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Abstract

Concepts are introduced and applied for analyzing and selecting arbitrage portfolios in the face of ambiguity about risk preferences and initial positions. A Stochastic Arbitrage Opportunity is defined as a zero-cost overlay that enhances every feasible benchmark portfolio for all admissible utility functions. The alternative to the existence of such opportunities is the solvability of a dual system of asset pricing restrictions based on a class of stochastic discount factors. Feasible approaches to numerical optimization and statistical inference are discussed. Empirical results suggest that equity factor investing is appealing for all risk-averse stock investors with sufficiently low transactions costs, by mixing multiple factor portfolios with high after-cost Information Ratio, low mutual correlation and negative downside beta. The findings weaken the case for risk-based explanations for the profitability of factor investing.

Key words: Finance; Portfolio analysis; Arbitrage portfolios; Asset pricing; Factor investing.

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1 Introduction

ARBITRAGE PORTFOLIOS play a pivotal role in Investments. Arbitrageurs can exploit mispricing without making a net investment by buying underpriced securities and short selling overpriced securities. Arbitrage portfolios are also useful for creating financial leverage and hedging systematic risk exposures through short selling. The present study introduces and applies new concepts and methods for analyzing and selecting arbitrage portfolios.

The standard concept of a Pure Arbitrage Opportunity (PAO) is based on self-financing portfolios with payoffs that are never non-positive but sometimes non-negative. Although this concept is generally accepted, it often lacks discriminatory power. PAOs are rare in many applications and their non-existence is generally places only loose restrictions on asset prices. To improve the power of the analysis, generalized arbitrage concepts have been developed.

Bernardo and Ledoit (2000) and Cochrane and Saa-Requejo (2000) propose pricing bounds for a new base asset which are based on bounds on the gain-loss ratio and bounds on the Sharpe ratio, respectively. Dybvig (1987), Constantinides, Jackwerth and Perrakis (2009), Beare (2011) and Post and Rodríguez Longarela (2021) consider joint restrictions on the prices of the base assets which ensure that the market index is not dominated by Stochastic Dominance criteria.

The present study proposes a more general approach based on partial information about investor risk preferences and initial portfolio positions (endowments). In the proposed framework, a Stochastic Arbitrage Opportunity (SAO) is a zero-cost overlay portfolio that enhances all benchmark portfolios that can be constructed from a set of base assets, for all admissible utility functions.

The analysis acknowledges possible ambiguity about existing portfolio holdings or benchmark portfolios of investors in addition to ambiguity about risk preferences. Such ambiguity notably arises for hedge fund managers and empirical researchers of market efficiency.

Hedge funds (as opposed to mutual fund managers) generally do not have an explicit benchmark and invest opportunistically in a broad universe. In addition, estimates of a hedge fund's investment style based on historical returns tend to be inaccurate, due to the use of concentrated and dynamic positions, limited disclose, and valuation problems for illiquid assets owned by the fund.

Academic researchers generally have only partial information about the portfolio holdings of individual investors. The adverse effect of benchmark error on hypothesis testing in empirical asset pricing research is well known. The S&P500 stock index (SPX) is an imperfect proxy for the global market portfolio, as it ignores bonds, real estate and international securities. It also does not represent domestic stock investors who actively deviate from the passive market capitalization weights, focus on certain stock market segments or eschew certain market segments.

The SAO concept is based on Expected Utility and allows for general classes of utility functions, including the general Second-degree Stochastic Dominance (SSD) class used in Constantinides, Jackwerth and Perrakis (2009) and Post and Rodríguez Longarela (2021). By allowing for general specifications of the risk preferences and return distribution, the analysis can account for asymmetric risk and tail risk. This feature is particularly relevant for arbitrage portfolios (as opposed to standard portfolio), because long-short positions can feature extreme levels of skewness and excess kurtosis, through financial leverage effects.

By contrast, Korkie and Turtle (2002) analyze arbitrage portfolios by considering only the mean and the variance of the distribution. The mean-variance approach is analytically convenient but it can lead to false conclusions and suboptimal choice if returns are not Gaussian.

The analysis also allows for general investment restrictions on the benchmarks and overlays. The restrictions may be externally defined by regulators and clients of money managers. They may also be used to model proportional transaction costs, as in Jouini and Kallal (2001), enhance the robustness for estimation error, as in DeMiguel, Garlappi, Nogales and Uppal (2009), or mitigate default risk and price impact, as in Post and Rodriguez Longarela

(2021).

The SAO concept is designed for the analysis of incomplete capital markets with market frictions; the analysis therefore eschews assumptions about the existence of a unique Stochastic Discount Factor (SDF) and risk-neutral distribution. The existence of SAOs defies equilibrium in a class of asset pricing models. A novel system of asset pricing restrictions is derived which is satisfied if and only if SAOs do not exist. These pricing restrictions define a class of SDFs which obey economic regularity conditions for representative investor models.

The focus is on the cross-sectional choice from a multitude of risky base assets in a single-period model. Multi-period models can be represented or approximated through the inclusion of conditional portfolios (Hansen and Richard (1987)) and/or timing portfolios (Brandt and Santa-Clara (2006)) as base assets in a single-period problem. Furthermore, the existence of arbitrage opportunities in a single-period model generally is a sufficient condition for the existence of arbitrage opportunities in a multi-period model, as investors generally can pursue buy-and-hold strategies even when portfolio rebalancing is allowed.

For practical implementation, we will demonstrate that methods for numerical optimization and statistical inference which have been tried and tested for simpler portfolio problems can also be used to evaluate overlays and identify SAOs. For discrete estimators of the joint distribution function, empirical SAOs can be identified by solving finite and convex optimization problems. In addition, Empirical Likelihood Ratio tests can be used to test whether a given overlay is a SAO or to test whether an empirical SAO is a population SAO out of sample.

The proposed framework is applied to equity factor investing. The empirical asset pricing literature forwards a number of factor portfolios such as the ubiquitous Small-Minus-Big (SMB) and High-Minus-Low (HML) portfolios by Fama and French (1996). A question of theoretical and practical importance is whether the investment returns to these portfolios reflect active trading opportunities or systematic risk premiums. We analyze whether existing factor portfolios are SAOs for a broad class of risk preferences and initial positions. Following

DeMiguel, Martin-Utrera, Nogales and Uppal (2020), we also take a portfolio perspective and ask whether multiple factor portfolios can be combined to form a SAO.

In this application, PAOs do not occur (which motivates the use of a generalized arbitrage concept), and, in addition, skewness and non-linear dependence play an important role (which motivates the use of Expected Utility in addition to the mean-variance approach). The analysis accounts for trading costs (Frazzini, Israel and Moskowitz (2015), Novy-Marx and Velikov (2016), Patton and Weller (2020)) and declining profitability over time (McLean and Pontiff (2016) and Jacobs and Müller (2020)).

2 Analytical Framework

In this section, the concept of SAO is introduced (Section 2.2) together with a system of dual restrictions on the market prices of the base assets (Section 2.3), and discussion of overlay choice (Section 2.4). Subsection 2.5 discusses empirical counterparts of the theoretical concepts. The Appendix includes the proofs of Thm 2.3.1 and Lemma 2.4.3.

2.1 Preliminaries

The focus is on a single-period investment problem with N base assets with payoffs at the investment horizon, $\boldsymbol{x} := (x_1 \cdots x_N)^{\mathrm{T}} \in \mathbb{R}^N$, $N < \infty$. The base assets may be individual securities or portfolios of securities. To represent or approximate certain multi-period problems, the base assets may include dynamic portfolios which are periodically re-balanced based on conditioning information. Riskfree base assets may be included.

The joint probability distribution function of the payoffs is given by $\mathcal{F} : \mathcal{X} \to [0,1]$, where $\mathcal{X} := [a, b]$, $-\infty < a < b < \infty$,[1] includes the supports of all base assets. The distribution

[1]Gross investment returns are naturally bounded from below by zero ($a \geq 0$), and net returns by -100% ($a \geq -1$). An upper bound $b < \infty$ is assumed here for numerical purposes, and the empirical application uses a discrete distribution estimator $\mathcal{F}_T$ with upper bound b equal to the sample maximum return. The

function is generally conditional on the prevailing stage of the business cycle and/or financial market conditions.

The prices of the base assets are taken as given, and are collected in the price vector $\boldsymbol{p} \in \mathbb{R}^N$. Payoffs may be converted to gross returns $\boldsymbol{r} := \mathrm{diag}^{-1}(\boldsymbol{p})\,\boldsymbol{x}$. The analysis can account for proportional transaction costs in the form of a bid-ask spread by introducing two sets of prices, $\boldsymbol{p}^{Bid}$ and $\boldsymbol{p}^{Ask}$, as in Jouini and Kallal (2001) and Constantinides, Jackwerth and Perrakis (2009); see the end of this subsection.

Risk preferences are represented by utility functions $u : \mathbb{R} \to \mathbb{R}$. Instead of specifying a particular functional form, a set of functions, $\mathcal{U}$, is defined using general functional properties. The functions are assumed to be continuously differentiable, strictly increasing and strictly concave. Additional assumptions can be employed to improve the discriminatory power, e.g., the assumptions about higher-order risk aversion properties for higher-degree SD rules. For numerical purposes, it is generally useful to transform the utility functions to avoid that $u(x) \approx c$ for all $x \in \mathcal{X}$.

The set $\mathcal{U}$ is assumed to be uniformly bounded and convex. This assumption is not very restrictive, given that the support of the payoff distribution is bounded and the proposed stochastic order is invariant to linear transformation and mixing of utility functions. For example, the set of all aforementioned utility functions can be represented using $\mathcal{U}_2$, the bounded set of strictly positive mixtures of elementary Russell and Seo (1989) functions $v_{2;\phi}(x) = -(\phi - x)_+$, $\phi \in \mathcal{X}$.

The portfolio possibilities are described by two distinct portfolio sets: a set of benchmark portfolios, $\mathrm{K} \subseteq \mathbb{R}^N$, and a set of arbitrage portfolios, $\Delta \subseteq \mathbb{R}^N$. Korkie and Turtle (2002) use a similar dichotomy and refer to the two portfolio sets as the Investment Opportunity Set and the Self-Financing Investment Opportunity Set, respectively. The asset positions $\boldsymbol{\kappa} \in \mathrm{K}$ and $\boldsymbol{\delta} \in \Delta$ translate into portfolio payoffs $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\kappa}$ and $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\delta}$.

The benchmark set K is assumed to be closed and bounded but it need not be convex.

statistical theory however does not require an upper bound for the population distribution $\mathcal{F}$, and the assumption $b < \infty$ can be relaxed by placing second moment existence conditions for the return process.

It may consist of a single benchmark index, e.g., a general market index or a tailor-made style index. It could also include a multitude of benchmarks, e.g., the universe of all passive mutual funds for a specific asset class. The benchmark set could also consist of a continuum of convex mixtures of the base assets. In that case, we use $\mathrm{K}^{(0)}$ to denote the set of extreme elements. Since the benchmarks are standard portfolios, the position vector $\boldsymbol{\kappa} \in \mathrm{K}$ must obey the budget restriction $\boldsymbol{p}^{\mathrm{T}}\boldsymbol{\kappa} = c$, for budget $c > 0$. Asset positions may be converted to portfolio weights $\boldsymbol{\lambda} := c^{-1}(\boldsymbol{p} \odot \boldsymbol{\kappa})$.

