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Block Empirical Likelihood Inference for Stochastic Bounding: Large Deviations Asymptotics Under *m*-Dependence

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Block Empirical Likelihood Inference for Stochastic Bounding: Large Deviations Asymptotics Under m-Dependence

Stelios Arvanitis,[*] *Nikolas Topaloglou*[†]

Abstract

The present note is occupied with the issue of generalized Neyman-Pearson optimality, for a testing procedure for the determination of stochastic bounding, that is based on data blocking and the minimization of the Kullback-Liebler divergence, in a time series context of m-dependence. Optimality is established via an extension of Sanov's Theorem on empirical measures for blocks of data of temporal dependence that becomes asymptotically negligible at sufficiently fast rates. A large deviation property for the-subsequent to the derivation of the test statistic-BEL estimator, and a corresponding confidence region are also obtained.
Key words: Generalized Neyman-Pearson Optimality; Large Deviations Property; Sanov's Theorem; Contraction Principle; Block Empirical Likelihood Ratio; Maximum BELE; Confidence Region; Conservativeness; Stochastic Dominance; Stochastic Bound; Portfolio analysis.

1 Introduction

The present note is occupied with a notion of fixed critical value optimality for statistical tests about order properties of stochastic dominance relations based on the Empirical Likelihood principle. Specifically, the issue of generalized Neyman-Pearson optimality is investigated, for a testing procedure for the determination of stochastic

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bounding, that is based on data blocking and the minimization of the Kullback-Liebler divergence, in a time series context of a finite horizon temporal dependence.

Stochastic dominance relations are preorders on spaces of Borel probability measures on the real line (see Fishburn (1976) (18)). They are widely used in the fields of economics, finance and statistics/econometrics (see inter alia Kroll and Levy (1980) (23), McFadden (1989) (27), Levy (1992) (24), Levy (2015) (25), Mosler and Scarsini (1993) (29), Gayant and Le Pape (2017) (19)), since they among others enable robust inference on issues regarding optimal choice under uncertainty, uniformly for large classes of preferences inside the expected utility paradigm.

This is due to that many widely used stochastic dominance relations have characterizations in terms of classes of utility functions (see Fishburn (1976) (18), Levy (1992) (24), Levy (2015) (25), Levy and Levy (2002) (26)). Thereby dominance w.r.t. such a relation implies preference by any utility in the class and vice versa. Hence, order properties of such stochastic dominance relations are directly connected to robust properties of optimal choices w.r.t. sets of preferences.

This note is specifically occupied with the notion of stochastic bounding introduced in Arvanitis, Post and Topaloglou (2021) (5); two sets-not necessarily disjoint-of the order are considered with a view towards the determination of whether the first contains a distribution that dominates every element of the second set. Latency of the underlying distributions, and-given a sample-approximability by the corresponding empirical distributions, enable the construction of statistical test for the determination of the existence of a stochastic bound for the given preorder. Due to the expected utility characterization of the bound, the underlying statistical/econometric model lies in the scope of set identification, as it is comprised by a-potentially infinite-set of moment inequalities.

Canay (2010) (12) introduced the Empirical Likelihood principle and the subsequent empirical likelihood test for set identified models of moment inequalities, based on previous work of Kitamura (2001) (21) on identified models of moment equalities. Both papers work in iid frameworks. Canay (2010) (12)-again extending Kitamura (2001) (21)-also shows the generalized Neyman-Pearson optimality of the empirical likelihood ratio test based on fixed critical values. The derivations rely on Sanov's Theorem concerning the large deviations properties of empirical measures in iid settings.

Thus in order to extend the optimality results in time series settings of temporal dependence, the extension of Sanov's Theorem in frameworks of stationarity and mixing is required; this is already known, see the results of Bryc and Dembo (1996) (11) and the references therein on appropriately mixing processes that involve super-geometric strong mixing coefficients. In order for those results to be connected to the ELR principle, the good rate function that appears in those extensions has to be identified as the Kullback-Liebler evaluated at the limiting distribution of the empirical measure and defined over a set of appropriate measures. In this note a first step towards such an identification is provided; in the context of m-dependence (see for example Bradley (1986) (10)) under blocking schemes such that the blocks exhibit temporal dependence that becomes asymptotically negligible at sufficiently fast rates, relative entropies are obtained as good rate functions.

Section 2 establishes some notation and discusses the probabilistic framework. Section 3 establishes the extension of Sanov's Theorem under m-dependence and a class of appropriate blocking scheme. Section 4 discusses a general stochastic dominance relation, the subsequent notion of stochastic bounding, designs a relevant-asymptotically conservative- Block Empirical Likelihood Test with data dependent rejection regions, and establishes its limit theory. Section 5 derives the generalized Neyman-Pearson optimality for versions of the aforementioned test based on fixed critical values. Section 6 presents the results of a Monte Carlo simulation. Section 7 utilizes the contraction principle in order to obtain a large deviation property for the subsequent to the derivation of the test statistic BEL estimator, and a corresponding confidence region. Section 8 discusses issues of further research.

2 Notation and framework

U is a closed and convex subset of a Polish vector space and $\mathcal{B}_U$ its Borel algebra. If $A \subseteq U$, A^o denotes its interior, and $\bar{A}$ its closure w.r.t. the underlying topology. $M_1(U)$ denotes the space of Borel probability measures on $\mathcal{B}_U$, equipped with the τ-topology; this is generated by $\left\{\mu \in M_1(U) : \left|\int f d\mu - x\right| < \delta\right\}$, $x \in \mathbb{R}, \delta > 0, f \in B(U, \mathbb{R})$, with $B(U, \mathbb{R})$ the set of bounded, Borel measurable real valued functions on U. By Theorem 1.7.2 of van der Vaart and Wellner (35) the τ-topology concides with the weak topology in $M_1(U)$.

$\mathbf{X} := (\mathbf{X}_t)_{t\in\mathbb{N}}$ is a stationary process with values in U. $\mathbf{b}$ denotes a blocking scheme on the process; $\mathbf{b} = ((k_t)_{t\in\mathbb{N}}, b)$ where $(k_t)_{t\in\mathbb{N}}$ is a strictly increasing $\mathbb{N}$-valued sequence, and $b : \mathbb{N} \to \mathbb{N}^\star$. Then $\mathbf{b}(\mathbf{X}) := (\frac{1}{b(k_t)} \sum_{j=1}^{b(k_t)} \mathbf{X}_{k(t)+j})_{t\in\mathbb{N}}$. If $n \in \mathbb{N}^\star$, and $\mathbf{X}_n := (\mathbf{X}_t)_{t\leq n}$ is an n-sample from $\mathbf{X}$, then $\mathbf{b}_n(\mathbf{X}) := (\frac{1}{b(k_t)} \sum_{j=1}^{b(k_t)} \mathbf{X}_{k(t)+j})_{t\in\mathbb{N}:k(t)+b(k_t)\leq n}$. Given the sample and the blocking scheme consider the empirical cdf $\mathbb{F}_{\mathbf{b},n} := \frac{1}{|\mathbf{b}_n(\mathbf{X})|} \sum_{i=1}^{|\mathbf{b}_n(\mathbf{X})|} \delta_{\mathbf{b}_n(\mathbf{X})_i}$, with $|\mathbf{b}_n(\mathbf{X})|$ denoting the cardinality of $\mathbf{b}_n(\mathbf{X})$, and δ_u, $u \in U$ denoting the degenerate probability measure at u. The strict monotonicity of $(k_t)_{t\in\mathbb{N}}$ implies that as $n \to \infty$, $|\mathbf{b}_n| := |\mathbf{b}_n(\mathbf{X})| \to \infty$.

