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Limit Theory for Martingale Transforms with Heavy-Tailed Multiplicative Noise

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Abstract

We establish a limit theorem for partial sums of martingale transforms with multiplicative noise and ergodic transform processes, resulting to regularly varying rates and stable limits. We first derive an extension of Breiman's Theorem to empirical distributions, and then obtain the limit theorem by combining the former with the Principle of Conditioning.

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1 Introduction

Similarly to the Gaussian limit theory, stationarity and ergodicity are not by themselves sufficient to extend to dependent processes, i.i.d. limit theorems to stable laws. Such results for weakly dependent processes exist in the literature; for two quite general formulations see Davis and Hsing [13] for a point processes based approach, and Bartkiewicz et al. [7] for an approach based on characteristic functions. When the process is a martingale transform, those results depend on conditions that may be either tedious to verify and/or more restrictive than necessary. In this paper we discuss weaker conditions for a limit theorem to stable laws for martingale transforms with multiplicative i.i.d. noise and ergodic transform processes. We utilize the so-called Principle of Conditioning (see Jakubowski [24]) which allows for deriving limit theorems for sums of dependent random variables from existing limit theory for independent processes.

Our motivation stems from the GARCH models literature (see for example the references in Ch. 3-7 of Straumann [35]) and the determination of the asymptotic properties of the computationally convenient Gaussian Quasi Maximum Likelihood Estimator (QMLE), in the context of such models. Under mild conditions, the score process there assumes the form of a multiplicative martingale transform where the squared innovations act as noise. When

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the i.i.d. innovations possess fourth moments, the estimator is known to be $\sqrt{n}$-consistent and asymptotically normal (see Ch. 5 of Straumann [35]). However, the empirically relevant possibility of the non-existence of the fourth moment (see e.g. Rachev and Mittnik [32] and Mittnik et al. [29]) raised the issue of its implications on the limit theory of the QMLE.

In this paper we extend previous results connected to this literature (see Hall and Yao [20], and Mikosch and Straumann [28]) to a limit theorem for martingale transforms with regularly varying rates and stable limits under weaker conditions regarding temporal dependence and existence of moments. Our derivations exploit the multiplicative structure of the transform and the local representations of the characteristic function of the noise (see Ibragimov and Linnik [23] and Aaronson and Denker [1]). In several cases they are facilitated by our asymptotic extension of the Denisov and Zwart [14] version of Breiman's Theorem (see Breiman [10]) to empirical distributions. Our approach does not require the verification of extremal index conditions (see Davis and Hsing [13]) in order to establish asymptotic non-degeneracy. It also avoids use of restrictions on the mixing rates and/or the existence of higher order moments for the transform process, at the cost of restrictions on the regular variation properties of the martingale difference process.

Our results fully characterize the limiting distributions beyond the index of stability in terms of the relevant parameters of the noise and moments of the transform process. They are readily extendable to multivariate transform processes via Cramér's Theorem. The case where the index of stability equals 2 yields Gaussian limits. Thus our results incorporate asymptotic normality, with rates of convergence potentially slower than $\sqrt{n}$. This occurs when the truncated second moment of the noise slowly diverges to infinity. In this case we only require the existence of the second moments for the transform process, something weaker than the condition implied in Hall and Yao [20], while we also avoid the conditional Lindberg condition (see Jeganathan [26]) which would fail in this framework. Finally our results constitute an essentially unified limit theory since they also incorporate the classical case involving stationary and ergodic square integrable transforms.

The remaining paper is organized as follows. In the following section, we establish notation as well as derive and discuss our main result. The final section contains the proofs.

2 Notation, Assumption Framework and Results

We work in the context of a complete probability space $(\Omega, \mathcal{G}, \mathbb{P})$. The abbreviation $\mathbb{P}$ a.s. signifies almost sure events with respect to $\mathbb{P}$. We denote convergence in distribution of sequences of random elements with $\rightsquigarrow$ and equality in distribution with $\overset{d}{=}$. All limits are considered as $n \to +\infty$ unless otherwise specified. The stochastic processes under consideration are defined on $\mathbb{Z}$ or $\mathbb{N}$. The results are presented for the first case, yet they also hold for the second with appropriate modifications that involve initial values, etc. $\mathcal{F}$ will denote some filtration on $\mathcal{G}$, i.e. an increasing double sequence $(\mathcal{G}_i)_{i\in\mathbb{Z}}$ of $\mathcal{G}$ sub-σ-algebras. Given a non empty set A, $\ell^{\infty}(A)$ denotes the set of bounded real functions on A equipped with the uniform metric. $\|\cdot\|$ denotes the Euclidean norm.

We are interested in the asymptotic behavior of the partial sums of the process $((\xi_i - \gamma) V_i)_{i\in\mathbb{Z}}$ where $(\xi_i)_{i\in\mathbb{Z}}$ is an i.i.d. process, γ is a location parameter, and $(V_i)_{i\in\mathbb{Z}}$ is a stationary ergodic process. We employ measurability properties of the constituent processes w.r.t. $\mathcal{F}$, that enable characterization of the pointwise product $((\xi_i - \gamma) V_i)_{i\in\mathbb{Z}}$ as a martingale transform of $(\xi_i)_{i\in\mathbb{Z}}$ by $(V_i)_{i\in\mathbb{Z}}$. The term is inaccurate in the cases where $\mathbb{E}[|\xi_0 V_0|] = +\infty$, yet we universally adopt it in the spirit of Mikosch and Straumann [28].

In order to derive the limit theory of the partial sums of $((\xi_i - \gamma) V_i)_{i\in\mathbb{Z}}$, we first present our assumption framework on properties of the constituent processes.

Let $S_\alpha(\beta, c, \gamma)$ be the (univariate) stable distribution with parameters α, β, c, γ denoting stability, skewness, scale and location respectively (see Ibragimov and Linnik [23]). When $\alpha = 2$, then $s = 0$ and $S_2(0, c, \gamma) = N(\gamma, c)$.

Assumption 1. *For some $\alpha \in (0, 2]$, $\beta \in [-1, 1]$, $c > 0$ and $\gamma \in \mathbb{R}$, $(\xi_i)_{i\in\mathbb{Z}}$ is i.i.d. and the distribution of ξ_0 lies in the domain of attraction (DoA) of $S_\alpha(\beta, c, \gamma)$.*

Assumption 2. *For some filtration $\mathcal{F} \equiv (\mathcal{G}_i)_{i\in\mathbb{Z}}$, $(\xi_i V_i)_{i\in\mathbb{Z}}$ is $\mathcal{F}$-adapted, ξ_i is independent of $\mathcal{G}_{i-1}$ and V_i is $\mathcal{G}_{i-1}$-measurable for all $i \in \mathbb{Z}$.*

Assumption 3. *$(V_i)_{i\in\mathbb{Z}}$ is stationary and ergodic with $\mathbb{E}[|V_0|^\alpha] < \infty$ and $\mathbb{P}(|V_0| > x) = o(\mathbb{P}(|\xi_0| > x))$ as $x \to \infty$.*

Remark 1. Our motivation behind Assumption 1 is the issue of the fourth moment existence for the innovations of GARCH-type models in the framework of empirical finance. In this respect $(\xi_i)_{i\in\mathbb{Z}}$ represents the squared innovations-potentially translated by $-\gamma = -1$, as those appear in parts of the Gaussian quasi log-likelihood function and its derivatives. It encompasses the usual case where $\alpha = 2$ and $\mathbb{E}[\xi_0^2] < +\infty$. It also allows for cases where $\mathbb{E}[\xi_0^2] = +\infty$ and $\mathbb{E}\left[\xi_0^2 1_{|\xi_0|\leq x}\right]$ is either regularly varying at infinity with index $1 - \frac{\alpha}{2}$ or it is slowly varying, whereas the usual ergodic square integrable martingale difference CLT is inapplicable to $\sum_{i=1}^n (\xi_i - \gamma) V_i$.

Remark 2. For any $\alpha \in (0, 2]$ the distribution of ξ_0 belongs to the DoA of $S_\alpha(\beta, c, \gamma)$ if and only if its log-characteristic function has the following representation as $t \to 0$:

$$\begin{cases} \gamma \mathbf{i}t - c|t|^\alpha L(|t|^{-1}) \left(1 - \mathbf{i}\beta \operatorname{sgn}(t) \tan\left(\frac{1}{2}\pi\alpha\right)\right) + o\left(|t| L\left(|t|^{-1}\right)\right) & , \alpha \neq 1 \\ \left(\gamma + H(|t|^{-1})\right) \mathbf{i}t - c|t| L(|t|^{-1}) \left(1 - 2C\mathbf{i}\frac{\beta}{\pi} \operatorname{sgn}(t)\right) + o\left(|t| L\left(|t|^{-1}\right)\right) & , \alpha = 1 \end{cases}, \tag{1}$$

where $\mathbf{i} := \sqrt{-1}$, $L(x)$ is a slowly varying function in the sense of Karamata (see for example Bingham et al. [8]), $H(\lambda) = \int_0^\lambda \frac{x}{1+x^2} L(x)(2\beta c\pi^{-1} + k(x))dx$ and $k(x) \to 0$ as $x \to \infty$ and $-C$ is the Euler-Mascheroni constant. This is due to Theorem 2.6.5 in Ibragimov and Linnik [23], with the case $\alpha = 1$ clarified in Aaronson and Denker [1] (see Theorems 1 and 2 there).

Remark 3. In some applications $\mathcal{F}$ represents the history of the $(\xi_i)_{i\in\mathbb{Z}}$ process, i.e. it is defined by $\mathcal{G}_i = \sigma(\xi_{i-j}, j \geq 0)$. In such cases the independence of ξ_i from $\mathcal{G}_{i-1}$ in Assumption 2 follows readily from Assumption 1. The $\mathcal{G}_{i-1}$-measurability of V_i could among others follow when this is defined as some measurable transformation of causal solutions to stochastic recurrence equations (SRE) involving solely elements of $\mathcal{G}_{i-1}$. In any case Assumption 2 along with the Principle of Conditioning (see Jakubowski [24]) implies that the limiting distribution of $\sigma_n^{-1}\left(\sum_{i=1}^n (\xi_i - \gamma) V_i - \tau_n\right)$ for the appropriate choice of (τ_n, σ_n), if any, will be determined by the limiting behavior of $\exp\left(\frac{\tau_n}{\sigma_n}\right) \prod_{i=1}^n \mathbb{E}\left[\exp\left(\mathbf{i}t\sigma_n^{-1}(\xi_i - \gamma) V_i\right)/\mathcal{G}_i\right]$, $t \neq 0$.

