicability of stochastic programming regarding financial applications was first implied by Bradley and Crane (see Bradley et al. [1972]) and Ziemba and Vickson (see Ziemba et al. [1975]). Soon after the nineties stochastic programming gained its popularity as a decision support tool for asset and liability management (see Kouwenberg and Zenios [2001]).

Stochastic programs are becoming more and more popular and widely accepted by the time because of their flexibility and the advantageous characteristics they provide.

Stochastic programming is one of the most appropriate tools in order to model diverse financial problems and provide solutions under uncertainty.

Therefore, multistage stochastic programs could be applied in order to purvey an efficacious framework to model and clarify dynamic portfolio management issues through which the dynamic features of a problem are captured and thus allowing for a number of comprehensions regarding the uncertain factors (see Mulvey and Vladimirou [1992]).

Many practical features such as portfolios of many assets, transaction costs, liquidity, trading and turnover constraints, limits on holdings in particular assets along with managerial and regulatory requirements can be modelled as constraints on the decision variables and combined by such programs and produce tremendously results. The elasticity of those programs makes them appropriate instruments for serving the risk management field. Because Derivatives are adequately for those reasons so they should be incorporated to portfolio optimization models to ameliorate the risk management tools. Recent upgrades and adjustments of models and applications of stochastic programs for management issues or other related conflicts are observed in Ziemba and Mulvey [1998], Zenios and Ziemba [2006, 2007] and Topaloglou et al. [2008].

It should be noted that even if some of this works and models are combined it is surely impossible to find a model that could give equivalent results to those of real market prices.

Chapter 2

2.1. Options

Options compose the second biggest category of derivatives after Futures and they are widely used all over the world either to defend the uncovered positions of hedgers or as a favorite tool for speculators. Despite they were invented long time ago, their growth was limited due to technical issues regarding their valuation and also because of the low liquidity that existed in non-organized Stock Exchanges where they were traded. Simultaneously both the discovery of the famous Black & Scholes model and the establishment of the Chicago Board Options Exchange gave rise to the option market and in nearly three decades options covered all the underlying assets and commodities that they are offered for trade (see Mylonas [2005]).

The first organized market of options was created in 1973 on Chicago from the Chicago Board Options Exchange (CBOE) with underlying securities trade on shares in the two stock exchanges of New York; the New York Stock Exchange and the American Stock exchange. After their invention, they were widely used and adapted from many Stock exchanges; in Greece the formal and organized market of options was institutionalized in 1997 and their function started in August 1999.

Options are financial tools which value depends on the values of the other underlying securities where they are addressed to. As mentioned in the introduction options are divided into two categories: American and European options. Most of the contracts involving options are American type because of the flexibility they give for exercise. It is a fact that European options are much easier to analyze and therefore in this thesis we will focus on European options but some features about American options are also displayed.

American options can be exercised at any time until the expiration date in contrast to European options that can be exercised only on a specific future date. The price which is paid when the option is exercised is called “Strike price” or “Exercise price”. The last day when the option could be exercised is called “maturity date” or “expiration date”.

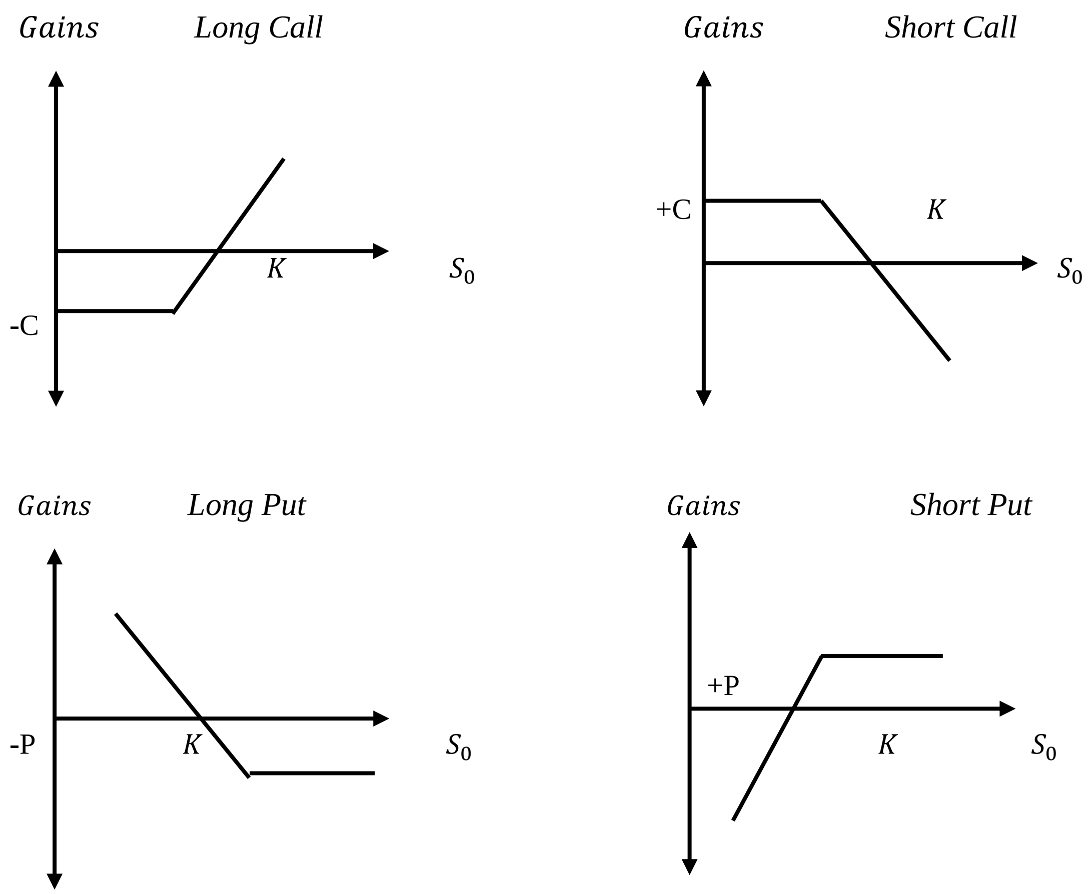
There are two kinds of options; the call option and the put option. Call option gives one the right to buy a single share of common stock or an asset in a given price and put option gives the right to sell a single share of common stock or an asset in a given price. However, options are connected with nearly all underlying instruments such as exchange rates, interests, commodities and indices. Moreover, there are also options addressed to future contracts. When there is a future contract on some underlying asset, there is usually an option contract which refers to the future contract. Commonly a future option contract matures just before the expiration date of the future contract.

Another distinction that must be made comes from the side of buyers (holders) and sellers (writers) of the options. Buyers (long position) have the right but not the obligation to exercise any option. Respectively sellers (short position) have the obligation to buy or sell any underlying asset that the option is addressed with whenever they are exercised from the buyers. Those who have taken short position (sellers or writers of the option) receive some cash front (call or put premium), but they have later potential responsibilities in case they are exercised by those who have taken long position (buyers).

Hence there are four types of options positions and below are their payoff diagrams:

1. Long position in call option
2. Short position in call option
3. Long position in put option
4. Short position in put option

Payoff Diagrams



Whereas S_0 is the spot price, K is the exercise price, c and p are the call and put premium respectively.

Below follows a descriptive analysis about the two types of options.

2.2. European Call Option

The call option gives the buyer (holder) the right but not the obligation to buy an underlying asset in a predetermined price (strike price). There are two possible scenarios for the strike price of the option: one scenario is the strike price to be above the spot price and the other is the strike price to be below the spot price. At the time the option expires if the strike price is above the spot price of the underlying security ($K > S_T$) the buyer will not exercise in any case the option and so the option will be left unexercised and the buyer will lose the premium he paid for purchasing the option.

In the latter case, where the spot price exceeds the strike price ($S_T > K$) the buyer will surely exercise the option and acquire a profit. The option will be exercised due to the fact that the buyer had agreed to buy the underlying security in a price smaller than the spot price and so the difference will be his profit. In this point we should infer that in order to yield profits from a long position to a call option the spot price must exceeds the strike price plus the premium that were paid. In the case the spot price is above the strike price but not with the premium added, any individual would exercise this call option in order to minimize his losses by capturing a fraction of the premium that was paid initially. Furthermore, the gains from a call option grow linearly and depend on the amount the spot price exceed the strike price.

Payoffs of European Call Options

The payoff of a buyer at the expiration date of a call option who has taken long position is:

$$Payoff\ Long\ Call = \max\,[S_T - K, 0]$$

And this shows that the option will be exercised if only $S_T - K > 0 \Rightarrow S_T > K$ and if $S_T < K$ the option will be left unexercised.

Net profit is referred to gains after taking into account the call premium and is:

$$Net\ profit = \max\ [S_T - K, 0] - c$$

Where c is the price of the call option (call premium).

An investor who had taken short position in a European call option expects the spot price of the underlying security to be kept either constant or go below the exercise price so it will not be worth exercised and obtain the call premium. Thus, sellers of options are addressed with huge levels of risks because there are no limits to stop their losses, since they have obligation to pay the buyers if they exercise the options.

The payoff of an investor who has taken short position in a European call option is:

$$Payoff\ Short\ Call = \min[K - S_T\,, 0]$$

Or

$$-\max[S_T - K, 0]$$

The net profits of both positions are diagrammatically depicted above:

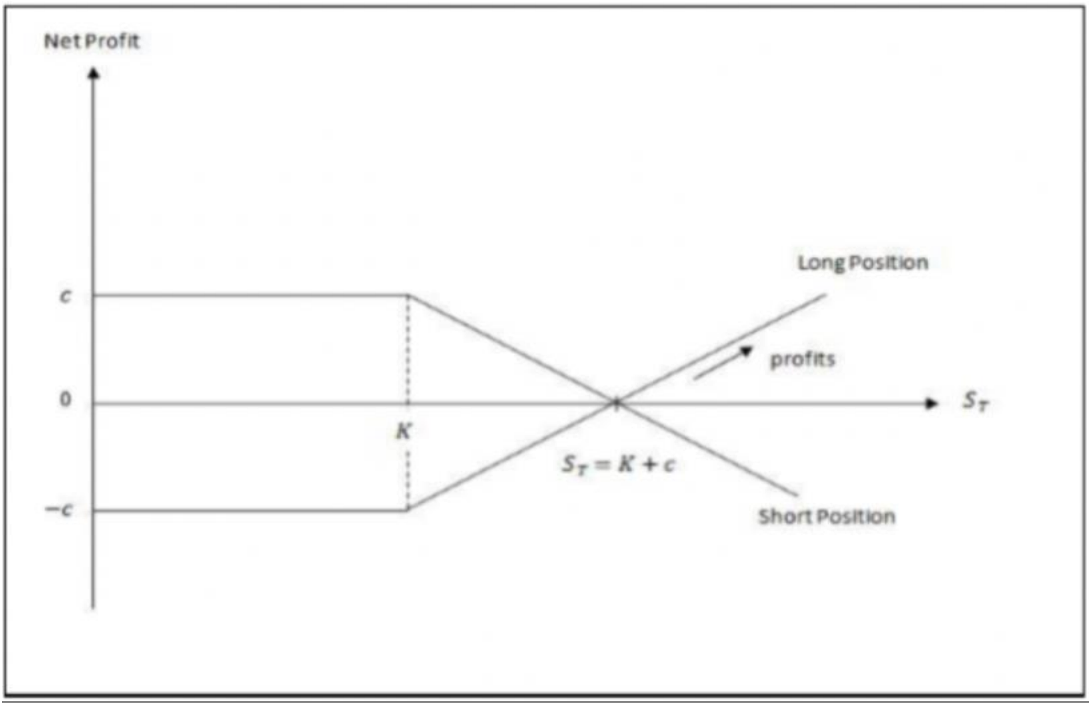


Figure 2.2.1: The net profits of long and short position from European Call options

From the above diagram it can be inferred that options are a sub-zero game; the profits that arise for the buyers if exercised are the equivalent losses for the seller and respectively if the option is not exercised the call premium from the buyer is transferred to the seller. Thus, a long position on a call option accentuates the upside potential in the event of an upward market movement.