The overlays $\boldsymbol{\delta} \in \Delta$ are combinations of short and long positions with a net investment of zero, or $\boldsymbol{p}^{\mathrm{T}}\boldsymbol{\delta} = 0$. The overlays can be added to a benchmark $\boldsymbol{\kappa} \in \mathrm{K}$ to form a combined portfolio $\boldsymbol{\lambda} = (\boldsymbol{\kappa} + \boldsymbol{\delta})$ that is a standard portfolio ($\boldsymbol{p}^{\mathrm{T}}\boldsymbol{\lambda} = c$). Overlay portfolio positions may equivalently be expressed using weights in the combined portfolio, or $\boldsymbol{\gamma} := c^{-1}(\boldsymbol{p} \odot \boldsymbol{\delta})$. An overlay could be constructed using active investing by the owner or manager of the benchmark portfolio or, alternatively, by an external specialized intermediary such as a hedge fund.

It is assumed that the overlay set is a bounded and convex polytope which is characterized by R linear restrictions: $\Delta := \left\{\boldsymbol{\delta} \in \mathbb{R}^N : \mathbf{A}\boldsymbol{\delta} \leq \boldsymbol{a}\right\}$; here, $\mathbf{A}$ is a $(R \times N)$ matrix of left-hand-side coefficients, and $\boldsymbol{a}$ is $(R \times 1)$. For example, proportional transactions costs can be modeled by replacing the self-financing constraint $\boldsymbol{p}^{\mathrm{T}}\boldsymbol{\delta} = 0$ with a system of inequalities that preempts buying at bid prices or selling at ask prices: $\boldsymbol{\delta}_1^{\mathrm{T}}\boldsymbol{p}^{Ask} - \boldsymbol{\delta}_2^{\mathrm{T}}\boldsymbol{p}^{Bid} = 0$; $\boldsymbol{\delta} = \boldsymbol{\delta}_1 + \boldsymbol{\delta}_2$; $\boldsymbol{\delta}_1, \boldsymbol{\delta}_2 \geq \mathbf{0}_N$. The overlay set is assumed to include the 'passive' solution, or $\mathbf{0}_N \in \Delta$; this solution is assumed to be the default choice if qualified arbitrage opportunities do not exist.

The set of all feasible combined portfolios is given by the vector subspace sum $\Lambda_0 := \mathrm{K} + \Delta$. The analysis does not attempt to identify individual standard portfolios $\boldsymbol{\lambda} \in \Lambda_0$ which stand out as being particularly appealing, for two reasons. First, it is generally not possible to identify a combined portfolio which is superior to all benchmarks for all relevant utility functions. Second, the benchmark set is a superset of the relevant portfolios for multiple investors; some elements of this superset may be infeasible for some of the investors. Instead, the analysis identifies arbitrage portfolios $\boldsymbol{\delta} \in \Delta$ which are appealing overlays for all feasible benchmarks and all relevant utility functions.

2.2 Stochastic Arbitrage Opportunities

Definition 2.2.1 (Pure Arbitrage Opportunities). *An overlay $\boldsymbol{\delta} \in \Delta$ is a pure arbitrage opportunity if its payoff is positive with certainty and non-negative with non-zero probability. The set of all feasible PAOs is given by*

$$\Delta_{\mathcal{F}}^{\mathrm{PA}} := \{\boldsymbol{\delta} \in \Delta : \mathbb{P}_{\mathcal{F}}[\boldsymbol{x}'\boldsymbol{\delta} \geq 0] = 1; \mathbb{P}_{\mathcal{F}}[\boldsymbol{x}'\boldsymbol{\delta} > 0] > 0\}. \tag{1}$$

The set of PAOs is based on minimal assumptions and it avoids specification error for risk preferences and/or initial investment positions. The flip side is that it is empty or small in many applications. To improve the discriminatory power, generalizations can be developed based on general assumptions about preferences and endowments.

A partial order for arbitrage portfolios is introduced here to generalize the notion of PAO to SAO. To allow for compact notation, the expected utility increment is denoted by $D(u, \boldsymbol{\kappa}, \boldsymbol{\delta}, \mathcal{F}) := \mathbb{E}_{\mathcal{F}}\left[u(\boldsymbol{x}^{\mathrm{T}}(\boldsymbol{\kappa} + \boldsymbol{\delta}))\right] - \mathbb{E}_{\mathcal{F}}\left[u(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\kappa})\right]$, for $(u, \boldsymbol{\kappa}, \boldsymbol{\delta}) \in \mathcal{U} \times \mathrm{K} \times \Delta$.

Definition 2.2.2 (Strict Stochastic Arbitrage Opportunities). *An overlay $\boldsymbol{\delta} \in \Delta$ (strictly) stochastically enhances a given benchmark $\boldsymbol{\kappa} \in \mathrm{K}$, or $(\boldsymbol{\kappa} + \boldsymbol{\delta}) \succ_{(\mathcal{U},\mathcal{F})} \boldsymbol{\kappa}$, if $D(u, \boldsymbol{\kappa}, \boldsymbol{\delta}, \mathcal{F}) > 0$ for all $u \in \mathcal{U}$. It is a (strict) stochastic arbitrage opportunity if such enhancement is achieved for all $\boldsymbol{\kappa} \in \mathrm{K}$. The set of all feasible (strict) SAOs is given by*

$$\Delta_{(\mathcal{U},\mathrm{K},\mathcal{F})}^{\succ} := \{\boldsymbol{\delta} \in \Delta : D(u, \boldsymbol{\kappa}, \boldsymbol{\delta}, \mathcal{F}) > 0 \; \forall \, (u, \boldsymbol{\kappa}) \in \mathcal{U} \times \mathrm{K}\}. \tag{2}$$

If a SAO $\boldsymbol{\delta} \in \Delta_{(\mathcal{U},\mathrm{K},\mathcal{F})}^{\succ}$ exists, then every benchmark $\boldsymbol{\kappa} \in \mathrm{K}$, regardless of its efficiency inside the benchmark set K, is inefficient in the combined portfolio set Λ_0, in the sense that it is not optimal for any utility function, and inferior to the combined portfolio $\boldsymbol{\kappa}+\boldsymbol{\delta}$.

The set of SAOs naturally increases as the restrictions on the overlays (Δ) are loosened and/or the restrictions on the benchmarks (K) and/or risk preferences ($\mathcal{U}$) are tightened.

If a riskless benchmark asset is feasible, then the SAO set equals the PAO set, under general conditions. Let $\mathcal{U}_2(c) := \{u \in \mathrm{cl}(\mathcal{U}_2) : u(x) < u(c)\, \forall x < c; u(x) = u(c)\, \forall x \geq c\}$, $c \in \mathbb{R}$. This set includes the aforementioned elementary utility functions $v_{2;\phi}(x)$, $\phi \in [a,b]$ as well as all increasing and concave transformations of those functions.

Proposition 2.2.3 (SAO vs. PAO). *If the benchmark set includes a riskless alternative* $\boldsymbol{\kappa}_0 \in \mathrm{K}$ *with riskless return* $\boldsymbol{x}'\boldsymbol{\kappa}_0 = c \in [a,b]$*, and* $\mathrm{cl}\,(\mathcal{U}) \cap \mathcal{U}_2(c) \neq \emptyset$, $c \in \mathbb{R}$*, then the set of SAOs set reduces to the set of PAOs, or* $\Delta^{\succ}_{(\mathcal{U},\mathrm{K},\mathcal{F})} = \Delta^{\mathrm{PA}}_{\mathcal{F}}$.

The condition $\mathrm{cl}\,(\mathcal{U}) \cap \mathcal{U}_2(c) \neq \emptyset$ obviously holds for $\mathcal{U} = \mathcal{U}_2$, as $\mathrm{cl}\,(\mathcal{U}_2)$ includes the elementary utility functions $v_{2;\phi}(x)$, $\phi \in [a,b]$.

In case of the singleton specification $\mathrm{K} = \{\boldsymbol{\kappa}\}$ and $\mathcal{U} = \mathcal{U}_2$, the enhanced portfolios $\boldsymbol{\lambda} \in \left(\{\boldsymbol{\kappa}\} + \Delta^{\succ}_{(\mathcal{U},\mathrm{K},\mathcal{F})}\right)$ dominate the benchmark by SSD, or $\boldsymbol{\lambda} \succ_{(\mathcal{U}_2,\mathcal{F})} \boldsymbol{\kappa}$, and the simpler risk arbitrage concept of Post and Rodriguez Longarela (2021) is obtained as a special case. The definition of SAO is more general and can expand the set of arbitrage opportunities by imposing additional risk preference assumptions, and avoid specification error for the initial positions by using multiple benchmarks.

The analysis is invariant to the inclusion of convex mixtures of the benchmark portfolios, that is, $\Delta^{\succ}_{(\mathcal{U},\mathrm{K},\mathcal{F})} = \Delta^{\succ}_{(\mathcal{U},\mathrm{conv(K)},\mathcal{F})}$, due to the convexity of the stochastic enhancement relation with respect to the benchmark portfolio positions.[2] This property facilitates a convenient discretization of the problem of searching for SAOs; see Section 2.4.

Perhaps surprisingly, the analysis is not invariant to the inclusion of non-optimal elements in K.[3] For this reason, the benchmark set generally cannot be replaced without consequences

[2]Using the Independence Axiom for Expected Utility, the two pairwise relations $\lambda\,(\boldsymbol{\kappa}_1+\boldsymbol{\delta}) \succsim_{(\mathcal{U},\mathcal{F})} \lambda\boldsymbol{\kappa}_1$ and $(\boldsymbol{\kappa}_2+\boldsymbol{\delta}) \succsim_{(\mathcal{U},\mathcal{F})} \boldsymbol{\kappa}_2$ imply $(\lambda\boldsymbol{\kappa}_1 + (1-\lambda)\boldsymbol{\kappa}_2 + \boldsymbol{\delta}) \succsim_{(\mathcal{U},\mathcal{F})} (\lambda\boldsymbol{\kappa}_1 + (1-\lambda)\boldsymbol{\kappa}_2)$, for any $\lambda \in [0,1]$.

[3]The transitivity of the dominance relation entails $\left((\boldsymbol{\kappa}_1+\boldsymbol{\delta}) \succ_{(\mathcal{U},\mathcal{F})} \boldsymbol{\kappa}_1\right) \wedge \left(\boldsymbol{\kappa}_1 \succ_{(\mathcal{U},\mathcal{F})} \boldsymbol{\kappa}_2\right) \Rightarrow \left((\boldsymbol{\kappa}_1+\boldsymbol{\delta}) \succ_{(\mathcal{U},\mathcal{F})} \boldsymbol{\kappa}_2\right)$, but not $\left((\boldsymbol{\kappa}_1+\boldsymbol{\delta}) \succ_{(\mathcal{U},\mathcal{F})} \boldsymbol{\kappa}_1\right) \wedge \left(\boldsymbol{\kappa}_1 \succ_{(\mathcal{U},\mathcal{F})} \boldsymbol{\kappa}_2\right) \Rightarrow \left((\boldsymbol{\kappa}_2+\boldsymbol{\delta}) \succ_{(\mathcal{U},\mathcal{F})} \boldsymbol{\kappa}_2\right)$, as the enhancement relation relies on the dependence structure between the overlay and the benchmark in addition to the two marginal distributions.

by the subset of its optimal elements. Nevertheless, non-optimal elements may deliberately be excluded to sharpen the analysis, if the relevant benchmarks are believed to be constructed using optimization. Riskless alternatives are a case in point. Although some base assets and arbitrage portfolios may be riskless, it is generally desirable to exclude riskless benchmarks (and avoid the set reduction $\Delta^{\succ}_{(\mathcal{U},\mathrm{K},\mathcal{F})} = \Delta^{\mathrm{PA}}_{\mathcal{F}}$), because riskless alternatives are generally not optimal for any investor in the face of risk premiums or pricing errors.