Remark 4. Stationarity and ergodicity for the $(V_i)_{i\in\mathbb{Z}}$ process, whenever this is defined as some measurable transformation of solutions of stationary and ergodic SREs, can be established via conditions on the Liapunov exponents of the relevant dynamical systems; see for example Bougerol and Picard [9]. The existence of the α moment for the stationary distribution can in a similar set up be derivable via results like the ones in Goldie [17]. The final part of the assumption employs a comparison between the tails of the stationary distributions of ξ_0 and V_0. In view of Assumption 1, it certainly holds whenever $\mathbb{E}\left[|V_0|^{\alpha+\delta}\right] < +\infty$ for some $\delta > 0$, but this is not necessary. Regular variation of the tails of V_0 with index α would suffice as long as the slowly varying part is asymptotically dominated by L. This obviously allows for cases where $\mathbb{E}\left[|V_0|^{\alpha+\delta}\right] = +\infty$ for any $\delta > 0$ even when $\mathbb{E}\left[|\xi_0|^{\alpha}\right] < +\infty$.

Define r_n by the asymptotic relation $L(n^{\frac{1}{\alpha}}r_n^{\frac{1}{\alpha}})/r_n \to 1$, and notice that $(r_n)_{n\in\mathbb{N}}$ is a well defined slowly varying sequence-see Paragraph 1.9 of [8] and Proposition 1.(iv) of [6]. Assumption 3 controls the behavior of the maximum order statistic for the $(|V_i|)_{i\in\mathbb{Z}}$ process; the result below shows that the latter cannot diverge at a rate faster than $n^{\frac{1}{\alpha}}r_n^{\frac{1}{\alpha}}$ with probability converging to one.

Lemma 1. *Suppose that Assumptions 1-3 hold. Then for any $M > 0$,*

$$\mathbb{P}\left(n^{-\frac{1}{\alpha}}r_n^{-\frac{1}{\alpha}}\max_{1\leq i\leq n}|V_i| > M\right) \to 0. \tag{2}$$

Remark 5. Assumption 1 implies that for any $M > 0$,

$$\mathbb{P}\left(n^{-\frac{1}{\alpha}}r_n^{-\frac{1}{\alpha}}\max_{1\leq i\leq n}|\xi_i| > M\right) \to \Phi_\alpha(M),$$

where Φ_α denotes the Fréchet distribution; see Embrechts et al. [16]. This and Lemma 1 partially imply that the index of the tail variation properties for the limiting distribution of the appropriately scaled partial sum of the transform is essentially determined by the distribution of ξ_0.

In view of the final observation in Remark 3, Lemma 1 implies that for $\sigma_n = n^{\frac{1}{\alpha}}r_n^{\frac{1}{\alpha}}$, $\mathbb{E}\left[\exp\left(\mathbf{i}t\sigma_n^{-1}(\xi_i-\gamma)V_i\right)/\mathcal{G}_i\right]$ can be approximated with high probability by the local representation

$$\begin{aligned}-\tfrac{c|t|^\alpha}{nr_n}|V_i|^\alpha L\left(n^{1/\alpha}r_n^{1/\alpha}|tV_i|^{-1}\right) + 1\{\alpha=1\}H(n^{1/\alpha}r_n^{1/\alpha}|tV_i|^{-1})\mathbf{i}t\tfrac{V_i}{n^{1/\alpha}r_n^{1/\alpha}} \\ \tfrac{|t|^\alpha}{nr_n}\mathbf{i}\beta c\,\mathrm{sgn}(t)\tan\left(\tfrac{1}{2}\pi\alpha\right)\mathrm{sgn}(V_i)|V_i|^\alpha L\left(n^{1/\alpha}r_n^{1/\alpha}|tV_i|^{-1}\right).\end{aligned}$$

This is due to Assumption 1 and the fact that $tn^{-\frac{1}{\alpha}}r_n^{-\frac{1}{\alpha}}\max_{1\leq i\leq n}|V_i|$ lies in the neighborhood of validity of the representation in (1) with probability converging to one, for any t. A simple calculation (see the proof of Theorem 1) then shows that the limit theory would be greatly facilitated by the determination of the asymptotic behavior of terms similar to $\frac{1}{nr_n}\sum_{i=1}^n|V_i|^\alpha L\left(n^{\frac{1}{\alpha}}r_n^{\frac{1}{\alpha}}/(|t||V_i|)\right)$. Given Assumption 3, and Birkhoff's LLN, if there exists some Tauberian-type result that would allow for the asymptotic factoring out from the sum of the slowly varying component, the required limits would be easily derivable from ergodicity. Towards the establishment of such a result, the Denisov and Zwart [14] representation of a slowly varying function at infinity in terms of pairs of long-tailed random variables will be useful. Remember that a random variable U is said to be long tailed iff $\mathbb{P}(U > x) \sim \mathbb{P}(U > x + y)$ as x tends to ∞ for any y, which then implies that $\mathbb{P}(U > \log x)$ is slowly varying:

Lemma 2. *(Lemma 2.1 of Denisov and Zwart [14]) If L is slowly varying then it admits one of the following four representations:*

1. $L(x) = c(x)$,
2. $L(x) = c(x)/\mathbb{P}(U > \log x)$,
3. $L(x) = c(x)\mathbb{P}(U > \log x)$,
4. $L(x) = c(x)\mathbb{P}(U > \log x)/\mathbb{P}(U^\star > \log x)$,

where $c(x)$ is a function converging to a strictly positive constant, while U and $U^\star$ are independent long-tailed random variables with hazard rates converging to 0.

When $\alpha < 2$, due to Assumptions 1-3 $\frac{1}{nr_n}\sum_{i=1}^{n}|V_i|^\alpha L\left(n^{\frac{1}{\alpha}}r_n^{\frac{1}{\alpha}}/(|t||V_i|)\right)$ is $\mathbb{P}$ a.s. asymptotically equivalent to $\mathbb{P}_n^\star\left[v_n|\xi_0|>\frac{1}{|t|}n^{\frac{1}{\alpha}}r_n^{\frac{1}{\alpha}}\right]/\mathbb{P}\left[|\xi_0|>\frac{1}{|t|}n^{\frac{1}{\alpha}}r_n^{\frac{1}{\alpha}}\right]$, where $\mathbb{P}_n^\star$ denotes the stochastic product measure between the empirical distribution of $(|V_i|)_{1\leq i\leq n}$ and $\mathbb{P}$, and v_n is a random variable that follows this empirical distribution and is independent of ξ_0. This ratio is partially an empirical analogue of the random variables product appearing in Breiman's Theorem (see Breiman [10]). This in our setting states that if, in addition to Assumption 3, $\mathbb{E}\left[|V_0|^{\alpha+\delta}\right] < +\infty$ for some $\delta > 0$, then as $n \to \infty$, $\mathbb{P}\left[|V_0\xi_0|>\frac{1}{|t|}n^{\frac{1}{\alpha}}r_n^{\frac{1}{\alpha}}\right]/\mathbb{P}\left[|\xi_0|>\frac{1}{|t|}n^{\frac{1}{\alpha}}r_n^{\frac{1}{\alpha}}\right] \to \mathbb{E}\left[|V_0|^\alpha\right]$. Denisov and Zwart [14] extend this result by essentially assuming the second part of Assumption 3, while imposing further conditions on the properties of L and/or the comparison between the tails of the involved distributions. Hence the marginal distribution of the transform has regularly varying tails with index equal to α, while the properties of V_0 only affect the scale, symmetry and location parameters.

Our first result extends the Theorem of Breiman (see Breiman [10]) and the corresponding extension of Denisov and Zwart [14] to be valid for $\mathbb{P}_n^\star$. This result is to our knowledge new and enables handling of the limiting behavior of the $\mathbb{P}_n^\star\left[v_n|\xi_0|>\frac{1}{|t|}n^{\frac{1}{\alpha}}r_n^{\frac{1}{\alpha}}\right]/\mathbb{P}\left[|\xi_0|>\frac{1}{|t|}n^{\frac{1}{\alpha}}r_n^{\frac{1}{\alpha}}\right]$ ratio. It employs conditions partially stronger to the ones of Denisov and Zwart [14], and it also in some cases requires restrictions on the weak dependence of the $(V_i)_{i\in\mathbb{Z}}$ process given the fact that it involves empirical distributions.

Proposition 1. *Suppose that Assumptions 1-3 hold for $\alpha < 2$. Suppose furthermore that either one of the following conditions hold:*

1. $\limsup_{x\to\infty} \sup_{1\leq y\leq x} L(y)/L(x) < \infty$, *or*
2. *for some* $\delta > 0$, $\mathbb{E}\left[|V_0|^{\alpha+\delta}\right] < \infty$.

Then, for any real sequence $m_n \to \infty$, as $n \to \infty$,

$$\frac{\mathbb{P}_n^\star\left[v_n\left|\xi_0\right|>m_n\right]}{\mathbb{P}\left[\left|\xi_0\right|>m_n\right]} \to \mathbb{E}\left[\left|V_0\right|^\alpha\right], \mathbb{P} \text{ a.s.} \tag{3}$$

where $\mathbb{P}_n^\star$ denotes the product measure between the empirical distribution of $(|V_i|)_{1\leq i\leq n}$ and $\mathbb{P}$, and v_n is a random variable that follows this empirical distribution and is independent of ξ_0.

Now suppose that neither of Conditions 1-2 hold, and, in addition to Assumptions 1-3, assume that $(V_i)_{i\in\mathbb{Z}}$ is strongly mixing and either one of the following conditions hold:

3. *L assumes the representation (3) or (4) of Lemma 2, $\lim_{x\to\infty}\int_0^x \frac{\mathbb{P}[U>x-y]}{\mathbb{P}[U>x]}\mathbb{P}\left[U>y\right]dy = 2\int_0^\infty \mathbb{P}\left[U>y\right]dy < +\infty$ and for some $0<\epsilon<1$, $\frac{x^\alpha \mathbb{P}^\epsilon[|V_0|>x]}{\mathbb{P}[U>\log x]} \to 0$ as $x\to\infty$, or*

4. *$\limsup_{x\to\infty}\sup_{\sqrt{x}\leq y\leq x}\frac{L(y)}{L(x)} < \infty$ and for some $0<\epsilon<1$, $\frac{\mathbb{P}^\epsilon[|V_0|>x]}{\mathbb{P}[|\xi_0|>x]}\int_0^x t^\alpha d\mathbb{P}\left[|\xi_0|\leq t\right] \to 0$ as $x\to\infty$.*

Then for any real sequence $m_n \to \infty$, as $n\to\infty$,

$$\frac{\mathbb{P}_n^\star\left[v_n\left|\xi_0\right|>m_n\right]}{\mathbb{P}\left[\left|\xi_0\right|>m_n\right]} \to \mathbb{E}\left[\left|V_0\right|^\alpha\right], \text{ in probability.} \tag{4}$$

Remark 6. Condition 1 implies the first two cases of Lemma 2. Hence it among others covers normal DoAs or eventual monotonically diverging slow variation for ξ_0 (for sufficient conditions on the existence of monotone versions of L see Buldygin et al. [11]). It corresponds to the empirical distribution extension of Proposition 2.1 of Denisov and Zwart [14]. Condition 2 requires the existence of $\alpha+\delta$ moment for $|V_0|$. It corresponds to the empirical distribution extension of the original result of Breiman [10]. Both imply the strong array LLN; $\frac{1}{nr_n}\sum_{i=1}^n |V_i|^\alpha L\left(n^{\frac{1}{\alpha}}r_n^{\frac{1}{\alpha}}/(|t||V_i|)\right) \to \mathbb{E}\left[|V_0|^\alpha\right]\mathbb{P}$ a.s. without any further dependence restrictions for $(V_i)_{i\in\mathbb{Z}}$.

