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Inconsistency for the Gaussian QMLE in GARCH-type models with infinite variance

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Abstract

We are occupied with the issue of consistency of the Gaussian QMLE in GARCH-type models with very heavy tailed squared innovations. We show that the appropriately scaled likelihood function weakly epi-converges to a stochastic process that is a.s. lower semi-continuous and proper. When moreover the volatility filter is increasing w.r.t. the parameter, inconsistency follows due to that the true parameter value misses the set of minimizers of the limit. This holds for models like the AGARCH the Augmented GARCH and the GQARCH.
Keywords: Heavy-tailed distribution, GARCH-type Models, Gaussian QMLE , AGARCH, Augmented GARCH, GQARCH, Inconsistency.
JEL codes: C00, C01, C13, C14, C58.

1 Introduction

A strand of the empirical finance literature considers the use of heavy tailed distributions for the squared innovation process in GARCH-type models-see for example Rachev and Mittnik, 1988 and Mittnik, Rachev, and Paolella, 1998. The implications of heavy tails on the limit theory of the commonly used Gaussian QMLE for the GARCH(p,q) model were analyzed by Mikosch and Straumann, 2006. They employed a martingale limit theorem to derive stable limiting distributions and regularly varying rates of convergence, when the extremal index of the innovations is strictly greater than one. Analogous results under similar frameworks are expected to hold for other GARCH-type models. However, the case where the index is less than (or in some instances equal to) one presents ambiguity even on the issue of consistency of the QMLE for such-like models. This is associated to the different asymptotic behavior of the log-likelihood process due to the required scaling.

In the present paper we are occupied with the issue of consistency of the QMLE when the index is strictly less than one. Under familiar assumptions, the likelihood function weakly epi-converges to a stochastic process that is almost surely

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lower semi-continuous (lsc) and proper. We derive inconsistency when there exists a parameter value at which the limit is finite and smaller compared to the value attained at the true value. This precludes the true parameter value from lying inside the (possibly stochastic) set of minimizers of the limit. This is true for models where the volatility filter has an almost sure monotonicity property; it implies that the likelihood limit almost surely achieves smaller values when the true parameter is partially translated to pointwise larger values. Examples are the AGARCH and Augmented GARCH and the GQARCH models.

In the following sections we describe our main framework, provide with our assumptions, obtain the results along with some examples, and describe potential extensions. Finally, we provide our proofs. In the Appendix we present a Martingale Limit Theorem that differs from the one of Mikosch and Straumann, 2006, first, by not requiring mixing conditions beyond ergodicity, and second, by allowing for continuous convergence w.r.t. Euclidean parameters.

2 Framework

Given an underlying probability space $(\Omega, \mathcal{G}, \mathbb{P})$, consider the following general model

$$\begin{cases} y_i = \sigma_i z_i \\ H(\sigma_i^2) = g_{\theta_0}\left(z_{i-1}, \ldots, z_{i-p}, H(\sigma_{i-1}^2), \ldots, H(\sigma_{i-\max(p,q)}^2)\right) \end{cases}, \; i \in \mathbb{Z} \tag{1}$$

where the parameter θ_0 lies in some non-empty compact parameter space $\Theta \subset \mathbb{R}^d$, the function g_θ is a measurable function on $\Theta \times [0, \infty)^q$, H is an invertible measurable self function on $\mathbb{R}$, and the sequence $(z_i)_{i \in \mathbb{Z}}$ is i.i.d.

Illustrative examples of known models that adhere to the formulation in (1) are (for further examples see among others Paragraph 3.3 in Straumann, 2004):

- AGARCH(p, q), see Ding, Granger, and Engle, 1993, where

$$\sigma_i^2 = \omega + \sum_{j=1}^{p} \left(a_{0,j} \left|y_{i-j}\right| - \gamma_0 y_{i-j}\right)^2 + \sum_{k=1}^{q} \beta_{0,k} \sigma_{i-k}^2,$$

- Augmented GARCH$(1, 1)$, see Francq and Zakoıan, 2013, where

$$\sigma_i^2 = \omega_0 + a_{0+}[\max(0, y_{i-1})]^2 + a_{0-}[\min(0, y_{i-1})]^2 + \beta_0 \sigma_{i-1}^2,$$

- GQARCH$(1, 1)$, see Sentana, 1995, where

$$\sigma_i^2 = \omega_0 + a(y_{i-1} + \frac{\gamma}{2a})^2 + \beta_0 \sigma_{i-1}^2.$$

The examples contains as a special case the celebrated symmetric GARCH(1,1) model (see Bollerslev, 1986). They are all dynamically asymmetric in the sense that they allow for non-zero covariances between y_i and σ_{i-1}^2 without requiring asymmetry for z_i. The unitary order of the last two examples is chosen for brevity; auxiliary properties for the unitary specifications are already known in the literature. Our main results can be readily extended to arbitrary orders for those two cases at the cost of heavier notation.

Using a stochastic recurrence equation (SRE) approach, Straumann and Mikosch, 2006 give sufficient conditions for the existence of a unique (up to a limiting argument), solution to (1), say $((y_i, \sigma_i^2))_{i\in\mathbb{Z}}$, that is stationary, ergodic and non-anticipative. This is ensured by, among others, imposing conditions on the Lipschitz coefficient of the random map that generates σ_i^2. In order to infer the unknown parameter θ_0, and since in practice only $(y_i)_{i=1,\dots,n}$ is observable, the volatility process (σ_i^2) is reconstructed under the parameter hypothesis θ using the following volatility filter for some random vector of initial values $(\varsigma_0, \varsigma_1, \dots, \varsigma_{q-1})' \in \mathbb{R}^q$,

$$H(\hat{h}_i(\theta)) = \begin{cases} \varsigma_i, & q-1 \leq i \leq 0, \\ g_\theta^\star\left(y_{i-1}, \dots, y_{i-p}; H(\hat{h}_{i-1}(\theta)), \dots, H(\hat{h}_{i-\max(p,q)}(\theta))\right), & i > 0, \end{cases} \tag{2}$$