A SAO should not be confused with a stochastic bound (Arvanitis, Post and Topaloglou (2021)), or a combined portfolio $\boldsymbol{\lambda} \in \Lambda_0$ which dominates every benchmark, or $\boldsymbol{\lambda} \succ_{(\mathcal{U}_2,\mathcal{F})} \boldsymbol{\kappa}$ for all $\boldsymbol{\kappa} \in \mathrm{K}$. A SAO is an arbitrage portfolio $\boldsymbol{\delta} \in \Delta$, while a bound is a standard portfolio $\boldsymbol{\lambda} \in \Lambda_0$. In addition, exact bounds generally do not exist if K includes multiple portfolios with diverse risk profiles, which is the reason for the use of approximate bounds in Arvanitis, Post and Topaloglou (2021). The definition of SAO is much less demanding than the definition of a stochastic bound, because the benchmark portfolio of a SAO enters both on the left-hand side and the right-hand-side of the pairwise order $(\boldsymbol{\kappa} + \boldsymbol{\delta}) \succ_{(\mathcal{U},\mathcal{F})} \boldsymbol{\kappa}$, so that the combined portfolio is 'updated' if the benchmark is changed. Instead of looking for a standard portfolio which is superior to every benchmark, the search is for a portfolio adjustment which is an improvement for every benchmark. This subtle adjustment yields a large improvement in discriminatory power in relevant applications.

2.3 Asset pricing restrictions

One of the first principles of asset pricing states that a PAO does not exist ($\Delta^{\mathrm{PA}}_{\mathcal{F}} = \emptyset$) if and only if a non-negative SDF $m : \mathbb{R}^N \to \mathbb{R}$ exists; see, e.g., Dybvig and Ross (2008). This section extends this principle from PAOs to SAOs. The non-existence of SAOs can be shown to be equivalent to the existence of a SDF which obeys economic regularity conditions beyond non-negativity.

Some additional concepts are introduced to facilitate the derivation and interpretation.

A common specification of the SDF takes the shape of an Intertemporal Marginal Rate of Substitution: $m(\boldsymbol{x}) = d(u, \boldsymbol{\kappa})u'(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\kappa})$, $(u, \boldsymbol{\kappa}) \in \mathcal{U} \times \mathrm{K}$, where the scalar $d(u, \boldsymbol{\kappa}) > 0$ captures time preferences and the marginal utility function $u'(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\kappa})$ captures risk preferences. The set $\mathcal{M}_{(\mathcal{U},\mathrm{K})} := \left\{m(\boldsymbol{x}) = d(u, \boldsymbol{\kappa})u'(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\kappa}); (u, \boldsymbol{\kappa}) \in \mathcal{U} \times \mathrm{K}\right\}$ includes all candidate SDFs which take this shape for some permissible utility function and some feasible benchmark. To invoke standard results from game theory and duality theory, the analysis uses the convex hull $\mathrm{conv}\left(\mathcal{M}_{(\mathcal{U},\mathrm{K})}\right)$ and the polar cone $\Delta^* := \left\{\boldsymbol{q} = \mathbf{A}^{\mathrm{T}}\boldsymbol{\sigma} : \boldsymbol{a}^{\mathrm{T}}\boldsymbol{\sigma} \leq 0; \boldsymbol{\sigma} \geq \mathbf{0}_R\right\}$, which restricts the shadow prices of the investment constraints $\mathbf{A}\boldsymbol{\delta} \leq \boldsymbol{a}$.

Theorem 2.3.1 (Existential Condition). *(Strict) SAOs do not exist for a given utility function class* $\mathcal{U}$*, benchmark set* K *and overlay set* Δ*, or* $\Delta^{\succ}_{(\mathcal{U},\mathrm{K},\mathcal{F})} = \emptyset$*, if and only if*

$$\left(\mathbb{E}_{\mathcal{F}}\left[m(\boldsymbol{x})\boldsymbol{x}\right] \in \Delta^*\right) \text{ for some } m \in \mathrm{conv}\left(\mathcal{M}_{(\mathcal{U},\mathrm{K})}\right). \tag{3}$$

In other words, the Present Value of the base asset payoffs, or $\mathbb{E}_{\mathcal{F}}\left[m(\boldsymbol{x})\boldsymbol{x}\right]$, must lie inside the polar cone Δ^*, for some permissible SDF which take the shape of the IMRS of investors with the assumed preferences and endowments or mixtures of the IMRS of multiple investors, if SAOs do not exist.[4]

The investment constraints relax the pricing restrictions. For example, in the case with proportional transactions costs, the polar cone is $\Delta^* = \left\{\boldsymbol{q} \in \mathbb{R}^N_+ : \boldsymbol{p}^{Bid} \leq \boldsymbol{q} \leq \boldsymbol{p}^{Ask}\right\}$ and the Present Values must lie inside the bid-ask intervals.

The pricing restriction (3) reduces to existence of a non-negative SDF $m : \mathbb{R}^N \to \mathbb{R}$ if a riskless benchmark $\boldsymbol{\kappa}_0 \in \mathrm{K}$ with riskless return $c \in [a, b]$ is feasible and some elementary utility function $u \in \mathcal{U}_2(c)$ is admissible, as in Proposition 2.2.3. Else, the pricing restriction is tighter due to the restrictions placed on preferences and endowments.

Since the SDF is not unique, the pricing restrictions could possibly be tightened further

[4] The convex hull $\mathrm{conv}\left(\mathcal{M}_{(\mathcal{U},\mathrm{K})}\right)$ is used here instead of $\mathcal{M}_{(\mathcal{U},\mathrm{K})}$, to establish a general necessary and sufficient condition. Without the convexification, a duality gap may arise between the arbitrage conditions and pricing conditions.

by extending the single-period portfolio problem to a multi-period portfolio problem. This approach is particularly useful in applications with a small number of base assets for which the joint stochastic process can be estimated with reasonable accuracy, for example, the index option pricing problems considered by Cochrane and Saa-Requejo (2000, Section III), Constantinides, Jackwerth and Perrakis (2009, Appendix A) and Perrakis and Oancea (2022). An common approximation to multi-period problems in high-dimensional applications is to include managed portfolios as base assets. For example, the benchmark portfolios and factor portfolios in our empirical application are periodically re-balanced and introduce risk-return combinations and pricing restrictions which are not feasible in a pure single-period model.

2.4 Overlay choice

The attention is now turned to the selection of a specific SAO which optimizes a given objective function. Using the strict SAO set $\Delta^{\succ}_{(\mathcal{U},\mathrm{K},\mathcal{F})}$ as the feasible set can be analytically inconvenient, as it may be an empty set or an open set, as a result of the definition based on strict inequalities and strict risk aversion. To facilitate numerical analysis, the feasible set is closed by using weak inequalities ($D(u,\boldsymbol{\kappa},\boldsymbol{\delta},\mathcal{F}) \geq 0$) and closure of the utility function class ($\mathrm{cl}\,(\mathcal{U})$) using an alternative definition based on weak inequalities and weak risk aversion.

Definition 2.4.1 (Weak Stochastic Arbitrage Opportunities). *An overlay $\boldsymbol{\delta} \in \Delta$ (weakly) stochastically enhances a given benchmark $\boldsymbol{\kappa} \in \mathrm{K}$, or $(\boldsymbol{\kappa}+\boldsymbol{\delta}) \gtrsim_{(\mathcal{U},\mathcal{F})} \boldsymbol{\kappa}$, if $D(u,\boldsymbol{\kappa},\boldsymbol{\delta},\mathcal{F}) \geq 0$ for all $u \in \mathrm{cl}(\mathcal{U})$. It is a (weak) stochastic arbitrage opportunity if such enhancement is achieved for all $\boldsymbol{\kappa} \in \mathrm{K}$. The set of all feasible (weak) SAOs is given by*

$$\Delta^{\gtrsim}_{(\mathcal{U},\mathrm{K},\mathcal{F})} := \{\boldsymbol{\delta} \in \Delta : D(u,\boldsymbol{\kappa},\boldsymbol{\delta},\mathcal{F}) \geq 0 \; \forall \, (u,\boldsymbol{\kappa}) \in \mathrm{cl}\,(\mathcal{U}) \times \mathrm{K}\} \,. \tag{4}$$

By construction, the weak set is a superset of the strict set: $\Delta^{\gtrsim}_{(\mathcal{U},\mathrm{K},\mathcal{F})} \supseteq \Delta^{\succ}_{(\mathcal{U},\mathrm{K},\mathcal{F})}$. It is convex and closed. Since the passive solution $\{\mathbf{0}_N\}$ is included, the weak SAO set is non-empty ($\Delta^{\gtrsim}_{(\mathcal{U},\mathrm{K},\mathcal{F})} \neq \emptyset$). Furthermore, $\boldsymbol{\delta} \in \Delta^{\succeq}_{(\mathcal{U},\mathrm{K},\mathcal{F})} \Rightarrow c\boldsymbol{\delta} \in \Delta^{\succeq}_{(\mathcal{U},\mathrm{K},\mathcal{F})}$ for all $c \in [0,1]$, so that SAOs are not disconnected portfolios but part of a continuous neighborhood of SAOs.

The enlargement $(\Delta^{\gtrsim}_{(\mathcal{U},\mathrm{K},\mathcal{F})} - \Delta^{\succ}_{(\mathcal{U},\mathrm{K},\mathcal{F})})$ is inconsequential for the analysis. One specific subset consists of trivial solutions: $\Delta^{\equiv}_{\mathcal{F}} := \left\{\boldsymbol{\delta} \in \Delta : \boldsymbol{x}^{\mathrm{T}}\boldsymbol{\delta} = 0\, \forall \boldsymbol{x} : \mathcal{F}(\boldsymbol{x}) > 0\right\}$, which includes the passive solution $\{\mathbf{0}_N\}$.[5] Including these trivial solutions is inconsequential, as they cannot be optimal if strict SAOs exist; moreover, if strict SAOs do not exist, then the default choice is the passive solution $\{\mathbf{0}_N\}$ which is equivalent to any trivial solution $\boldsymbol{\delta} \in \Delta^{\equiv}_{\mathcal{F}}$. Including the non-trivial weak SAOs $\left(\Delta^{\gtrsim}_{(\mathcal{U},\mathrm{K},\mathcal{F})} - \Delta^{\succ}_{(\mathcal{U},\mathrm{K},\mathcal{F})} - \Delta^{\equiv}_{\mathcal{F}}\right)$ is also inconsequential. Any of these portfolios is accompanied by strict SAOs its close proximity which are indistinguishable from it in terms of the value for continuous objective functions.[6]

By Def. 2.4.1, (weak) SAOs are solutions to the following inequality system:

$$D(u, \boldsymbol{\kappa}, \boldsymbol{\delta}, \mathcal{F}) \geq 0\ \forall\, (u, \boldsymbol{\kappa}) \in \mathrm{cl}(\mathcal{U}) \times \mathrm{K}; \tag{5}$$

$$\boldsymbol{\delta} \in \Delta.$$

This system is analytically challenging for two reasons. First, closed-form solutions generally do not exist for the portfolio payoff distribution $\mathcal{F}_{\boldsymbol{\lambda}}(x) := \int_{\{\boldsymbol{x}:\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\lambda} \leq x\}} d\mathcal{F}(\boldsymbol{x})$ and the portfolio-level expected utility $\mathbb{E}_{\mathcal{F}}\left[u(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\lambda})\right] = \int u(x) d\mathcal{F}_{\boldsymbol{\lambda}}(x)$, even for relatively simple distributions such as the multivariate log-normal distribution. Second, the system has a semi-infinite structure due to the need to evaluate the first set of model restrictions for every $(u, \boldsymbol{\kappa}) \in \mathrm{cl}(\mathcal{U}) \times \mathrm{K}$, although the number of decision variables (M) is finite.

