Cooperative oligopoly games with boundedly rational firms

Paraskevas V. Lekeas* Giorgos Stamatopoulos†

October 29, 2012

Abstract

We analyze a cooperative Cournot game with boundedly rational agents. Due to cognitive constraints, the members of a coalition cannot accurately predict the coalitional structure of the non-members. Thus, they compute their value following simple heuristics. In particular, they assign various non-equilibrium probability distributions over the outsiders' set of partitions. We construct the characteristic function of a coalition and analyze the core of the corresponding game. We show that the core is non-empty provided the number of firms in the market is sufficiently large. Moreover, we show that if two distributions are related via first-order dominance, then the core of the game under the dominated distribution is a subset of the core under the dominant one. In particular, this implies that our core is contained in the γ core.

Keywords: Cooperative game; externalities; Cournot market; core; bounded rationality

JEL Classification: C71, L2

1 Introduction

The issue of cooperation among firms in oligopolistic markets is a wide-spread phenomenon and constantly attracts the interest of economists. By colluding, firms can restrict output and raise prices in the market, thus extracting

*Department of Applied Mathematics, University of Crete, 71409 Heraklion, Crete, Greece; email: plekeas@gmail.com

†Department of Economics, University of Crete, 74100 Rethymno, Crete, Greece; email: gstamato@econ.soc.uoc.gr

a higher surplus from the consumers. From a methodological point of view, economists analyze this market phenomenon using either non-cooperative games (when agreements among firms are non-enforceable by an outside entity) or cooperative games (whenever the signing of enforceable agreements is possible).[1] Under the last approach, the focus usually lies on the core of an appropriately defined cooperative game. The core consists of all allocations of total market profits that cannot be blocked by any coalition of firms. Non-empty core means that cooperation among all firms in the market is a priori feasible.

When a coalition contemplates breaking-off from the set of the other firms, it has to calculate its payoff. In a market environment such a calculation is not a trivial task, as the coalition's worth depends on how the non-members act. It particular, it depends on the partition (i.e., the coalition structure) that the outsiders will form. This calls for the formation of beliefs about non-members' behavior.

Different conjectures about the reaction of the outsiders lead to different coalitional worths and thus to different notions of core. The α and β cores (Aumann 1959) are based on min-max behavior on behalf of the non-members; the γ core (Chander & Tulkens 1997) is based on the assumption that outsiders play individual best replies to the deviant coalition; the δ core scenario (Hart and Kurz 1983) assumes that outsiders form a single coalition. Various authors applied these core notions to the study of Cournot markets. Rajan (1989) used the concept of γ core and showed that it is non-empty for a market with 4 firms. A more general result for any number of firms is provided by Chander (2009). Currarini & Marini (1998) built a refinement of the γ core by assuming that the deviant coalition acts as a Stackelberg leader in the product market. Zhao (1999) showed that the α and β cores of oligopolistic markets are non-empty.

The seminal work of Ray & Vohra (1999) goes one step further, as the worth of a coalition is deduced via arguments that satisfy a consistency criterion: a deviant coalition takes into account the fact that after its deviation, other deviations might follow, with the newly deviant coalitions thinking in a similar forward way. For games where binding agreements are feasible, Huang & Sjostrom (1998, 2003) and Koczy (2007) developed the recursive core. The recursive core is constructed under the assumption that the members of a coalition compute their value by looking recursively on the cores of the sub-games played among the outsiders.

Predicting the optimal coalitional formation in a game with many players is computationally cumbersome. Sandholm et.al (1999) showed that for

[1] For a discussion on legal cartels, we refer the reader to Dick (1996), Motta (2007) or Haucap et.al (2010).

an n-player game the number of different coalition structures is $O(n^n)$ and $\omega(n^{\frac{n}{2}})$. Hence, computing the coalition structure that the outsiders form is a particularly difficult task (at least, for games with a large number of players). As a matter of fact, the problem of finding the coalition structure that maximizes the sum of all players' payoffs is NP-hard (Sandholm et.al 1999). Even finding sub-optimal solutions requires the search of an exponential number of cases.