which is generally non-stationary. Here, $g_\theta^\star$ is obtained by g_θ by replacing z_{i-j} with $\frac{y_{i-j}}{\sqrt{\hat{h}_{i-j}(\theta)}}$ for all $j = 1, \dots, p$. Given the above, the Gaussian quasi likelihood function is proportional to

$$\hat{c}_n(\theta) = \frac{1}{n}\sum_{i=1}^{n} \hat{\ell}_i(\theta),$$

where $\hat{\ell}_i(\theta) = \log \hat{h}_i(\theta) + \frac{y_i^2}{\hat{h}_i(\theta)}$, and then the Gaussian QMLE θ_n of θ_0 is defined by

$$\hat{c}_n(\theta_n) = \inf_{\theta\in\Theta} \hat{c}_n(\theta).$$

Standard lower semi-continuity and measurability conditions imply existence of the estimator (see for example Definition 3.5 and Proposition 3.6 in Chapter 5 of Molchanov, 2011), while the definition-and the subsequent results-can be easily extended so that approximate/numerical optimization is allowed.

3 Assumptions, Results and Examples

We are interested in the issue of consistency of θ_n when z_0^2 belongs to the domain of attraction of a stable distribution with tails heavy enough so that it does not have a first moment. We first present our assumption framework. There, $S_\alpha(s, c, \gamma)$ denotes the (univariate) stable distribution with α, s, c, γ denoting stability, skewness, scale and location respectively (see Chapter 2 of Ibragimov and Linnik, 1971).

A.1 The distribution of z_0^2 lies in the domain of attraction (DoA) of $S_\alpha(1, c, 1)$ with $\alpha \in (0, 1)$.

The assumption implies that $\mathbb{E}[z_0^2] = +\infty$ (see Chapter 2 of Ibragimov and Linnik, 1971) and in fact allows for very heavy right tail for the distribution of z_0^2. Its unit location is in accordance to the innovations' unit variance normalization condition ($\mathbb{E}[z_0^2] = 1$) that is typical in such models when the innovations have enough moments. We do not consider the case where $\alpha > 1$ (as well as the case of $\alpha = 1$ and $\mathbb{E}[z_0^2] < +\infty$) since consistency then readily follows by Theorem 5.3.1 of Straumann, 2004 under A.2-4 below, and a usual identification condition. The restriction of the skewness parameter to one follows naturally from the fact that the support of $z_0^2 - 1$ is bounded from below.

Assumptions A.2-A.4 that follow are similar to the ones used in the literature in the case where $\mathbb{E}[z_0^2] < +\infty$ (see Straumann, 2004, ch. 5, C.1, C.3-C.4 and Wintenberger, 2013). They among others enable the approximation of (2) by the process $(h_i(\theta))_{i\in\mathbb{Z}}$, $\theta \in \Theta$ defined by

$$H(h_i(\theta)) = g_\theta^{\star\star}\left(y_{i-1}, \dots, y_{i-p}; H(h_{i-1}(\theta)), \dots, H(h_{i-\max(p,q)}(\theta))\right), i \in \mathbb{Z}, \qquad (3)$$

which is independent of initial conditions. The form $g_\theta^{\star\star}$ here, is obtained from g_θ by replacing z_{i-j} with $\frac{y_{i-j}}{\sqrt{h_{i-j}(\theta)}}$ for all $j = 1, \dots, p$.

A.2 The model (1) admits a unique stationary ergodic non-anticipative solution $((y_i, \sigma_i^2))_{i\in\mathbb{Z}}$, with $\mathbb{E}[\ln^+(\sigma_0^2)] < +\infty$.

A.3 The model (3) is continuously invertible in Θ. The functions $\ln \circ H^{-1}$, $\frac{1}{H^{-1}}$ are well defined and Lipschitz continuous on the range of g_θ for all $\theta \in \Theta$. Finally, the model (3) is $\mathbb{P}$ a.s. continuous in θ.

A.4 $\inf_{\theta\in\Theta} \inf_{\mathbf{x}\in\mathbb{R}^p\times[0,\infty)^q} H^{-1} \circ g_\theta(\mathbf{x}) > 0$.

Assumption A.2 holds if there exists $\mathbf{u}$, for which $\mathbb{E}[\ln^+(|\psi_0(\mathbf{u})|)] < +\infty$, $\mathbb{E}[\ln^+(\Lambda(\psi_0))] < +\infty$ and $\mathbb{E}[\ln(\Lambda(\psi_0))] < 0$-see Theorem 2.6.1 in Straumann, 2004. For $\mathbf{u} \in \mathbb{R}^{\max(p,q)}$, $\psi_t(\mathbf{u})$ is defined as $g_{\theta_0}(z_{i-1}, \dots, z_{i-p}, \mathbf{u})$, and for a real Lipschitz continuous self-function f, $\Lambda(f)$ denotes its Lipschitz coefficient. The form of (1), the functional invertibility of H, and A.2 imply that $h_i(\theta_0) = \sigma_i^2$, $\mathbb{P}$ a.s., $\forall i \in \mathbb{Z}$.

Given A.2, continuous invertibility enables the almost sure convergence of $H(\hat{h}_i(\theta)$ to $H(h_i(\theta)$ as $i \to \infty$ uniformly in Θ, given Definition 3 of Wintenberger, 2013. From Theorem 2 of Wintenberger, 2013, this follows whenever there exists $\mathbf{u}$, for which $\mathbb{E}[\ln^+(\sup_{\theta\in\Theta} |\phi_T(\mathbf{u}, \theta)|)] < +\infty$, $\mathbb{E}[\ln^+(\sup_{\theta\in\Theta} \Lambda(\phi_0(\theta))] < +\infty$ and $\mathbb{E}[\ln(\Lambda(\phi_0(\theta))] < 0$, $\forall \theta \in \Theta$. $\phi_0(\mathbf{u}, \theta)$ is $g_\theta^{\star\star}(y_{i-1}, \dots, y_{i-p}, \mathbf{u})$. The Lipschitz continuity conditions of the second part of A.3 enable then the analogous approximation of $\ln(\hat{h}_i(\theta))$ by $\ln(h_i(\theta))$ and of $\frac{1}{\hat{h}_i(\theta)}$ by $\frac{1}{h_i(\theta)}$. Hence, A.3 enables among others, asymptotic validity of the substitution of $\hat{h}_i(\theta)$ by $h_i(\theta)$ in the likelihood function, and thereby the limiting independence of the likelihood process of the initial conditions. The final part of A.3 along with continuous invertibility, the Lipschitz continuity conditions and the compactness of Θ imply the $\mathbb{P}$ a.s. continuity of $h_i(\theta)$ in θ.

