ΟΙΚΟΝΟΜΙΚΟ ΠΑΝΕΠΙΣΤΗΜΙΟ ΑΘΗΝΩΝ
ATHENS UNIVERSITY OF ECONOMICS AND BUSINESS
ΣΧΟΛΗ ΟΙΚΟΝΟΜΙΚΩΝ ΕΠΙΣΤΗΜΩΝ
SCHOOL OF ECONOMIC SCIENCES
ΤΜΗΜΑ ΟΙΚΟΝΟΜΙΚΗΣ ΕΠΙΣΤΗΜΗΣ
DEPARTMENT OF ECONOMICS

Department of Economics

Athens University of Economics and Business

WORKING PAPER no. 15-2025

Universal Preference Spaces for Expected Utility via a Banach-type Functorial Fixed Point

Stelios Arvanitis, Pantelis Argyropoulos, and Spyros Vassilakis

August 2025

Πατησίων 76, 104 34 Αθήνα. Τηλ.: 210 8203303-5. E-mail: econ@aueb.gr / www.aueb.gr
76, Patission Street, Athens 104 34 Greece. Tel.: (+30) 210 8203303-5

Universal Preference Spaces for Expected Utility via a Banach-type Functorial Fixed Point

Stelios Arvanitis[*], Pantelis Argyropoulos[†], Spyros Vassilakis[‡]

August 29, 2025

Abstract

This paper develops universal preference spaces in the Expected Utility paradigm using functorial fixed point theory in enriched categories. We work within the CMS-enriched category of compact metric spaces with embedding-retraction morphisms and define the endofunctor $\mathbf{vNM}_\lambda$, which maps each object to a space of Lipschitz-continuous von Neumann–Morgenstern utilities. Under conditions, $\mathbf{vNM}_\lambda$ admits a canonical fixed point, representing a universal space of recursively structured preferences. This construction is extended to multi-utility preferences by composing $\mathbf{vNM}_\lambda$ with the Hausdorff functor. Both universal spaces are also characterized as colimits of their respective ω-chains, starting from the terminal object. These constructions provide a categorical framework for representing layered decision models under ambiguity. Possible extensions include dropping embedding requirements and organizing preferences via coend-based constructions.

JEL Codes: D81.
Keywords: Expected utility, ambiguity of preferences, infinite regress, enriched category, endofunctor, canonical fixed point, colimit, universal preference space, multi-utility ordering.

[*]Department of Economics, Queen's University, Canada, and Department of Economics, Athens University of Economics and Business, Greece. Email: stelios@aueb.gr

[†]Department of Economics, Athens University of Economics and Business, Greece. Email: argyroppant@aueb.gr

[‡]Department of IEES, Athens University of Economics and Business, Greece. Email: svasilak@aueb.gr

1 Introduction

A central challenge in decision theory under uncertainty is modeling preferences when agents are uncertain not only about outcomes, but also about their own evaluative criteria. Classical expected utility theory represents preferences over probabilistic outcomes by a von Neumann–Morgenstern (vNM) utility function, defined as a linear functional on the space of probability measures over a compact outcome space X. This theory assumes that agents possess a stable and uniquely defined utility function over X. Yet in many economic environments—such as contexts involving preference ambiguity—this assumption may not hold.

To accommodate such phenomena, one may consider recursive layers of uncertainty, where an agent is unsure about her utility function, about how to resolve that uncertainty, and so on. This motivates studying fixed-point constructions that stabilize such recursive ambiguity. This idea leads to the formulation of a universal preference space: a mathematical object that contains all consistent levels of meta-preferences structured by repeated application of a functor that organizes the set of preferences definable on a related outcome space.

This notion is related to the concept of a universal choice space studied, for example, in Vassilakis (1991); Lipman (1991); Moss and Viglizzo (2004); Galeazzi and Marti (2023), where recursive selections are performed over sequences of transformed choice sets. A universal choice space is a more general concept as it can incorporate procedures involving agent computations and actions across the different levels of uncertainty-see again Vassilakis (1991); Lipman (1991), while as mentioned above, the universal preference space is a mathematical description of the entities that represent preferences across those layers-see, for example, Pivato (2024a,b). Both concepts can be obtained as appropriate limits of the relevant infinite regresses.

Our constructions employ tools from category theory-see the Supplemental Appendix Arvanitis et al. (2025) for some introductory notions and mathematical references. The use of categorical language to address problems of recursive structures in economic theory is obviously not novel here. For example, and in the context of games with incomplete information, Mertens and Zamir (1985) proved the existence of a space that contains all possible hierarchies of beliefs that agents can form in a game. This space is frequently called the universal type space, and it is constructed as a (what is termed direct) limit of a suitable sequence that accommodates hierarchies of beliefs and belief interactions between agents. The aforementioned result essentially allows the formulation of the problem through category theory. Indeed, many authors have used category theory to provide comprehensive accounts regarding the problem of existence of the Universal Type Space. See for example Vassilakis (1992), Moss and Viglizzo (2004), Pintér (2010), Blumensath and Winschel (2013), Heinsalu (2014), Fukuda (2024), Guarino (2024).

In the context of decision problems with absence of strategic interactions-as the one we consider here, suchlike recursive structures forming infinite regresses similar to the above, were considered by Vassilakis (1989) and Lipman (1991), who gave sufficient conditions for the existence of a universal choice space; Vassilakis (1991) showed that this was realizable as a minimal fixed point of a functor defined on a suitable category. Pivato (2024a,b) developed the idea of a universal preference space, among others motivated by concerns over bounded awareness, higher-order reasoning and autonomy. Our results indicate that in a framework involving vNM expected utility this can be realized as a simultaneously minimal and maximal (this is termed canonical) fixed point of a functor that associates a class of vNM utilities to any compact metric space of primitive events. This can be also characterized as a limit of self-iterations of the particular functor initiated on singleton metric spaces-terminal objects in the categorical language-and are thus up to isomorphism independent from the original space of primitive events.

More precisely we work within the context of enriched category theory, particularly the setting developed in Adámek et al. (2018) and extended in Adamek and Reiterman (1994), where categories are enriched over complete metric spaces and endofunctors can be contractive. We work in the CMS-enriched category $\mathbf{CompMet}_{\mathrm{CMS}}^{\mathrm{Emb}}$ of compact metric spaces with Lipschitz morphisms, further restricted to embedding–retraction pairs to support stable interpretations of model expansions. The key functors we consider are:

- the probability functor Δ, assigning to each space the set of Borel probability measures (endowed with the Wasserstein metric), and
- the utility functor $\mathbf{vNM}_\lambda$, mapping each space to its vNM utility functionals formed by uniformly Lipschitz Bernoulli utility functions.

We define recursive preference structures via the endofunctor $\mathbf{vNM}_\lambda$, and show that it admits a canonical fixed point—denoted $\mathbb{U}_{\mathbf{vNM}_\lambda}$—in the sense of Adamek and Reiterman (1994), using their fixed-point theorem for contractive endofunctors. This fixed point contains, in a well-defined and internally coherent sense, all levels of iterated utility uncertainty that are metrically stable. We also extend this framework to preferences represented by multi-utilities, by composing $\mathbf{vNM}_\lambda$ with the Hausdorff hyperspace functor $\mathcal{H}$, yielding the endofunctor $\mathbf{MvNM}_\lambda := \mathcal{H} \circ \mathbf{vNM}_\lambda$, and establish the existence of its canonical fixed point $\mathbb{U}_{\mathbf{MvNM}_\lambda}$; an entity that summarizes all layers of iterated uncertainty over preferences representable by vNM multi-utilities formed by compact sets of uniformly Lipschitz Bernoulli utilities. This space among others contains information about standard stochastic dominance relations, and (multi-)utilities that represent preferences over them under ambiguity.

The remainder of the paper proceeds as follows. Section 2 introduces the CMS-enriched categorical framework, defines the key functors and provides the main result; i.e. proves the existence of the canonical fixed point $\mathbb{U}_{\mathbf{vNM}_\lambda}$ and interprets its structure. Section 3 extends the construction to compact multi-utility preferences and shows that the functor $\mathbf{MvNM}_\lambda$ also admits a canonical fixed point. The final section discusses the semantic implications of the construction, contrasts it with alternative formulations of recursive preferences, and outlines future directions, including possible applications of coend constructions in non-retractive categories. The main technical apparatus—contractivity, enrichment, and the fixed-point construction—is explained in the Supplementary Appendix (Arvanitis et al. (2025)).