Remark 7. In Condition 3 the integrability requirement for $\mathbb{P}\left[U>x\right]$ is equivalent to that the tail distribution function is a sub-exponential density, or equivalently that the random variable U belongs to the class S^* (see Klüppelberg [27]). In the case 3 of Lemma 2 this is equivalent to $\int_0^x \frac{L\left(e^{x-y}\right)}{L(e^x)}L\left(e^y\right)dy = 2\int_0^\infty L\left(e^y\right)dy < +\infty$ (see Denisov and Zwart [14]). Whenever L assumes the representation 3 or 4 of Lemma 2, Condition 4 essentially handles cases where $\int_0^\infty \mathbb{P}\left[U>y\right]dy$ diverges. A sufficient condition for the asymptotic boundedness required in Condition 4 is that $L\circ\exp$ is of bounded variation, while it is easy to see that the result holds if $\sqrt{x}$ is replaced with x^β for any $\beta\in(0,1)$ (see again Denisov and Zwart [14]). The remaining asymptotic negligibility requirements in both Conditions 3-4 are stronger analogues to the ones in Propositions 2.2-3 of Denisov and Zwart [14]-the latter are allowed to hold for $\epsilon=1$. They essentially constitute more refined comparisons between the tails of the stationary distributions of ξ_0 and V_0. The stronger requirement that $\epsilon<1$ along with strong mixing for $(V_i)_{i\in\mathbb{Z}}$, allow for asymptotic handling of a scaled approximation error of the empirical distribution of $(|V_i|)_{1\leq i\leq n}$ to the stationary distribution of $|V_0|$ uniformly over appropriate classes of events. Strong mixing can be substituted by a more general mixingale restriction (see Theorem 1 of Hill [22] and the derivation of (14) in the proof of the proposition) without affecting the validity of the results. A weak array LLN is then obtained; $\frac{1}{nr_n}\sum_{i=1}^n |V_i|^\alpha L\left(n^{\frac{1}{\alpha}} r_n^{\frac{1}{\alpha}}/(|t||V_i|)\right) \to \mathbb{E}\left[|V_0|^\alpha\right]$ in probability.

The asymptotic representation of $\frac{1}{nr_n}\sum_{i=1}^n |V_i|^\alpha L\left(n^{\frac{1}{\alpha}} r_n^{\frac{1}{\alpha}}/(|t||V_i|)\right)$ as the probability ratio in eq. (3) or (4), is not valid when $\alpha = 2$. In this case a simple calculation shows that the sum equals
$\sum_{i=1}^n \mathbb{E}\left[(\xi_i-\gamma)^2 V_i^2 1_{|(\xi_i-\gamma)V_i|>\frac{1}{|t|}}/\mathcal{G}_{i-1}\right]$. Our second result handles the asymptotic properties of this representation.

Proposition 2. *Suppose that Assumptions 1-3 hold for $\alpha = 2$. Then for any $M > 0$,*

$$\sum_{i=1}^n \mathbb{E}\left[\frac{1}{nr_n}(\xi_i-\gamma)^2 V_i^2 1_{|(\xi_i-\gamma)V_i|\leq M\sqrt{nr_n}}/\mathcal{G}_{i-1}\right] \to \mathbb{E}\left[V_0^2\right], \mathbb{P}\ a.s. \tag{5}$$

Remark 8. The proof of this proposition closely parallels the proof of Proposition 1 under Condition 1, due to the monotonicity of L. Notice that when r_n converges, then (5) implies the conditional Lindberg condition typically encountered in several martingale limit theorems (see for example Hall and Heyde [19], Jeganathan [26]).

Hence, Propositions 1-2 provide with the Tauberian type result required above. Modulo further details exemplified in the proof, they facilitate the derivation of our martingale transform limit theorem that we now state:

Theorem 1. *Under the premises of Propositions 1-2: If $\alpha \neq 1$, then*

$$\frac{1}{n^{1/\alpha} r_n^{1/\alpha}}\sum_{i=1}^n (\xi_i-\gamma) V_i \rightsquigarrow S_\alpha\left(\beta\frac{\mathbb{E}\left[|V_0|^\alpha \operatorname{sgn}(V_0)\right]}{\mathbb{E}\left[|V_0|^\alpha\right]}, c\mathbb{E}\left[|V_0|^\alpha\right], 0\right). \tag{6}$$

If $\alpha = 1$ and

$$\mathbb{E}\left[|V_0||\log(|V_0|)|\right] < \infty,\ \textit{when}\ \beta \neq 0$$

then,

$$\frac{1}{nr_n}\sum_{i=1}^n (\xi_i-\gamma-H(nr_n)) V_i - 2\beta c\pi^{-1}\left(C\mathbb{E}(V_0) - \mathbb{E}\left[V_0|\log(|V_0|)|\right]\right)$$
$$\rightsquigarrow S_1\left(\beta\frac{\mathbb{E}\left[V_0\right]}{\mathbb{E}\left[|V_0|\right]}, c\mathbb{E}\left[|V_0|\right], 0\right). \tag{7}$$

If $\alpha < 1$ and either for any $M > 0$,

$$\mathbb{P}\left(\max_{1\leq i\leq n}|V_i| > M q_n^{\frac{1}{\alpha}} n^{\frac{1}{\alpha}}\right) \to 0, \tag{8}$$

where $q_n = O(r_n^{1/(1-\alpha)})$, or for some $\delta > 0$,

$$\mathbb{E}\left[|V_0|^{\alpha+\delta}\right] < \infty, \tag{9}$$

then,

$$\frac{1}{n^{1/\alpha}r_n^{1/\alpha}}\sum_{i=1}^{n}\xi_i V_i \rightsquigarrow S_\alpha\left(\beta\frac{\mathbb{E}\left[|V_0|^\alpha \operatorname{sgn}(V_0)\right]}{\mathbb{E}\left[|V_0|^\alpha\right]}, c\mathbb{E}\left[|V_0|^\alpha\right], 0\right). \tag{10}$$

Remark 9. In unison with Remark 5 the rate $n^{1/\alpha}r_n^{1/\alpha}$ reflects tail variation patterns of the distribution of ξ_0 while it is not affected by properties of the $(V_i)_{i\in\mathbb{Z}}$ process. The limiting distribution is stable with stability parameter strictly determined by Assumption 1. The distribution of V_0 affects only the scale and symmetry parameters of the limit. It also affects the form of the translating constants in the case where $\alpha = 1$. Notice that in this case, we do not impose conditions that restrict the translating sequence to zero as in Theorem 1-Condition 5 of Bartkiewicz et al. [7]. When the attractor is symmetric, V_0 affects only the limiting scale, in which case $(V_i)_{i\in\mathbb{Z}}$ can be characterized as a stochastic scaling process for the transform. In this case and for $\alpha = 1$, the term $2\beta c\pi^{-1}\left(C\mathbb{E}\left(V_0\right) - \mathbb{E}\left[V_0|\log(|V_0|)|\right]\right)$ disappears from the centering sequence, thus (7) coincides with (6) and the limit becomes a Cauchy distribution. When $\alpha < 1$, and due to the zero location, when $c_2 = 0$ (resp. $c_1 = 0$), the limiting distribution is supported on $[0, +\infty)$ (resp. $(-\infty, 0]$). Furthermore, when $\alpha < 1$ and either (8)-that strengthens (2), or the stricter (9) hold, then the translating sequence $\frac{\gamma}{n^{1/\alpha}r_n^{1/\alpha}}\sum_{i=1}^{n}V_i$ becomes asymptotically negligible. Finally, when $V_i = 1$, $\mathbb{P}$ a.s. for all i, then the classical i.i.d. results are recovered for all α.

Remark 10. When $\alpha = 2$ and r_n converges-necessarily to $\mathbb{E}\left[\xi_0^2\right]$-then we obtain a version of the classical CLT for stationary and ergodic martingale difference sequences with limit the $N\left(0, \mathbb{E}\left[V_0^2\right]\right)$ distribution. When r_n diverges we still obtain a CLT, albeit with slower rates $\sqrt{\frac{n}{r_n}}$ and the same limiting distribution. The rate reflects the divergence properties of the truncated second moment of ξ_0. Due to the monotonicity of the mapping $x \to \mathbb{E}\left[\xi_0^2 1_{|\xi_0|\leq x}\right]$, this case is derivable without the need for existence of $\mathbb{E}\left[|V_0|^{\alpha+\delta}\right]$ for some $\delta > 0$, or any mixing conditions for $(V_i)_{i\in\mathbb{Z}}$. Furthermore, then, the conditional Lindberg condition requiring that $\sum_{i=1}^{n}\mathbb{E}\left[\frac{1}{nr_n}\left(\xi_i-\gamma\right)^2 V_i^2 1_{|(\xi_i-\gamma)V_i|>M\sqrt{nr_n}}/\mathcal{G}_{i-1}\right] \to 0$ in probability for any $M > 0$, fails. Jeganathan [26] considers it as in some sense necessary condition for the martingale convergence, hence such cases constitute almost counter-examples. For a simple example suppose that $\xi_0 \sim t_2$. Then the result assumes the form $\frac{1}{\sqrt{n\log n}}\sum_{i=1}^{n}\xi_i V_i \rightsquigarrow N\left(0, \mathbb{E}\left[V_0^2\right]\right)$ by a simple calculation. This essentially generalizes the results of Abadir and Magnus [2] to dependent processes.