The question that is addressed in the following section concerns the establishment of sufficient conditions under which: (a.) the $M_1(U), \tau$-valued sequence $(\mathbb{F}_{\mathbf{b},n})_{n\in\mathbb{N}^\star}$ satisfies the Large Deviation Principle (LDP-see for example Dembo and Zeitouni (2009) (15)); this means that there exists a good rate function, i.e. an $I : M_1(U) \to \mathbb{R}$ that is lower semi-continuous and inf-compact (see Ch. 1 of Rockafellar and Wetts (2009) (32)) such that for any measurable $A \subseteq M_1(U)$,

$$-\inf_{\mu\in A^o} I(\mu) \leq \lim\inf_{n\to\infty} \frac{1}{|\mathbf{b}_n|} \ln \mathbb{P}(\mathbb{F}_{\mathbf{b},n} \in A) \leq \lim\sup_{n\to\infty} \frac{1}{|\mathbf{b}_n|} \ln \mathbb{P}(\mathbb{F}_{\mathbf{b},n} \in A) \leq -\inf_{\mu\in\bar{A}} I(\mu), \tag{1}$$

and (b.) whether I can be obtained as the Kullback-Liebler divergence (also known as relative entropy-see for example Cover and Thomas (2006) (14)) partially evaluated at appropriate measures inside $M_1(U)$. Such a characterization is relatable to possible optimality properties of statistical tests constructed via statistics that resemble the relative entropy.

3 A Sanov-type theorem with blocking

Whenever (k_t) is the identity, b is constant at 1 and the stationary process is comprised by independent random elements the LDP principle, with rate equal to the Kullback-Liebler divergence evaluated at the stationary measure of the process is provided by the classical Sanov's Theorem (see Paragraph 3.2 of Dembo and Zeitouni (2009) (15)). A slight extension is provided by the main result of the present section by considering processes that are m-dependent and blocking schemes that mix random elements of $\mathbf{X}$ producing asymptotically independent blocks at a sufficient rate; the following assumptions are considered-for the notion of m-dependence see Bradley (1986) (10), and for the notion of a slowly varying sequence see Paragraph 1.9 in Bingham, Goldie and Teugels (1989) (9):

Assumption 1. *For some $m \in \mathbb{N}^\star$, the process $\mathbf{X}$ is stationary m-dependent.*

m-dependence is a restrictive assumption on the dynamics of the process. It can however accommodate time series models that exhibit finite order moving average structures at their conditional moments; e.g. (Vector) MA processes, or the stochastic volatility processes of the form:

$$y_t = z_t\sqrt{h_t},\ h(t) = \exp(\omega + \sum_{i=1}^{\bar{m}} \alpha_i u_{t-i});$$

where, $\bar{m} \leq m$, $\omega, \alpha_i \in \mathbb{R}$, $i = 1, \ldots, m$, z_t are iid with zero mean and unit variance, and the u_t are stationary and s-dependent, with $m = s\bar{m}$ and well defined moment generating functions.

Assumption 2. *The function b is bounded. For a real sequence $(\ell_n)_{n\in\mathbb{N}}$ and some $\mathbb{N}^\star \ni m^\star \leq m$, $\sup_{t\leq n} |\{\mathbf{X}_{k_t+j}, j=1,\ldots,b(k_t)\} \triangle \{\mathbf{X}_{tm-1+j}, j=1,\ldots,m^\star\}| \leq \ell_n$, for large enough n, with $\ell_n = o(|\mathbf{b}_n|)$, and $|\mathbf{b}_n(\mathbf{X})| \sim \lfloor n/m^\star \rfloor$.*

The assumption restricts the blocking scheme in such a way so that the dependence between the blocks vanishes at a rate that is dominated by the rate that appears in the LDP that is sought. The blocks' sequence can thus be approximated by a sequence of independent random elements-with stationary distribution that resides in $M_1(U)$, enabling the identification of the rate function at work via the classical form of the Sanov's Theorem.

The following result is thus obtained:

Theorem 1. *Under Assumptions 1, and 2, the LDP in (9) holds for the empirical measures $(\mathbb{F}_{\mathbf{b},n})_{n\in\mathbb{N}^\star}$, with $I(\mu) := \mathrm{KL}(\mathbb{Q}_{m^\star}\|\mu)$, with $\mathbb{Q}_{m^\star}$ the stationary distribution of the random element $\frac{1}{m^\star}\sum_{i=1}^{m^\star}\mathbf{X}_{i-1}$, with $m^\star$ as in Assumption 2.*

Proof. Notice first that the boundedness of b along with Assumption 1 imply that the blocking process $\mathbf{b}(\mathbf{X})$ is also $2l$-dependent for $l \leq \max(m, \max_t b(t))$. Hence it is also α-mixing (see Bradley (1986) (10)), that conforms to Proposition 2 of Bryc and Dembo (1996) (11). Thereby, by Theorem 1 of Bryc and Dembo (1996) (11), and Theorem 1.12.4 of van der Vaart and Wellner (1996) (36), the LDP holds with good rate function given by $I(\mu) := \sup_{f\in B(U,\mathbb{R})}(\int_U f d\mu - \Xi(f))$, with $\Xi(f) := \lim_{n\to\infty}\frac{1}{|\mathbf{b}_n|}\ln \mathbb{E}(\exp(\sum_{j=1}^{|\mathbf{b}_n|} f(\mathbf{b}_n(\mathbf{X})_j)))$. Then, due to Assumptions 1 and 2,

$$\mathbb{E}(\exp(\sum_{j=1}^{|\mathbf{b}_n|} f(\mathbf{b}_n(\mathbf{X})_j))) = \mathbb{E}^{\lfloor n/m^\star \rfloor}(\exp(f(\frac{1}{m^\star}\sum_{i=1}^{m^\star}\mathbf{X}_{i-1})))\exp(A_n),$$

where $A_n \leq 2\ell_n \max_{u\in U} f(u)$. Due to Assumption 2, $\exp(A_n) = 1 + o(1)$. The result then follows as in the proof of the classical Sanov's Theorem (see Theorem 6.2.10 of Dembo and Zeitouni (1992) (15)). Specifically, via the Donsker-Varadhan variational representation of the Kullback-Liebler divergence. □

Remark 1. When $m^\star = m = 1$ the original Sanov's Theorem is retrieved even with blocking that becomes asymptotically negligible sufficiently fast.

Remark 2. The result supports blocking schemes that eventually recover m and adapt accordingly.

Remark 3. Using conditional arguments, it is be possible to prove an analogous result for m stochastic, yet independent of $\mathbf{X}$, that converges a.s. to some positive integer as

$n \to \infty$. The LDP rate and the good rate function would be identical to the current result. This could be relevant to the recovery of m via out of sample statistical inference.

Remark 4. A similar to the previous remark extension, can be established by allowing for the blocking scheme to be stochastic. The results would remain intact if the blocking scheme is independent of the sample, and it obeys the rate restrictions that appear in Assumption 2 in probability.

