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Inconsistency for the Gaussian QMLE in GARCH-type models with infinite variance

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Abstract

We are occupied with the issue of consistency of the Gaussian QMLE in GARCH-type models with very heavy tailed squared innovations. We show that the appropriately scaled likelihood function weakly epi-converges to a stochastic process that is a.s. lower semi-continuous and proper. When moreover the volatility filter is increasing w.r.t. the parameter, inconsistency follows due to that the true parameter value misses the set of minimizers of the limit. This holds for models like the AGARCH, the Augmented GARCH, and the GQARCH.

Keywords: Heavy-tailed distribution, GARCH-type Models, Gaussian QMLE , AGARCH, Augmented GARCH, GQARCH, Inconsistency.

1 Introduction

A strand of the empirical finance literature considers the use of heavy tailed distributions for the squared innovation process in GARCH-type models-see for example Rachev and Mittnik, 1988 and Mittnik, Rachev, and Paolella, 1998. The implications of heavy tails on the limit theory of the commonly used Gaussian QMLE for the GARCH(p,q) model were analyzed by the seminal papers of Hall and Yao, 2003 and of Mikosch and Straumann, 2006. The resulting limit theory derived consistency, as well as, stable limiting distributions and regularly varying rates of convergence, when the tail index of the stationary distribution of the squared innovations is strictly greater than one. The slower rates and the possibly non Gaussian limiting distributions motivated a strand of literature for the development of alternative Quasi likelihood functions, or other semi-parametric methods of estimation (see for example Preminger and Storti, 2017, Fan, Qi, and Xiu, 2014, Peng and

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Yao, 2003) as well as the references therein. Even then, some of those methods, e.g. Preminger and Storti, 2017, and Fan, Qi, and Xiu, 2014 may involve first steps that depend on the Gaussian QMLE, and therefore may crucially rely on its consistency properties.

However, the case where the index of stability of the squared innovations is less than (or in some instances equal to) one presents ambiguity on the issue of consistency of the QMLE for such-like models. This is due to the different asymptotic behavior of the log-likelihood process associated with the required scaling. Such very heavy tailed cases can be expected to be relevant in return time series from technological innovations (see Silverberg and Verspagen, 2007), or associated to economic losses and operational risks from natural disasters (see R. Ibragimov, Jaffee, and Walden, 2009). Therefore, it is of both theoretical and empirical interest to establish whether consistency holds and if not, whether the asymptotic properties of multi-step estimation procedures that involve Non-Gaussian likelihoods (Fan, Qi, and Xiu, 2014), or OLS regressions (Preminger and Storti, 2017), that rely on the Gaussian QMLE, also hold in such very heavy tailed cases.

Thus, in the present paper we are occupied with the issue of consistency of the QMLE for GARCH-type models, when the aforementioned index is strictly less than one. Our contributions consists of the following: i) under familiar assumptions, the likelihood function weakly epi-converges (see the relevant discussion just before Lemma 1) to a stochastic process that is almost surely lower semi-continuous (lsc) and proper. This is not the typical behavior whenever the index is greater than one; the limit then is deterministic. (ii) We derive inconsistency when there exists a parameter value at which the limit is finite and smaller compared to the value attained at the true value. This precludes the true parameter value from lying inside the (possibly stochastic) set of minimizers of the limit. In such cases the usual identification assumption (see for example Wintenberger, 2013) holds, but consistency fails. (iii) We show that this is true for models where the volatility filter has an almost sure monotonicity property; it implies that the likelihood limit almost surely achieves smaller values when the true parameter is partially translated to pointwise larger values. Examples are the standard GARCH, the AGARCH and Augmented GARCH and the GQARCH models. Hence our derivations involve a quite wide class of volatility models that are commonly employed in the empirical literature. (iv) As a by-product we derive in the Appendix, a Martingale Limit Theorem that differs from the one of Mikosch and Straumann, 2006, first, by not requiring mixing conditions beyond ergodicity, and second, by allowing for continuous convergence w.r.t. Euclidean parameters.

In the following sections we describe our main framework, provide with our assumptions, obtain the results along with some examples, and describe potential extensions. Finally, we provide our proofs.

2 Framework

Given an underlying probability space $(\Omega, \mathcal{G}, \mathbb{P})$, consider the following general model

$$\begin{cases} y_i = \sigma_i z_i \\ H(\sigma_i^2) = g_{\theta_0}\left(z_{i-1}, \dots, z_{i-p}, H(\sigma_{i-1}^2), \dots, H(\sigma_{i-\max(p,q)}^2)\right) \end{cases}, \; i \in \mathbb{Z} \qquad (1)$$

where the parameter θ_0 lies in some non-empty compact parameter space $\Theta \subset \mathbb{R}^d$, the function g_θ is a measurable function on $\Theta \times [0, \infty)^q$, H is an invertible measurable self function on $\mathbb{R}$, and the sequence $(z_i)_{i\in\mathbb{Z}}$ is i.i.d.

Illustrative examples of known models that adhere to the formulation in (1) are (for further examples see among others Paragraph 3.3 in Straumann, 2004):

- AGARCH(p, q), see Ding, Granger, and Engle, 1993, where

$$\sigma_i^2 = \omega + \sum_{j=1}^{p} \left(a_{0,j} \left|y_{i-j}\right| - \gamma_0 y_{i-j}\right)^2 + \sum_{k=1}^{q} \beta_{0,k} \sigma_{i-k}^2,$$

- Augmented GARCH$(1, 1)$, see Francq and Zakoıan, 2013, where

$$\sigma_i^2 = \omega_0 + a_{0+}[\max(0, y_{i-1})]^2 + a_{0-}[\min(0, y_{i-1})]^2 + \beta_0 \sigma_{i-1}^2,$$

- GQARCH$(1, 1)$, see Sentana, 1995, where

$$\sigma_i^2 = \omega_0 + a_0(y_{i-1} + \frac{\gamma_0}{2a_0})^2 + \beta_0 \sigma_{i-1}^2.$$

The examples contain as a special case the celebrated symmetric GARCH(1,1) model (see Bollerslev, 1986). They are all dynamically asymmetric in the sense that they allow for non-zero covariances between y_i and σ_{i-1}^2 without requiring asymmetry for z_i. The unitary order of the last two examples is chosen for brevity; auxiliary properties for the unitary specifications are already known in the literature. Our main results can be readily extended to arbitrary orders for those two cases at the cost of heavier notation.

Using a stochastic recurrence equation (SRE) approach, Straumann and Mikosch, 2006, give sufficient conditions for the existence of a unique (up to a limiting argument), solution to (1), say $((y_i, \sigma_i^2))_{i\in\mathbb{Z}}$, that is stationary, ergodic and non-anticipative. This is ensured by, among others, imposing conditions on the Lipschitz coefficient of the random map that generates σ_i^2. In order to infer the unknown parameter θ_0, and since in practice only $(y_i)_{i=1,\dots,n}$ is observable, the volatility process (σ_i^2) is reconstructed under the parameter hypothesis θ using the following volatility filter for some random vector of initial values $(\varsigma_0, \varsigma_1, \dots, \varsigma_{q-1})' \in \mathbb{R}^q$,

$$H(\hat{h}_i(\theta)) = \begin{cases} \varsigma_i, & q-1 \leq i \leq 0, \\ g_\theta^\star\left(y_{i-1}, \dots, y_{i-p}; H(\hat{h}_{i-1}(\theta)), \dots, H(\hat{h}_{i-\max(p,q)}(\theta))\right), & i > 0, \end{cases} \qquad (2)$$

which is generally non-stationary. Here, $g_\theta^\star$ is obtained by g_θ by replacing z_{i-j} with $\frac{y_{i-j}}{\sqrt{\hat{h}_{i-j}(\theta)}}$ for all $j = 1, \dots, p$. Given the above, the Gaussian quasi likelihood function is proportional to

$$\hat{c}_n(\theta) = \frac{1}{n}\sum_{i=1}^{n} \hat{\ell}_i(\theta),$$

where $\hat{\ell}_i(\theta) = \log \hat{h}_i(\theta) + \frac{y_i^2}{\hat{h}_i(\theta)}$, and then the Gaussian QMLE θ_n of θ_0 is defined by

$$\hat{c}_n(\theta_n) = \inf_{\theta\in\Theta} \hat{c}_n(\theta).$$

Standard lower semi-continuity and measurability conditions imply existence of the estimator (see for example Definition 3.5 and Proposition 3.6 in Chapter 5 of Molchanov, 2011), while the definition-and the subsequent results-can be easily extended so that approximate/numerical optimization is allowed.