2.3. European Put Options

The put option gives the buyer (holder) the right to sell an underlying asset in a certain price on a certain period. The buyer of the put option expects the spot price of the asset to go below the strike price in order to yield a profit in contrast to the holder of a call option who expects the opposite. This implies that the option will not be exercised if the strike price is below the spot price ($S_T > K$) because it is disadvantageous to sell something that is cheap and buy it immediately more expensive. In the case which the strike price is above the spot price ($S_T < K$) at the end of its maturity the holder will surely exercise the option and acquire an immediate profit due to the fact that he sells something expensive and simultaneously buy it in the prevailing smaller price of the market. In this point we should infer that in order to yield profits from a long position to a put option the strike price must exceeds the spot price plus the premium that were paid initially. In the case the strike price is above the spot price but not with the premium added, any individual would exercise this put option in order to minimize his losses by capturing a fraction of the premium that was paid initially. Profits grow linearly as the spot price of the underlying asset declines and moreover a put option gives protection in the event that the price of the underlying security plunges.

Payoff of European Put Options

Therefore, the payoff of an investor who had taken long position on a European put option is:

$$Payoff\ Long\ Put = \max[K - S_T, 0]$$

And this shows that the option will be exercised only if $K - S_T > 0 \Rightarrow K > S_T$ and not exercised if $K < S_T$.

Net profit here is:

$$Net\ profit = \max[K - S_T, 0] - p$$

Where p is the price of the put option (put premium).

The writer of a put option expects the price to be kept either constant or go above the strike price so the option will not be worth exercised and obtain the put premium. Hence, sellers of options are addressed with huge risks because there are no limits to stop their losses, since they have obligation to pay the buyers if they exercise the options.

The payoff of an investor who has taken short position in a European put option is:

$$Payoff\ Short\ Put = \min[S_T - K, 0]$$

Or

$$-\max[K - S_T, 0]$$

The net profits of both positions on a European put option are depicted below:

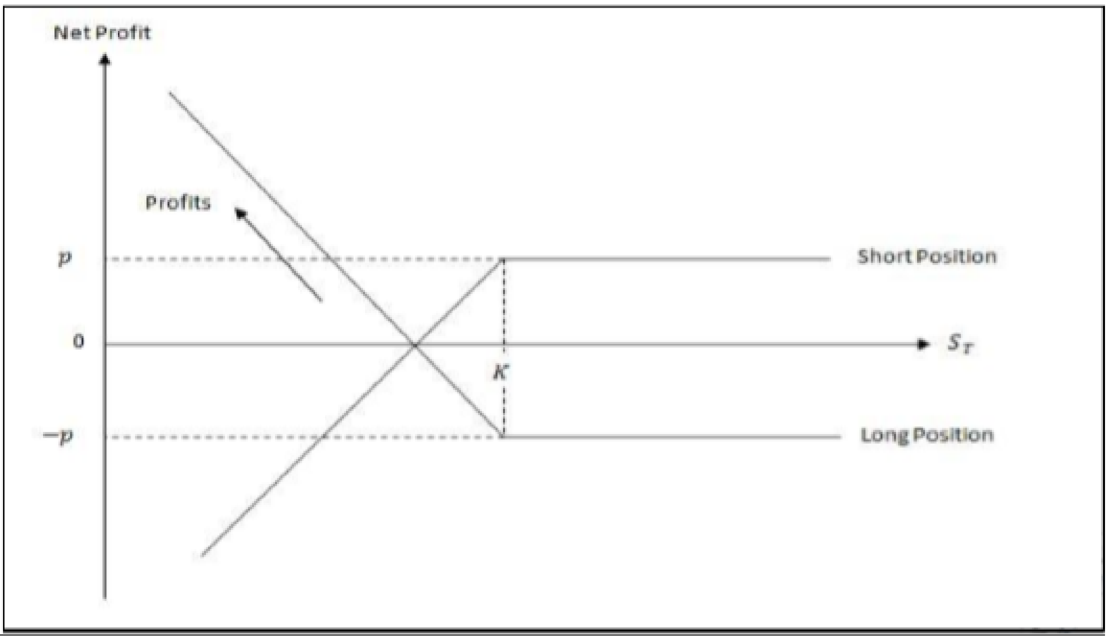


Figure 2.3.1: The net profits of long and short position from European Put options

A long position on a short option accentuates the upside potential in the event of a downward market movement. We can conclude from the two diagrams of payoffs of European Options that the gains or losses of someone who has taken the short position (writer) is the reverse of someone who has taken long position (buyer).

2.4. How factors affect the option prices particular

Options are being affected by six factors: the current price of the underlying asset S_0, the strike price K, the time to expiration T, the volatility of the asset σ, the risk free interest rate r and the dividends D that are expected to be paid. Below is a table that shows in which way every factor affects the price of the option both in American and European options and after there is a sort analysis for each one factor.

Table 2.4.1: How factors affect the prices of options

Variable	European Call	European Put	American Call	American Put
Current stock price	+	-	+	-
Exercise price	-	+	-	+
Time to expiration	?	?	+	+
Volatility	+	+	+	+
Risk-free interest rate	+	-	+	-
Dividends	-	+	-	+
* + indicates that an increase in the variable will directly make the option price to increase				
*- indicates that an increase in the variable will directly make the option price to decrease				
* ? Indicates that the relationship is addressed with uncertainty				

This table illustrates the effect on the price of a stock option when one variable increase and the others are kept constant.

It can be concluded that the value of an American and European call option increases as the time to expiration (this increase is addressed only to the American option), the volatility, the risk-free interest rate and the current stock price increase. Accordingly, the value of an American and European call option decreases when the exercise price or the expected dividends increase. The value of a put option increases as the time to expiration (this increase is referred only to the American put option), the volatility, the strike price and the future expected dividends increase. Moreover, an increase in the current stock price or the risk- free interest rate will decrease the value of both American and European put option.

Price of the underlying asset and Exercise price

When a call option is exercised the payoff will be the amount by which the price of the underlying asset exceeds the exercise price. For the put option when it is exercised the payoff will be the amount by which the exercise price exceeds the price of the underlying asset. The value of the call options increases when the price of the underlying asset increases and become less valuable when the exercise price increases. The opposite stands for the put options where their value increase when the exercise price increase and decrease when the price of the underlying asset increases.

Volatility

The volatility of an underlying asset can be defined as a measure to explain how uncertain or unpredictable the future price movements are. Stocks volatility is usually fluctuating between 15%-60% (see Hull 8th ed.) There is no particular reason for what creates the volatility. Many researchers agree that volatility comes up from the new information produced by the market and as a result people need to review their opinion about the underlying asset. Volatility has a positive impact in the value of options both call and put; this is due to the fact that the bigger the volatility is the greater the possibility the price has to exceed the strike price in the case of a call option and the opposite stands for the put option.

Time to expiration

American call and put options become more valuable as the time to expiration increases. If we take two options either call or put that are different only in the expiration date, we can easily observe that the owner of the long-life option has more exercise opportunities than the owner of the short life. Hence the long-life option must have at least as much value as the short life option. But if we take into account the dividends that are given from the stocks the result could be altered. For example, consider two European calls on a stock: the first with expiration date in 2 months and the second with expiration date in 4 months. If large dividends are expected to be given, this will have as a result a decline to the stock price and consequently the short life option could be more valuable than the long-life option (see Hull 8th ed.)

Risk-free interest rate

Interest rate plays a crucial role in order to evaluate all kind of derivatives. One issue that arises is which rate is the most suitable for traders to use as the risk-free interest rate. Generally, it was thought for many that derivatives traders would adopt the interest rates implied by Treasury bill and bonds as the risk-free interest rates, but this is not the case for a couple of reasons. The most important reasons are addressed with tax or regulatory issues which have the indirect consequence of causing Treasury rates[1] to be very low. As a result, many financial institutions and derivatives traders use the LIBOR[2] rate as the risk-free interest rate.

Dividends

Dividends have indirect consequences on the price of stock options. When dividends are given, the stock price tend to decline and as a result the value of call options decrease and the value of put options increase. Generally, the value of a put option is positively related to the number of dividends and the value of a call option is negatively related to the number of dividends.

Intrinsic and Time value

During the maturity date T of an option, in an efficient market the value of an option will be zero provided that it is not offered for exercise or the difference between the spot price of the underlying asset and the exercise price if it is a call option or respectively the opposite if it is a put option. This is named as Intrinsic value and is depicted as follows:

Call option: $$Intrinsic\ value = \max\,[0; S_t - K] \quad \textbf{(2.4.1)}$$

Put option: $$Intrinsic\ value = \max\,[0; K - S_t] \quad \textbf{(2.4.2)}$$

According to the unit price rule (see Mylonas [2005]), during the day of expiration the speculators will drive the prices of the options to their intrinsic value because any other price would give the opportunity for arbitrage[3]. Below there is an example in order to see the arbitrage opportunity:

Example

There is a call option on a share with X=12$ (X=strike price) and S=13.5$ (S=Spot price). In the expiration date the premium of the call option must worth 1.5$ as its intrinsic value (eq. 2.4.1). If its value was 2$, then the speculators would benefit from this opportunity selling the option and obtaining 2$ and after they would buy the share for 13.5$ in order to deliver to the buyer of the option (in case it is exercised). So, without any risk and investment the speculator

[1] Treasury rates are the rates which are paid by every government for borrowing on its own currency

[2] LIBOR are the rates by which banks are being borrowed from the internal bank market.

[3] Arbitrage is an investment opportunity that brings profit without capital needed and being exposed to any risk

made a profit of 0.50$ per share. This incentive would actuate many investors to sell call options and buy shares; as a result, the price of the options would tend to equalize with the intrinsic value of the option. Respectively this can occur with put options with reverse positions.

During the days before the option expires it is observed that the option shows up with value greater than the intrinsic value. This added value is called Time value and reflects the expectations of traders about the shift of the price of the underlying asset so that its exercise be beneficial at his maturity. It can be seen that the greater the duration of the option is the greater the Time value of the option would be. In the last day of the option maturity, Time value is equivalent to zero and buyer cannot be more benefited from the fluctuations of the price of underlying asset.

Time value is computed as follows:

$$Time\ Value = Value\ of\ the\ Option - Intrinsic\ Value$$

2.5 Moneyness

According to the sizes of Time value and Intrinsic value, options are divided into three categories:

The first category is called In the Money and includes all the options that have greater value than their intrinsic value and they are profitable if exercised. In this category call options with strike prices below spot price and put options with strike prices above spot price are called In the Money.

The second category is called At the Money and includes all the options that does not have intrinsic value (equivalent to zero), buyers (holder of the option) are uninterested either to exercise them or not and the options have only Time value.

The third category is called Out of the Money and includes all the options that does not have intrinsic value (equivalent to zero), are unprofitable to be exercised and the option has only Time value. In this category call options with strike prices above spot price and put options with strike prices below spot price are called Out of the Money.

2.6 Upper and Lower bounds for European Options

In this part we infer the upper and lower bounds of option prices. If the option price of an underlying asset is higher than the upper bound or below the lower bound, then arbitrage opportunities arise. Before analyzing the upper and lower bounds we will use the following notations:

S_0 : Current stock price

K : Exercise price of the option

T : Time to expiration of the option

S_T : Stock price on the expiration date

R : Continuously compounded risk-free interest rate for an investment maturing in

Time T

c : Value of European call option

p : Value of European put option

Upper bounds of European Call options

A European call option gives the buyer the right but not the obligation to buy one share of a stock or an underlying asset in a determined price. So, in any case the price of the option cannot be worth more than the stock or the underlying asset. Therefore, the stock price is the upper bound to the option price:

$$c \leq S_0$$

If these conditions were not satisfied, an arbitrageur could make a riskless profit by selling the call option and buying the stock.