The last considerations give the motivation of the current paper. We analyze an n-firm cooperative Cournot oligopoly assuming that no group of firms has the cognitive ability to accurately deduce the coalition structure that the non-members will form. As a result of this constraint, the members of a coalition cannot compute their value with precision. Instead, they compute it by following simple procedures or heuristics.

Clearly, the number of different heuristics than one can adopt is very large. Computer scientists, for example, model similar situations via search algorithms that give solutions within certain bounds from the optimal coalition structure (Sandholm et.al 1999, Dang & Jennings 2004). On the other hand, the economists' toolbox of heuristics includes models with players of various degrees of cognitive abilities (Stahl & Wilson 1994, Camerer 2003, Camerer et.al 2004, Haruvy & Stahl 2007), models with probabilistic choice rules (McKelvey & Palfrey 1995, Chen et.al 1997, Anderson et.al 2002), to name only a few.

In our paper, the heuristics are based on the assignment of non-equilibrium probability distributions over the set of coalition structures that the opponents can form. I.e, when contemplating a deviation from the grand coalition, the members of a coalition make the simplifying assumption that the reactions of the outsiders follow various plausible -but not necessarily optimal-probability distributions.

Our benchmark case assumes that the probability of a coalition structure is proportional to the profitability that the structure induces for the outsiders. Namely, the deviant coalition assumes that it is more likely that its opponents will manage to coordinate and partition according to the more efficient structures. This approach is in the spirit of the quantal response approach of McKelvey & Palfrey (1995) in non-cooperative games, where the probability of choosing a strategy depends on its relative payoff, with the probability being positive even if the strategy is inferior. We derive the characteristic function of a coalition under such a proportional distribution and we examine the core of the corresponding game. We show that if the number of firms in the market is sufficiently large then the core is non-empty. Hence, bounded rationality supports cooperation among all firms in the market.

In the second part of the paper, we extend our analysis by considering more general probability distributions. In particular we consider a pair of distributions that are related by first-order stochastic dominance. We show that the the core of the game under the dominated distribution is contained in the core of the game under the dominant one. In particular, this implies that the core of the game under the proportional distribution is contained in the core of a game under any distribution that first-order dominates it. Thus we indirectly show that our game has non-empty core for a large class of probability distributions.[2]

In particular, the above inclusion holds for the case of γ core. Namely, the core under the proportional distribution is contained in the core of the game constructed under the assumption that outsiders remain separate entities (in our terminology, the γ scenario corresponds to the degenerate distribution that assigns probability one to the singletons partition). Hence our core refines the γ core.

In what follows, we present the basic model in section 2 and in section 3 we present our results. Section 4 provides extensions and section 5 concludes.

2 The model

We consider a market with the set $N = \{1, 2, ..., n\}$ of firms. Firms produce a homogeneous product. They face the inverse demand function $P = max\{a - Q, 0\}$ where P is the market price, $Q = q_1 + q_2 + ... + q_n$ is the market quantity, q_l is the quantity of firm l, $l = 1, 2, ..., n$ and $a > 0$. Firm l produces with the cost function $C(q_l) = cq_l$, where $0 < c < a$.

Let $S \subset N$ denote a coalition with $|S| = s$ members and let $N \setminus S$ denote the complementary set of S, where $|N \setminus S| = n - s$. The worth or value of S is the sum of its members' profits. In order to compute this value, the members of S need to predict how the members of $N \setminus S$ partition themselves into coalitions. The set $N \setminus S$ can be partitioned into disjoint subsets in B_{n-s} ways, where B_{n-s} is Bell's $(n-s)^{th}$ number (Bell 1934). The B_{n-s} different partitions define B_{n-s} different coalition structures that coalition S might face in the market.

What matters for a deviant coalition in a constant returns, symmetric Cournot market is the number of the opponent coalitions and not their synthesis. Consider the example $N = \{1, 2, 3, 4, 5\}$, $S = \{1\}$. Then the partitions $\{\{2, 3\}, \{4, 5\}\}$ and $\{\{2, 3, 4\}, \{5\}\}$ are equivalent for S (and so are all partitions with two members) as both would induce the same profit for S.

[2] A distribution that dominates the proportional (or any other) distribution at first-order gives higher weight to partitions that consist of many (small) coalitions. Its use is justified if S assumes that the formation of large coalitions is more costly compared to the formation of small ones.