4 A test for stochastic bounding

U is henceforth considered a subset of $\mathbb{R}^d$. The random vector $\mathbf{X}_t$ represents some uncertain economic phenomenon occurring at time t; e.g. the one period stationary returns of d financial assets, and U is the support of its' joint distribution, say $\mathbb{P}$.

A prospect on $\mathbf{X}$ is any real linear function on $\mathbb{R}^d$. Hence it is represented by a unique element of the latter due to the Riesz representation theorem-see for example Aliprantis and Border (2006) (1). In the context of financial returns any prospect can be perceived as a financial portfolio constructed by the assets that participate in $\mathbf{X}$; the elements of its representing vector are the respective portfolio weights.

Alternative prospects are evaluated via expected utility, using utility functions $u : \mathbb{R} \to \mathbb{R}$ that are increasing, continuous, and concave. Instead of specifying a particular functional form, a uniformly bounded and convex set of continuous functions, say $\mathcal{U}$, is considered for the analysis.

The analysis involves two sets of prospects $\mathbf{\Lambda}$, $\boldsymbol{K} \subseteq \mathbb{R}^d$. They need not be disjoint, and they are both considered convex and compact. In what follows $\boldsymbol{\lambda}$, $\boldsymbol{\kappa}$ denote typical elements of $\mathbf{\Lambda}$ and $\boldsymbol{K}$ respectively.

The above enable the definition of a stochastic dominance relation on the sets of prospects, via $\mathcal{U}$: in the stationary framework considered, $\boldsymbol{\lambda}$ is said to dominate $\boldsymbol{\kappa}$ w.r.t. the utility class $\mathcal{U}$ iff $\mathbb{E}(u(\boldsymbol{\lambda}'\mathbf{X}_0)) \geq \mathbb{E}(u(\boldsymbol{\kappa}'\mathbf{X}_0))$, $\forall u \in \mathcal{U}$; i.e. $\boldsymbol{\lambda}$ is preferred over $\boldsymbol{\kappa}$-in the expected utility paradigm-by every utility in the considered class. The generality of the particular framework implies that it incorporates several of the widely known stochastic dominance relations; e.g. when $\mathcal{U}$ is the set of the Russell-Seo utilities-see Russell and Seo (1989) (33), and U is compact, the second order stochastic dominance relation is retrieved.

In this context, a stochastic bound of $\boldsymbol{K}$, is any prospect-not necessarily inside $\boldsymbol{K}$, that dominates every prospect in $\boldsymbol{K}$-see Arvanitis, Post and Topaloglou (2021) (5). It may be of interest to inquire whether $\mathbf{\Lambda}$ contains a stochastic bound of $\boldsymbol{K}$; the analysis in the aforementioned paper and the present framework imply that this is

the case iff
$$\xi(\mathbb{P}) := \sup_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}} \inf_{\boldsymbol{\kappa}\in\boldsymbol{K}, u\in\mathcal{U}} D(u, \boldsymbol{\lambda}, \boldsymbol{\kappa}, \mathbb{P}) \geq 0,$$
with $D(u, \boldsymbol{\lambda}, \boldsymbol{\kappa}, \mathbb{P})$ denoting the "moment differential"-or the expected utility difference $\mathbb{E}(u(\boldsymbol{\lambda}'\mathbf{X}_0)) - \mathbb{E}(u(\boldsymbol{\kappa}'\mathbf{X}_0))$; whenever $\mathcal{U}$ is the set of the Russell-Seo utilities-see Russell and Seo (1989) (33), and U is compact, the differentials can be represented via differences of appropriate integrals of the underlying cdfs.

In order to avoid (manageable-see Arvanitis and Post (2023) (8)) complications with potential infinities for the number of moment differentials that are zero at bounds when bounds exist, we hereafter assume that $\boldsymbol{\Lambda}$ is finite, and that $\boldsymbol{K}$ and $\mathcal{U}$ are polyhedral, so that the analysis can be restricted due to convexity on the finite set of their extreme points. In order for notational clutter to be avoided, the $\boldsymbol{K}$ and $\mathcal{U}$ symbols are also hereafter used for the respective finite sets of extreme points.

The latency of $\mathbb{P}$, along with the availability of the sample $\mathbf{X}_n$, imply that the above question on the existence of a bound, can be expressed via the hypothesis structure, $\mathbf{H}_0 : \xi(\mathbb{P}) \geq 0$ vs $\mathbf{H}_1 : \xi(\mathbb{P}) < 0$, which can be tested statistically.

Given the blocking scheme $\mathbf{b}$, a potential test statistic is the Block Empirical Likelihood Ratio (BELR) which is defined as follows: for $\mathbb{Q}_{\mathbf{b},n}$ denoting the solution to the optimization problem
$$\min_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}} \min_{\mathbb{Q}\in M_1(U)} \mathrm{KL}(\mathbb{F}_{\mathbf{b},n} \| \mathbb{Q}), \text{ s.t. } \inf_{u\in\mathcal{U}, \boldsymbol{\kappa}\in\boldsymbol{K}} D(u, \boldsymbol{\lambda}, \boldsymbol{\kappa}, \mathbb{Q}) \geq 0. \tag{2}$$
the statistic is $\mathrm{ELR}_{\mathbf{b},n} := 2\mathrm{KL}(\mathbb{F}_{\mathbf{b},n} \| \mathbb{Q}_{\mathbf{b},n})$.

The finiteness of $\boldsymbol{\Lambda}$ implies that the numerical aspects of the above optimization problem are not particularly complicated; for each $\boldsymbol{\lambda}$ the optimization of a profile likelihood-see for example Section 3.1 of Canay (2010) (12)-in order to obtain the optimal Lagrange multipliers vector is a problem of convex programming. Given its' solution for each $\boldsymbol{\lambda}$, the determination of $\mathbb{Q}_{\mathbf{b},n}$ is straightforward. In particular, the variational representation of the $\mathrm{ELR}_{\mathbf{b},n}$ statistic is:

$$2 \min_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}} \max_{\boldsymbol{\mu}\in\mathbb{R}^{\#(\mathcal{U}\times\boldsymbol{K})+}} \sum_{j=1}^{\lfloor n/m^\star \rfloor} \log(1 + \boldsymbol{\mu}'(u(\frac{1}{m^\star}\boldsymbol{\lambda}' \sum_{i=1}^{m^\star} \mathbf{X}_{j,i-1}) - u(\frac{1}{m^\star}\boldsymbol{\kappa}' \sum_{i=1}^{m^\star} \mathbf{X}_{j,i-1}))_{(u,\boldsymbol{\kappa})\in\mathcal{U}\times\boldsymbol{K}}), \tag{3}$$
where $\mathbf{X}_{j,i}$ denotes the i^{th} element of the j^{th} block.

A decision procedure could be then constructed as follows: given a significance level $\alpha \in (0, 1)$, let c_n be a random variable that converges in probability to a non-stochastic limit that is greater than or equal to the $1 - \alpha$ quantile of the null limiting distribution of $\mathrm{ELR}_{\mathbf{b},n}$.

The rejection regions designated by c_n is indicatively constructed by a modification of the conservative approach of Arvanitis and Post (2023) (7). Other procedures based on subsampling or block bootstrap resampling are also possible-see for example Canay (2010) (12) for an analogy in the iid setting.