2 Main Result

2.1 Categorical Framework

Let $\lambda \in (0,1)$ be fixed. We work in the category **CompMet** whose objects are compact metric spaces with distances uniformly bounded by 1, and whose morphisms are 1-Lipschitz maps. Following Adámek et al. (2018), we enrich this category by endowing each hom-set with a complete metric space structure. We achieve this by defining for any $X, Y \in \mathbf{CompMet}$ and $f, g \in \mathrm{Hom}(X, Y)$, the uniform metric

$$d_{X,Y}(f,g) \coloneqq \sup_{x \in X} d_Y(f(x), g(x)) \ .$$

Moreover, in the enriched category composition is non-expansive in both variables. This means that,

$$d_{X',Y'}(p \circ f \circ q,\, p \circ g \circ q) \leq d_{X,Y}(f,g)$$

for any $p\colon Y \to Y'$, $q\colon X' \to X$. We denote the resulting enriched category by $\mathbf{CompMet}_{\mathrm{CMS}}$. In general, any category whose hom-sets are complete metric spaces with composition non-expansive in both arguments is called a CMS-enriched category.

We focus further on the category of embeddings, denoted $\mathbf{CompMet}^{\mathrm{Emb}}_{\mathrm{CMS}}$ (see Adámek et al. (2018); America and Rutten (1989), and Part III of Adamek and Reiterman (1994)). Its objects coincide with those of $\mathbf{CompMet}_{\mathrm{CMS}}$, but morphisms are pairs (f, g_f), where $f\colon X \to Y$ and $g_f\colon Y \to X$ are morphisms in $\mathbf{CompMet}_{\mathrm{CMS}}$ such that $g_f \circ f = \mathrm{id}_X$. Such pairs define a section–retraction structure and embed X into Y as a retract.

Each hom-set is equipped with the max-sup metric:

$$d\big((f, g_f), (h, g_h)\big) := \max\left\{\sup_{x\in X} d_Y(f(x), h(x)),\ \sup_{y\in Y} d_X(g_f(y), g_h(y))\right\},$$

which also makes the space complete. By Adamek and Reiterman (1994), see Sec. III there, $\mathbf{CompMet}^{\mathrm{Emb}}_{\mathrm{CMS}}$ is also CMS-enriched.

Objects of $\mathbf{CompMet}^{\mathrm{Emb}}_{\mathrm{CMS}}$ model compact choice spaces, while morphisms represent interpretable model expansions and retractions. Requiring both the embedding f and its retraction g_f to be non-expansive ensures that preferences remain metrically stable under such expansions. The enrichment supports reasoning about approximation and stability of utility representations under outcome spaces refinements.

2.2 Two Useful Endofunctors

We now introduce two key endofunctors: the probability functor Δ and the utility functor $\mathbf{vNM}_\lambda$. The former assigns to any object its space of Borel probability measures, and the latter assigns the space of von Neumann–Morgenstern utilities formed by λ-contractive Bernoulli utilities.

The probability functor Δ. For any $X \in \mathbf{CompMet}_{\mathrm{CMS}}$, $\Delta(X)$ is the set of Borel probability measures on X, equipped with the 1-Wasserstein metric:

$$d_{1W}(\mathbb{P}, \mathbb{Q}) := \inf_{\gamma \in \Gamma(\mathbb{P},\mathbb{Q})} \int_{X \times X} d_X(x, x')\, d\gamma(x, x'),$$

where $\Gamma(\mathbb{P}, \mathbb{Q})$ is the set of couplings with marginals $\mathbb{P}, \mathbb{Q}$. This metric topologizes weak convergence (see Rahimian and Mehrotra (2019)).

Lemma 1. *Δ is a well-defined locally non-expansive endofunctor on* $\mathbf{CompMet}^{\mathrm{Emb}}_{\mathrm{CMS}}$.

Well-defined means that it satisfies the categorical definition of a functor and locally non-expansive means that it satisfies the standard contracting inequality with constant equal to 1 (see the Supplementary Appendix Arvanitis et al. (2025) for details on both).

Proof. Since X is compact, $\Delta(X)$ is also compact (Theorem 15.11 of Aliprantis and Border (1999)). For any $f\colon X \to Y$ in $\mathbf{CompMet}_{\mathrm{CMS}}$, define $\Delta f\colon \Delta X \to \Delta Y$ via $\Delta f(\mathbb{P}) := \mathbb{P} \circ f^{-1}$. The change of variables formula and the non-expansiveness of f ensure that Δf is non-expansive in the Wasserstein metric. The functoriality of Δ follows from the obvious $\Delta \mathbf{1}_X(\mathbb{P}) = \mathbf{1}_{\Delta X}(\mathbb{P})$, and that $\Delta(g \circ f)(\mathbb{P}) = \mathbb{P} \circ (f^{-1} \circ g^{-1}) = (\mathbb{P} \circ f^{-1}) \circ g^{-1} = \Delta g(\mathbb{P} \circ f^{-1}) = (\Delta g \circ \Delta f)(\mathbb{P})$.

Finally, for any two morphisms $(f, g_f), (h, g_h)$ from X to Y, we have:

$$\sup_{\mathbb{P} \in \Delta(X)} d_{1W}\left(\mathbb{P} \circ f^{-1}, \mathbb{P} \circ h^{-1}\right) \leq \sup_{x \in X} d_Y(f(x), h(x)),$$

and likewise for the retractions g_f, g_h, showing local non-expansiveness. □

The utility functor $\mathbf{vNM}_\lambda$. Define $\mathbf{vNM}_\lambda(X)$ to be the set of linear functionals $v_u\colon \Delta X \to \mathbb{R}$ given by $v_u(\mathbb{P}) := \int u\, d\mathbb{P}$, where $u\colon X \to \mathbb{R}$ is a λ-Lipschitz Bernoulli utility bounded above by 1. This space is compact in the uniform topology due to total boundedness and completeness of the Lipschitz functions on X; see Proposition 4.1 in Ok and Weaver (2023) for non-emptiness, compactness then follows from that X is the set of extreme points of $\Delta(X)$, as well as from the fact that the set of Bernoulli utilities that form the elements of $\mathbf{vNM}_\lambda(X)$ is uniformly Lipschitz and X is compact.

Lemma 2. $\mathbf{vNM}_\lambda$ *is a well-defined locally contractive endofunctor on* $\mathbf{CompMet}^{\mathrm{Emb}}_{\mathrm{CMS}}$.

Proof. Let $(f, g_f) \in \mathrm{Hom}(X, Y)$. For each $v_u \in \mathbf{vNM}_\lambda(X)$, define $f^\star(v_u)(\mathbb{Q}) := \int_Y u(g_f(y))\, d\mathbb{Q}(y)$, which lies in $\mathbf{vNM}_\lambda(Y)$ because the composition $u \circ g_f$ remains λ-Lipschitz and bounded. Similarly define $g_f^\star$.

We set $\mathbf{vNM}_\lambda(f, g_f) := (f^\star, g_f^\star)$, and verify that this defines a morphism in $\mathbf{CompMet}^{\mathrm{Emb}}_{\mathrm{CMS}}$. The functorial properties follow by associativity and identity of integrals.

The contraction property is verified as:

$$\sup_{\mathbb{Q}\in\Delta(Y)} |f^\star(v_u)(\mathbb{Q}) - f^\star(v'_u)(\mathbb{Q})| \leq \lambda \sup_{y\in Y} |g_f(y) - g_h(y)|,$$

and similarly on the other side, confirming that the functor contracts by the fixed factor λ. □

The functor Δ can be interpreted to reflect probabilistic beliefs formation over the measurable parts of outcome spaces, while $\mathbf{vNM}_\lambda$ represents boundedly rational preferences under uncertainty. The contractiveness of $\mathbf{vNM}_\lambda$ ensures diminishing sensitivity to model refinements, and recursive application supports multi-level preference hierarchies—such as agents uncertain about their own utilities.

2.3 Universal vNM Preference Space

The endofunctor $\mathbf{vNM}_\lambda$ captures the representation of preferences over distributions in terms of λ-contractive Bernoulli utilities. Since agents may be uncertain about their preferences, iterated applications of $\mathbf{vNM}_\lambda$ can model higher-order beliefs or meta-preferences. This leads to considering the sequence $X, \mathbf{vNM}_\lambda(X), \mathbf{vNM}_\lambda^2(X), \ldots$.

The limit object of this iteration, if it exists, should embody a universal structure capable of representing all such levels of preference ambiguity in a coherent way. The following result, based on the canonical fixed point theorem of Adamek and Reiterman (1994)-see their Theorem 3, establishes the existence of a canonical fixed point of the $\mathbf{vNM}_\lambda$ endofunctor.