A.4 is a technical condition of strict positivity and in several cases follows from restrictions on Θ that facilitate the positivity of the volatility process.

A.1-A.4 suffice for the establishment of the weak epi-convergence of the scaled likelihood process to a lsc and proper stochastic process driven by a very heavy tailed random variable. We first establish the required notation. Due to Theorem 2.6.5 of Ibragimov and Linnik Ibragimov and Linnik, 1971, the DoA assumption is equivalent to that the characteristic function of z_0^2 has for some $t_0 > 0$, the representation $\mathbf{i}t - c|t|^a\mathbf{h}(|t|^{-1})\left(1 - \mathbf{i}\mathrm{sgn}(t)\tan(\frac{1}{2}\pi\alpha)\right)$ on $(-t_0, t_0)$, where $\mathbf{h}$ is a slowly varying function at infinity. Define r_T by the asymptotic relation $\frac{\mathbf{h}(T^{\frac{1}{\alpha}} r_T^{\frac{1}{\alpha}})}{r_T} \to 1$, and notice that r_T is a well defined slowly varying sequence (see Paragraph 1.9 of Bingham, Goldie, and Teugels, 1989, and Proposition 1.(iv) of Astrauskas, 1983). Futhermore, $\overset{\text{epi}}{\rightsquigarrow}$ denotes epi-convergence in distribution (see Knight, 1999). An extended real valued function is called proper when it never assumes the value $-\infty$, and is not identically equal to $+\infty$. A stochastic process with extended real valued sample paths is called proper, when almost all its sample paths are proper functions.

Lemma 1. *Suppose that A.1-A.4 hold. Then, as $T \to \infty$*

$$\frac{1}{T^{1/\alpha} r_T^{1/\alpha}} \sum_{i=1}^{T} \hat{\ell}_i(\theta) \overset{epi}{\rightsquigarrow} Z\mathbb{E}^{1/\alpha}[(\frac{\sigma_i^2}{h_i(\theta)})^{\alpha}], \tag{4}$$

where $Z \sim S_\alpha(1, c, 0)$ for some $c > 0$. The limiting process is $\mathbb{P}$ a.s. lsc and proper.

The derivation utilizes among others the martingale limit theorem (see Theorem 2) that we establish in the Appendix, and the fact that Skorokhod representations are available in our framework. This is due to that the epi-convergence topology on the set of lsc and proper extended real valued functions defined on Θ is separable (see Knight, 1999). The limiting process has positive sample paths since $\alpha < 1$. Furthermore, $\mathbb{E}[(\frac{\sigma_0^2}{h_0(\theta)})^{\alpha}] \in (0, +\infty]$. The epi-convergence controls the limiting behavior of the minimizers of the likelihood function. It implies that every weak accumulation point must lie inside the set of minimizers of the limiting process. When the latter does not contain θ_0 inconsistency is obtained due to non-identification.

The final assumption concerns the behavior of the moment appearing in the deterministic part of the limiting process, and it essentially ensures asymptotic non-identification in the framework of A.1, due to the positivity of the limit in 4.

A.5 $\exists \theta \neq \theta_0 : \mathbb{E}[(\frac{\sigma_0^2}{h_0(\theta)})^{\alpha}] < 1$.

A.5 can be valid even in cases where condition $h_0(\theta) = \sigma_0^2 \Leftrightarrow \theta = \theta_0$, $\mathbb{P}$ a.s. (ID) holds. When $\mathbb{E}[z_0^2]$ exists and equals to one, the latter is the usual identification condition-see for example Wintenberger, 2013. Under A.1, when A.5 holds, (ID) becomes insufficient as an identification condition.

The following auxiliary result says that A.5 holds when the stationary version of the filter in (3) has a monotonicity property w.r.t. some part of the parameter that can be adequately translated inside the parameter space. In this respect suppose that Θ is factored as $\Phi \times \Psi$, so that $\theta_0 := (\phi_0, \varphi_0)$, and let $\theta(\phi) := (\phi, \psi_0)$ and denote with K_Φ some neighborhood of ϕ_0. ∂_ϕ denotes partial derivation w.r.t. ϕ.

Lemma 2. *Suppose that*

$$\hat{g}_\phi \circ \mathbf{u} := g^{\star\star}_{\theta(\phi)}(y_{i-1}, \ldots, y_{i-p}, \mathbf{u})$$

is $\mathbb{P}$ a.s. continuously differentiable on $K_\Phi \times (0, +\infty)$ (by potentially extending Φ), and that K_Φ is compact and coincides with the closure of its interior. Suppose that the SRE

$$m_i = \partial_\phi(\hat{g}_\phi \circ (m_{i-1}, \ldots, m_{i-\max(p,q)})), \tag{5}$$

has a unique, stationary and ergodic solution $(m_i)_{i\in\mathbb{N}}$ such that $m_i > 0$, $\mathbb{P}$ a.s. for all i, and that H^{-1} is strictly increasing. Then A.5 holds.

The existence of a unique stationary and ergodic solution to (5) can be established by conditions similar to the conditions D.1-2 of Straumann, 2004 (see Ch. 5) restricted to K_ϕ. In models like the AGARCH(p,q) we can choose $\phi = \omega$ in which case the SRE admits a simple form, and positivity is ensured due to the positivity of the GARCH autoregressive parameters and the constant. Furthermore, in such cases the potentially needed extension of Φ (when for example ϕ_0 lies in its boundary) is usually empirically innocuous.

We can now state our main result.