3 Assumptions, results and examples

We are interested in the issue of consistency of θ_n when z_0^2 belongs to the domain of attraction of a stable distribution with tails heavy enough so that it does not have a first moment. We first present our assumption framework. There, $S_\alpha(s, c, \gamma)$ denotes the (univariate) stable distribution with α, s, c, γ denoting stability index, skewness, scale and location parameters respectively (see Chapter 2 of I. A. Ibragimov and Linnik, 1971).

A.1 *The distribution of z_0^2 lies in the domain of attraction (DoA) of $S_\alpha(1, c, 1)$ with $\alpha \in (0, 1)$.*

The assumption implies that $\mathbb{E}[z_0^2] = +\infty$ (see Chapter 2 of I. A. Ibragimov and Linnik, 1971) and in fact allows for very heavy right tail for the distribution of z_0^2. Its unit location is in accordance to the innovations' unit variance normalization condition ($\mathbb{E}[z_0^2] = 1$) that is typical in such models when the innovations have enough moments. We do not consider the case where $\alpha > 1$ (as well as the case of $\alpha = 1$ and $\mathbb{E}[z_0^2] < +\infty$) since consistency then readily follows by Theorem 5.3.1 of Straumann, 2004 under A.2-4 below, and a usual identification condition. The restriction of the skewness parameter to one follows naturally from the fact that the support of $z_0^2 - 1$ is bounded from below.

Assumption A.1 directly implies that $\mathbb{P}[|z_0| > x] \sim C\frac{l(x)}{x^{2\alpha}}$, as $x \to \infty$, where l is a slowly varying function at infinity (see Bingham, Goldie, and Teugels, 1989) and $C > 0$. If l converges to zero sufficiently fast, then Theorem 5 of Kesten, 1973 and Proposition 2.1 of Denisov and Zwart, 2007 imply that for each of the three examples considered above, there exist parameter values and a constant $C^\star > 0$, such that $\mathbb{P}[|y_0| > x] \sim C^\star\frac{1}{x^{2\alpha}}$ as $x \to \infty$. This implies that Assumption A.1 allows for stationary processes with unconditional tail indices associated with very heavy tailed time series that can be relevant in empirical economics (see for example Silverberg and Verspagen, 2007).

Assumptions A.2-A.4 that follow are similar to the ones used in the literature in the case where $\mathbb{E}[z_0^2] < +\infty$ (see Straumann, 2004, ch. 5, C.1, C.3-C.4 and

Wintenberger, 2013). They among others enable the approximation of (2) by the process $(h_i(\theta))_{i\in\mathbb{Z}}$, $\theta\in\Theta$ defined by

$$H(h_i(\theta)) = g_\theta^{\star\star}\left(y_{i-1},\ldots,y_{i-p};H(h_{i-1}(\theta)),\ldots,H(h_{i-\max(p,q)}(\theta))\right), i\in\mathbb{Z}, \quad (3)$$

which is independent of initial conditions. The form $g_\theta^{\star\star}$ here, is obtained from g_θ by replacing z_{i-j} with $\frac{y_{i-j}}{\sqrt{h_{i-j}(\theta)}}$ for all $j=1,\ldots,p$.

A.2 *The model* (1) *admits a unique stationary ergodic non-anticipative solution* $((y_i,\sigma_i^2))_{i\in\mathbb{Z}}$, *with* $\mathbb{E}[\ln^+(\sigma_0^2)]<+\infty$.

A.3 *The model* (3) *is continuously invertible in* Θ. *The functions* $\ln\circ H^{-1}$, $\frac{1}{H^{-1}}$ *are well defined and Lipschitz continuous on the range of* g_θ *for all* $\theta\in\Theta$. *Finally, the model* (3) *is* $\mathbb{P}$ *a.s. continuous in* θ.

A.4 $\inf_{\theta\in\Theta}\inf_{\mathbf{x}\in\mathbb{R}^p\times[0,\infty)^q} H^{-1}\circ g_\theta(\mathbf{x})>0$.

Assumption A.2 holds if there exists $\mathbf{u}$, for which $\mathbb{E}[\ln^+(|\psi_0(\mathbf{u})|)]<+\infty$, $\mathbb{E}[\ln^+(\Lambda(\psi_0))]<+\infty$ and $\mathbb{E}[\ln(\Lambda(\psi_0))]<0$-see Theorem 2.6.1 in Straumann, 2004. For $\mathbf{u}\in\mathbb{R}^{\max(p,q)}$, $\psi_t(\mathbf{u})$ is defined as $g_{\theta_0}(z_{i-1},\ldots,z_{i-p},\mathbf{u})$, and for a real Lipschitz continuous self-function f, $\Lambda(f)$ denotes its Lipschitz coefficient. The form of (1), the functional invertibility of H, and A.2 imply that $h_i(\theta_0)=\sigma_i^2$, $\mathbb{P}$ a.s., $\forall i\in\mathbb{Z}$.

Given A.2, continuous invertibility enables the almost sure convergence of $H(\hat{h}_i(\theta)$ to $H(h_i(\theta)$ as $i\to\infty$ uniformly in Θ, given Definition 3 of Wintenberger, 2013. From Theorem 2 of Wintenberger, 2013, this follows whenever there exists $\mathbf{u}$, for which $\mathbb{E}[\ln^+(\sup_{\theta\in\Theta}|\phi_T(\mathbf{u},\theta)|)]<+\infty$, $\mathbb{E}[\ln^+(\sup_{\theta\in\Theta}\Lambda(\phi_0(\theta))]<+\infty$ and $\mathbb{E}[\ln(\Lambda(\phi_0(\theta))]<0$, $\forall\theta\in\Theta$. $\phi_0(\mathbf{u},\theta)$ is $g_\theta^{\star\star}(y_{i-1},\ldots,y_{i-p},\mathbf{u})$. The Lipschitz continuity conditions of the second part of A.3 enable then the analogous approximation of $\ln(\hat{h}_i(\theta))$ by $\ln(h_i(\theta))$ and of $\frac{1}{\hat{h}_i(\theta)}$ by $\frac{1}{h_i(\theta)}$. Hence, A.3 enables among others, asymptotic validity of the substitution of $\hat{h}_i(\theta)$ by $h_i(\theta)$ in the likelihood function, and thereby the limiting independence of the likelihood process of the initial conditions. The final part of A.3 along with continuous invertibility, the Lipschitz continuity conditions and the compactness of Θ imply the $\mathbb{P}$ a.s. continuity of $h_i(\theta)$ in θ.

A.4 is a technical condition of strict positivity and in several cases follows from restrictions on Θ that facilitate the positivity of the volatility process.