Briefly, this theorem states that if a CMS-enriched category $\mathbf{C}_{\mathrm{CMS}}$ is connected and has a terminal object and inverse limits of contracting sequences, then each endofunctor defined on $\mathbf{C}_{\mathrm{CMS}}^{\mathrm{Emb}}$ with the properties of being hom-contracting and AR-contracting, has a canonical fixed-point. For definitions of these properties see Adamek and Reiterman (1994) as well as the Supplementary Appendix Arvanitis et al. (2025). This theorem is applied on the von-Neumann Morgenstern utility endofunctor $\mathbf{vNM}_\lambda$. We have the following result.

Theorem 1. *The functor* $\mathbf{vNM}_\lambda$ *admits a canonical fixed point.*

Proof. Lemma 2 established the local contractivity of the endofunctor defined on $\mathbf{CompMet}_{\mathrm{CMS}}^{\mathrm{Emb}}$. This directly implies that the functor is hom-contracting. The AR-contractivity property also follows from the uniform λ-Lipschitz property of the Bernoulli utilities involved; using partly the notation of Adamek and Reiterman (1994) and using the notation in Sec. III of Adamek and Reiterman (1994), we obtain that for any pair X, Y of objects in the category, and morphisms pair f, g_f, $\delta_{\mathbf{vNM}_\lambda \phi} \leq \sup_{v_u^\star, \mathbb{P}^\star} |\int_Y u^\star(f(g_f(y))) - u^\star(y) d\mathbb{P}^\star| \leq \lambda \sup_{y \in Y} d(f(g_f(y)), \mathrm{id}_Y(y)) = \lambda \delta_\phi$.

Moreover, the underlying category $\mathbf{CompMet}_{\mathrm{CMS}}$ is connected, which means that all hom-sets are non empty, due to the existence of constant functions. Hence, the existence of a canonical fixed point would be then obtained by Theorem 3 of Adamek and Reiterman (1994), as long as $\mathbf{CompMet}_{\mathrm{CMS}}$ has inverse limits of contracting sequences (see the Definition in Par. II of Adamek and Reiterman (1994)). Since $\mathbf{CompMet}_{\mathrm{CMS}}$ has also a terminal object (the one point space), it suffices to show that $\mathbf{CompMet}_{\mathrm{CMS}}$ has inverse limits and this is proven in Sec. 2.2 of the Supplementary Appendix (Arvanitis et al. (2025)). □

This canonical fixed point, denoted $\mathbb{U}_{\mathbf{vNM}_\lambda}$, can be characterized as the colimit of the ω-sequence-see :

$$1 \overset{(f,g_f)}{\to} \mathbf{vNM}_\lambda(1) \overset{\mathbf{vNM}_\lambda(f,g_f)}{\to} \mathbf{vNM}^2_\lambda(1) \overset{\mathbf{vNM}^2_\lambda(f,g_f)}{\to} \cdots,$$

where 1 denotes the terminal object (the one-point space) of the category, and the pair (f, g_f) is an arbitrary morphism emanating from the terminal object-see the first part of the proof of Th. 3 of Adamek and Reiterman (1994). Each map in the sequence is given by the image of the embedding induced by the functor. By construction, $\mathbb{U}_{\mathbf{vNM}_\lambda}$ contains all iterated structures of λ-contractive vNM utilities on probability measures of previously constructed spaces, forming a universal receptacle for all such preference levels.

The space $\mathbb{U}_{\mathbf{vNM}_\lambda}$ provides a formal setting to accommodate agents with deep uncertainty about their utility functions. It allows internal summarization of utility perspectives that encode ambiguity about the agent's "true" utility function. Through the recursive structure of the vNM endofunctor, the universal fixed point space captures all possible levels of utility uncertainty—representing not just preferences over outcomes, but preferences over lotteries of utilities, and so on—while preserving axiomatic coherence. The categorical construction ensures stability under model expansion and supports rationalization of a broad class of behaviors. Thus, $\mathbb{U}_{\mathbf{vNM}_\lambda}$ acts as a canonical universal preference space under von Neumann–Morgenstern rationality extended to meta-preference hierarchies.

3 Compact Multi-Utilities

We extend the construction of the universal expected utility space by considering multi-utility preferences (see, e.g., Ok and Weaver (2023)). In particular, we examine the composition $\mathcal{H} \circ \mathbf{vNM}_\lambda$, where $\mathbf{vNM}_\lambda$ is the contractive endofunctor defined previously and $\mathcal{H}$ denotes the Hausdorff functor that maps each compact metric space to the hyper-space of its nonempty compact subsets endowed with the Hausdorff metric. Each such compact subset of vNM utilities represents a multi-utility preference, interpreted as the product of the preference relations defined by its members. We show that this composition yields a contractive endofunctor on the metric-enriched category $\mathbf{CompMet}^{\mathrm{Emb}}_{\mathrm{CMS}}$, whose canonical fixed point admits a natural interpretation as a universal space of compact multi-utility preferences.

3.1 The Compact Multi-Utility Functor

Consider the Hausdorff functor defined in Example 5.2.6.(4) of Ch. 5 in Adámek et al. (2025). For X be an object in $\mathbf{CompMet}_{\mathrm{CMS}}$, $\mathcal{H}(X)$ is the hyper-space of the non-empty compact subsets of X, equipped with the Hausdorff metric-a compact metric space (see Remark 5.2.1 of Adámek et al. (2023)). On morphisms $f : X \to Y$ operates by sending each non-empty compact subset of X to its f-image, a compact subset of Y due to the continuity of f. The analysis in Example 5.2.6.(4) of Ch. 5 in

Adámek et al. (2025) implies that the functor also respects pairs of sections-retractions morphisms and that it is a well defined non expansive endofunctor on $\mathbf{CompMet}^{\mathrm{Emb}}_{\mathrm{CMS}}$.

Then the composed functor $\mathbf{MvNM}_\lambda := \mathcal{H} \circ \mathbf{vNM}_\lambda$ is a well defined endofunctor on $\mathbf{CompMet}^{\mathrm{Emb}}_{\mathrm{CMS}}$. This categorically organizes the compact λ-Lipschitz multi-utility orderings that among others appear in Theorem 4.1 of Ok and Weaver (2023). Notice that if X is any compact non empty subset of the real line equipped with the usual metric, then several stochastic dominance relations are included in $\mathbf{MvNM}_\lambda(X)$, like for example tthose that structurally depend on the set of Russell-Seo ramp functions that can be scaled to become λ-Lipschitz (see Russell and Seo (1989)).

3.2 Universal MvNM Preference Space

In economic environments where agents face non-totality of underlying preferences, the latter would typically fail to be rationalized by real-valued utility functions-see for example Sec. 3.4 of Ok and Weaver (2023). Instead, agents may reason via compact sets of plausible utilities, each encoding a coherent view of the world. The existence of a canonical fixed point of $\mathbf{MvNM}_\lambda$ would provide a universal preference space that accommodates not only such ambiguity-aware non-total preferences but also their higher-order recursive structure—representing preferences over sets of preferences, and so on—while preserving rational consistency under refinement. Similarly to the ambiguity considerations of the previous section, this leads to considering sequences of the form $X, \mathbf{MvNM}_\lambda(X), \mathbf{MvNM}^2_\lambda(X), \ldots$, and subsequently the issue of existence of a canonical fixed point for the functor at hand.

The aforementioned properties of the factor functors, namely the local and the AR-contractivity of $\mathbf{vNM}_\lambda$, and the non expansiveness of $\mathcal{H}$ directly imply the applicability of Theorem 3 of Adamek and Reiterman (1994) in this case:

Theorem 2. *The functor* $\mathbf{MvNM}_\lambda$ *admits a canonical fixed point.*

Proof. The non-expasiveness of $\mathcal{H}$, and the local contractivity of $\mathbf{vNM}_\lambda$ along with the composition property of Lipschitz functions imply that $\mathbf{MvNM}_\lambda$ is both AR and locally contractive. □

Similarly to the previous case, this canonical fixed point, say $\mathbb{U}_{\mathbf{MvNM}_\lambda}$, is characterizable as the colimit of the ω-sequence-see :

$$1 \stackrel{(f,g_f)}{\rightarrow} \mathbf{MvNM}_\lambda(1) \stackrel{\mathbf{MvNM}_\lambda(f,g_f)}{\rightarrow} \mathbf{MvNM}^2_\lambda(1) \stackrel{\mathbf{MvNM}^2_\lambda(f,g_f)}{\rightarrow} \cdots,$$

where again 1 denotes the terminal object (the one-point space) of the category, and the pair (f, g_f) is an arbitrary morphism emanating from the terminal object-see the first part of the proof of Th. 3 of Adamek and Reiterman (1994). Thus by construction the fixed point encompasses every potential ambiguity present at any level of preferences characterized via compact sets of vNM utilities.