Specifically, c_n is connected to the $1-\alpha$ quantile of a χ^2 distribution with degrees of freedom equal to the latent number $N(\mathbb{P}, \boldsymbol{\lambda})$ of non-trivial contacts of elements of the set of maximizers-see the following theorem-under the null of $\inf_{u\in\mathcal{U},\boldsymbol{\kappa}\in\boldsymbol{K}} D(u, \boldsymbol{\lambda}, \boldsymbol{\kappa}, \mathbb{P})$; there $N(\mathbb{P}, \boldsymbol{\lambda})$ is the cardinality of the "contact" set $\mathrm{CS}(\mathbb{P}, \boldsymbol{\lambda}) := \{(u, \boldsymbol{\lambda}, \boldsymbol{\kappa}) : D(u, \boldsymbol{\lambda}, \boldsymbol{\kappa}) = 0, \mathrm{Var}_{\mathbb{P}}(u(\boldsymbol{\lambda}'\mathbf{X}_0) - u(\boldsymbol{\kappa}'\mathbf{X}_0)) > 0\}$. This latent number of degrees of freedom can be asymptotically approximated from below via the number of empirical moment conditions that are approximately binding regarding the empirical bounds, i.e. via $N(\mathbb{Q}_{\mathbf{b},n}, \boldsymbol{\lambda}, m_n)$, which is the cardinality of $\left\{(u, \boldsymbol{\lambda}, \boldsymbol{\kappa}) : |D(u, \boldsymbol{\lambda}, \boldsymbol{\kappa}, \mathbb{Q}_{\boldsymbol{b},n})| \leq m_n,\ \mathrm{Var}_{\mathbb{Q}_{\mathbf{b},n}}(u(\boldsymbol{\lambda}'\mathbf{X}_0) - u(\boldsymbol{\kappa}'\mathbf{X}_0)) > 0\right\}$, where $\boldsymbol{\lambda} \in \arg\max_{\boldsymbol{\Lambda}} \inf_{u\in\mathcal{U},\boldsymbol{\kappa}\in\boldsymbol{K}} D(u, \boldsymbol{\lambda}, \boldsymbol{\kappa}, \mathbb{Q}_{\boldsymbol{b},n})$; $m_n > 0$ is a-potentially data dependent-slack variable that converges to zero at an appropriate rate. The relevant analysis in the proof of Theorem 4.2.2 of Arvanitis et al. (2021) (6) shows that (a) $D(u, \boldsymbol{\lambda}, \boldsymbol{\kappa}, \mathbb{Q}_{\boldsymbol{b},n}) = \frac{1}{\lfloor n/m^\star} \sum_{j=1}^{\lfloor n/m^\star \rfloor} \frac{(u(\frac{1}{m^\star}\boldsymbol{\lambda}' \sum_{i=1}^{m^\star} \mathbf{X}_{j,i-1}) - u(\frac{1}{m^\star}\boldsymbol{\kappa}' \sum_{i=1}^{m^\star} \mathbf{X}_{j,i-1}))_{(u,\boldsymbol{\kappa})\in\mathcal{U}\times\boldsymbol{K}})}{1+\boldsymbol{\mu}_n'(u(\frac{1}{m^\star}\boldsymbol{\lambda}' \sum_{i=1}^{m^\star} \mathbf{X}_{j,i-1}) - u(\frac{1}{m^\star}\boldsymbol{\kappa}' \sum_{i=1}^{m^\star} \mathbf{X}_{j,i-1}))_{(u,\boldsymbol{\kappa})\in\mathcal{U}\times\boldsymbol{K}})}$, where $\boldsymbol{\mu}_n$ is the optimal Lagrange multiplier that solves (3) and, (b) due to the lower semi-continuity of the arg min operator-see Rockafellar and Wetts (2009) (32)-the distributional cluster points of the $1-\alpha$ quantile of $\chi^2_{N(\mathbb{Q}_{\mathbf{b},n}, \boldsymbol{\lambda}, m_n)}$ are greater than or equal to the $1-\alpha$ quantile of the null limiting distribution of the test statistic. Given the present assumption framework it can be also seen that $N(\mathbb{P}, \boldsymbol{\lambda})$ can be also consistently estimated by $\{(u, \boldsymbol{\lambda}, \boldsymbol{\kappa}) : |D(u, \boldsymbol{\lambda}, \boldsymbol{\kappa}, \mathbb{P}_n)| \leq m_n,\ \mathrm{Var}_{\mathbb{P}_n}(u(\boldsymbol{\lambda}'\mathbf{X}_0) - u(\boldsymbol{\kappa}'\mathbf{X}_0)) > 0\}$. Then the decision rule is that $\mathbf{H}_0$ is rejected iff $\mathrm{ELR}_{\mathbf{b},n} > c_n$, where c_n is finally the $1-\alpha$ quantile of the $\chi^2_{N(\mathbb{Q}_{\mathbf{b},n}, m_n)}$ distribution, and the construction of the rejection region signals limiting conservatism.

The following result derives the limit theory of the testing procedure described above. There, for an arbitrary non-empty $B \subset \mathcal{U} \times K$, $\mathcal{LV}((\mathbb{P}, \lambda, \mathrm{B})$ denotes the matrix $\mathrm{Cov}_{\mathbb{P}}\left[\left(u\left(\boldsymbol{\lambda}'\mathbf{X}_0\right) - u\left(\boldsymbol{\kappa}'\mathbf{X}_0\right)\right)_{(u,\boldsymbol{\kappa})\in B}, \left(u^\star\left(\boldsymbol{\lambda}^{\star\prime}\mathbf{X}_0\right) - u^\star\left(\boldsymbol{\kappa}^{\star\prime}\mathbf{X}_0\right)\right)_{(u^\star,\boldsymbol{\kappa}^\star)\in B}\right]$. The test is-as expected-asymptotically conservative as well as consistent:

Theorem 2. *Suppose that Assumptions 1 and 2 hold with $m^\star = m$. Furthermore, (a) U is compact, (b) there exists some $\epsilon > 0$ such that,*

$$\inf_{\boldsymbol{\lambda} \text{ is a bound}} \lambda_{\min}\left(\mathcal{LV}(\mathbb{P}, \lambda, \mathrm{CS}(\mathbb{P}, \lambda))\right) > \epsilon,$$

where $\lambda_{\min}\left(\mathcal{LV}((\mathbb{P}, \lambda))\right)$ denotes the matrix's minimum eigenvalue, and, (c) the slacks satisfy $m_n \to 0$, while, $\sqrt{n}m_n \to +\infty$ almost surely.

