Precautionary Savings and Debt Stabilization in the Euro Area

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Abstract:
The interactions between debt stabilization and precautionary savings in the Euro Area are analyzed by using a square-root Ornstein-Uhlenbeck stochastic process with mean-reverting income to capture the dynamics and constraints inflicted upon households by the crisis. Austerity programs are modeled as an additional Poisson process to reflect the abrupt implementation and their large effect on households' uncertainty and precautionary savings. A closed-form approximation of the savings function is derived and then it is used to analyze the recessionary impact of austerity policies currently implemented in the Euro Area. A more gradual consolidation is found to be relaxing the precautionary motive, thus resulting in less depressed consumption and output.

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1.Introduction

In the aftermath of the 2008 global crisis, several advanced economies faced two threats: on the one hand aggregate demand was collapsing and recession was arriving at an alarming pace; on the other hand, public debt was accumulating fast hence disabling Governments from responding to the slump by invoking traditional demand management tools. The reason behind Governments' incapacitation was not as much ideological as it was practical: the room for fiscal expansion was very limited because public debt was already escalating fast as a result of either massive bailouts to write off banking failures or their own previous excesses in public finances. Though the need to tackle both recession and public indebtedness was widely recognized, sharp divisions emerged regarding prioritization between the two and the intensity of policy tools that should be used.

For example, Akerlof and Shiller (2009, p18) advised from early on that - in addressing the crisis - the Government should not just adopt a monetary and fiscal policy to restore full employment, but at the same time it should aim to provide credit at a scale that is capable to restore the badly tarnished confidence in the economy. At the other end, Reinhart and Rogoff (2010) argued that a public debt exceeding a threshold around 80% of GDP is detrimental to growth and Governments should rather engage in consolidation programs to speed up their exit from recession. The idea was fervently taken up by economists in the European Central Bank (ECB) who argued that in such a case Governments should be "… in favour of swiftly implementing ambitious strategies for debt reduction"; e.g. see Checherita and Rother (2010).

But in practice, a 'swift consolidation' can never be too swift as to bypass households' anxiety about their future incomes, thus the severity of contractionary policies turned to be counter-productive. One way to assess the degree of uncertainty prevailing in a group of economies is to compare the volatility of consumption over that of income. In normal times, the ratio is expected to be low as rational households disregard temporary fluctuations and choose their consumption portfolio according to a smooth profile envisaged in the seminal paper by Muth (1960). The ratio may well be kept low even in the presence of substantial income shocks if households can rely on accumulated endowments or easy credit lines to overcome the current shortage.

In Fig. 1, the excess volatility of consumption is examined in connection with the intensity of fiscal adjustment that took place in the main Euro Area economies.[2] The extent of adjustment is simply measured by the improvement in primary balances between 2009 and 2013.

[Figure 1, here]

By simple inspection, it is evident that there are two different constellations: in economies with a relatively low fiscal consolidation, volatility of consumption is small and keeps the same degree of 'smoothing' as before the crisis. In contrast, volatility rises sharply for the economies of Greece, Ireland, Portugal and Spain that were subjected in intensive fiscal consolidation as a condition for being bailed-out by the so-called 'troika' of International Monetary Fund (IMF), the ECB and the European Commission (EC), witnessed a further deterioration of aggregate activity and uncertainty multiplied, rather than being subdued. This reveals a dual pattern of post-crisis reactions in the Euro Area economies that will be investigated throughout.

The issue of consumption volatility in the presence of unusually persisting shocks has long been discussed in the literature. In contrast to the 'smoothing pattern' of the permanent income hypothesis, Hall and Mishkin (1982) noted the *"excess sensitivity"* of consumption to changes in transitory income, while Chah *et al* (1995) mentioned that even forward-looking households faced with predictable changes in income may experience large variations in consumption. If income cuts are combined with liquidity constraints they may trigger wild fluctuations in consumption and output. The reason is that liquidity constraints unable several households to follow an intertemporal optimization process and force them to consume only out of current income.

Hubbard and Judd (1986) examined a two sector model with and without liquidity constraints and find that the aggregate income multiplier increases and makes the effects of tax cuts to be more pronounced than envisaged in the unconstrained case. According to Zeldes (1989), this happens because they make "…the marginal propensity to consume out of transitory changes in income to be much larger for households holding few assets relative to future income than for the rest of the population"; In the same spirit, Deaton

[2] The analysis proceeds with the twelve economies of the Euro Area that took part in the Economic and Monetary Union from early on. This avoids idiosyncratic aspects related with the adjustment period for the countries that joined the common currency area at a later stage.

(1991) argued that the propensity to consume rises under uncertainty because "…the combination of the persistence of the random walk and the binding liquidity constraints precludes the accumulation of assets."

All such warnings were manifestly ignored; for a critical account see Christodoulakis (2013). The fear that a rapid consolidation might entrap Euro Area economies further into deflation was conveniently waived by the assumption of low fiscal multipliers, according to which a fiscal correction by 1% of GDP would reduce output by only 0.50% and raise unemployment by a mere 0.30%; see IMF WEO (2010, p94). Moreover, as the growth potential would be unleashed by market reforms and swift public debt reduction, any recession was expected to bottom-out in late 2010 and rebound afterwards; (*ibid*, p165).