The canonical fixed point $\mathbb{U}_{\mathbf{MvNM}\lambda}$ provides a universal semantic domain for modeling agents whose preferences and meta-preferences are not pinned down by a single utility function, but rather by coherent, compact sets of vNM utilities formed by Bernoulli λ-Lipschitz utilities. This could accommodate multi-criteria decision-making under ambiguity, or incomplete preferences. Notably, the canonical fixed point $\mathbb{U}_{\mathbf{vNM}\lambda}$ constructed in the previous section embeds naturally into $\mathbb{U}_{\mathbf{MvNM}\lambda}$ via the singleton map $\eta : u \mapsto u$, which preserves the recursive structure. This embedding admits a natural retraction given by a selection operator or representative utility within a compact set—when such selection is warranted—allowing analysts to recover crisp preference representations from ambiguous ones in well-posed cases. The present enriched categorical framework thus encodes both the usual vNM rational utility theory and its multi-utility generalizations, providing a mathematically robust and behaviorally meaningful foundation for modeling recursive preference uncertainty.

4 Discussion

This paper proposes a categorical framework for representing recursive preference structures under uncertainty, using contractive functors in a metric-enriched setting. Within the CMS-enriched category $\mathbf{CompMet}_{\mathrm{CMS}}^{\mathrm{Emb}}$, we defined two key endofunctors: the von Neumann–Morgenstern utility functor $\mathbf{vNM}_\lambda$, assigning λ-Lipschitz utilities over probability measures, and its multi-utility extension $\mathbf{MvNM}_\lambda := \mathcal{H} \circ \mathbf{vNM}_\lambda$, which captures compact sets of such utilities. Both functors were shown to be locally contractive, and their canonical fixed points were constructed using a general fixed point theorem from the enriched categorical setting. These fixed points were interpreted as universal preference spaces that internalize recursive levels of utility reasoning—either via single utilities or compact subsets thereof.

A related construction is provided in Pivato (2024b), where universal recursive preference structures are defined over $\Delta(X)$ for a compact metric space X, as the terminal coalgebra of an endofunctor that produces recursive preference structures. Our framework identifies a specific subclass of such recursive preferences—those representable via compact sets of vNM utilities—and a part of the universal space that arises as the canonical fixed point of the contractive multi-utility functor. This construction yields a space that is independent (up to isomorphism) of the underlying domain X, and realized both as a colimit of iterated functor applications and as a canonical fixed point. It seems therefore to shed further categorical light on a portion of Pivato's universal preference space, exploiting the additional categorical structure used here.

The main constructions presented here may be relevant for modeling economic agents who are uncertain about their utility functions or operate with multiple evaluative dimensions. In this setting, compact sets of utilities can serve as proxies for ambiguity-robust or incomplete preference representations. The universal fixed point of $\mathbf{MvNM}_\lambda$ offers a structured space capable of supporting such representations across multiple levels of recursive reasoning. Moreover, the canonical embedding of the single-utility fixed point $\mathbb{U}_{\mathbf{vNM}_\lambda}$ into the multi-utility space provides a principled way to interpret standard expected utility as a special case.

The framework could be helpful for the investigation of hierarchical decision making under uncertainty and ambiguity without strategic interactions. For example in certain multilevel organizations—such as administrative agencies or policy hierarchies—decisions made at one level often treat objectives from superior levels as fixed inputs, resolving internal ambiguity through level-specific evaluative schemes. Klein and Kozlowski (2000) emphasize how such nested decision architectures function with loosely coupled, non-strategic levels. Also, for an example originating in the cognitive sciences, the dynamical systems perspective of Varona and Rabinovich (2016) provides a useful analogy: they explore how the brain may resolve informational conflicts hierarchically, through dynamic preference stabilization without inter-layer strategic modeling. Though not institutional in scope, this layered form of information processing suggests broader potential for the further investigation of the structure of universal preference spaces that represent ambiguity resolution in hierarchies without invoking game-theoretic interdependence.

A natural direction for further investigation is to explore the formulation of recursive preference spaces in categories without embedding–retraction pairs. In such settings, and given the then resulting contravariant nature of the analogous **vNM** functor, the fixed point machinery seems inapplicable. Instead the machinery of coends may provide an alternative method for aggregating structured families of preferences. Specifically, given a functor that assigns to each compact metric space its corresponding space of preferences (single or multi-utility), one may define an object which intuitively aggregates all ways that preferences over various spaces X can be transported into a target space Z-see Ch. X of Mac Lane (1978) and the supplementary material (Arvanitis et al. (2025)). Such a coend construction generalizes the idea of "gluing together" preference representations across spaces via morphisms, and may offer a way to systematically encode meta-preferences, transformations of preferences, or polymorphic preference updates. In the absence of embeddings, coends can preserve the functorial structure needed to reason about preference composition, while avoiding the rigidity of retractions. A precise understanding of whether this construction yields a meaningful universal preference space is left for future research.

Conflict of Interest Statement

All authors declare that they have no conflicts of interest.

References

Adámek, J., Milius, S., Moss, L. S., 2023. On kripke, vietoris and hausdorff polynomial functors. arXiv preprint arXiv:2303.11071.

Adamek, J., Reiterman, J., 1994. Banach's fixed-point theorem as a base for data-type equations. Applied Categorical Structures 2, 77–90.

Adámek, J., Milius, S., Moss, L. S., 2018. Fixed points of functors. Journal of Logical and Algebraic Methods in Programming 95, 41–81.

Adámek, J., Milius, S., Moss, L. S., 2025. Initial Algebras and Terminal Coalgebras: The Theory of Fixed Points of Functors. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press.

Aliprantis, C. D., Border, K. C., 1999. Infinite Dimensional Analysis, A Hitchhiker's Guide (2nd ed.). Springer.

America, P., Rutten, J., 1989. Solving reflexive domain equations in a category of complete metric spaces. Journal of Computer and System Sciences 39 (3), 343–375.

Arvanitis, S., Argyropoulos, P., Vassilakis, S., 2025. Supplementary appendix to: "universal preference spaces and expected utility via a banach-type functorial fixed point". Supplementary appendix to the main paper.

Blumensath, A., Winschel, V., 2013. A coalgebraic framework for games in economics. In 19th international conference on computing in economics and finance.

Fukuda, S., 2024. The existence of universal qualitative belief spaces. Journal of Economic Theory 216 (105784).

Galeazzi, P., Marti, J., 2023. Choice structures in games. Games and Economic Behavior 140, 431–455.

Guarino, P., 2024. Topology-free type structures with conditioning events. Econ Theory (2024).

Heinsalu, S., 2014. Universal type structures with unawareness. Games and Economic Behavior 83, 255–266.

Klein, K. J., Kozlowski, S. W., 2000. A multilevel approach to theory and research in organizations: Contextual, temporal, and emergent processes. Multilevel theory, research, and methods in organizations: Foundations, extensions, and new directions, 3–90.

Lipman, B. L., 1991. How to decide how to decide how to...: Modeling limited rationality. Econometrica: Journal of the Econometric Society, 1105–1125.

Mac Lane, S., 1978. Categories for the Working Mathematician. Graduate Texts in Mathematics.

Mertens, J. F., Zamir, S., 1985. Formulation of bayesian analysis for games with incomplete information. International journal of game theory 14, 1–29.

Moss, L., Viglizzo, I., 2004. Harsanyi type spaces and final coalgebras constructed from satisfied theories. Electronic notes in theoretical computer science 106, 279–295.

Ok, E. A., Weaver, N., 2023. Lipschitz bernoulli utility functions. Mathematics of Operations Research 48 (2), 728–747.

Pintér, M., 2010. The non-existence of a universal topological type space. Journal of Mathematical Economics 46 (2), 223–229.

Pivato, M., 2024a. Categorical decision theory and global subjective expected utility representations. Working paper.

Pivato, M., 2024b. Universal recursive preference structures. Working paper.

Rahimian, H., Mehrotra, S., 2019. Distributionally robust optimization: A review. arXiv preprint arXiv:1908.05659.

Russell, W. R., Seo, T. K., 1989. Representative sets for stochastic dominance rules. In: Studies in the economics of uncertainty: in Honor of Josef Hadar. Springer, pp. 59–76.

Varona, P., Rabinovich, M. I., 2016. Hierarchical dynamics of informational patterns and decision-making. Proceedings of the Royal Society B: Biological Sciences 283 (1832), 20160475.

Vassilakis, S., 1989. Economic data types. Working Paper No. 252, University of Pittsburgh.