Then, the following hold: (i)

$$\mathrm{ELR}_{\mathbf{b},n} \rightsquigarrow \begin{cases} \inf_{\boldsymbol{\lambda}\text{ is a bound}} \inf_{v\in\mathbb{R}_+^{\mathrm{N}(\mathbb{P},\lambda)}} \left(\mathbb{C}(\boldsymbol{\lambda})-v\right)' \mathcal{V}_{\mathbb{C}}^{-1}\left(\mathbb{C}(\boldsymbol{\lambda})-v\right), & \mathbf{H}_0 \wedge \sum_{\boldsymbol{\lambda}\text{ is a bound}} \mathrm{N}(\mathbb{P},\lambda)\neq 0 \\ 0, & \mathbf{H}_0 \wedge \sum_{\boldsymbol{\lambda}\text{ is a bound}} \mathrm{N}(\mathbb{P},\lambda)= 0\,, \\ +\infty, & \mathbf{H}_1 \end{cases} \tag{4}$$

where $\mathbb{C}(\boldsymbol{\lambda})$ is a zero-mean Gaussian vector with covariance matrix

$$\mathcal{V}_{\mathbb{C}} := \mathcal{L}\mathcal{V}+2\sum_{t=1}^{m}\mathbb{E}\left[\left(u\left(\boldsymbol{\lambda}'\mathbf{X}_0\right)-u\left(\boldsymbol{\kappa}'\mathbf{X}_0\right)\right)_{(u,\boldsymbol{\kappa})\in\mathrm{CS}(\mathbb{P},\lambda)}\left(u^\star\left(\boldsymbol{\lambda}^{\star\prime}\mathbf{X}_t\right)-u^\star\left(\boldsymbol{\kappa}^{\star\prime}\mathbf{X}_t\right)\right)_{(u^\star,\boldsymbol{\kappa}^\star)\in\mathrm{CS}(\mathbb{P},\lambda)}\right];$$

(ii) Under $\mathbf{H}_0 \wedge \sum_{\boldsymbol{\lambda}\text{ is a bound}} \mathrm{N}(\mathbb{P},\lambda)\neq 0$,

$$\limsup_{n\to\infty}\mathbb{P}\left(\mathrm{ELR}_{\mathbf{b},n}\geq c_n\right)\leq\alpha, \tag{5}$$

while under $\mathbf{H}_0 \wedge \sum_{\boldsymbol{\lambda}\text{ is a bound}} \mathrm{N}(\mathbb{P},\lambda)= 0$,

$$\lim_{n\to\infty}\mathbb{P}\left(\mathrm{ELR}_{\mathbf{b},n}\geq c_n\right)=0; \tag{6}$$

(iii) Finally, under $\mathbf{H}_1$,

$$\lim_{n\to\infty}\mathbb{P}\left(\mathrm{ELR}_{\mathbf{b},n}\geq c_n\right)=1. \tag{7}$$

Proof. (4) follows as in the proof of Theorem 4.2.1 of Arvanitis, Post, Poti and Karabati (2021) (6) combined-regarding the contact set calculus-with the proof of Proposition 4.1.2 of Arvanitis, Post and Topaloglou (2021) (5). Consider the case $\mathbf{H}_0 \wedge \sum_{\boldsymbol{\lambda}\text{ is a bound}} \mathrm{N}(\mathbb{P},\lambda)\neq 0$; uniformly w.r.t. the $(u,\boldsymbol{\kappa})\notin\mathrm{CS}(\mathbb{P},\boldsymbol{\lambda})$ and the set of stochastic bounds, it is found that, due to the definition of slacks and the Birkhoff's ULLN, $\mathbb{E}_{\mathbb{Q}_{\boldsymbol{b},n}}\left[u\left(\boldsymbol{\lambda}'\boldsymbol{X}_0\right)-u\left(\boldsymbol{\kappa}'\boldsymbol{X}_0\right)\right]>m_n$, eventually, almost surely. Using Skorokhod representations, we also have that, since $\sqrt{n}m_n$ diverges to infinity almost surely, uniformly w.r.t. the elements of the contact sets, $\left|\sqrt{n}\mathbb{E}_{\mathbb{Q}_{\boldsymbol{b},n}}\left[u\left(\boldsymbol{\lambda}'\boldsymbol{X}_0\right)-u\left(\boldsymbol{\kappa}'\boldsymbol{X}_0\right)\right]\right|\leq\sqrt{n}m_n$, eventually, almost surely. The previous imply that $\mathrm{N}(\mathbb{Q}_{\boldsymbol{b},n},\boldsymbol{\lambda},m_n)\rightsquigarrow\mathrm{N}(\mathbb{P},\boldsymbol{\lambda}$, uniformly w.r.t. the set of bounds, jointly with $\mathrm{ELR}_{\mathbf{b},n}$, and thereby we obtain that

$$\begin{gathered}\mathrm{ELR}_{\mathbf{b},n}\rightsquigarrow\inf_{\boldsymbol{\lambda}\text{ is a bound}}\inf_{v\in\mathbb{R}_+^{\mathrm{N}(\mathbb{P},\boldsymbol{\lambda})}}\left(\mathbb{C}-v\right)'\mathcal{V}_{\mathbb{C}}^{-1}\left(\mathbb{C}-v\right)\\ =\inf_{\boldsymbol{\lambda}\text{ is a bound}}\inf_{v\in\mathbb{R}_+^{\mathrm{N}(\mathbb{P},\boldsymbol{\lambda})}}\left[\begin{array}{c}\mathbb{C}'\mathcal{V}_{\mathbb{C}}^{-1}\mathbb{C}\\ -\inf_{v\in C^o}\left(\mathbb{C}-v\right)'\mathcal{V}_{\mathbb{C}}^{-1}\left(\mathbb{C}-v\right)\end{array}\right]\leq\inf_{\boldsymbol{\lambda}\text{ is a bound}}\mathbb{C}'\mathcal{V}_{\mathbb{C}}^{-1}\mathbb{C},\end{gathered}$$

due to Proposition 3.4.1 of Silvapulle and Sen (2005) (34), where C^o denotes the polar cone of $\mathbb{R}_+^{\mathrm{N}(\mathbb{P},\boldsymbol{\lambda})}$. Then the Portmanteau Theorem establishes (5). For the case

$\mathbf{H}_0 \wedge \sum_{\boldsymbol{\lambda} \text{ is a bound}} \mathrm{N}(\mathbb{P}, \lambda) = 0$, (4) implies that $\mathrm{ELR}_{\mathbf{b},n}$ is eventually zero w.h.p., hence (6) follows. Finally, when the null hypothesis does not hold, then the implication of the previous is that $\mathrm{ELR}_{\mathbf{b},n}$ diverges to $+\infty$, while the $1-\alpha$ quantile of $\chi^2_{N(\mathbb{Q}_{\mathbf{b},n},\boldsymbol{\lambda},m_n)}$ is almost surely bounded; (7) then follows. $\square$

The minimum eigenvalue condition holds whenever the random vector $(u(\boldsymbol{\lambda}'\boldsymbol{X}) - u(\boldsymbol{\kappa}'\boldsymbol{X}))_{\mathrm{CS}(\mathbb{P},\boldsymbol{\lambda})}$ consists of linearly independent random variables. The restriction of the asymptotic behavior of slacks is standard in the literature; see Andrews and Soares (2010) (2) and the relevant references therein.

The test is asymptotically conservative, even in cases where the associated contact sets of the bounds are empty. This is due to the way that the rejection region is constructed. Conservativeness implies potential poor power properties on the boundary of the null hypothesis. Whenever no-non trivial contacts exist conservativenss is maximal. The test is also consistent; the probability of type II error converges to zero for fixed alternatives.

The aforementioned conservativeness and consistency would also hold whenever the test was performed utilizing blocking schemes for which $m^\star < m$. The difference in the limit theory would lie in the form of the Gaussian process involved; its covariance kernel would not equal $\mathcal{V}_{\mathbb{C}}$ and conservativeness would then be more severe.

