A DSGE MODEL OF INSIDERS – OUTSIDERS CAPITALISM: EMPIRICAL RESULTS FROM SEVEN EUROPEAN COUNTRIES.

APPENDIX

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A1. Sequential linearization.

Consider the functions f, h where moreover $h = h(x,y)$. Their composition is $f\left[h(x,y)\right]$. Linearizing $f\left[h(x,y)\right]$ around $q_0 = (x_0, y_0)$ we get

$$f\left[h(x,y)\right] \;=\; f\left[h(q_0)\right] \;+\; f'\left[h(q_0)\right]\frac{\partial h(q_0)}{\partial x}(x-x_0) + f'\left[h(q_0)\right]\frac{\partial h(q_0)}{\partial y}(y-y_0) + R_1.$$

Consider now sequential linearization. First linearize f around some value h_0

$$f(h) \;=\; f(h_0) \;+\; f'(h_0)(h-h_0) + R_1'.$$

Second, linearize $h(x,y)$ **around *the same*** $q_0 = (x_0, y_0)$ as before,

$$h(x,y) = \; h(q_0) \;+\; \frac{\partial h(q_0)}{\partial x}(x-x_0) + \frac{\partial h(q_0)}{\partial y}(y-y_0) + R_1''.$$

Substitute in the previous expression

$$f(h) \;=\; f(h_0) \;+\; f'(h_0)\left(h(q_0) \;+\; \frac{\partial h(q_0)}{\partial x}(x-x_0) + \frac{\partial h(q_0)}{\partial y}(y-y_0) + R_1'' - h_0 \right) + R_1'$$

$$\Rightarrow f\left[h(x,y)\right] \;=\; f(h_0) + f'(h_0)h(q_0) \;-\; f'(h_0)h_0$$

$$+\; f'(h_0)\frac{\partial h(q_0)}{\partial x}(x-x_0) \;+\; f'(h_0)\frac{\partial h(q_0)}{\partial y}(y-y_0) \;+\; \left(f'(h_0)R_1'' + R_1'\right)$$

Compare this with the expression from the one-step linearization,

$$f\left[h(x,y)\right] \;=\; f\left[h(q_0)\right] \;+\; f'\left[h(q_0)\right]\frac{\partial h(q_0)}{\partial x}(x-x_0) + f'\left[h(q_0)\right]\frac{\partial h(q_0)}{\partial y}(y-y_0) + R_1$$

We see that **if** $h_0 = h(q_0)$ **the linearized parts become identical** (and then necessarily the Taylor remainder parts also).

Under this condition, sequential linearization as above is identical to one-step linearization around q_0 directly.

Namely, instead of linearizing $f\left[h(x,y)\right]$ directly around $q_0 = (x_0, y_0)$ we can:

1) Linearize $f\left[h(x,y)\right]$ around $h_0 = h(q_0)$ and express f as a linear function of h

2) Linearize $h(x,y)$ around $q_0 = (x_0, y_0)$

3) Substitute the linearized part of $h(x,y)$ into the linearized f and obtain the linearization of f around (x_0, y_0).

Now, use the following notation for log-deviations from the steady state: $\hat{x}_t \equiv \ln x_t - \ln \bar{x}$, and note that $x_t = \bar{x}\exp\{\hat{x}_t\}$.

In our case, f stands for any of the initial equations of the empirical model. Log-linearization around the steady-state, means that we want to linearize around, among other variables, $\hat{B}_t = \hat{\chi}_t = 0$, in a one-step linearization. Instead we start by linearizing around $\hat{Z}_t = 0$ and f is expressed as a linear function of $\hat{Z}_t$. So "h" is the $\hat{Z}_t$ variable in our case, and $h_0 = 0$.

Then we take the non-linear expression for $Z_t = Z_t\left(B_t\,,\,\chi_t\right)$, and we manipulate it in order to bring in the surface the $\hat{Z}_t$ variable,

$$Z_t = \frac{\Delta_t\left(B_t\,,\,\chi_t\right)}{1+\Delta_t\left(B_t\,,\,\chi_t\right)} \Rightarrow \bar{Z}\exp\left\{\hat{Z}_t\right\} = \frac{\Delta_t\left(\bar{B}\;\exp\left\{\hat{B}_t\right\},\,\bar{\chi}\exp\left\{\chi_t\right\}\right)}{1+\Delta_t\left(\bar{B}\;\exp\left\{\hat{B}_t\right\},\,\bar{\chi}\exp\left\{\chi_t\right\}\right)}$$

$$\Rightarrow \hat{Z}_t = \ln\left[\Delta_t\left(\bar{B}\;\exp\left\{\hat{B}_t\right\},\,\bar{\chi}\exp\left\{\chi_t\right\}\right)\right] - \ln\left[1+\Delta_t\left(\bar{B}\;\exp\left\{\hat{B}_t\right\},\,\bar{\chi}\exp\left\{\chi_t\right\}\right)\right] - \ln\bar{Z}.$$

If we linearize the RHS at $\hat{B}_t = \hat{\chi}_t = 0$,

a) we linearize around the center of expansion that we would have wanted in one-step linearization, and

b) we essentially linearize around $\hat{Z}_t = 0$, so we respect the condition for sequential linearization, $h_0 = h(q_0)$**.** Then the linearized expression that we will obtain for $\hat{Z}_t$ can be inserted into the linearized f function, and obtain the same results as with one-step linearization.

A2. Log-linearizing the auxiliary variable Z_t.

We have

$$Z_t = \frac{\Delta_t\left(B_t\,,\,\chi_t\right)}{1+\Delta_t\left(B_t\,,\,\chi_t\right)} \Rightarrow \bar{Z}\exp\left\{\hat{Z}_t\right\} = \frac{\Delta_t\left(\bar{B}\exp\left\{\hat{B}_t\right\},\,\bar{\chi}\exp\left\{\hat{\chi}_t\right\}\right)}{1+\Delta_t\left(\bar{B}\exp\left\{\hat{B}_t\right\},\,\bar{\chi}\exp\left\{\hat{\chi}_t\right\}\right)}$$

$$\Rightarrow \hat{Z}_t = -\ln\bar{Z} + \ln\left[\Delta_t\left(\bar{B}\exp\left\{\hat{B}_t\right\},\,\bar{\chi}\exp\left\{\hat{\chi}_t\right\}\right)\right] - \ln\left[1+\Delta_t\left(\bar{B}\exp\left\{\hat{B}_t\right\},\,\bar{\chi}\exp\left\{\hat{\chi}_t\right\}\right)\right]$$

From eq. (II.11) of the Technical Appendix 2024-03-11 we have

$$\Delta_t = \left(\frac{\varphi_1}{1-\varphi_1}\right)^{1/(1-\phi)}\left[\theta^{\alpha}\xi^{(1-\alpha)}\right]^{\phi/(1-\phi)}\left(B_t\right)^{\phi/(1-\phi)}\left(\frac{\chi_t}{1-\chi_t}\right)^{\frac{\theta\varphi_2+\phi(1-\theta)}{\theta(1-\phi)}}$$

Setting $b = \dfrac{\theta\varphi_2+\phi(1-\theta)}{\theta(1-\phi)}$, $\quad \bar{\Delta} \equiv \left(\dfrac{\varphi_1}{1-\varphi_1}\right)^{1/(1-\phi)}\left[\theta^{\alpha}\xi^{(1-a)}\right]^{\phi/(1-\phi)}\left(\bar{B}\right)^{\phi/(1-\phi)}\left(\dfrac{\bar{\chi}}{1-\bar{\chi}}\right)^{b}$

we can write

$$\begin{aligned}\Delta_t &\equiv \left(\frac{\varphi_1}{1-\varphi_1}\right)^{1/(1-\phi)}\left[\theta^{\alpha}\xi^{(1-a)}\bar{B}\right]^{\phi/(1-\phi)}\exp\left\{\frac{\phi}{1-\phi}\hat{B}_t\right\}\left(\frac{\bar{\chi}^{b}\exp\left\{b\hat{\chi}_t\right\}}{\left(1-\bar{\chi}\exp\left\{\hat{\chi}_t\right\}\right)^{b}}\right)\\ &= \bar{\Delta}\cdot\exp\left\{\frac{\phi}{1-\phi}\hat{B}_t\right\}\frac{\left(1-\bar{\chi}\right)^{b}\exp\left\{b\hat{\chi}_t\right\}}{\left(1-\bar{\chi}\exp\left\{\hat{\chi}_t\right\}\right)^{b}}\end{aligned}$$

So

$$\hat{Z}_t = -\ln\bar{Z} + \ln\bar{\Delta} + b\ln\left(1-\bar{\chi}\right) + \frac{\phi}{1-\phi}\hat{B}_t + b\hat{\chi}_t - b\ln\left(1-\bar{\chi}\exp\left\{\hat{\chi}_t\right\}\right)$$

$$-\ln\left[1+\Delta_t\left(\bar{B}\ \exp\left\{\hat{B}_t\right\},\,\bar{\chi}\exp\left\{\hat{\chi}_t\right\}\right)\right]$$

We now linearize the remaining non-linear terms around $\hat{B}_t = \hat{\chi}_t = 0$ to get

$$\hat{Z}_t = \left[-\ln\bar{Z} + \ln\bar{\Delta} + b\ln\left(1-\bar{\chi}\right) - b\ln\left(1-\bar{\chi}\right) - \ln\left(1+\bar{\Delta}\right)\right]$$

$$+ \frac{\phi}{1-\phi}\hat{B}_t + b\hat{\chi}_t + b\frac{\bar{\chi}}{\left(1-\bar{\chi}\right)}\hat{\chi}_t - \frac{\partial\ln\left(1+\Delta_t\right)}{\partial\hat{B}_t}\Big|_{ss}\hat{B}_t - \frac{\partial\ln\left(1+\Delta_t\right)}{\partial\hat{\chi}_t}\Big|_{ss}\hat{\chi}_t$$

The term in brackets is zero. Re-arranging,

$$\hat{Z}_t = \left[\frac{\phi}{1-\phi} - \frac{\partial \ln(1+\Delta_t)}{\partial \hat{B}_t}\Big|_{ss}\right]\hat{B}_t + \left[b + b\frac{\bar{\chi}}{(1-\bar{\chi})} - \frac{\partial \ln(1+\Delta_t)}{\partial \hat{\chi}_t}\Big|_{ss}\right]\hat{\chi}_t$$

We have

$$\frac{\partial \ln(1+\Delta_t)}{\partial \hat{B}_t}\Big|_{ss} = \frac{1}{1+\bar{\Delta}}\frac{\partial}{\partial \hat{B}_t}\left(\bar{\Delta}\cdot\exp\left\{\frac{\phi}{1-\phi}\hat{B}_t\right\}\frac{(1-\bar{\chi})^b\exp\{b\hat{\chi}_t\}}{(1-\bar{\chi}\exp\{\hat{\chi}_t\})^b}\right)\Big|_{ss} = \frac{\phi}{1-\phi}\cdot\frac{\bar{\Delta}}{1+\bar{\Delta}}$$

$$\frac{\partial \ln(1+\Delta_t)}{\partial \hat{\chi}_t}\Big|_{ss} = \frac{1}{1+\bar{\Delta}}\frac{\partial}{\partial \hat{\chi}_t}\left(\bar{\Delta}\cdot\exp\left\{\frac{\phi}{1-\phi}\hat{B}_t\right\}\frac{(1-\bar{\chi})^b\exp\{b\hat{\chi}_t\}}{(1-\bar{\chi}\exp\{\hat{\chi}_t\})^b}\right)\Big|_{ss} =$$

$$= \frac{\bar{\Delta}}{1+\bar{\Delta}}(1-\bar{\chi})^b\frac{\partial}{\partial \hat{\chi}_t}\left(\frac{\exp\{b\hat{\chi}_t\}}{(1-\bar{\chi}\exp\{\hat{\chi}_t\})^b}\right)\Big|_{ss} = \frac{\bar{\Delta}}{1+\bar{\Delta}}(1-\bar{\chi})^b\frac{b(1-\bar{\chi})^b + b(1-\bar{\chi})^{b-1}\bar{\chi}}{(1-\bar{\chi})^{2b}}$$

$$= \frac{\bar{\Delta}}{1+\bar{\Delta}}\frac{b(1-\bar{\chi})^{2b} + b(1-\bar{\chi})^{2b-1}\bar{\chi}}{(1-\bar{\chi})^{2b}} = \frac{\bar{\Delta}}{1+\bar{\Delta}}b\left(1+\frac{\bar{\chi}}{1-\bar{\chi}}\right) = \frac{\bar{\Delta}}{1+\bar{\Delta}}\frac{b}{1-\bar{\chi}}$$

So

$$\hat{Z}_t \approx \left[\frac{\phi}{1-\phi} - \frac{\bar{\Delta}}{1+\bar{\Delta}}\frac{\phi}{1-\phi}\right]\hat{B}_t + \left[b + b\frac{\bar{\chi}}{(1-\bar{\chi})} - \frac{\bar{\Delta}}{1+\bar{\Delta}}\frac{b}{1-\bar{\chi}}\right]\hat{\chi}_t$$

$$\Rightarrow \hat{Z}_t \approx (1-\bar{Z})\frac{\phi}{1-\phi}\hat{B}_t + \left[1 + \frac{\bar{\chi}}{1-\bar{\chi}} - \frac{\bar{Z}}{1-\bar{\chi}}\right]b\hat{\chi}_t$$

$$= (1-\bar{Z})\frac{\phi}{1-\phi}\hat{B}_t + \frac{(1-\bar{Z})}{1-\bar{\chi}}b\hat{\chi}_t$$

or forwarding once and re-instating the original coefficients,

or
$$\boxed{\hat{Z}_{t+1} \approx \frac{(1-\bar{Z})\phi}{1-\phi}\hat{B}_{t+1} + \frac{(1-\bar{Z})\phi}{1-\phi}\left(\frac{\theta\varphi_2+\phi(1-\theta)}{\theta\phi(1-\bar{\chi})}\right)\hat{\chi}_{t+1}}$$

After log-linearizing the measurement equation with respect to $\hat{Z}_{t+1}$ we can insert the above expression in place of $\hat{Z}_{t+1}$.