A.1-A.4 suffice for the establishment of the weak epi-convergence of the scaled likelihood process to a lsc and proper stochastic process driven by a very heavy tailed random variable. We first establish the required notation. Due to Theorem 2.6.5 of Ibragimov and Linnik I. A. Ibragimov and Linnik, 1971, the DoA assumption is equivalent to that the characteristic function of z_0^2 has for some $t_0>0$, the representation $\mathbf{i}t-c|t|^a\mathbf{h}\left(|t|^{-1}\right)\left(1-\mathbf{i}\mathrm{sgn}(t)\tan(\frac{1}{2}\pi\alpha)\right)$ on $(-t_0,t_0)$, where $\mathbf{h}$ is a slowly varying function at infinity. Define r_T by the asymptotic relation $\frac{\mathbf{h}(T^{\frac{1}{\alpha}}r_T^{\frac{1}{\alpha}})}{r_T}\to 1$, and notice that r_T is a well defined slowly varying sequence (see Paragraph 1.9 of Bingham, Goldie, and Teugels, 1989, and Proposition 1.(iv) of Astrauskas, 1983). Futhermore, $\overset{\text{epi}}{\rightsquigarrow}$ denotes epi-convergence in distribution; a sequence of deterministic lsc real functions (f_n) on a Polish space epi-converges to a limit function f, iff $\forall x,\forall x_n\to x,\liminf_{n\to\infty}f_n(x_n)\geq f(x)$, and $\forall x,\exists x_n\to x,\limsup_{n\to\infty}f_n(x_n)\leq f(x)$

(see Ch.3 of Molchanov, 2011 for this sequential characterization). The "in distribution" stochastic mode of epi-convergence is obtained by those sequential characterizations, along with the consideration of Skorokhod representations on suitably enlarged probability spaces that exist due to the separability of the associated metrizable topology, see Knight, 1999. See also Knight, 1999 for a characterization of epi-convergence in distribution, in terms of fidis and stochastic equi-semi-continuity. An extended real valued function is called proper when it never assumes the value $-\infty$, and is not identically equal to $+\infty$. A stochastic process with extended real valued sample paths is called proper, when almost all its sample paths are proper functions.

Lemma 1. *Suppose that A.1-A.4 hold. Then, as* $T \to \infty$

$$\frac{1}{T^{1/\alpha} r_T^{1/\alpha}} \sum_{i=1}^{T} \hat{\ell}_i(\theta) \overset{epi}{\rightsquigarrow} Z\mathbb{E}^{1/\alpha}[(\frac{\sigma_i^2}{h_i(\theta)})^\alpha], \tag{4}$$

where $Z \sim S_\alpha(1, c, 0)$ *for some* $c > 0$. *The limiting process is* $\mathbb{P}$ *a.s. lsc and proper.*

The derivation utilizes among others the martingale limit theorem (see Theorem 2) that we establish in the Appendix, and the aforementioned availability of Skorokhod representations. The limiting process has positive sample paths since $\alpha < 1$. Furthermore, $\mathbb{E}[(\frac{\sigma_0^2}{h_0(\theta)})^\alpha] \in (0, +\infty]$. The asymptotic location parameter is zero, thereby the limit does not contain information on γ that appears in A.1. For any θ for which $\mathbb{E}[(\frac{\sigma_0^2}{h_0(\theta)})^\alpha]$ is finite, ergodicity and the fact that $\alpha < 1$ ensures that the location related term associated with Theorem 2 is asymptotically negligible (see Corollary 1 and its proof). The epi-convergence controls the limiting behavior of the minimizers of the likelihood function. It implies that every weak accumulation point must lie inside the set of minimizers of the limiting process. When the latter does not contain θ_0 inconsistency is obtained due to non-identification.

The final assumption concerns the behavior of the moment appearing in the deterministic part of the limiting process, and it essentially ensures asymptotic non-identification in the framework of A.1, due to the positivity of the limit in 4.

A.5 $\exists\theta \neq \theta_0 : \mathbb{E}[(\frac{\sigma_0^2}{h_0(\theta)})^\alpha] < 1$.

A.5 can be valid even in cases where condition $h_0(\theta) = \sigma_0^2 \Leftrightarrow \theta = \theta_0$, $\mathbb{P}$ a.s. (ID) holds. When $\mathbb{E}[z_0^2]$ exists and equals to one, the latter is the usual identification condition-see for example Wintenberger, 2013. Under A.1, when A.5 holds, (ID) becomes insufficient as an identification condition.

The following auxiliary result says that A.5 holds when the stationary version of the filter in (3) has a monotonicity property w.r.t. some part of the parameter that can be adequately translated inside the parameter space. In this respect suppose that Θ is factored as $\Phi \times \Psi$, so that $\theta_0 := (\phi_0, \varphi_0)$, and let $\theta(\phi) := (\phi, \psi_0)$ and denote with K_Φ some neighborhood of ϕ_0. ∂_ϕ denotes partial derivation w.r.t. ϕ.

Lemma 2. *Suppose that*

$$\hat{g}_\phi \circ \mathbf{u} := g^{\star\star}_{\theta(\phi)}(y_{i-1}, \dots, y_{i-p}, \mathbf{u})$$

is $\mathbb{P}$ a.s. continuously differentiable on $K_\Phi \times (0, +\infty)$ (by potentially extending Φ), and that K_Φ is compact and coincides with the closure of its interior. Suppose that the SRE

$$m_i = \partial_\phi(\hat{g}_\phi \circ (m_{i-1}, \ldots, m_{i-\max(p,q)})), \tag{5}$$

has a unique, stationary and ergodic solution $(m_i)_{i\in\mathbb{N}}$ such that $m_i > 0$, $\mathbb{P}$ a.s. for all i, and that H^{-1} is strictly increasing. Then A.5 holds.

The existence of a unique stationary and ergodic solution to (5) can be established by conditions similar to the conditions D.1-2 of Straumann, 2004 (see Ch. 5) restricted to K_ϕ. In models like the AGARCH(p,q) we can choose $\phi = \omega$ in which case the SRE admits a simple form, and positivity is ensured due to the positivity of the GARCH autoregressive parameters and the constant. Furthermore, in such cases the potentially needed extension of Φ (when for example ϕ_0 lies in its boundary) is usually empirically innocuous.

We can now state our main result.

Theorem 1. *Under A.1-A.5, the Gaussian QMLE is inconsistent.*

Lemma 1 and A.5 imply that the QMLE has subsequential limits inside the set of minimizers of the limiting process that do not include θ_0 with probability one, hence inconsistency follows essentially from Theorem 7.31.b of Rockafellar and Wets, 2009 that relates epi-convergence to the limiting properties of minimizers. The result rests on a qualitatively different behavior compared to the classical identification failures. Those occur due to that either $\mathbb{E}[z_0^2] \neq 1$ but finite, and/or $\mathbb{E}[z_0^2] = 1$ but (ID) fails. In all those cases the likelihood limit is deterministic and θ_0 belongs to the non-singleton set of minimizers under A.2-A.4. Despite the randomness of the limiting likelihood, weak consistency would be obtained as long as $\forall \theta \neq \theta_0 : \mathbb{E}[(\frac{\sigma_0^2}{h_0(\theta)})^\alpha] > 1$, which would constitute the sufficient identification condition under A.1.

We finally examine the AGARCH, Augmented GARCH and GQARCH examples that were presented above.