5 Fixed critical value Neyman-Pearson optimality

The considerations below follow closely Paragraph 3 of Canay (2010) (12). The notion of the generalized Neyman-Pearson optimality is now considered for the BELR test based on the statistic established in the previous section, yet performed via a decision procedure based on a fixed critical value. Specifically, for some $\eta > 0$, the testing procedure $r_n = \mathbf{1}(\mathrm{ELR}_{\mathbf{b},n} > \eta)$, where the null hypothesis is rejected iff $r_n = 1$ is considered. The procedure can then be represented by a measurable (w.r.t. the weak topology) partition of $M_1(U)$, $(P_{0,n}, P^{\mathrm{c}}_{0,n})$, where $P_{0,n} := \left\{\mathbb{Q} \in M_1(U) : \inf_{\mathbb{G}\in \boldsymbol{P}_0(\mathbb{Q}} \mathrm{KL}(\mathbb{Q}\|\mathbb{G}) \leq \eta\right\}$, and $\mathbb{G} \in \boldsymbol{P}_0(\mathbb{Q}) := \{\mathbb{G} \in M_1(U) : \xi(\mathbb{G}) \geq 0, \mathbb{G} \gg \mathbb{Q}, \mathbb{Q} \gg \mathbb{G}\}$; the null hypothesis is rejected iff $\mathbb{F}_{\mathbf{b},n} \in P^{\mathrm{c}}_{0,n}$.

The following definition of the null parameter space, in the spirit of Definition 3.1 of Canay (2010) (12) is considered:

Definition 1. For arbitrary $\delta > 0$, the null distributions space $\mathbf{P}_{0,\delta}$, is the subset of $M_1(u)$ for which for any element $\mathbb{Q}$:

1. $\xi(\mathbb{Q}) \geq 0$,
2. $\inf_{\boldsymbol{\lambda}} |\left(\mathcal{LV}((\mathbb{Q}, \lambda, \mathcal{U} \times \mathrm{K}))\right)_{u\in\mathcal{U},\boldsymbol{\kappa}\in\boldsymbol{K}}| \geq \delta$,

$\mathbf{P}_{0,\delta}$ is then a set of null distributions for which-due to the second part of the definition-every bound has non empty sets of non-trivial contacts. This is analogous to Assumption 3.5 of Kitamura et al. (2012) (22). As in Canay (2010) (12) such degeneracies are avoided by conditions on the associated covariance matrices; those here assume the form of restrictions on the set of utilities employed.

The analysis is also complemented by the following compactness assumption:

Assumption 3. *U is compact.*

This directly implies that $M_1(U)$ is also compact-see Paragraph 15.3 of Alipratnis and Border (2006) (1). The assumption can be avoided if it is assumed that the maintained hypothesis is a compact-w.r.t. the weak topology-subset of $M_1(U)$.

The following generalized Neyman-Pearson optimality result is then obtained-for the Levy metric d_L employed there, that metrizes the weak topology on $M_1(U)$, see for example Appendix D of Dembo and Zeitouni (2009) (15):

Theorem 3. *Suppose that Assumptions 1, 2 and 3 hold. Then for the BELR statistic defined above we have that there exists some $\eta(\delta) > 0$ such that for any $0 < \eta \leq \eta(\delta)$:*

1. $\sup_{\mathbb{Q}\in\mathbf{P}_{0,\delta}} \limsup_{n\to\infty} \frac{1}{\lfloor n/m^\star \rfloor} \mathbb{Q}(\mathbb{F}_{\mathbf{b},n} \in P^{\mathrm{c}}_{0,n}) \leq \eta$,
2. *for any other test based on $\mathbb{F}_{\mathbf{b},n}$ that induces some measurable partition $(G_{0,n}, G^{\mathrm{c}}_{0,n})$ on $M_1(U)$, that satisfies $\sup_{\mathbb{Q}\in\mathbf{P}_{0,\delta}} \limsup_{n\to\infty} \frac{1}{\lfloor n/m^\star \rfloor} \mathbb{Q}(\mathbb{F}_{\mathbf{b},n} \in G^{\mathrm{c},\epsilon}_{0,n}) \leq \eta$, with $G^{\mathrm{c},\epsilon}_{0,n} := \{\mu \in M_1(U), \inf_{v\in G^{\mathrm{c}}} d_L(\mu, v) \leq \epsilon\}$, and d_L denotes the Levy metric, then*

$$\limsup_{n\to\infty} \frac{1}{\lfloor n/m^\star \rfloor} \mathbb{Q}_1(\mathbb{F}_{\mathbf{b},n} \in G_{0,n}) \geq \limsup_{n\to\infty} \frac{1}{\lfloor n/m^\star \rfloor} \mathbb{Q}_1(\mathbb{F}_{\mathbf{b},n} \in P_{0,n}),$$

for any $\mathbb{Q}_1 \in \{\mathbb{Q} \in M_1(U) : \inf_{\mathbf{P}_0} d_L(\mathbb{Q}, \mathbb{P}) \geq \eta/2\}$, with $\mathbf{P}_0 := \{\mathbb{P} \in M_1(U) : \xi(\mathbb{P}) \geq 0\}$.

Proof. The result follows as in the proof of Theorem 3.2 of Canay (2000) (12), due to the finiteness of $\mathbf{\Lambda}$. Specifically the only modifications needed require that (in the notation there) relation (A.10) is modified so that it is considered w.r.t. the supremum of the associated parameter θ-which is in our case $\boldsymbol{\lambda}$, relations (A.11)-(A.12) are similarly modified so that they are considered w.r.t. the infimum of θ, and Theorem 1 is invoked in place of the original Sanov's Theorem. □

The results establish that the BELR test considered in this section, controls uniformly over the occasional null distribution space, the rate at which the probability of type I error vanishes. Furthermore, the rate at which the probability of type II error vanishes is maximal w.r.t. to any test of similar type I error rate, with a test statistic that depends on $\mathbf{X}_n$ only via $\mathbb{F}_{\mathbf{b},n}$), w.r.t. to any alternative distribution that is sufficiently far-in the Levy metric sense-from the boundary of the null hypothesis.

The above holds true for the-generally infeasible-tests based on the fixed critical values obtained from the $1-\alpha$ quantile of the latent distribution of $\inf_{\boldsymbol{\lambda} \text{ is a bound}} \inf_{v\in\mathbb{R}_+^{N(\mathbb{P},\boldsymbol{\lambda})}} (\mathbb{C}-v)' \mathcal{V}_{\mathbb{C}}^{-1} (\mathbb{C}-v)$ or of $\inf_{\boldsymbol{\lambda} \text{ is a bound}} \mathbb{C}'\mathcal{V}_{\mathbb{C}}^{-1}\mathbb{C}$, for $\mathbb{C}$ the Gaussian process that is present in Theorem 2. The first would also be asymptotically exact.