More generally, fix some $j \in \{1, 2, ..., n-s\}$. Then all partitions with j coalitions induce the same profit for S, irrespective of how the $n-s$ outside firms are grouped among the j coalitions. We will call these structures *j-similar*. Given this similarity, we simply let $\mathcal{P}_j$ denote a coalition structure with j coalitions (without the need to specify the exact allocation of the $n-s$ firms among the j coalitions).

Denote the number of j-similar coalition structures by $K_{n-s,j}$, where $K_{n-s,j}$ gives the number of ways to partition a set of $n-s$ objects into j groups, or else the Stirling numbers of the second kind:

$$K_{n-s,j} = \frac{1}{j!} \sum_{i=0}^{j} (-1)^i \binom{j}{i} (j-i)^{n-s} \tag{1}$$

The basic assumption that underlies this paper is that S uses simple probabilistic models in order to predict the coalitional behavior of the non-members. In particular, the probability of a partition is proportional to the profitability that the partition induces for the outsiders. This approach is in line with the spirit of the logit quantal response model (McKelvey & Palfrey 1995) in non-cooperative games, where the probability of choosing a strategy depends on its relative payoff, with the probability being positive even if the strategy is inferior.

Fix a partition $\mathcal{P}_j$. Let Π_j denote the sum of the profits that the j outside coalitions would earn under this partition (this sum is constant over all j-similar partitions). Observe that Π_j are the total profits that j coalitions earn in a market with $j+1$ competing firms (the j coalitions plus S). Then define the function

$$f_{n,s}(j) = \frac{\exp(\Pi_j) K_{n-s,j}}{\sum_{m=1}^{n-s} \exp(\Pi_m) K_{n-s,m}} \tag{2}$$

Notice that $f_{n,s}(j) \in (0,1)$ and $\sum_{j=1}^{n-s} f_{n,s}(j) = 1$ (this is because the Stirling numbers of the second kind add up to the corresponding Bell number). Then, $f_{n,s}(j)$ gives the total probability that a coalition with s members assigns to all j-similar structures.[3] In one sense, the number $K_{n-s,j}$ expresses the "easiness" with which partitions with j members can occur. The results of

[3]In what follows (after Proposition 1 and in section 4) we elaborate on the use of more general distributions.

the paper hold even if the profitability of a partition with j members is not adjusted by $K_{n-s,j}$.

2.1 The game (N, v)

Let us now compute the characteristic function of a coalition. We use the j-similarity and focus for each j on one representative of the similar structures. So, let $q_i^{\mathcal{P}_j}$ denote the quantity that coalition i chooses $(i = 1, 2, ..., j)$ under structure $\mathcal{P}_j$; and let q_s denote the quantity of coalition S. The profit function that coalition S faces is then given by

$$\pi(S) = \sum_{j=1}^{n-s} f_{n,s}(j)(a - q_s - \sum_{i=1}^{j} q_i^{\mathcal{P}_j} - c)q_s \tag{3}$$

The profit function of coalition i under structure $\mathcal{P}_j$, $j = 1, 2, ..., n - s$, is

$$\pi_i^j = (a - q_s - \sum_{r=1, r\neq i}^{j} q_r^{\mathcal{P}_j} - q_i^{\mathcal{P}_j} - c)q_i^{\mathcal{P}_j}, \;\; i = 1, 2, ..., j$$

Hence the maximization problems to solve for are

$$max_{q_s} \pi(S)$$

and for $j = 1, 2, ..., n - s$,

$$max_{q_i^{\mathcal{P}_j}} \pi_i^j, \;\; i = 1, 2, ..., j$$

By symmetry, the solution of the above problems will involve $q_1^{\mathcal{P}_j} = q_2^{\mathcal{P}_j} = ... = q_j^{\mathcal{P}_j}$. Define

$$F_{f_{n,s}} = \sum_{j=0}^{n-s} \frac{j \cdot f_{n,s}(j)}{j+1} \tag{4}$$

Then it is easy to show that the solution of the maximization problems is given by

$$q_s = \frac{1 - F_{f_{n,s}}}{2 - F_{f_{n,s}}}(a - c) \tag{5}$$

and for $j = 1, 2, ..., n - s$,

$$q_i^{\mathcal{P}_j} = \frac{a - c}{(j+1)(2 - F_{f_{n,s}})}, \;\; i = 1, 2, ..., j \tag{6}$$