A timely warning on the unusual recession came from the rise in savings. Kerdrain et al (2010) found that household saving rates in the developed countries rose from their pre-crisis levels on account of tighter credit conditions and rising labour market uncertainty Similarly, the increases in the savings ratio in the Euro Area were mainly due to precautionary savings accumulated to count for future income uncertainties; see the special report EC (2011, p 25).

The reason for such a rise is that households prudentially engage in extra savings to face unforeseen circumstances in the future, and –by doing so- they deviate from the permanent income hypothesis as extensive research has demonstrated. For example, (1993) developed a mean-reverting process with noise and found that the motivation for precautionary savings increases with the persistence of shocks and the degree of uncertainty. In a similar vein, Smith (1998) adopted an overlapping- generations model in which income follows a simple Brownian motion without drift and showed that precautionary savings rise with uncertainty. . In a more general framework, Carroll and Kimball (1996) find that either liquidity constraints and/or precautionary motives make the consumption function to be concave to income and this creates asymmetric effects in a crisis as it implies that the marginal propensity to consume falls with a rising income. By analogy, it is increasing by income cuts, thus making the fiscal multiplier larger and the austerity effects more pronounced.

In the aftermath of the crisis, the rise in public debt in the Euro Area economies was a major source of uncertainty as likely to lead to future tax rises. Households reacted by raising precautionary savings, thus depressing consumption and further fuelling recession. Fig. 2 displays quarterly data of the savings ratio for the Euro Area and the quarterly changes in the public debt to GDP ratio. A positively correlated pattern is easily seen, especially in the years following the crisis in 2008.[3]

[Figure 2, here]

Though most Governments in the Euro Area did adopt fiscal consolidation plans to stave off rising indebtedness, their effect was not unambiguous in reducing uncertainty. As remarked in EC (2011, p26), "the risk of an increase in the households' savings rate with depressing consequences for private consumption remained significant." The reason was that most austerity programs included flat expenditure cuts and tax hikes, and this led lower-income households to suffer disproportionately more than upper-income households. With a larger marginal propensity to consume out of income, crisis-stricken households cut their consumption more than predicted by pre-crisis models, vindicating the concavity effect. With recession deepening in the Euro Area, another channel of increased precautionary savings and economic slowdown came from the postponement of reforms. Instead of a reform bonanza that would wake up the dormant forces of growth as envisaged in the IMF report (WEO, 2010), the fear of large income losses led to social protests and made reforms all the more difficult. According to Giavazzi and McMahon (2008), reform postponement fuels uncertainty about the conduct of policy in the future, leading to more precautionary savings and further recession.

The aim of the present paper is to analyze the interactions between debt stabilization and precautionary savings by using a stochastic process with mean-reverting income so as to better capture the dynamics and the constraints inflicted upon by the crisis. Austerity programs that were designed for public debt stabilization are modeled as an additional stochastic process to reflect their abrupt implementation as well as the effect on uncertainty and precautionary savings. A closed-form approximation of the precautionary savings

[3] Using country-specific data, it is also found that the rise in savings ratios during 2009-2010 is positively correlated with the deterioration of public debt positions; details available from the author.

function is obtained, and this helps to analyze the impact of austerity policies and found to be in accordance with actual developments in the Euro Area.

The rest of the paper is organized as follows: Section 2 describes a model of optimizing households with income following a stochastic mean-reverting process. Section 3 derives a closed-form solution for precautionary savings in the presence of austerity measures and a number of properties are outlined. Some key stylized facts concerning the impact of austerity in the Euro Area economies are analyzed in Section 4 and explained by appealing to the behavior of precautionary savings. It is also shown that precautionary savings could have been reduced by an optimal calculation of austerity measures. This is taken to imply that a more gradual adjustment could have had exerted a milder effect on recession as summarized by the conclusions in Section 5. Appendix A provides details on the solution and the propositions. To calibrate the model parameters, a mean reverting process of per capita GDP is estimated in Appendix B and data sources are outlined.

2.Austerity and income constraints

In this section the effect of austerity policies on consumption is examined in a framework that is suitable to reflect the post-crisis realities. To capture the unusual uncertainty and losses that incurred on incomes, households are assumed to optimize an intertemporal utility function with an income subject to stochastic shocks and impaired by credit constraints. A number of options are examined below.

Modeling income as a Brownian process

The evolution of income is represented by a Brownian process with a systematic trend that affects "permanent income" and a random part that represents "transitory" changes and allows volatility to grow over time. A general representation is given by:

$$dy = (\delta - \theta y) \cdot dt + [\sigma y^m] \cdot dz \quad (1)$$

For specific values of the parameters, the following stochastic Brownian patterns are obtained as special cases:

S1. For $\theta=m=0$, the process is simple Brownian, with drift (δ) and volatility (σ).

S2. For δ =*0* and m=*1,* it is geometric Brownian with drift rate (-*θ)* and growth rate volatility *(σ).*

S3. For m=*0,* the Ornstein-Uhlenbeck process is obtained that is reverting around a mean level $\mu = \delta/\theta$, at a speed (*θ).*

S4. For m=*1/2*, equation (1) becomes the square-root version of the Ornstein-Uhlenbeck process. It is mean-reverting as before but now volatility depends on the level of income or, in other words, variance per level of income remains constant, that is $[\sigma\sqrt{y}]^2/y = \sigma^2$.