Vassilakis, S., 1991. Functorial fixed points. Technical report no. 33, Stanford - Institute for Thoretical Economics.

Vassilakis, S., 1992. Some economic applications of scott domains. Mathematical Social Sciences 24 (2-3), 173–208.

Supplementary Appendix to: "Universal Preference Spaces and Expected Utility via A Banach-type Functorial Fixed Point".

Stelios Arvanitis*, Pantelis Argyropoulos†, Spyros Vassilakis‡

August 29, 2025

Abstract

This supplementary appendix gives an overview of the main technical apparatus that was employed in the paper. Basic categorical theoretic definitions are introduced and other categorical concepts, like functor contractivity, enrichment, and functorial fixed-points are explained. Moreover, the notion of a coend is briefly introduced given its potential as a more general alternative of formulating co-inductive representations, without relying on endo-functors and retractions. There are numerous resources one can refer to in order to study category theory and the theory of fixed-points of functors in greater depth. Two great classic textbooks are Mac Lane (1978) and Manes and Arbib (1986), while the very recent textbook of Adámek et al. (2025) is also an excellent source.

*Department of Economics, Queen's University, Canada, and Department of Economics, Athens University of Economics and Business, Greece. Email: stelios@aueb.gr

†Department of Economics, Athens University of Economics and Business, Greece. Email: argyroppant@aueb.gr

‡Department of IEES, Athens University of Economics and Business, Greece. Email: svasilak@aueb.gr

1 Basic Definitions

The central concept of category theory is the notion of a category. A category can be thought of as a universe of objects and relations between them. The definition is the following.

Definition: *A category* $\mathbf{C}$ *consists of a collection of objects, denoted by* $\mathbf{ob}(\mathbf{C})$ *and for each pair of objects* $A, B \in \mathbf{ob}(\mathbf{C})$*, a collection of morphisms from* A *to* B*, denoted by* $\mathbf{Hom}_{\mathbf{C}}(A, B)$*, that satisfy the following:*

Composition: *For* $A, B, C \in \mathbf{ob}(\mathbf{C})$*, there exists a function called composition,* $\circ : \mathbf{Hom}_{\mathbf{C}}(B, C) \times \mathbf{Hom}_{\mathbf{C}}(A, B) \to \mathbf{Hom}_{\mathbf{C}}(A, C)$ *that satisfies an associativity axiom: If* $f \in \mathbf{Hom}_{\mathbf{C}}(A, B)$*,* $g \in \mathbf{Hom}_{\mathbf{C}}(B, C)$ *and* $h \in \mathbf{Hom}_{\mathbf{C}}(C, D)$*, then* $(h \circ g) \circ f = h \circ (g \circ f)$*.*

Identity: *For* $A \in \mathbf{ob}(\mathbf{C})$*, there exists a morphism in* $\mathbf{Hom}_{\mathbf{C}}(A, A)$*, called the identity morphism, denoted by* $\mathbf{1}_A$*, that satisfies the following axiom: If* $f \in \mathbf{Hom}_{\mathbf{C}}(A, B)$*, then* $f \circ \mathbf{1}_A = f = \mathbf{1}_B \circ f$*.*

We will typically write $X \in \mathbf{C}$ to mean $X \in \mathbf{ob}(\mathbf{C})$ and also we will call sets like $\mathbf{Hom}_{\mathbf{C}}(A, B)$ as a hom-set of the category, and when the underlying category is understood, we will skip the index, i.e. we will write $\mathbf{Hom}(A, B)$. Finally, we will write $f \colon A \to B$ belongs to $\mathbf{C}$, or $f \in \mathbf{C}$, to mean that $f \in \mathbf{Hom}(A, B)$. The following definition will be useful:

Definition: *A category with non-empty hom-sets is called connected.*

It is natural to think that categories are objects as well, and there may be systematic relationships between them. This leads to functors, which are essentially maps between categories. Their definition is the following.

Definition: *Let* $\mathbf{C_1}$ *and* $\mathbf{C_2}$ *be categories.*

A functor $F\colon \mathbf{C_1} \to \mathbf{C_2}$ *consists of a function* $F\colon \mathbf{ob}(\mathbf{C}_1) \to \mathbf{ob}(\mathbf{C}_2)$*, and for each* $A, A' \in \mathbf{C_1}$*, a function* $F\colon \mathbf{Hom}_{\mathbf{C_1}}(A, A') \to \mathbf{Hom}_{\mathbf{C_2}}(F(A), F(A'))$*.*

The latter preserves composition and identity. This means that if $A, A', A'' \in \mathbf{C_1}$, $f\colon A \to A'$ and $f'\colon A' \to A''$ are in $\mathbf{C_1}$, then $F(f' \circ f) = F(f') \circ F(f)$, and if $A \in \mathbf{C_1}$, then $F(\mathbf{1}_A) = \mathbf{1}_{F(A)}$.

In the paper, we work with the category **CompMet** (see page 4 of the paper for the exact definition), modified in two ways as follows. First, hom-sets are equipped with a metric that provides them the structure of a complete metric space, with non-expansive composition in both variables. Obviously, this enrichment can be achieved to other categories as well. Hence, we denote by $\mathbf{C}_{\text{CMS}}$, the category $\mathbf{C}$ whose hom-sets are enriched with a complete metric space structure such that composition is non-expansive in both arguments.

Moreover, for reasons pertaining to the nature of the $\mathbf{vNM}_\lambda$ functor, based on Adámek et al. (2018) and Adamek and Reiterman (1994), we restricted our morphisms to be pairs of morphisms with a section–retraction structure. This means that morphisms are now pairs of morphisms (f, g_f), where $f\colon X \to Y$ and $g_f\colon Y \to X$ are in $\mathbf{CompMet}_{\text{CMS}}$ and satisfy $g_f \circ f = \text{id}_X$. This new category is known in the literature as the category of *embeddings*, and it is denoted by $\mathbf{CompMet}_{\text{CMS}}^{\text{Emb}}$ (objects are the same as before). As with the enrichment, any category $\mathbf{C}$, restricted to pairs with a section-retraction structure is denoted by $\mathbf{C}^{\text{Emb}}$.

Examples of Categories and Functors

We now present some standard examples that clarify the abstract definitions introduced above.

Examples of Categories

- ω: The category whose objects are the natural numbers $\{0, 1, 2, ...\}$. Morphisms for any two objects are described by the less than ($\leq$) relation. That is, there exists (unique) arrow/morphism from i to j if $i \leq j$. This category generalizes the idea of a partially ordered set.

- ω^{op}: Objects are the same but morphisms are now described by the greater than ($\geq$) relation. That is, there exists (unique) arrow/morphism from i to j if $i \geq j$.

- **Set**: The category whose objects are sets and whose morphisms are functions between sets.

- **Top**: The category of topological spaces with continuous maps as morphisms.

- **Met**: The category of metric spaces with Lipschitz or non-expansive maps as morphisms.

- **CompMet**: The category of compact metric spaces with continuous maps.

- $\mathbf{Vect}_{\mathbb{R}}$: The category of real vector spaces with linear transformations.

Examples of Functors

- The *power set functor* $\mathcal{P}\colon \mathbf{Set} \to \mathbf{Set}$, which sends each set to its power set and each function $f : X \to Y$ to the function $\mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y)$ given by $A \mapsto f(A)$.

- The *probability measure functor* $\Delta\colon \mathbf{CompMet} \to \mathbf{CompMet}$, which maps each compact metric space to the space of Borel probability measures on it (endowed with a suitable topology like the weak-* or Wasserstein), and continuous maps to their pushforwards.

- The *Hausdorff functor* $\mathcal{H}\colon \mathbf{CompMet} \to \mathbf{CompMet}$, assigning to each compact metric space X its hyperspace of non-empty compact subsets with the Hausdorff metric.

- The *forgetful functor* $U\colon \mathbf{Vect}_{\mathbb{R}} \to \mathbf{Set}$, which forgets the vector space structure and retains only the underlying set.

A standard example of a CMS-enriched category is **CMS** itself, whose objects are complete metric spaces and morphisms are non-expansive maps, with hom-sets metrized by the supremum distance-see Adámek et al. (2018).

2 Limits in Category Theory

The notion of limit in category theory is way more general than the usual corresponding notion in analysis and topology. This stems of course from the nature of Category Theory as a field. A limit in a category is first and foremost an object of the category. However, as in usual analysis, a limit in a space must satisfy a certain property with every other element on a sequence of elements, a limit in category theory must satisfy certain properties with other objects in a certain sequence of objects. Moreover, the uniqueness of limits in category theory is reflected by a property that is called universality.