Using (5) and (6) in (3), we obtain the worth function $v(S)$ as[4]

$$v(S) = (a-c)^2 \frac{1 - F_{f_{n,s}}}{(2 - F_{f_{n,s}})^2} \sum_{j=0}^{n-s} \frac{f_{n,s}(j)}{j+1} \tag{7}$$

Hence our game is the pair (N, v) where v is defined by (7). The worth of the grand coalition, $v(N)$, is the monopoly profit. An allocation is a vector $x = (x_1, x_2, ..., x_n)$ where $\sum_{i \in N} = v(N)$. The core of (N, v) is the set of all allocations that cannot be blocked by any coalition, i.e., the core is the set

$$\mathcal{C}_f = \{x = (x_1, ..., x_n) : \nexists S \ \ with \ \ v(S) > \sum_{i \in S} x_i\}$$

3 Results

With a slight abuse of notation, let $v^n(s)$ denote the worth of a coalition with s members in a game with n players (with the understanding that $v^n(n) = v(N)$ is independent of n).

Lemma 1 *For every positive integer k, the equality $v^n(s) = v^{n+k}(s+k)$ holds.*

Proof. Consider an outsiders' partition consisting of j members. Notice that the total profits of the j coalitions do not depend on the number of members of the deviant coalition S. Hence

$$f_{n,s}(j) = \frac{\exp(\Pi_j) K_{n-s,j}}{\sum\limits_{m=1}^{n-s} \exp(\Pi_m) K_{n-s,m}} =$$

$$\frac{\exp(\Pi_j) K_{n+k-(s+k),j}}{\sum\limits_{m=1}^{n+k-(s+k)} \exp(\Pi_m) K_{n+k-(s+k),m}} = f_{n+k,s+k}(j) \tag{8}$$

Expression (8) implies that $F_{f_{n,s}} = F_{f_{n+k,s+k}}$. This combined with (7) proves the result. ■

The intuition behind Lemma 1 is clear. Let S_{+k} denote a deviant coalition that has $s+k$ players in a game with $n+k$ players. Then S_{+k} faces the

[4]Normally, we should sum from $j = 1$ up to $n - s$ but for convenience we start from $j = 0$ with the understanding that $f_{n,s}(0) = 0$.

number $n+k-s-k=n-s$ of outsiders, which is equal to the number of outsiders that a coalition with s members faces in a game with n players. Hence the two coalitions face the same set of potential coalition structures and thus have the same value.

An almost immediate implication of Lemma 1 is the monotonicity of $v^n(S)$ in s.

Lemma 2 *For every n, $v^n(s)$ is strictly increasing in s.*

Proof. We will use induction on the number of players, n. For the base case, $n=2$, we have to prove that $v^2(2) > v^2(1) > v^2(0)$. Notice that $v^2(2) = \left(\frac{a-c}{2}\right)^2 > \left(\frac{a-c}{3}\right)^2 = v^2(1) > 0 = v^2(0)$, so we have the base case.

Assume for the induction hypothesis that in a game with n players and for an arbitrary s, $1 < s \leq n$ we have that $v^n(s) > v^n(s-1)$.

We will prove that $v^{n+1}(s) > v^{n+1}(s-1)$. But this is an immediate result of lemma 1 and the induction hypothesis since $v^{n+1}(s) = v^n(s-1) > v^n(s-2) = v^{n+1}(s-1)$. ■

Proposition 1 *The game (N,v) has a non-empty core if n is sufficiently large.*

Proof Since firms are identical, the core is non-empty if and only if for all $S : |S| = s \leq n$,

$$\frac{v^n(n)}{n} \geq \frac{v^s(s)}{s} \tag{9}$$

It is easy to verify that the inequality does not hold for $3 \leq n \leq 10$. So for these values of n the core is empty.[5] The inequality holds for $n = 11$ (Table 1 in the Appendix). We will prove the rest of the proposition using induction on n, $n \geq 11$.

Base: Table 1 in the Appendix establishes the base case ($n = 11$).