Theorem 1. *Under A.1-A.5, the Gaussian QMLE is inconsistent.*

Lemma 4 and A.5 imply that the QMLE has subsequential limits inside the set of minimizers of the limiting process that do not include θ_0 with probability one, hence inconsistency follows essentially from Theorem 7.31.b of Rockafellar and Wets, 2009 that relates epi-convergence to the limiting properties of minimizers. The result rests on a qualitatively different behavior compared to the classical identification failures. Those occur due to that either $\mathbb{E}[z_0^2] \neq 1$ but finite, and/or $\mathbb{E}[z_0^2] = 1$ but (ID) fails. In all those cases the likelihood limit is deterministic and θ_0 belongs to the non-singleton set of minimizers under A.2-A.4. Despite the randomness of the limiting likelihood, weak consistency would be obtained as long as $\forall \theta \neq \theta_0 : \mathbb{E}[(\frac{\sigma_0^2}{h_0(\theta)})^\alpha] > 1$, which would constitute the sufficient identification condition under A.1.

We finally examine the AGARCH, Augmented GARCH and GQARCH examples that were presented above.

Lemma 3. *Suppose that A.1 holds. (i). For the AGARCH(p,q) suppose that* $a_{0,j} > 0$ *for some* $j > 0$, $(a_{0,p}, \beta_{0,q}) \neq (0,0)$, *and the polynomials* $\sum_{j=1}^{p} a_{0,j} z^j$ *and* $1 - \sum_{j=1}^{q} \beta_{0,j} z^j$ *do not have common roots. (ii). For the Augmented GARCH(1,1) case, suppose that* $\mathbb{E}[\ln(a_{0+}[\max(0,z_0)]^2 + a_{0-}[\min(0,z_0)]^2 + \beta_0)] < 0$ *and* $\beta < 1$, *for all* $\theta \in \Theta$. *(iii). For the GQARCH(1,1) case, suppose that* $\mathbb{E}[\ln(\sup_{s \in [\frac{\omega_0}{1-\beta_0}, +\infty)} |a_0 z_0^2 + \beta_0 + \frac{\gamma_0}{2\sqrt{s}}|)] < 0$ *and* $\beta < 1$, *for all* $\theta \in \Theta$. *In each case (i),(ii),(iii), the Gaussian QMLE is inconsistent.*

Besides A.1, the conditions that appear in the lemma are identical to Theorem 5.4.6 of Straumann, 2004, Theorem 3.1.(i) of Francq and Zakoıan, 2013, and Proposition 3.2 of Arvanitis and Louka, 2015 that establish consistency for each model respectively. In all cases A.2-A.4 and condition (ID) hold, however A.5 is also valid due to the linearity in volatility of (3) for those models. This implies monotonicity for the choice $\phi = \omega$, due to the positivity of the GARCH autoregressive parameters, and the strict positivity of the constants, via the application of Lemma 2.

The inconsistency of the Gaussian QMLE is mitigated by the existence of consistent semi-parametric estimators for θ_0. Under an assumption framework that contains similar conditions to A.1-4, and additionally involves conditions of mean boundedness for the derivatives of the volatility filter, the LAD estimator for θ_0 of Chan and Ng, 2009, is strongly consistent when $\ln(z_0^2)$ has zero median and a bounded density that is continuous at zero (see Lemma A.1 of Chan and Ng, 2009).

4 Further research

The filter monotonicity property of Lemma 2 does not seem to hold for models with more complex volatility filters like the EGARCH (see Nelson, 1991). There, A.5 may not be the case and identification could hold, if for example $\forall\theta \neq \theta_0$, $\mathbb{E}[(\frac{\sigma_0^2}{h_0(\theta)})^\alpha] = +\infty$ under A.1. If this is true, then a slight modification of A.2-3 that takes care of the non-existence of logarithmic moments when $\alpha \leq 0.5$ would imply the weak consistency of the QMLE. This potentially interesting investigation is delegated to further research.

Moreover, the results do not cover the cases where $\alpha = 1$ yet $\mathbb{E}\left[z_0^2\right] = +\infty$. We suspect that these can be handled under the premises of a (locally uniform) LLN for stationary strong mixing sequences with potentially diverging slowly varying moments. To our knowledge such LLNs are currently available only under uniform mixing conditions (see for example Corollary 3 of Szewczak, 2010) that are not generally compatible to the examined models in the current framework. Hence such a consideration is also delegated to further research.

5 Proofs

Proof of Lemma 1. First notice that for arbitrary $K > 0$,

$$\frac{1}{T^{1/\alpha}r_T^{1/\alpha}}\sup_{\theta\in\Theta}|\sum_{i=1}^{T}(\log\hat{h}_i(\theta) + \frac{z_i^2(\sigma_i^2\wedge K)}{\hat{h}_i(\theta)} - \log h_i(\theta) - \frac{z_i^2(\sigma_i^2\wedge K)}{h_i(\theta)})|$$
$$\leq \frac{1}{T^{1/\alpha}r_T^{1/\alpha}}C\sum_{i=1}^{T}(1 + z_i^2(\sigma_i^2\wedge K))\sup_{\theta\in\Theta}|\hat{h}_i(\theta) - h_i(\theta)|$$

for some $C > 0$ independent of K that exists due to the Lipschitz continuity conditions in A.3, and A.4. Then by Definition 2 of Wintenberger, 2013 and due to A.3, $\sup_{\theta\in\Theta}|\hat{h}_i(\theta) - h_i(\theta)|$ converges $\mathbb{P}$ a.s. to zero as $i \to \infty$. Hence, $(\sigma_i^2\wedge K)\sup_{\theta\in\Theta}|\hat{h}_i(\theta) - h_i(\theta)|$ also converges $\mathbb{P}$ a.s. to zero. Then the dominant term of the previous display weakly converges to zero due to the Cezaro sum Theorem, the fact that by a trivial application of Theorem 2 and Corollary 1, $\frac{1}{T^{1/\alpha}r_T^{1/\alpha}}\sum_{i=1}^{T}z_i^2$ weakly converges to an α-stable random variable, Theorem 1.10.4 of Vaart and Wellner, 2012 and the fact that $\alpha < 1$. Letting $T \to \infty$ and then $K \to \infty$ we obtain that $\frac{1}{T^{1/\alpha}r_T^{1/\alpha}}\sup_{\theta\in\Theta}|\sum_{i=1}^{T}(\log\hat{h}_i(\theta) + \frac{z_i^2\sigma_i^2}{\hat{h}_i(\theta)} - \log h_i(\theta) - \frac{z_i^2\sigma_i^2}{h_i(\theta)})|$ weakly converges to zero.