Despite the appeal of Brownian processes in capturing volatile patterns of income, their application in problems of intertemporal optimization of households is limited due to the complexity of stochastic calculus. Even when applied, most solutions are obtained numerically and this hinders an analytic examination of how consumption behavior is shaped. For example, Seater (1996) solved the Euler equation in non-closed-form for the liquidity-unconstrained case and Park (2006) derived a closed-form solution only for the inverse function. The special case of process S1 with δ =*0* has been employed by Smith (2006) to examine the effect of precautionary savings in consumption, while Travaglini (2008) used process S2 to obtain a closed-form solution when income has an upper bound.

Though processes S1 and S2 are the most tractable among the above, their main shortcoming is that income eventually may fall below zero. In the presence of liquidity constraints, a negative level of income will cause consumption to be negative and this severely limits the appeal of the model. Even if variables are considered as deviations from a long run income path, negative consumption cannot be ruled out if disturbance is large. Another limitation for process S2 is that zero is an absorption point; if the stochastic variable happens to hit zero both the drift and the variance follow attune and the process remains stuck.

The two versions of the Ornstein-Uhlenbeck process look more promising to overcome these drawbacks. Process S3 is pulling the stochastic variable toward the mean, thus avoiding the downward bias and making the probability of reaching a negative value less likely. However, such an eventuality is still probable and the same criticism applies as before. It is only the square-root Ornstein-Uhlenbeck process S4 (SROU for short) that has zero as a reflecting boundary. A well-known property (e.g. Shreve, 2004) is that if the process is not excessively volatile so that

$$\sigma^2 < 2\theta\mu \tag{2}$$

the *probability* of income hitting a non-positive level is zero. Of course, the *possibility* cannot be excluded in a random realization. The main properties are described in Appendix A3. After the pioneering work by Cox, Ingersoll and Ross (1985), the SROU process has been widely employed for modeling movements in nominal interest rates to capture the zero bound property. In other applications, the fact that variance increases by the level of income makes the process suitable to study economies in which the dispersion of income rises at the higher cohorts of population.

In the present context, a SROU income process is most suitable to capture the volatility of income after the global crisis and incorporate the front-loaded austerity programs implemented in the Euro Area economies, though at the cost of cumbersome calculations. To the best of my knowledge, this is the first time that it is used in an optimal consumption framework and closed-form solutions are derived. Households' income is assumed to vary according to the augmented pattern:

$$dy = \theta(\mu - y) \cdot dt + \left[\sigma\sqrt{y}\,\right] \cdot dz - k \cdot dw \tag{3}$$

where *(μ)* is the mean to which income is reverting at a speed *(θ)*, *{dz}* a random process normally distributed with zero mean and variance *dt*, and *{dw}* is the process of implementing austerity measures.

Crisis effects

Two further aspects are considered to make the process suitable in examining the effects of the credit crunch and austerity policies on income and consumption: First, a lower mean *(μ)* may represent the reduction in household earnings, while a lower value for the mean-reversion speed *(θ)* indicates stronger persistence of the shocks impinging on income. Additionally, tax hikes *(k)* are imposed on households through a separate process *{dw}* and this significantly alters the dynamics of income as will be examined shortly.

Second, an upper bound on income is imposed. This is meant to reflect both the reduction of employment and earnings opportunities brought about by the contraction of activity, as well as various types of liquidity shortages that may inhibit households from extending their expenditure plans beyond a certain level. Conducting a household survey in Euro Area countries on the extent of liquidity constraints in the aftermath of the crisis, Teppa et al (2014) found that the level of income is positively related with the probability to take out new formal loans, thus low-wage earners face a more stringent borrowing constraint. To reflect this, the upper bound is modeled as a margin *(Ω)* to the reversion mean, thus income varies as $\{ 0 \leq y \leq \Omega\mu, \Omega > 1\}$.

Households

Apart from the specific income structure, the rest of the assumptions are kept as simple as possible.[4] Following Christodoulakis (2014), the economy is populated by identical households which in each period *(t)* receive income (y_t) exclusive of interest payments, consume (c_t) and have a discount rate equal to the exogenously set real rate of interest *(r)*. With an over-dot denoting the time derivative, savings (a_t) accumulate as

$$\dot{a} = ra_t + y_t - c_t \tag{4}$$

Private sector savings (a_t) are deposited in the banking sector and subsequently invested either in domestic Government bonds (b_t) or in foreign assets (v_t) abroad. A balance sheet condition implies:

$$a_t = b_t + v_t \tag{5}$$

The stream of consumption $\{c_{t+s}, 0 \leq s < \infty\}$ is chosen so as to maximize the intertemporal utility, i.e.

$$\max_{\{c\}} \int_0^{\infty} e^{-rs}\, u(c_{t+s}) ds \tag{6a}$$

subject to the usual transversality condition for wealth

$$\lim_{s \to \infty} e^{-rs}\, a_{t+s} = 0 \tag{6b}$$

[4] A limitation of the model is that it lacks a production sector thus the effect of tax policies on investment decisions cannot be captured. To lessen the problem, tax effects are assumed to be lump-sums as described later.