Induction hypothesis: For all $S : |S| = s \leq n$, $\frac{v^n(n)}{n} \geq \frac{v^n(s)}{s}$.

[5] For $3 \leq n \leq 10$ it holds that $v^n(1) > \frac{v^n(n)}{n}$ (see Table 2 in the Appendix). The relevant calculations were made using the Maple program and they are available by the authors upon request.

Induction step: We will show that for all $S : |S| = s \leq n+1$,

$$\frac{v^{n+1}(n+1)}{n+1} \geq \frac{v^{n+1}(s)}{s}$$

By Lemma 1 we have that $v^{n+1}(s) = v^{n+1}((s-1)+1) = v^n(s-1)$ and also that $v^{n+1}(n+1) = v^n(n)$. So we have to show that

$$\frac{v^n(n)}{n+1} \geq \frac{v^n(s-1)}{s} \tag{10}$$

From the Induction hypothesis we have

$$v^n(n) \geq \frac{n}{s-1} v^n(s-1)$$

and thus

$$(s-1)v^n(n) \geq nv^n(s-1) \tag{11}$$

Using Lemma 2,

$$v^n(n) > v^n(s-1) \tag{12}$$

Adding (11) and (12) we have

$$sv^n(n) > (n+1)v^n(s-1)$$

which implies that (10) holds. So we have the proof for $n+1$ and thus the proposition is proved. ■

The monopoly profit is independent of n. This is due to our assumption of constant returns to scale. On the other hand, $v^n(S)$ decreases in n. As a result, for sufficiently large n the difference $v^n(n)/n - v^n(S)/s$ becomes positive for all s and the core is non-empty.

4 Extensions

The goal of this section is to compare the cores of different games, i.e., games defined by different probability distributions. To this end, we first express the characteristic function in terms of a concept which we call probabilistic harmonic number. Then we offer a comparison of cores that is based on these numbers (subsection 4.1). In particular, we compare the core under the proportional distribution with the core of a game defined by a distribution that first-order dominates it. A specific comparison relates our core with one of the most widely used core notions, the γ core (subsection 4.2).

4.1 A representation of $v(S)$

Let us give a useful representation of $v(S)$ that will be used later on. The representation is based on harmonic numbers. Recall that the k-th harmonic number is defined as

$$h^k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} = \sum_{j=0}^{k-1} \frac{1}{1+j} \tag{13}$$

where k is a natural number. Let us now define

$$h_f^k = \sum_{j=0}^{k-1} \frac{f(j)}{1+j} \tag{14}$$

as the *k-th probabilistic harmonic number induced by f*, where f is a probability distribution on $\{0, 1, 2, \cdots, k\}$. To make a connection between this concept and our game, notice that we can write (7) as

$$v(S) = \frac{1 - F_{f_{n,s}}}{(2 - F_{f_{n,s}})^2}(a-c)^2 h_{f_{n,s}}^{n-s+1} \tag{15}$$

where $h_{f_{n,s}}^{n-s+1} = \sum_{j=0}^{n-s} \frac{f_{n,s}(j)}{1+j}$. Notice next that

$$F_{f_{n,s}} = \sum_{j=0}^{n-s} \frac{j \cdot f_{n,s}(j)}{j+1} = \sum_{j=0}^{n-s} [1 - \frac{1}{j+1}] f_{n,s}(j) = 1 - \sum_{j=0}^{n-s} \frac{f_{n,s}(j)}{j+1}$$

Hence

$$F_{f_{n,s}} = 1 - h_{f_{n,s}}^{n-s+1} \tag{16}$$

Then combining (15) and (16) we get

$$v(S) = \frac{(h_{f_{n,s}}^{n-s+1})^2}{(1 + h_{f_{n,s}}^{n-s+1})^2}(a-c)^2 \tag{17}$$

Representations of the form (17) hold for any probability distribution that a coalition assigns over the set of coalition structures. So consider two such distributions $g_{n,s}$ and $z_{n,s}$. Let (N, v_g) and (N, v_z) denote the corresponding games and let C_g and C_z denote the cores of the two games. We have the following:

Lemma 3 *Consider two probability distributions $g_{n,s}$ and $z_{n,s}$ such that $h_{g_{n,s}}^{n-s+1} > h_{z_{n,s}}^{n-s+1}$, for all s. If $C_g \neq \emptyset$ then $C_z \neq \emptyset$ as well.*

Proof Since $h_{g_{n,s}}^{n-s+1} > h_{z_{n,s}}^{n-s+1}$ then $v_g(S) > v_z(S)$. Let $x \in C_g$. Then $\sum_{i \in S} x_i \geq v_g(S) > v_z(S)$ for any S. Hence $x \in C_z$ and $C_z \neq \emptyset$. ■

We can re-state the above result by saying that if $h_{g_{n,s}}^{n-s+1} > h_{z_{n,s}}^{n-s+1}$ then $C_g \subseteq C_z$ (provided that $C_z \neq \emptyset$). Lemma 3 is useful as it provides a simple method to compare the cores of two different games. If we know that the core of one of the two games is non-empty, Lemma 3 gives us a (computationally efficient) sufficient condition for the non-emptiness of the core of the other: we simply need to compare two numbers, i.e., the probabilistic harmonic numbers induced by the corresponding distributions.

4.2 First-order stochastic dominance

Let $z_{n,s}$ be a probability distribution that the members of S assign to the set of all possible coalition structures of the non-members. Assume that $z_{n,s}$ dominates $f_{n,s}$ at first-order, i.e., $\sum_{j \leq j^*} z_{n,s}(j) \leq \sum_{j \leq j^*} f_{n,s}(j)$, for all j^*.

Compared to the cumulative distribution of $f_{n,s}$, the cumulative distribution of $z_{n,s}$ assigns higher probabilities to events that include coalition structures with many coalitions. Such a distribution is plausible if we accept the assumption that a structure with few coalitions is in general less likely to form (few coalitions means that the $n-s$ players form large coalitions; and large coalitions require more effort, time, coordination, etc).[6]

Denote by $v_z(S)$ the worth of S under $z_{n,s}$; and let C_z denote the core of the resulting game (N, v_z).

Proposition 2 *Assume that $z_{n,s}$ dominates $f_{n,s}$ at first-order. The game (N, v_z) has non-empty core if n is sufficiently large. Furthermore, $C_f \subseteq C_z$.*

Proof We shall use Proposition 1 and Lemma 3. First we show that the probabilistic harmonic number induced by $f_{n,s}$ is larger than that induced by $z_{n,s}$. To see this, notice that $h_{f_{n,s}^{n-s+1}} > h_{z_{n,s}^{n-s+1}}$ iff

$$\sum_{j=0}^{n-s} \frac{1}{1+j}[f_{n,s}(j) - z_{n,s}(j)] > 0 \tag{18}$$

[6] The cost of forming coalitions is not modeled here. It is simply reflected in the cumulative distribution of a probability scheme.

Since $f_{n,s}(0) = z_{n,s}(0) = 0$, we have

$$\sum_{j=0}^{n-s} \frac{1}{1+j}[f_{n,s}(j) - z_{n,s}(j)] = \sum_{j=1}^{n-s} \frac{1}{1+j}[f_{n,s}(j) - z_{n,s}(j)] =$$

$$\frac{\overbrace{f_{n,s}(1) - z_{n,s}(1)}^{\geq 0}}{2} + \frac{f_{n,s}(2) - z_{n,s}(2)}{3} + \sum_{j=3}^{n-s} \frac{1}{1+j}[f_{n,s}(j) - z_{n,s}(j)] >$$

$$\frac{1}{3}\overbrace{[\sum_{j=1}^{2} f_{n,s}(j) - \sum_{j=1}^{2} z_{n,s}(j)]}^{\geq 0} + \sum_{j=3}^{n-s} \frac{1}{1+j}[f_{n,s}(j) - z_{n,s}(j)] =$$

$$\frac{1}{3}\overbrace{[\sum_{j=1}^{2} f_{n,s}(j) - \sum_{j=1}^{2} z_{n,s}(j)]}^{\geq 0} + \frac{f_{n,s}(3) - z_{n,s}(3)}{4} + \sum_{j=4}^{n-s} \frac{1}{1+j}[f_{n,s}(j) - z_{n,s}(j)] >$$

$$\frac{1}{4}\overbrace{[\sum_{j=1}^{3} f_{n,s}(j) - \sum_{j=1}^{3} z_{n,s}(j)]}^{\geq 0} + \sum_{j=4}^{n-s} \frac{1}{1+j}[f_{n,s}(j) - z_{n,s}(j)]$$