Now consider the term, $\frac{1}{T^{1/\alpha}r_T^{1/\alpha}}\sum_{i=1}^{T}z_i^2\frac{\sigma_i^2}{h_i(\theta)}1\left\{\sup_{\theta\in\Theta}\frac{\sigma_i^2}{h_i(\theta)} \leq K\right\}$. Under A.1-A.4 and due to Theorem 2 and Corollary 1, it weakly converges uniformly (over Θ) to $S_\alpha\left(1, c\mathbb{E}[(\frac{\sigma_i^2}{h_i(\theta)})^\alpha 1\left\{\sup_{\theta\in\Theta}\frac{\sigma_i^2}{h_i(\theta)} \leq K\right\}], 0\right)$, which is equal in distribution to $\mathbb{E}^{1/\alpha}[(\frac{\sigma_i^2}{h_i(\theta)})^\alpha 1\left\{\sup_{\theta\in\Theta}\frac{\sigma_i^2}{h_i(\theta)} \leq K\right\}]S_\alpha(1, c, 0)$. Theorem 1.10.4 and Addendum 1.10.5 of Vaart and Wellner, 2012 implies the existence of an enhanced probability space and measurable mappings defined on it with values on the original probability space, say $\phi_{T,K}$, ϕ_K, such that $\frac{1}{T^{1/\alpha}r_T^{1/\alpha}}\sum_{i=1}^{T}z(\phi_{K,T})_i^2\frac{\sigma_i^2}{h_i(\theta)}(\phi_{T,K})1\left\{\sup_{\theta\in\Theta}\frac{\sigma_i^2}{h_i(\theta)}(\phi_{T,K}) \leq K\right\}$ is equal in distribution to $\frac{1}{T^{1/\alpha}r_T^{1/\alpha}}\sum_{i=1}^{T}z_i^2\frac{\sigma_i^2}{h_i(\theta)}1\left\{\sup_{\theta\in\Theta}\frac{\sigma_i^2}{h_i(\theta)} \leq K\right\}$, and converges almost surely to $\mathbb{E}^{1/\alpha}[(\frac{\sigma_i^2}{h_i(\theta)}(\phi_K))^\alpha 1\left\{\sup_{\theta\in\Theta}\frac{\sigma_i^2}{h_i(\theta)}(\phi_K) \leq K\right\}]S_\alpha(1, c, 0)(\phi_K)$. The latter is equal in distribution to $\mathbb{E}^{1/\alpha}[(\frac{\sigma_i^2}{h_i(\theta)})^\alpha 1\left\{\sup_{\theta\in\Theta}\frac{\sigma_i^2}{h_i(\theta)}\leq K\right\}]S_\alpha(1,c,0)$. Applying again Theorem 1.10.4 and Addendum 1.10.5 of Vaart and Wellner, 2012, we have that for any T (as well as for $T=+\infty$), and letting $K\to\infty$, there exist measurable mappings defined on a potentially further enhancement of the aforementioned probability space, with values on the original probability space, say ϕ_T, ϕ, such that: a. $\frac{1}{T^{1/\alpha}r_T^{1/\alpha}}\sum_{i=1}^T z(\phi_{K,T})_i^2\frac{\sigma_i^2}{h_i(\theta)}(\phi_{T,K})1\left\{\sup_{\theta\in\Theta}\frac{\sigma_i^2}{h_i(\theta)}(\phi_{T,K})\leq K\right\}$ converges almost surely to $\frac{1}{T^{1/\alpha}r_T^{1/\alpha}}\sum_{i=1}^T z(\phi_T)_i^2\frac{\sigma_i^2}{h_i(\theta)}(\phi_T)$ which is equal in distribution to $\frac{1}{T^{1/\alpha}r_T^{1/\alpha}}\sum_{i=1}^T z_i^2\frac{\sigma_i^2}{h_i(\theta)}$, and b. $\mathbb{E}^{1/\alpha}[(\frac{\sigma_i^2}{h_i(\theta)}(\phi_K))^\alpha 1\left\{\sup_{\theta\in\Theta}\frac{\sigma_i^2}{h_i(\theta)}(\phi_K)\leq K\right\}]\times S_\alpha(1,c,0)(\phi_K)$ converges almost surely to $\mathbb{E}^{1/\alpha}[(\frac{\sigma_i^2}{h_i(\theta)}(\phi))^\alpha S_\alpha(1,c,0)(\phi)$, which is equal in distribution to $\mathbb{E}^{1/\alpha}[(\frac{\sigma_i^2}{h_i(\theta)})^\alpha 1\left\{\sup_{\theta\in\Theta}\frac{\sigma_i^2}{h_i(\theta)}\leq K\right\}]S_\alpha(1,c,0)$. The previous convergences along with Definition 7.12, Theorem 7.14 and Proposition 7.15 of Rockafellar and Wets, 2009 imply that as $T\to\infty$, $\frac{1}{T^{1/\alpha}r_T^{1/\alpha}}\sum_{i=1}^T z(\phi_T)_i^2\frac{\sigma_i^2}{h_i(\theta)}(\phi_T)$, epi-converges almost surely to $\mathbb{E}^{1/\alpha}[(\frac{\sigma_i^2}{h_i(\theta)}(\phi))^\alpha S_\alpha(1,c,0)(\phi)$. Reverting to the original probability space we obtain that $\frac{1}{T^{1/\alpha}r_T^{1/\alpha}}\sum_{i=1}^T z_i^2\frac{\sigma_i^2}{h_i(\theta)}$ epi-converges in distribution to the limit that appears in the rhs of (4).

Finally, it suffices to show that the term $\frac{1}{T^{1/\alpha}r_T^{1/\alpha}}\sum_{i=1}^T\log\frac{\sigma_i^2}{h_i(\theta)}$ weakly converges to zero uniformly over θ. But notice that due to A.4, this is less than or equal to $C\frac{1}{T^{1/\alpha}r_T^{1/\alpha}}\sum_{i=1}^T\log\sigma_i^2$, where C is independent of θ, and the latter converges to zero, from Birkoff's LLN which is applicable due to the existence of the log-moment in A.2., and since $\alpha<1$.