A quadratic utility function is defined as in Zeldes (1989):

$$u(c_t) = c_t - \frac{\gamma}{2} {c_t}^2 \tag{7}$$

where $\gamma > 0$ and $u'(c) = \ 1 - \gamma c,\ u''(c) = \ -\gamma, u'''(c) = 0$. The specific choice is made for ensuring a closed-form solution under a stochastic process for income. Though its third derivative is zero, prudence and precautionary savings are induced by the uncertainties entailed in the Brownian process as will become clear in the next Section.

Optimality implies that expected consumption is constant over time, that is $c_t = E_t\{c_{t+s}\}$ with operator $E_t\{\cdot\}$ describing the expectation based on information available at time *t*. Taking (7) into account and solving (6a, 6b), optimal consumption is obtained as the well-known function of savings and human capital:

$$c_t = r \cdot (a_t + H_t) \tag{8}$$

Human capital is defined as the present value of expected future incomes:

$$H_t = \int_0^{\infty} e^{-rs} E_t\{y_{t+s}\} ds \tag{9}$$

As in Dixit and Pindyck (1994, p. 87) one can treat (H) as an asset and equate the return on it to the sum of the dividend (i.e. the current income) and the expected capital gain:

$$rH_t \cdot dt = y_t \cdot dt + E_t(dH) \tag{10a}$$

or after rearranging and omitting subscripts for simplicity:

$$rH = y + \frac{1}{dt} E_t(dH) \tag{10b}$$

To solve this dynamic stochastic equation in the presence of austerity measures and income constraints, the Government sector should be described first.

Government

In every period, the Government runs ordinary primary surpluses *(l_t)*, generated by non-distortionary lump-sums net of transfers and public expenditure. Public debt (b_t) accumulates according to the budget constraint:

$$\dot{b} = rb_t - l_t \tag{11a}$$

Public debt sustainability requires that

$$\lim_{s\to\infty} e^{-rs}\, b_{t+s} \leq 0 \tag{11b}$$

Note that the possibility of the banks investing abroad makes the households' transversality condition (6b) not to be sufficient for automatically ensuring the respective condition for the public sector. The present value of surplus-generating capacity in the future is defined as:

$$f_t = \int_0^\infty e^{-rs} E_t\{l_{t+s}\} ds \tag{12}$$

Solving (11a), condition (11b) is ensured as long as $b_t \leq f_t$, i.e. future surplus-generating capacity covers current outstanding debt; for a discussion see Giammaroli (2007).

By analogy, sustainability breaks down whenever fiscal capacity collapses *($\Delta f<0$)* or an imbalance appears in (5). For example, if the off-shore assets turn to be toxic for some extraneous reason (i.e. *$dv<0$*), the Government has to cover the losses in the banking sector by a new issuance *($\Delta b>0$)*.[5] Such developments augment indebtedness and trigger a series of austerity measures as examined next.

Austerity

To restore sustainability of public finances, the Government imposes an austerity program for a period *(T)* with present value *p(T)* such as to ensure that:

[5] For example, in Greece public deficit plummeted to -16% of GDP in 2009, Portugal faced a long recession that weakened revenue capacity, while Ireland and Spain suffered major losses in commercial banks that prompted rescue operations by issuance of Government bonds.

$$p(T) = \Delta b_t - \Delta f_t \quad (13)$$

Austerity is implemented in the form of non-distortionary tax hikes *(k)* which are assumed to follow a Poisson process *{dw}* with arrival rate equal to *(λ)*, i.e.

$$dw = \begin{cases} 1 & \text{with probability } (\lambda dt) \\ 0 & \text{with probability } (1 - \lambda dt) \end{cases} \quad (14)$$

As argued by Toche (2005), a Poisson process is more suitable for modeling the uncertainty associated with rare, large and permanent income losses, and such indeed have been the consolidation programs applied in several Euro Area economies. If, for example, the module of time *(dt)* is a month, a rate λ=1/12 implies Government interventions occurring once a year in average. Tax hikes are modeled as impulse functions, thus we have that $E_t\{k_{t+s} dw_{t+s}\} = \lambda k$. The present value of the consolidation program is calculated as: [6]

$$p(T) = \int_0^T e^{-rs} E_t\{k_{t+s} dw_{t+s}\} ds = \frac{1}{r} \lambda k \left[1 - e^{-rT}\right] \quad (15a)$$

Rearranging, the tax intensity is obtained as:

$$k = \frac{r \cdot p(T)}{\lambda \cdot [1 - e^{-rT}]} \quad (15b)$$

The tax hike increases with the debt burden (or the loss in revenue capacity) that has to be corrected and is inversely affected by the duration period *(T)* and the frequency *(λ)* at which policy measures are implemented. This creates room for policy trade-offs between intensity and duration of the program, and their effects on consumption and savings are investigated below.

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[6] This expression differs from the well-known formula derived by Merton (1971) where random increments in income are assumed to be step functions and, therefore, the compounding factor is squared *(1/r^2)*.