These complications generally call for some sort of discretization of $\mathcal{F}$, $\mathcal{U}$ and K to allow for numerical analysis using finite mathematical programming problems. A number of

[5]In fact, the passive solution $\{\mathbf{0}_N\}$ is the only element of $\Delta^{\equiv}_{\mathcal{F}}$ if the payoffs to the base assets are linearly independent, as is true in our empirical application.

[6]Mixing $\boldsymbol{\delta} \in \left(\Delta^{\gtrsim}_{(\mathcal{U},\mathrm{K},\mathcal{F})} - \Delta^{\succ}_{(\mathcal{U},\mathrm{K},\mathcal{F})} - \Delta^{\equiv}_{\mathcal{F}}\right)$ with $\mathbf{0}_N$ always yields a feasible strict SAO: for any $(u, c) \in \mathcal{U} \times (0, 1)$, we have that $D\left(u, \boldsymbol{\kappa}, \left((c\boldsymbol{\delta} + (1-c)\mathbf{0}_N\right), \mathcal{F}\right) > cD\left(u, \boldsymbol{\kappa}, \boldsymbol{\delta}, \mathcal{F}\right) \geq 0$, and thus, $(c\boldsymbol{\delta} + (1-c)\mathbf{0}_N) \in \Delta^{\succ}_{(\mathcal{U},\mathrm{K},\mathcal{F})}$. As a result, $\left(\Delta^{\gtrsim}_{(\mathcal{U},\mathrm{K},\mathcal{F})} - \Delta^{\equiv}_{\mathcal{F}}\right) \neq \emptyset \Leftrightarrow \Delta^{\succ}_{(\mathcal{U},\mathrm{K},\mathcal{F})} \neq \emptyset$. In addition, as $c \uparrow 1$, the mixture becomes indistinguishable from the real thing.

general approaches are discussed below.

Continuous distributions $\mathcal{F}$ can be discretized using, e.g., Monte-Carlo simulation methods or lattice models. The trade-off between the achieved accuracy and the required computer burden naturally depends on the dimensions of the distribution.

If the benchmark set K is continuous, then it may be replaced without loss by its set of extreme elements $\mathrm{K}^{(0)}$, for example, the vertices of a polytope. This replacement is allowed due to the convexity property of the stochastic enhancement relation with respect to the benchmark portfolio positions; see Section 2.2.

Since expected utility is a linear function of $u \in \mathrm{cl}(\mathcal{U})$, and $\mathrm{cl}(\mathcal{U})$ is a convex set, $\mathrm{cl}(\mathcal{U})$ may be replaced with the set of its extreme elements, or $\mathcal{U}^{(0)} : \mathrm{cl}(\mathcal{U}) = \mathrm{conv}\left(\mathcal{U}^{(0)}\right)$, without loss of generality. For many relevant specifications of $\mathcal{U}$, the extreme elements $u \in \mathcal{U}^{(0)}$ are low-dimensional functions, which reduces the numerical complexity of searching over $\mathrm{cl}(\mathcal{U})$.

For SSD, the elementary function set $\mathcal{U}_2^{(0)}$ consists of the elementary Russell and Seo (1989) utility functions $v_{2;\phi}(x) = -(\phi - x)_+$, $\phi \in [a, b]$. The one-dimensional parameter space $[a, b]$ can easily be discretized with an arbitrary level of precision. The enhancement constraints $D(v_{2;\phi}, \boldsymbol{\kappa}, \boldsymbol{\delta}, \mathcal{F}) \geq 0$ are not smooth. However, they can be relaxed to an equivalent system of linear inequalities, for discrete distributions, as in Rockafellar and Uryasev (2000), Kuosmanen (2004) and Dentcheva and Ruszczynski (2006), or, evaluated using iterative Mixed-Integer Linear Programming algorithms as Fábián, Mitra and Roman (2011). Hodder, Jackwerth and Kolokolova (2015) and Constantinides, Czerwonko and Perrakis (2020) are examples of large-scale applications of this approach.

Similarly, for general n-th degree Stochastic Dominance, the elementary utility functions are $v_{n;\phi}(x) = -(\phi - x)_+^{n-1}$, and $D(v_{n;\phi}, \boldsymbol{\kappa}, \boldsymbol{\delta}, \mathcal{F}) \geq 0$ is a convex non-smooth polynomial constraint which can be relaxed to set of Convex Polynomial Programming constraints.

Combining these insights, the semi-infinite system of inequalities (5) can be reduced to the following system:

$$D(u,\boldsymbol{\kappa},\boldsymbol{\delta},\mathcal{F}) \geq 0 \;\forall\,(u,\boldsymbol{\kappa}) \in \mathcal{U}^{(0)} \times \mathrm{K}^{(0)}; \tag{6}$$
$$\boldsymbol{\delta} \in \Delta.$$

This system is finite in our application, due to the finite number of elements of $\mathrm{K}^{(0)}$ and $\mathcal{U}_2^{(0)}$.

Maximizing a concave objective function $G(\boldsymbol{\delta},\mathcal{F})$ subject to this system yields a finite convex optimization problem. The choice of the objective function naturally is application specific, just as the choice of $\mathcal{U}$ and K.

In our empirical application, the objective is the expected payoff of the overlay: $G_1(\boldsymbol{\delta},\mathcal{F}) := \mathbb{E}_{\mathcal{F}}\left[\boldsymbol{x}\right]'\boldsymbol{\delta}$. The optimal value $G_1(\boldsymbol{\delta}^*,\mathcal{F})$ seems a natural measure of risk-adjusted or abnormal performance, as it measures the largest improvement of expected payoff that can be is achieved without deteriorating expected utility for any admissible utility function and benchmark portfolio. This measure is comparable with standard alphas, or expected abnormal returns, computed relative to risk factor models. Clearly, $G_1(\boldsymbol{\delta}^*,\mathcal{F}) > 0$ implies that $\boldsymbol{\delta}^*$ is a weak SAO; $G_1(\boldsymbol{\delta}^*,\mathcal{F}) = 0$ implies that strict SAOs do not exist.

One possible alternative specification based on ambiguity aversion uses the maxmin objective $G_2(\boldsymbol{\delta},\mathcal{F}) := \min_{\mathcal{U},\mathrm{K}} D(u,\boldsymbol{\kappa},\boldsymbol{\delta},\mathcal{F})$ or the smallest improvement across all utility functions and benchmark portfolios. Clearly, $G_2(\boldsymbol{\delta}^*,\mathcal{F}) > 0$ implies that $\boldsymbol{\delta}^*$ is a strict SAO; $G_2(\boldsymbol{\delta}^*,\mathcal{F}) = 0$ implies that strict SAOs do not exist. For numerical purposes, this approach requires that the utility functions are normalized to avoid that $u(x) \approx c$ for.all $x \in \mathcal{X}$ to avoid that $G_2(\boldsymbol{\delta}^*,\mathcal{F}) \approx 0$ even if strict SAOs exist. If $\mathrm{K}^{(0)}$ and $\mathcal{U}_2^{(0)}$ are discrete, then this approach could be implemented by using the finite formulation $G_2(\boldsymbol{\delta},\mathcal{F}) := \min\left\{\theta : \theta \leq D(u,\boldsymbol{\kappa},\boldsymbol{\delta},\mathcal{F}),\ \forall(u,\boldsymbol{\kappa}) \in \mathcal{U}^{(0)} \times \mathrm{K}^{(0)}\right\}$ or an equivalent dual maximization problem.

2.5 Empirical counterparts

Empirical counterparts are introduced for the theoretical concepts. We discuss how the asymptotic statistical theory of Post, Karabatı and Arvanitis (2018) for optimization with SSD constraints (that is, the special case with $\mathrm{K} = \{\boldsymbol{\kappa}\}$ and $\mathcal{U} = \mathcal{U}_2$) can be generalized to SAO choice for a general benchmark set K and utility class $\mathcal{U} \subseteq \mathcal{U}_2$. We also discuss how statistical inference about membership to the SAO set can be performed using an Empirical Likelihood Ratio (ELR) test in the spirit of Davidson and Duclos (2014), Post and Poti (2017) and Post (2017).

Sampling variation with high probability does not cause violations of inequalities which are not binding under the population distribution. An important role in the statistical theory of the estimation of SAO sets is therefore played by equivalence relations, or binding weak stochastic enhancement relations $D(u, \boldsymbol{\kappa}, \boldsymbol{\delta}, \mathcal{F}) = 0$, $(u, \boldsymbol{\kappa}) \in \mathcal{U} \times \mathrm{K}$. Introducing multiple benchmarks increases the occurrence of such equivalence relations, which requires an extension of the original theory from a single benchmark to a benchmark set.

2.5.1 Empirical SAO set properties

Following Post, Karabatı and Arvanitis (2018), we assume that the latent CDF $\mathcal{F}$ is estimated by the empirical CDF (ECDF) $\mathcal{F}_T$ from a time series sample $(\boldsymbol{x}_t)_{t=1,\ldots,T}$, as in the empirical application in Section 3. Since payoffs generally are non-stationary random variables, the analysis can benefit from transforming payoffs to returns $\boldsymbol{r} := \mathrm{diag}^{-1}(\boldsymbol{p})\,\boldsymbol{x}$ and positions to weights $\boldsymbol{\lambda} := c^{-1}(\boldsymbol{p} \odot \boldsymbol{\kappa})$ and $\boldsymbol{\gamma} := c^{-1}(\boldsymbol{p} \odot \boldsymbol{\delta})$, which does not alter the economic meaning. In order to focus on the effect of sampling variation on the stochastic enhancement constraints, it is furthermore assumed that the constraints in K and Δ are deterministic, as is true in our empirical application to factor investing. Extensions are briefly discussed at the end of this subsection.