A3. Log-linearizing the measurement equation.

A. Proportion of time working.

$$\frac{\vartheta(1-\alpha)}{1-\vartheta}(1-\tau_{t+1}^{L})\left[1-\left(1-\xi\right)Z_{t+1}\right]\left(1-h_{t+1}\right)=\frac{c_{t+1}h_{t+1}}{y_{t+1}}$$

$$\Rightarrow \ln\left(\frac{\vartheta(1-\alpha)}{1-\vartheta}\right)+\ln(1-\tau_{t+1}^{L})+\ \ln\left[1-\left(1-\xi\right)Z_{t+1}\right]+\ln\left(1-h_{t+1}\right)=\ln c_{t+1}+\ \ln h_{t+1}-\ln y_{t+1}$$

At the steady-state,

$$\ln\left(\frac{\vartheta(1-\alpha)}{1-\vartheta}\right)+\ln\left(1-\overline{\tau}^{L}\right)+\ln\left[1-\left(1-\xi\right)\overline{Z}\right]+\ln\left(1-\overline{h}\right)=\ln\overline{c}+\ln\overline{h}-\ln\overline{y}$$

Subtracting the steady-state relation from both sides of the current relation, and using

$\hat{x}_t=\ln x_t-\ln\overline{x}\Rightarrow x_t=\overline{x}\exp\left\{\hat{x}_t\right\}$, we get

$$\begin{aligned}&\ln(1-\tau_{t+1}^{L})+\ \ln\left[1-\left(1-\xi\right)Z_{t+1}\right]+\ln\left(1-h_{t+1}\right)\\&\qquad-\ln\left(1-\overline{\tau}^{L}\right)-\ln\left[1-\left(1-\xi\right)\overline{Z}\right]-\ln\left(1-\overline{h}\right)=\hat{c}_{t+1}+\ \hat{h}_{t+1}-\hat{y}_{t+1}\end{aligned}$$

Linearize the components that need linearizing,

Linearize around zero log-deviations from the steady-state $\hat{x}_t=0$,

$$\ln(1-\tau_{t+1}^{L})=\ln\left(1-\overline{\tau}^{L}\exp\left\{\hat{\tau}_{t+1}^{L}\right\}\right)\approx\ln\left(1-\overline{\tau}^{L}\right)\ -\ \frac{\overline{\tau}^{L}}{1-\overline{\tau}^{L}}\hat{\tau}_{t+1}^{L}$$

$$\ln\left[1-\left(1-\xi\right)Z_{t+1}\right]=\ \ln\left[1-\left(1-\xi\right)\overline{Z}\exp\left\{\hat{Z}_{t+1}\right\}\right]\approx\ln\left[1-\left(1-\xi\right)\overline{Z}\right]\ -\ \frac{\left(1-\xi\right)\overline{Z}}{1-\left(1-\xi\right)\overline{Z}}\hat{Z}_{t+1}$$

$$\ln\left(1-h_{t+1}\right)=\ln\left(1-\overline{h}\exp\left\{\hat{h}_{t+1}\right\}\right)\approx\ln\left(1-\overline{h}\right)\ -\ \frac{\overline{h}}{1-\overline{h}}\hat{h}_{t+1}$$

Insert these into the equation, cancelling out terms, and then re-arranging,

$$-\ \frac{\overline{\tau}^{L}}{1-\overline{\tau}^{L}}\hat{\tau}_{t+1}^{L}\ -\ \frac{\left(1-\xi\right)\overline{Z}}{1-\left(1-\xi\right)\overline{Z}}\hat{Z}_{t+1}\ -\ \frac{\overline{h}}{1-\overline{h}}\hat{h}_{t+1}\ =\ \hat{c}_{t+1}+\ \hat{h}_{t+1}-\hat{y}_{t+1}$$

$$\Rightarrow\frac{1}{1-\overline{h}}\hat{h}_{t+1}-\hat{y}_{t+1}+\ \hat{c}_{t+1}\ =\ -\ \frac{\overline{\tau}^{L}}{1-\overline{\tau}^{L}}\hat{\tau}_{t+1}^{L}-\ \frac{\left(1-\xi\right)\overline{Z}}{1-\left(1-\xi\right)\overline{Z}}\hat{Z}_{t+1}$$

...which is the equation in log-deviations from the steady state. We have re-arranged to obtain the order of variables in the general form $\Theta\mathbf{y}_{t+1} = \mathbf{d} + \Lambda\mathbf{y}_t + \mathrm{M}\mathbf{x}_{t+1} + \Xi\mathbf{s}_{t+1}$.

We insert now the expression for $\hat{Z}_{t+1}$,

$$\frac{1}{1-\bar{h}}\hat{h}_{t+1} - \hat{y}_{t+1} + \hat{c}_{t+1} = -\frac{\bar{\tau}^L}{1-\bar{\tau}^L}\hat{\tau}^L_{t+1} - \frac{(1-\xi)\bar{Z}}{\left[1-(1-\xi)\bar{Z}\right]}\frac{\left(1-\bar{Z}\right)\phi}{1-\phi}\hat{B}_{t+1}$$

$$-\frac{(1-\xi)\bar{Z}}{\left[1-(1-\xi)\bar{Z}\right]}\frac{\left(1-\bar{Z}\right)\phi}{(1-\phi)}\left[\frac{\theta\varphi_2 + (1-\theta)\phi}{\theta\phi(1-\bar{\chi})}\right]\hat{\chi}_{t+1}$$

Note that since the steady state of the data-variables will be taken to be their empirical mean, they will remain in log-deviation-from-the-steady-state form, but not the unobservables.

We assign matrix coefficients, we separate the SS values, we use the omega symbols for compactness and we get

(L.1)

$$\theta_{11}\hat{h}_{t+1} + \theta_{12}\hat{y}_{t+1} + \theta_{14}\hat{c}_{t+1} = d_h + \mathrm{m}_{13}\hat{\tau}^L_{t+1} + \xi_{12}\ln B_{t+1} + \xi_{13}\ln\chi_{t+1}$$

$$\theta_{11} = \frac{1}{\left(1-\bar{h}\right)},\quad \theta_{12} = -1,\quad \theta_{14} = 1,$$

$$\mathrm{m}_{13} = -\frac{\bar{\tau}^L}{1-\bar{\tau}^L}$$

$$\xi_{12} = -\frac{1-\bar{\bar{\omega}}}{\bar{\bar{\omega}}}\frac{\left(1-\bar{Z}\right)\phi}{1-\phi},\quad \xi_{13} = \xi_{12}\frac{\theta\varphi_2 + (1-\theta)\phi}{\theta\phi\left(1-\bar{\chi}\right)}$$

$$d_h = -\left(\xi_{12}\ln\bar{B} + \xi_{13}\ln\bar{\chi}\right)$$

Note that the structural coefficients θ_{11}, m_{13} are fixed since the steady states of the observables will not be estimated.

B. Output.

$y_{t+1} = \bar{\omega}_{t+1} \cdot k_{t+1}^{\alpha} h_{t+1}^{(1-\alpha)} \Rightarrow \ln y_{t+1} = \ln \bar{\omega}_{t+1} + \alpha \ln k_{t+1} + (1-\alpha) \ln h_{t+1}$.

At the steady state

$\ln \bar{y} = \ln \bar{\bar{\omega}} + \alpha \ln \bar{k} + (1-\alpha) \ln \bar{h}$.

Subtracting the SS relation from both sides of the current relation, and re-arranging,

$$-(1-\alpha)\hat{h}_{t+1} + \hat{y}_{t+1} = \alpha \hat{k}_{t+1} + \hat{\bar{\omega}}_{t+1}$$

Linearize what needs to be linearized which is only the omega variable,

$$\bar{\omega}_{t+1} \equiv \varphi_1^{1/\phi} \theta^{\alpha} \xi^{(1-\alpha)} \frac{\tilde{A}_{t+1} \cdot B_{t+1} \cdot \chi_{t+1}^{\frac{\theta\varphi_2 + \phi(1-\theta)}{\theta\phi}}}{\left[1-(1-\theta)Z_{t+1}\right]^{\alpha} \left[1-(1-\xi)Z_{t+1}\right]^{1-\alpha} Z_{t+1}^{(1-\phi)/\phi}}$$

$$\ln \bar{\omega}_{t+1} = \ln\left(\varphi_1^{1/\phi} \theta^{\alpha} \xi^{(1-a)}\right) + \ln \tilde{A}_{t+1} + \ln B_{t+1} + \frac{\theta\varphi_2 + (1-\theta)\phi}{\phi\theta} \ln \chi_{t+1} - \frac{1-\phi}{\phi} \ln Z_{t+1}$$

$$-\alpha \ln\left[1-(1-\theta)Z_{t+1}\right] - (1-\alpha)\ln\left[1-(1-\xi)Z_{t+1}\right].$$

We have

$$\ln\left[1-(1-\theta)Z_{t+1}\right] = \ln\left(1-(1-\theta)\bar{Z}\exp\left\{\hat{Z}_{t+1}\right\}\right) \approx \ln\left(1-(1-\theta)\bar{Z}\right) - \frac{(1-\theta)\bar{Z}}{1-(1-\theta)\bar{Z}} \hat{Z}_{t+1}$$

$$\ln\left[1-(1-\xi)Z_{t+1}\right] = \ln\left(1-(1-\xi)\bar{Z}\exp\left\{\hat{Z}_{t+1}\right\}\right) \approx \ln\left(1-(1-\xi)\bar{Z}\right) - \frac{(1-\xi)\bar{Z}}{1-(1-\xi)\bar{Z}} \hat{Z}_{t+1}$$

Inserting into $\ln \bar{\omega}_{t+1}$,

$$\begin{aligned} \ln \bar{\omega}_{t+1} = {} & \ln\left(\varphi_1^{1/\phi} \theta^{\alpha} \xi^{(1-a)}\right) + \ln \tilde{A}_{t+1} + \ln B_{t+1} + \frac{\theta\varphi_2 + (1-\theta)\phi}{\phi\theta} \ln \chi_{t+1} - \frac{1-\phi}{\phi} \ln Z_{t+1} \\ & -\alpha \ln\left(1-(1-\theta)\bar{Z}\right) + \alpha \frac{(1-\theta)\bar{Z}}{1-(1-\theta)\bar{Z}} \hat{Z}_{t+1} \\ & -(1-\alpha)\ln\left(1-(1-\xi)\bar{Z}\right) + (1-\alpha)\frac{(1-\xi)\bar{Z}}{1-(1-\xi)\bar{Z}} \hat{Z}_{t+1} \end{aligned}$$

and compacting and rearranging, and subtracting from both sides the SS expression for omega bar,

$$\begin{aligned} \hat{\bar{\omega}}_{t+1} = {} & \hat{\tilde{A}}_{t+1} + \hat{B}_{t+1} + \frac{\theta\varphi_2 + (1-\theta)\phi}{\phi\theta} \hat{\chi}_{t+1} \\ & + \left[\alpha \frac{(1-\theta)\bar{Z}}{1-(1-\theta)\bar{Z}} + (1-\alpha)\frac{(1-\xi)\bar{Z}}{1-(1-\xi)\bar{Z}} - \frac{1-\phi}{\phi}\right] \hat{Z}_{t+1} \end{aligned}$$