6 Monte Carlo Experiment

In this section Monte Carlo experiments are designed and performed to evaluate actual finite sample size and power of the proposed Block Empirical Likelihood Ratio (BELR) Bounding test in finite samples. The size and power of the proposed tests is also compared, with the Bounding testing procedure of Arvanitis, Post and Topaloglou (2021) (5). Three financial assets are considered in a time series context. It is assumed assume that their vector returns' process behaves like a stationary vector MA(1) process of order one $\boldsymbol{X}_t = \boldsymbol{A} + \boldsymbol{B}\boldsymbol{\nu}_{t-1} + \boldsymbol{\nu}_t$, where $\boldsymbol{\nu}_t \sim N(0, \boldsymbol{\Sigma})$ and $\boldsymbol{B}$ is set equal to the relevant identity matrix. The process is 2-dependent, hence conformable to the general framework discussed above. The marginal distribution of $\boldsymbol{X}$ is zero mean Gaussian, hence the $\boldsymbol{A}$ vector is set equal to zero, and the covariance matrix $\boldsymbol{\Sigma}$ is set equal to the empirical variance covariance matrix of the actual data of three financial assets, namely the S&P500 Index, the Russel 2000 stock index, and the Barcklays bond Index; it is noted that the latter has the minimal empirical variance. The actual data set used for $\boldsymbol{\Sigma}$ spans the period from January 1990 to December 2020.

6.1 Design

The aforementioned VMA DGP is used to draw realizations of the three asset returns. We generate $R = 500$ original samples with size $n = 200$, 500 and 1000. These samples are then used to evaluate the actual size and power of the Bounding tests. For the BEL Bounding tests, for each original sample we generate blocks (non-overlapping and independent) of two return observations and calculate the average return for each block. These average returns are then used to estimate the size and power of the BEL Bounding test.

Two sets of prospects are considered, namely $\boldsymbol{\Lambda}$, $\boldsymbol{K}$, and $\boldsymbol{\lambda}$, $\boldsymbol{\kappa}$ are portfolios of $\boldsymbol{\Lambda}$ and $\boldsymbol{K}$ respectively.

Moreover, it is assumed that $\mathcal{U}$ is a finitely discretized set of the Russell-Seo utilities-see Russell and Seo (1989) (33), and thus U is compact as required by Assumption 3. Thus, the utility functions can be expressed as positive linear combinations of elementary utility functions with different kink points, for a finite number of kink points (or thresholds). It is noted that the set of thresholds used does not contain the smallest sample minimum of the portfolios' returns since that would invalidate Theorem 2.(b). The design thus fully supports the the aforementioned theoretical assumption framework.

For the Arvanitis, Post and Topaloglou (2021) (5) Bounding test, the decision to reject the null of the existence of a stochastic bound, is obtained by via a computationally intense sub-sampling procedure as described in section 4.2 of their paper.

For the BELR Bounding test, for the rejection region the $1-\alpha$ quantile of a χ^2 distribution is used, with degrees of freedom equal to the the cardinality of $\{(u, \boldsymbol{\lambda}, \boldsymbol{\kappa}) : |D(u, \boldsymbol{\lambda}, \boldsymbol{\kappa}, \mathbb{P}_n)| \leq m_n,\ \mathrm{Var}_{\mathbb{P}_n}(u(\boldsymbol{\lambda}'\mathbf{X}_0) - u(\boldsymbol{\kappa}'\mathbf{X}_0)) > 0\}$ which as noted before is a consistent estimator of $N(\mathbb{P}, \boldsymbol{\lambda})$. There the slacks used are set equal to $m_n = \frac{0{,}1}{\ln(n)}$, which is also conformable to their theoretically required asymptotic behavior.

6.1.1 Size

To evaluate the actual size, we test whether $\boldsymbol{\Lambda}$ contains a stochastic bound of $\boldsymbol{K}$, when both sets $\boldsymbol{\Lambda}$ and $\boldsymbol{K}$ are equal and comprised by all the three assets. In this case a stochastic bound exists, due to the existence of a minimal variance and the assumption of zero means.

6.1.2 Power

To evaluate the actual power, we test whether $\boldsymbol{\Lambda}$ contains a stochastic bound of $\boldsymbol{K}$, when the set $\boldsymbol{\Lambda}$ contains the two stock indices which have higher variance than the third bond index, while the set $\boldsymbol{K}$ contains all the three assets. In this case a stochastic bound does not exist, since the set $\boldsymbol{\Lambda}$ contains the two assets with the higher variance.

6.2 Computational Issues

The optimization problems of the Arvanitis, Post and Topaloglou (2021) (5) Bounding test, as well as the convex optimization of the BELR test statistic, are solved using the General Algebraic Modeling System (GAMS), which is a high-level modeling system for mathematical programming and optimization. This language calls special solvers (Gurobi and Conopt) that are specialized in linear and convex programs. The Matlab code (where the simulations run) calls the specific GAMS program, which calls the appropriate solver to solve each optimization. The optimizations are performed on a number of computers (with i7 processors, 16 core, 3.2 GHz Power, 128Gb of RAM). The almost exponential increase in solution time with the increasing number of sample size is noted.

6.3 Results

The Monte Carlo results are presented in Table 1. It is observed that both tests perform well with respect to both size and power, a finding conformable to Theorem 2. It is also observe a minor improvement in the power of BEL Bounding test over the Arvanitis, Post and Topaloglou (2021) Bounding test. This is also reminiscent of the spirit of the power properties results in Theorem 3, even though the test performed does not use a fixed and data independent rejection region.

Tab. 1: Monte Carlo Experiment. The experiment is based on a problem with N=3 assets that follow an MA(1) process, and n=300, 500, 1000 time series observations. Actual size and power of the Bounding test statistic of Arvanitis, Post and Topaloglou (2021), and the Block Empirical Likelihood (BEL) test statistic.

Sample size n	200	500	1000
Size			
Bounding			
$\hat{R}P_1(5\%)$	7.63%	6.54%	4.88%
BELR Bounding			
$\hat{R}P_1(5\%)$	7.46%	6.62%	4.92%
Power			
Bounding			
$\hat{R}P_2(5\%)$	88.22%	90.76%	93.19%
BELR Bounding			
$\hat{R}P_2(5\%)$	89.33%	91.75%	94.16%

7 Large deviations confidence regions for bounds

The LDP asymptotics-in the spirit of Arcones (2006) (3)-of the empirical bound maximum BEL estimator is now considered-the latter is naturally:

$$\boldsymbol{\lambda}_n \in \arg\min_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}} \min_{\mathbb{Q}\in M_1(U)} \mathrm{KL}(\mathbb{F}_{\mathbf{b},n}\|\mathbb{Q}), \text{ s.t. } \inf_{u\in\mathcal{U},\boldsymbol{\kappa}\in\boldsymbol{K}} D(u,\boldsymbol{\lambda},\boldsymbol{\kappa},\mathbb{Q}) \geq 0. \tag{8}$$

Suppose that the null hypothesis is true, and the following identification condition holds:

Assumption 4. $\boldsymbol{\lambda}_0 := \arg\max_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}} \inf_{u\in\mathcal{U},\boldsymbol{\kappa}\in\boldsymbol{K}} D(u,\boldsymbol{\lambda},\boldsymbol{\kappa},\mathbb{P})$ *is a singleton.*

The identification condition is quite strong, yet in enables the invocation of the contraction principle in the Large Deviations Theory. When $\mathcal{U}$ and/or $\boldsymbol{K}$ are large, whence the existence of bounds becomes less plausible, it may not be very restrictive.