Properness follows from A.2-A.4 and Proposition 5.2.12 of Straumann, 2004 which imply that $\mathbb{E}[(\frac{\sigma_0^2}{h_0(\theta_0)})^\alpha]=1$. The lsc property follows by the final part of A.3 which implies that the scaled likelihood function is $\mathbb{P}$ a.s. lsc and from the metrizability of epi-convergence on the set of lsc proper functions on Euclidean domains as separable spaces (see Knight, 1999), Theorem 1.10.4 of Vaart and Wellner, 2012 and Theorem 7.15.(a) of Rockafellar and Wets, 2009. □

Proof of Lemma 2. From the chain rule and Theorem 2.6.1 of Straumann on the form of the stationary solution to (3), m_0 is identified as the $\mathbb{P}$ a.s. derivative (w.r.t. ϕ) at θ_0 of $H\circ h_0$. Since m_0 is $\mathbb{P}$ a.s. strictly positive and H^{-1} is strictly increasing, $h_0(\theta(\phi))$ is $\mathbb{P}$ a.s. strictly increasing w.r.t. the pointwise product order on K_ϕ. This implies that there exists some ϕ-possibly in an extension of K_ϕ-such that $\frac{\sigma_0^2}{h_0(\theta(\phi))}<1$ $\mathbb{P}$ a.s., which is sufficient for A.5. □

Proof of Theorem 1. First notice that the QMLE by construction minimizes $\frac{1}{T^{1/\alpha}r_T^{1/\alpha}}\sum_{i=1}^T\hat{\ell}_i(\theta)$. Then due to Lemma 1, the separability of Θ, and Theorem 1.10.4 of Vaart and Wellner, 2012, we obtain by 7.31.b of Rockafellar and Wets, 2009 that the accumulation points of the QMLE lie inside the set of minimizers of the limiting likelihood process. This set does not contain θ_0 due to A.5, and the fact that $Z\sim S_\alpha(1,c,0)$ and thereby Z has positive support. □

Proof of Lemma 3. For all cases A.2-A.4 can be established from the proofs and the intermediate results that lead to Theorem 5.4.6 of Straumann, 2004, Theorem 3.1.(i) of Francq and Zakoıan, 2013, and Proposition 3.2 of Arvanitis and Louka, 2015 (the existence of log-moments follows directly from A.1). Choosing $\phi = \omega$ it is easy to establish that the conditions of Lemma 2 hold for all cases since then (5) assumes a very simple form, that of a non-homogeneous linear difference equation of order equal to $\max j : \beta_{0,j} > 0$, with unitary constant term and positive autoregressive coefficients. The uniqueness, stationarity-ergodicity of the solution follows from the restrictions on the modulus on the matrix of autoregressive parameters that hold due to the referred conditions, while strict positivity follows from the positivity restrictions on the GARCH autoregressive parameters and the strict positivity of the constant term. H^{-1} is strictly increasing since it is the identity in all cases. □

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Appendix

The Appendix establishes a (uniform) martingale limit theorem for stationary and ergodic martingale transforms without use of mixing conditions. In what follows $\rightsquigarrow$ denotes convergence in distribution.

Theorem 2. *Suppose that $(\xi_i)_{i\in\mathbb{Z}}$ is an iid sequence and such that ξ_0 belongs to the domain of attraction (DoA) of $S_\alpha(s,c,\gamma)$, where $\alpha\in(0,1)$, $c>0$, $s\in\mathbb{R}$. Furthermore, for Θ a compact subset of some Euclidean space, $(V_i(\theta))_{i\in\mathbb{Z}}$ is a stationary and ergodic, sequence adapted to the filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}}$ with $\mathcal{F}_i:=\sigma(z_{i-j},\ j>0)$ for all $\theta\in\Theta$, $V_0(\theta))$ is continuous, and $\mathbb{E}[\sup_{\theta\in\Theta}|V_0(\theta)|^{\alpha+\delta}]<+\infty$ for some $\delta>0$. Then, for some slowly varying real sequence $(r_i)_{i\in\mathbb{N}}$, and any $\theta\in\Theta$, and deterministic $\theta_T\to\theta$, as $T\to\infty$*

$$\frac{1}{T^{1/\alpha}r_T^{1/\alpha}}\sum_{i=1}^{T}(\xi_i-\gamma)\,V_i(\theta_T)\rightsquigarrow S_\alpha\left(s\frac{\mathbb{E}[\mathrm{sgn}(V_0(\theta))|V_0(\theta)|^\alpha]}{\mathbb{E}[(|V_0(\theta)|^\alpha]},c\mathbb{E}[|V_0(\theta)|^\alpha],0\right). \quad (6)$$

Proof. By Theorem 1.1 along with Paragraph 3 of Jakubowski, 1986) the result would follow if for all $t\in\mathbb{R}$

$$\prod_{i=1}^{T}\mathbb{E}\left(\exp\left(it\frac{1}{T^{\frac{1}{\alpha}}r_T^{\frac{1}{\alpha}}}\rho_{i,\alpha}\right)/\mathcal{F}_i\right) \quad (7)$$

converges pointwise $\mathbb{P}$ a.s. to the characteristic function of the limit, for some appropriate real slowly varying sequence $(r_i)_{i\in\mathbb{N}}$, where $\rho_{i,\alpha,\theta_T}:=(\xi_i-\gamma)V_i(\theta_T)$. Due to Theorem 2.6.5 of Ibragimov and Linnik Ibragimov and Linnik, 1971, the DoA assumption is equivalent to that the characteristic function of ξ_0 has for some $t_0>0$, the representation $\mathbf{i}\gamma t-c|t|^a\mathbf{h}\left(|t|^{-1}\right)\left(1-\mathbf{i}\beta\mathrm{sgn}(t)\tan(\frac{1}{2}\pi\alpha)\right)$ on $(-t_0,t_0)$, where $\mathbf{h}$

is a slowly varying function at infinity. Define r_T by

$$\frac{\mathbf{h}(T^{\frac{1}{\alpha}} r_T^{\frac{1}{\alpha}})}{r_T} \to 1, \tag{8}$$