3. Precautionary savings

To solve (10b), Ito's lemma is applied for human capital as in Dixit and Pindyck (1994, p86) giving:

$$\frac{1}{dt}E_t(dH) = \frac{\partial H}{\partial t} + \theta(\mu - y)\frac{\partial H}{\partial y} + \frac{1}{2}(\sigma^2 y)\frac{\partial^2 H}{\partial y^2} + E_w\{\lambda \cdot [H(y-k) - H(y)]\} \quad (16)$$

with $E_w\{\cdot\}$ denoting the expected value of process *{dw}*. For simplicity, time subscripts are omitted and variables refer to the present time unless stated otherwise. By Taylor's rule the second-order approximation of the term due to the Poisson process is:

$$H(y-k) - H(y) \approx -k\frac{\partial H}{\partial y} + \frac{1}{2}k^2\frac{\partial^2 H}{\partial y^2} \quad (17)$$

Substituting (17) into (16) and (10b), the following stochastic differential equation for human capital is obtained:

$$\frac{1}{2}(\sigma^2 y + \lambda k^2)H'' + [\theta(\mu - y) - \lambda k]H' - rH + y = 0 \quad (18)$$

In the above formulation, the variance and the drift of income in the Brownian process have been adjusted by (λk^2) and (λk) respectively[7] due to the austerity policies.

To obtain a closed-form general solution of (18), a number of linear transformations are required so that it takes the form of the so-called Kummer equation as described by Dixit and Pindyck (1994, p. 161-163) for the simple Ornstein-Uhlenbeck process. Namely:

$$\alpha = \frac{r}{\theta} \quad (19a)$$

$$\beta(k) := \frac{2\theta}{\sigma^2}\cdot[\mu - \frac{\lambda k}{\theta} + \frac{\lambda k^2}{\sigma^2}] \quad (19b)$$

$$q(y) := \frac{2\theta}{\sigma^2}\cdot[y + \frac{\lambda k^2}{\sigma^2}] \quad (19c)$$

[7] Note that for a Poisson process the mean and the variance are equal to (λ).

The closed-form solution for consumption is finally obtained in Appendix A1 as:

$$c = ra + \frac{\theta\mu - \lambda k}{r+\theta} + \frac{r}{r+\theta} y - S(k) \tag{20}$$

Function *S(k)* denotes precautionary savings due to the presence of uncertainty and the effects of austerity. At a given level of income precautionary savings are calculated as

$$S(k) \; = \frac{1}{2} \cdot \frac{\sigma^2}{r+\theta} \cdot \frac{\beta \cdot F[\alpha;\beta;q(y)]}{F[\alpha+1;\beta+1;q(\mu\Omega)]} \tag{21}$$

In the above expression, *F(α;β;x)* denotes the so-called confluent hyper-geometric function (CHF) of the first kind, defined by the infinite series:

$$F(\alpha;\beta;q) = 1 + \frac{\alpha}{\beta} q + \frac{\alpha(\alpha+1)}{\beta(\beta+1)} \frac{q^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)} \cdot \frac{q^3}{3!} + \cdots \tag{22}$$

Though obtained in closed-form, the above expression of precautionary savings is analytically intractable and any intuition on its properties will be mired with numerical simulations. To get a notion of how the CHF behaves, note that for $\alpha=\beta$ it collapses to the simple exponential function and this motivates the linearization described below. The CHF can be approximated by some more tractable forms if some of its arguments happen to be inter-depended and this is indeed the case for the functional forms *β(.)* and *q(.)*.With the tax hike variable *(k)* sufficiently above zero, the square term dominates in both and eventually they converge to the same level. Their proportional deviation is given by *(ε)*:

$$\varepsilon = \varepsilon(k) = \frac{q(\mu\Omega)}{\beta} - 1 = \left[\mu - \frac{\lambda k}{\theta} + \frac{\lambda k^2}{\sigma^2}\right]^{-1} \cdot \left[\mu\Omega + \frac{\lambda k^2}{\sigma^2}\right] - 1 \tag{23}$$

For a small *(ε)* and after some tedious algebraic transformations, a convenient log-linear approximation *s(k)* of the savings function is obtained as follows:

$$S(k) \approx s(k) := \frac{\sigma^2 r}{2(r+\theta)\theta\varphi} \cdot \exp\left[-\frac{2\theta\varphi}{\sigma^2} \cdot (\mu\Omega - y)\right] \tag{24}$$

where

$$\varphi := \frac{\varepsilon}{2}\left(\frac{\varepsilon}{2} + 1\right) \quad (25)$$

Full details are given in Appendix A2. The functional $\varphi=\varphi(k)$ depends only on the tax hike (k) and the original model parameters and this allows the approximate savings function to be examined with respect to the structural features as well as the austerity effects.

Expression (24) renders some well-known results as special cases. For example, if there is no effective upper bound on income ($\Omega \to \infty$) or there is no persistence ($\theta \to \infty$), precautionary savings tend to zero in agreement with the permanent income hypothesis. For income close to its upper bound ($y \to \mu\Omega$) and in the absence of tax hikes $(k=0)$, we have that $\varepsilon=\Omega-1$, and precautionary savings become a simple proportion to the variance (σ^2):

$$s(0) = \eta \cdot \frac{\sigma^2}{2}\ , \quad \text{with } \eta = \frac{4r}{\theta(r+\theta)(\Omega^2-1)} \quad (26)$$

With (η) loosely interpreted as a measure of "absolute prudence", the above formula is similar to the Equivalent Precautionary Premium (EPP) derived by Kimball (1990, equation 4). In the present context, "prudence" increases as the real interest rate (r) rises, shocks become more persisting (i.e. lower θ), or the upper income margin gets stiffer ($\Omega \to 1$). For calibrated parameters as in Appendix B1, it takes the value of 0.24, close to upper end of the range (0.063…0.243) of EPP estimated by Carroll (1996, Table D) for various sectors in the US economy for the period 1981-1987.