The following empirical weak SAO set is constructed using the ECDF:

$$\Delta^{\gtrsim}_{(\mathcal{U},\mathrm{K},\mathcal{F}_T)} := \left\{ \boldsymbol{\delta} \in \triangle : (\boldsymbol{\kappa} + \boldsymbol{\delta}) \gtrsim_{(\mathcal{U},\mathcal{F}_T)} \boldsymbol{\kappa},\ \forall \boldsymbol{\kappa} \in \mathrm{K} \right\}. \tag{7}$$

This empirical set can be shown to possess a number of favorable consistency properties under two high-level assumptions which are motivated using detailed lower-level assumptions in footnotes below. The first assumption concerns the limiting properties of the relevant empirical processes: (i) the process $m_T D(u, \boldsymbol{\kappa}, \boldsymbol{\delta}, \mathcal{F}_T - \mathcal{F})$ is convergent in distribution, for some positive real sequence $m_T \to \infty$, and (ii) the process $G(\boldsymbol{\delta}, \mathcal{F}_T) - G(\boldsymbol{\delta}, \mathcal{F})$ converges uniformly to zero with high probability.[7]

The second high-level assumption is that the goal $G(\boldsymbol{\delta}, \mathcal{F})$ is Lipschitz continuous as a function of $\boldsymbol{\delta}$, and, furthermore, the neighborhood of every weak SAO $\boldsymbol{\delta}_1 \in \Delta^{\gtrsim}_{(\mathcal{U},\mathrm{K},\mathcal{F})}$ contains strict SAOs $\boldsymbol{\delta}_2 \in \Delta^{\succ}_{(\mathcal{U},\mathrm{K},\mathcal{F})}$ whenever strict SAOs exist, so that the strict SAO set $\Delta^{\succ}_{(\mathcal{U},\mathrm{K},\mathcal{F})}$ is equivalent to the larger weak SAO set $\Delta^{\gtrsim}_{(\mathcal{U},\mathrm{K},\mathcal{F})}$ in terms of the feasible values of the objective function. [8]

Under these two assumptions, it can be shown that the empirical optimal value of the goal function $G(\boldsymbol{\delta}^*, \mathcal{F}_T)$ approximates its latent population value $G(\boldsymbol{\delta}^*, \mathcal{F})$ with high probability, and that any portfolio at which the empirical solutions accumulate will be a population optimal SAO. Among other things, these consistency results imply that the probability of two relevant types of decision errors vanishes asymptotically: (I) selecting an empirical SAO which is not a population SAO and (II) selecting a suboptimal SAO due to the false exclusion

[7]Low-level statistical assumptions that are standard in empirical finance can be used in order to ensure (i) and (ii). Stationarity and ergodicity of the return process, coupled with the compactness of $\mathcal{X} \times \Delta$ and the Lipschitz continuity of G with continuous modulus, would imply the uniform convergence required in (ii), due to Birkhoff's ULLN. If ergodicity is strengthened to strong mixing with sufficiently rapidly vanishing mixing coefficients, then along with the compactness of $\mathcal{X}$, the subsequent boundedness of $D(\cdot,\cdot,\cdot,\mathcal{F})$ and the uniform (in u) Lipschitz continuity property of $D(u,\cdot,\cdot,\mathcal{F})$, it would imply that (i) holds with m_T being the standard $\sqrt{T}$ rate and with limiting Gaussianity.

[8]A mild sufficient condition for the neighborhood assumption can be obtained by generalizing the Weak Independence assumption of Post, Karabatı and Arvanitis (2018) to the assumption that, for any non constant, locally linear v, there exists some $\boldsymbol{\delta}(v) \in \Delta^{\gtrsim}_{(\mathcal{U},\mathrm{K},\mathcal{F})}$, such that $D(v, \boldsymbol{\kappa}, \boldsymbol{\delta}(v), \mathcal{F}) > 0$, for any $\boldsymbol{\kappa} \in K$. This 'Joint Enhancement' condition is much weaker than the strict SAO condition, as it allows for different overlays $\boldsymbol{\delta}(v)$ for different utility functions v on the boundary of $\mathrm{cl}\,(\mathcal{U})$.

of better SAOs.

For Type I error, the reasoning is as follows. An overlay which is not a weak SAO, or $\boldsymbol{\delta} \notin \Delta^{\succeq}_{(\mathcal{U},\mathrm{K},\mathcal{F})}$, features at least one strict inequality $D(u, \boldsymbol{\kappa}, \boldsymbol{\delta}, \mathcal{F}) < 0$, for some $(u, \boldsymbol{\kappa}) \in \mathcal{U} \times \mathrm{K}$. Since strict inequalities are robust to sampling variation, the probability that such an overlay is falsely included in the empirical SAO set, or $\boldsymbol{\delta} \in \Delta^{\succeq}_{(\mathcal{U},\mathrm{K},\mathcal{F}_T)}$, is asymptotically negligible.

For analyzing Type II error, we face a non-vanishing probability of false exclusion of population non-strict SAOs $\boldsymbol{\delta} \in \left(\Delta^{\succeq}_{(\mathcal{U},\mathrm{K},\mathcal{F})} - \Delta^{\succ}_{(\mathcal{U},\mathrm{K},\mathcal{F})}\right)$, which feature equivalence $D(u, \boldsymbol{\kappa}, \boldsymbol{\delta}, \mathcal{F}) = 0$, for some $(u, \boldsymbol{\kappa}) \in \mathcal{U} \times \mathrm{K}$. However, these non-strict SAOs can be approximated by equivalent sequences of strict SAOs $\boldsymbol{\delta}_T \in \Delta^{\succ}_{(\mathcal{U},\mathrm{K},\mathcal{F})}$ which feature asymptotically strict (scaled) empirical inequalities $m_T D(u, \boldsymbol{\kappa}, \boldsymbol{\delta}_T, \mathcal{F}_T) > 0$ for all $(u, \boldsymbol{\kappa}) \in \mathcal{U} \times \mathrm{K}$, and are therefore included in $\Delta^{\succeq}_{(\mathcal{U},\mathrm{K},\mathcal{F}_T)}$ with high probability. Convergence of $G(\boldsymbol{\delta}, \mathcal{F}_T) - G(\boldsymbol{\delta}, \mathcal{F})$ and the continuity of the goal imply then that the relevant approximating sequences of optimal elements of $\Delta^{\gtrsim}_{(\mathcal{U},\mathrm{K},\mathcal{F})}$ are also approximately empirically optimal.

The statistical theory can be extended to cases where $\mathcal{F}$ is a conditional distribution and/or estimated using alternative non-parametric methods or (semi-)parametric models. The estimation may be based on a well-specified (semi-)parametric model or a fully non-parametric method, depending on the data properties and available side information. (Quasi) Maximum Likelihood or Nadaraya-Watson type estimators could be employed respectively. Standard assumptions that involve smoothness of the associated likelihood function, or regularity properties of the unknown densities involved and the associated kernels, would suffice to obtain consistency.

It is also possible to include stochastic constraints in K and Δ, for example, constraints on Value-at-Risk and risk factor loadings. The statistical analysis can be extended to cover such additional constraints incorporating in the asymptotic analysis the additional sources of stochasticity.

2.5.2 Empirical Likelihood Ratio (ELR) test

Beyond consistent set estimation, our framework can also be used for statistical inference about the set of SAOs. The null hypothesis $\mathbf{H}_0 : \boldsymbol{\delta} \in \Delta^{\gtrsim}_{(\mathcal{U},\mathrm{K},\mathcal{F})}$ can be tested in sample for a given arbitrage portfolio $\boldsymbol{\delta} \in \Delta^{\gtrsim}_{(\mathcal{U},\mathrm{K},\mathcal{F})}$ or out of sample for an arbitrage portfolio which is optimal in a given sample. A consistent and asymptotically conservative test about $\mathbf{H}_0$ can be based on an ELR statistic, in the spirit of Davidson and Duclos (2014), Post and Poti (2017) and Post (2017). The ELR test is based on the smallest adjustments to the probability mass of the ECDF which suffice to qualify the evaluated arbitrage portfolio as a weak SAO, using the Kullback-Leibler divergence from the optimal distribution to the historical ECDF.

The test statistic can be computed by solving the following Minimum Relative Entropy problem:

$$
\begin{aligned}
&\min \mathrm{KL}\left(\mathcal{F}_T || F\right) \\
&\text{s.t. } \boldsymbol{\delta} \in \Delta^{\gtrsim}_{(\mathcal{U},\mathrm{K},F)}; \\
&F \in \Phi.
\end{aligned} \tag{8}
$$

Here, the divergence is defined as $\mathrm{KL}\left(\mathcal{F}_T || F\right) := \sum_{t=1}^{T} f_T(\boldsymbol{x}_t) \ln(f_T(\boldsymbol{x}_t)/f(\boldsymbol{x}_t))$, where Φ is the set of discrete distributions with the same support as the ECDF. Since the distribution function F is discrete, only the probability mass levels $f(\boldsymbol{x}_t)$, $t = 1, \cdots, T$, need to be determined, and the Minimum Relative Entropy problem reduces to a standard, finite Convex Optimization problem, if also $\mathcal{U}^{(0)}$ and $\mathrm{K}^{(0)}$ are discrete.

The test statistic is defined as $\mathrm{ELR} = 2T \cdot \mathrm{KL}\left(\mathcal{F}_T || F\left(\boldsymbol{\delta}\right)\right)$, with $F\left(\boldsymbol{\delta}\right)$ the solution of (8). The exact null distribution of ELR is not known, but we obtain an asymptotically conservative test using the central chi-squared distribution with degrees of freedom equal to

$N_K := \text{card}(\text{K}^{(0)})$, the number of extreme points of K. The number of degrees of freedom reflects that stochastic enhancement of the vertices of the benchmark set is a necessary and sufficient condition for being a SAO.

The $\chi^2_{N_K}$ distribution is the exact asymptotic distribution for an ELR test statistic for standard moment equalities $D(u, \boldsymbol{\kappa}_i, \boldsymbol{\delta}, \mathcal{F}) = 0 \; i = 1, \cdots, N_K$, for a given arbitrage portfolio $\boldsymbol{\delta} \in \Delta$ and given utility function $u \in \mathcal{U}$, if the data are serially IID distributed. This known distribution majorizes the latent distribution of the ELR test statistic in large samples, and can be used for conservative inference, if the moment conditions are inequalities and the utility function is endogenously selected from $\mathcal{U}$.

When the number of vertices is large, a potentially less conservative test can be obtained by choosing the degrees of freedom by moment selection procedures; see for example Andrews and Soares (2010). Exact tests can be constructed by combining moment selection and bootstrap resampling in the spirit of Canay (2010).

In our application to monthly returns of diversified equity portfolios, temporal dependence seems limited compared to mutual dependence. For applications with more pronounced temporal dependence, the ELR test can be efficiently modified using a blockwise approach.

Smoothed Empirical Likelihood and parametric Likelihood Ratio tests can respectively replace the ELR test used for statistical inference, in cases where $\mathcal{F}$ is estimated using kernel based non-parametric methods or parametric models.

3 Factor Investing

Our framework is used to analyze historical investment returns to equity factor portfolios from the empirical asset pricing literature. We analyze whether individual factor portfolios are SAOs and whether multiple factor portfolios can be combined to form SAOs. The focus is on all risk-averse investors, or $\mathcal{U} = \mathcal{U}_2$, or the SSD criterion.

Monthly percentage total returns to various benchmark portfolios and factor portfolios are taken from Kenneth French' online library.[9] The sample period is from July 1963 through December 2021 (702 months), the longest period for which data for all factor portfolios is available at the time of writing. The CDF is estimated using the ECDF, $\mathcal{F}_T$. The analysis considers three non-overlapping sub-periods (1963-1982, 1983-2002, 2003-2021), to perform statistical tests out of sample.