Upper bounds of European Put options

A European put option gives the right to sell one share of a stock or an underlying asset for *K*. Regardless how low the price of the stock becomes, the price of the option cannot be worth more than *K*. Hence, it follows that it cannot be worth more than the present value of *K* and the upper bound to the option price is:

$$p \leq K$$

Lower bounds of Call options

The price of the option reflects the difference between the future expected value of an underlying asset and the exercise price, so the future expected value of the underlying asset can never be less than $S_0\ e^{RT}$. This is due to the fact that if the future expected value is less than $S_0\ e^{RT}$, it is conceivable to short-sell the underlying asset, invest the amount of money form short-selling at risk-free interest rate and take a profit at time *T*. Hence the future expected value of an underlying asset cannot be less than the current price compounded at risk free interest rate, and therefore the value of the option cannot be less than the strike price discounted at risk free interest rate. If the latter doesn't hold, then one would buy a call option, short sell the underlying asset, invest the money at risk-free interest rate and then be left with a riskless profit no matter whether the future price of the underlying asset goes up or down. Therefore, the lower bound for a European call option is:

$$c \ \geq S - Ke^{-RT} \ \ or \ \ c \ \ \geq \max \ (S - K\ e^{-RT}, 0 \ )$$

Lower Bounds of Put Options

As inferred before the future expected price of an underlying asset cannot be less than the compounded value of its current price, compounded at the risk-free interest rate. In other words, the present price of the underlying asset cannot be less than the discounted value of future expected price or the discounted value of the exercise price. In case it is, it should be reflected in the price of the put option and so the lower bound for a European put option is:

$$p \ \geq Ke^{-RT} - \ S_0 \ \ or \ \ p \geq \max \ (Ke^{-RT} - \ S_0, 0 \ )$$

2.7 Why the investors use options

There are mainly two reasons about the usage and importance of options.

The first reason is about speculation. In the field of options, while one using them is like betting on the future move of the underlying asset, which either follows a downward movement and profit is acquired by taking long position to a put option or follows an upward movement and profit is acquired by taking long position to a call option. The biggest advantage of options is that there is no limit in making profits by just paying a premium which is not so expensive in accordance with the possible profits that could arise.

The second reason concerns hedging. They are plenty of reasons why investors choose to hedge but its primary purpose is about reducing or even easing the potential losses which can arise from the fluctuations of a stock or a portfolio of stocks that the investors own. Options can be used either from an individual investor or from a financial institution in order to protect their portfolios or ensure an investment from the big variances that the prices of the underlying securities are addressed.

Chapter 3

Pricing of Financial Derivatives

One of the most important characteristics of financial derivatives is their pricing methods. The most crucial feature of pricing derivatives is to find the most appropriate and suitable method in order to receive results which are closer to the observed market prices. In this Chapter the characteristics of the two pricing formulas are displayed and analyzed; the Black & Scholes model with the differential equation produced by Merton and the one-step and two-step Binomial trees. Also, some deficiencies of the models are reported.

3.1 The Black & Scholes model

The fundamental papers on option pricing by Fisher Black, Myron Scholes [1973] and Robert Merton [1973] not only gave solution for valuing any individual financial instrument but also showed that these applications could be applied in a wide range of issues regarding computing corporate liabilities. Black, Scholes and Merton made through their papers three admirable inventions. Firstly, assuming frictionless trading and that asset prices follows Ito lemma processes, they deducted that it was feasible to create a perfect hedge position between the option and the underlying security and due to the fact that the position was riskless they suggested that the required return of the riskless position should be the riskless interest rate. Secondly, they revealed the famous option pricing formula which connects the price of the option with the price of the underlying asset and this was the solution to the valuation formula for European options. Thirdly, they indicated that many types of contracts with some mutual and common characteristics such as corporate debt and warrants could be considered as options and thus be valued in a similar way (see Shaefer [1998]). Apart from the main use of the model regarding options pricing, these methods of Black, Scholes and Merton are extensively used for maintaining financial risk management.

The Black & Scholes model refers to European options and within its construction they made the following assumptions (see Black and Scholes [1973]):

1) The options are European, so that they can be exercised only in a predetermined date
2) There are no dividends paid by the stock or other distributions
3) The interest rate is accounted to be constant and known through time[4]
4) The stock price follows a random walk in continuous time with a variance rate proportional to the square of the stock price. Moreover, the distribution of possible stock prices at the end of any finite interval is log-normal and the variance rate of the return of the stock is constant.
5) There are no penalties for short selling

[4] they tried the assumption that the stocks expected return was equivalent to the interest rate, (this meant that short term and long term interest rates were equal); in other words they implied that the beta of the stock was zero, thus all of its risk could be diversified or eliminated

6) It is feasible to borrow any fraction of the price of the security either to buy it or hold it at the short-term interest rate
7) There are no transaction costs in buying or selling the stock or the option

They demonstrated that it is possible to create a hedged position constituted by long position on a stock and short position to an option the value of which will not depend on the value of the stock price (see Black and Scholes [1973]).

Moreover, the Black & Scholes model hypothesize that the prices of the underlying assets follow a geometric Brownian motion with constant mean and variance.

Geometric Brownian motion is referred as a Wiener process and it is a particular type of Markov[5] stochastic process with a mean change of zero and a variance rate of 1 per year (see Hull 8th ed.). A variable x is said to follow a Brownian motion if the following two properties are satisfied:

1. The change Δx during a small period of time Δt is

   $$\Delta x = z\sqrt{\Delta t} \qquad \textbf{(3.1.1)}$$

   Where z has a standardized normal distribution $\varphi(0,1)$

   It follows that Δx has a normal distribution with mean of $\Delta x = 0$ and

   variance of $\Delta x = \Delta t$

2. The values of Δx for any two different short intervals of time Δt are independent, implying that x follows a Markov process.

Another important assumption of the Black-Scholes model is the assumption of the log-normal distribution (see Hull 8th edition).

Two of the conditions for an asset price to have a lognormal distribution are:

1. The volatility of the asset is constant.

2. The price of the asset changes smoothly with no jumps.

In practice, neither the first nor the second of these conditions is satisfied for an exchange rate. The volatility of an exchange rate is far from constant, and exchange rates frequently exhibit jumps.

Let's define:

μ: is the expected return on underlying asset per year

σ: is the volatility of the underlying asset per year

In time Δt the mean of the return is $\mu\,\Delta t$ and the standard deviation is $\sigma\sqrt{T}$ and so it follows that:

[5] A Markov process is a distinct stochastic process which infers that only the current value of a variable is appropriate about predicting the future expected value. The past movements of the variable and the way that emerged from the past are uncorrelated. Stock prices usually follow a Markov process.

$$\frac{\Delta S}{S} \sim \varphi\ (\mu\ \Delta t, \sigma^2\ \Delta t) \quad \textbf{(3.1.2)}$$

Where ΔS is the change of the price of the stock in time Δt and $\varphi\ (x, \sigma)$ implies a normal distribution with mean x and variance σ. The model indicates that:

$$\ln S_T - \ln S_0 \sim \varphi\ [\ (\ \mu - \sigma^2/2)\ T, \sigma^2\ T]$$

$$\ln \frac{S_T}{S_0} \sim \varphi\ [\ (\mu - \sigma^2/2)\ T, \sigma^2\ T] \quad \textbf{(3.1.3)}$$

And so

$$\ln S_T \sim \varphi\ [\ln S_0 + (\mu - \sigma^2/2)\ T, \sigma^2\ T] \quad \textbf{(3.1.4)}$$

Where S_t is the stock price at the time *T* and S_0 is the initial stock price at time *T=0*. Equation 3.1.4 denotes that $\ln S_t$ is normally distributed with the mean being equivalent to $\ln S_0 + (\mu - \sigma^2/2)$ T and the variance being $\sigma^2\ T$ and so it follows that S_T is lognormal distributed.

By respect to these assumptions the Black & Scholes formula for a European Call option with dividends equivalent to zero is derived below:

$$c = S_0 N(d_1) - X e^{-rT} N(d_2) \quad \textbf{(3.1.5)}$$

The model is separated into two fragments: The first part is $S_0 N(d_1)$; the price is multiplied by the change in the price of the call premium regarding a variation to the underlying price. The second part $X e^{-rT} N(d_2)$, depicts the prevailing value of paying the strike price during maturity. Then the value of European Call option is computed by taking the differences of those parts as it is noted to the equation.

Hence, the valuation of a European Put option with dividends equivalent to zero is:

$$p = X\ e^{-rT} N(-d_2) - S_0 N(-d_1) \quad \textbf{(3.1.6)}$$

Where:

$$d_1 = \frac{ln(^{S_0}/_X) + \left(r + ^{\sigma^2}/_2\right) T}{\sigma\sqrt{T}} \quad \textbf{(3.1.7)}$$

$$d_2 = \frac{ln(^{S_0}/_X) + \left(r - ^{\sigma^2}/_2\right) T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \quad \textbf{(3.1.8)}$$

Whereas *c* is the value of the call option (call premium), *p* is the value of the put option (put premium), S_0 is the spot price of the underlying asset, *X* is the strike price or exercise price, *r* is the short term rate which is constant, *T* is the time until the expiration date, σ^2 is the variance rate of return for the underlying asset and $N(d_i)$ is the cumulative normal density function evaluated for every d_i.

3.2 The Black & Scholes-Merton Model with Dividend Policy

Merton extended the Black & Scholes model making major adjustments in several ways (see Duffie [1998]). The most important concerns his contribution about valuation of American options and options with a dividend policy (see Merton [1973]), his extension to the occasion of the discontinuous stock prices processes (see Merton [1976]) which implied that the model would not be confined in the future to the setting of Brownian motion and the conversion of the original Black-Scholes-Merton no-arbitrage pricing argument from one based on instantaneous returns to one based on dynamic replicating strategies (see Merton [1977]). Below there is a short analysis about dividends and the Black & Scholes formula involving dividends is derived.

Dividends have a direct result on the stock prices. When dividends are given, the stock prices tend to decrease by the amount of the dividends paid. In other words, if a stock price with a dividend yield of d widen from S_0 today at S_T at time *T*, then in the absence of dividends it would grow from S_0 today to $S_T e^{dT}$ at time *T*. Respectively, in the absence of dividends the stock price would grow from $S_0 e^{-dT}$ today to S_T at time *T*.

In order to derive the Black-Scholes-Merton formula we substitute $S_T e^{dT}$ instead of S_0 to the equations (3.1.5) and (3.1.6); the value of European Call and Put options with dividend rate *d* are depicted below:

$$c = S_0 e^{-dT} N(d_1) - X e^{-rT} N(d_2) \quad \textbf{(3.2.1)}$$

$$p = K\, e^{-rT} N(-d_2) - S_0 e^{-dT} N(-d_1) \quad \textbf{(3.2.2)}$$

So it follows that *d1* and *d2* are given by

$$d_1 = \frac{ln(^{S_0}/_X) + \left(r - d + ^{\sigma^2}/_2\right)T}{\sigma\sqrt{T}} \quad \textbf{(3.2.3)}$$

$$d_2 = \frac{ln(^{S_0}/_X) + \left(r - d - ^{\sigma^2}/_2\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \quad \textbf{(3.2.4)}$$

The results given were first introduced by Merton (see Merton [1973]).

3.3 Derivation of the Black-Scholes-Merton differential equation

The Black-Scholes-Merton differential equation is an equation that can be applied to price any derivative which depends in a stock without dividends being paid. It should be noticed that unlike the portfolio used in pricing with binomial trees as we will show in next section, the Black-Scholes-Merton portfolio which is applied in order to derive the equation is riskless only for a short time of period. Thus, the portfolio will be riskless if only appropriate changes on the portions of the derivatives and the stock in the portfolio are made regularly (see Hull 8th edition).