Remark 11. Theorem 1 can be readily extended when V_0 is an $\mathbb{R}^d$-valued random vector via the use of the Cramer's Theorem. Suppose that for any $\lambda \in \mathbb{R}^d$ different than zero, when $\alpha \neq 0$, we have that $\frac{1}{n^{1/\alpha}r_n^{1/\alpha}}\sum_{i=1}^{n}\left(\xi_i-\gamma\right)\lambda^T V_i$ converges in distribution to $S_\alpha\left(\beta\frac{\mathbb{E}\left[\left|\lambda^T V_0\right|^\alpha \operatorname{sgn}\left(\lambda^T V_0\right)\right]}{\mathbb{E}\left[\left|\lambda^T V_0\right|^\alpha\right]}, c\mathbb{E}\left[\left|\lambda^T V_0\right|^\alpha\right], 0\right)$, while when $\alpha = 1$ we obtain the same result by re-centering with

$$2\beta c\pi^{-1}\left(C\mathbb{E}\left(\lambda^T V_0\right) - \mathbb{E}\left[\lambda^T V_0|\log(|\lambda^T V_0|)|\right]\right).$$

When $\alpha \geq 1$, Example 3.3.4 of Samorodnitsky and Taqqu [33] implies that the multivariate limits are identified as multivariate α-stable distributions with spectral measures Γ determined by

$$\begin{array}{c}\beta\left(\lambda\right)=\frac{\int_{\mathbb{S}^{d-1}}\left|s^{T}\lambda\right|^{\alpha}\operatorname{sgn}(s^{T}\lambda)\Gamma(ds)}{\int_{\mathbb{S}^{d-1}}\left|s^{T}\lambda\right|^{\alpha}\Gamma(ds)}\\ c^{\alpha}\left(\lambda\right)=\int_{\mathbb{S}^{d-1}}\left|s^{T}\lambda\right|^{\alpha}\Gamma\left(ds\right)\end{array}, \tag{11}$$

and

$$\gamma\left(\lambda\right)=\begin{cases}\gamma^{T}\lambda & ,\alpha\neq 1\\ \gamma^{T}\lambda-\frac{2}{\pi}\int_{\mathbb{S}^{d-1}}\left|s^{T}\lambda\right|\log\left(\left|s^{T}\lambda\right|\right)\Gamma\left(ds\right) & ,\alpha=1\end{cases}, \tag{12}$$

where $\mathbb{S}^{d-1}$ denotes the $d-1$ dimensional sphere. Theorem 2.3 of Gupta et al. [18] and the zero location in (6) implies that the same is true for $\alpha<1$. Notice though that our results cannot accommodate the case where ξ_i is an $\mathbb{R}^d$-valued random vector since this would require a non trivial extension of results like the ones in Proposition 1 involving spectral measures. Such a consideration is delegated to future research.

Remark 12. (The multivariate analogue of) Theorem 1 extends the results of Mikosch and Straumann [28] (and partly the ones in the GARCH type model specific framework of Hall and Yao [20]). This is due to that: i) it avoids simultaneously requiring the existence of $\mathbb{E}\left[\left|V_0\right|^{\alpha+\delta}\right]$ for some $\delta>0$ and mixing conditions for the $(V_i)_{i\in\mathbb{Z}}$ process. A fortiori in the case corresponding to Condition 1 of Proposition 1 neither mixing nor the existence of $\mathbb{E}\left[\left|V_0\right|^{\alpha+\delta}\right]$ is required. ii) It avoids conditions requiring strict positivity for the extremal index of the process $\left(\left(\xi_i-\gamma\right)V_i\right)_{i\in\mathbb{Z}}$. Remember that this index reflects information on the clustering of the process above large thresholds-see for example Davis and Hsing [13], and its evaluation need not be trivial in applications. Its positivity ensures that the limiting distribution is not degenerate at zero. This, in our case follows simply when $\mathbb{E}\left[\left|V_0\right|^{\alpha}\right]$ is strictly positive (in the multivariate case when $\mathbb{E}\left[\left|\lambda^T V_0\right|^{\alpha}\right]$ is strictly positive for some allowable λ). iii) It is more informative on the characterization of the limiting distributions. iv) It generally allows for $\alpha\leq 1$. Theorem 1 extends the results in Surgailis [36] since it allows for $\mathbb{E}\left[\left|V_0\right|\right]^{\alpha+\delta}=+\infty$ for any $\delta>0$, $\alpha\leq 1$ and does not require normal DoAs-consider for example Condition 1 of Proposition 1 for L monotonically diverging. Due to the latter it analogously extends the results of Jakubowski [25].

Remark 13. Theorem 1 can be also useful for processes that admit decompositions involving multiplicative martingale transforms. Suppose for example that $\left(X_i,\mathcal{G}_i\right)_{i\in\mathbb{Z}}$ is a stationary L_1-mixingale of size -1 (see Ch. 16 of Davidson [12]). By the proof of Theorem 5.4 of Hall and Heyde [19], it admits the decomposition $X_i=Y_i+Z_i-Z_{i-1}$, where $\left(Y_i,\mathcal{G}_i\right)_{i\in\mathbb{Z}}$ is a martingale difference sequence and $\left(Z_i\right)_{i\in\mathbb{Z}}$ is stationary with $\mathbb{E}\left[\left|Z_0\right|\right]<+\infty$. Suppose that for all $i\in\mathbb{Z}$, Y_i can be factored as $\left(\xi_i-\gamma\right)V_i$, for which the premises of Propositions 1-2 are valid for $\alpha>1$. Then due to the previous we obtain that $\max_{i\leq n}\mathbb{P}\left[\frac{\left|Z_i\right|}{n^{1/\alpha}r_n^{1/\alpha}}>M\right]\to 0$ for all $M>0$, and thereby due to Theorem 1, $\frac{1}{n^{1/\alpha}r_n^{1/\alpha}}\sum_{i=1}^{n}X_i\rightsquigarrow S_\alpha\left(\beta\frac{\mathbb{E}\left[\left|V_0\right|^{\alpha}\operatorname{sgn}(V_0)\right]}{\mathbb{E}\left[\left|V_0\right|^{\alpha}\right]},c\mathbb{E}\left[\left|V_0\right|^{\alpha}\right],0\right)$.

Remark 14. In the case where V_0 is $\mathbb{P}$ a.s. positive and the transform process is independent of the noise, $\left(\xi_i V_i\right)_{i\in\mathbb{Z}}$ can be characterized as a stochastic volatility process (see Andersen et al. [4]). Then Theorem 1 extends the results of Proposition 2 of Bartkiewicz et al. [7] without assuming that $(\ln V_i)$ is a Gaussian ARMA process, or that, when $\alpha=1$, ξ_0 is symmetric, while it allows for Gaussian limits even if the martingale difference process has infinite variance.

Remark 15. Theorem 1 is inapplicable in cases where $\prod_{i=1}^{n}\mathbb{E}\left[\exp\left(\mathbf{i}t\sigma_n^{-1}\left(\xi_i-\gamma V_i\right)\right)/\mathcal{G}_i\right]$ can only converge weakly to some non-degenerate limit. Such a result would generalize Wang [37] and in turn could be useful in several applications including non linear co-integration (see for example Wang [37]), or the limit theory of the OLSE in frameworks of moderate deviations from a unit root (see for example Phillips and Magdalinos [31], etc. We delegate this to further research.

Remark 16. Suppose that (Θ, d) is a totally bounded metric space, $(V_i(\theta))_{i\in\mathbb{Z}}$ is stationary ergodic and $\mathbb{E}\left[|V_0|^{\alpha+\delta}(\theta)\right]<+\infty$ for all $\theta\in\Theta$ and some $\delta>0$, $|V_i(\theta)-V_i(\theta^\star)|\leq v_i d(\theta,\theta^\star)$, for all $i\in\mathbb{Z}$ and θ, $\theta^\star\in\Theta$, where $(v_i)_{i\in\mathbb{Z}}$ is stationary ergodic, and $\mathbb{E}\left[v_0^{\alpha+\delta}\right]$ exists. Suppose furthermore that $\gamma=0$ if $\alpha\geq 1$, and $\beta=0$ if $\alpha=1$. Then, for any $\varepsilon,\eta>0$ there exists $\delta>0$ small enough, such that

$$\begin{aligned}\mathbb{P}&\left(\sup_{\theta,\theta^\star\in\Theta,d(\theta,\theta^\star)<\delta}\frac{1}{n^{1/\alpha}r_n^{1/\alpha}}\sum_{i=1}^{n}\xi_i\left|V_i(\theta)-V_i(\theta^\star)\right|>\varepsilon\right)\\&\leq\mathbb{P}\left(\sup_{\theta,\theta^\star\in\Theta,d(\theta,\theta^\star)<\delta}\frac{d(\theta,\theta^\star)}{n^{1/\alpha}r_n^{1/\alpha}}\sum_{i=1}^{n}\xi_i v_i>\varepsilon\right)\\&\leq\mathbb{P}\left(O_p(1)>\frac{\varepsilon}{\delta}\right)\leq\eta,\end{aligned}\tag{13}$$

where the $O_p(1)$ term in the previous display is obtained by the application of Theorem 1 (in the case where $\alpha<1$ then 10 holds) to $\frac{1}{n^{1/\alpha}r_n^{1/\alpha}}\sum_{i=1}^{n}\xi_i v_i$. (13) then implies stochastic equicontinuity and therefore along with the application of Theorem 1 to the FIDIs of $\frac{1}{n^{1/\alpha}r_n^{1/\alpha}}\sum_{i=1}^{n}\xi_i V_i(\theta)$, the weak convergence of the latter to a stochastic process with α-stable marginals in $\ell^\infty[\Theta]$. In the special case where $\alpha<1$ and $\min\{\xi_0,V_0\}\geq 0$ $\mathbb{P}$ a.s., the limiting process is equal in distribution to $Z\mathbb{E}[V_0^\alpha(\theta)]$, where $Z\sim S_\alpha(1,c,0)$ for some $c>0$. In that case, suppose also that $\arg\max_{\theta\in\Theta}\mathbb{E}\left[V_0^\alpha(\theta)\right]=\{\theta_0\}\subset\Theta$. Then the CMT implies that $\arg\max_\Theta\frac{1}{n^{1/\alpha}r_n^{1/\alpha}}\sum_{i=1}^{n}\xi_i V_i(\theta)\rightsquigarrow\{\theta_0\}$ since Z has positive support. This could be useful for establishing consistency of M-estimators in very heavy tailed cases even when the appropriately scaled objective functions have stochastic limits, as long as parameter identification holds.

The consideration of heavy-tailed distributions for the squared innovation process in GARCH-type models became of interest in financial applications (see e.g. Rachev and Mittnik [32] and Mittnik et al. [29]). Focusing on the issue of its implications on the limit theory of the Gaussian QMLE for the GARCH(p,q) model, Mikosch and Straumann [28] employ their martingale limit theorem which among others depends on a mixing condition for the volatility process and strict positivity for the extremal index of the associated martingale transform appearing in the score vector. They indicate that such conditions seem indispensable and thereby have to be in each case confirmed, in order for their results to be extended to other GARCH-type models. Our martingale limit theory avoids mixing in cases 1 and 2 of Proposition 1, as well as the consideration of the relevant extremal index. Case 2 is relevant in many such models, hence one obvious application of our main result concerns the subsequent extension of the Mikosch and Straumann [28] limit theory in such examples. When the squared innovation process for such models obeys i) Assumption 1 for $\alpha > 1$, and ii) the volatility gradient is stationary, ergodic and admits $\alpha + \delta$ moments, as well as iii) the remaining standard assumption framework that appears in the literature (see for example Ch. 4 in [35]), holds, employing Theorem 1, we can extend the existing results for the Gaussian QMLE in the GARCH(p, q) case to a variety of conditionally heteroskedastic models. Examples are the GQARCH$(1, 1)$, AGARCH(p, q) and EGARCH$(1, 1)$ models-for their definitions see Sentana [34], Ding et al. [15] and Nelson [30] respectively, as well as Arvanitis and Louka [5], Straumann [35] and Wintenberger and Cai [38] for establishing the consistency between i), ii) and iii) in each model respectively.