Alternatively, if expression $\eta(r+\theta)/r$ is taken to imply a measure of risk aversion, expression (26) gives the precautionary savings as derived by Weil (1993, equation 2.1). Finally, if the margin is set to vary progressively with income (e.g. $\Omega = \Omega(\mu),\ \Omega' > 0$), the "prudence" measure in (26) becomes regressive for higher incomes confirming the finding by Lee and Sawada (2007) that people with more financial wealth are less likely to face a liquidity constraint and therefore take bigger risks by saving less.

The marginal propensity to consume

Substituting (24) into (8), consumption is approximately given by:

$$c_t \approx \frac{\theta\mu - \lambda k}{r+\theta} + \frac{r}{r+\theta} \cdot \left\{ y - \frac{\sigma^2}{2\theta\varphi} \cdot \exp\left[-\frac{2\theta\varphi}{\sigma^2} \cdot (\mu\Omega - y) \right] \right\} \quad (27)$$

The marginal propensity to consume (MPC) is derived by differentiating (27) w.r.t. income:

$$MPC = \frac{\partial c}{\partial y} = \frac{r}{r+\theta} \left\{ 1 - \exp\left[-\frac{2\theta\varphi}{\sigma^2} (\mu\Omega - y) \right] \right\} \approx \frac{r}{r+\theta} \cdot \frac{2\theta\varphi}{\sigma^2} (\mu\Omega - y) \quad (28)$$

Using the above expressions, a number of properties are readily established for precautionary savings and the marginal propensity to consume. Proofs are in Appendix A3:

Proposition 1. Precautionary savings rise with income and income variance, as the income margin gets narrower or the mean-reverting process becomes more persisting, i.e. speed parameter *(θ)* falls.

This follows from (24) by checking that *s(k)* is increasing in *(y)*, *(μ)* and *(σ^2)*, and decreasing in *(θ)* and *(Ω)*. These findings are in line with Weil (1993) and Smith (1998) where an AR process and a zero-drift process are employed respectively. With the income ceiling unchanged, a rising income implies that households are more likely to face a limitation in their future consumption, so they behave more prudentially today. With a higher income ceiling households are less concerned that their future expenditure will be constrained, thus they become less prudent today.

Proposition 2. Precautionary savings rise with interest rates.
This is obvious from (24). The result is in agreement with Weil (1993) who notes that a rise in interest rates increases labour income risk and motivates a higher precaution in savings.

Proposition 3. The marginal propensity to consume is decreasing in income and is concave w.r.t. the tax hike *(k)*. It is also falling as precautionary savings rise.
The first part is obvious by differentiating w.r.t. income. Concavity is obtained since *$\varphi(k)$* is concave w.r.t. *(k)*. The second part follows from the fact that $y \leq \mu\Omega$ and precautionary savings is decreasing in *$\varphi(k)$*. An explanation is that as income increases,

households' precautionary motive strengthens to avoid being confronted with the upper bound constraint. Thus, they cut from the implied rise in consumption and MPC falls.

Proposition 4. Precautionary savings, the marginal propensity to precautionary savings (MPPS) and the variability of savings are convex w.r.t. the tax hike *(k)*.

For a detailed proof see Appendix A3. This implies that variability of savings is high when the intensity of fiscal consolidation is either too weak or too strong. In the former case, households fear that debt stabilization is not sufficiently accomplished, new cuts may be imposed in the future and this increases income risk and uncertainty. In the strong consolidation case, households face severe cuts in income and –given liquidity constraints- this upturns their consumption behavior and increases volatility of savings.

4.Stylised facts of austerity

The implications of the above Propositions are important for explaining some stylized facts of the current recession in the Euro Area. The uncertainty of income increased substantially after the crisis and this led households to raise their precautionary savings as a hedging strategy against future disturbances. On top of that, uncertainty was multiplied by the implementation of austerity programs and this led to further precautionary savings in the distressed economies. Given that the aggregate output multiplier moves in the same direction with MPC, it follows that as incomes were curtailed after the global crises output multipliers increased, thus making the recessionary effects of fiscal consolidation more pronounced.

For each Euro Area country, the recessionary impact *(R_j , j=1,…,12)* is defined as the average annual rate of output losses impacting upon each specific economy, namely:

$$R_j = \left[1 + \sum_{t=2009}^{T=2014} \frac{\hat{y}_{j,t} - y_{j,t}}{y_{j,2008}}\right]^{1/6} - 1 \qquad (29)$$

In the above expression $y_{j,t}$ denotes per capita real GDP at period *t*, $\hat{y}_{j,t}$ is a simple time-trend projection[8] evaluated over the period 2001-2008, and their difference is expressed as a ratio to per capita GDP level at 2008. The respective tax hikes *(*k_j*, j=1,…12)* are roughly approximated by the change in tax rates affecting households during 2009-2013. To avoid country-specific compositions, the simple sum of increases in VAT and income tax rates is considered as sourced in Appendix B2. This underestimates the intensity of fiscal consolidation in each country as cuts in wages and transfers are not taken into account, but, nevertheless, provides a tentative indication of the extent of austerity. The following stylized facts are examined:

Stylized fact 1: The recessionary impact is convex to the intensity of austerity imposed in the Euro Area economies.