Categories are fundamentally defined by the "direction of dependence" between the objects that "live" within them. For any two objects, either there exists an arrow that represents a certain relation between those objects, or there is not. From this perspective, we can see limits as objects for which, for every appropriately-defined sequence of objects, there is a certain systematic relationship between any object of the sequence and the limit. Moreover, the universality means that for every other candidate-object that satisfies the same properties with all objects of the sequence, there is again a certain systematic relationship between the limit and the candidate-object. In what follows, we formalize these ideas. Most of the material presented here is from Leinster (2014).

Consider a category C and the category ω^{op} that we defined above. A functor $D\colon \omega^{op} \to \mathrm{C}$ is called a diagram in C of shape ω^{op}. This functor represents what we would think of as an inverse sequence of objects. Diagrammatically, it has the following form:

$D(0) \xleftarrow{D(1\to 0)} D(1) \xleftarrow{D(2\to 1)} D(2) \xleftarrow{D(3\to 2)} D(3)$ ...

Then, we have the following definition.

Definition: A *Cone* on D is an object $L \in \mathrm{C}$, together with a family $(L \xrightarrow{f_i} D(i))_{i\in\omega^{op}}$ of morphisms in C, such that for all morphisms $u\colon i \to j$ in ω^{op}, it is true that $D(u) \circ f_i = f_j$.

The object L is called the vertex of the Cone.

Then, L is an (inverse) Limit of D, if it is a Universal Cone on D. That is, for any other cone $(A \xrightarrow{g_i} D(i))_{i\in\omega^{op}}$ on D, there exists unique morphism $\bar{f}\colon A \to L$ in

C, such that for any $i \in \omega^{op}$, it is true that $f_i \circ \bar{f} = g_i$.

Why do we call this an inverse limit? This has to do with the choice of the category ω^{op} and the resulting direction of the arrows in the diagram D. Clearly, we could choose to work with the category ω instead, and this would lead us to direct limits (or colimits), the dual notion of limit in category theory. The definition of colimits stems from dualization of the one above and we present it for completeness.

Definition: If we have a diagram $D\colon \omega \to \mathrm{C}$ of shape ω, then a *Cocone* on D is an object $C \in \mathrm{C}$ (the vertex of the Cocone), together with a family $(D(i) \xrightarrow{f_i} C)_{i\in\omega}$ of morphisms in C, such that for all morphisms $u\colon i \to j$ in ω, it is true that $f_j \circ D(u) = f_i$.

Then, C is a colimit of D, if for any other Cocone $(D(i) \xrightarrow{g_i} A)_{i\in\omega}$ on D, there exists unique morphism $\bar{f}\colon C \to A$ in C, such that for any $i \in \omega$, it is true that $\bar{f} \circ f_i = g_i$.

2.1 Examples of Limits and Colimits

The following is an indicative list of examples of limits and colimits, relevant to the exemplary categories met in the current text:

- **Product in Set**: Products are special cases of limits. Given sets A and B, the product is $A \times B$ equipped with projection maps π_A, π_B. This is the limit of the discrete diagram $A \leftarrow \cdot \rightarrow B$. This diagram denotes a functor from a two-object category with only identity morphisms that assigns to the two objects, the objects A and B in **Set**.

- **Inverse Limit in CompMet**: For a diagram $\cdots \to X_2 \to X_1 \to X_0$ of compact metric spaces with continuous maps, the inverse limit is the closed subset of the product $\prod X_n$ satisfying compatibility:

$$\varprojlim X_n = \left\{ (x_n) \in \prod X_n \mid \forall n, f_n(x_{n+1}) = x_n \right\}.$$

- **Coproduct in Set**: The coproduct of two sets is their disjoint union $A \sqcup B$ with canonical injections $i_A \colon A \to A \sqcup B$, $i_B \colon B \to A \sqcup B$. This is the colimit of the discrete diagram $A \leftarrow \cdot \rightarrow B$.

- **Colimit in $\mathbf{CompMet}_{\mathrm{CMS}}^{\mathrm{Emb}}$**: Given a diagram of compact metric spaces and 1-Lipschitz embedding-retraction pairs $(f_n, g_n) \colon X_n \leftrightarrows X_{n+1}$, the colimit can be constructed as the metric quotient of the disjoint union $\bigsqcup_n X_n$ under the identifications $x \sim f_n(x)$, equipped with the canonical quotient metric.

2.2 $\mathbf{CompMet}_{\mathrm{CMS}}$ has inverse Limits

$\mathbf{CompMet}_{\mathrm{CMS}}$ has inverse limits since for any ω^{op}-sequence in this category, represented by the set of non-expansive morphisms $\{X_{n+1} \overset{h_n}{\to} X_n\}_{n\in\omega}$, the product $\prod_{n\in\omega} X_n$ equipped with the maximum metric, is compact. The inverse limit is defined as the subset $\mathrm{IL} = \{(x_0, x_1, ...) \in \prod_{n\in\omega} X_n \mid x_n = h_n(x_{n+1}), n \in \omega\}$, equipped with the set of projections $\{\mathrm{IL} \overset{\pi_n}{\to} X_n\}_{n\in\omega}$, where $\pi_n(x) = x_n$, which is obviously non-expansive, since the following two conditions are satisfied by the construction. First, $\pi_n = h_n \circ \pi_{n+1}$ must hold for all $n \in \omega$. It is easy to see to see that this holds by the definitions. The second condition is that for any other set of non-expansive morphisms $\{X \overset{f_n}{\to} B_n\}_{n\in\omega}$ such that $f_n = h_n \circ f_{n+1}$ for all $n \in \omega$, where A is an object in the category, a unique morphism $f \colon X \to \mathrm{IL}$ such that $f_n = \pi_n \circ f$ for all $n \in \omega$ must exist. The proof of latter condition follows easily if we define f in the only possible way, that is $f(y) = (f_0(y), f_1(y), ...)$, and notice that $f(y) \in \mathrm{IL}$ due to the way each morphism f_n is defined. Due to the definition of the max-metric and the properties of the associated functions f is readily seen to be non-expansive, and the uniform distances between any such functions are bounded by 1.

3 A Functorial Fixed-Point Theorem

Our aim now is to describe in detail the theorem that we used for the main result of our paper. That is, theorem 3 of Adamek and Reiterman (1994). However, in order to state this theorem, we first need some additional definitions. These are general definitions for any category $\mathbf{C}$.

Any functor $F\colon \mathbf{C} \to \mathbf{C}$ is called an endofunctor. In what follows in this section, unless otherwise noted, we assume that we have such an endofunctor. We now have the following definitions.

Definition: *A morphism* $f\colon A \to B$ *in* $\mathbf{C}$ *is called an isomorphism if there exists a morphism* $g\colon B \to A$ *in* $\mathbf{C}$ *such that* $g \circ f = \mathbf{1}_A$ *and* $f \circ g = \mathbf{1}_B$*.*

Definition: *A fixed-point of* F *is an object* $A \in \mathbf{C}$ *together with an isomorphism between* A *and* $F(A)$*.*

Definition: *An object* $A \in \mathbf{C}$ *is called initial if for each object* $B \in \mathbf{C}$*, there exists unique morphism from* A *to* B*. Dually, an object* $A \in X$ *is called terminal if for each object* $B \in \mathbf{C}$ *there exists unique morphism morphism from* B *to* A*.*

Both initial and terminal objects are unique up to isomorphism. We denote the initial object by $\mathbf{0}$ and the terminal object by $\mathbf{1}$.

Definition: *A pair* (A, α)*, where* $A \in \mathbf{C}$ *and* $\alpha\colon FA \to A$ *is in the category* $\mathbf{C}$*, is called an* $F-$*algebra, or an algebra of* F*.*

We can show that $F-$algebras constitute a category where the objects are $F-$algebras and the morphisms between two objects, say (A, α) and (B, β), are the morphisms $m: A \to B$ in the category $\mathbf{C}$, that are homeomorphisms, i.e. they satisfy $m \circ \alpha = \beta \circ Fm$.

Dually, an $F-$coalgebra, or in other words, a coalgebra of F, consists of an object A and a morphism $\alpha: A \to FA$ in the category. In a completely analogous way, the category of coalgebras is constructed, with objects the coalgebras, and morphisms the homeomorphisms $m: A \to B$ in $\mathbf{C}$, for any two coalgebras (A, α) and (B, β).

It follows directly from the definitions, that a fixed-point of F can be viewed as either an algebra or a coalgebra of F.