Inserting the expression for $\hat{Z}_{t+1}$ we get

$$\hat{\widehat{\omega}}_{t+1} = \hat{\hat{A}}_{t+1} + \left\{1 + \frac{\left(1-\bar{Z}\right)\phi}{1-\phi}\left[\frac{\alpha\left(1-\theta\right)\bar{Z}}{1-\left(1-\theta\right)\bar{Z}} + \frac{\left(1-\alpha\right)\left(1-\xi\right)\bar{Z}}{1-\left(1-\xi\right)\bar{Z}} - \frac{1-\phi}{\phi}\right]\right\}\hat{B}_{t+1}$$
$$+ \left\{\frac{\theta\varphi_2 + (1-\theta)\phi}{\phi\theta} + \frac{\left(1-\bar{Z}\right)\phi}{1-\phi}\left[\frac{\theta\varphi_2 + (1-\theta)\phi}{\theta\phi\left(1-\bar{\chi}\right)}\right]\left[\alpha\frac{\left(1-\theta\right)\bar{Z}}{1-\left(1-\theta\right)\bar{Z}} + \left(1-\alpha\right)\frac{\left(1-\xi\right)\bar{Z}}{1-\left(1-\xi\right)\bar{Z}} - \frac{1-\phi}{\phi}\right]\right\}\hat{\chi}_{t+1}$$

We manipulate the coefficient expressions,

$$1 + \frac{\left(1-\bar{Z}\right)\phi}{1-\phi}\left[\frac{\alpha\left(1-\theta\right)\bar{Z}}{1-\left(1-\theta\right)\bar{Z}} + \frac{\left(1-\alpha\right)\left(1-\xi\right)\bar{Z}}{1-\left(1-\xi\right)\bar{Z}} - \frac{1-\phi}{\phi}\right]$$

$$= 1 + \frac{\alpha\left(1-\theta\right)\bar{Z}}{1-\left(1-\theta\right)\bar{Z}}\frac{\left(1-\bar{Z}\right)\phi}{1-\phi} + \frac{\left(1-\alpha\right)\left(1-\xi\right)\bar{Z}}{1-\left(1-\xi\right)\bar{Z}}\frac{\left(1-\bar{Z}\right)\phi}{1-\phi} - 1 + \bar{Z}$$

$$= \bar{Z} + \frac{\left(1-\bar{Z}\right)\phi}{1-\phi}\left[\frac{\alpha\left(1-\theta\right)\bar{Z}}{1-\left(1-\theta\right)\bar{Z}} + \frac{\left(1-\alpha\right)\left(1-\xi\right)\bar{Z}}{1-\left(1-\xi\right)\bar{Z}}\right]$$

Using for compactness the omega variables

$$\widehat{\omega}_t = 1-\left(1-\theta\right)Z_t \Rightarrow \alpha\widehat{\omega}_t = \alpha - \alpha\left(1-\theta\right)Z_t \Rightarrow \alpha\left(1-\theta\right)Z = \alpha - \alpha\widehat{\omega}_t$$
$$\breve{\omega}_t = 1-\left(1-\xi\right)Z_t \Rightarrow \left(1-\alpha\right)\breve{\omega}_t = \left(1-\alpha\right) - \left(1-\alpha\right)\left(1-\xi\right)Z_t \Rightarrow \left(1-\alpha\right)\left(1-\xi\right)Z_t = \left(1-\alpha\right) - \left(1-\alpha\right)\breve{\omega}_t$$

$$\ldots = \bar{Z} + \frac{\left(1-\bar{Z}\right)\phi}{1-\phi}\left[\frac{\alpha - \alpha\bar{\widehat{\omega}}}{\bar{\widehat{\omega}}} + \frac{\left(1-\alpha\right) - \left(1-\alpha\right)\bar{\breve{\omega}}}{\bar{\breve{\omega}}}\right]$$
$$= \bar{Z} + \frac{\left(1-\bar{Z}\right)\phi}{1-\phi}\left[\frac{\alpha}{\bar{\widehat{\omega}}} - \alpha + \frac{\left(1-\alpha\right)}{\bar{\breve{\omega}}} - 1 + \alpha\right]$$
$$= \bar{Z} + \frac{\left(1-\bar{Z}\right)\phi}{1-\phi}\left[\frac{\alpha}{\bar{\widehat{\omega}}} + \frac{\left(1-\alpha\right)}{\bar{\breve{\omega}}} - 1\right]$$

Next,

$$\left\{\frac{\theta\varphi_2 + (1-\theta)\phi}{\phi\theta} + \frac{\left(1-\bar{Z}\right)\phi}{1-\phi}\left[\frac{\theta\varphi_2 + (1-\theta)\phi}{\theta\phi\left(1-\bar{\chi}\right)}\right]\left[\alpha\frac{\left(1-\theta\right)\bar{Z}}{1-\left(1-\theta\right)\bar{Z}} + \left(1-\alpha\right)\frac{\left(1-\xi\right)\bar{Z}}{1-\left(1-\xi\right)\bar{Z}} - \frac{1-\phi}{\phi}\right]\right\}$$

$$= \frac{\theta\varphi_2 + (1-\theta)\phi}{\phi\theta}\left\{1 + \frac{1}{(1-\bar{\chi})}\frac{(1-\bar{Z})\phi}{1-\phi}\left[\frac{\alpha(1-\theta)\bar{Z}}{1-(1-\theta)\bar{Z}} + \frac{(1-\alpha)(1-\xi)\bar{Z}}{1-(1-\xi)\bar{Z}} - \frac{1-\phi}{\phi}\right]\right\}$$

$$= \frac{\theta\varphi_2 + (1-\theta)\phi}{\phi\theta}\frac{1}{(1-\bar{\chi})}\left\{1-\bar{\chi} + \frac{\alpha(1-\theta)\bar{Z}}{1-(1-\theta)\bar{Z}}\frac{(1-\bar{Z})\phi}{1-\phi} + \frac{(1-\alpha)(1-\xi)\bar{Z}}{1-(1-\xi)\bar{Z}}\frac{(1-\bar{Z})\phi}{1-\phi} - 1 + \bar{Z}\right\}$$

$$= \frac{\theta\varphi_2 + (1-\theta)\phi}{\phi\theta(1-\bar{\chi})}\left\{\bar{Z} + \frac{(1-\bar{Z})\phi}{1-\phi}\left[\frac{\alpha(1-\theta)\bar{Z}}{1-(1-\theta)\bar{Z}} + \frac{(1-\alpha)(1-\xi)\bar{Z}}{1-(1-\xi)\bar{Z}}\right] - \bar{\chi}\right\}$$

Using the results just previously, $\ldots = \dfrac{\theta\varphi_2 + (1-\theta)\phi}{\phi\theta(1-\bar{\chi})}\left\{\bar{Z} + \dfrac{(1-\bar{Z})\phi}{1-\phi}\left[\dfrac{\alpha}{\bar{\bar{\omega}}} + \dfrac{(1-\alpha)}{\bar{\bar{\omega}}} - 1\right] - \bar{\chi}\right\}$

Inserting back into the expression for omega-bar,

$$\hat{\bar{\omega}}_{t+1} = \hat{\bar{A}}_{t+1} + \left[\bar{Z} + \frac{(1-\bar{Z})\phi}{1-\phi}\left(\frac{\alpha}{\bar{\bar{\omega}}} + \frac{(1-\alpha)}{\bar{\bar{\omega}}} - 1\right)\right]\hat{B}_{t+1}$$
$$+ \frac{\theta\varphi_2 + (1-\theta)\phi}{\phi\theta(1-\bar{\chi})}\left[\bar{Z} + \frac{(1-\bar{Z})\phi}{1-\phi}\left(\frac{\alpha}{\bar{\bar{\omega}}} + \frac{(1-\alpha)}{\bar{\bar{\omega}}} - 1\right) - \bar{\chi}\right]\hat{\chi}_{t+1}$$

Substituting into the main equation

$$-(1-\alpha)\hat{h}_{t+1} + \hat{y}_{t+1} = \alpha\hat{k}_{t+1} + \hat{\bar{A}}_{t+1} + \left[\bar{Z} + \frac{(1-\bar{Z})\phi}{1-\phi}\left(\frac{\alpha}{\bar{\bar{\omega}}} + \frac{(1-\alpha)}{\bar{\bar{\omega}}} - 1\right)\right]\hat{B}_{t+1}$$
$$+ \frac{\theta\varphi_2 + (1-\theta)\phi}{\phi\theta(1-\bar{\chi})}\left[\bar{Z} + \frac{(1-\bar{Z})\phi}{1-\phi}\left(\frac{\alpha}{\bar{\bar{\omega}}} + \frac{(1-\alpha)}{\bar{\bar{\omega}}} - 1\right) - \bar{\chi}\right]\hat{\chi}_{t+1}$$

We assign matrix coefficients and we separate the SS values of the unobservables only,

[L.2]

$$\theta_{21}\hat{h}_{t+1} + \theta_{22}\hat{y}_{t+1} = d_y + m_{21}\hat{k}_{t+1} + \xi_{21}\ln\tilde{A}_{t+1} + \xi_{22}\ln B_{t+1} + \xi_{23}\ln\chi_{t+1}$$

$$\theta_{21} = -(1-\alpha), \quad \theta_{22} = 1$$

$$m_{21} = \alpha,$$

$$\xi_{21} = 1, \quad \xi_{22} = \bar{Z} + \frac{(1-\bar{Z})\phi}{1-\phi}\left(\frac{\alpha}{\bar{\bar{\omega}}} + \frac{(1-\alpha)}{\bar{\bar{\omega}}} - 1\right), \quad \xi_{23} = (\xi_{22} - \bar{\chi})\frac{\theta\varphi_2 + (1-\theta)\phi}{\phi\theta(1-\bar{\chi})},$$

$$d_y = -\left(\xi_{21}\ln\bar{\bar{A}} + \xi_{22}\ln\bar{B} + \xi_{23}\ln\bar{\chi}\right)$$

C. Government expenses

Here, g_{t+1} is government outflows (government consumption, investment and transfers) as a %GDP. The postulated covariance relation is

$$g_{t+1}=\left\{\tau_{t+1}^{K}\left[\alpha\widehat{\omega}_{t+1}-\delta\left(\frac{k_{t+1}}{y_{t+1}}\right)\right]+(1-\alpha)\tau_{t+1}^{L}\tilde{\omega}_{t+1}\right\}\;+\;\hat{\psi}\chi_{t+1}+\;\widehat{\psi}\chi_{t+1}^{2}$$

$$=\tau_{t+1}^{K}\left[\alpha\widehat{\omega}_{t+1}-\delta\left(\frac{k_{t+1}}{y_{t+1}}\right)\right]\;+\;(1-\alpha)\tau_{t+1}^{L}\tilde{\omega}_{t+1}+\;\hat{\psi}\chi_{t+1}+\;\widehat{\psi}\chi_{t+1}^{2}$$

Note that

$$\tilde{\omega}_t\equiv1+\frac{\alpha}{1-\alpha}\left(1-\theta\right)Z_t\Rightarrow(1-\alpha)\tilde{\omega}_t=(1-\alpha)+\alpha\left(1-\theta\right)Z_t=1-\alpha\left[1-\left(1-\theta\right)Z_t\right]=1-\alpha\widehat{\omega}_t$$

So, bringing also all terms in one side

$$g_{t+1}-\tau_{t+1}^{K}\left[\alpha\widehat{\omega}_{t+1}-\delta\left(\frac{k_{t+1}}{y_{t+1}}\right)\right]\;-\;(1-\alpha)\tau_{t+1}^{L}\tilde{\omega}_{t+1}-\;\hat{\psi}\chi_{t+1}-\widehat{\psi}\chi_{t+1}^{2}=0$$

or using the Z_{t+1} variable,

$$g_{t+1}-\tau_{t+1}^{K}\left[\alpha\left(1-\left(1-\theta\right)Z_{t+1}\right)-\delta\frac{k_{t+1}}{y_{t+1}}\right]\;-\;\tau_{t+1}^{L}\left[1-\alpha\left(1-\left(1-\theta\right)Z_{t+1}\right)\right]$$
$$-\;\hat{\psi}\chi_{t+1}-\widehat{\psi}\chi_{t+1}^{2}=0$$

Steady state

$$\bar{g}-\bar{\tau}^{K}\left(\alpha\bar{\bar{\omega}}-\delta\frac{\bar{k}}{\bar{y}}\right)\;-\;\bar{\tau}^{L}\left(1-\alpha\bar{\bar{\omega}}\right)-\;\hat{\psi}\bar{\chi}-\;\widehat{\psi}\bar{\chi}^{2}=0$$