Continuing the iterations, we eventually get that

$$\sum_{j=0}^{n-s} \frac{1}{1+j}[f_{n,s}(j) - z_{n,s}(j)] > \frac{1}{n-s}\overbrace{[\sum_{j \leq n-s-1} f_{n,s}(j) - \sum_{j \leq n-s-1} z_{n,s}(j)]}^{\geq 0} +$$

$$\frac{f_{n,s}(n-s) - z_{n,s}(n-s)}{1+n-s} =$$

$$\frac{1}{n-s}\overbrace{[\sum_{j=1}^{n-s-1} f_{n,s}(j) - \sum_{j=1}^{n-s-1} z_{n,s}(j)]}^{\geq 0} + \frac{1}{1+n-s}[-\sum_{j=1}^{n-s-1} f_{n,s}(j) + \sum_{j=1}^{n-s-1} z_{n,s}(j)] =$$

$$\overbrace{[\sum_{j \leq n-s-1} f_{n,s}(j) - \sum_{j \leq n-s-1} z_{n,s}(j)]}^{\geq 0}\left(\frac{1}{n-s} - \frac{1}{1+n-s}\right) > 0$$

So, (18) is proved. Furthermore, by Proposition 1 we know that $C_f \neq \emptyset$ whenever n is large ($n \geq 11$). Hence by Lemma 3, $C_z \neq \emptyset$ and $C_f \subseteq C_z$. ■

Under $z_{n,s}$, the core is non-empty more often. The reason is that in comparison to $f^{\lambda}_{n,s}$, distributions like $z_{n,s}$ "penalize" the structures that are more favorable for a deviant coalition (i.e, structures with few coalitions) and give more weight to less favorable structures (i.e., structures with many coalitions).

A particular probability distribution that dominates $f_{n,s}$ is the distribution defined by $\tilde{z}_{n,s}(j) = 0$, for $j = 1, 2, ..., n-s-1$ and $\tilde{z}_{n,s}(n-s) = 1$. This degenerate distribution corresponds to the γ core scenario. It is known that the latter core is non-empty for Cournot oligopolies (Chander 2009). Denote by $v_{\gamma}(S)$ the characteristic function of coalition S under the γ scenario and by $\mathcal{C}_{\gamma}$ the corresponding core. We have the following.

Corollary 1 *The inclusion $\mathcal{C}_f \subset \mathcal{C}_{\gamma}$ holds.*

Proof Notice that $h_{\tilde{z}^{n-s+1}_{n,s}} = \frac{1}{1+n-s}$ and hence $h_{f^{n-s+1}_{n,s}} > h_{\tilde{z}^{n-s+1}_{n,s}}$, as the number $h_{f^{n-s+1}_{n,s}}$ is a weighted average of the list of numbers $(1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{1+n-s})$. Hence $v_f(S) > v_{\gamma}(S)$, implying that $\mathcal{C}_f \subset \mathcal{C}_{\gamma}$. ∎

The γ core is based on the worst scenario for the members of the deviant coalition S: all $n-s$ firms remain separate entities. Under $f_{n,s}$, the singleton coalitions structure is just one of the structures that the members of S take into account. Other, more favorable, structures occur with positive probability. Hence, the worth of S in our game is higher than its worth under the γ theory, which explains the relation between the two cores.

5 Conclusions

This paper analyzed a family of cooperative oligopoly games. The analysis is based on the assertion that when a coalition contemplates a deviation from the grand coalition, it assigns various non-equilibrium distributions on the set of coalition structures that the outsiders can form. This assumption is justified by imposing cognitive constraints on behalf of the firms in the market. Provided that the number of firms in the market is sufficiently high, the corresponding game has a non-empty core for a large class of probability distributions.

Let us mention a few extensions of the current work. The analysis of oligopolistic markets with general demand and cost functions and/or other modes of competition (e.g., product differentiation, price competition) are natural future directions. Further, the application of the current framework to other economic environments (e.g, environmental agreements, etc.) is of interest.