Lemma 3. *Suppose that A.1 holds. (i). For the AGARCH(p,q) suppose that $a_{0,j} > 0$ for some $j > 0$, $(a_{0,p}, \beta_{0,q}) \neq (0, 0)$, and the polynomials $\sum_{j=1}^p a_{0,j} z^j$ and $1 - \sum_{j=1}^q \beta_{0,j} z^j$ do not have common roots. (ii). For the Augmented GARCH(1,1) case, suppose that $\mathbb{E}[\ln(a_{0+}[\max(0, z_0)]^2 + a_{0-}[\min(0, z_0)]^2 + \beta_0)] < 0$ and $\beta < 1$, for all $\theta \in \Theta$. (iii). For the GQARCH(1,1) case, suppose that $\mathbb{E}[\ln(\sup_{s\in[\frac{\omega_0}{1-\beta_0},+\infty)} |a_0 z_0^2 + \beta_0 + \frac{\gamma_0}{2\sqrt{s}}|)] < 0$ and $\beta < 1$, for all $\theta \in \Theta$. In each case (i),(ii),(iii), the Gaussian QMLE is inconsistent.*

Besides A.1, the conditions that appear in the lemma are identical to Theorem 5.4.6 of Straumann, 2004, Theorem 3.1.(i) of Francq and Zakoıan, 2013, and Proposition 3.2 of Arvanitis and Louka, 2015 that establish consistency for each model respectively. In all cases A.2-A.4 and condition (ID) hold, however A.5 is also valid due to the linearity in volatility of (3) for those models. This implies monotonicity for the choice $\phi = \omega$, due to the positivity of the GARCH autoregressive parameters, and the strict positivity of the constants, via the application of Lemma 2.

In cases where Lemma 1 holds with $\Phi = \Theta$, and thereby the whole model is increasing w.r.t. the parameter, and Θ has a greatest element, say $\theta_{\max}$, then $\theta_n \rightsquigarrow$

$\theta_{\max}$. This is due to that at $\theta_{\max}$, the expected ratio $\mathbb{E}[(\frac{\sigma_0^2}{h_0(\theta)})^\alpha] < 1$ attains its lowest value. Notice that this monotonicity holds for the Augmented GARCH example, and the parameter space has a greatest element when it is a product of compact intervals. Analogous pseudo-true values for the QMLE can be established for models that are not fully monotonic, via furter restrictions on the distribution of z_0.

Lemma 3 implies difficulties for the asymptotic properties of multi-step estimation procedures employed in models like the above, when the first step is the derivation of the Gaussian QMLE: Assumption 3 coupled with Assumption 2 Fan, Qi, and Xiu, 2014 makes their Non-Gaussian QMLE inconsistent. Analogously, whenever the mean of $\ln(z_0^2)$ is unknown, and estimated by the residuals obtained by the Gaussian QMLE, the least squares estimator of Preminger and Storti, 2017 becomes also inconsistent (for example it is possible to show that Lemma A.2.(a) in the Appendix of Preminger and Storti, 2017 holds under A.1, when $p < \alpha$. This and the usual identification condition for the GARCH(1,1) model imply inconsistency).

The inconsistency of the Gaussian QMLE is mitigated by the existence of consistent semi-parametric estimators for θ_0. For an example consider the LAD estimator for θ_0 of Peng and Yao, 2003. Under an assumption framework that contains similar conditions to A.1-4, and additionally involves conditions of mean boundedness for the derivatives of the volatility filter, the estimator, is strongly consistent when $\ln(z_0^2)$ has zero median and a bounded density that is continuous at zero (see Lemma A.1 of Peng and Yao, 2003).

4 Monte Carlo study

We perform a set of Monte Carlo experiments to assess the deviation of the QMLE from θ_0 in the GARCH(1,1) model. We set $\theta_0 := (\omega_0, a_0, \beta_0) = (4.5, 0.27, 0.5)$. We assume that z_0 follows a Student's t_v distribution with $v = 1.95$, or $v = 2$ or $v = 2.5$ degrees of freedom, hence z_0^2 belongs to the DoA of a stable distribution with $\alpha = \frac{v}{2}$ (see Lemma 3 in the Supplement of Arvanitis and Anyfantaki, 2020). The first case conforms to A.1; Lemma 3 and the subsequent discussion suggest that the QMLE converges in probability to $\theta_{\max} := (8, 2, 0.8)$. The third case is well known in the literature (see for example Straumann and Mikosch, 2006). For the second case, and to the best of our knowledge, the issue of consistency for the QMLE is not yet resolved. We use $T = 500$, 1000, 10000, 100000 and 1000000 and the number of Monte Carlo replications is set to 1000 for all T.

The results are reported in Figures 1-3 below. For each of the examined v, the respective figure presents the Monte Carlo probabilities that $\|\text{QMLE} - \theta_\star\| > \varepsilon$, for $\theta_\star = \theta_0$, $\theta_{\max}$, $0 \leq \varepsilon \leq 5$, and for every examined T, where $\|\cdot\|$ denotes the ℓ^2 norm.

The results confirm the respective limit theories when $v = 1.95$ and $v = 2.5$. Specifically, Figure 1 suggests inconsistency and convergence to $\theta_{\max}$ as expected. Figure 3 suggests consistency as expected, see for example Straumann and Mikosch, 2006, albeit at a slow rate of convergence of magnitude $T^{0.2}$. For the unknown case where $\alpha = 2$, Figure 2 seems to suggest the possibility of a similar behavior to the one predicted when $\alpha < 1$. For a discussion on the difficulties associated with $\alpha = 1$ see the following section. Similar results, available upon request, for other values of v are not presented for economy of space.

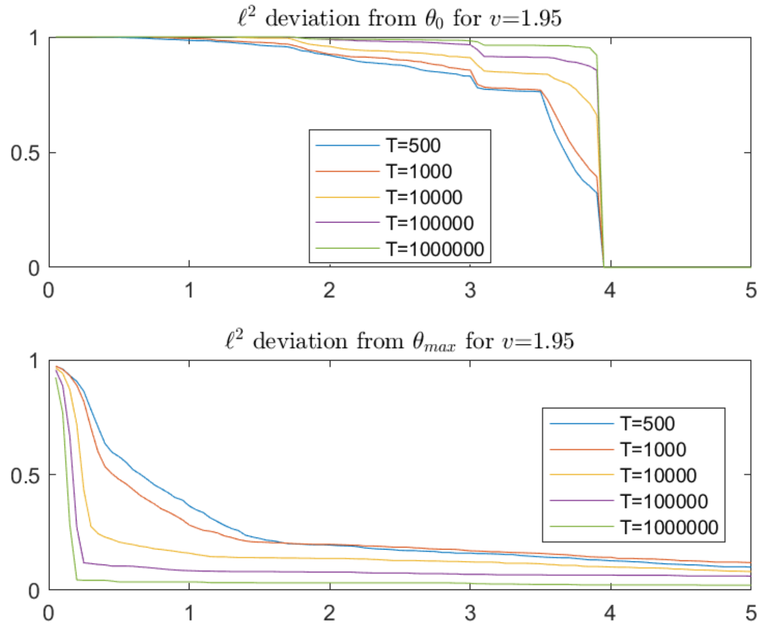

Fig. 1: Monte Carlo empirical probabilities that $\|\text{QMLE} - \theta_\star\| > \varepsilon$, for $\theta_\star = \theta_0, \theta_{\max}$, $0 \le \varepsilon \le 5$, for the ℓ^2 norm, with $\alpha = 0.9475$.