Log-linearization

$$\bar{g}\exp\left\{\hat{g}_{t+1}\right\}-\bar{\tau}^{K}\exp\left\{\hat{\tau}_{t+1}^{K}\right\}\left[\alpha\left(1-\left(1-\theta\right)\bar{Z}\exp\left\{\hat{Z}_{t+1}\right\}\right)-\delta\frac{\bar{k}\exp\left\{\hat{k}_{t+1}\right\}}{\bar{y}\exp\left\{\hat{y}_{t+1}\right\}}\right]$$

$$-\;\bar{\tau}^{L}\exp\left\{\hat{\tau}_{t+1}^{L}\right\}\left[1-\alpha\left(1-\left(1-\theta\right)\bar{Z}\exp\left\{\hat{Z}_{t+1}\right\}\right)\right]$$

$$-\;\hat{\psi}\bar{\chi}\exp\left\{\hat{\chi}_{t+1}\right\}\;-\;\widehat{\psi}\bar{\chi}^{2}\exp\left\{2\hat{\chi}_{t+1}\right\}\quad=\;0.$$

We have

$$\bar{g}\hat{g}_{t+1} - \bar{\tau}^K\left(\alpha\bar{\bar{\omega}} - \frac{\delta\bar{k}}{\bar{y}}\right)\hat{\tau}^K_{t+1} - \bar{\tau}^L\left(1-\alpha\bar{\bar{\omega}}\right)\hat{\tau}^L_{t+1} - \hat{\psi}\bar{\chi}\hat{\chi}_{t+1} - 2\widehat{\psi}\bar{\chi}^2\hat{\chi}_{t+1}$$

$$+\bar{\tau}^K\alpha\left(1-\theta\right)\bar{Z}\hat{Z}_{t+1} - \bar{\tau}^L\alpha\left(1-\theta\right)\bar{Z}\hat{Z}_{t+1} + \frac{\bar{\tau}^K\delta\bar{k}}{\bar{y}}\hat{k}_{t+1} - \frac{\bar{\tau}^K\delta\bar{k}}{\bar{y}}\hat{y}_{t+1} = 0$$

Compact and bring in the proper order,

$$-\frac{\bar{\tau}^K\delta\bar{k}}{\bar{y}}\hat{y}_{t+1} + \bar{g}\hat{g}_{t+1} = -\frac{\bar{\tau}^K\delta\bar{k}}{\bar{y}}\hat{k}_{t+1} + \bar{\tau}^K\left(\alpha\bar{\bar{\omega}} - \frac{\delta\bar{k}}{\bar{y}}\right)\hat{\tau}^K_{t+1} + \bar{\tau}^L\left(1-\alpha\bar{\bar{\omega}}\right)\hat{\tau}^L_{t+1}$$
$$+ \left(\hat{\psi}\bar{\chi} + 2\widehat{\psi}\bar{\chi}^2\right)\hat{\chi}_{t+1} + \bar{\psi}\left(\bar{\tau}^L - \bar{\tau}^K\right)\alpha\left(1-\theta\right)\bar{Z}\hat{Z}_{t+1}$$

Decomposing $\hat{Z}_{t+1}$

$$-\frac{\bar{\tau}^K\delta\bar{k}}{\bar{y}}\hat{y}_{t+1} + \bar{g}\hat{g}_{t+1} = -\frac{\bar{\tau}^K\delta\bar{k}}{\bar{y}}\hat{k}_{t+1} + \bar{\tau}^K\left(\alpha\bar{\bar{\omega}} - \frac{\delta\bar{k}}{\bar{y}}\right)\hat{\tau}^K_{t+1} + \bar{\tau}^L\left(1-\alpha\bar{\bar{\omega}}\right)\hat{\tau}^L_{t+1}$$
$$+ \left(\hat{\psi}\bar{\chi} + 2\widehat{\psi}\bar{\chi}^2\right)\hat{\chi}_{t+1}$$
$$+ \left(\bar{\tau}^L - \bar{\tau}^K\right)\alpha\left(1-\theta\right)\bar{Z}\left[\frac{\left(1-\bar{Z}\right)\phi}{1-\phi}\hat{B}_{t+1} + \frac{\left(1-\bar{Z}\right)\phi}{1-\phi}\left[\frac{\theta\varphi_2 + (1-\theta)\phi}{\theta\phi\left(1-\bar{\chi}\right)}\right]\hat{\chi}_{t+1}\right]$$

$$-\frac{\bar{\tau}^K\delta\bar{k}}{\bar{y}}\hat{y}_{t+1} + \bar{g}\hat{g}_{t+1} = -\frac{\bar{\tau}^K\delta\bar{k}}{\bar{y}}\hat{k}_{t+1} + \bar{\tau}^K\left(\alpha\bar{\bar{\omega}} - \frac{\delta\bar{k}}{\bar{y}}\right)\hat{\tau}^K_{t+1} + \bar{\tau}^L\left(1-\alpha\bar{\bar{\omega}}\right)\hat{\tau}^L_{t+1}$$
$$+ \left(\bar{\tau}^L - \bar{\tau}^K\right)\alpha\left(1-\theta\right)\bar{Z}\frac{\left(1-\bar{Z}\right)\phi}{1-\phi}\hat{B}_{t+1}$$
$$+\left[\left(\bar{\tau}^L - \bar{\tau}^K\right)\alpha\left(1-\theta\right)\bar{Z}\frac{\left(1-\bar{Z}\right)\phi}{1-\phi}\left[\frac{\theta\varphi_2 + (1-\theta)\phi}{\theta\phi\left(1-\bar{\chi}\right)}\right] + \hat{\psi}\bar{\chi} + 2\widehat{\psi}\bar{\chi}^2\right]\hat{\chi}_{t+1}$$

We allocate matrix coefficients and separate SS values of the unobservables only, using also the omega variables,

[L.3]

$$-\frac{\bar{\tau}^K \delta \bar{k}}{\bar{y}}\hat{y}_{t+1} + \bar{g}\hat{g}_{t+1} = -\frac{\bar{\tau}^K \delta \bar{k}}{\bar{y}}\hat{k}_{t+1} + \bar{\tau}^K\left(\alpha\bar{\bar{\omega}} - \frac{\delta\bar{k}}{\bar{y}}\right)\hat{\tau}^K_{t+1} + \bar{\tau}^L\left(1-\alpha\bar{\bar{\omega}}\right)\hat{\tau}^L_{t+1}$$

$$+ \left(\bar{\tau}^L - \bar{\tau}^K\right)\alpha\left(1-\theta\right)\bar{Z}\frac{\left(1-\bar{Z}\right)\phi}{1-\phi}\hat{B}_{t+1}$$

$$+\left[\left(\bar{\tau}^L - \bar{\tau}^K\right)\alpha\left(1-\theta\right)\bar{Z}\frac{\left(1-\bar{Z}\right)\phi}{1-\phi}\left[\frac{\theta\varphi_2 + (1-\theta)\phi}{\theta\phi\left(1-\bar{\chi}\right)}\right] + \hat{\psi}\bar{\chi} + 2\widehat{\psi}\bar{\chi}^2\right]\hat{\chi}_{t+1}$$

$$\theta_{32}\hat{y}_{t+1} + \theta_{33}\hat{g}_{t+1} = d_g + m_{31}\hat{k}_{t+1} + m_{32}\hat{\tau}^K_{t+1} + m_{33}\hat{\tau}^L_{t+1} + \xi_{32}\ln B_{t+1} + \xi_{33}\ln\chi_{t+1}$$

$$\theta_{32} = -\frac{\bar{\tau}^K \delta \bar{k}}{\bar{y}}, \quad \theta_{33} = \bar{g},$$

$$m_{31} = \theta_{32}, \quad m_{32} = \bar{\tau}^K\left(\alpha\bar{\bar{\omega}} - \frac{\delta\bar{k}}{\bar{y}}\right), \; m_{33} = \bar{\tau}^L\left(1-\alpha\bar{\bar{\omega}}\right)$$

$$\xi_{32} = \left(\bar{\tau}^L - \bar{\tau}^K\right)\alpha\left(1-\bar{\bar{\omega}}\right)\frac{\left(1-\bar{Z}\right)\phi}{1-\phi}, \quad \xi_{33} = \left[\xi_{32}\frac{\theta\varphi_2 + (1-\theta)\phi}{\theta\phi\left(1-\bar{\chi}\right)} + \hat{\psi}\bar{\chi} + 2\widehat{\psi}\bar{\chi}^2\right]$$

$$d_g = -\left(\xi_{32}\ln\bar{B} + \xi_{33}\ln\bar{\chi}\right)$$

D. Consumption

Eq.: $$\frac{[c_t^{\vartheta}(1-h_t)^{1-\vartheta}]^{1-\gamma}}{c_t} = \frac{\beta}{(1+\eta)} E_t\left\{\frac{[c_{t+1}^{\vartheta}(1-h_{t+1})^{1-\vartheta}]^{1-\gamma}}{c_{t+1}}\left[1+(1-\tau_{t+1}^K)\left(\alpha\widehat{\omega}_{t+1}\frac{y_{t+1}}{k_{t+1}}-\delta\right)\right]\right\}$$

Simplify, compact and introduce an expectational error term to eliminate the conditional expected value:

$$c_t^{\vartheta(1-\gamma)-1}(1-h_t)^{(1-\vartheta)(1-\gamma)} = \frac{\beta}{(1+\eta)} c_{t+1}^{\vartheta(1-\gamma)-1}(1-h_{t+1})^{(1-\vartheta)(1-\gamma)}\left[1+(1-\tau_{t+1}^K)\left(\alpha\widehat{\omega}_{t+1}\frac{y_{t+1}}{k_{t+1}}-\delta\right)\right]\exp\left\{-e_{c,t+1}\right\}$$

Take logarithms,

$$\Rightarrow \left[\vartheta(1-\gamma)-1\right]\ln c_t \;+\; (1-\vartheta)(1-\gamma)\ln(1-h_t) \;=\; \ln\left(\frac{\beta}{(1+\eta)}\right) \;+\; \left[\vartheta(1-\gamma)-1\right]\ln c_{t+1} \;+$$

$$+(1-\vartheta)(1-\gamma)\ln(1-h_{t+1}) \;+\; \ln\left[1+(1-\tau_{t+1}^K)\left(\alpha\widehat{\omega}_{t+1}\frac{y_{t+1}}{k_{t+1}}-\delta\right)\right] \;-\; e_{c,t+1}$$

Linearize what needs linearization:

$$\ln\left(1-h_{t+1}\right) = \ln\left(1-\bar{h}\exp\left\{\hat{h}_{t+1}\right\}\right) \approx \ln\left(1-\bar{h}\right) \;-\; \frac{\bar{h}}{1-\bar{h}}\hat{h}_{t+1}$$

and analogously for h_t

$$\ln\left(1-h_t\right) = \ln\left(1-\bar{h}\exp\left\{\hat{h}_t\right\}\right) \approx \ln\left(1-\bar{h}\right) \;-\; \frac{\bar{h}}{1-\bar{h}}\hat{h}_t$$

Also, for the last term

$$\ln\left[1+\left(1-\bar{\tau}^{K}\exp\left\{\hat{\tau}_{t+1}^{K}\right\}\right)\left(\alpha\left[1-(1-\theta)\bar{Z}\exp\left\{\hat{Z}_{t+1}\right\}\right]\frac{\bar{y}\exp\left\{\hat{y}_{t+1}\right\}}{\bar{k}\exp\left\{\hat{k}_{t+1}\right\}}-\delta\right)\right]$$

$$\approx\ln\left[1+\left(1-\bar{\tau}^{K}\right)\left(\alpha\left[1-(1-\theta)\bar{Z}\right]\frac{\bar{y}}{\bar{k}}-\delta\right)\right]$$

$$\frac{-\left(\alpha\left[1-(1-\theta)\bar{Z}\right]\frac{\bar{y}}{\bar{k}}-\delta\right)\bar{\tau}^{K}\hat{\tau}_{t+1}^{K}-\left(1-\bar{\tau}^{K}\right)\alpha(1-\theta)\bar{Z}\frac{\bar{y}}{\bar{k}}\hat{Z}_{t+1}}{\left[1+\left(1-\bar{\tau}^{K}\right)\left(\alpha\left[1-(1-\theta)\bar{Z}\right]\frac{\bar{y}}{\bar{k}}-\delta\right)\right]}+$$

$$+\quad\frac{\left(1-\bar{\tau}^{K}\right)\alpha\left[1-(1-\theta)\bar{Z}\right]\frac{\bar{y}}{\bar{k}}\hat{y}_{t+1}-\left(1-\bar{\tau}^{K}\right)\alpha\left[1-(1-\theta)\bar{Z}\right]\frac{\bar{y}}{\bar{k}}\hat{k}_{t+1}}{\left[1+\left(1-\bar{\tau}^{K}\right)\left(\alpha\left[1-(1-\theta)\bar{Z}\right]\frac{\bar{y}}{\bar{k}}-\delta\right)\right]}$$

At the perfect-foresight steady state we will have

$$\frac{\beta}{(1+\eta)}=\frac{1}{\left[1+\left(1-\bar{\tau}^{K}\right)\left(\alpha\left[1-(1-\theta)\bar{Z}\right]\frac{\bar{y}}{\bar{k}}-\delta\right)\right]}$$