References

1. Anderson S., J. K. Goeree and C. A. Holt (2002) The logit equilibrium: A perspective on intuitive behavioral anomalies, Southern Economic Journal, 69, 21-47.

2. Aumann R. (1959) Acceptable points in general cooperative n-person games, Contributions to the theory of games IV, Annals of Mathematics Studies vol. 40, Princeton University Press, Princeton.

3. Bell E. T. (1934) Exponential Numbers. American Mathematical Monthly, 41, 411-419.

4. Camerer C. (2003) Behavioral studies of strategic thinking in games, TRENDS in Cognitive Sciences, 7, 225-231.

5. Camerer C., T-H Ho and J-K Chong (2004) A Cognitive hierarchy model of games, Quarterly Journal of Economics, 119, 861-898.

6. Chander P. (2009) Cores of games with positive externalities, Working Paper.

7. Chander P. and H. Tulkens (1997) A core of an economy with multilateral environmental externalities, International Journal of Game Theory, 26, 379-401.

8. Chen H.C, J. Friedman and J.-F. Thisse (1997) Boundedly rational Nash equilibrium: a probabilistic choice approach. Games and Economic Behavior, 18, 32-54.

9. Currarini S. and M. Marini (1998) The core of games with Stackelberg leaders, Working Paper, MPRA.

10. Dang V. D. and Jennings, N. R. (2004) Generating coalition structures with finite bound from the optimal guarantees. In: 3rd International Conference on Autonomous Agents and Multi-Agent Systems, New York, USA, 564-571.

11. Dick A. (1996) When are cartels stable contracts? Journal of Law and Economics, 39, 241-283.

12. Hart S. and Kurz, M. (1983) Endogenous formation of coalitions, Econometrica 52, 1047-1064.

13. Haruvy E. and D. Stahl (2007) Equilibrium selection and bounded rationality in symmetric normal form games, Journal of Economic Behavior and Organization, 62, 98-119.

14. Haucap J., U. Heimeshoff and L.M Schultz (2010) Legal and illegal

cartels in Germany between 1958-2004, Working Paper, Dusseldorf Institute for Competition Economics.

15. Huang C.Y. and T. Sjostrom (1998) The p-core, Working Paper.

16. Huang C.Y. and T. Sjostrom (2003) Consistent solutions for cooperative games with externalities, Games and Economic Behavior, 43, 196-213.

17. Koczy L. (2007) A recursive core for partition function form games, Theory and Decision, 63, 41-51.

18. McKelvey R. and T. R. Palfrey (1995) Quantal response equilibria for normal form games, Games and Economic Behavior 10, 6-38.

19. Motta M. (2009) Cartels in the European Union: Economics, Law, Practice, in Xavier Vives (ed.), Competition in the EU: Fifty Years on from the Treaty of Rome, Oxford University Press.

20. Rajan R. (1989) Endogenous coalition formation in cooperative oligopolies, International Economic Review, 30, 863-876.

21. Ray D. and R. Vohra (1999) A theory of endogenous coalition structures, Games and Economic Behavior, 26, 286-336.

22. Sandholm T., K. Larson, M. Andersson, O. Shehory, F. Tohmé (1999) Coalition structure generation with worst case guarantees. Artificial Intelligence 111, 209-238.

23. Stahl D. and P. Wilson (1994) On players' models of other players: Theory and experimental evidence, Games and Economic Behavior, 10, 218-254.

24. Zhao, J. (1999) A β-core existence result and its application to oligopoly markets, Games and Economic Behavior, 27, 153-168.

Appendix

s	$\frac{v(s)}{(a-c)^2}$
1	0.02261
2	0.02530
3	0.02858
4	0.03266
5	0.03785
6	0.04462
7	0.05386
8	0.06712
9	0.08633
10	0.11111
11	0.25

Table 1: values $v(S)$ in (N, v) with $n = 11$ players.

n	$\frac{v^n(\{i\})}{(a-c)^2}$
3	0.08633
4	0.06712
5	0.05386
6	0.04463
7	0.03785
8	0.03266
9	0.02858
10	0.02530

Table 2: values $v^n(\{i\})$, $n \in \{3, 4, \cdots, 10\}$