Definition: *An initial object in the category of $F-$algebras, if it exists, is called an initial algebra. Dually, a terminal coalgebra is, if it exists, the terminal object of the category of coalgebras. Both the initial algebra and the terminal coalgebra are unique up to isomorphism.*

Moreover, by Lambek's lemma, Lambek (1968), we know that if an initial $F-$algebra exists, then it is a fixed point of F. This is a fundamental result in category theory.

Examples: To fix ideas, consider the following standard examples-see for example Adámek et al. (2025). The functor $\mathcal{P}: \mathbf{Set} \to \mathbf{Set}$, mapping a set to its power set, is an endofunctor; $\mathcal{P}$ does not have an initial algebra since there are no onto maps from the power set to the underlying set. The initial algebra of the **Set** endofunctor $F(X) = 1 + X$, where 1 is a singleton, and + signifies disjoint union, is isomorphic to the set $\mathbb{N}$. Its terminal coalgebra also exists and is isomorphic to the set $\mathbb{N}_\top = \mathbb{N} \cup \{\top\}$, where $\top$ represents an infinite descent. Its structure map τ satisfies $\tau(0) = *, \tau(n) = n - 1$ for $n > 0$, and $\tau(\top) = \top$, capturing both finite and infinite behavior in a coinductive manner. Intuitively, the terminal coalgebra represents potentially infinite unfolding: finite elements eventually reach the base case 0, while $\top$ models an endless process. This coinductive structure allows one to reason about both finite and infinite dynamics within a unified framework.

Definition: *A canonical fixed-point of an endofunctor is an initial algebra, say* $(A, \alpha\colon FA \to A)$*, for which the inverse pair* $(A, \alpha^{-1}\colon A \to FA)$ *is a terminal coalgebra for the endofunctor. If it exists, it is unique up to isomorphism.*

Hence, a functorial fixed point is canonical iff it is isomorphic to an initial algebra and a terminal coalgebra at the same time.

The following notions of functor-contractivity play a major role in the paper.

Definition: *An endofunctor on a CMS-enriched category is* ϵ*-locally contractive if there exists* $\epsilon < 1$ *such that for any two morphisms* $f, g : X \to Y$ *in* $\mathbf{C}$*, where* $X, Y \in \mathbf{C}$ *we have that* $d(Ff, Fg) \leq \epsilon \cdot d(f, g)$*.*

The functor is also called locally non-expansive if $\epsilon = 1$. The property of local contractiveness essentially means that the functor is contractive on hom-sets.

Moreover, F is called hom-contracting, if it is ϵ-locally contractive on hom-sets of the form $\mathbf{Hom}(\mathbf{1}, A)$, for any $A \in \mathbf{C}$. Clearly, a locally contractive functor is hom-contracting. Moreover:

Definition: *Consider a CMS-enriched category with embeddings,* $\mathbf{C}_{\mathrm{CMS}}^{\mathrm{Emb}}$*, and with a terminal object* $\mathbf{1}$*. An endofunctor* F *on* $\mathbf{C}_{\mathrm{CMS}}^{\mathrm{Emb}}$ *is called AR-contracting, if there exists* $\epsilon < 1$*, such that for each pair of morphisms* (f, g_f) *where* $f\colon X \to Y$ *and* $g_f\colon Y \to X$ *in the category* $\mathbf{C}_{\mathrm{CMS}}^{\mathrm{Emb}}$*, it is true that* $d_{FY}(Ff \circ Fg_f, \mathbf{1}_{FY}) \leq \epsilon * d_Y(f \circ g_f, \mathbf{1}_Y)$*, where* d_{FY} *and* d_Y *are the metrics with which the hom-sets* $\mathbf{Hom}(FY, FY)$ *and* $\mathbf{Hom}(Y, Y)$ *are equipped, respectively.*

We can state now the main theorem that we use, that is Theorem 3 from Adamek and Reiterman (1994).

Theorem:*If a CMS-enriched category* $\mathbf{C}_{\mathrm{CMS}}$ *is connected and has a terminal object and inverse limits of contracting sequences, then each AR-contracting and hom-contracting endofunctor on* $\mathbf{C}^{\mathrm{Emb}}_{\mathrm{CMS}}$*, has a canonical fixed-point.*

4 Coends

This section provides a brief introduction to the notion of a coend. For more technical depth, see Mac Lane (1978).

Consider a functor $F\colon \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{D}$, where $\mathbf{C}^{op} \times \mathbf{C}$ is the product of the opposite category $\mathbf{C}^{op}$ and $\mathbf{C}$ itself. A morphism in this product category has the form (f^{op}, f) for a morphism $f\colon C \to C'$ in $\mathbf{C}$. The objects of this product are pairs (C, C), and the functor F maps such pairs to objects in $\mathbf{D}$.

Definition (Wedge): *Given a functor* $F\colon \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{D}$*, and an object* $d \in \mathbf{D}$*, a* wedge *from* F *to* d *is a family of morphisms* $\{\zeta_C\colon F(C,C) \to d\}_{C\in\mathbf{C}}$ *such that for every morphism* $f\colon C \to C'$ *in* $\mathbf{C}$*, the diagram*

$$\zeta_C \circ F(f, \mathbf{1}_C) = \zeta_{C'} \circ F(\mathbf{1}_{C'}, f)$$

commutes.

Definition (Coend): *A coend of* F *is a pair* (d, ζ)*, where* $d \in \mathbf{D}$ *and* $\zeta\colon F \Rightarrow d$ *is a wedge, such that for any other wedge* $\zeta'\colon F \Rightarrow d'$*, there exists a unique morphism*

$h\colon d \to d'$ with $\zeta'_C = h \circ \zeta_C$ for all $C \in \mathbf{C}$. The coend is often denoted symbolically as:

$$\int^{C \in \mathbf{C}} F(C, C).$$

In concrete categories such as **Set**, the coend $\int^C F(C, C)$ can be constructed as a coequalizer:

$$\coprod_{f\colon C \to C'} F(C', C) \rightrightarrows \coprod_{C \in \mathbf{C}} F(C, C) \to \int^C F(C, C),$$

where the two arrows identify $F(f, \mathrm{id}_C)$ and $F(\mathrm{id}_{C'}, f)$ for every morphism $f\colon C \to C'$. This means that coends "mod out" redundancy along morphisms, capturing the idea of identifying equivalent representations along morphisms in $\mathbf{C}$.

In the main paper, the universal preference spaces are objects capturing how preferences across multiple hierarchical compact choice sets can be coherently aggregated. While this construction in the main text relies on an endofunctor, one may ask whether a more general notion-such as the coend-could offer a unifying perspective even when the expected utility functor becomes contravariant (e.g., due to duality or retraction structures). The coend provides a formal apparatus to aggregate functorial data when domain and codomain are allowed to vary dually, thereby generalizing the fixed-point idea and enabling representation of preference systems that are not expressible as canonical fixed points of an endofunctor. This may prove especially relevant in cases where preferences are induced via contravariant or profunctorial processes, such as conditional expectations or adjoint correspondences between belief and act spaces.

References

Adamek, J., Reiterman, J., 1994. Banach's fixed-point theorem as a base for data-type equations. Applied Categorical Structures 2, 77–90.

Adámek, J., Milius, S., Moss, L. S., 2018. Fixed points of functors. Journal of Logical and Algebraic Methods in Programming 95, 41–81.

Adámek, J., Milius, S., Moss, L. S., 2025. Initial Algebras and Terminal Coalgebras - The Theory of Fixed Points of Functors. Cambridge University Press.

Lambek, J., 1968. A fixpoint theorem for complete categories. Mathematische Zeitschrift 103, 151–161.

Leinster, T., 2014. Basic Category Theory. Cambridge University Press.

Mac Lane, S., 1978. Categories for the Working Mathematician. Graduate Texts in Mathematics.

Manes, E., Arbib, M., 1986. Algebraic approaches to program semantics. Springer Science & Business Media.