We will begin our analysis by assuming that the stock price follows a geometric Brownian motion:

$$dS = \mu S dt + \sigma S dz \qquad \textbf{(3.3.1)}$$

Let *f* be the price of the call option or any other derivative subject to *S*, which also need to be function of *S* and *t*. So, we have:

$$df = \left(\frac{\vartheta f}{\vartheta S}\mu S + \frac{\vartheta f}{\vartheta t} + \frac{1}{2}\frac{\vartheta^2 f}{\vartheta S^2}\sigma^2 S^2\right) dt + \frac{\vartheta f}{\vartheta S}\sigma S dz \qquad \textbf{(3.3.2)}$$

Π is represented as the value of the portfolio and so by definition:

$$\Pi = -f + \frac{\vartheta f}{\vartheta S} S \qquad \textbf{(3.3.3)}$$

Hence the change *ΔΠ* in the value of portfolio in a short time interval *Δt* is given by:

$$\Delta\Pi = -\Delta f + \frac{\vartheta f}{\vartheta S}\Delta S \qquad \textbf{(3.3.4)}$$

Because of the assumptions reported above it is inferred that the portfolio must be riskless and be entitled with the same rate of returns such as other short-term risk-free underlying assets. If the interest rate of portfolio diverges by either being more or less, then arbitrage opportunities arise. So, it follows that:

$$\Delta\Pi = r\Pi\,\Delta t \qquad \textbf{(3.3.5)}$$

Where r is the risk-free interest rate. Substituting the equations above the Black-Scholes-Merton differential equation is derived:

$$\left(\frac{\vartheta f}{\vartheta t} + \frac{1}{2}\frac{\vartheta^2 f}{\vartheta S^2}\sigma^2 S^2\right)\Delta t = r\left(f - \frac{\vartheta f}{\vartheta S}S\right)\Delta t$$

So that $$\frac{\vartheta f}{\vartheta t} + rS\frac{\vartheta f}{\vartheta S} + \tfrac{1}{2}\sigma^2 S^2 \frac{\vartheta^2 f}{\vartheta S^2} = rf \qquad \textbf{(3.3.6)}$$

The above equation is the Black-Scholes-Merton differential equation which has many solutions being analogous to all the different derivatives that can be described with S as the underlying variable. One important advantage this model is accompanied with is that the results of the equation are not affected by the risk preferences which investors have. The deviations appearing in the equation are the spot price, time, risk-free interest rate and volatility of the stock price.

3.4 Deficiencies of the Black & Scholes Model

Many studies have been made on the assumptions the Black & Scholes model is based on. Investigating the discrepancies between the model assumptions and the real-world assumptions is essential. It can be easily observed that these assumptions when compared to the real world are violated. For instance, continuous process and normal distribution are often violated when real market data are tested. Embrechts et al. [1999] expressed that the financial data are more likely addressed to non-continuous to jump-like progress. Another strong prerequisite necessity for the derivation of the Black-Scholes model is the perfect derivative replication by the share and a risk-free instrument; however, this cannot be achieved without transaction costs (see Jankova [2018]). Grossman and Zhou [1996] discovered that volatility is not only correlated to the spot price of the underlying asset but also to the volume of trade and the existence of transactions costs. This led to the creation of new models who account for the existence of the transaction costs (see Davis et al. [1993]).

Above all, volatility parameter is the most imperative input for pricing an option and plays a crucial role on option pricing and trading. For investors who use options, volatility is a measure of the rate of return they acquire over the holding period and so its forecasting is essential in order to understand to what extent of risk is the investor faced by trading on options.

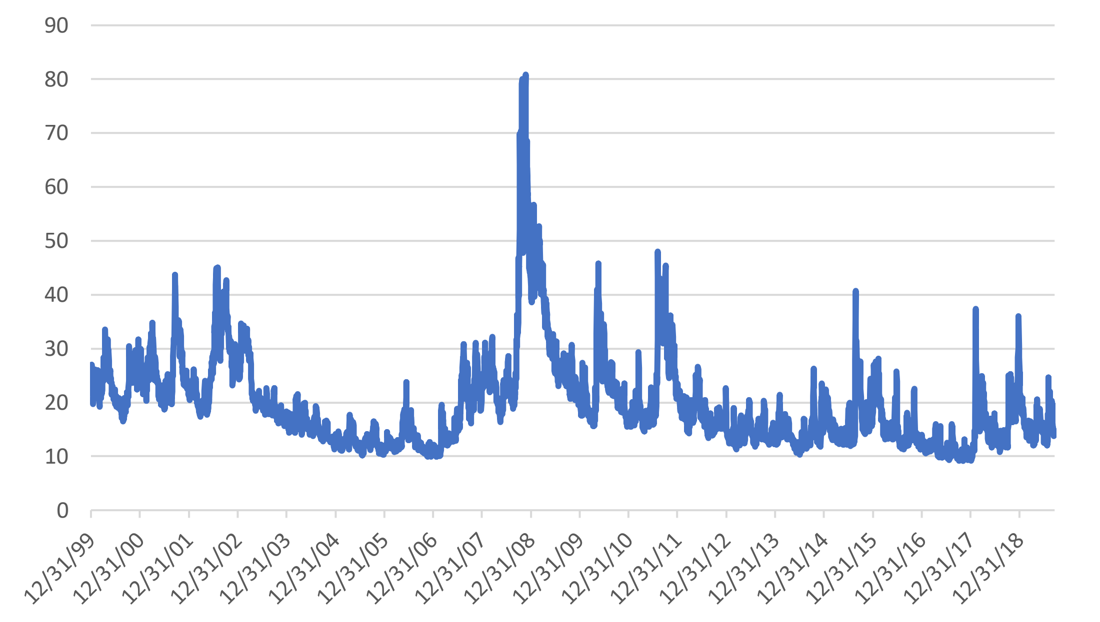
The variation of stocks has received much attention in the empirical literature; some researchers have constructed new methods for enhancing the accuracy of the computation of the variance from historical stock return data while others have used option prices to recover current estimates (implied standard deviation).

The most crucial parameter of the Black & Scholes model is the volatility. The model we described above assumes volatility to be constant; returns of the underlying security are thought to be constant but this is not the case. The higher value the volatility variable takes the more fluctuation the price will have and as a result high uncertainty occurs regarding the accomplishment of the expected return. Volatility estimation is the most important factor in order to price derivatives and acquire accurate results. In the Black & Scholes model volatility is estimated by taking historical data and has taken a lot of critics.

James D. Macbeth and Larry J. Merville [1979] stated that the biggest problem that emerges from the Black & Scholes model emanate from the variance rate. They extracted similar computation tests like Blattberg and Gonedes [1974], Latane and Rendlman [1976], Chiras and Manaster [1977]; all of these works were focused on calculation of the implied standard deviation which is derived by substituting the observed market prices of the options on the Black & Scholes model while keeping all the other variables constant the standard deviations is computed. Macbeth and Merville deduced from their empirical results that variance rate is not constant and changes over time and that these differences are also related to the differences of the stock prices, the exercise prices and the time to expiration (see Macbeth and Merville [1979]). In this Chapter it is not our intention to describe new alternative methods for better calculation of the variance rate; some are depicted on the Literature Chapter.

The fact that volatility is not constant can be seen also from the VIX Index which was established by the Chicago Board Options Exchange (CBOE); it is a market index that shows that markets expectation of 30-day forward-looking volatility which is calculated from the prices of the S&P 500 index options.

Figure 3.4.1:VIX Index



Source: Datastream, Price Index (PI)

From the diagram above we can observe that the volatility is far from constant, especially in periods where financial crisis occurs (2007-2008) or important economic turnovers exist.

Another assumption that is violated concerns the Brownian motion process. The Black & Scholes model suggests that stocks follow a random walk; this means that in any given time the price of the underlying asset can go either up or down with the same probability. But this does not stand because stock prices are ascertained by many factors which cannot be granted the same probability in the way they will affect the movement of the stock prices.

Moreover, the assumption of the log-normal distributions is also violated. Many have observed that asset returns have a finite variance and semi-heavy tails in contrast to the stable distributions such as log-normal with infinite variance and heavy tails. Hull inferred that returns are more likely to be leptokurtic, having more tendency to exhibit outliers than would be the case if they were normally distributed (see Yalincak [2012]).

Hence, this model suggests that interest rates are constant through time and known; such assumption cannot stand in a realistic world. The U.S. Government Treasury Bill 30-day rate is used as the risk-free interest rate, but even treasury rates change over time of high volatility.

3.5 Binomial trees

Another technique that is widely used for pricing derivatives is the construction of Binomial trees. Binomial Option Pricing Model was first introduced by Cox and Rubinstein [1979] and was suggested as a tool to explain the Black & Scholes model but soon after it became a more accurate model for pricing options. Binomial trees refer to the development of a diagram which represents different possible paths that the stock price might follow during the life of an option. In this technique a basic assumption is made that the stock price follows a random walk[6].

The Binomial trees can be applied by making the following assumptions:

- There are no arbitrage opportunities
- The interest rate for borrowing is the same as the interest rate for investing and it is constant
- The time is discrete
- There are no taxes, transaction costs or margin requirements
- Short selling is allowed
- Markets are competitive[7]

The binomial tree model produces a binomial distribution of all the possible paths that a stock price could follow during the life of the option. During each time step there is a determined probability the price will move up and a determined probability the price will move down. As the time steps become smaller, the Binomial tree shows a significant convergence with the results of the Black and Scholes model. In this model the underlying asset can take only one value between the two possible values that arise, which is in no way realistic because any asset can be worth of an infinite number of values within any given range and that is its main disadvantage.

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[6] During each step, there is a certain probability that the price moves up and a certain probability that the price moves down. This model entails a very important principle; it is used for valuing American options.

[7] Any particular investor can buy or sell as much of any underlying asset as he wishes without any consequence to the price of the asset.

3.6 The one step binomial model

In the following analysis we will assume that arbitrage opportunities do not exist. Let consider of a stock with price S_0 and an option addressed to the stock with current price *f*. We assume that the option will last for time *T* and that during the life of the option the stock price can either move up from S_0 to S_0u, where $u > 1$ or down from the current stock price S_0 to S_0d, where $d < 1$. The percentage increase when the stock follows an upward movement is $u - 1$; the percentage decrease when the stock follows a downward movement is $1 - d$. In the occasion the stock price moves to S_0u, we infer the payoff of the option as f_u and if the stock price moves down to S_0d we infer the payoff of the option to be f_d. The above situation is depicted with the following diagram:

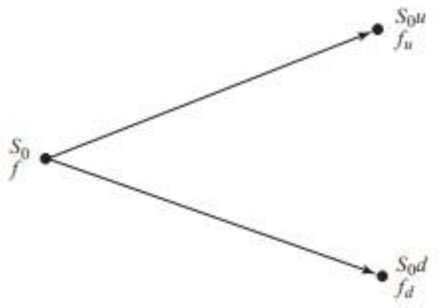

Figure 3.6.1: Stock and option prices represented by a one step-tree

We now consider of a portfolio consisting of a long position in Δ shares and a short position in one option. We will calculate the value of Δ in order to set the portfolio to be riskless.

When the stock price follows an upward movement, the value of portfolio at the end of the life of the option is

$$S_0u\Delta - f_u \quad \textbf{(3.6.1)}$$

If the stock price follows a downward movement, the value of the portfolio becomes

$$S_0d\Delta - f_d \quad \textbf{(3.6.2)}$$

In order the portfolio to be riskless equations (3.6.1) and (3.6.2) must be equal:

$$S_0u\Delta - f_u = S_0d\Delta - f_d \quad \textbf{(3.6.3)}$$

And as a result

$$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d} \quad \textbf{(3.6.4)}$$

The portfolio now is riskless, arbitrage opportunities do not exist, and we earn the risk-free interest rate. Equation 3.6.4 shows that Δ is the ratio of the change of the option price to the current stock price as we move between the nodes at time *T*.

We will denote the risk-free interest rate by *r* and so the present value of portfolio becomes:

$$(S_0 u\Delta - f_u)e^{-rT} \quad \textbf{(3.6.5)}$$

It should be noticed that we require $u > r > d$. If these inequalities do not hold, arbitrage opportunities arise involving the stock and the riskless borrowing and lending interest rate (see Cox et al. [1985]).