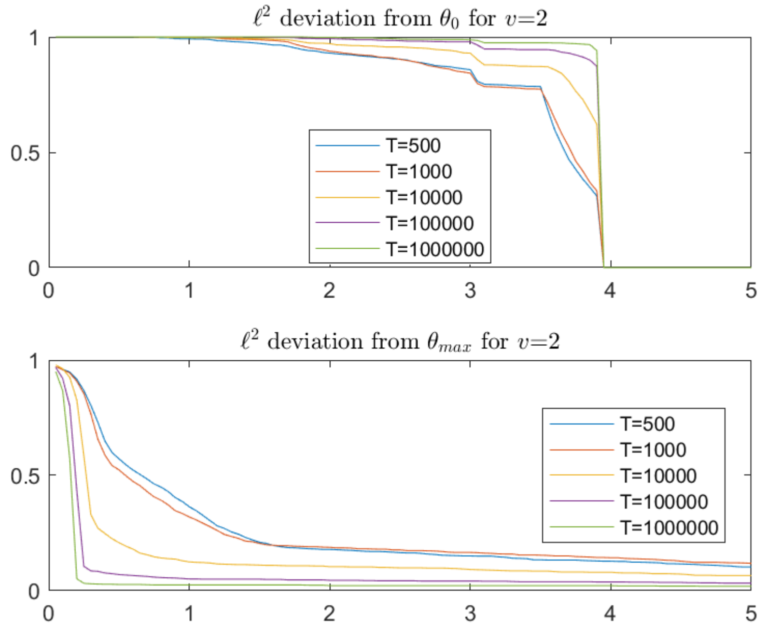

Fig. 2: Monte Carlo empirical probabilities that $\|\text{QMLE} - \theta_\star\| > \varepsilon$, for $\theta_\star = \theta_0, \theta_{\max}$, $0 \le \varepsilon \le 5$, for the ℓ^2 norm, with $\alpha = 1$.

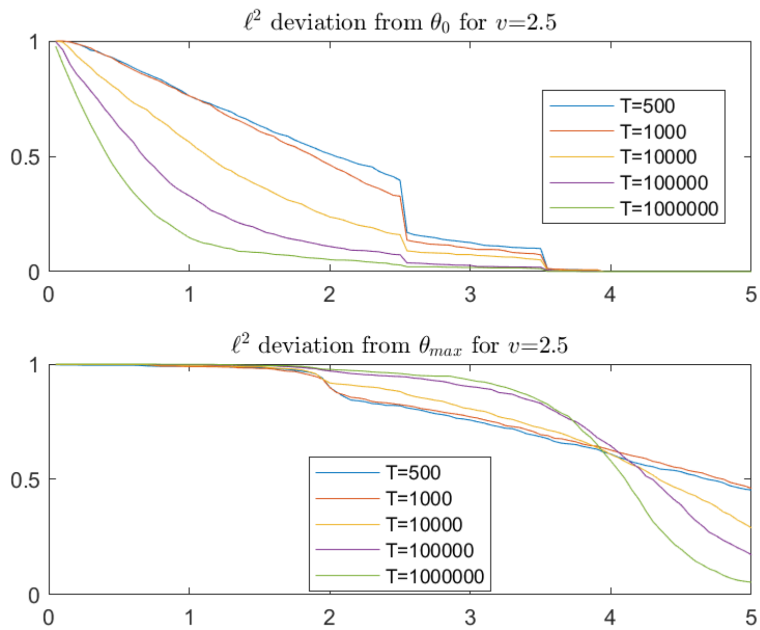

Fig. 3: Monte Carlo empirical probabilities that $\|\text{QMLE} - \theta_\star\| > \varepsilon$, for $\theta_\star = \theta_0, \theta_{\max}$, $0 \leq \varepsilon \leq 5$, for the ℓ^2 norm, with $\alpha = 1.25$.

5 Further research

The filter monotonicity property of Lemma 2 does not seem to hold for models with more complex volatility filters like the EGARCH (see Nelson, 1991). There, A.5 may not be the case and identification could hold, if for example $\forall\theta \neq \theta_0$, $\mathbb{E}[(\frac{\sigma_0^2}{h_0(\theta)})^\alpha] = +\infty$ under A.1. If this is true, then a slight modification of A.2-3 that takes care of the non-existence of logarithmic moments when $\alpha \leq 0.5$ would imply the weak consistency of the QMLE. This potentially interesting investigation is delegated to further research.

When a unique pseudo-true value exists, the issue of the establishing the rate of convergence of the QMLE to it, as well as the associated limiting distribution can be also of interest for further research.

The derivation of the limit theory for the NGQMLE of Fan, Qi, and Xiu, 2014, when the QMLE in the first step is replaced by the LAD estimator of Peng and Yao, 2003, under A.1, can be analogously delegated to further research.

Finally, as noted above, the results do not cover the cases where $\alpha = 1$ yet $\mathbb{E}\left[z_0^2\right] = +\infty$. We suspect that these can be handled under the premises of a (locally uniform) LLN for stationary strong mixing sequences with potentially diverging slowly varying moments. To our knowledge such LLNs are currently available only under uniform mixing conditions (see for example Corollary 3 of Szewczak, 2010) that are not generally compatible to the examined models in the current framework. Hence such a consideration is also delegated to further research.

6 Proofs

Proof of Lemma 1. First notice that for arbitrary $K > 0$,

$$\begin{aligned}\frac{1}{T^{1/\alpha}r_T^{1/\alpha}} \sup_{\theta\in\Theta} \left| \sum_{i=1}^{T} (\log \hat{h}_i(\theta) + \frac{z_i^2(\sigma_i^2\wedge K)}{\hat{h}_i(\theta)} - \log h_i(\theta) - \frac{z_i^2(\sigma_i^2\wedge K)}{h_i(\theta)}) \right| \\ \leq \frac{1}{T^{1/\alpha}r_T^{1/\alpha}} C \sum_{i=1}^{T} (1 + z_i^2(\sigma_i^2 \wedge K)) \sup_{\theta\in\Theta} |\hat{h}_i(\theta) - h_i(\theta)|\end{aligned}$$

for some $C > 0$ independent of K that exists due to the Lipschitz continuity conditions in A.3, and A.4. Then by Definition 2 of Wintenberger, 2013 and due to A.3, $\sup_{\theta\in\Theta} |\hat{h}_i(\theta) - h_i(\theta)|$ converges $\mathbb{P}$ a.s. to zero as $i \to \infty$. Hence, $(\sigma_i^2 \wedge K) \sup_{\theta\in\Theta} |\hat{h}_i(\theta) - h_i(\theta)|$ also converges $\mathbb{P}$ a.s. to zero. Then the dominant term of the previous display weakly converges to zero due to the Cezaro sum Theorem, the fact that by a trivial application of Theorem 2 and Corollary 1, $\frac{1}{T^{1/\alpha}r_T^{1/\alpha}} \sum_{i=1}^{T} z_i^2$ weakly converges to an α-stable random variable, Theorem 1.10.4 of Vaart and Wellner, 2012 and the fact that $\alpha < 1$. Letting $T \to \infty$ and then $K \to \infty$ we obtain that $\frac{1}{T^{1/\alpha}r_T^{1/\alpha}} \sup_{\theta\in\Theta} |\sum_{i=1}^{T} (\log \hat{h}_i(\theta) + \frac{z_i^2\sigma_i^2}{\hat{h}_i(\theta)} - \log h_i(\theta) - \frac{z_i^2\sigma_i^2}{h_i(\theta)})|$ weakly converges to zero.