Using this temporarily to compact a bit,

$$\ln\left[1+\left(1-\bar{\tau}^{K}\exp\left\{\hat{\tau}_{t+1}^{K}\right\}\right)\left(\alpha\left[1-(1-\theta)\bar{Z}\exp\left\{\hat{Z}_{t+1}\right\}\right]\frac{\bar{y}\exp\left\{\hat{y}_{t+1}\right\}}{\bar{k}\exp\left\{\hat{k}_{t+1}\right\}}-\delta\right)\right]$$

$$\approx-\ln\left[\frac{\beta}{(1+\eta)}\right]-\frac{\beta}{(1+\eta)}\left(\alpha\left[1-(1-\theta)\bar{Z}\right]\frac{\bar{y}}{\bar{k}}-\delta\right)\bar{\tau}^{K}\hat{\tau}_{t+1}^{K}-\frac{\beta}{(1+\eta)}\left(1-\bar{\tau}^{K}\right)\alpha(1-\theta)\bar{Z}\frac{\bar{y}}{\bar{k}}\hat{Z}_{t+1}$$

$$+\frac{\beta}{(1+\eta)}\left(1-\bar{\tau}^{K}\right)\alpha\left[1-(1-\theta)\bar{Z}\right]\frac{\bar{y}}{\bar{k}}\hat{y}_{t+1}-\frac{\beta}{(1+\eta)}\left(1-\bar{\tau}^{K}\right)\alpha\left[1-(1-\theta)\bar{Z}\right]\frac{\bar{y}}{\bar{k}}\hat{k}_{t+1}$$

So in all, the log-linearized (around the **perfect foresight** steady state, so $e_{c,t+1}=0$), Euler condition is written

$$\Rightarrow \left[\vartheta(1-\gamma)-1\right]\ln c_t \ + \ (1-\vartheta)(1-\gamma)\left[\ln\left(1-\bar{h}\right) \ - \ \frac{\bar{h}}{1-\bar{h}}\hat{h}_t \ \right] =$$

$$\ln\left(\frac{\beta}{(1+\eta)}\right) \ + \ \left[\vartheta(1-\gamma)-1\right]\ln c_{t+1} \ +$$

$$+(1-\vartheta)(1-\gamma)\left[\ln\left(1-\bar{h}\right) \ - \ \frac{\bar{h}}{1-\bar{h}}\hat{h}_{t+1}\right] \ -\ln\left[\frac{\beta}{(1+\eta)}\right]$$

$$-\frac{\beta}{(1+\eta)}\left(\alpha\left[1-\left(1-\theta\right)\bar{Z}\right]\frac{\bar{y}}{\bar{k}}-\delta\right)\bar{\tau}^K\hat{\tau}^K_{t+1} - \ \frac{\beta}{(1+\eta)}\left(1-\bar{\tau}^K\right)\alpha\left(1-\theta\right)\bar{Z}\,\frac{\bar{y}}{\bar{k}}\hat{Z}_{t+1}$$

$$+ \ \frac{\beta}{(1+\eta)}\left(1-\bar{\tau}^K\right)\alpha\left[1-\left(1-\theta\right)\bar{Z}\right]\frac{\bar{y}}{\bar{k}}\hat{y}_{t+1} - \frac{\beta}{(1+\eta)}\left(1-\bar{\tau}^K\right)\alpha\left[1-\left(1-\theta\right)\bar{Z}\right]\frac{\bar{y}}{\bar{k}}\hat{k}_{t+1}$$

and simplifying constants and terms equal on the two sides of the equation

$$\Rightarrow \left[\vartheta(1-\gamma)-1\right]\ln c_t \ - \ (1-\vartheta)(1-\gamma)\frac{\bar{h}}{1-\bar{h}}\hat{h}_t \ =$$

$$\left[\vartheta(1-\gamma)-1\right]\ln c_{t+1} \ -(1-\vartheta)(1-\gamma)\frac{\bar{h}}{1-\bar{h}}\hat{h}_{t+1}$$

$$-\frac{\beta}{(1+\eta)}\left(\alpha\left[1-\left(1-\theta\right)\bar{Z}\right]\frac{\bar{y}}{\bar{k}}-\delta\right)\bar{\tau}^K\hat{\tau}^K_{t+1} - \ \frac{\beta}{(1+\eta)}\left(1-\bar{\tau}^K\right)\alpha\left(1-\theta\right)\bar{Z}\,\frac{\bar{y}}{\bar{k}}\hat{Z}_{t+1}$$

$$+ \ \frac{\beta}{(1+\eta)}\left(1-\bar{\tau}^K\right)\alpha\left[1-\left(1-\theta\right)\bar{Z}\right]\frac{\bar{y}}{\bar{k}}\hat{y}_{t+1} - \frac{\beta}{(1+\eta)}\left(1-\bar{\tau}^K\right)\alpha\left[1-\left(1-\theta\right)\bar{Z}\right]\frac{\bar{y}}{\bar{k}}\hat{k}_{t+1}$$

Bring in the log-linearization of the Zeta variable

$$\Rightarrow \left[\vartheta(1-\gamma)-1\right]\ln c_t - (1-\vartheta)(1-\gamma)\frac{\bar{h}}{1-\bar{h}}\hat{h}_t =$$

$$\left[\vartheta(1-\gamma)-1\right]\ln c_{t+1} - (1-\vartheta)(1-\gamma)\frac{\bar{h}}{1-\bar{h}}\hat{h}_{t+1}$$

$$-\frac{\beta}{(1+\eta)}\left(\alpha\left[1-(1-\theta)\bar{Z}\right]\frac{\bar{y}}{\bar{k}}-\delta\right)\bar{\tau}^K\hat{\tau}^K_{t+1}$$

$$-\frac{\beta}{(1+\eta)}\left(1-\bar{\tau}^K\right)\alpha(1-\theta)\bar{Z}\frac{\bar{y}}{\bar{k}}\left[\frac{(1-\bar{Z})\phi}{1-\phi}\hat{B}_{t+1} + \frac{(1-\bar{Z})\phi}{1-\phi}\left[\frac{\theta\varphi_2+(1-\theta)\phi}{\theta\phi(1-\bar{\chi})}\right]\hat{\chi}_{t+1}\right]$$

$$+\frac{\beta}{(1+\eta)}\left(1-\bar{\tau}^K\right)\alpha\left[1-(1-\theta)\bar{Z}\right]\frac{\bar{y}}{\bar{k}}\hat{y}_{t+1}$$

$$-\frac{\beta}{(1+\eta)}\left(1-\bar{\tau}^K\right)\alpha\left[1-(1-\theta)\bar{Z}\right]\frac{\bar{y}}{\bar{k}}\hat{k}_{t+1}$$

Bring in the proper order and divide throughout by $\left[\vartheta(1-\gamma)-1\right]$

$$\Rightarrow -\frac{(1-\vartheta)(1-\gamma)}{\left[\vartheta(1-\gamma)-1\right]}\frac{\bar{h}}{1-\bar{h}}\hat{h}_{t+1} + \frac{\beta}{(1+\eta)}\frac{\left(1-\bar{\tau}^K\right)\alpha\left[1-(1-\theta)\bar{Z}\right]}{\left[\vartheta(1-\gamma)-1\right]}\frac{\bar{y}}{\bar{k}}\hat{y}_{t+1} + \ln c_{t+1}$$

$$=$$

$$-\frac{(1-\vartheta)(1-\gamma)}{\left[\vartheta(1-\gamma)-1\right]}\frac{\bar{h}}{1-\bar{h}}\hat{h}_t + \ln c_t$$

$$+\frac{\beta}{(1+\eta)}\left(1-\bar{\tau}^K\right)\alpha\frac{\left[1-(1-\theta)\bar{Z}\right]}{\left[\vartheta(1-\gamma)-1\right]}\frac{\bar{y}}{\bar{k}}\hat{k}_{t+1}$$

$$+\frac{\beta}{(1+\eta)}\frac{\bar{\tau}^K}{\left[\vartheta(1-\gamma)-1\right]}\left(\alpha\left[1-(1-\theta)\bar{Z}\right]\frac{\bar{y}}{\bar{k}}-\delta\right)\hat{\tau}^K_{t+1}$$

$$+\frac{\beta}{(1+\eta)}\frac{\left(1-\bar{\tau}^K\right)}{\left[\vartheta(1-\gamma)-1\right]}\alpha(1-\theta)\bar{Z}\frac{\bar{y}}{\bar{k}}\frac{(1-\bar{Z})\phi}{1-\phi}\hat{B}_{t+1}$$

$$+\frac{\beta}{(1+\eta)}\frac{\left(1-\bar{\tau}^K\right)}{\left[\vartheta(1-\gamma)-1\right]}\alpha(1-\theta)\bar{Z}\frac{\bar{y}}{\bar{k}}\frac{(1-\bar{Z})\phi}{1-\phi}\left[\frac{\theta\varphi_2+(1-\theta)\phi}{\theta\phi(1-\bar{\chi})}\right]\hat{\chi}_{t+1}$$

Assign matrix coefficients and separate SS values while canceling off those who appear on both sides.

One more tweak is that we can dispense with $\frac{\beta}{(1+\eta)}$ by using the steady state relation

$$\frac{\beta}{(1+\eta)} = \frac{1}{1+\left(1-\bar{\tau}^K\right)\left(\alpha\bar{\bar{\omega}}\left(\bar{y}/\bar{k}\right)-\delta\right)}$$

The efficiency parameter η is calibrated and not estimated, so by doing this we reduce by one the number of unknown parameters. With this we have

[L.4]

$$\theta_{41}\hat{h}_{t+1} + \theta_{42}\hat{y}_{t+1} + \theta_{44}\hat{c}_{t+1} = d_c + \lambda_{41}\hat{h}_t + \lambda_{44}\hat{c}_t$$

$$+ \mathrm{m}_{41}\hat{k}_{t+1} + \mathrm{m}_{42}\hat{\tau}^K_{t+1} + \xi_{42}\ln B_{t+1} + \xi_{43}\ln\chi_{t+1}$$

$$\theta_{41} = \lambda_{41} = -\frac{(1-\vartheta)(1-\gamma)}{\left[\vartheta(1-\gamma)-1\right]}\frac{\bar{h}}{\left(1-\bar{h}\right)} \quad , \quad \theta_{42} = \frac{\left(1-\bar{\tau}^K\right)\alpha\bar{\bar{\omega}}\left(\bar{y}/\bar{k}\right)}{\left[1+\left(1-\bar{\tau}^K\right)\left(\alpha\bar{\bar{\omega}}\left(\bar{y}/\bar{k}\right)-\delta\right)\right]\left[\vartheta(1-\gamma)-1\right]},$$

$$\theta_{44} = \lambda_{44} = 1$$

$$\mathrm{m}_{41} = \theta_{42}, \qquad \mathrm{m}_{42} = \frac{\bar{\tau}^K\left(\alpha\bar{\bar{\omega}}\left(\bar{y}/\bar{k}\right)-\delta\right)}{\left[1+\left(1-\bar{\tau}^K\right)\left(\alpha\bar{\bar{\omega}}\left(\bar{y}/\bar{k}\right)-\delta\right)\right]\left[\vartheta(1-\gamma)-1\right]}$$

$$\xi_{42} = \frac{\left(1-\bar{\tau}^K\right)\alpha\left(1-\bar{\bar{\omega}}\right)\left(\bar{y}/\bar{k}\right)}{\left[1+\left(1-\bar{\tau}^K\right)\left(\alpha\bar{\bar{\omega}}\left(\bar{y}/\bar{k}\right)-\delta\right)\right]\left[\vartheta(1-\gamma)-1\right]}\frac{\left(1-\bar{Z}\right)\phi}{1-\phi}, \qquad \xi_{43} = \xi_{42}\left[\frac{\theta\varphi_2+(1-\theta)\phi}{\theta\phi\left(1-\bar{\chi}\right)}\right]$$

$$d_c = -\left(\xi_{42}\ln\bar{B} + \xi_{43}\ln\bar{\chi}\right)$$

Note that the coefficients $\theta_{41} = \lambda_{41}$ will be fixed since they depend on the steady states of observables and on utility parameters that will be calibrated.