3.1 Equity benchmark portfolios

Three alternative specifications of the benchmark set are used: $\mathrm{K}_1, \mathrm{K}_{10}$ and K_{49}. K_1 considers only the value-weighted market index – the traditional approach to enhanced benchmarking. The two alternative specifications consider all convex mixtures of multiple standard portfolios that are formed and annually re-balanced by grouping individual stocks based on their four-digit Standard Industrial Classification (SIC) codes. K_{10} considers 10 value-weighted industry portfolios; K_{49} considers 49 value-weighted industry portfolios.

We will distinguish between the benchmark set $\mathrm{K} = \mathrm{K}_1, \mathrm{K}_{10}, \mathrm{K}_{49}$ that is used for constructing SAOs and the benchmark set $\mathrm{K}' = \mathrm{K}_1, \mathrm{K}_{10}, \mathrm{K}_{49}$ that is used for evaluating whether a given overlay is a SAO. This dichotomy allows us to analyze how far an overlay that is constructed for one benchmark set (K) is from being a SAO for another benchmark set ($\mathrm{K}' \neq \mathrm{K}$).

Table 1 gives summary statistics for the 10 industry portfolios; for brevity, statistics for the 49 industry portfolios are not tabulated here. The portfolios show substantial variation in average return, standard deviation, skewness and annualized Sharpe Ratio (SR), and hence represent a diverse range of initial positions. A SAO will have to enhance expected utility for all combinations of these portfolios.

[9]http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.

Table 1: Equity Industry Benchmark Portfolios: 1963-2021

		NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other
Marg.	Mean	1.043	1.046	0.989	0.977	1.104	0.851	1.085	1.083	0.832	0.961
Distr.	StDev	4.235	6.803	4.914	5.933	6.336	4.615	5.115	4.799	4.003	5.289
(Net)	Skew	-0.298	0.677	-0.498	0.064	-0.240	-0.198	-0.218	0.030	-0.140	-0.492
	SR (ann.)	0.553	0.344	0.437	0.356	0.401	0.364	0.485	0.517	0.403	0.388
Corr.	NoDur	1	0.63	0.81	0.49	0.59	0.63	0.82	0.75	0.62	0.82
Matrix	Durbl	0.63	1	0.80	0.49	0.66	0.57	0.73	0.51	0.40	0.76
	Manuf	0.81	0.80	1	0.63	0.77	0.65	0.82	0.71	0.52	0.90
	Enrgy	0.49	0.49	0.63	1	0.44	0.44	0.44	0.43	0.51	0.60
	HiTec	0.59	0.66	0.77	0.44	1	0.61	0.72	0.62	0.31	0.71
	Telcm	0.63	0.57	0.65	0.44	0.61	1	0.63	0.54	0.50	0.68
	Shops	0.82	0.73	0.82	0.44	0.72	0.63	1	0.68	0.46	0.82
	Hlth	0.75	0.51	0.71	0.43	0.62	0.54	0.68	1	0.46	0.70
	Utils	0.62	0.40	0.52	0.51	0.31	0.50	0.46	0.46	1	0.56
	Other	0.82	0.76	0.90	0.60	0.71	0.68	0.82	0.70	0.56	1
(Down)	ALL	0.78	1.21	1.03	0.87	1.23	0.77	0.99	0.82	0.51	1.10
Market	Perc50	0.78	1.16	1.04	0.94	1.13	0.79	0.93	0.73	0.53	1.14
Beta	Perc25	0.81	1.20	1.04	0.92	1.13	0.80	0.94	0.74	0.52	1.12
	Perc10	1.01	1.41	1.22	0.98	1.01	0.58	1.11	0.74	0.54	1.14

The correlation matrix shows generally high positive correlations, as is typical for diversified long-only portfolios in the same asset class and the same country. The possibilities for risk reduction through diversification across the benchmarks are therefore limited. Negative correlations could however be introduced through short positions, notably by adding properly chosen arbitrage portfolios as overlays to the benchmarks.

The table also shows the regular market beta and the downside beta for the 50%, 25% and 10% of worst market returns. A SAO will have to enhance both benchmarks for cyclical industries such as Consumer Durables (Durbl) and benchmarks for non-cyclical industries such as Utilities (Utls). It will also have to account for the non-linear dependency that is reflected in the divergence between the downside betas and the regular betas.

3.2 Equity factor portfolios

The overlay set Δ consists of all convex combinations of seven factor portfolios. The factor portfolios are arbitrage portfolios which consist of long positions in one equity market segment and short positions in another segment. The market segments are defined as the intersections of two groups of stocks based on market capitalization of equity and three groups based on a particular second stock characteristic (which is different for different factor portfolios).

The first four factor portfolios are based on the non-market factors of Fama and French (2015): Small Minus Big (SMB), High Minus Low (HML), Robust Minus Weak (RMW) and Conservative Minus Aggressive (CMA).[10] Three additional factor portfolios are based on the research on trading strategies based on past realized return by Jegadeesh (1990), Jegadeesh and Titman (1993), and De Bondt and Thaler (1987): Short-Term Reversal (STR), Intermediate-Term Momentum (ITM) and Long-Term Reversal (LTR). We were reluctant to include a larger number of factor portfolios, to limit the risk of overfitting to the data.

Since factor portfolios take positions in small cap stocks and are periodically rebalanced, trading costs must be taken into account for practical implementation; see, for example, Frazzini, Israel and Moskowitz (2015), Novy-Marx and Velikov (2016), and Patton and Weller (2020). The costs naturally depend on the portfolio size and the trading infrastructure and brokerage services available to a specific investor.

In this study, we use cost estimates by Frazzini, Israel and Moskowitz (2015, Table VIII Panel A). They use proprietary transaction data to estimate the lowest costs at which factor portfolios can be replicated. For a Net Asset Value of USD 100 million, the minimum costs to replicate SMB, HML, STR and ITM subject to a 1% tracking error restriction is estimated at 0.21%, 0.57%, 2.14% and 6.17% per annum, respectively. We assume here that RMW, CMA and LTR have the same costs as HML (0.57%), because these portfolios have similar

[10] The market factor (RMRF) is not included here, as it can only be used to change the level of financial leverage and market exposure and not to take concentrated positions in a particular stock market segment. Unreported results show that including RMRF does not affect our results and conclusions.

turnover. We study the robustness of our results to doubling, tripling or quadrupling the cost estimates.

It should be stressed that the trading costs of a mixture of multiple factor portfolios is likely to be lower than computed using linear interpolation, because individual stocks may appear in the long leg of some factor portfolios and in the short leg of other factor portfolios, which reduces the overall portfolio turnover and trading costs, as pointed out by DeMiguel, Martin-Utrera, Nogales and Uppal (2020).

Table 2 gives summary statistics for the seven factor portfolios. Also included is the Equal-Weighted Portfolio (EWP) which can be seen as a heuristic and robust solution to the arbitrage portfolio selection problem. The average return, standard deviation, skewness and annualized Information Ratio (IR) vary considerably across the factor portfolios. CMA and STR stand out among these portfolios, with an appealing combination of positive skewness and high IR. The low correlations point at significant possibilities for risk reduction through diversification across multiple factors. Indeed, the EWP achieves an impressive IR of 1.013 (before tradings costs).

Attractively, RMW and CMA have negative downside betas. These factor portfolios are expected to reduce downside risk when they are added as an overlay to a standard stock portfolio. By contrast, the other factor portfolios and the EWP have a positive downside beta, which makes these portfolios less suitable for downside risk reduction.

The bottom panel shows p-values for the ELR test for being a SAO (after trading costs) for $\mathrm{K}' = \mathrm{K}_1, \mathrm{K}_{10}, \mathrm{K}_{49}$ and the full sample period. If the value-weighted index is the only benchmark ($\mathrm{K}' = \mathrm{K}_1$), RMW and CMA are the only factor portfolios that do not significantly violate the SAO conditions. If the benchmark set is enlarged to $\mathrm{K}' = \mathrm{K}_{49}$, then only CMA passes the ELR test (p-value: 0.133). The EWP is a borderline case which comes close to being a SAO (p-value: 0.099). Despite its high gross IR, the positive downside beta and relatively high trading costs limit the appeal of this arbitrage portfolio. Of course, it is possible that transactions costs for EWP are lower than assumed here after the netting of

Table 2: Equity Factor Portfolios: 1963-2021

		SMB	HML	RMW	CMA	STR	ITM	LTR	EWP
Marg.	Mean	0.229	0.269	0.270	0.262	0.481	0.629	0.204	0.335
Distr.	StDev	3.034	2.901	2.210	1.977	3.133	4.208	2.572	1.145
(Gross)	Skew	0.343	0.007	-0.296	0.324	0.431	-1.281	0.548	0.950
	IR (ann.)	0.262	0.321	0.423	0.460	0.532	0.518	0.275	1.013
Corr.	SMB	1	-0.02	-0.35	-0.09	0.17	-0.06	0.34	0.40
Matrix	HML	-0.02	1	0.09	0.67	0.04	-0.22	0.48	0.60
	RMW	-0.35	0.09	1	-0.03	-0.09	0.09	-0.28	0.09
	CMA	-0.09	0.67	-0.03	1	-0.11	-0.03	0.48	0.54
	STR	0.17	0.04	-0.09	-0.11	1	-0.31	0.12	0.30
	ITM	-0.06	-0.22	0.09	-0.03	-0.31	1	-0.11	0.28
	LTR	0.34	0.48	-0.28	0.48	0.12	-0.11	1	0.65
(Down)	ALL	0.192	-0.138	-0.097	-0.168	0.215	-0.151	0.011	-0.019
Market	Perc50	0.285	-0.057	-0.121	-0.169	0.241	-0.021	-0.002	0.022
Beta	Perc25	0.277	-0.065	-0.090	-0.174	0.264	-0.042	-0.030	0.020
	Perc10	0.404	0.062	-0.042	-0.093	0.266	0.019	0.113	0.104
Costs (ppa)		0.21	0.57	0.57	0.57	2.14	6.17	0.57	1.54
P-values	$\mathrm{K}' = \mathrm{K}_1$	0.000	0.368	1	1	0.000	0.072	0.011	0.157
	$\mathrm{K}' = \mathrm{K}_{10}$	0.001	0.050	0.191	0.256	0.000	0.057	0.060	0.120
	$\mathrm{K}' = \mathrm{K}_{49}$	0.000	0.000	0.004	0.133	0.000	0.001	0.000	0.099

off-setting stock positions.

Table 3 summarizes results for the three subperiods. The appeal of the factor portfolios clearly declines over time. In the third subperiod, five out of seven portfolios have negative average returns, and none of the portfolios is a SAO for $\mathrm{K}' = \mathrm{K}_{49}$. This pattern is consistent with existing evidence about out-of-sample and post-publication decline of return predictability in the US stock market due to data snooping effects and/or increased arbitrage activity; see for example, McLean and Pontiff (2016) and Jacobs and Müller (2020).