(see Paragraph 1.9 of Bingham, Goldie, and Teugels, 1989) and notice that this defines a slowly varying sequence. Fix $t \neq 0$ and define the event

$$C_{T,K,\theta_T} := \left\{ \omega \in \Omega : |V_i(\theta_T)| \leq K_t \left(T r_T\right)^{\frac{1}{\alpha}} \quad , \forall i = 1, \ldots, n \right\}$$

where $K_t < \frac{t_0}{|t|}$. From to the moment existence condition above and the inequality of Markov we obtain

$$\begin{aligned} \mathbb{P}[C^c_{T,K,\theta_T}] &\leq \sum_{i=1}^{T} \mathbb{P}[\sup_{\theta \in \Theta} |V_i(\theta)| > K_t \left(T r_T\right)^{\frac{1}{\alpha}}] \\ &\leq \frac{\mathbb{E}[\sup_{\theta \in \Theta} |V_i(\theta)|^{\alpha+\delta}]}{K_t^{\alpha} T^{\frac{\delta}{\alpha}} r_T^{1+\frac{\delta}{\alpha}}} = o(1). \end{aligned}$$

Now, if $\omega \in C_{T,K,\theta_T}$ then $\log \mathbb{E}[\exp\left(\mathbf{i} t \frac{1}{T^{\frac{1}{\alpha}} r_T^{\frac{1}{\alpha}}} \sum_{i=1}^{T} \left[\xi_i - \gamma\right] V_i(\theta_T))/\mathcal{F}_i\right)]$ equals

$$\begin{aligned} & -\frac{c|t|^{\alpha}}{T r_T} \sum_{i=1}^{T} |V_i(\theta_T)|^{\alpha} \mathbf{h}\left(T^{1/\alpha} r_T^{1/\alpha} \left|V_i(\theta_T)\right|^{-1}\right) \left(1 - \mathbf{i}\beta \operatorname{sgn}(t V_i(\theta_T)) \tan(\frac{1}{2}\pi\alpha)\right) \\ = \; & -\frac{c|t|^{\alpha}}{T r_T} \sum_{i=1}^{T} |V_i(\theta_T)|^{\alpha} \mathbf{h}\left(T^{1/\alpha} r_T^{1/\alpha} \left|V_i(\theta_T)\right|^{-1}\right) \\ & + \frac{|t|^{\alpha}}{T r_T} \mathbf{i}\beta c \operatorname{sgn}(t) \tan\left(\frac{1}{2}\pi\alpha\right) \sum_{i=1}^{T} |V_i(\theta_T)|^{\alpha} \mathbf{h}\left(T^{1/\alpha} r_T^{1/\alpha} \left|V_i(\theta_T)\right|^{-1}\right) \operatorname{sgn}(V_i(\theta_T)) \end{aligned}$$

Examining the first term, define $g(x) := x^{-\alpha}\mathbf{h}(x)$. Then for any $\varepsilon > 0$

$$\begin{aligned} & \frac{1}{T r_T} \sum_{i=1}^{T} |V_i(\theta_T)|^{\alpha} \mathbf{h}\left(T^{1/\alpha} r_T^{1/\alpha} \left|V_i(\theta_T)\right|^{-1}\right) 1\left\{|V_i(\theta_T)| \leq \varepsilon\right\} \\ = \; & \sum_{i=1}^{n} g\left(T^{1/\alpha} r_T^{1/\alpha} |V_i(\theta_T)|^{-1}\right) 1\left\{|V_i(\theta_T)| \leq \varepsilon\right\} \\ = \; & \frac{\mathbf{h}\left(T^{1/\alpha} r_T^{1/\alpha}\right)}{r_T} \frac{1}{T} \sum_{i=1}^{T} \frac{g\left(T^{1/\alpha} r_T^{1/\alpha} |V_i(\theta_T|^{-1}\right)}{g\left(T^{1/\alpha} r_T^{1/\alpha}\right)} 1\left\{|V_i(\theta_T)| \leq \varepsilon\right\}. \end{aligned}$$

Since $\frac{\mathbf{h}\left(n^{1/\alpha} r_n^{1/\alpha}\right)}{r_n} \to 1$ by construction, the last term of the previous display is asymptotically equivalent to

$$\frac{1}{T} \sum_{i=1}^{T} |V_i(\theta_T)|^{\alpha} 1\left\{|V_i(\theta_T)| \leq \varepsilon\right\} + \frac{1}{T} \sum_{i=1}^{T} \left[\frac{g\left(T^{1/\alpha} r_T^{1/\alpha} |V_i(\theta_T)|^{-1}\right)}{g\left(T^{1/\alpha} r_T^{1/\alpha}\right)} - |V_i(\theta_T)|^{\alpha} \right] 1\left\{|V_i(\theta_T)| \leq \varepsilon\right\}$$

But,

$$\frac{1}{T}\sum_{i=1}^{T}\left[\frac{g\left(T^{1/\alpha}r_T^{1/\alpha}|V_i(\theta_T)|^{-1}\right)}{g\left(T^{1/\alpha}r_T^{1/\alpha}\right)}-|V_i(\theta_T)|^{\alpha}\right]1\left\{|V_i(\theta_T)|\leq\varepsilon\right\}\leq\sup_{|x|\leq\varepsilon}\left|\frac{g\left(T^{1/\alpha}r_T^{1/\alpha}|x|^{-1}\right)}{g\left(T^{1/\alpha}r_T^{1/\alpha}\right)}-|x|^{\alpha}\right|\to 0.$$

by an application of the Uniform Convergence Theorem for regularly varying functions. Thus,

$$\frac{1}{Tr_T}\sum_{i=1}^{T}|V_i(\theta_T)|^{\alpha}\,\mathbf{h}\left(T^{1/\alpha}r_T^{1/\alpha}\,|V_i(\theta_T)|^{-1}\right)1\left\{|V_i(\theta_T)|\leq\varepsilon\right\}\to\mathbb{E}\left[|V_0(\theta)|^{a}\,1\left\{|V_0(\theta)|\leq\varepsilon\right\}|\right],\mathbb{P}\text{a.s.},$$