A4. Inverting the matrix Θ

I. Determinant

From the previous results, we have the matrix
$$\Theta = \begin{bmatrix} \theta_{11} & -1 & 0 & 1 \\ \theta_{21} & 1 & 0 & 0 \\ 0 & \theta_{32} & \theta_{33} & 0 \\ \theta_{41} & \theta_{42} & 0 & 1 \end{bmatrix}$$

Its determinant is (expanding based on the 3rd column)

$$|\Theta| = \theta_{33} \cdot \begin{vmatrix} \theta_{11} & -1 & 1 \\ \theta_{21} & 1 & 0 \\ \theta_{41} & \theta_{42} & 1 \end{vmatrix} = \theta_{33} \cdot \begin{vmatrix} \theta_{21} & 1 \\ \theta_{41} & \theta_{42} \end{vmatrix} + \theta_{33} \cdot \begin{vmatrix} \theta_{11} & -1 \\ \theta_{21} & 1 \end{vmatrix} = \theta_{33}\left(\theta_{21}\theta_{42} - \theta_{41}\right) + \theta_{33}\left(\theta_{11} + \theta_{21}\right)$$

$$\Rightarrow |\Theta| = \theta_{33}\left(\theta_{11} + \theta_{21} + \theta_{21}\theta_{42} - \theta_{41}\right)$$

II. Invertibility

$$\theta_{11} = \frac{1}{\left(1-\bar{h}\right)}, \quad \theta_{21} = -\left(1-\alpha\right), \quad \theta_{32} = \frac{\bar{\tau}^K \delta\left(\bar{k}/\bar{y}\right)}{\bar{\tau}^L\left(1-\alpha\bar{\bar{\omega}}\right)}, \quad \theta_{33} = \bar{g}$$

$$\theta_{41} = -\frac{(1-\vartheta)(1-\gamma)}{\left[\vartheta(1-\gamma)-1\right]}\frac{\bar{h}}{\left(1-\bar{h}\right)} \quad , \quad \theta_{42} = \frac{\beta}{(1+\eta)}\frac{\left(1-\bar{\tau}^K\right)\alpha\bar{\bar{\omega}}\left(\bar{y}/\bar{k}\right)}{\left[\vartheta(1-\gamma)-1\right]}.$$

Since $\bar{g} > 0$ always, we need only examine

$$\theta_{11} + \theta_{21} + \theta_{21}\theta_{42} - \theta_{41} =$$

$$= \frac{1}{\left(1-\bar{h}\right)} - \left(1-\alpha\right) - \left(1-\alpha\right)\frac{\beta}{(1+\eta)}\frac{\left(1-\bar{\tau}^K\right)\alpha\bar{\bar{\omega}}\left(\bar{y}/\bar{k}\right)}{\left[\vartheta(1-\gamma)-1\right]} + \frac{(1-\vartheta)(1-\gamma)}{\left[\vartheta(1-\gamma)-1\right]}\frac{\bar{h}}{\left(1-\bar{h}\right)}$$

Note that for well-behaved utility we require $0 < \vartheta < 1$ while for bounded utility we need $\gamma > 1$. To make clear the sign of each component we re-write

$\theta_{11}+\theta_{21}+\theta_{21}\theta_{42}-\theta_{41}=$

$$=\frac{1}{\left(1-\bar{h}\right)} \;-\left(1-\alpha\right)+\left(1-\alpha\right)\frac{\beta}{(1+\eta)}\frac{\left(1-\bar{\tau}^{K}\right)\alpha\bar{\bar{\omega}}\left(\bar{y}/\bar{k}\right)}{\left[\vartheta(\gamma-1)+1\right]}+\frac{(1-\vartheta)(\gamma-1)}{\left[\vartheta(\gamma-1)+1\right]}\frac{\bar{h}}{\left(1-\bar{h}\right)}$$

The last two components of the sum are therefore positive. Moreover, since $0<\bar{h}<1,\;\; 0<\alpha<1$

we have that $\dfrac{1}{\left(1-\bar{h}\right)}\;-\left(1-\alpha\right)>0$.

So we conclude that the determinant of Θ is always strictly positive, given a priori restrictions on the parameters to make economic sense. So the matrix is invertible.

III. Inversion

We will invert it using its **cofactors** (minors including the sign).

$$|C_{11}|=\begin{vmatrix}1&0&0\\ \theta_{32}&\theta_{33}&0\\ \theta_{42}&0&1\end{vmatrix}=1\cdot\begin{vmatrix}\theta_{33}&0\\ 0&1\end{vmatrix}=\theta_{33},\quad |C_{12}|=-\begin{vmatrix}\theta_{21}&0&0\\ 0&\theta_{33}&0\\ \theta_{41}&0&1\end{vmatrix}=-\theta_{21}\begin{vmatrix}\theta_{33}&0\\ 0&1\end{vmatrix}=-\theta_{21}\theta_{33},$$

$$|C_{13}|=\begin{vmatrix}\theta_{21}&1&0\\ 0&\theta_{32}&0\\ \theta_{41}&\theta_{42}&1\end{vmatrix}=\theta_{32}\cdot\begin{vmatrix}\theta_{21}&0\\ \theta_{41}&1\end{vmatrix}=\theta_{21}\theta_{32},\quad |C_{14}|=-\begin{vmatrix}\theta_{21}&1&0\\ 0&\theta_{32}&\theta_{33}\\ \theta_{41}&\theta_{42}&0\end{vmatrix}=-(-\theta_{33})\begin{vmatrix}\theta_{21}&1\\ \theta_{41}&\theta_{42}\end{vmatrix}=\theta_{33}\left(\theta_{21}\theta_{42}-\theta_{41}\right).$$

$$|C_{21}|=-\begin{vmatrix}-1&0&1\\ \theta_{32}&\theta_{33}&0\\ \theta_{42}&0&1\end{vmatrix}=-\theta_{42}\begin{vmatrix}0&1\\ \theta_{33}&0\end{vmatrix}-1\cdot\begin{vmatrix}-1&0\\ \theta_{32}&\theta_{33}\end{vmatrix}=\theta_{42}\theta_{33}+\theta_{33}=\theta_{33}\left(1+\theta_{42}\right),$$

$$|C_{22}|=\begin{vmatrix}\theta_{11}&0&1\\ 0&\theta_{33}&0\\ \theta_{41}&0&1\end{vmatrix}=\theta_{33}\cdot\begin{vmatrix}\theta_{11}&1\\ \theta_{41}&1\end{vmatrix}=\theta_{33}\left(\theta_{11}-\theta_{41}\right),$$

$$|C_{23}|=-\begin{vmatrix}\theta_{11}&-1&1\\ 0&\theta_{32}&0\\ \theta_{41}&\theta_{42}&1\end{vmatrix}=-\theta_{32}\begin{vmatrix}\theta_{11}&1\\ \theta_{41}&1\end{vmatrix}=-\theta_{32}\left(\theta_{11}-\theta_{41}\right),$$

$$|C_{24}|=\begin{vmatrix}\theta_{11}&-1&0\\ 0&\theta_{32}&\theta_{33}\\ \theta_{41}&\theta_{42}&0\end{vmatrix}=-\theta_{33}\cdot\begin{vmatrix}\theta_{11}&-1\\ \theta_{41}&\theta_{42}\end{vmatrix}=-\theta_{33}\theta_{11}\theta_{42}\;-\theta_{33}\theta_{41}=-\theta_{33}\left(\theta_{11}\theta_{42}\;+\theta_{41}\right)$$

$$|C_{31}| = \begin{vmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ \theta_{42} & 0 & 1 \end{vmatrix} = 0, \qquad |C_{32}| = -\begin{vmatrix} \theta_{11} & 0 & 1 \\ \theta_{21} & 0 & 0 \\ \theta_{41} & 0 & 1 \end{vmatrix} = 0,$$

$$|C_{33}| = \begin{vmatrix} \theta_{11} & -1 & 1 \\ \theta_{21} & 1 & 0 \\ \theta_{41} & \theta_{42} & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} \theta_{21} & 1 \\ \theta_{41} & \theta_{42} \end{vmatrix} + 1 \cdot \begin{vmatrix} \theta_{11} & -1 \\ \theta_{21} & 1 \end{vmatrix} = \theta_{21}\theta_{42} - \theta_{41} + \theta_{11} + \theta_{21} = \frac{|\Theta|}{\theta_{33}}$$

$$|C_{34}| = -\begin{vmatrix} \theta_{11} & -1 & 0 \\ \theta_{21} & 1 & 0 \\ \theta_{41} & \theta_{42} & 0 \end{vmatrix} = 0$$

$$|C_{41}| = -\begin{vmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ \theta_{32} & \theta_{33} & 0 \end{vmatrix} = -(1) \cdot \begin{vmatrix} 1 & 0 \\ \theta_{32} & \theta_{33} \end{vmatrix} = -\theta_{33}, \quad |C_{42}| = \begin{vmatrix} \theta_{11} & 0 & 1 \\ \theta_{21} & 0 & 0 \\ 0 & \theta_{33} & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} \theta_{21} & 0 \\ 0 & \theta_{33} \end{vmatrix} = \theta_{21}\theta_{33},$$

$$|C_{43}| = -\begin{vmatrix} \theta_{11} & -1 & 1 \\ \theta_{21} & 1 & 0 \\ 0 & \theta_{32} & 0 \end{vmatrix} = -1 \cdot \begin{vmatrix} \theta_{21} & 1 \\ 0 & \theta_{32} \end{vmatrix} = -\theta_{21}\theta_{32},$$

$$|C_{44}| = \begin{vmatrix} \theta_{11} & -1 & 0 \\ \theta_{21} & 1 & 0 \\ 0 & \theta_{32} & \theta_{33} \end{vmatrix} = \theta_{33} \begin{vmatrix} \theta_{11} & -1 \\ \theta_{21} & 1 \end{vmatrix} = \theta_{33}\left(\theta_{11} + \theta_{21}\right).$$

Then

$$\boxed{\begin{gathered} \Theta^{-1} = |\Theta|^{-1} \begin{bmatrix} \theta_{33} & \theta_{33}\left(1+\theta_{42}\right) & 0 & -\theta_{33} \\ -\theta_{21}\theta_{33} & \theta_{33}\left(\theta_{11}-\theta_{41}\right) & 0 & \theta_{21}\theta_{33} \\ \theta_{21}\theta_{32} & -\theta_{32}\left(\theta_{11}-\theta_{41}\right) & |\Theta|/\theta_{33} & -\theta_{21}\theta_{32} \\ \theta_{33}\left(\theta_{21}\theta_{42}-\theta_{41}\right) & -\theta_{33}\left(\theta_{11}\theta_{42}+\theta_{41}\right) & 0 & \theta_{33}\left(\theta_{11}+\theta_{21}\right) \end{bmatrix}, \\ |\Theta| = \theta_{33}\left(\theta_{11} + \theta_{21} + \theta_{21}\theta_{42} - \theta_{41}\right) \end{gathered}}$$

A5. Obtaining the matrix Θ from the reduced-form estimates.

We take into account that that the parameters $\eta,\ \delta,\ \gamma,\ \varphi_1,\ \varphi_2$, if and when they appear in the expressions below, are fixed outside the estimation procedure and of known value.

We have

$$\Gamma = \frac{1}{|\Theta|}\begin{bmatrix} -\theta_{33}\lambda_{41} & 0 & 0 & -\theta_{33} \\ \theta_{21}\theta_{33}\lambda_{41} & 0 & 0 & \theta_{21}\theta_{33} \\ -\theta_{21}\theta_{32}\lambda_{41} & 0 & 0 & -\theta_{21}\theta_{32} \\ \theta_{33}\left(\theta_{11}+\theta_{21}\right)\lambda_{41} & 0 & 0 & \theta_{33}\left(\theta_{11}+\theta_{21}\right) \end{bmatrix} = \begin{bmatrix} \gamma_{11} & 0 & 0 & \gamma_{14} \\ \gamma_{21} & 0 & 0 & \gamma_{24} \\ \gamma_{31} & 0 & 0 & \gamma_{34} \\ \gamma_{41} & 0 & 0 & \gamma_{44} \end{bmatrix}$$

and

$$\Theta = \begin{bmatrix} \theta_{11} & -1 & 0 & 1 \\ \theta_{21} & 1 & 0 & 0 \\ 0 & \theta_{32} & \theta_{33} & 0 \\ \theta_{41} & \theta_{42} & 0 & 1 \end{bmatrix} \text{ with determinant } |\Theta| = \theta_{33}\left(\theta_{11}+\theta_{21}+\theta_{21}\theta_{42}-\theta_{41}\right)$$

We have the relations

$$\theta_{11} = \frac{1}{\left(1-\bar{h}\right)},\quad \theta_{12} = -1,\qquad \theta_{14} = 1,$$

$$\theta_{21} = -\left(1-\alpha\right),\qquad \theta_{22} = 1$$

$$\theta_{32} = -\frac{\bar{\tau}^K \delta \bar{k}}{\bar{y}},\qquad \theta_{33} = \bar{g},$$

$$\theta_{41} = \lambda_{41} = -\frac{(1-\vartheta)(1-\gamma)}{\left[\vartheta(1-\gamma)-1\right]}\frac{\bar{h}}{\left(1-\bar{h}\right)}\quad,\qquad \theta_{42} = \frac{\left(1-\bar{\tau}^K\right)\alpha\bar{\bar{\omega}}\left(\bar{y}/\bar{k}\right)}{\left[1+\left(1-\bar{\tau}^K\right)\left(\alpha\bar{\bar{\omega}}\left(\bar{y}/\bar{k}\right)-\delta\right)\right]\left[\vartheta(1-\gamma)-1\right]},$$

$$\theta_{44} = \lambda_{44} = 1$$

- $\theta_{11} = \dfrac{1}{\left(1-\bar{h}\right)}$. It can be computed using the empirical time-series mean of *h*
- $\theta_{21} = -\left(1-\alpha\right)$. θ_{21} can be computed from $\quad -\dfrac{\gamma_{24}}{\gamma_{14}} = -\dfrac{|\Theta|^{-1}}{|\Theta|^{-1}}\dfrac{\theta_{21}\theta_{33}}{-\theta_{33}} = \theta_{21}$
- $\theta_{32} = -\dfrac{\bar{\tau}^K \delta \bar{k}}{\bar{y}}$. It can be computed from empirical time series mean of the data, and the calibrated depreciation parameter.