Fig. 3 reveals a strong non-linear correlation between the impact of recession and tax-surges (R_j, k_j). This is explained by the convexity of precautionary savings as outlined in Proposition 4. If fiscal consolidation is too weak, the precautionary motive is strong as households fear that eventually they will be subjected to austerity policies to correct debt imbalances.

[Fig. 3 here]

This extends the analysis by Christodoulakis (2014) where a linear relation between the impact of consolidation programs on recession and the intensity of austerity was established by examining only the countries in the Euro Area periphery. Apparently, the linear effect is the right-hand branch of the non-linear fitting in Fig. 3, which is now referring to the twelve Euro Area economies.

Including the economies of the northern part of the Euro Area, a strong non-linearity is revealed. By looking at the left-hand branch, the implication is that consolidation should have been more decisive in Finland, Italy, Netherlands and Luxemburg, and marginally so in Belgium and Austria. Germany and France seem to be close to the optimal level, by applying a program of low intensity with only a mild recession. On the other end of the spectrum, Ireland, Greece and Spain seem to suffer most by applying harsh tax measure, while Portugal could have managed with less austerity as well.

[8] The simple time-trend is more suitable for short periods. Other measures of recessions, such as Hodrick-Prescott filters or output gaps, were also used with similar results.

Another measure of recession, more closely related to precautionary savings, is the suppression of consumption growth rates by uncertain households. Calculating the average annual deterioration of per capita consumption during 2009-2013 from the pre-crisis period, Fig. 4 displays a similar non-linear effect as before, (albeit concave this time).

[Fig. 4 here]

By looking at a country level, the implications are the same for Germany, France, Finland and Portugal but less pronounced for Austria, Belgium, Italy and Luxemburg. On the falling side, Greece outpaces Ireland and appears as the country most disproportionately hit by austerity.

Stylized fact 2: Savings rate volatility is convex to the austerity of intensity imposed in the Euro Area economies.
Fig. 5 plots the standard deviation of households' savings rates during 2009-2013 versus the tax hikes $(k_j, j=1,\dots 12)$ and a strong non-linear correlation is obtained. This is exactly in line with Proposition 4 and can be explained by similar arguments as before.

[Fig. 5 here]

A noticeable difference with Fig. 4 is that volatility is found to be mild for Luxemburg, Italy and Netherlands possibly because households in these countries are less liquidity-constrained. Volatility is high for Spain and Ireland, though not much so for Portugal. Of all Euro Area economies, Greece has experienced the highest volatility ($\sigma=5.13$), thus confirming the convexity. However, it is not reported in Fig. 5 as savings rates appear to be strongly negative during the period in apparent contradiction with the behavior in other countries.[9] The paradox is explained by the massive capital outflow that took place during the period of examination due to fear of exiting the Euro Area.

[9] Besides, no Eurostat data are reported for Greece prior to 2009 so it is difficult to compare with previous trends.

Optimal austerity

By virtue of Proposition 4, precautionary savings are minimized when *φ(k),* or for that matter *ε(k),* reach a maximum. To maximize *ε(k)* the following monotonic transformation is considered:

$$\frac{1}{\varepsilon} = 1 + \frac{1}{\rho} \tag{30a}$$

with

$$\rho = \frac{\sigma^2}{\theta} \cdot \left[k^2 + \frac{\sigma^2\mu}{\lambda}\Omega\right]^{-1} \cdot \left[k + \frac{\theta\mu}{\lambda}(\Omega - 1)\right] \tag{30b}$$

Setting $\partial\rho/\partial k = 0$, the optimal tax hike is obtained as the positive root of the equation

$$k^2 + 2\frac{\theta\mu}{\lambda}(\Omega - 1) - \frac{\sigma^2\mu}{\lambda}\Omega \tag{31}$$

The calculation of *k** gives:

$$k = k^* = -\frac{\theta\mu}{\lambda}(\Omega - 1) + \left[\frac{\theta^2\mu^2}{\lambda^2}(\Omega - 1)^2 + \frac{\sigma^2\mu}{\lambda}\Omega\right]^{1/2} \tag{32}$$

It is easily checked that *φ* is monotonically increasing with *(ε)* and (*ρ*), while the latter is concave w.r.t. (*k*); see Appendix A3 for details. Therefore the policy *k=k** yields a maximum for *ρ(k)* and *φ(k)* and a minimum for precautionary savings.

For parameters calibrated as in Appendix B1, the exact solution and the closed-form approximation are plotted in Fig. 6.

[Figure 6, here]

The optimal tax hike is calculated from (32) to be *k*=10%*, and the expected value of the tax consolidation is *λk*=5%*. Though of similar magnitude, the figure is expectedly higher than the sum of tax changes that corresponds to the lowest recession as shown in Fig. 3, since austerity here is assumed to take place only through tax hikes.