Then, Assumptions 1, 2, 3, 4, the Sanov's-type Theorem 9, the contraction principle (see for example Lemma 2.1.4 in Deuschel and Stroock (1989) (16)), the lower semi-continuity property of the Kullback-Liebler divergence, the finiteness of $\boldsymbol{\Lambda}$ and $\mathcal{U}\times\boldsymbol{K}$, Corollary 4.7 of Rio (2017) (31), and Theorem 7.11 of Rockafellar and Wetts (2009) (32) and Theorem 3.4 of Molchanov (2006) (28) imply the following LDP:

Theorem 4. *Under Assumptions 1, 2, 3, 4, and for any* $A\subseteq\boldsymbol{\Lambda}-\boldsymbol{\lambda}_0$,

$$\lim_{n\to\infty}\frac{1}{\lfloor n/m^\star\rfloor}\ln\mathbb{P}(\boldsymbol{\lambda}_n-\boldsymbol{\lambda}_0\in A) = -\inf_{x\in A} I^\star(x), \tag{9}$$

where $I^\star:\boldsymbol{\Lambda}-\boldsymbol{\lambda}_0\to\mathbb{R}$ *is a well defined convex good rate function, defined by:*

$$I^\star(x) := \inf_{\mu}\left\{\mathrm{KL}(\mathbb{Q}_{m^\star}\|\mu) : \arg_{\boldsymbol{\lambda}} \inf_{u\in\mathcal{U},\boldsymbol{\kappa}\in\boldsymbol{K}} D(u,\boldsymbol{\lambda},\boldsymbol{\kappa},\mu) - \boldsymbol{\lambda_0} = x\right\}.$$

Then using the analysis in Section 4 of Arcones (2006) (3), it is easy to see that, for $\beta\in(0,+\infty)$, the region $C_{\boldsymbol{\lambda}_n}(\beta) := \{\boldsymbol{\lambda}\in\boldsymbol{\Lambda} : I_\star(\boldsymbol{\lambda}_n)<\beta\}$, where $I_\star:\boldsymbol{\Lambda}\to\mathbb{R}$ is a well defined convex good rate function, defined by:

$$I_\star(x) := \inf_{\mu}\left\{\mathrm{KL}(\mathbb{Q}_{m^\star}\|\mu) : \inf_{u\in\mathcal{U},\boldsymbol{\kappa}\in\boldsymbol{K}} D(u,\boldsymbol{\lambda},\boldsymbol{\kappa},\mu) = \inf_{u\in\mathcal{U},\boldsymbol{\kappa}\in\boldsymbol{K}} D(u,x,\boldsymbol{\kappa},\mu)\right\},$$

is a β-confidence region for the estimator, in the sense that $\limsup_{n\to\infty}\mathbb{P}(\boldsymbol{\lambda}_n\notin C_{\boldsymbol{\lambda}_n}(\beta))\leq-\beta$. A consistent estimator for the latent region w.r.t. the upper topology on the space of non-empty subsets of $\mathbf{\Lambda}$-see for example Proposition 19.1 and the Appendix A.1 of Arvanitis (2013) (4) is

$$\bar{C}_{\boldsymbol{\lambda}_n}(\beta) := \left\{ \boldsymbol{\lambda} \in \mathbf{\Lambda} : \inf_{\mu} \left\{ \mathrm{KL}(\mathbb{F}_{\boldsymbol{b},n} \| \mu) : \inf_{u \in \mathcal{U}, \boldsymbol{\kappa} \in \boldsymbol{K}} D(u, \boldsymbol{\lambda}, \boldsymbol{\kappa}, \mu) = \inf_{u \in \mathcal{U}, \boldsymbol{\kappa} \in \boldsymbol{K}} D(u, \boldsymbol{\lambda}_n, \boldsymbol{\kappa}, \mathbb{Q}_{\mathbf{b},n}) \right\} < \beta \right\}.$$

Upper consistency follows from Corollary 4.7 of Rio (2017) (31), Theorem 7.11 of Rockafellar and Wetts (2009) (32), and Theorem 3.4 of Molchanov (2006) (28), as well as from the semi-continuity properties of the Kullback-Liebler divergence-see for example Lemma 1.4.3 of Dupuis and Ellis (1997) (17).

8 Discussion

Several extensions of the results above could be the subject of further research. A very important one is the extension of Theorem 9 to strong mixing processes with appropriate mixing rates. Given the results appearing in Bryc and Dembo (1996) (11), where a faster than geometric strong mixing rate suffices for an LDP property to hold, this extension faces a twofold inquiry: (a.) whether the former rate is sufficient, and (b.) whether standard blocking schemes-like the maximally overlapping one-would imply the identification of the good rate function as an appropriate relative entropy.

The extension of the optimality results presented in Theorem 3 in several instances missed by the present framework could be also of interest. For example the extension of the result regarding the fixed critical value BELR test in cases where the underlying contact sets are potentially infinite-this could be true for tests employing the alternative hypothesis (see Arvanitis and Post (2023) (8)) could be useful.

The extension of the results to notions of optimality that are more relevant to tests that utilize data-dependent rejection regions-see for example the Hodges-Lehmann optimality of Canay and Otsu (2012) (13) is also interest given the complexity of the tests employed in Stochastic Dominance. Specifically, any testing procedure involving a critical value q_n, that may be data dependent, is termed Hodges-Lehmann optimal at a distribution, say $\mathbb{P}_1$ in the alternative hypothesis, iff (a) the test has asymptotic size α bounded above by α for any distribution in the null hypothesis, and (b) the test has a limiting power at $\mathbb{P}_1$ greater than or equal to any other test of asymptotic size α.

It is pointed out that a version of the general Theorem 2.1 of Canay and Otsu (2012) (13), that identifies sufficient conditions for Hodges-Lehman optimality in iid frameworks, is readily extended in our framework of m-dependence and blocking. Specifically:

Theorem 5. *Suppose that Assumptions 1, 2 and 3 hold. If the (a). the null hypothesis is comprised of distributions at which the limiting statistic equals zero, and (b). the test statistic is lower semi-continuous at every distribution in the maintained hypothesis, then the test is Hodges-Lehman optimal at any distribution in the alternative hypothesis that is mutually absolutely continuous with at least one distribution inside the null hypothesis.*

Proof. The result follows exactly as in the proof of Theorem 2.1 of Canay and Otsu (2012) (13), with a minor modification; Theorem 1 is invoked in place of the original Sanov's Theorem. □

Under the aforementioned assumptions, and if moreover Theorem 2.b holds and $\sum_{\boldsymbol{\lambda} \text{ is a bound}} \mathrm{N}(\mathbb{P}, \lambda) \neq 0$, then the BELR test has asymptotic level α due to the result in 2.2. Given the lower-semicontinuity properties of the Kullback-Liebler divergence (see again Lemma 2.1.4 in Deuschel and Stroock (1989) (16)), if the null hypothesis is a singleton, then a modification of the proof of Theorem 3.1.(ii) of Canay and Otsu (2012) (13) would imply that the BELR test satisfies Theorem 5, and therefore the particular testing procedure is Hodges-Lehman optimal.

Finally, the consideration of LDP properties for the associated estimators of bounds without strong identification conditions, and/or Fell consistent estimators (see again Proposition 19.1 and the Appendix A.1 of Arvanitis (2013) (4)) for the associated confidence region estimator could be also useful.

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