- $\theta_{33} = \bar{g}$ It is equal to the empirical time series mean of government expenses.

- $\theta_{41} = \lambda_{41} = -\dfrac{(1-\vartheta)(1-\gamma)}{\left[\vartheta(1-\gamma)-1\right]}\dfrac{\bar{h}}{\left(1-\bar{h}\right)}$ can be computed from $\dfrac{\gamma_{11}}{\gamma_{14}} = \dfrac{|\Theta|^{-1}}{|\Theta|^{-1}}\dfrac{-\theta_{33}\lambda_{41}}{-\theta_{33}} = \lambda_{41} = \theta_{41}$

- From this we can obtain the value of ϑ using

$$\theta_{41} = -\frac{(1-\vartheta)(1-\gamma)}{\left[\vartheta(1-\gamma)-1\right]}\frac{\bar{h}}{\left(1-\bar{h}\right)} = -\frac{(1-\vartheta)(\gamma-1)}{\left[\vartheta(\gamma-1)+1\right]}\frac{\bar{h}}{\left(1-\bar{h}\right)}$$

$$\Rightarrow \theta_{41}\left(1-\bar{h}\right)\left[\vartheta(\gamma-1)+1\right] = -(\gamma-1)\bar{h} \;+\; \vartheta(\gamma-1)\bar{h}$$

$$\Rightarrow \theta_{41}\left(1-\bar{h}\right) + \theta_{41}\left(1-\bar{h}\right)\vartheta(\gamma-1) = -(\gamma-1)\bar{h} \;+\; \vartheta(\gamma-1)\bar{h}$$

$$\Rightarrow \theta_{41}\left(1-\bar{h}\right) + (\gamma-1)\bar{h} = \vartheta(\gamma-1)\bar{h} - \theta_{41}\left(1-\bar{h}\right)\vartheta(\gamma-1)$$

$$\Rightarrow \vartheta = \frac{(\gamma-1)\bar{h} + \theta_{41}\left(1-\bar{h}\right)}{(\gamma-1)\bar{h} - \theta_{41}\left(1-\bar{h}\right)(\gamma-1)} = \frac{\bar{h} + \theta_{41}\left[\left(1-\bar{h}\right)/(\gamma-1)\right]}{\bar{h} - \theta_{41}\left(1-\bar{h}\right)}$$

- $$\theta_{42} = \frac{\left(1-\bar{\tau}^{K}\right)\alpha\bar{\bar{\omega}}\left(\bar{y}/\bar{k}\right)}{\left[1+\left(1-\bar{\tau}^{K}\right)\left(\alpha\bar{\bar{\omega}}\left(\bar{y}/\bar{k}\right)-\delta\right)\right]\left[\vartheta(1-\gamma)-1\right]}$$

It can be obtained using the previous results and the expression of the determinant $|\Theta|$,

$$\gamma_{14} = \frac{-\theta_{33}}{|\Theta|} = \frac{-\theta_{33}}{\theta_{33}\left(\theta_{11}+\theta_{21}+\theta_{21}\theta_{42}-\theta_{41}\right)} \Rightarrow \theta_{11}+\theta_{21}+\theta_{21}\theta_{42}-\theta_{41} = -\frac{1}{\gamma_{14}}$$

$$\Rightarrow \theta_{21}\theta_{42} = -\frac{1}{\gamma_{14}} - \theta_{11} - \theta_{21} + \theta_{41}$$

$$\Rightarrow \theta_{42} = -\frac{1}{\theta_{21}\gamma_{14}} - \frac{\theta_{11}}{\theta_{21}} - 1 + \frac{\theta_{41}}{\theta_{21}}$$

This concludes the recovery of the elements of matrix Θ .

A6. Recovering steady states and model unknowns.

1. Recovering the steady-state values of the unobservables.

With the matrix Θ available, we can obtain the matrix Ξ by pre-multiplication $\Xi = \Theta\Pi$.

With Ξ available we can obtain the steady states of the unobservable state variables as follows:

We have from the structural system of equations

$$d_h = -\left(\xi_{12} \ln \bar{B} + \xi_{13} \ln \bar{\chi}\right)$$

$$d_y = -\left(\xi_{21} \ln \bar{\tilde{A}} + \xi_{22} \ln \bar{B} + \xi_{23} \ln \bar{\chi}\right)$$

$$d_g = -\left(\xi_{32} \ln \bar{B} + \xi_{33} \ln \bar{\chi}\right)$$

$$d_c = -\left(\xi_{42} \ln \bar{B} + \xi_{43} \ln \bar{\chi}\right)$$

This can be written as an over-identified system

$$\begin{bmatrix} 0 & \xi_{12} & \xi_{13} \\ 1 & \xi_{22} & \xi_{23} \\ 0 & \xi_{32} & \xi_{33} \\ 0 & \xi_{42} & \xi_{43} \end{bmatrix} \begin{bmatrix} \ln \bar{\tilde{A}} \\ \ln \bar{B} \\ \ln \bar{\chi} \end{bmatrix} = -\begin{bmatrix} d_h \\ d_y \\ d_g \\ d_c \end{bmatrix}$$

We have in generic notation

$$\Xi \mathbf{w} = -\mathbf{d} \Rightarrow \mathbf{w} = -\left(\Xi'\Xi\right)^{-1} \Xi' \mathbf{d}$$

The vector $\mathbf{w}$ contains the logarithms of the steady-state values we want.

Exponentiating element-by-element $\mathbf{w}$ we will obtain the SS values of the unobservables, $\bar{\tilde{A}}, \bar{B}, \bar{\chi}$

$$\begin{bmatrix} \bar{\tilde{A}} \\ \bar{B} \\ \bar{\chi} \end{bmatrix} = \exp\left\{-\left(\Xi'\Xi\right)^{-1} \Xi' \mathbf{d}\right\}$$

2. Recovering the model unknowns.

- From $\theta_{21} = -(1-\alpha)$ we can compute $\alpha = 1+\theta_{21}$

- From the obtained value for θ_{42} we can retrieve the value of $\bar{\bar{\omega}}$

$$\theta_{42} = \frac{\left(1-\bar{\tau}^K\right)\alpha\bar{\bar{\omega}}\left(\bar{y}/\bar{k}\right)}{\left[1+\left(1-\bar{\tau}^K\right)\left(\alpha\bar{\bar{\omega}}\left(\bar{y}/\bar{k}\right)-\delta\right)\right]\left[\vartheta(1-\gamma)-1\right]}$$

$$\Rightarrow \left[1+\left(1-\bar{\tau}^K\right)\left(\alpha\bar{\bar{\omega}}\left(\bar{y}/\bar{k}\right)-\delta\right)\right]\left[\vartheta(1-\gamma)-1\right]\theta_{42} = \left(1-\bar{\tau}^K\right)\left(\bar{y}/\bar{k}\right)\alpha\bar{\bar{\omega}}$$

$$\Rightarrow \left[\vartheta(1-\gamma)-1\right]\theta_{42} + \left[\vartheta(1-\gamma)-1\right]\theta_{42}\left(1-\bar{\tau}^K\right)\left(\alpha\bar{\bar{\omega}}\left(\bar{y}/\bar{k}\right)-\delta\right) = \left(1-\bar{\tau}^K\right)\left(\bar{y}/\bar{k}\right)\alpha\bar{\bar{\omega}}$$

$$\Rightarrow -\left[\vartheta(\gamma-1)+1\right]\theta_{42} - \left[\vartheta(\gamma-1)+1\right]\theta_{42}\left(1-\bar{\tau}^K\right)\alpha\left(\bar{y}/\bar{k}\right)\bar{\bar{\omega}} + \left[\vartheta(\gamma-1)+1\right]\theta_{42}\left(1-\bar{\tau}^K\right)\delta$$

$$= \left(1-\bar{\tau}^K\right)\left(\bar{y}/\bar{k}\right)\alpha\bar{\bar{\omega}}$$

$$\Rightarrow -\left[\vartheta(\gamma-1)+1\right]\left[1-\left(1-\bar{\tau}^K\right)\delta\right]\theta_{42} = \left[1+\left[\vartheta(\gamma-1)+1\right]\theta_{42}\right]\left(1-\bar{\tau}^K\right)\left(\bar{y}/\bar{k}\right)\alpha\bar{\bar{\omega}}$$

$$\Rightarrow \bar{\bar{\omega}} = \frac{-\left[1-\left(1-\bar{\tau}^K\right)\delta\right]\left[\vartheta(\gamma-1)+1\right]\theta_{42}}{\left[1+\left[\vartheta(\gamma-1)+1\right]\theta_{42}\right]\left(1-\bar{\tau}^K\right)\left(\bar{y}/\bar{k}\right)\alpha}$$

We next use the coefficients of the Ξ matrix,

$$\xi_{12} = -\frac{1-\bar{\bar{\omega}}}{\bar{\bar{\omega}}}\frac{\left(1-\bar{Z}\right)\phi}{1-\phi}, \quad \xi_{13} = \xi_{12}\frac{\theta\varphi_2 + (1-\theta)\phi}{\theta\phi\left(1-\bar{\chi}\right)}$$

$$\xi_{21} = 1, \qquad \xi_{22} = \bar{Z} + \frac{\left(1-\bar{Z}\right)\phi}{1-\phi}\left(\frac{\alpha}{\bar{\bar{\omega}}} + \frac{(1-\alpha)}{\bar{\bar{\omega}}} - 1\right), \qquad \xi_{23} = \left(\xi_{22}-\bar{\chi}\right)\frac{\theta\varphi_2 + (1-\theta)\phi}{\phi\theta\left(1-\bar{\chi}\right)}$$

$$\xi_{32} = \left(\bar{\tau}^L - \bar{\tau}^K\right)\alpha\left(1-\bar{\bar{\omega}}\right)\frac{\left(1-\bar{Z}\right)\phi}{1-\phi}, \quad \xi_{33} = \left[\xi_{32}\frac{\theta\varphi_2 + (1-\theta)\phi}{\theta\phi\left(1-\bar{\chi}\right)} + \hat{\psi}\bar{\chi} + 2\hat{\psi}\bar{\chi}^2\right]$$

$$\xi_{42} = \frac{\left(1-\bar{\tau}^K\right)\alpha\left(1-\bar{\bar{\omega}}\right)\left(\bar{y}/\bar{k}\right)}{\left[1+\left(1-\bar{\tau}^K\right)\left(\alpha\bar{\bar{\omega}}\left(\bar{y}/\bar{k}\right)-\delta\right)\right]\left[\vartheta(1-\gamma)-1\right]}\frac{\left(1-\bar{Z}\right)\phi}{1-\phi}, \qquad \xi_{43} = \xi_{42}\left[\frac{\theta\varphi_2 + (1-\theta)\phi}{\theta\phi\left(1-\bar{\chi}\right)}\right]$$

We start by computing the composite $\frac{\left(1-\bar{Z}\right)\phi}{1-\phi}$

$$\xi_{32}=\left(\bar{\tau}^{L}-\bar{\tau}^{K}\right)\alpha\left(1-\bar{\bar{\omega}}\right)\frac{\left(1-\bar{Z}\right)\phi}{1-\phi}$$

$$\Rightarrow\frac{\left(1-\bar{Z}\right)\phi}{1-\phi}=\frac{\xi_{32}}{\left(\bar{\tau}^{L}-\bar{\tau}^{K}\right)\alpha\left(1-\bar{\bar{\omega}}\right)}.$$