3.3 Optimized overlay portfolios

The next question is whether the factor portfolios can be combined to form robust SAOs, given the low correlation between the factor portfolios. To answer this question, we engineer

Table 3: Equity Factor Portfolios: Three Subperiods

			SMB	HML	RMW	CMA	STR	ITM	LTR	EWP
	Marg.	Mean	0.541	0.404	-0.018	0.269	0.292	0.741	0.464	0.385
	Distr.	StDev	3.195	2.602	1.627	2.003	2.688	3.639	2.568	0.974
	(Net)	Skew	0.207	-0.156	0.062	0.027	0.956	-0.480	0.875	0.446
'63-'82		IR (ann.)	0.586	0.537	-0.037	0.465	0.377	0.706	0.626	1.369
	P-values	$K' = K_1$	0.000	1	0.048	1	0.007	0.165	1	1
		$K' = K_{10}$	0.011	0.593	0.012	1	0.010	0.139	0.727	0.952
		$K' = K_{49}$	0.000	0.216	0.000	1	0.000	0.078	0.106	0.241
	Marg.	Mean	-0.044	0.418	0.490	0.399	-0.039	0.758	0.258	0.320
	Distr.	StDev	3.274	3.208	2.883	2.297	3.583	4.462	2.466	1.282
	(Net)	Skew	0.524	0.321	-0.647	0.331	0.309	-0.730	0.702	2.105
'83-'02		IR (ann.)	-0.046	0.451	0.589	0.602	-0.037	0.588	0.363	0.865
	P-values	$K' = K_1$	0.077	1	1	1	0.013	0.261	0.337	0.739
		$K' = K_{10}$	0.046	0.017	0.286	0.173	0.001	0.063	0.254	0.393
		$K' = K_{49}$	0.000	0.003	0.032	0.173	0.000	0.063	0.013	0.177
	Marg.	Mean	0.140	-0.174	0.188	-0.035	-0.364	-0.172	-0.270	-0.098
	Distr.	StDev	2.543	2.824	1.870	1.515	3.028	4.426	2.639	1.100
	(Net)	Skew	0.184	-0.379	0.207	0.376	0.329	-2.301	0.181	-0.482
'03-'21		IR (ann.)	0.190	-0.213	0.347	-0.080	-0.417	-0.135	-0.354	-0.310
	P-values	$K' = K_1$	0.012	0.037	1	0.570	0.002	0.288	0.001	0.060
		$K' = K_{10}$	0.001	0.016	0.320	0.208	0.000	0.156	0.000	0.005
		$K' = K_{49}$	0.000	0.000	0.031	0.001	0.000	0.001	0.000	0.000

empirical SAOs using numerical optimization.

The objective is to maximize the average return subject to the empirical SAO conditions and the position limits $\delta_i \geq 0$, $i = 1, \cdots, 7$, and $\sum_{i=1}^{7} \delta_i \leq 1$. The position limits are imposed to enhance the robustness of the solutions and the comparability with the individual factor portfolios (which obey the same restrictions). Since the factor portfolios already include short positions in particular stock market segments, negative positions are not required to allow for short sales in this application.

The overlay choice problem is solved for $\mathrm{K} = \mathrm{K}_1, \mathrm{K}_{10}, \mathrm{K}_{49}$. We use the numerical strategy for $\mathcal{U} = \mathcal{U}_2$ and polyhedral K that is described in Section 2.4.[11]Transactions costs for mixed factor portfolios are computed in a conservative way using linear interpolation (ignoring the netting of off-setting stock positions). The ELR test that is described in Section 2.5.2 is used to test whether the overlays that are constructed for $\mathrm{K} = \mathrm{K}_1, \mathrm{K}_{10}, \mathrm{K}_{49}$ are SAO for $\mathrm{K}' = \mathrm{K}_1, \mathrm{K}_{10}, \mathrm{K}_{49}$.

For the sake of comparison, a second set of overlays is formed based on the Mean-Variance Dominance (MVD) criterion by maximizing average return subject to lowering the standard deviation for every benchmark portfolio. Comparison is also made with the 'naive' EWP.

Table 4 summarizes the results for the full sample period. Shown are the overlay composition, performance measures, market betas, and p-values for the ELR test for $\mathrm{K}' = \mathrm{K}_1, \mathrm{K}_{10}, \mathrm{K}_{49}$.

If the value-weighted market index is the only benchmark ($\mathrm{K} = \mathrm{K}_1$), the optimal SSD overlay diversifies across four factor portfolios: HML, RMW, CMA and ITM. As the benchmark set is enlarged to $\mathrm{K} = \mathrm{K}_{10}, \mathrm{K}_{49}$, the optimal overlay becomes more concentrated in RMW and CMA, the results become more negatively skewed and downside beta becomes

[11]The full optimization problem is $\max \left\{ \mathbb{E}_{\mathcal{F}_T} \left[\boldsymbol{x}'\boldsymbol{\delta} \right] : \boldsymbol{\delta} \in \Delta^{\succeq}_{(\mathcal{U},\mathrm{K},\mathcal{F}_T)} \right\}$. The continuous benchmark set $\mathrm{K} = \mathrm{K}_1, \mathrm{K}_{10}, \mathrm{K}_{49}$ is replaced by the set of its extreme elements $\mathrm{K}^{(0)}$, that is, the 1, 10 or 49 benchmark portfolios. The utility function set $\mathcal{U}_2$ is replaced by $\mathcal{U}_2^{(0)}$, the set of elementary Russell and Seo (1989) utility functions $v_{2;\phi}(x) = -(\phi - x)_+$, $\phi \in [a, b]$, where the sample range $[a, b]$ is discretized using 50 equally-spaced grid points. The non-smooth enhancement constraints $D(v_{2;\phi}, \boldsymbol{\kappa}, \boldsymbol{\delta}, \mathcal{F}_T) \geq 0$ are relaxed to an equivalent system of linear inequalities, for discrete distributions, as in Rockafellar and Uryasev (2000).

negative.

The MVD overlays differ in important ways from the SSD overlays. For $\mathrm{K} = \mathrm{K}_{10}, \mathrm{K}_{49}$, the MVD overlay struggles to reduce the risk of non-cyclical industries such as Utilities. Instead of mixing RMW and CMA, the MVD overlay lowers variance by shrinking the overlay portfolio; the sum of the weights is only 0.28. The resulting arbitrage portfolio has a lower mean, lower skewness and lower IR than the SSD overlay.

The above analysis uses fixed portfolio weights for the entire sample period and, in addition, the statistical tests were performed in sample which has the effect of inflating the p-values. To explore robustness and statistical significance, the overlay choice problem for $\mathrm{K} = \mathrm{K}_{10}$ was also solved for each of the three subperiods. For the first two subperiods (1963-1982 and 1983-2002), we can test whether the optimal overlay is a SAO for $\mathrm{K}' = \mathrm{K}_1, \mathrm{K}_{10}, \mathrm{K}_{49}$ in the subsequent subperiod (1983-2002 and 2003-2021, respectively).

The results are shown in Table 5. Despite the general decline in profitability of factor investing, the SSD overlay is a SAO out of sample; even in the third subperiod (2003-2021) and for $\mathrm{K}' = \mathrm{K}_{49}$, the ELR test cannot be rejected (p-value: 0.267). The composition of the overlay changes substantially over time, with less focus on past-return factors (STR and ITM) and more focus on fundamental factors (RMW and CMA). Despite these changes, the overlays which are optimized in a given subperiod do not significantly violate the SAO conditions in the subsequent period. Since the individual factor portfolios are not SAOs (see Table 3), the results critically depend on diversification across multiple factors.

The MVD overlay trails the SSD overlay in terms of average return in every subperiod. The table also reveals that the MVD portfolio for the second subperiod is not a SAO for $\mathrm{K}' = \mathrm{K}_{49}$ out of sample (p-value: 0.000). In the third subperiod, the MVD criterion struggles to find arbitrage opportunities and the optimal overlay becomes small.

The cost estimates by Frazzini, Israel and Moskowitz (2015) can be seen as a lower bound for the costs of replicating the individual factor portfolios, for most investors. Table 6 analyzes the robustness of the results to doubling, tripling or quadrupling the estimated

Table 4: Optimized Overlay Portfolios: 1963-2021

	$\mathcal{U}$		SSD			MVD		EWP
	K	K_1	K_{10}	K_{49}	K_1	K_{10}	K_{49}	
Weights	SMB		0.05	0.00		0.02		0.14
	HML	0.32	0.01	0.02	0.22		0.00	0.14
	RMW	0.09	0.34	0.34		0.13	0.00	0.14
	CMA	0.16	0.30	0.28	0.21	0.05	0.17	0.14
	STR		0.03					0.14
	ITM	0.43	0.16	0.13	0.57	0.13	0.11	0.14
	LTR		0.02	0.00		0.11		0.14
	Sum	1	0.91	0.78	1	0.44	0.28	1
Marg.	Mean	0.319	0.228	0.201	0.352	0.120	0.086	0.206
Distr.	StDev	1.996	1.146	1.109	2.452	0.669	0.560	0.145
(Net)	Skew	-0.951	0.436	0.440	-1.256	-0.464	-0.847	0.950
	IR	0.554	0.689	0.627	0.497	0.622	0.533	0.624
(Downside)	ALL	-0.15	-0.09	-0.10	-0.15	-0.04	-0.05	-0.02
Market	Perc50	-0.07	-0.07	-0.09	-0.06	-0.02	-0.03	0.02
Beta	Perc25	-0.08	-0.07	-0.09	-0.07	-0.03	-0.03	0.02
	Perc10	0.01	-0.01	-0.03	0.01	0.01	-0.01	0.10
P-values	$K' = K_1$	1	1	1	0.448	0.841	1	0.322
	$K' = K_{10}$	0.272	1	1	0.218	0.325	0.846	0.120
	$K' = K_{49}$	0.163	0.361	1	0.161	0.295	0.342	0.099

costs. As the trading costs increase, and the net IRs of all factor portfolios fall, the SSD overlay becomes more concentrated in CMA, which maintains positive IR, positive skewness and negative downside beta, for every cost level. The MVD overlay becomes more similar to the SSD overlay as costs are increased, by reducing the relative weight of the high-cost ITM, but it increases the relatively weight of the negatively-skewed RMW.

The profitability of both overlays goes to zero for the highest cost level (four times the original estimates), which underlines the importance of cost-efficient implementation. Both overlays are however superior to the EWP which is partly committed to the high-turnover strategies based on past returns (STR and ITM).