The cost of creating the portfolio is:

$$S_0\Delta - f$$

And so, it follows that:

$$S_0\Delta - f = (S_0 u\Delta - f_u)e^{-rT}$$

Or:

$$f = S_0\Delta(1 - ue^{-rT}) + f_u e^{-rT}$$

After we make the right substitutions, we deduce the follow equation:

$$f = e^{-rT}[pf_u + (1-p)f_d] \quad \textbf{(3.6.6)}$$

Where:

$$p = \frac{e^{-rT} - d}{u - d} \quad \textbf{(3.6.7)}$$

The two equations above allow us to price an option where the stock price movements are given by the one step binomial tree with the only assumption that arbitrage opportunities do not exist. This formula includes an important problem because the probabilities of the movements of the stock price are not included and it can be easily derived that when the probability of an upward move increases it affects the prices of the options: the price of a call option will increase and the price of the put option will decrease.

This formula entails some interested features that are depicted below:

- The value of a call option is not affected by the investor's preferences toward risk. Hence, we would acquire the same formula even if investors are risk-averse, risk-lovers or risk-neutral

- The only stochastic value on which the value of the call option depends is the stock price. More specific it does not depend on the random prices of other underlying assets or portfolios

3.7 Two-step Binomial Trees

Now we will expand the above formula for more than one period; we will consider a call option with two periods remaining before tis maturity date. By keeping the binomial process, the stock can now take three possible values after two periods as it is shown on Figure 3.7.1**.** The assumptions that were made are the same for every stage. The stock spot price is S_0 and during each time step the stock price either moves up to *u* times its initial value or moves down to *d* times its initial value. The length of the time step is *Δt* years and *r* is the risk-free interest rate. Because the value of the length has changed from *T* to *Δt* the equations (3.6.6) and (3.6.7) are transformed and we resulted to the following equations:

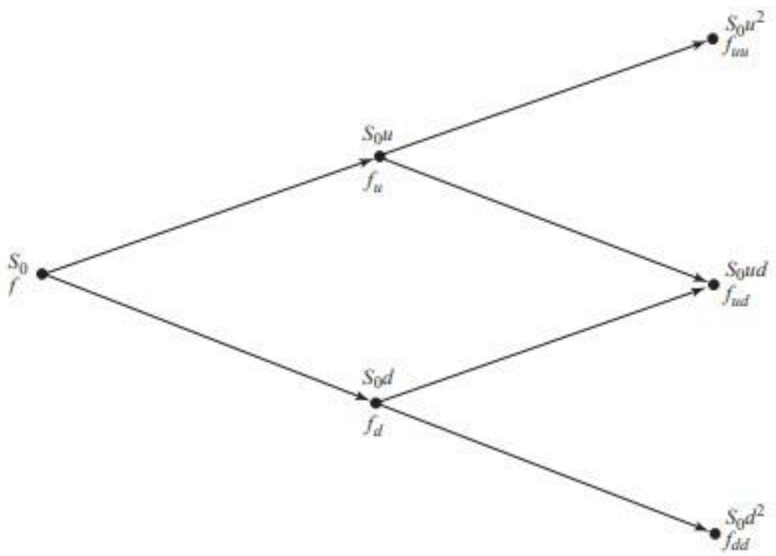

Figure 3.7.1: Stock and option prices represented by a two step-tree

$$f = e^{-r\Delta t}[pf_u + (1-p)f_d] \qquad \textbf{(3.7.1)}$$

$$p = \frac{e^{-r\Delta t} - d}{u-d} \qquad \textbf{(3.7.2)}$$

The above equations after some substitutions and calculations give us:

$$f_u = e^{-r\Delta t}[pf_{uu} + (1-p)f_{ud}] \quad \textbf{(3.7.3)}$$

$$f_d = e^{-r\Delta t}[pf_{ud} + (1-p)f_{dd}] \quad \textbf{(3.7.4)}$$

$$f = e^{-r\Delta t}[pf_u + (1-p)f_d] \quad \textbf{(3.7.5)}$$

After substituting from equations (3.7.3) and (3.7.4) into equation (3.7.5), we take the following equation:

$$f = e^{-2r\Delta t}[p^2 f_{uu} + 2p(1-p)f_{ud} + (1-p)^2 f_{dd}] \quad \textbf{(3.7.6)}$$

Where p^2, $2p(1-p)$ and $(1-p)^2$ are the probabilities that the upper, middle and lower final nodes will be reached respectively.

The latter equation gives us the price of a call option where two periods have remained and thus it could be expanded for more time steps.

3.8 Volatility estimation

An important feature that we should take into consideration when developing a binomial tree is the parameter of volatility. We choose the parameters u and d in order to match with the volatility of the stock price. A crucial question that arises is whether we should use volatility in the real world or the risk-neutral world. We will deduce that volatility is the same for both the real world and the risk-neutral world. Let's denote that the expected return of the stock price is μ and the volatility of the stock price is σ. We consider p^* to be the probability of an upward movement in the real world and p to be the probability of an upward movement in a risk-neutral world. In the real world the expected stock price is $S_0 e^{\mu\Delta t}$.

The expected stock price at this time on the tree is:

$$p^* S_0 u + (1-p^*)S_0 d \quad \textbf{(3.8.1)}$$

In order the expected return of the stock to be in accordance with the trees parameters we need these two equations to be equal. So, we have:

$$p^* S_0 u + (1-p^*)S_0 d = S_0 e^{\mu\Delta t}$$

Or

$$p^* = \frac{e^{\mu\Delta t} - d}{u-d} \quad \textbf{(3.8.2)}$$

Cox, Ross and Rubinstein [1979] proposed the values of *u* and *d* in order to be consistent with the volatility and these are derived below:

$$u = e^{\sigma\sqrt{\Delta t}} \quad \textbf{(3.8.3)}$$

$$d = e^{-\sigma\sqrt{\Delta t}} \quad \textbf{(3.8.4)}$$

Moreover, binomial trees formula can be applied also on options which are connected with stocks paying a continuous dividend yield, stock indices, currencies and futures.

A fundamental principle of this model is that we can assume the world is risk-neutral when valuing an option and thus the results we obtain are equivalent to the same option prices as it would be in the real world.

This model when compared to the Black & Scholes model can more precisely price American style options because it accounts the possibilities of early exercise and other factors like dividends. Nevertheless, there are also some disadvantages compared to the Black & Scholes Model. Binomial Option Pricing Model is much more complicated, slower and also not useful for calculating thousands of options on a short time in contrast to the Black & Scholes Model.

Chapter 4

4.1 Empirical Application

In this section we implement and empirically validate methods for pricing European options subject to underlying assets which follow discrete distributions based on scenario trees. We are concentrated on European options and apply two methods that have been implemented by Topaloglou et al. [2008]. These methods take into account the effects of higher moments (skewness and kurtosis) as they appear on the empirical distribution of the returns of the underlying asset.

The first method is an expansion of the Black & Scholes model with two adjustments which take into consideration the effects of higher moments. This enable us to compute the option price as a sum of three segments: the price resulted from the Black & Scholes model plus two additional parameters that refer to skewness and kurtosis as those appear in the scenario sets of assets prices.

In the second method, we define an equivalent risk-neutral probability derived from the physical probability measure that is associated with the prices on each node of the scenario tree. A crucial factor for pricing European options is the distribution of their asset's prices at the time the option expires. In this approach the distribution is illustrated by a subset of nodes of a scenario tree. After the risk-neutral probabilities of these nodes are calculated, pricing the option is a simple case. The price of an option reflects the expectation of its discounted payoff and this expectation is taken under the risk-neutral probability measure over the nodes of the scenario tree that represent the discrete support of the asset prices at the expiration date of the options.

It should be noticed that the above procedures can be used for pricing European options at any node of the scenario tree and another important principle of these approaches is the fact that they can be applied without making any assumptions about the distribution of the underlying asset price; they can incorporate any arbitrary discrete distribution.

These methods will be displayed more descriptive below and numerical tests will be implemented in order to test the viability and the performance of these methods. Hence, we get estimates from the Black & Scholes model which we assume it to be our basis for comparisons. Our aim is to compare the results from the two pricing methods and the Black & Scholes model with the real market data and deduce which methods produce more consistent results with the observed market prices. It should be inferred that it is impossible any of these methods to predict outputs that are exactly equivalent to the observed market data. Some of the deviations can be partially explained by Bates [2003], Jarrow and Madan [2000], Jacquier and Jarrow [2000].

4.2 Data Description

Our focus is on European call and put options addressed to the S&P 500 index with time to maturity of 40-days. We have collected historical prices on the S&P 500 index starting from 1/1/2000 until 31/12/2010. Thus, we computed the returns for every 40-days of the index during our tests and based on these returns we calculated the descriptive statistics as shown in the table below. After those data are collected, pricing the option following Method 1 is just some calculations in Excel. From the market data we generated 167 scenarios of index prices implying that the stock might follow at the end of the time-period testing and matched exactly the empirical moments that are observed on the returns of the index; this technique of moment-matching permit us to create scenarios which are in accordance with the empirical distribution of the index. We price both call and put options on two different dates; 10 July 2007 where $S_0 = 1510{,}12$ and 11 January 2010 where $S_0 = 1146{,}98$. These two days have a major difference: investors have different expectations about the trend of the market; one is referred to a period that investors suggest that the market will follow a downward movement (2007) and the other to a period that the market is expecting to follow an upward movement.

In this point it is proper to infer that we used a quadratic program which was solved and implemented by GAMS using CONOPT solver for NLP (Non-Linear Programming) problems in order the risk-neutral probabilities be derived.

Table 4.2.1: Descriptive statistics of monthly returns of the S&P 500 index

Mean	0,000344334
Standard Error	0,004132722
Standard Deviation	0,047301161
Sample Variance	0,0022374
Kurtosis	0,691390865
Skewness	-0,535215991

In the above table the statistical characteristics of monthly returns of the S&P 500 index are displayed. It can be inferred that the monthly returns of the index does not follow a normal distribution due to the fact that skewness is different than zero ($skewness \neq 0$) and kurtosis is different than 3 ($kurtosis \neq 3$). More precisely, the variance of the monthly returns exhibits the mean by a significant level. Skewness is a statistical feature which express the asymmetry that exists from the normal distribution in a set of data; it can take both positive and negative values depending on whether the data are skewed to the right or left side of the average respectively. As we can see hear negative skewness exist. Kurtosis is a statistical feature that is used to depict the distribution of the regarded data around the mean. In this set of data low kurtosis is observed ($kurtosis < 3$) which infers that thinner tails exist in the postulated distribution and it is inferred to be platykurtic.

The fact that the distribution of the returns is not normal justify the usage of alternative methods than Black & Scholes, which don’t assume initially any distribution.

Before the methods are analyzed we will make a formulation of the evolution of stochastic asset prices based on a scenario tree.

4.3 Formulation of stochastic asset prices based on a scenario tree

A major issue in any application of multistage stochastic programming is the representation of the underlying random data process. In this section we will infer the structure of a scenario tree that models uncertain (stochastic) prices. The dynamic evolution of an underlying asset will be represented in form of a scenario tree as shown in the figure below. The root node is addressed to the current state at time

$t = 0$ from which a number of branches originate until the next period. Every one of the branches leads to an immediate successor node that expresses the outcome of the prices of the underlying assets that are followed jointly at the next decision stage, with a certain associated probability. The same rule is applied to every node of the scenario tree. For each node the group of immediate successors model the conditional discrete distribution of the underlying asset prices for the next time period, based on their mutual predecessor node. Decisions are being made on every node for the time period $t = 0,1,..T$.