due to the compactness of Θ, the moment existence condition $\mathbb{E}[\sup_{\theta\in\Theta}|V_0(\theta)|^{\alpha+\delta}]<+\infty$, Theorem 2.2.1 of Straumann, 2004, dominated convergence, and Theorem 7.14 of Rockafellar and Wets, 2009. Similarly, and applying the Uniform Convergence Theorem for regularly varying functions on $\frac{h\left(T^{1/\alpha}r_T^{1/\alpha}|V_i(\theta_T)|^{-1}\right)}{h\left(T^{1/\alpha}r_T^{1/\alpha}\right)}$ whenever $|V_i(\theta_T)|>\varepsilon$ we obtain that

$$\frac{1}{Tr_T}\sum_{i=1}^{T}|V_i(\theta_T)|^{\alpha}\,\mathbf{h}\left(T^{1/\alpha}r_T^{1/\alpha}\,|V_i(\theta_T)|^{-1}\right)1\left\{|V_i(\theta_T)|>\varepsilon\right\}\to\mathbb{E}\left[|V_0(\theta)|^{a}\,1\left\{|V_0(\theta)|>\varepsilon\right\}|\right],\mathbb{P}\text{a.s.}$$

Letting $\varepsilon\to\infty$ we obtain due to the moment existence condition

$$\frac{1}{Tr_T}\sum_{i=1}^{T}|V_i(\theta_T)|^{\alpha}\,\mathbf{h}\left(T^{1/\alpha}r_T^{1/\alpha}\,|V_i(\theta_T)|^{-1}\right)\to\mathbb{E}\left[|V_0(\theta)|^{a}\right],\ \mathbb{P}\text{ a.s.}$$

Treating the second term analogously, we obtain

$$\begin{aligned}&\frac{|t|^{\alpha}}{Tr_T}\mathbf{i}\beta c\operatorname{sgn}(t)\tan\left(\frac{1}{2}\pi\alpha\right)\sum_{i=1}^{T}|V_i(\theta_T)|^{\alpha}\,\mathbf{h}\left(T^{1/\alpha}r_T^{1/\alpha}\,|V_i(\theta)|^{-1}\right)\operatorname{sgn}(V_i(\theta))\\&\to\mathbf{i}\beta c|t|^{\alpha}\operatorname{sgn}(t)\tan\left(\frac{1}{2}\pi\alpha\right)\mathbb{E}\left[|V_0(\theta)|^{\alpha}\operatorname{sgn}(V_0(\theta))\right],\ \mathbb{P}\text{ a.s.}\end{aligned}$$

establishing the result due to the form of the characteristic function of an α stable distribution (see Ch. 2 of Ibragimov and Linnik, 1971). □

The moment existence condition in the previous theorem also implies the following corollary.

Corollary 1. *Under the premises of Theorem 2, as $T\to\infty$*

$$\frac{1}{T^{1/\alpha}r_T^{1/\alpha}}\sum_{i=1}^{T}\xi_iV_i(\theta_T)\rightsquigarrow S_{\alpha}\left(s\frac{\mathbb{E}[\operatorname{sgn}(V_0(\theta))|V_0(\theta)|^{\alpha}]}{\mathbb{E}[|V_0(\theta)|^{\alpha}]},c\mathbb{E}[|V_0(\theta)|^{\alpha}],0\right).(9)$$

Proof. Due to the compactness of Θ, the moment existence condition $\mathbb{E}[\sup_{\theta\in\Theta}|V_0(\theta)|^{\alpha+\delta}]<+\infty$, Theorem 2.2.1 of Straumann, 2004, dominated convergence, and Theorem 7.14 of Rockafellar and Wets, 2009, we have that

$\frac{1}{T}\sum_{i=1}^{T}\left|V_i(\theta_T)\right|^{\alpha+\delta} \to \mathbb{E}[|V_0(\theta)|^{\alpha+\delta}]$, $\mathbb{P}$ a.s. Since δ can be chosen small enough so that $\alpha+\delta<1$, and due to the reverse Minkowski inequality, we obtain the bound $(\frac{1}{T^{1/\alpha}r_T^{1/\alpha}}\sum_{i=1}^{T}|V_i(\theta_T)|)^{\alpha+\delta} \leq \frac{1}{T^{\delta/\alpha}r_T^{(\alpha+\delta)/\alpha}}\frac{1}{T}\sum_{i=1}^{T}\left|V_i(\theta_T)\right|^{\alpha+\delta}$. This, along with the almost sure convergence established above and the fact that $\frac{\delta}{\alpha}>0$ imply that

$$\frac{1}{T^{1/\alpha}r_T^{1/\alpha}}\sum_{i=1}^{T} V_i(\theta_T) \to 0, \ \mathbb{P} \text{ a.s.},$$

The result follows then from Theorem 2. $\square$

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Department of Economics
Athens University of Economics and Business

The Department is the oldest Department of Economics in Greece with a pioneering role in organising postgraduate studies in Economics since 1978. Its priority has always been to bring together highly qualified academics and top quality students. Faculty members specialize in a wide range of topics in economics, with teaching and research experience in world-class universities and publications in top academic journals.

The Department constantly strives to maintain its high level of research and teaching standards. It covers a wide range of economic studies in micro-and macroeconomic analysis, banking and finance, public and monetary economics, international and rural economics, labour economics, industrial organization and strategy, economics of the environment and natural resources, economic history and relevant quantitative tools of mathematics, statistics and econometrics.

Its undergraduate program attracts high quality students who, after successful completion of their studies, have excellent prospects for employment in the private and public sector, including areas such as business, banking, finance and advisory services. Also, graduates of the program have solid foundations in economics and related tools and are regularly admitted to top graduate programs internationally. Three specializations are offered:1. Economic Theory and Policy, 2. Business Economics and Finance and 3. International and European Economics. The postgraduate programs of the Department (M.Sc and Ph.D) are highly regarded and attract a large number of quality candidates every year.

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