3 Proofs

Proof of Lemma 1. We have that $\mathbb{P}\left(\max_{1\leq i\leq n}|V_i| > Mr_n^{\frac{1}{\alpha}}n^{\frac{1}{\alpha}}\right)$ equals

$$P(\cup_{i=1}^{n}\left\{|V_i| > Mr_n^{\frac{1}{\alpha}}n^{\frac{1}{\alpha}}\right\}) \leq \sum_{i=1}^{n}\mathbb{P}\left(|V_i| > Mr_n^{\frac{1}{\alpha}}n^{\frac{1}{\alpha}}\right) = n\mathbb{P}\left(|V_0| > Mr_n^{\frac{1}{\alpha}}n^{\frac{1}{\alpha}}\right),$$

due to the stationarity of $(V_i)_{i\in\mathbb{Z}}$. Then the result follows by Assumptions 1 and 3. □

Proof of Proposition 1. As in Denisov and Zwart [14] (see the proofs of Propositions 2.1, 2.2 and 2.3) we can work by replacing $|V_0|$, $|\xi_0|$ and m_n with $|V_0|^{\alpha}$, $|\xi_0|^{\alpha}$ and $m_n^{\frac{1}{\alpha}}$ respectively, and thereby we can assume that $\alpha = 1$, and without loss of generality we can assume that $\mathbb{P}(|V_0| = 0) = 0$. Notice first that

$$\liminf_{n\to\infty}\frac{\mathbb{P}_n^{\star}[v_n|\xi_0| > m_n]}{\mathbb{P}[|\xi_0| > m_n]} = \liminf_{n\to\infty}\int_0^{\infty}\frac{\mathbb{P}[t|\xi_0| > m_n]}{\mathbb{P}[|\xi_0| > m_n]}d\mathbb{P}_n[v_n \leq t] \geq \int_0^{\infty}td\mathbb{P}[|V_0| \leq t],$$

where the last inequality in the previous display follows from Fatou's lemma with varying measures, Birkhoff's LLN, the regular variation of index -1 of the tail of $|\xi_0|$. For an upper bound consider

$$\mathbb{P}_n^{\star}[v_n|\xi_0| > m_n] = \sum_{j=1}^{3}\mathbb{P}_n^{\star}[v_n|\xi_0| > m_n, v_n \in A_{i,n}],$$

with $A_{1,n} = (0, \varepsilon)$, $A_{2,n} = [\varepsilon, g_n m_n)$, and $A_{3,n} = [g_n m_n, \infty)$ for some $\varepsilon > 0$, and, $g_n \downarrow 0$ with $g_n m_n \to \infty$, such that $\mathbb{P}[|V_0| > g_n m_n] = o(\mathbb{P}[|\xi_0| > m_n])$. Denote the respective terms as $\mathcal{I}_{1,n}, \mathcal{I}_{2,n}, \mathcal{I}_{3,n}$. Due to the UCT for regularly varying varying functions with negative index (see Theorem 1.5.2 of Bingham et al. [8]) and Birkhoff's LLN we obtain

$$\limsup_{n\to\infty}\frac{\mathcal{I}_{1,n}}{\mathbb{P}[|\xi_0| > m_n]} = \mathbb{E}[|V_0|1(0 < |V_0| < \varepsilon)], \text{ a.s.}$$

Furthermore by the construction of g_n and the Glivenko-Cantelli Theorem (see Theorem 1 of Adams and Nobel [3])

$$\limsup_{n\to\infty}\frac{\mathcal{I}_{3,n}}{\mathbb{P}[|\xi_0| > m_n]} \leq \limsup_{n\to\infty}\frac{\mathbb{P}_n[v_n > g_n m_n]}{\mathbb{P}[|\xi_0| > m_n]} = \limsup_{n\to\infty}\frac{\mathbb{P}_n[|V_0| > g_n m_n]}{\mathbb{P}[|\xi_0| > m_n]} = 0, \text{ a.s.},$$

since $\mathbb{P}\left[|V_0| > g_n m_n\right] = o\left(\mathbb{P}\left[|\xi_0| > m_n\right]\right)$. Using the above we have that

$$\limsup_{n\to\infty} \frac{\mathbb{P}_n^{\star}\left[v_n\left|\xi_0\right| > m_n\right]}{\mathbb{P}\left[|\xi_0| > m_n\right]} \leq \mathbb{E}\left[|V_0|\, 1\left(0 < |V_0| < \varepsilon\right)\right] + \limsup_{n\to\infty} \frac{\mathcal{I}_{2,n}}{\mathbb{P}\left[|\xi_0| > m_n\right]}.$$

Thus, it suffices for (3) that the last term in the rhs of the previous display converges a.s. to zero as $n \to \infty$ and then $\varepsilon \to \infty$. Notice that for this term we have that

$$\frac{\mathcal{I}_{2,n}}{\mathbb{P}\left[|\xi_0| > m_n\right]} = \int_{\varepsilon}^{g_n m_n} \frac{L(\frac{m_n}{t})}{L(m_n)} t d\mathbb{P}_n\left[v_n \leq t\right].$$

Suppose that Condition (1) holds. Then for $C_n := \sup_{y\in[1,m_n]} \frac{L\left(y\right)}{L\left(m_n\right)} = \sup_{t\in[1,m_n]} \frac{L(\frac{m_n}{t})}{L\left(m_n\right)}$, and due to Birkhoff's LLN we obtain

$$\limsup_{n\to\infty} \frac{\mathcal{I}_{2,n}}{\mathbb{P}\left[|\xi_0| > m_n\right]} \leq \limsup_{n\to\infty} C_n \mathbb{E}_n\left[u_n 1\left(u_n > \varepsilon\right)\right] \leq \mathbb{E}\left[|V_0|\, 1\left(|V_0| > \varepsilon\right)\right] \limsup_{n\to\infty} C_n,\ \mathbb{P}\text{ a.s.}$$

Since $\limsup_{n\to\infty} C_n < \infty$ and $\mathbb{E}\left[|V_0|\right] < \infty$ the result follows.

Suppose that Condition (2) holds. Given δ, we can choose ε large enough so that Potter's Theorem (see Theorem 1.5.6 (i)-(ii) of Bingham et al. [8]) is applicable and via Birkhoff's LLN we obtain that

$$\limsup_{n\to\infty} \frac{\mathcal{I}_{2,n}}{\mathbb{P}\left[|\xi_0| > m_n\right]} \leq \int_{\varepsilon}^{\infty} t^{1+\frac{\delta}{\alpha}} d\mathbb{P}\left[|V_0| \leq t\right] = \mathbb{E}\left[|V_0|^{1+\frac{\delta}{\alpha}}\, 1\left(|V_0| > \varepsilon\right)\right],$$

and the result follows since $\mathbb{E}\left[|V_0|^{1+\frac{\delta}{\alpha}}\right] < \infty$.

For establishing (4) notice first that due to the previous, it suffices that $\frac{\mathcal{I}_{2,n}}{\mathbb{P}[|\xi_0|>m_n]}$ converges in probability to zero. Secondly, notice that due to Markov's, the generalized von Bahr-Esseen (see Theorem 1 of Hill [22]) and the C_r inequalities, for any $\delta > 0$, and for $\eta > 0$ such that $(1+\eta)\,\epsilon < 1$, we have that for some $C > 0$ and $r = 1+\eta$

$$\mathbb{P}\left[\frac{\left|\mathbb{P}_n\left[v_n > t^-\right] - \mathbb{P}\left[|V_0| > t\right]\right|}{\mathbb{P}^{\epsilon}\left[|V_0| > t\right]} > \delta\right] \leq \frac{C}{\delta^{1+\eta} n^{\eta}} \left(\mathbb{P}^{1-(1+\eta)\epsilon}\left[|V_0| > t\right] + \mathbb{P}^{1-\epsilon}\left[|V_0| > t\right]\right), \quad (14)$$

and the rhs of the previous display converges to zero uniformly in t. Hence $\frac{\mathbb{P}_n\left[v_n>t^-\right]-\mathbb{P}[|V_0|>t]}{\mathbb{P}^{\epsilon}[|V_0|>t]}$ is bounded with probability converging to one uniformly in t.

Suppose now that Condition (3) holds. Define $c_n^{\star} := \sup_{t\in[\varepsilon, g_n m_n]} c\left(\frac{m_n}{t}\right)/c\left(m_n\right)$ and notice that $\left(c_n^{\star}\right)$ is bounded since by construction $c\left(x\right)$ converges to a positive real number as $x \to \infty$. Then

$$\frac{\mathcal{I}_{2,n}}{\mathbb{P}\left[|\xi_0| > m_n\right]} \leq c_n^{\star} \int_{\varepsilon}^{g_n m_n} \frac{\mathbb{P}\left[U > \ln m_n - \ln t\right]}{\mathbb{P}\left[U > \ln m_n\right]} t d\mathbb{P}_n\left[v_n \leq t\right].$$

Hence it suffices to show that integral in the rhs of the previous display a.s. converges to zero as $n \to \infty$ and then $\varepsilon \to \infty$. Let $s_{\epsilon}\left(x\right) := \frac{x\mathbb{P}^{\epsilon}[|V_0|>x]}{\mathbb{P}[U>\ln x]}$ which converges to zero as $x \to \infty$ due to (2). Using the integration by parts formula for the Lebesgue-Stieljes integral-see Theorem 21.67 in Hewitt and Stromberg [21]-we have that

$$\frac{1}{\mathbb{P}\left[U > \ln m_n\right]} \int_{\varepsilon}^{g_n m_n} \mathbb{P}\left[U > \ln m_n - \ln t\right] t d\mathbb{P}_n\left[v_n \leq t\right]$$

$$= -\mathbb{P}_n\left[v_n > g_n m_n\right] \frac{\mathbb{P}\left[U > -\ln g_n\right]}{\mathbb{P}\left[U > \ln m_n\right]} g_n m_n + \mathbb{P}_n\left[v_n > \varepsilon^-\right] \frac{\mathbb{P}\left[U > \ln m_n - \ln \varepsilon\right]}{\mathbb{P}\left[U > \ln m_n\right]} \varepsilon$$

$$+ \int_{\varepsilon}^{g_n m_n} \frac{\mathbb{P}\left[U > \ln m_n - \ln t\right]}{\mathbb{P}\left[U > \ln m_n\right]} \mathbb{P}_n\left[v_n > t^-\right] \mathrm{d}t + \int_{\varepsilon}^{g_n m_n} \mathbb{P}_n\left[v_n > t^-\right] t d_t \frac{\mathbb{P}\left[U > \ln m_n - \ln t\right]}{\mathbb{P}\left[U > \ln m_n\right]}.$$