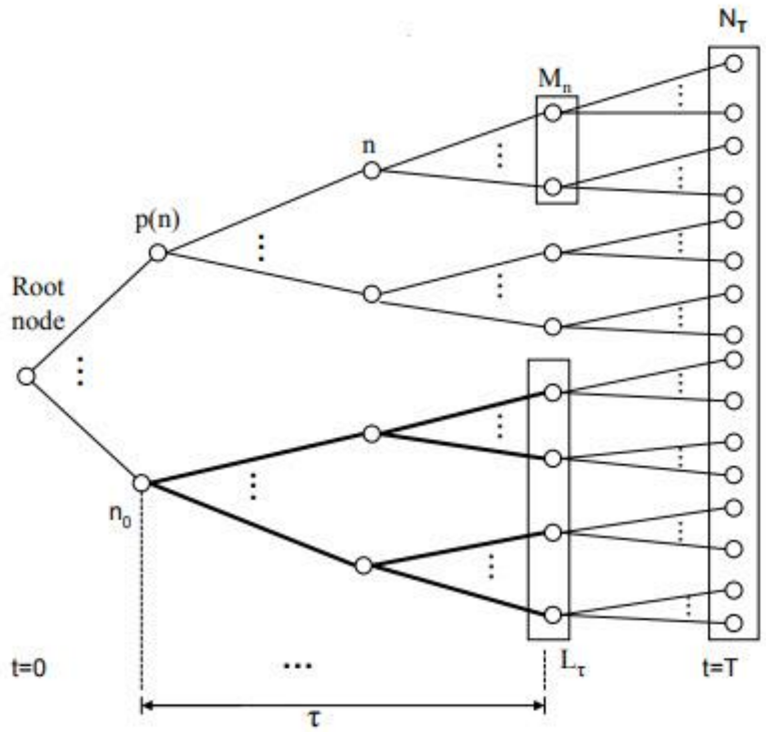

Figure 4.3.1: Illustration of a scenario tree

Let define the following notation:

N are the set of nodes of the scenario tree;

$n \in N$ represent a typical node of the scenario tree (n=0 indicates the root node at t=0);

$N_T \subset N$ is the set of terminal nodes at the last period T that uniquely recognize the complete set of scenarios over the planning horizon. For every period, a scenario is addressed to a unique path from the initial root to a distinct node at the equivalent period;

$p(n) \in N$ Is the unique predecessor node of node $n \in N$;

$M_n \subset N$ is the set of the immediate successor nodes of node n. This set of nodes creates the discrete outcomes of the stochastic asset prices at the representative time period, conditional on the predecessor node n;

S_i^n is the price of the asset i, where $i = 1,2, \dots I$, at node $n \in N$

Commonly the scenario trees include more than two nodes and it is not as depicted in the figure above. The tree indicates the dynamics evolution of stochastic process for the underlying asset prices. There are many ways in order to construct scenarios generations processes (continuous stochastic processes, econometric models, subjective forecasts etc.). Dupacova et al. [2000] studied scenario generation methods and deduced that there is no unique way for a scenario generation method. Depending on the underlying asset the most appropriate method should be applied. The creation of this scenario tree is done by using the method developed by Hoyland et al. [2003]. This method produces scenarios so that the first four marginal moments (mean, variance, skewness and kurtosis) are equivalent and in accordance with the moments of the empirical historical distribution. Initially we choose both the number of the scenarios that we want to set up and the number of the immediate stages, where in this specific case we assume that the time is one month. The scenarios are being produced on every immediate stage until the whole tree is constructed.

4.4 Method 1: Black and Scholes Model augmented with Skewness and Kurtosis

The first method is an expansion of the Black and Scholes model with two additional terms accounting for effects of higher moments; skewness and kurtosis. The theoretical part of this approach was first displayed by Jarrow and Rudd [1982] and further investigated by Corrado and Su [1996]. Before we begin to analyze this method, we will make the following notations:

τ is the term of the European Option;

S_0 is the Spot price of the underlying asset at the time of the option valuation;

S_τ is the random price of the underlying asset at the options maturity date;

r_f is the risk-free interest rate;

K is the strike price of the option.

So, the log-return of the underlying asset during the holding period is:

$$r_\tau = \ln S_\tau - \ln S_0 = \ln S_\tau / S_0 \quad \textbf{(4.4.1)}$$

Hence,

$$S_\tau = S_0 e^{r_\tau} \quad \textbf{(4.4.2)}$$

The conditional distribution of S_τ depends on that of the log-return r_τ.

In the risk-neutral environment, the price, C_0, of a call option is calculated as follows:

$$C_0 = (e^{-r_f})^t \, E[(S_\tau - K)^+] = (e^{-r_f})^t \int_{\ln K/S_0}^{\infty} (S_0 e^X - K) \, f(x) \, dx, \quad \textbf{(4.4.3)}$$

Where $f(\cdot)$ is the conditional density of r_τ .

Corrado and Su [1996] used a Gram-Charlier series expansion to approach the empirical probability density of the asset log-returns in order to construct an option pricing formula.

The series expansion equates the underlying distribution with an alternate distribution, this of the log normal. The coefficients of the series expansion are functions of the moments of both the original and the implied distribution. The first two terms of the series expansion refer to the Black and Scholes pricing formula. Further terms adjusted in the series expansion take into account the repercussions of higher moments of the asset returns distribution on the option price. Corrado and Su depicted the pricing formula by applying a four-term Gram-Charlier series expansion in order to take into consideration the skewness and kurtosis of the asset returns.

Applying a four-term Gram-Charlier series expansion for the conditional density $f(\cdot)$, Corrado and Su solved and came across with the following formula for the price of a call option:

$$C_0 = C_{BS} + \gamma_1 Q_3 + (\gamma_2 - 3)Q_4 \quad \textbf{(4.4.4)}$$

Where C_{BS} is the price of the call option obtained by the Black & Scholes model, Q_3 and Q_4 illustrate the adjustments for nonzero skewness and kurtosis respectively and γ_1, γ_2 are the Fisher parameters for skewness and kurtosis respectively.

The above variables are being calculate below:

$$C_{BS} = S_0 N(d) - K e^{-r_f t} N(d - \sigma_\tau), \quad \textbf{(4.4.5)}$$

$$Q_3 = \frac{1}{3} S_0 \sigma_\tau [(2\sigma_\tau - d)\varphi(d) + \sigma_\tau^2 N(d)], \quad \textbf{(4.4.6)}$$

$$Q_4 = \frac{1}{4} S_0 \sigma_\tau [(d^2 - 3d\sigma_\tau + 3\sigma_\tau^2 - 1)\varphi(d) + \sigma_\tau^3 N(d)], \quad \textbf{(4.4.7)}$$

$$d = \frac{\ln(S_0/K) + r_f \tau + \sigma_\tau^2/2}{\sigma_\tau} \quad \textbf{(4.4.8)}$$

Where $\varphi(\cdot)$ is the standard normal density, $N(\cdot)$ is the cumulative normal distribution and σ_τ is the standard deviation of the underlying asset calculated by historical data. Parameters γ_1, γ_2 have been calculated likewise by historical data.

Hence to derive the price P_0 of a put option with the same strike price K, we will use the Put-Call parity (see Stoll [1969] and Merton [1973]):

$$P_0 = C_0 + K e^{-r_f \tau} - S_0$$

The above method we described enhances the price resulted from the Black & Scholes formula with two adjustments added which take into account the effect of skewness and kurtosis of the distribution of the asset returns. The implementation of this method is based on the computation of the effect of higher moments. From the begin of the node n_0 of the scenario tree till the last nodes n where $n \in L_T$, the first four higher moments of the discrete distribution of the underlying asset are being estimated. It should be noticed that this method is easier to implement in contrast to the other method that will be analyzed in next section because it does not involve any solutions from optimization programs or non-linear equations; it is straightforward to implement and require just simple calculations.

4.5 Method 2: Determining risk-neutral probabilities

The risk-neutral probability measure results when we assume that the future expected value of all financial assets is equivalent to the future payoff discounted with the risk-free interest rate. This measure is used in pricing of options due to the fact that all the underlying assets have the same expected return (risk-free interest rate), whatever risk is addressed to the underlying security. The crucial input to price the option concerns the distribution of the underlying assets price at the option maturity, conditional on node n_0. The discrete support of the distribution of the underlying assets price is:

$$\Omega = \{\omega^n = S^n : n \in L_T\}$$

The equivalent conditional probabilities of the physical distribution are:

$$P = \{p_n = \pi_n / \pi_{n0} : n \in L_\tau\}$$

Where π_n is the unconditional probability mass for node n.

It should be noted that every citation to the underlying's random price at the option maturity, the distribution of the price, its discrete support and the probabilities are all conditional with respect to the initial underlying's price being at node n_0 on the option valuations date.

Before we continue our analysis, we will use the following notations:

τ is the term of the European option priced at state n_0;

S_0 is the price of the underlying asset at node n_0;

S^n is the price of the underlying asset at node $n \in L_\tau$ of the subtree. These prices in conjunction with their probabilities over all leaf nodes L_τ, represent the discrete and conditional distribution of the asset's prices at the option's maturity date;

$\tilde{S}_\tau$ is the random price the underlying asset will have at the expiration date of the option, depending on the price S_0 at the valuation day of the option;

r_f is the suitable riskless rate conditional to the lifetime of the option;

K is the strike price of the option;

P is the physical probability measure for the conditional discrete distribution of the underlying's price at the expiration of the option;

$\tilde{P}$ is an equivalent risk-neutral probability measure for the same discrete distribution.

A basic theorem in pricing of options suggests that there are no arbitrage opportunities. Harrison and Kreps [1979] implied that in the absence of arbitrage opportunities there is a risk-neutral probability measure under which the price of any underlying asset is martingale. The martingale condition, given the risk-neutral probability measure $\tilde{P}$, requires that for any time interval the return of the asset is equal to the risk-free interest rate during the same time interval and this is shown below:

$$E_{\tilde{P}}\left[e^{-r_f \tau}\, \tilde{S}_\tau | S_0\right] = S_0 \qquad \textbf{(4.5.1)}$$

Taking into account the above equation, we will derive the following system of linear equations and inequalities that the martingale measure must satisfy:

$$\sum_{n\in L_\tau} \tilde{p}_n S_i^n = S_0 (e^{r_f})^\tau, \quad i = 1,2, \dots I, \qquad \textbf{(4.5.2)}$$

$$\sum_{n\in L_\tau} \tilde{p}_n = 1 \qquad \textbf{(4.5.3)}$$

$$\tilde{p}_n > 0 \ \forall \ n \in L_\tau \qquad \textbf{(4.5.4)}$$

The equality (5.2.3) make certain that $\tilde{P}$ is an appropriate measure and the inequality (5.2.4) ensure us that there is consistent equivalence between the physical, P, and the risk-neutral $\tilde{P}$ probability measure. The essential difference on those two measures is that the physical measure entails underlying assets which have greater risk accompanied with greater future returns. The above martingale conditions (5.2.2-5.2.4) do not give us a unique solution, instead they produce a huge number of risk-neutral probability measures. In order this system to be completed and a unique martingale measure to be originated, we need to enrich the above system with an additional condition. Thus, we can make developments concerning market equilibrium conditions to option pricing so to augment the martingale conditions and deduce the implied risk-neutral probabilities for the discrete price outcomes.

In equilibrium, the physical probability measure is connected with the risk-neutral probability measure through a stochastic discount factor; the pricing kernel ξ:

$$S_0 = (e^{-r_f})^\tau \ E_P[\xi \tilde{S}_\tau], \qquad \textbf{(4.5.5)}$$

Where $E_P[\cdot]$ express the expectation operator under the physical probability measure P.