Now consider the term, $\frac{1}{T^{1/\alpha}r_T^{1/\alpha}} \sum_{i=1}^{T} z_i^2 \frac{\sigma_i^2}{h_i(\theta)} 1\left\{\sup_{\theta\in\Theta} \frac{\sigma_i^2}{h_i(\theta)} \leq K\right\}$. Under A.1-A.4 and due to Theorem 2 and Corollary 1, it weakly converges uniformly (over Θ) to $S_\alpha\left(1, c\mathbb{E}[(\frac{\sigma_i^2}{h_i(\theta)})^\alpha 1\left\{\sup_{\theta\in\Theta} \frac{\sigma_i^2}{h_i(\theta)} \leq K\right\}], 0\right)$, which is equal in distribution to $\mathbb{E}^{1/\alpha}[(\frac{\sigma_i^2}{h_i(\theta)})^\alpha 1\left\{\sup_{\theta\in\Theta} \frac{\sigma_i^2}{h_i(\theta)} \leq K\right\}] S_\alpha(1, c, 0)$. Theorem 1.10.4 and Addendum 1.10.5 of Vaart and Wellner, 2012 implies the existence of an enhanced probability space and measurable mappings defined on it with values on the original probability space, say $\phi_{T,K}, \phi_K$, such that $\frac{1}{T^{1/\alpha}r_T^{1/\alpha}} \sum_{i=1}^{T} z(\phi_{K,T})_i^2 \frac{\sigma_i^2}{h_i(\theta)}(\phi_{T,K}) 1\left\{\sup_{\theta\in\Theta} \frac{\sigma_i^2}{h_i(\theta)}(\phi_{T,K}) \leq K\right\}$ is equal in distribution to $\frac{1}{T^{1/\alpha}r_T^{1/\alpha}} \sum_{i=1}^{T} z_i^2 \frac{\sigma_i^2}{h_i(\theta)} 1\left\{\sup_{\theta\in\Theta} \frac{\sigma_i^2}{h_i(\theta)} \leq K\right\}$, and converges almost surely to $\mathbb{E}^{1/\alpha}[(\frac{\sigma_i^2}{h_i(\theta)}(\phi_K))^\alpha 1\left\{\sup_{\theta\in\Theta} \frac{\sigma_i^2}{h_i(\theta)}(\phi_K) \leq K\right\}] S_\alpha(1, c, 0)(\phi_K)$. The latter is equal in distribution to $\mathbb{E}^{1/\alpha}[(\frac{\sigma_i^2}{h_i(\theta)})^\alpha 1\left\{\sup_{\theta\in\Theta} \frac{\sigma_i^2}{h_i(\theta)} \leq K\right\}] S_\alpha(1, c, 0)$. Applying again Theorem 1.10.4 and Addendum 1.10.5 of Vaart and Wellner, 2012, we have that for any T (as well as for $T = +\infty$), and letting $K \to \infty$, there exist measurable mappings defined on a potentially further enhancement of the aforementioned probability space, with values on the original probability space, say ϕ_T, ϕ, such that: a. $\frac{1}{T^{1/\alpha}r_T^{1/\alpha}} \sum_{i=1}^{T} z(\phi_{K,T})_i^2 \frac{\sigma_i^2}{h_i(\theta)}(\phi_{T,K}) 1\left\{\sup_{\theta\in\Theta} \frac{\sigma_i^2}{h_i(\theta)}(\phi_{T,K}) \leq K\right\}$ converges almost surely to $\frac{1}{T^{1/\alpha}r_T^{1/\alpha}} \sum_{i=1}^{T} z(\phi_T)_i^2 \frac{\sigma_i^2}{h_i(\theta)}(\phi_T)$ which is equal in distribution to $\frac{1}{T^{1/\alpha}r_T^{1/\alpha}} \sum_{i=1}^{T} z_i^2 \frac{\sigma_i^2}{h_i(\theta)}$, and b. $\mathbb{E}^{1/\alpha}[(\frac{\sigma_i^2}{h_i(\theta)}(\phi_K))^\alpha 1\left\{\sup_{\theta\in\Theta} \frac{\sigma_i^2}{h_i(\theta)}(\phi_K) \leq K\right\}] \times S_\alpha(1, c, 0)(\phi_K)$ converges almost surely to $\mathbb{E}^{1/\alpha}[(\frac{\sigma_i^2}{h_i(\theta)}(\phi))^\alpha S_\alpha(1, c, 0)(\phi)$, which is equal in distribution to $\mathbb{E}^{1/\alpha}[(\frac{\sigma_i^2}{h_i(\theta)})^\alpha 1\left\{\sup_{\theta\in\Theta} \frac{\sigma_i^2}{h_i(\theta)} \leq K\right\}] S_\alpha(1, c, 0)$. The previous convergences along with Definition 7.12, Theorem 7.14 and Proposition 7.15 of Rockafellar and Wets, 2009 imply that as $T \to \infty$, $\frac{1}{T^{1/\alpha}r_T^{1/\alpha}} \sum_{i=1}^{T} z(\phi_T)_i^2 \frac{\sigma_i^2}{h_i(\theta)}(\phi_T)$, epi-converges almost surely to $\mathbb{E}^{1/\alpha}[(\frac{\sigma_i^2}{h_i(\theta)}(\phi))^\alpha S_\alpha(1, c, 0)(\phi)$. Reverting to the original probability space we obtain that $\frac{1}{T^{1/\alpha}r_T^{1/\alpha}}\sum_{i=1}^{T} z_i^2 \frac{\sigma_i^2}{h_i(\theta)}$ epi-converges in distribution to the limit that appears in the rhs of (4).

Finally, it suffices to show that the term $\frac{1}{T^{1/\alpha}r_T^{1/\alpha}}\sum_{i=1}^{T} \log \frac{\sigma_i^2}{h_i(\theta)}$ weakly converges to zero uniformly over θ. But notice that due to A.4, this is less than or equal to $C\frac{1}{T^{1/\alpha}r_T^{1/\alpha}}\sum_{i=1}^{T} \log \sigma_i^2$, where C is independent of θ, and the latter converges to zero, from Birkoff's LLN which is applicable due to the existence of the log-moment in A.2., and since $\alpha < 1$.

Properness follows from A.2-A.4 and Proposition 5.2.12 of Straumann, 2004 which imply that $\mathbb{E}[(\frac{\sigma_0^2}{h_0(\theta_0)})^{\alpha}] = 1$. The lsc property follows by the final part of A.3 which implies that the scaled likelihood function is $\mathbb{P}$ a.s. lsc and from the metrizability of epi-convergence on the set of lsc proper functions on Euclidean domains as separable spaces (see Knight, 1999), Theorem 1.10.4 of Vaart and Wellner, 2012 and Theorem 7.15.(a) of Rockafellar and Wets, 2009. □

Proof of Lemma 2. From the chain rule and Theorem 2.6.1 of Straumann on the form of the stationary solution to (3), m_0 is identified as the $\mathbb{P}$ a.s. derivative (w.r.t. ϕ) at θ_0 of $H \circ h_0$. Since m_0 is $\mathbb{P}$ a.s. strictly positive and H^{-1} is strictly increasing, $h_0(\theta(\phi))$ is $\mathbb{P}$ a.s. strictly increasing w.r.t. the pointwise product order on K_ϕ. This implies that there exists some ϕ-possibly in an extension of K_ϕ-such that $\frac{\sigma_0^2}{h_0(\theta(\phi))} < 1$ $\mathbb{P}$ a.s., which is sufficient for A.5. □

Proof of Theorem 1. First notice that the QMLE by construction minimizes $\frac{1}{T^{1/\alpha}r_T^{1/\alpha}}\sum_{i=1}^{T} \hat{\ell}_i(\theta)$. Then due to Lemma 1, the separability of Θ, and Theorem 1.10.4 of Vaart and Wellner, 2012, we obtain by 7.31.b of Rockafellar and Wets, 2009 that the accumulation points of the QMLE lie inside the set of minimizers of the limiting likelihood process. This set does not contain θ_0 due to A.5, and the fact that $Z \sim S_\alpha(1, c, 0)$ and thereby Z has positive support. □