The first term is the rhs of the previous display is less than or equal to zero and thereby can be ignored for the construction of an upper bound. For the second term since $U \in S^*$ (i.e. U has a sub-exponential tail distribution function; see Klüppelberg [27]) and due to the Glivenko-Cantelli theorem we have that

$$\limsup_{n\to\infty} \mathbb{P}_n\left[v_n > \varepsilon^-\right] \frac{\mathbb{P}\left[U > \ln m_n - \ln \varepsilon\right]}{\mathbb{P}\left[U > \ln m_n\right]} \varepsilon \leq \varepsilon \mathbb{P}\left[|V_0| > \varepsilon\right], \text{ a.s.}$$

For the third term we have that since $\epsilon < 1$

$$\int_{\varepsilon}^{g_n m_n} \frac{\mathbb{P}\left[U > \ln m_n - \ln t\right]}{\mathbb{P}\left[U > \ln m_n\right]} \mathbb{P}_n\left[v_n > t^-\right] \mathrm{d}t$$

$$\leq \sup_{t\geq\varepsilon} s_\epsilon(t) \int_{\varepsilon}^{g_n m_n} \frac{\mathbb{P}\left[U > \ln m_n - \ln t\right] \mathbb{P}\left[U > \ln t\right]}{\mathbb{P}\left[U > \ln m_n\right]} \frac{\mathbb{P}_n\left[v_n > t^-\right]}{\mathbb{P}^\epsilon\left[|V_0| > t\right]} d\ln t, \tag{15}$$

and thereby the rhs of (15) is bounded from above with probability converging to one by

$$\left(1 + O_p(1)\right) \sup_{t\geq\varepsilon} s_\epsilon(t) \int_{\varepsilon}^{g_n m_n} \frac{\mathbb{P}\left[U > \ln m_n - \ln t\right] \mathbb{P}\left[U > \ln t\right]}{\mathbb{P}\left[U > \ln m_n\right]} d\ln t \leq 2\mathbb{E}\left[U\right] \sup_{t\geq\varepsilon} s_\epsilon(t),$$

since $U \in S^*$. For the fourth term we analogously have that with probability converging to one

$$\int_{\varepsilon}^{g_n m_n} \mathbb{P}_n\left[v_n > t^-\right] t d_t \frac{\mathbb{P}\left[U > \ln m_n - \ln t\right]}{\mathbb{P}\left[U > \ln m_n\right]}$$

$$\leq \sup_{t\geq\varepsilon} s_\epsilon(t) \int_{\varepsilon}^{g_n m_n} \frac{\mathbb{P}_n\left[v_n > t^-\right]}{\mathbb{P}^\epsilon\left[|V_0| > t\right]} d_t \mathbb{P}\left[U > \ln m_n - \ln t\right]$$

$$= \left(1 + O_p(1)\right) \sup_{t\geq\varepsilon} s_\epsilon(t) \int_{\varepsilon}^{g_n m_n} d_t \mathbb{P}\left[U > \ln m_n - \ln t\right] \leq 2 \sup_{t\geq\varepsilon} s(t).$$

The previous imply that with probability converging to one

$$\frac{1}{\mathbb{P}\left[U > \ln m_n\right]} \int_{\varepsilon}^{g_n m_n} \mathbb{P}\left[U > \ln m_n - \ln t\right] t d\mathbb{P}_n\left[v_n \leq t\right]$$

$$\leq \varepsilon \mathbb{P}\left[|V_0| \geq \varepsilon\right] + 2\left(\mathbb{E}\left[U\right] + 1\right)\left(1 + O_p(1)\right) \sup_{t\geq\varepsilon} s_\epsilon(t),$$

and the latter converges to zero as $\varepsilon \to \infty$.

Suppose finally that Condition (4) holds. As in the proof of Proposition 2.3 of Denisov and Zwart [14] we can assume that eventually $g_n > \frac{1}{\sqrt{m_n}}$. Using this we have that $\frac{\mathcal{I}_{2,n}}{\mathbb{P}[|\xi_0| > m_n]} = \int_{\varepsilon}^{\sqrt{m_n}} \frac{L(\frac{m_n}{t})}{L(m_n)} t \mathrm{d}\mathbb{P}_n\left[v_n \leq t\right] + \int_{\sqrt{m_n}}^{g_n m_n} \frac{L(\frac{m_n}{t})}{L(m_n)} t \mathrm{d}\mathbb{P}_n\left[v_n \leq t\right]$. Then, due to Birkhoff's LLN

$$\limsup_{n\to\infty} \int_{\varepsilon}^{\sqrt{m_n}} \frac{L(\frac{m_n}{t})}{L(m_n)} t \mathrm{d}\mathbb{P}_n\left[v_n \leq t\right] \leq \mathbb{E}\left[|V_0| \, 1\left(|V_0| > \varepsilon\right)\right] \limsup_{n\to\infty} \sup_{\sqrt{m_n}\leq t\leq m_n} \frac{L(\frac{m_n}{t})}{L(m_n)}, \text{ a.s.},$$

and as $\varepsilon \to \infty$, the term in the rhs of the previous display converges to zero. Furthermore, for the second integral above, using the integration by parts formula for the Lebesgue-Stieljes integral we obtain

$$\int_{\sqrt{m_n}}^{g_n m_n} \frac{L(\frac{m_n}{t})}{L(m_n)} t d\mathbb{P}_n\left[v_n \leq t\right] = -\frac{L(\frac{1}{g_n})}{L(m_n)}\mathbb{P}_n\left[v_n > g_n m_n\right] g_n m_n$$

$$+\frac{L(\sqrt{m_n})}{L(m_n)}\mathbb{P}_n\left[v_n > \sqrt{m_n}^{-}\right]\sqrt{m_n} + \frac{1}{L(m_n)}\int_{\sqrt{m_n}}^{g_n m_n}\mathbb{P}_n\left[v_n > t^{-}\right] d_t\left(tL\left(\frac{m_n}{t}\right)\right).$$

As previously the first term in the rhs of the previously display is less than or equal to zero and therefore can be ignored. For the second term we have that it is less than or equal to

$$\frac{1}{n}\sum_{i=1}^{n}|V_0|\, 1\left(|V_0| > \sqrt{m_n}\right) \sup_{\sqrt{m_n}\leq t\leq m_n}\frac{L(\frac{m_n}{t})}{L(m_n)},$$

and the latter converges a.s. to zero as $n \to \infty$ since $\mathbb{E}\left[|V_0|\right] < \infty$. For the final term let $Q\left(x\right) := \int_0^x t^{\alpha} d\mathbb{P}\left[|\xi_0| \leq x\right]$. Due to the previous we have that as $n \to \infty$ with probability tending to 1

$$\frac{1}{L(m_n)}\int_{\sqrt{m_n}}^{g_n m_n}\mathbb{P}_n\left[v_n > t^{-}\right] d_t\left(tL\left(\frac{m_n}{t}\right)\right) \leq$$

$$o(1)\left(1 + O_p\left(1\right)\right)\int_{\sqrt{m_n}}^{g_n m_n} m_n \frac{\mathbb{P}\left[|\xi_0| > t\right]}{L\left(m_n\right)Q\left(t\right)} d\mathbb{P}\left[|\xi_0| \leq \frac{m_n}{t}\right],$$

and the integral in the rhs of the previous display is bounded from above exactly as in the proof of Proposition 2.3 of Denisov and Zwart [14]. □

Proof of Proposition 2. Notice first that since as $x \to \infty$, $L\left(x\right) \sim \mathbb{E}\left[\left(\xi_i - \gamma\right)^2 1_{|\xi_i - \gamma| > x}\right]$, L has an equivalent monotone version. Assumption 3 and Birkhoff's LLN implies that $\sum_{i=1}^n \mathbb{E}\left[\frac{1}{nr_n}\left(\xi_i - \gamma\right)^2 V_i^2 1_{|(\xi_i - \gamma)V_i| > M\sqrt{nr_n}}/\mathcal{G}_{i-1}\right]$ is asymptotically equivalent to $\frac{L(M\sqrt{nr_n})}{nr_n}\sum_{i=1}^n V_i^2 \frac{L\left(\frac{M\sqrt{nr_n}}{|V_i|}\right)}{L(M\sqrt{nr_n})}$. The same argument implies that we can replace L by its monotone equivalent version. Denote the latter with L for brevity. Let $\varepsilon > 0$ and consider

$$\frac{L\left(M\sqrt{nr_n}\right)}{nr_n}\sum_{i=1}^{n} V_i^2 1_{|V_i| > \varepsilon}\frac{L\left(\frac{M\sqrt{nr_n}}{|V_i|}\right)}{L\left(M\sqrt{nr_n}\right)} \leq \frac{L\left(\frac{M\sqrt{nr_n}}{\varepsilon}\right)}{L\left(M\sqrt{nr_n}\right)}\frac{L\left(M\sqrt{nr_n}\right)}{nr_n}\sum_{i=1}^{n} V_i^2 1_{|V_i| > \varepsilon}.$$

Due to Assumption 3, Birkhoff's LLN, the slow variation of L, the defining property $L(n^{\frac{1}{\alpha}} r_n^{\frac{1}{\alpha}})/r_n \to 1$ of $(r_n)_{n\in\mathbb{N}}$ implying that it is a slowly varying sequence-see Paragraph 1.9 of [8] and Proposition 1.(iv) of [6], the rhs of the previous display converges $\mathbb{P}$ a.s. to $\mathbb{E}\left[V_0^2 1_{|V_0| > \varepsilon}\right]$. Furthermore, let $g(x) = x^{-2}L(x)$ and notice that

$$\frac{1}{n}\sum_{i=1}^{n}\left|\frac{g(M\sqrt{nr_n}|V_i|^{-1})}{g(M\sqrt{nr_n})} - |V_i|^{-2}\right| 1_{|V_i|\leq\varepsilon}$$

$$\leq \sup_{\lambda\in[\varepsilon^{-1},\infty)}\left|\frac{g(\lambda M\sqrt{nr_n})}{g(M\sqrt{nr_n})} - \lambda^{-2}\right|\frac{1}{n}\sum_{i=1}^{n} 1_{|V_i|\leq\varepsilon}.$$

Due to Assumption 3 and Birkhoff's LLN $\frac{1}{n}\sum_{i=1}^{n} 1_{|V_i|>\varepsilon} = O(1)$ $\mathbb{P}$ a.s. Due to the UCT for regularly varying varying functions with negative index (see Theorem 1.5.2 of Bingham et al. [8]) $\sup_{\lambda\in[\varepsilon^{-1},\infty)} \left|\frac{g(\lambda n^{1/\alpha}r_n^{1/\alpha}|t|^{-1})}{g(n^{1/\alpha}r_n^{1/\alpha}|t|^{-1})} - \lambda^{-\alpha}\right| = o(1)$. Hence

$$\frac{L\left(M\sqrt{nr_n}\right)}{nr_n}\sum_{i=1}^{n} V_i^2 1_{|V_i|\leq\varepsilon}\frac{L\left(\frac{M\sqrt{nr_n}}{|V_i|}\right)}{L\left(M\sqrt{nr_n}\right)} = \frac{L\left(M\sqrt{nr_n}\right)}{r_n}\left(\frac{1}{n}\sum_{i=1}^{n} V_i^2 1_{|V_i|\leq\varepsilon} + o_{\mathbb{P}\text{ a.s.}}(1)\right).$$