Bakshi et al. [2003] constructed a method in order to link the physical and risk-neutral probability measure through a pricing kernel. Following their method (see Bakshi et al. [2003]) it can be derived that:

$$S_0 = (e^{-r_f})^\tau \ E_P[\xi \tilde{S}_\tau] = (e^{-r_f})^\tau \int_\Omega \tilde{S}_\tau(\omega) \, \xi(\omega) dP(\omega) = (e^{-r_f})^\tau \int_\Omega \tilde{S}_\tau(\omega) d\tilde{P}(\omega) \,, \qquad \textbf{(4.5.6)}$$

Where

$$\xi(\omega) dP(\omega) = \ d\tilde{P}(\omega) \Rightarrow \frac{d\tilde{P}(\omega)}{dP(\omega)} = \xi(\omega) \qquad \textbf{(4.5.7)}$$

The result produced above can be altered in the case the distribution of the asset price is a finite discrete set as:

$$\xi(\omega^n) = \frac{\tilde{P}(\omega^n)}{P(\omega^n)} \Rightarrow \tilde{P}(\omega^n) = \xi(\omega^n) P(\omega^n) \quad \forall \ \omega^n \ \in \ \Omega.$$

Bakshi et al. [2003] determined a transformation between the physical and the risk-neutral probabilities in the special case of the existence of a discrete support and it takes the following form:

$$\tilde{p}_n = \frac{E_P[\xi|S^n]p_n}{\sum_{n\in L_\tau} E_P[\xi|S^n]p_n}\ ,\ \ n \in L_\tau\ , \tag{4.5.8}$$

Where ξ is a general change of measure pricing kernel.

Under the hypothesis of the dynamic power utility function, the stochastic discount factor can be written as follows:

$$E_P[\xi|S^n] = (S^n)^{-\gamma} = e^{-\gamma(\ln S^n)}\ \ , n \in\ L_\tau \tag{4.5.9}$$

Where γ we symbolize the coefficient of relative risk aversion.

Substituting the (5.2.9) in (5.2.8) and divide both parts by $S_0^{-\gamma}$, we end up with the following relationship:

$$\tilde{p}_n = \frac{e^{-\gamma(\ln S^n/S_0)}p_n}{\sum_{n\epsilon L_\tau} e^{-\gamma(\ln S^n/S_0)}p_n} = \frac{e^{-\gamma R^n}p_n}{\sum_{n\epsilon L_\tau} e^{-\gamma R^n}p_n}\ \ ,\ n\epsilon L_\tau, \tag{4.5.10}$$

Where $R^n = \ln S^n/S_0$ is the return of the underlying asset at leaf node n whereas $n\epsilon L_\tau$, conditional to the initial price S_0.

The modification in (5.2.10) has the intention to develop a risk-neutral probability measure under the equilibrium principles. It can be seen that this alternation is being affected by the parameter of the risk aversion γ. Topaloglou et al. [2008] calculate this parameter by obtaining real option prices and used an unconstrained quadratic program, where they minimized the squared distance between the observed prices and the theoretical produced by the method above over a set of options with the same maturity and different strike prices. Topaloglou et al. [2008] computed the following equation:

$$\gamma = argmin_{\gamma'} \sum_{i=1}^{m} \left(\frac{CP_i(\gamma') - MP_i}{MP_i}\right)^2 \tag{4.5.11}$$

From the above equation the parameter γ is approximately 2[8].

Hence, the risk-neutral probabilities of the price at state $n\epsilon L_\tau$ at the options maturity date are given below from the quadratic program:

$$minimize_{\tilde{p}_n \epsilon L_\tau} \quad \sum_{n\epsilon L_\tau} (\tilde{p}_n - \hat{p}_n)^2,$$

[8] Topaloglou et al. [2008] tried variations of the estimated parameter γ taking prices from 1 to 3 and deduced that these alternations do not affect the price of options on a significant way. In our tests we will take $\gamma = 2$ for all the options

s.t. $\sum_{n\in L_\tau}\tilde{p}_n S^n = S_0(e^{r_f})^\tau,$ **(4.5.12)**

$\sum_{n\in L_\tau}\tilde{p}_n = 1,$

$\tilde{p}_n > 0 \quad \forall\; n \in L_\tau.$

where parameter $\tilde{p}_n$ is computed by the modification on (5.2.10).

The risk-neutral measure $\hat{p}_n$ produced by the solution of (5.2.10) are those which are the closest (in a Euclidean sense) to the risk-neutral probabilities as they are given from (5.2.12) and simultaneously satisfy the martingale constraint as it is depicted on the solutions (5.2.2-5.2.4).

Since we have determined the risk-neutral probabilities $\tilde{p}_n$ for the discrete outcomes,S^n,

$n \in L_\tau$, of the underlying asset's price, pricing the options is given by simple calculations.

The price of a European call option at node n, with strike price K, is the expected value of its payoff multiplied by the risk-neutral measure and discounted by the risk-free interest rate:

$$c_0(S_0, K) = (e^{-r_f})^\tau \sum_{n\in L_\tau}\tilde{p}_n\,[\,\max(S^n - K, 0)\,]. \qquad \textbf{(4.5.13)}$$

The price of a European put option is calculated identically as:

$$p_0(S_0, K) = (e^{-r_f})^\tau \sum_{n\in L_\tau}\tilde{p}_n\,[\,max(K - S^n, 0)\,]. \qquad \textbf{(4.5.14)}$$

This method is quite different from the first that was described, and its advantage is found on its generality. It can be applied for any discrete representation of the stochastic path the prices might follow and consequently values the options by just taking into consideration these discrete distributions.

4.6 Empirical results

We test from empirical results the performance of the Black & Scholes model compared to the other two methods we described above. We assume the Black & Scholes model to be our basis for the comparisons that follow. The most essential parameter of the Black & Scholes is the volatility of the underlying return. It can be calculated either from historical observations of the underlying assets prices or by taking the real prices of options into the formula and while keeping all other variables constant, solve for volatility (ISD). In this thesis we use only the first method.

For our empirical test we have used European call and put options addressed to the S&P 500 stock index with time to maturity 40 days. All the data were acquired from Thomson Reuters Datastream database. We are pricing the options at two different dates: 10 July 2007 and 11 January 2010. Four tables will be presented from which the two of them display the results from call options and the remaining two the results from put options.

Table 4.6.1: Observed vs. estimated prices of European call options on S&P 500 index (10 July 2007)

Strike Price	Market Prices	Black & Scholes	Error BS	Method 1	Error M1	Method 2	Error M2
1390	129,70	121,68	-6,19%	121,46	-6,36%	138,52	6,80%
1410	111,50	102,83	-7,78%	103,08	-7,55%	125,92	12,93%
1425	98,30	89,24	-9,22%	90,05	-8,39%	116,58	18,60%
1450	77,00	68,09	-11,57%	70,04	-9,03%	101,03	31,21%
1490	45,90	39,80	-13,29%	42,77	-6,81%	76,20	66,01%
1510	33,00	28,81	-12,71%	31,37	-4,93%	63,77	93,24%
1560	9,70	10,73	10,61%	10,22	5,41%	33,15	241,75%
1565	7,70	9,58	24,38%	8,75	13,58%	30,10	290,91%
1570	6,50	8,52	31,15%	7,39	13,63%	27,04	316,00%
1575	5,50	7,57	37,58%	6,14	11,70%	23,99	336,18%
1585	3,40	5,91	73,90%	4,00	17,65%	17,89	426,18%
Average Error			**23,22%**		**9,55%**		**167,26%**

Table 4.6.2: Observed vs. estimated prices of European put options on S&P 500 index (10 July 2007)

Strike Price	Market Prices	Black & Scholes	Error BS	Method 1	Error M1	Method 2	Error M2
1390	5,60	1,08	-80,68%	0,86	-84,60%	11,99	114,11%
1410	7,40	2,23	-69,92%	2,48	-66,45%	19,30	160,81%
1425	9,00	3,63	-59,67%	4,45	-50,58%	24,89	176,56%
1450	12,40	7,47	-39,72%	9,43	-23,97%	34,22	175,97%
1490	20,80	19,17	-7,82%	22,15	6,47%	49,21	136,59%
1510	27,40	28,17	2,81%	30,74	12,18%	56,80	107,30%
1560	53,40	60,08	12,50%	59,57	11,56%	75,84	42,02%
1565	56,70	63,92	12,74%	63,09	11,27%	77,76	37,14%
1570	60,30	67,87	12,55%	66,73	10,66%	79,68	32,14%
1575	64,10	71,91	12,18%	70,49	9,96%	81,61	27,32%
1585	72,10	80,25	11,31%	78,34	8,65%	85,46	18,53%
Average Error			**29,26%**		**26,94%**		**93,50%**

Table 4.6.3: Observed vs. estimated prices of European call options on S&P 500 index (11 January 2010)

Strike Price	Market Prices	Black & Scholes	Error BS	Method 1	Error M1	Method 2	Error M2
1070	79,80	78,63	-1,46%	78,81	-1,24%	93,26	16,87%
1085	66,70	65,10	-2,39%	65,86	-1,26%	84,46	26,63%
1095	58,60	56,57	-3,47%	57,78	-1,40%	78,60	34,13%
1105	50,60	48,52	-4,11%	50,18	-0,83%	72,74	43,75%
1130	32,50	31,04	-4,51%	33,29	2,44%	58,13	78,86%
1145	23,70	22,63	-4,51%	24,63	3,94%	49,39	108,40%
1180	8,90	9,28	4,29%	9,20	3,41%	29,14	227,42%
1185	7,50	8,02	6,96%	7,60	1,37%	26,26	250,13%
1190	6,30	6,90	9,52%	6,16	-2,23%	23,38	271,11%
1195	5,20	5,90	13,47%	4,87	-6,33%	20,51	294,42%
1200	4,40	5,03	14,32%	3,73	-15,23%	18,49	320,23%
Average Error			**4,95%**		**2,24%**		**152,00%**

Table 4.6.4: Observed vs. estimated prices of European put options on S&P 500 index (11 January 2010)

Strike Price	Market Prices	Black & Scholes	Error BS	Method 1	Error M1	Method 2	Error M2
1070	5,80	1,65	-71,59%	1,82	-68,55%	13,90	139,66%
1085	7,80	3,12	-60,03%	3,87	-50,38%	20,07	157,31%
1095	9,30	4,58	-50,74%	5,79	-37,72%	24,19	160,11%
1105	11,20	6,53	-41,68%	8,19	-26,86%	28,31	152,77%
1130	18,20	14,05	-22,81%	16,31	-10,40%	38,64	112,31%
1145	24,00	20,64	-13,98%	22,65	-5,64%	44,87	86,96%
1180	43,80	42,29	-3,44%	42,22	-3,62%	59,54	35,94%
1185	47,60	46,03	-3,29%	45,62	-4,17%	61,65	29,52%
1190	51,40	49,91	-2,89%	49,17	-4,33%	63,76	24,05%
1195	55,30	53,92	-2,50%	52,88	-4,38%	65,88	19,13%
1200	59,40	58,04	-2,29%	56,74	-4,48%	67,26	13,23%
Average Error			**27,86%**		**21,33%**		**84,63%**

Tables[9] 4.6.1, 4.6.2, 4.6.3, 4.6.4 present the results of the Black & Scholes model and the proposed methods compared to the real market prices of options both on call and put options on 10 July 2007 and 11 January 2010 respectively. We estimated a number of 40-days call and put options with different strike prices. More precisely, we calculated for every day five

[9] In the above tables it should be inferred that Average Errors have been calculated by taking the absolute values of the estimated Errors

options in the money, five options out of the money and one at the money. Comparing the Black & Scholes results with that of method 1, we can observe that the proposed method performed better in almost all of the cases. More specific, this method produced options prices which are closer to the real observed prices and especially in deep out of the money options compared with the Black & Scholes. Method 2 generally produces better results compared to Black & Scholes but here it didn't yield the expected results. This could be due to three reasons. The first accounts to the scenario generation; we create a discrete distribution with 167 scenarios of index prices matching the first four moments of the index's returns. The number of the scenarios produced account for the inconsistent results and if more scenarios produced, better results would be accomplished. The second reason is addressed to the probabilities of the scenarios; in our scenario we produced equivalent probabilities for each price scenario, and this could be the case for the improperly results, because the expectations regarding the behavior of the market were not been incorporated. For instance, if the market is followed by a tendency for upward movements, the probability for the price to exceed the current spot price is essentially bigger than the probability for the price to go below the spot price, and in no case equivalent as we assumed here. The third reason is linked to the fact that we used historical data and assumed that the returns from the past years are those that will occur in the future and this is not so realistic scenario because the price of an underlying asset is affected by many factors and in order to construct such scenarios, many characteristics of the markets should be examined explicitly.