Proof of Lemma 3. For all cases A.2-A.4 can be established from the proofs and the intermediate results that lead to Theorem 5.4.6 of Straumann, 2004, Theorem 3.1.(i) of Francq and Zakoıan, 2013, and Proposition 3.2 of Arvanitis and Louka, 2015 (the existence of log-moments follows directly from A.1). Choosing $\phi = \omega$ it is easy to establish that the conditions of Lemma 2 hold for all cases since then (5) assumes a very simple form, that of a non-homogeneous linear difference equation of order equal to $\max j : \beta_{0,j} > 0$, with unitary constant term and positive autoregressive coefficients. The uniqueness, stationarity-ergodicity of the solution follows from the restrictions on the modulus on the matrix of autoregressive parameters that hold due to the referred conditions, while strict positivity follows from the positivity restrictions on the GARCH autoregressive parameters and the strict positivity of the constant term. H^{-1} is strictly increasing since it is the identity in all cases. □

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Appendix

The Appendix establishes a (uniform) martingale limit theorem for stationary and ergodic martingale transforms without use of mixing conditions. In what follows $\rightsquigarrow$ denotes convergence in distribution.

Theorem 2. *Suppose that $(\xi_i)_{i\in\mathbb{Z}}$ is an iid sequence and such that ξ_0 belongs to the domain of attraction (DoA) of $S_\alpha(s,c,\gamma)$, where $\alpha\in(0,1)$, $c>0$, $s\in\mathbb{R}$. Furthermore, for Θ a compact subset of some Euclidean space, $(V_i(\theta))_{i\in\mathbb{Z}}$ is a stationary and ergodic, sequence adapted to the filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}}$ with $\mathcal{F}_i := \sigma(z_{i-j},\ j>0)$ for all $\theta\in\Theta$, $V_0(\theta))$ is continuous, and $\mathbb{E}[\sup_{\theta\in\Theta}|V_0(\theta)|^{\alpha+\delta}]<+\infty$ for some $\delta>0$. Then, for some slowly varying real sequence $(r_i)_{i\in\mathbb{N}}$, and any $\theta\in\Theta$, and deterministic $\theta_T\to\theta$, as $T\to\infty$*

$$\frac{1}{T^{1/\alpha}r_T^{1/\alpha}}\sum_{i=1}^{T}(\xi_i-\gamma)\,V_i(\theta_T)\rightsquigarrow S_\alpha\left(s\frac{\mathbb{E}[\mathrm{sgn}(V_0(\theta))|V_0(\theta)|^\alpha]}{\mathbb{E}[(|V_0(\theta)|^\alpha]},c\mathbb{E}[|V_0(\theta)|^\alpha],0\right). \quad (6)$$

Proof. By Theorem 1.1 along with Paragraph 3 of Jakubowski, 1986) the result would follow if for all $t\in\mathbb{R}$

$$\prod_{i=1}^{T}\mathbb{E}\left(\exp\left(it\frac{1}{T^{\frac{1}{\alpha}}r_T^{\frac{1}{\alpha}}}\rho_{i,\alpha}\right)/\mathcal{F}_i\right) \quad (7)$$

converges pointwise $\mathbb{P}$ a.s. to the characteristic function of the limit, for some appropriate real slowly varying sequence $(r_i)_{i\in\mathbb{N}}$, where $\rho_{i,\alpha,\theta_T}:=(\xi_i-\gamma)V_i(\theta_T)$. Due to Theorem 2.6.5 of Ibragimov and Linnik I. A. Ibragimov and Linnik, 1971, the DoA assumption is equivalent to that the characteristic function of ξ_0 has for some

$t_0 > 0$, the representation $\mathbf{i}\gamma t - c|t|^a \mathbf{h}\left(|t|^{-1}\right)\left(1 - \mathbf{i}\beta \operatorname{sgn}(t)\tan(\frac{1}{2}\pi\alpha)\right)$ on $(-t_0, t_0)$, where $\mathbf{h}$ is a slowly varying function at infinity. Define r_T by

$$\frac{\mathbf{h}(T^{\frac{1}{\alpha}} r_T^{\frac{1}{\alpha}})}{r_T} \to 1, \tag{8}$$

(see Paragraph 1.9 of Bingham, Goldie, and Teugels, 1989) and notice that this defines a slowly varying sequence. Fix $t \neq 0$ and define the event

$$C_{T,K,\theta_T} := \left\{\omega \in \Omega : |V_i(\theta_T)| \leq K_t \left(T r_T\right)^{\frac{1}{a}} \quad, \forall i = 1, \ldots, n\right\}$$

where $K_t < \frac{t_0}{|t|}$. From to the moment existence condition above and the inequality of Markov we obtain

$$\begin{aligned}
\mathbb{P}[C^c_{T,K,\theta_T}] &\leq \sum_{i=1}^{T} \mathbb{P}[\sup_{\theta\in\Theta} |V_i(\theta)| > K_t \left(T r_T\right)^{\frac{1}{\alpha}}] \\
&\leq \frac{\mathbb{E}[\sup_{\theta\in\Theta} |V_i(\theta)|^{\alpha+\delta}]}{K_t^\alpha T^{\frac{\delta}{\alpha}} r_T^{1+\frac{\delta}{\alpha}}} = o(1).
\end{aligned}$$

Now, if $\omega \in C_{T,K,\theta_T}$ then $\log \mathbb{E}[\exp\left(\mathbf{i}t \frac{1}{T^{\frac{1}{\alpha}} r_T^{\frac{1}{\alpha}}} \sum_{i=1}^T \left[\xi_i - \gamma\right] V_i(\theta_T))/\mathcal{F}_i\right)]$ equals

$$\begin{aligned}
&-\frac{c|t|^\alpha}{T r_T} \sum_{i=1}^{T} |V_i(\theta_T)|^\alpha\, \mathbf{h}\left(T^{1/\alpha} r_T^{1/\alpha} \,|V_i(\theta_T)|^{-1}\right)\left(1 - \mathbf{i}\beta \operatorname{sgn}(tV_i(\theta_T))\tan(\frac{1}{2}\pi\alpha)\right) \\
= \quad &-\frac{c|t|^\alpha}{T r_T} \sum_{i=1}^{T} |V_i(\theta_T)|^\alpha\, \mathbf{h}\left(T^{1/\alpha} r_T^{1/\alpha} \,|V_i(\theta_T)|^{-1}\right) \\
&+\frac{|t|^\alpha}{T r_T} \mathbf{i}\beta c \operatorname{sgn}(t)\tan\left(\frac{1}{2}\pi\alpha\right) \sum_{i=1}^{T} |V_i(\theta_T)|^\alpha\, \mathbf{h}\left(T^{1/\alpha} r_T^{1/\alpha} \,|V_i(\theta_T)|^{-1}\right) \operatorname{sgn}(V_i(\theta_T))
\end{aligned}$$