Again, due to Assumption 3, Birkhoff's LLN, the slow variation of L, and the defining property of $(r_n)_{n\in\mathbb{N}}$, the rhs of the previous display converges $\mathbb{P}$ a.s. to $\mathbb{E}\left[V_0^2 1_{|V_0|>\varepsilon}\right]$. The result then follows from Assumption 3 by letting $\varepsilon\to\infty$, since $\mathbb{E}\left[V_0^2\right] < +\infty$. □

Proof of Theorem 1. By the Main Lemma for Sequences in Jakubowski [25] (see, equivalently, Theorem 1.1 along with Paragraph 3 of Jakubowski [24]) the result would follow if we would prove that for all $t\in\mathbb{R}$

$$\prod_{i=1}^{n}\mathbb{E}\left(\exp\left(\mathbf{i}t\frac{1}{n^{\frac{1}{\alpha}}r_n^{\frac{1}{\alpha}}}\rho_{i,\alpha}\right)/\mathcal{F}_i\right), \tag{16}$$

for $\mathbf{i} := \sqrt{-1}$, pointwise converges $\mathbb{P}$ a.s. to the characteristic function of $S_\alpha\left(\beta\left(\frac{\mathbb{E}[|V_0|^\alpha \operatorname{sgn}(V_0)]}{\mathbb{E}(|V_0|^\alpha)}\right), c\mathbb{E}\left(|V_0|^\alpha\right), 0\right)$, where

$$\rho_{i,\alpha} = \left\{\begin{array}{c} (\xi_i-\gamma)V_i, \quad \alpha\neq 1 \\ (\xi_i-\gamma-H(nr_n))V_i - r_n 2\beta c\pi^{-1}\left(C\mathbb{E}\left(V_0\right) - \mathbb{E}\left[V_0|\log(|V_0|)|\right]\right), \quad \alpha = 1 \end{array}\right.$$

Given the local representation in Assumption 1 hold for all $t\in(-t_0,t_0)$, where $t_0>0$. Then notice that for any $t\neq 0$ by defining the event

$$C_{n,K} \equiv \left\{\omega\in\Omega : |V_i| \leq K_t\left(nr_n\right)^{\frac{1}{\alpha}} \quad, \forall i = 1,\ldots,n\right\}$$

where $K_t < \frac{t_0}{|t|}$, we have that $\mathbb{P}(C^c_{n,K})$ which by Lemma 1 tends to 0. When $\alpha\neq 1$, due to Assumption 1 if $\omega\in C_{n,K}$ then the logarithm of (16) equals

$$\begin{gathered} -\frac{c|t|^\alpha}{nr_n}\sum_{i=1}^{n}|V_i|^\alpha L\left(n^{1/\alpha}r_n^{1/\alpha}\left|tV_i\right|^{-1}\right) + \\ \frac{|t|^\alpha}{nr_n}\mathbf{i}\beta c\operatorname{sgn}(t)\tan\left(\tfrac{1}{2}\pi\alpha\right)\sum_{i=1}^{n}\operatorname{sgn}(V_i)\left|V_i\right|^\alpha L\left(n^{1/\alpha}r_n^{1/\alpha}\left|tV_i\right|^{-1}\right). \end{gathered} \tag{17}$$

Notice that when $\alpha<2$, due to Lemma 1, Assumption 1, and the asymptotic representation of the tail of the distribution of $|\xi_0|$ (see Appendix 1 in Ibragimov and Linnik [23]), and the the defining property $L(n^{\frac{1}{\alpha}}r_n^{\frac{1}{\alpha}})/r_n \to 1$ of $(r_n)_{n\in\mathbb{N}}$ implying that it is a slowly varying sequence-see Paragraph 1.9 of [8] and Proposition 1.(iv) of [6], we have that

$$\begin{aligned} \frac{\mathbb{P}^\star_n[v_n|\xi_0|>n^{1/\alpha}r_n^{1/\alpha}]}{\mathbb{P}[|\xi_0|>n^{1/\alpha}r_n^{1/\alpha}]} &= \frac{\mathbb{E}^\star_n\left[\mathbb{P}\left[|\xi_0|>\frac{n^{1/\alpha}r_n^{1/\alpha}}{u_n}\right]\right]}{\mathbb{P}\left[|\xi_0|>n^{1/\alpha}r_n^{1/\alpha}\right]} \\ &= \frac{nr_n}{(1+o(1))L\left(n^{1/\alpha}r_n^{1/\alpha}\right)}\mathbb{E}^\star_n\left[\frac{(1+o(1))u_n^\alpha}{nr_n}L\left(n^{1/\alpha}r_n^{1/\alpha}\left|V_i\right|^{-1}\right)\right] \\ &= \frac{(1+o(1))}{(1+o(1))^2 r_n}\frac{1}{n}\sum_{i=1}^{n}|V_i|^\alpha L\left(n^{1/\alpha}r_n^{1/\alpha}\left|V_i\right|^{-1}\right), \end{aligned}$$

and the rhs of the previous display is asymptotically equivalent to $\frac{1}{nr_n}\sum_{i=1}^n |V_i|^\alpha L\left(n^{1/\alpha}r_n^{1/\alpha}|V_i|^{-1}\right)$, which, due to Lemma 1, that t is fixed and non-zero, and that L is slowly varying to infinity, is asymptotically equivalent to $\frac{1}{nr_n}\sum_{i=1}^n |V_i|^\alpha L\left(n^{1/\alpha}r_n^{1/\alpha}|tV_i|^{-1}\right)$. Analogously it is easy to show that $\frac{\mathbb{P}_n^\star[v_n 1\{v_n\geq 0\}|\xi_0|>n^{1/\alpha}r_n^{1/\alpha}]}{\mathbb{P}[|\xi_0|>n^{1/\alpha}r_n^{1/\alpha}]} - \frac{\mathbb{P}_n^\star[|v_n|1\{v_n<0\}|\xi_0|>n^{1/\alpha}r_n^{1/\alpha}]}{\mathbb{P}[|\xi_0|>n^{1/\alpha}r_n^{1/\alpha}]}$ is asymptotically equivalent to $\frac{1}{nr_n}\sum_{i=1}^n \operatorname{sgn}(V_i)|V_i|^\alpha L\left(n^{1/\alpha}r_n^{1/\alpha}|tV_i|^{-1}\right)$. Hence (17) is asymptotically equivalent to

$$\begin{array}{c} -c|t|^\alpha \frac{\mathbb{P}_n^\star[v_n|\xi_0|>n^{1/\alpha}r_n^{1/\alpha}]}{\mathbb{P}[|\xi_0|>n^{1/\alpha}r_n^{1/\alpha}]}+ \\ |t|^\alpha \mathbf{i}\beta c\operatorname{sgn}(t)\tan\left(\frac{1}{2}\pi\alpha\right)\left[\frac{\mathbb{P}_n^\star[v_n 1\{v_n\geq 0\}|\xi_0|>n^{1/\alpha}r_n^{1/\alpha}]}{\mathbb{P}[|\xi_0|>n^{1/\alpha}r_n^{1/\alpha}]} - \frac{\mathbb{P}_n^\star[|v_n|1\{v_n<0\}|\xi_0|>n^{1/\alpha}r_n^{1/\alpha}]}{\mathbb{P}[|\xi_0|>n^{1/\alpha}r_n^{1/\alpha}]}\right] \end{array}.$$

When $\alpha=2$, (17) is asymptotically equivalent to

$$-c|t|^2\sum_{i=1}^n \mathbb{E}\left[\frac{1}{nr_n}(\xi_i-\gamma)^2 V_i^2 1_{|(\xi_i-\gamma)V_i|>M\sqrt{nr_n}}/\mathcal{G}_{i-1}\right].$$

The result then follows from Propositions 1 and 2 respectively.

When $\alpha=1$, by Assumption 1, if $\omega\in C_{n,K}$ then the logarithm of (16) equals

$$\begin{aligned} -c|t|\frac{1}{nr_n}\sum |V_i| L\left(nr_n|tV_i|^{-1}\right) + \mathbf{i}2\beta c\pi^{-1}Ct\frac{1}{nr_n}\sum V_i L\left(nr_n|tV_i|^{-1}\right) \\ + \mathbf{i}t\frac{1}{nr_n}\sum V_i\left[H\left(nr_n|tV_i|^{-1}\right) - H(nr_n)\right], \end{aligned}$$

where the first two terms of the above expression can be treated analogously to obtain their $\mathbb{P}$ a.s. limit as

$$-c|t|\mathbb{E}\left[|V_0|\right] + i2\beta c\pi^{-1}Ct\mathbb{E}\left[V_0\right] = -c\mathbb{E}\left[|V_0|\right]|t|\left[1 - i2\beta\pi^{-1}C\operatorname{sgn}(t)\frac{\mathbb{E}\left[V_0\right]}{\mathbb{E}\left[|V_0|\right]}\right].$$

For the third term, first notice that

$$H(k\lambda) - H(\lambda) = \int_1^k \frac{\lambda^2 x}{1+\lambda^2x^2}\left(c_1 - c_2 + k(\lambda x)\right)L(\lambda x)dx.$$

Then we have that

$$\begin{aligned} &\frac{1}{nr_n}\sum V_i\left[H\left(nr_n|tV_i|^{-1}\right) - H(nr_n)\right] \\ &= \frac{L(nr_n)}{r_n}\left(2\beta c\pi^{-1} + o(1)\right)\frac{1}{n}\sum V_i\int_1^{|tV_i|^{-1}} \frac{x}{\frac{1}{n^2r_n^2}+x^2}\frac{L(xnr_n)}{L(nr_n)}dx, \end{aligned}$$

since for any constant A, $\sup_{x\in[(\max|V_i|)^{-1},A]} k(nr_n|t|^{-1}x) = k(nr_n|t|^{-1}x_n^*)$ for some x_n^*. Note that Lemma 1 implies that $(\max|V_t|)^{-1}n^{1/\alpha}r_n^{1/\alpha}\to\infty$ in $\mathbb{P}$ probability, hence we have that

$k(nr_n|t|^{-1}x_n^*) = o(1)$. Furthermore using a similar to the above truncation argument we have that

$$\frac{1}{n}\sum_{i=1}^n V_i 1\{|V_i| \le \varepsilon\} \int_1^{|tV_i|^{-1}} \frac{x}{\frac{1}{n^2r_n^2} + x^2} \frac{L(xnr_n)}{L(nr_n)} dx$$

$$= \frac{1}{n}\sum_{i=1}^n V_i 1\{|V_i| \le \varepsilon\} \int_1^{|tV_i|^{-1}} \frac{x}{\frac{1}{n^2r_n^2} + x^2} dx$$

$$+ \frac{1}{n}\sum_{i=1}^n V_i 1\{|V_i| \le \varepsilon\} \int_1^{|tV_i|^{-1}} \frac{x}{\frac{1}{n^2r_n^2} + x^2} \left(\frac{L(xnr_n)}{L(nr_n)} - 1\right) dx.$$