- Then we can obtain the value of $\bar{\bar{\omega}}$ by

$$\xi_{12}=-\frac{1-\bar{\bar{\omega}}}{\bar{\bar{\omega}}}\frac{\left(1-\bar{Z}\right)\phi}{1-\phi}=-\frac{1-\bar{\bar{\omega}}}{\bar{\bar{\omega}}}\frac{\xi_{32}}{\left(\bar{\tau}^{L}-\bar{\tau}^{K}\right)\alpha\left(1-\bar{\bar{\omega}}\right)}$$

$$\Rightarrow\bar{\bar{\omega}}\left(\bar{\tau}^{L}-\bar{\tau}^{K}\right)\alpha\left(1-\bar{\bar{\omega}}\right)\xi_{12}=-\xi_{32}+\bar{\bar{\omega}}\xi_{32}$$

$$\Rightarrow\bar{\bar{\omega}}=\frac{\xi_{32}}{\xi_{32}-\left(\bar{\tau}^{L}-\bar{\tau}^{K}\right)\alpha\left(1-\bar{\bar{\omega}}\right)\xi_{12}}$$

- Now that we have obtain the $\bar{\bar{\omega}}$ value and we also have the value of the composite $\frac{\left(1-\bar{Z}\right)\phi}{1-\phi}$ we can get the value for $\bar{Z}$ by

$$\xi_{22}=\bar{Z}+\frac{\left(1-\bar{Z}\right)\phi}{1-\phi}\left(\frac{\alpha}{\bar{\bar{\omega}}}+\frac{\left(1-\alpha\right)}{\bar{\bar{\omega}}}-1\right)\Rightarrow$$

$$\bar{Z}=\xi_{22}-\frac{\xi_{32}}{\left(\bar{\tau}^{L}-\bar{\tau}^{K}\right)\alpha\left(1-\bar{\bar{\omega}}\right)}\left(\frac{\alpha}{\bar{\bar{\omega}}}+\frac{\left(1-\alpha\right)}{\bar{\bar{\omega}}}-1\right)$$

- We can then obtain ϕ from

$$\frac{\left(1-\bar{Z}\right)\phi}{1-\phi}=\frac{\xi_{32}}{\left(\bar{\tau}^{L}-\bar{\tau}^{K}\right)\alpha\left(1-\bar{\bar{\omega}}\right)}\Rightarrow\left(1-\bar{Z}\right)\left(\bar{\tau}^{L}-\bar{\tau}^{K}\right)\alpha\left(1-\bar{\bar{\omega}}\right)\phi=\xi_{32}-\phi\xi_{32}$$

$$\Rightarrow\phi=\frac{\xi_{32}}{\xi_{32}+\left(1-\bar{Z}\right)\left(\bar{\tau}^{L}-\bar{\tau}^{K}\right)\alpha\left(1-\bar{\bar{\omega}}\right)}$$

- Then we can obtain θ from

$$\xi_{23} = (\xi_{22} - \bar{\chi})\frac{\theta\varphi_2 + (1-\theta)\phi}{\phi\theta(1-\bar{\chi})} \Rightarrow \phi\theta(1-\bar{\chi})\xi_{23} = (\xi_{22} - \bar{\chi})\theta\varphi_2 + (\xi_{22} - \bar{\chi})\phi - \phi\theta(\xi_{22} - \bar{\chi})$$

$$\left[\phi(1-\bar{\chi})\xi_{23} - (\xi_{22} - \bar{\chi})\varphi_2 + \phi(\xi_{22} - \bar{\chi})\right]\theta = (\xi_{22} - \bar{\chi})\phi$$

$$\Rightarrow \theta = \frac{(\xi_{22} - \bar{\chi})\phi}{\phi(1-\bar{\chi})\xi_{23} + (\phi - \varphi_2)(\xi_{22} - \bar{\chi})}$$

- The parameter ξ can be obtained from

$$\bar{\bar{\omega}} = 1-(1-\xi)\bar{Z} = 1- \bar{Z} + \xi\bar{Z}$$

$$\Rightarrow \xi = \frac{\bar{Z} - 1 + \bar{\bar{\omega}}}{\bar{Z}}$$

Note: In the theoretical model, we have the relation

$$\xi = \frac{(1-\alpha)\theta + \{1-\alpha\theta - [(1-\alpha\theta) + (1-\theta)]\lambda\}\left(\frac{\mu}{1-\mu}\right)}{(1-\alpha)\left[1 + (1-\lambda)\left(\frac{\mu}{1-\mu}\right)\right]}$$

where the parameters λ, μ are calibrated outside the model. By using the obtained estimates for ξ, α, θ we can determine the range of permissible combinations of λ, μ that are consistent with the sample.

- We can compute $\breve{\bar{\omega}} = 1 + \frac{\alpha}{1-\alpha}(1-\theta)\bar{Z}$

- To compute $\bar{\bar{\omega}}$ we first compute $\bar{\Delta}$ and then use its definition.

- **3. Alternative formulas**

We can also compute the following alternative formulas for some of the above

- An alternative way to compute θ is from

$$\bar{\bar{\omega}} \equiv 1-(1-\theta)\bar{Z} \Rightarrow \bar{\bar{\omega}} = 1 - \bar{Z} + \theta\bar{Z}$$

$$\Rightarrow \theta = \frac{\bar{Z} - 1 + \bar{\bar{\omega}}}{\bar{Z}}$$

- We can compute $\bar{Z}$ after computing $\bar{\Delta}$.

A7. Going from yearly to quarterly frequency in autoregressive schemes

Assume a yearly relation (we care about the variances, so there is no need to include a drift) of an index

$$x_{t+1} = \rho x_t + u_{t+1}$$

Break the period to go from *t* to *t*+1 in four quarters,

$$\begin{aligned}
&x_t \\
&x_{q1} = \rho_q x_t + u_{q1} \\
&x_{q2} = \rho_q x_{q1} + u_{q2} = \rho_q^2 x_t + \rho_q u_{q1} + u_{q2} \\
&x_{q3} = \rho_q x_{q2} + u_{q3} = \rho_q^3 x_t + \rho_q^2 u_{q1} + \rho_q u_{q2} + u_{q3} \\
&x_{t+1} = x_{q4} = \rho_q x_{q3} + u_{q4} = \rho_q^4 x_t + \rho_q^3 u_{q1} + \rho_q^2 u_{q2} + \rho_q u_{q3} + u_{q4}
\end{aligned}$$

So

$$\rho x_t = \rho_q^4 x_t, \qquad u_{t+1} = \rho_q^3 u_{q1} + \rho_q^2 u_{q2} + \rho_q u_{q3} + u_{q4}$$

We get

$$\rho_q = \rho^{1/4}$$

$$\operatorname{var}\left(u_{yearly}\right) = \left(\rho_q^6 + \rho_q^4 + \rho_q^2 + 1\right)\operatorname{var}\left(u_q\right) \Rightarrow \operatorname{var}\left(u_q\right) = \frac{\operatorname{var}\left(u_{yearly}\right)}{\left(\rho_q^6 + \rho_q^4 + \rho_q^2 + 1\right)}$$

A8. Expressions of reduced form coefficients in terms of structural coefficients.

This is just for purposes of scripting. We will use the notational convention to use upperscripts for the elements of an inverse matrix so in this section we have

$$\boxed{\Theta^{-1} = \left|\Theta\right|^{-1} \begin{bmatrix} \theta^{11} & \theta^{12} & 0 & \theta^{14} \\ \theta^{21} & \theta^{22} & 0 & \theta^{24} \\ \theta^{31} & \theta^{32} & \theta^{33} & \theta^{34} \\ \theta^{41} & \theta^{42} & 0 & \theta^{44} \end{bmatrix}}$$

The constant term

$$\mathbf{c} = |\Theta|^{-1}\begin{bmatrix} \theta^{11} & \theta^{12} & 0 & \theta^{14} \\ \theta^{21} & \theta^{22} & 0 & \theta^{24} \\ \theta^{31} & \theta^{32} & \theta^{33} & \theta^{34} \\ \theta^{41} & \theta^{42} & 0 & \theta^{44} \end{bmatrix}\begin{bmatrix} d_h \\ d_y \\ d_g \\ d_c \end{bmatrix} = |\Theta|^{-1}\begin{bmatrix} \theta^{11}d_h + \theta^{12} d_y + \theta^{14}d_c \\ \theta^{21}d_h + \theta^{22}d_y + \theta^{24}d_c \\ \theta^{31}d_h + \theta^{32}d_y + \theta^{33}d_g + \theta^{34}d_c \\ \theta^{41}d_h + \theta^{42}d_y + \theta^{44}d_c \end{bmatrix} = \begin{bmatrix} c_h \\ c_y \\ c_g \\ c_c \end{bmatrix}$$

Matrix $\Gamma = \Theta^{-1}\Lambda$.

$$\boxed{\Gamma = \frac{1}{|\Theta|}\begin{bmatrix} \theta^{11} & \theta^{12} & 0 & \theta^{14} \\ \theta^{21} & \theta^{22} & 0 & \theta^{24} \\ \theta^{31} & \theta^{32} & \theta^{33} & \theta^{34} \\ \theta^{41} & \theta^{42} & 0 & \theta^{44} \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda_{41} & 0 & 0 & 1 \end{bmatrix} = \frac{1}{|\Theta|}\begin{bmatrix} \theta^{14}\lambda_{41} & 0 & 0 & \theta^{14} \\ \theta^{24}\lambda_{41} & 0 & 0 & \theta^{24} \\ \theta^{34}\lambda_{41} & 0 & 0 & \theta^{34} \\ \theta^{44}\lambda_{41} & 0 & 0 & \theta^{44} \end{bmatrix}}$$

Matrix $\Psi = \Theta^{-1}\mathrm{M}$**.**

$$\Psi = \frac{1}{|\Theta|}\begin{bmatrix} \theta^{11} & \theta^{12} & 0 & \theta^{14} \\ \theta^{21} & \theta^{22} & 0 & \theta^{24} \\ \theta^{31} & \theta^{32} & \theta^{33} & \theta^{34} \\ \theta^{41} & \theta^{42} & 0 & \theta^{44} \end{bmatrix}\begin{bmatrix} 0 & 0 & \mathrm{m}_{13} \\ \mathrm{m}_{21} & 0 & 0 \\ \mathrm{m}_{31} & \mathrm{m}_{32} & \mathrm{m}_{33} \\ \mathrm{m}_{41} & \mathrm{m}_{42} & 0 \end{bmatrix}$$

$$= |\Theta|^{-1}\begin{bmatrix} \theta^{12}\mathrm{m}_{21} + \theta^{14}\mathrm{m}_{41} & \theta^{14}\mathrm{m}_{42} & \theta^{11}\mathrm{m}_{13} \\ \theta^{22}\mathrm{m}_{21} + \theta^{24}\mathrm{m}_{41} & \theta^{24}\mathrm{m}_{42} & \theta^{21}\mathrm{m}_{13} \\ \theta^{32}\mathrm{m}_{21} + \theta^{33}\mathrm{m}_{31} + \theta^{34}\mathrm{m}_{41} & \theta^{33}\mathrm{m}_{32} + \theta^{34}\mathrm{m}_{42} & \theta^{31}\mathrm{m}_{13} + \theta^{33}\mathrm{m}_{33} \\ \theta^{42}\mathrm{m}_{21} + \theta^{44}\mathrm{m}_{41} & \theta^{44}\mathrm{m}_{42} & \theta^{41}\mathrm{m}_{13} \end{bmatrix}$$

Matrix $\Pi = \Theta^{-1}\Xi$**.**

$$\Pi = |\Theta|^{-1}\begin{bmatrix} \theta^{11} & \theta^{12} & 0 & \theta^{14} \\ \theta^{21} & \theta^{22} & 0 & \theta^{24} \\ \theta^{31} & \theta^{32} & \theta^{33} & \theta^{34} \\ \theta^{41} & \theta^{42} & 0 & \theta^{44} \end{bmatrix}\begin{bmatrix} 0 & \xi_{12} & \xi_{13} \\ 1 & \xi_{22} & \xi_{23} \\ 0 & \xi_{32} & \xi_{33} \\ 0 & \xi_{42} & \xi_{43} \end{bmatrix} =$$

$$= |\Theta|^{-1}\begin{bmatrix} \theta^{12} & \theta^{11}\xi_{12} + \theta^{12}\xi_{22} + \theta^{14}\xi_{42} & \theta^{11}\xi_{13} + \theta^{12}\xi_{23} + \theta^{14}\xi_{43} \\ \theta^{22} & \theta^{21}\xi_{12} + \theta^{22}\xi_{22} + \theta^{24}\xi_{42} & \theta^{21}\xi_{13} + \theta^{22}\xi_{23} + \theta^{24}\xi_{43} \\ \theta^{32} & \theta^{31}\xi_{12} + \theta^{32}\xi_{22} + \theta^{33}\xi_{32} + \theta^{34}\xi_{42} & \theta^{31}\xi_{13} + \theta^{32}\xi_{23} + \theta^{33}\xi_{33} + \theta^{34}\xi_{43} \\ \theta^{42} & \theta^{41}\xi_{12} + \theta^{42}\xi_{22} + \theta^{44}\xi_{42} & \theta^{41}\xi_{13} + \theta^{42}\xi_{23} + \theta^{44}\xi_{43} \end{bmatrix}$$