It can be deduced that the Black & Scholes model tend to misprice deep in the money options and overprice deep out of the money options as shown in the tables above.

Chapter 5

Conclusions

In the present dissertation an overview literature of the Black & Scholes model with the advancements made by the years were displayed. The model has various deficiencies, with the most important being the assumption of the constant volatility. Our main focus was the presentation of two alternative methods for pricing options as well as the advantages that come up with these methods compared to the widespread Black & Scholes model.

We implemented and validate two pricing methods where the underlying asset follows a discrete distribution. The discrete distribution of the underlying asset where in this case is the S&P 500 index was constructed by a scenario tree. The proposed methods take into account the effects of higher moments into the distribution of the returns of the underlying asset.

We analyze the viability of those methods by making empirical tests regarding their performance. We compare the price of both call and put options addressed to S&P 500 index with the two proposed methods between them as also with the prices resulting from the Black & Scholes. We showed that the effects of higher moments play a significant role on pricing of options and can partially explain the differences that occur between the observed market prices and the Black & Scholes model, which takes into account only the first two moments.

The first method yields option prices which are closer to the observed prices in contrast to the prices produced by the Black & Scholes and the second method did not produce the expected results due to the inefficiencies that displayed. A crucial conclusion that can be derived for the Black & Scholes model is that it misprices deep in the money and overprice deep out of the money options.

Options constitutes appropriate and suitable tools for maintaining risk management due to their asymmetric payoffs. The presence of the category of options market fulfills the financial markets. Options are connected to every underlying asset such as indices, currencies, futures and commodities that offered for trade and so they can be part of portfolios of all investors either for hedging or speculation. Hence, efficiency in pricing of options is imperative and this will have a direct result to increase the confidence of investors and moreover be an impulsion for corporates to hedge.

References

1. Aït-Sahalia, Y. and Lo, A. (1998). Nonparametric Estimation of State-Price Densities Implicit in Financial Asset Prices. The Journal of Finance, 53(2), pp.499-547.
2. Ayres, H. (1963). Risk aversion in Warrant Markets. Dissertation for MSc. Massachusetts Institute of Technology.
3. Bakshi, G., Cao, C. and Chen, Z. (1997). Empirical Performance of Alternative Option Pricing Models. The Journal of Finance, 52(5), p.2003.
4. Bakshi, G., Kapadia, N. and Madan, D. (2003). Stock Return Characteristics, Skew Laws, and the Differential Pricing of Individual Equity Options. Review of Financial Studies, 16(1), pp.101-143.
5. Bates, D. (2003). Empirical option pricing: a retrospection. Journal of Econometrics, 116(1-2), pp.387-404.
6. Black, F. (1989). How we came up with the option formula. The Journal of Portfolio Management, 15(2), pp.4-8.
7. Black, F. and Scholes, M. (1973). The Pricing of Options and Corporate Liabilities. Journal of Political Economy, 81(3), pp.637-654.
8. Blattberg, R. and Gonedes, N. (1974). A Comparison of the Stable and Student Distributions as Statistical Models for Stock Prices. The Journal of Business, 47(2), p.244.
9. Boness, A. (1962). A Theory of measurement of stock option value. Ph. D. Dissertation. University of Chicago.
10. Boness, A. (1964). Elements of a Theory of Stock-Option Value. Journal of Political Economy, 72(2), pp.163-175.
11. Boyle, P. (2000). Worldwide Asset and Liability Modeling. Edited by William T. Ziemba and John M. Mulvey (Cambridge University Press, 1998). British Actuarial Journal, 6(2), pp.433-434.
12. Bradley, S. and Crane, D. (1972). A Dynamic Model for Bond Portfolio Management. Management Science, 19(2), pp.139-151.
13. Britten-Jones, M. and Neuberger, A. (2000). Option Prices, Implied Price Processes, and Stochastic Volatility. The Journal of Finance, 55(2), pp.839-866.
14. Chiras, D. and Manaster, S. (1978). The information content of option prices and a test of market efficiency. Journal of Financial Economics, 6(2-3), pp.213-234.
15. Corrado, C. and Su, T. (1996). Skewness and kurtosis in S&P 500 index returns implied by option prices. Journal of Financial Research, 19(2), pp.175-192.
16. Cox, J. and Ross, S. (1976). The valuation of options for alternative stochastic processes. Journal of Financial Economics, 3(1-2), pp.145-166.
17. Cox, J. and Rubinstein, M. (1985). Options Markets. United States of America: Prentice Hall, Englewood Cliffs, New Jersey 07632.
18. Cox, J., Ross, S. and Rubinstein, M. (1979). Option pricing: A simplified approach. Journal of Financial Economics, 7(3), pp.229-263.
19. Davis, M., Panas, V. and Zariphopoulou, T. (1993). European Option Pricing with Transaction Costs. SIAM Journal on Control and Optimization, 31(2), pp.1299-1304.
20. Galai, D. (1978). On the Boness and Black-Scholes Models for Valuation of Call Options. The Journal of Financial and Quantitative Analysis, 13(1), p.15.
21. Geske, R. (1979). The valuation of compound options. Journal of Financial Economics, 7(1), pp.63-81.
22. Harrison, J. and Kreps, D. (1979). Martingales and arbitrage in multiperiod securities markets. Journal of Economic Theory, 20(3), pp.381-408.
23. Helex. (2019). Αγορά Παραγώγων-athexgroup.gr. [online] Available at: http://www.helex.gr/el/web/guest/markets-derivatives [Accessed 13 Sep. 2019].
24. Hull, J. (2012). Options, futures, and other derivatives. 8th ed. Boston (Mass): Pearson.
25. Hull, J. (2015). Options, futures, and other derivatives. 9th ed. Boston: Pearson.
26. Hull, J. and White, A. (1987). The Pricing of Options on Assets with Stochastic Volatilities. The Journal of Finance, 42(2), p.281.
27. Investopedia. (2019). Backtesting Definition. [online] Available at: https://www.investopedia.com/terms/b/backtesting.asp [Accessed 5 Sep. 2019].
28. Investopedia. (2019). CBOE Volatility Index (VIX) Definition. [online] Available at: https://www.investopedia.com/terms/v/vix.asp [Accessed 15 Sep. 2019].

29. Investopedia. (2019). Derivative—How the Ultimate Hedge Play Works. [online] Available at: https://www.investopedia.com/terms/d/derivative.asp [Accessed 7 Sep. 2019].
30. Investopedia. (2019). The History Of Options Contracts. [online] Available at: https://www.investopedia.com/articles/optioninvestor/10/history-options-futures.asp [Accessed 7 Sep. 2019].
31. J. Madegowda, P. (2019). Forecasting Volatility in Options Trading - Nexus between Historical Volatility and Implied Volatility. [online] Papers.ssrn.com. Available at: https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2881154 [Accessed 9 Sep. 2019].
32. Jackwerth, J. and Rubinstein, M. (1996). Recovering Probability Distributions from Option Prices. *The Journal of Finance*, 51(5), p.1611.
33. Jacquier, E. and Jarrow, R. (2000). Bayesian analysis of contingent claim model error. Journal of Econometrics, 94(1-2), pp.145-180.
34. Janková, Z. (2018). Drawbacks and Limitations of Black-Scholes Model for Options Pricing. Journal of Financial Studies and Research, 2018, pp.1-7.
35. Jarrow, R. and Madan, D. (2000). Arbitrage, martingales, and private monetary value. The Journal of Risk, 3(1), pp.73-90.
36. Jarrow, R. and Rudd, A. (1982). Approximate option valuation for arbitrary stochastic processes. Journal of Financial Economics, 10(3), pp.347-369.
37. Kouwenberg, R. (2001). Scenario generation and stochastic programming models for asset liability management. European Journal of Operational Research, 134(2), pp.279-292.
38. Kraus, A. and Litzenberger, R. (1976). Skewness Preference and the Valuation of Risk Assets. The Journal of Finance, 31(4), p.1085.
39. Latane, H. and Rendleman, R. (1976). Standard Deviations of Stock Price Ratios Implied in Option Prices. The Journal of Finance, 31(2), p.369.
40. Litzenberger, R. (1991). William F. Sharpe's Contributions to Financial Economics. The Scandinavian Journal of Economics, 93(1), p.37.
41. Macbeth, J. and Merville, L. (1979). An Empirical Examination of the Black-Scholes Call Option Pricing Model. The Journal of Finance, 34(5), p.1173.
42. Madan, D., Carr, P. and Chang, E. (1998). The Variance Gamma Process and Option Pricing. Review of Finance, 2(1), pp.79-105.
43. Merton, R. (1973). The Relationship between Put and Call Option Prices: Comment. The Journal of Finance, 28(1), p.183.
44. Merton, R. (1973). Theory of Rational Option Pricing. The Bell Journal of Economics and Management Science, 4(1), p.141.
45. Merton, R. (1976). Option pricing when underlying stock returns are discontinuous. Journal of Financial Economics, 3(1-2), pp.125-144.
46. Merton, R. (1977). On the pricing of contingent claims and the Modigliani-Miller theorem. Journal of Financial Economics, 5(2), pp.241-249.
47. Mulvey, J. and Vladimirou, H. (1992). Stochastic Network Programming for Financial Planning Problems. Management Science, 38(11), pp.1642-1664.
48. Mylonas, N. (2005). Markets and Derivatives Products. Athens University of Economic and Business: Dardanos.
49. N., Shabarisha and J, Madegowda. Forecasting Volatility in Options Trading - Nexus between Historical Volatility and Implied Volatility (2016). International Journal of Research in Finance and Marketing, Vol. 6 pp. 1-14.
50. Rubinstein, M. (1983). Displaced Diffusion Option Pricing. The Journal of Finance, 38(1), p.213.
51. Rubinstein, M. (1985). Nonparametric Tests of Alternative Option Pricing Models Using All Reported Trades and Quotes on the 30 Most Active CBOE Option Classes from August 23, 1976 Through August 31, 1978. The Journal of Finance, 40(2), p.455.
52. Rubinstein, M. (1994). Implied Binomial Trees. The Journal of Finance, 49(3), p.771.
53. Samuelson, P. (1965). Rational Theory of Option Pricing. Industrial Management Review, 39.
54. Schaefer, S. (1998). Robert Merton, Myron Scholes and the Development of Derivative Pricing. Scandinavian Journal of Economics, 100(2), pp.425-445.
55. Scott, L. (1987). Option Pricing when the Variance Changes Randomly: Theory, Estimation, and an Application. The Journal of Financial and Quantitative Analysis, 22(4), p.419.

56. Stoll, H. (1969). The Relationship between Put and Call Option Prices. The Journal of Finance, 24(5), p.801.
57. Topaloglou, N., Vladimirou, H. and Zenios, S. (2008). Pricing options on scenario trees. Journal of Banking & Finance, 32(2), pp.283-298.
58. Yalincak, H. (2012). Criticism of the Black-Scholes Model: But Why is it Still Used? (The Answer is Simpler than the Formula). SSRN Electronic Journal
59. Zachary, S. (1999). Modelling Extremal Events. By P. Embrechts, C. Kluppelberg and T. Mikosch (Springer-Verlag, 1997). British Actuarial Journal, 5(2), pp.465-465.
60. Zenios, S. and Ziemba, W. (2008). Handbook of asset and liability management. Amsterdam: North Holland.
61. Ziemba, W. T. and R. G. Vickson (Eds.) (1975). Stochastic Optimization Models in Finance, San Diego: Academic Press.
62. Ευρετήριο Οικονομικών Όρων. (2019). Παράγωγα Χρηματοοικονομικά Προϊόντα (Derivatives) - ορισμός | Ευρετήριο Οικονομικών Όρων. [online] Available at: https://www.euretirio.com/paragoga-xrimatooikonomika-proionta/ [Accessed 8 Sep. 2019].