Then notice that for some $A_2 = [a_1, a_2]$ with $0 < a_1 \le a_2$ and possibly dependent on the choice of ε

$$\frac{1}{n}\sum_{i=1}^n V_i 1\{|V_i| \le \varepsilon\} \int_1^{|tV_i|^{-1}} \frac{x}{\frac{1}{n^2r_n^2} + x^2} \left(\frac{L(xnr_n)}{L(nr_n)} - 1\right) dx$$

$$\le \int_{x\in A_2} \frac{x}{\frac{1}{n^2r_n^2} + x^2} \left|\frac{L(xnr_n)}{L(nr_n)} - 1\right| dx \frac{1}{n}\sum_{i=1}^n |V_i|\, 1\{|V_i| \le \varepsilon\}$$

and the dominant part of the previous display converges to zero $\mathbb{P}$ a.s. via the use of the Dominated Convergence Theorem and Assumption 2. Regarding the first term, first notice that

$$\int_1^{|tV_i|^{-1}} \frac{x}{\frac{1}{n^2r_n^2} + x^2} dx 1\{|V_i| \le \varepsilon\} = \frac{1}{2}\left[\log\left(1 + n^2r_n^2x^2\right)\right]_1^{|tV_i|^{-1}} 1\{|V_i| \le \varepsilon\}$$

$$= \frac{1}{2}\log\left(\frac{1 + n^2r_n^2|tV_i|^{-2}}{1 + n^2r_n^2}\right) 1\{|V_i| \le \varepsilon\} = \log|tV_i|^{-1} 1\{|V_i| \le \varepsilon\} + o(1),$$

where the $o(1)$ term is independent of V_i using the fact that

$$\sup_{x\in[(t\varepsilon)^{-1},\infty)} \left|\log\left(\frac{1 + \lambda^2 x}{1 + \lambda^2}\right) - \log x\right| \to 0 \text{ as } \lambda \to +\infty.$$

Therefore

$$\frac{1}{n}\sum_{i=1}^n V_i 1\{|V_i| \le \varepsilon\} \int_1^{|tV_i|^{-1}} \frac{x}{\frac{1}{n^2r_n^2} + x^2} \frac{h(xnr_n)}{h(nr_n)} dx$$

$$= \log\frac{1}{|t|}\frac{1}{n}\sum_{i=1}^n V_i 1\{|V_i| \le \varepsilon\} - \frac{1}{n}\sum_{i=1}^n V_i \log|V_i| 1\{|V_i| \le \varepsilon\} + o(1).$$

Next, we treat the analogous term obtained by truncating $V_i 1\{|V_i| > \varepsilon\}$, i.e.

$$\frac{1}{n}\sum_{i=1}^n V_i 1\{|V_i| >\} \int_1^{|tV_i|^{-1}} \frac{x}{\frac{1}{n^2r_n^2} + x^2} \frac{L(xnr_n)}{L(nr_n)} dx$$

by noticing that $|tV_i|^{-1} < 1$, since ε can be chosen large enough. Suppose first that Condition C.1 holds. Notice that,

$$\left|\int_{|tV_i|^{-1}}^{1} \frac{x}{\frac{1}{n^2r_n^2}+x^2}\left(\frac{L(xnr_n)}{L(nr_n)}-1\right)dx\right| \leq \int_{|tV_i|^{-1}}^{1} \frac{x^{1-\delta}}{\frac{1}{n^2r_n^2}+x^2}x^{\delta}\left|\frac{L(xnr_n)}{L(nr_n)}-1\right|dx$$
$$= o\left(1\right)\int_{|tV_i|^{-1}}^{1} \frac{x^{1-\delta}}{\frac{1}{n^2r_n^2}+x^2}dx \leq o\left(1\right)\int_{|tV_i|^{-1}}^{1} x^{-(1+\delta)}dx = o\left(1\right)\left(1+|V_i|^{\delta}\right),$$

where the second inequality follows from the UCT for regularly varying functions with positive index (see Theorem 1.5.2 of Bingham et al. [8]). Hence, and due to that $\frac{x}{\frac{1}{n^2r_n^2}+x^2} \leq \frac{1}{x}$ for all x, we obtain that

$$\left|\frac{1}{n}\sum_{i=1}^{n} V_i 1\{|V_i|>\varepsilon\}\int_{1}^{|tV_i|^{-1}} \frac{x}{\frac{1}{n^2r_n^2}+x^2}\frac{h(xnr_n)}{h(nr_n)}dx\right|$$
$$\leq o\left(1\right)\frac{1}{n}\sum_{i=1}^{n}|V_i|\left(1+|V_i|^{\delta}\right)1\{|V_i|>\varepsilon\} + \frac{|\ln|t||}{n}\sum_{i=1}^{n}|V_i|\ln|tV|\,1\{|V_i|>\varepsilon\},$$

and the result follows by letting $n\to\infty$ and then $\varepsilon\to\infty$.

Suppose now that C.2 holds. Notice that,

$$\int_{|tV_i|^{-1}}^{1} \frac{x}{\frac{1}{n^2r_n^2}+x^2}\frac{L(xnr_n)}{L(nr_n)}dx = \frac{1}{L(nr_n)}\int_{|tV_i|^{-1}}^{1} \frac{x^2}{\frac{1}{n^2r_n^2}+x^2}nr_n\frac{1}{xnr_n}L(xnr_n)dx$$
$$= \frac{nr_n}{L(nr_n)}\int_{|tV_i|^{-1}}^{1} \frac{x^2}{\frac{1}{n^2r_n^2}+x^2}\mathbb{P}\left[|\xi_0|>xnr_n\right]dx$$
$$= \frac{1}{L(nr_n)}\int_{nr_n|tV_i|^{-1}}^{nr_n} \frac{u^2}{1+u^2}\mathbb{P}\left[|\xi_0|>u\right]du = \int_{nr_n|tV_i|^{-1}}^{nr_n} \frac{u}{1+u^2}\frac{L(u)}{L(nr_n)}du. \tag{18}$$

Then, since with probability converging to one $nr_n\,|tV_i|^{-1} \geq 1$, $\frac{u}{1+u^2} \leq \frac{1}{u}$ for all u, the rhs integral in (18) is for ε large enough with the same probability less than or equal to

$$\limsup_{n\to\infty}\sup{}_{1\leq y\leq nr_n}\frac{L(y)}{L(nr_n)}\int_{nr_n|tV_i|^{-1}}^{nr_n}\frac{1}{u}du \leq C\ln|tV_i|\,.$$

Hence, as before we obtain that with probability converging to one

$$\left|\frac{1}{n}\sum_{i=1}^{n} V_i 1\{|V_i|>\varepsilon\}\int_{1}^{|tV_i|^{-1}} \frac{x}{\frac{1}{n^2r_n^2}+x^2}\frac{h(xnr_n)}{h(nr_n)}dx\right| \leq C\frac{\ln|t|}{n}\sum_{i=1}^{n}|V_i|\,|\ln|V_i||\,1\{|V_i|>\varepsilon\},$$

and the result follows again by letting $n\to\infty$ and then $\varepsilon\to\infty$.

For the cases where either C.3 or C.4 holds consider the following. First we have that

$$H\left(nr_n\,|tV_i|^{-1}\right) - H\left(nr_n\right) = -\left(2\beta c\pi^{-1}+o(1)\right)\int_{1}^{|tV_i|} \frac{x}{\frac{|tV_i|^2}{n^2r_n^2}+x^2}\frac{L(xnr_n\,|tV_i|^{-1})}{L(nr_n)}dx,$$

and that due to that $\frac{x^2}{\frac{|tV_i|^2}{n^2 r_n^2}+x^2} \leq 1$ uniformly in n, x and i,

$$\int_1^{|tV_i|} \frac{x}{\frac{|tV_i|^2}{n^2 r_n^2}+x^2} \frac{L(xnr_n |tV_i|^{-1})}{L(nr_n)} dx \leq |V_i|^{-1} \int_1^{|tV_i|} \frac{\mathbb{P}\left[|V_i| |\xi_0| > \frac{x}{|t|} nr_n\right]}{\mathbb{P}\left[|\xi_0| > \frac{x}{|t|} nr_n\right]} dx.$$

Notice that the results in the proofs of Propositions 2.2-3 of Denisov and Zwart [14], concerning upper bounds for $\frac{\mathbb{P}[|V_i||\xi_0|>\delta nr_n]}{\mathbb{P}[|\xi_0|>\delta nr_n]}$ hold uniformly in δ as long as this is bounded away from zero. Hence the rhs of the previous display is less than or equal to $\mathbb{E}\left[|V_0|\right]\left(1+|V_i|^{-1}\right)$, and thereby,

$$\left|\frac{1}{n}\sum_{i=1}^{n} V_i 1\{|V_i| > \varepsilon\} \int_1^{|tV_i|^{-1}} \frac{x}{\frac{1}{n^2 r_n^2}+x^2} \frac{h(xnr_n)}{h(nr_n)} dx\right|$$
$$\leq \mathbb{E}\left[|V_0|\right] \frac{1}{n}\sum_{i=1}^{n} |V_i| \left(1+|V_i|^{-1}\right) 1\{|V_i| > \varepsilon\},$$

and again the result follows by letting first $n \to \infty$, and then $\varepsilon \to \infty$.Combining the above results we obtain (7).

Finally, when $\alpha < 1$ and under (8), observe that

$$\frac{1}{n^{1/\alpha} r_n^{1/\alpha}} \sum_{i=1}^{n} |V_i| \leq \frac{1}{n^{1/\alpha-1} r_n^{1/\alpha}} \max_{1\leq i\leq n} |V_i|^{1-\alpha} \frac{1}{n}\sum_{i=1}^{n} |V_i|^{\alpha} \leq O_{\text{a.s.}}(1) \frac{q_n^{\frac{1-\alpha}{\alpha}}}{r_n^{1/\alpha}} M,$$

with $\mathbb{P}$ probability approaching 1 as $n \to \infty$. The result follows as we can choose M arbitrarily small. Under (9), for δ small enough (so that $\alpha+\delta < 1$), $\left(\frac{1}{n^{1/\alpha} r_n^{1/\alpha}} \sum_{i=1}^{n} |V_i|\right)^{\alpha+\delta} \leq \frac{1}{n^{\frac{\delta}{\alpha}} r_n^{1+\frac{\delta}{\alpha}}} \frac{1}{n} \sum_{i=1}^{n} |V_i|^{\alpha+\delta}$ and the result in (10) follows since $n^k r_n \to \infty$ for any $k > 0$. □

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