Department of Economics
Athens University of Economics and Business

List of Recent Working Papers

2023

01-23 Real interest rate and monetary policy in the post Bretton Woods United States, George C. Bitros and Mara Vidali
02-23 Debt targets and fiscal consolidation in a two-country HANK model: the case of Euro Area, Xiaoshan Chen, Spyridon Lazarakis and Petros Varthalitis
03-23 Central bank digital currencies: Foundational issues and prospects looking forward, George C. Bitros and Anastasios G. Malliaris
04-23 The State and the Economy of Modern Greece. Key Drivers from 1821 to the Present, George Alogoskoufis
05-23 Sparse spanning portfolios and under-diversification with second-order stochastic dominance, Stelios Arvanitis, Olivier Scaillet, Nikolas Topaloglou
06-23 What makes for survival? Key characteristics of Greek incubated early-stage startup(per)s during the Crisis: a multivariate and machine learning approach, Ioannis Besis, Ioanna Sapfo Pepelasis and Spiros Paraskevas
07-23 The Twin Deficits, Monetary Instability and Debt Crises in the History of Modern Greece, George Alogoskoufis
08-23 Dealing with endogenous regressors using copulas; on the problem of near multicollinearity, Dimitris Christopoulos, Dimitris Smyrnakis and Elias Tzavalis
09-23 A machine learning approach to construct quarterly data on intangible investment for Eurozone, Angelos Alexopoulos and Petros Varthalitis
10-23 Asymmetries in Post-War Monetary Arrangements in Europe: From Bretton Woods to the Euro Area, George Alogoskoufis, Konstantinos Gravas and Laurent Jacque
11-23 Unanticipated Inflation, Unemployment Persistence and the New Keynesian Phillips Curve, George Alogoskoufis and Stelios Giannoulakis
12-23 Threshold Endogeneity in Threshold VARs: An Application to Monetary State Dependence, Dimitris Christopoulos, Peter McAdam and Elias Tzavalis
13-23 A DSGE Model for the European Unemployment Persistence, Konstantinos Giakas
14-23 Binary public decisions with a status quo: undominated mechanisms without coercion, Efthymios Athanasiou and Giacomo Valletta
15-23 Does Agents' learning explain deviations in the Euro Area between the Core and the Periphery? George Economides, Konstantinos Mavrigiannakis and Vanghelis Vassilatos
16-23 Mild Explocivity, Persistent Homology and Cryptocurrencies' Bubbles: An Empirical Exercise, Stelios Arvanitis and Michalis Detsis
17-23 A network and machine learning approach to detect Value Added Tax fraud, Angelos Alexopoulos, Petros Dellaportas, Stanley Gyoshev, Christos Kotsogiannis, Sofia C. Olhede, Trifon Pavkov
18-23 Time Varying Three Pass Regression Filter, Yiannis Dendramis, George Kapetanios, Massimiliano Marcellino

19-23 From debt arithmetic to fiscal sustainability and fiscal rules: Taking stock, George Economides, Natasha Miouli and Apostolis Philippopoulos
20-23 Stochastic Arbitrage Opportunities: Set Estimation and Statistical Testing, Stelios Arvanitis and Thierry Post
21-23 Behavioral Personae, Stochastic Dominance, and the Cryptocurrency Market, Stelios Arvanitis, Nikolas Topaloglou, and Georgios Tsomidis
22-23 Block Empirical Likelihood Inference for Stochastic Bounding: Large Deviations Asymptotics Under *m*-Dependence, Stelios Arvanitis and Nikolas Topaloglou
23-23 A Consolidation of the Neoclassical Macroeconomic Competitive General Equilibrium Theory via Keynesianism (Part 1 and Part 2), Angelos Angelopoulos
24-23 Limit Theory for Martingale Transforms with Heavy-Tailed Noise, Stelios Arvanitis and Alexandros Louka

2024

01-24 Market Timing & Predictive Complexity, Stelios Arvanitis, Foteini Kyriazi, Dimitrios Thomakos
02-24 Multi-Objective Frequentistic Model Averaging with an Application to Economic Growth, Stelios Arvanitis, Mehmet Pinar, Thanasis Stengos, Nikolas Topaloglou
03-24 State dependent fiscal multipliers in a Small Open Economy, Xiaoshan Chen, Jilei Huang, Petros Varthalitis
04-24 Public debt consolidation: Aggregate and distributional implications in a small open economy of the Euro Area, Eleftherios-Theodoros Roumpanis
05-24 Intangible investment during the Global Financial Crisis in the EU, Vassiliki Dimakopoulou, Stelios Sakkas and Petros Varthalitis
06-24 Time will tell! Towards the construction of instantaneous indicators of different agent-types, Iordanis Kalaitzoglou, Stelios Arvanitis
07-24 Norm Constrained Empirical Portfolio Optimization with Stochastic Dominance: Robust Optimization Non-Asymptotics, Stelios Arvanitis
08-24 Asymptotics of a QLR-type test for optimal predictive ability, Stelios Arvanitis
09-24 Strongly Equitable General Equilibrium Allocations, Angelos Angelopoulos
10-24 The Greek macroeconomy: A note on the current situation and future outlook, Apostolis Philippopoulos
11-24 Evolution of Greek Tax System, A Survey of Legislated Tax Changes from 1974 to 2018, Panagiotis Asimakopoulos
12-24 Macroeconomic Impact of Tax Changes, The case of Greece from 1974 to 2018, Panagiotis Asimakopoulos
13-24 `Pareto, Edgeworth, Walras, Shapley' Equivalence in a Small Economy, Angelos Angelopoulos
14-24 Stimulating long-term growth and welfare in the U.S, James Malley and Apostolis Philippopoulos
15-24 Distributionally Conservative Stochastic Dominance via Subsampling, Stelios Arvanitis
16-24 Before and After the Political Transition of 1974. Institutions, Politics, and the Economy of Post-War Greece, George Alogoskoufis
17-24 Gaussian Stochastic Volatility, Misspecified Volatility Filters and Indirect Inference Estimation, Stelios Arvanitis, Antonis Demos
18-24 Endogenous Realtor Intermediation and Housing Market Liquidity, Miroslav Gabrovski, Ioannis Kospentaris, Victor Ortego-Marti
19-24 Universal Choice Spaces and Expected Utility: A Banach-type Functorial Fixed Point, Stelios Arvanitis, Pantelis Argyropoulos, Spyros Vassilakis

2025

01-25 Democracy, redistribution, and economic growth: Some evidence from post-1974 Greece, George C. Bitros
02-25 Assessing Downside Public Debt Risks in an Environment of Negative Interest Rates Growth Differentials, Yiannis Dendramis, Georgios Dimitrakopoulos and Elias Tzavalis
03-25 Wicksellian General Equilibrium, Angelos Angelopoulos
04-25 Institutions and Ethics, Angelos Angelopoulos
05-25 General Equilibrium in a Capitalist Economy, Angelos Angelopoulos
06-25 Public investment multipliers revisited: The role of production complementarities, Vasiliki Dimakopoulou, George Economides, and Apostolis Philippopoulos
07-25 The heterogeneous causal effects of the EU's Cohesion Fund, Angelos Alexopoulos, Ilias Kostarakos, Christos Mylonakis, and Petros Varthalitis
08-25 Gaussian Invariant Markov Chain Monte Carlo, Michalis K. Titsias, Angelos Alexopoulos , Siran Liu, and Petros Dellaportas
09-25 Financial Intermediation, Investment, and Monetary Policy, Ioannis Kospentaris and Florian Madison
10-25 Temperature as a premium factor. Insurance incorporating climate change. A comparative case study between Europe and North America, Nomikou-Lazarou Irene
11-25 How High will our Buildings Become? Angelos Angelopoulos
12-25 Emergency Room General Equilibrium, Angelos Angelopoulos
13-25 Updating Density Estimates using Conditional Information Projection: Stock Index Returns and Stochastic Dominance Relations, Stelios Arvanitis, Richard McGee, and Thierry Post
14-25 Measuring Textual Cohesion via Graph Curvature, Weighted Dominance, and Embedding-Based Coherence, Stelios Arvanitis
15-25 Universal Preference Spaces for Expected Utility via a Banach-type Functorial Fixed Point Stelios Arvanitis, Pantelis Argyropoulos, and Spyros Vassilakis

Department of Economics
Athens University of Economics and Business

The Department is the oldest Department of Economics in Greece with a pioneering role in organising postgraduate studies in Economics since 1978. Its priority has always been to bring together highly qualified academics and top quality students. Faculty members specialize in a wide range of topics in economics, with teaching and research experience in world-class universities and publications in top academic journals.

The Department constantly strives to maintain its high level of research and teaching standards. It covers a wide range of economic studies in micro-and macroeconomic analysis, banking and finance, public and monetary economics, international and rural economics, labour economics, industrial organization and strategy, economics of the environment and natural resources, economic history and relevant quantitative tools of mathematics, statistics and econometrics.

Its undergraduate program attracts high quality students who, after successful completion of their studies, have excellent prospects for employment in the private and public sector, including areas such as business, banking, finance and advisory services. Also, graduates of the program have solid foundations in economics and related tools and are regularly admitted to top graduate programs internationally. Three specializations are offered:1. Economic Theory and Policy, 2. Business Economics and Finance and 3. International and European Economics. The postgraduate programs of the Department (M.Sc and Ph.D) are highly regarded and attract a large number of quality candidates every year.

For more information:

https://www.dept.aueb.gr/en/econ/