Table 5: Out-of-sample tests for in-sample SAOs ($K = K_{10}$)

	$\mathcal{U}$	SSD			MVD			EWP		
	Period	'63-'82	'83-'02	'03-'21	'63-'82	'83-'02	'03-'21	'63-'82	'83-'02	'03-'21
Weights	SMB					0.17	0.00	0.14	0.14	0.14
	HML				0.07			0.14	0.14	0.14
	RMW		0.26	0.85	0.16	0.10	0.20	0.14	0.14	0.14
	CMA		0.34		0.15	0.10		0.14	0.14	0.14
	STR	0.40	0.21		0.06	0.03		0.14	0.14	0.14
	ITM	0.60	0.20	0.15	0.30	0.08	0.07	0.14	0.14	0.14
	LTR				0.25	0.12		0.14	0.14	0.14
	Sum	1	1	1	1	0.61	0.28	1	1	1
In-sample	Mean	0.875	0.574	0.200	0.543	0.228	0.049	0.385	0.320	-0.098
Distr.	StDev	1.987	1.505	1.792	1.213	0.878	0.517	0.974	1.282	1.100
(Net)	Skew	0.000	0.973	-0.224	-0.518	1.185	-0.674	0.446	2.105	-0.482
	IR	1.525	1.322	0.386	1.550	0.902	0.325	1.369	0.865	-0.310
Out-of-sample	Mean	0.753	0.097		0.577	0.029		0.320	-0.098	
Distr.	StDev	2.641	1.092		1.776	0.679		1.282	1.100	
(Net)	Skew	0.230	-2.046		0.224	0.040		2.105	-0.482	
	IR	0.988	0.307		1.126	0.149		0.865	-0.310	
P-values	$K' = K_1$	0.284	1.000		0.533	0.305		0.739	0.060	
Out-of-sample	$K' = K_{10}$	0.209	0.321		0.360	0.159		0.393	0.005	
	$K' = K_{49}$	0.153	0.267		0.314	0.000		0.177	0.000	

Table 6: The Effect of Trading Costs: 1963-2021; $K = K_{10}$

	$\mathcal{U}$	SSD				MVD				EWP			
	Costs	1x	2x	3x	4x	1x	2x	3x	4x	1x	2x	3x	4x
Weights	SMB	0.05	0.09	0.07	0.026	0.02	0.03	0.05	0.06	0.14	0.14	0.14	0.14
	HML	0.01		0.01						0.14	0.14	0.14	0.14
	RMW	0.34	0.28	0.13	0.087	0.13	0.14	0.14	0.12	0.14	0.14	0.14	0.14
	CMA	0.30	0.35	0.30	0.251	0.05	0.06	0.08	0.09	0.14	0.14	0.14	0.14
	STR	0.03								0.14	0.14	0.14	0.14
	ITM	0.16	0.11	0.08	0.012	0.13	0.12	0.10	0.06	0.14	0.14	0.14	0.14
	LTR	0.02	0.01			0.11	0.10	0.07	0.03	0.14	0.14	0.14	0.14
	Sum	0.91	0.84	0.58	0.38	0.44	0.45	0.43	0.36	1	1	1	1
Marg.	Mean	0.228	0.156	0.072	0.028	0.120	0.083	0.049	0.020	0.206	0.078	-0.051	-0.179
Distr.	StDev	1.146	1.006	0.721	0.517	0.669	0.633	0.561	0.416	0.145	1.145	1.145	1.145
(Net)	Skew	0.436	0.874	0.720	0.809	-0.464	-0.291	0.021	0.577	0.950	0.950	0.950	0.950
	IR	0.689	0.537	0.348	0.189	0.622	0.455	0.301	0.169	0.624	0.235	-0.145	-0.543
(Down)	ALL	-0.09	-0.09	-0.06	-0.05	-0.04	-0.04	-0.03	-0.03	-0.02	-0.02	-0.02	-0.02
Market	Perc50	-0.07	-0.07	-0.05	-0.05	-0.02	-0.02	-0.02	-0.02	0.02	0.02	0.02	0.02
Beta	Perc25	-0.07	-0.06	-0.05	-0.05	-0.03	-0.02	-0.02	-0.02	0.02	0.02	0.02	0.02
	Perc10	-0.01	-0.01	-0.01	-0.02	0.01	0.01	0.02	0.01	0.10	0.10	0.10	0.10
P-values	$K' = K_1$	1	1	1	1	0.841	0.771	0.724	0.821	0.322	0.226	0.000	0.000
	$K' = K_{10}$	1	1	1	1	0.325	0.307	0.301	0.300	0.120	0.086	0.000	0.000
	$K' = K_{49}$	0.361	0.317	0.311	0.508	0.295	0.273	0.218	0.138	0.099	0.001	0.000	0.000

4 Concluding Remarks

The application to factor investing uncovers robust SAOs for all risk-averse investors who are endowed with an equity portfolio with an arbitrary industry composition and who have sufficiently low tradings costs. Combinations of multiple factor portfolios are more likely to be a SAO than individual factor portfolios, due to the low correlation between factor portfolios.

Particularly appealing seems the combination of the RMW and CMA factor portfolios due to their favorable properties of high IR, weak mutual correlation and negative downside beta. SMB, HML and LTR don't play a significant role due to their lower IR and positive downside beta. STR and ITM also play a limited role due to the relatively high trading costs; they may however become more important if the cost calculation takes into account the effect of the netting of offsetting positions in individual stocks in the various factor portfolios.

The results point at active trading opportunities for investors with the assumed preferences, endowments and cost levels. The results also suggest that admissible SDFs for the factor portfolios do not take the shape of the IMRS for a risk-averse stock investor, which challenges a class of equilibrium models, unless sufficiently high transactions costs are assumed. Given that the optimal overlays are not PAOs, the results are however consistent with the existence of non-negative SDFs with non-standard shapes.

Given the out-of-sample and post-publication decline in return predictability, naive extrapolation should however be avoided and active investors in search of SAOs are probably well-advised to periodically update the investment universe of factor portfolios based on the prevailing economic conditions and contemporaneous research findings.

The analysis also shows that skewness and non-linear dependence play an important role for factor portfolios, and that an analysis based on mean and variance alone may lead to false conclusions and suboptimal choice (witness the results for the subperiods 1983-2002 and 2003-2021 in Table 5). The SSD overlays are also superior out of sample to a heuristic equal-weighted strategy which ignores average returns, risk levels, trading costs and the dependency structure of the factor portfolios.

An earlier version of this study included an empirical application to S&P500 equity index (SPX) options combinations in the spirit of Constantinides, Czerwonko and Perrakis (2000) and Post and Rodriguez Longarela (2001). In that application, the case for multiple benchmarks is weaker than for the present application to factor investing, for a number of reasons. First, the S&P500 index seems an obvious benchmark in that case, as SPX options are generally used for altering the risk profile of stock portfolios which resemble the S&P500 equity index. Second, the payoffs of SPX options depend only on the S&P500 index, and hence the SDF for options is also expected to depend on the index. Third, the SPX options market is relatively close to being dynamically complete (which limits the possible shapes of the SDF) due to the possibility to replicate index options using dynamic combinations of stocks and bills.

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Appendix: Proofs

Proof of Proposition 2.2.3. Proof by contradiction. If some $\boldsymbol{\delta} \in \Delta$ is a SAO but not a PAO, then it must have either (i) $\mathbb{P}_{\mathcal{F}}[\boldsymbol{x}'\boldsymbol{\delta} = 0]$=1 or (ii) $\mathbb{P}_{\mathcal{F}}[\boldsymbol{x}'\boldsymbol{\delta} < 0]$>0. Case (i) contradicts strict stochastic enhancement of any $\boldsymbol{\kappa} \in \mathrm{K}$; case (ii) implies $\mathbb{P}_{\mathcal{F}}[\boldsymbol{x}'(\boldsymbol{\kappa}_0 + \boldsymbol{\delta}) < c] > 0$ and $\mathbb{E}_{\mathcal{F}}[u(\boldsymbol{x}'(\boldsymbol{\kappa}_0 + \boldsymbol{\delta}))] < u(c)$, which contracts stochastic enhancement of the riskfree $\boldsymbol{\kappa}_0$, for all $u \in \mathrm{cl}(\mathcal{U}) \cap \mathcal{U}_2(c)$.□

Proof of Theorem 2.3.1. By Definition 2.2.2, $\boldsymbol{\delta}$ is not a strict SAO, $\boldsymbol{\delta} \notin \Delta^{\succ}_{(\mathcal{U},\mathrm{K},\mathcal{F})}$, iff $\left(\mathbb{E}_{\mathcal{F}}\left[u(\boldsymbol{x}^{\mathrm{T}}(\boldsymbol{\kappa} + \boldsymbol{\delta})) - u(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\kappa})\right] \leq 0\right)$, for some $(u, \boldsymbol{\kappa}) \in \mathcal{U} \times \mathrm{K}$. Due to concavity of the utility function, this condition is equivalent to $\boldsymbol{\lambda} = \boldsymbol{\kappa}$ being the optimum for $\sup_{\boldsymbol{\lambda} \in \Theta} \mathbb{E}_{\mathcal{F}}\left[u(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\lambda})\right]$, for feasible set $\Theta := \{\boldsymbol{\lambda} \in \Lambda_0 : \boldsymbol{\lambda} = \boldsymbol{\kappa} + \theta\boldsymbol{\delta}; \theta \in [0, 1]\}$. The necessary and sufficient Karush-Kuhn-Tucker condition therefore implies $\boldsymbol{\delta} \notin \Delta^{\succ}_{(\mathcal{U},\mathrm{K},\mathcal{F})}$ iff $\mathbb{E}_{\mathcal{F}}\left[u'(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\kappa})\boldsymbol{x}\right]^{\mathrm{T}}\boldsymbol{\delta} \leq 0$ for some

$(u, \boldsymbol{\kappa}) \in \mathcal{U} \times \mathrm{K}$. Since the scaling of marginal utility by the scalar $d(u, \boldsymbol{\kappa})$ does not affect the optimality condition, this condition is equivalent to $\mathbb{E}_{\mathcal{F}}\left[m(\boldsymbol{x})\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\delta}\right] \leq 0$ for some $m \in \mathcal{M}_{(\mathcal{U},\mathrm{K})}$. Hence, $\Delta^{\succ}_{(\mathcal{U},\mathrm{K},\mathcal{F})} = \emptyset$ if and only if

$$\sup_{\Delta} \inf_{\mathcal{M}_{(\mathcal{U},\mathrm{K})}} \left(\mathbb{E}_{\mathcal{F}}\left[m(\boldsymbol{x})\boldsymbol{x}^{\mathrm{T}}\right]\boldsymbol{\delta}\right) \leq 0 \Leftrightarrow \tag{9}$$

$$\sup_{\Delta} \inf_{\mathrm{conv}\left(\mathcal{M}_{(\mathcal{U},\mathrm{K})}\right)} \left(\mathbb{E}_{\mathcal{F}}\left[m(\boldsymbol{x})\boldsymbol{x}^{\mathrm{T}}\right]\boldsymbol{\delta}\right) \leq 0 \Leftrightarrow \tag{10}$$

$$\inf_{\mathrm{conv}\left(\mathcal{M}_{(\mathcal{U},\mathrm{K})}\right)} \sup_{\Delta} \left(\mathbb{E}_{\mathcal{F}}\left[m(\boldsymbol{x})\boldsymbol{x}^{\mathrm{T}}\right]\boldsymbol{\delta}\right) \leq 0 \Leftrightarrow \tag{11}$$

$$\inf_{\mathrm{conv}\left(\mathcal{M}_{(\mathcal{U},\mathrm{K})}\right)} \left\{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{\sigma} : \mathbb{E}_{\mathcal{F}}\left[m(\boldsymbol{x})\boldsymbol{x}\right] = \mathbf{A}^{\mathrm{T}}\boldsymbol{\sigma}; \boldsymbol{\sigma} \geq \mathbf{0}_R\right\} \leq 0 \Leftrightarrow \tag{12}$$

$$\left(\mathbb{E}_{\mathcal{F}}\left[m(\boldsymbol{x})\boldsymbol{x}\right] \in \Delta^{*}\right) \text{ for some } m \in \mathrm{conv}\left(\mathcal{M}_{(\mathcal{U},\mathrm{K})}\right). \tag{13}$$

The convexification of the feasible set of inner minimization problem, $\mathcal{M}_{(\mathcal{U},\mathrm{K})}$, in (10) is allowed because the objective function is a bilinear map $\Delta \times \mathcal{M}_{(\mathcal{U},\mathrm{K})} \rightarrow \mathbb{R}$. The reversal of the order of the optimization operators in (11) is allowed by the Kneser-Fan Minimax Theorem, because the feasible sets for both optimization operators are convex. The equivalent formulation (12) is based on the dual formulation of the embedded, linear maximization problem over Δ.□

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