Examining the first term, define $g\left(x\right) := x^{-\alpha}\mathbf{h}\left(x\right).$ Then for any $\varepsilon > 0$

$$\begin{aligned}
&\frac{1}{T r_T} \sum_{i=1}^{T} |V_i(\theta_T)|^\alpha\, \mathbf{h}\left(T^{1/\alpha} r_T^{1/\alpha} \,|V_i(\theta_T)|^{-1}\right) 1\left\{|V_i(\theta_T)| \leq \varepsilon\right\} \\
= \quad &\sum_{i=1}^{n} g\left(T^{1/\alpha} r_T^{1/\alpha} |V_i(\theta_T)|^{-1}\right) 1\left\{|V_i(\theta_T)| \leq \varepsilon\right\} \\
= \quad &\frac{\mathbf{h}\left(T^{1/\alpha} r_T^{1/\alpha}\right)}{r_T} \frac{1}{T} \sum_{i=1}^{T} \frac{g\left(T^{1/\alpha} r_T^{1/\alpha} |V_i(\theta_T|^{-1}\right)}{g\left(T^{1/\alpha} r_T^{1/\alpha}\right)} 1\left\{|V_i(\theta_T)| \leq \varepsilon\right\}.
\end{aligned}$$

Since $\frac{\mathbf{h}\left(n^{1/\alpha} r_n^{1/\alpha}\right)}{r_n} \to 1$ by construction, the last term of the previous display is asymptotically equivalent to

$$\frac{1}{T} \sum_{i=1}^{T} |V_i(\theta_T)|^\alpha 1\left\{|V_i(\theta_T)| \leq \varepsilon\right\} + \frac{1}{T} \sum_{i=1}^{T} \left[\frac{g\left(T^{1/\alpha} r_T^{1/\alpha} |V_i(\theta_T)|^{-1}\right)}{g\left(T^{1/\alpha} r_T^{1/\alpha}\right)} - |V_i(\theta_T)|^\alpha\right] 1\left\{|V_i(\theta_T)| \leq \varepsilon\right\}$$

But,

$$\frac{1}{T}\sum_{i=1}^{T}\left[\frac{g\left(T^{1/\alpha}r_T^{1/\alpha}|V_i(\theta_T)|^{-1}\right)}{g\left(T^{1/\alpha}r_T^{1/\alpha}\right)}-|V_i(\theta_T)|^{\alpha}\right]1\left\{|V_i(\theta_T)|\leq\varepsilon\right\}\leq\sup_{|x|\leq\varepsilon}\left|\frac{g\left(T^{1/\alpha}r_T^{1/\alpha}|x|^{-1}\right)}{g\left(T^{1/\alpha}r_T^{1/\alpha}\right)}-|x|^{\alpha}\right|\to 0.$$

by an application of the Uniform Convergence Theorem for regularly varying functions. Thus,

$$\frac{1}{Tr_T}\sum_{i=1}^{T}|V_i(\theta_T)|^{\alpha}\,\mathbf{h}\left(T^{1/\alpha}r_T^{1/\alpha}\,|V_i(\theta_T)|^{-1}\right)1\left\{|V_i(\theta_T)|\leq\varepsilon\right\}\to\mathbb{E}\left[|V_0(\theta)|^{a}\,1\left\{|V_0(\theta)|\leq\varepsilon\right\}|\right],\mathbb{P}\text{a.s.},$$

due to the compactness of Θ, the moment existence condition $\mathbb{E}[\sup_{\theta\in\Theta}|V_0(\theta)|^{\alpha+\delta}]<+\infty$, Theorem 2.2.1 of Straumann, 2004, dominated convergence, and Theorem 7.14 of Rockafellar and Wets, 2009. Similarly, and applying the Uniform Convergence Theorem for regularly varying functions on $\frac{h\left(T^{1/\alpha}r_T^{1/\alpha}|V_i(\theta_T)|^{-1}\right)}{h\left(T^{1/\alpha}r_T^{1/\alpha}\right)}$ whenever $|V_i(\theta_T)|>\varepsilon$ we obtain that

$$\frac{1}{Tr_T}\sum_{i=1}^{T}|V_i(\theta_T)|^{\alpha}\,\mathbf{h}\left(T^{1/\alpha}r_T^{1/\alpha}\,|V_i(\theta_T)|^{-1}\right)1\left\{|V_i(\theta_T)|>\varepsilon\right\}\to\mathbb{E}\left[|V_0(\theta)|^{a}\,1\left\{|V_0(\theta)|>\varepsilon\right\}|\right],\mathbb{P}\text{a.s.}$$

Letting $\varepsilon\to\infty$ we obtain due to the moment existence condition

$$\frac{1}{Tr_T}\sum_{i=1}^{T}|V_i(\theta_T)|^{\alpha}\,\mathbf{h}\left(T^{1/\alpha}r_T^{1/\alpha}\,|V_i(\theta_T)|^{-1}\right)\to\mathbb{E}\left[|V_0(\theta)|^{a}\right],\ \mathbb{P}\text{ a.s.}$$

Treating the second term analogously, we obtain

$$\frac{|t|^{\alpha}}{Tr_T}\mathbf{i}\beta c\text{sgn}(t)\tan\left(\frac{1}{2}\pi\alpha\right)\sum_{i=1}^{T}|V_i(\theta_T)|^{\alpha}\,\mathbf{h}\left(T^{1/\alpha}r_T^{1/\alpha}\,|V_i(\theta)|^{-1}\right)\text{sgn}(V_i(\theta))$$
$$\to\mathbf{i}\beta c|t|^{\alpha}\text{sgn}(t)\tan\left(\frac{1}{2}\pi\alpha\right)\mathbb{E}\left[|V_0(\theta)|^{\alpha}\,\text{sgn}(V_0(\theta))\right],\ \mathbb{P}\text{ a.s.}$$

establishing the result due to the form of the characteristic function of an α stable distribution (see Ch. 2 of I. A. Ibragimov and Linnik, 1971). □

The moment existence condition in the previous theorem also implies the following corollary.

Corollary 1. *Under the premises of Theorem 2, as $T\to\infty$*

$$\frac{1}{T^{1/\alpha}r_T^{1/\alpha}}\sum_{i=1}^{T}\xi_iV_i(\theta_T)\rightsquigarrow S_{\alpha}\left(s\frac{\mathbb{E}[\text{sgn}(V_0(\theta))|V_0(\theta)|^{\alpha}]}{\mathbb{E}[|V_0(\theta)|^{\alpha}]},c\mathbb{E}[|V_0(\theta)|^{\alpha}],0\right).(9)$$

Proof. Due to the compactness of Θ, the moment existence condition $\mathbb{E}[\sup_{\theta\in\Theta}|V_0(\theta)|^{\alpha+\delta}]<+\infty$, Theorem 2.2.1 of Straumann, 2004, dominated convergence, and Theorem 7.14 of Rockafellar and Wets, 2009, we have that

$\frac{1}{T}\sum_{i=1}^{T}\left|V_i(\theta_T)\right|^{\alpha+\delta} \to \mathbb{E}[|V_0(\theta)|^{\alpha+\delta}]$, $\mathbb{P}$ a.s. Since δ can be chosen small enough so that $\alpha+\delta<1$, and due to the reverse Minkowski inequality, we obtain the bound $\left(\frac{1}{T^{1/\alpha}r_T^{1/\alpha}}\sum_{i=1}^{T}|V_i(\theta_T)|\right)^{\alpha+\delta} \leq \frac{1}{T^{\delta/\alpha}r_T^{(\alpha+\delta)/\alpha}}\frac{1}{T}\sum_{i=1}^{T}\left|V_i(\theta_T)\right|^{\alpha+\delta}$. This, along with the almost sure convergence established above and the fact that $\frac{\delta}{\alpha}>0$ imply that

$$\frac{1}{T^{1/\alpha}r_T^{1/\alpha}}\sum_{i=1}^{T}V_i(\theta_T)\to 0,\ \mathbb{P}\text{ a.s.},$$

The result follows then from Theorem 2. □

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Athens University of Economics and Business

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