Though the above findings should be interpreted only tentatively, their implication may be important for the pace of fiscal consolidation in the Euro Area. Plugging the results into

(15b), one obtains that for a period of implementation of five years (*T=5*), it would have implied a reduction of indebtedness by -23% of GDP for the Euro Area as a whole. The mechanism would have worked by keeping precautionary savings as low as possible, thus letting consumption to be less depressed and recession to be milder. Fig. 6 suggests that consumption could have been 2-3% higher per year and, with output less contracted, the debt to GDP ratio would have improved and uncertainty about future incomes could have been dissipated rather than surging. In practice, however, the public debt to GDP ratio in EA12 rose from 80.66% in 2009 to 96.66% amid fears of a prolonged recession in the Euro Area.

The analysis may also help to explain why the early assumptions of low fiscal multipliers went off the mark. Since aggregate output multipliers move in the same direction with MPC, it follows from Proposition 3that with post-crisis incomes depressed, MPC and consequently the output multipliers increased substantially from their pre-crisis levels. This made the recessionary effects of fiscal consolidation more pronounced. By now, it is widely recognized that the severity of front-loaded austerity programs in the Euro Area has actually accentuated recession and indebtedness rather than dissipating them.
It was only after recession reached a depth threatening social stability in several Euro Area countries that the above views were revised, even from within the policy-making organizations that initially endorsed them. For example, Batini et al (2012) -in agreement with the above findings- report that "frontloaded consolidations tend to be more contractionary and, hence, delay the reduction in the debt-to-GDP ratio relative to smoother consolidations". Blanchard and Leigh (2013) found that the depressing effect of fiscal consolidation was seriously underestimated and the size of the fiscal multiplier is likely to be three times the level assumed by the early estimates.

5. Conclusions

Despite optimistic assumptions that a swift fiscal consolidation applied in the Euro Area economies in the aftermath of the global crisis would have small and only transient recessionary effects, the outcome proved to be a lot more contractionary and lasting. Using a stochastic mean-reversing framework for labor income, the paper derived closed-form solutions for households' precautionary savings and examined their behavior under austerity and liquidity constraints. Precautionary savings are found to rise with regards to the uncertainty prevailed in the Euro Area after the crisis and to be further augmented by the austerity measures that were applied to stabilize the debt to output ratio.

They are also found to rise in economies which are characterized by an insufficient degree of fiscal consolidation. As the intensity of consolidation varied and the economies were at the same time characterized by a different degree of liquidity availability, a dual pattern is revealed in the Euro Area: a gradual consolidation in the northern part caused only a mild recession even in countries with a high debt-to-GDP ratio in the beginning of the crisis, (such as Belgium or Italy). By contrast, in the economies of the European periphery where more stringent liquidity constraints[10] were coupled with intensive and front-loaded consolidation programs as part of their bail-out agreements, recession was more severe and lasting.

Results obtained for calibrated parameter values for the Euro Area suggest that an optimal austerity level should have been more evenly distributed across economies: the result would be a more systematic consolidation in the countries of the core and a more back-loaded program in the periphery where recession reached unprecedented depths. Overall it suggests that a more gradual adjustment could have had milder recessionary effects all over the Euro Area and would be perhaps politically more acceptable and socially sustainable.
Future research will try to apply the findings for each particular member of the Euro Area and extend the analysis of the austerity impact on employment and investment decisions.

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[10] Teppa et al (2014) report various indices of loan availability as a measure of liquidity constrained households in the Euro Area. Mediterranean countries (excluding Italy) are found substantially more constrained than continental economies.

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Appendix A

A1. Solving the stochastic Kummer equation

To find the general solution of (20), it is necessary that income is scaled to a new variable *q(y)* as in (19c). It is easy to verify that the new human capital function $L(q) := H(y)$ satisfies the so called Kummer equation:

$$qL'' + (\beta - q)L' - \alpha L = 0 \qquad (33)$$

Coefficients are defined as in (19a, b, c). A general solution is given by[11]

$$L(q) = \xi \cdot F(\alpha;\beta;q) + \zeta \cdot U((\alpha;\beta;q) \qquad (34)$$

In the above expression, *(ξ,ζ)* are constants to be determined by terminal conditions and *F(.), U(.)* denote the confluent hypergeometric functions (CHF) of the first and second kind respectively; for details see Daalhuis (2010). The following property is readily established for the first-order derivative:[12]

$$F'(\alpha;\beta;q) \coloneqq \frac{\partial F}{\partial q} = \frac{\alpha}{\beta} \cdot F(\alpha+1;\beta+1;q) \qquad (35a)$$

For determining the constants in (34), we look at the behavior of consumption as income approaches the two boundary levels *[0,Ω]*. According to Daalhuis (2010, equation 13.2.26) a satisfactory approximation near the origin is

$$U((\alpha;\beta;q) \approx [q^{1-\beta}] \cdot F(\alpha - \beta + 1; 2 - \beta; q) \qquad (35b)$$

Note that to ensure zero as a lower bound for process (3), condition (2) now demands that $\sigma^2 < 2(\theta\mu - \lambda k)$, that is β>1. With a non-zero *(ζ)*, this means that *L(q)* in (34) has a branch point at *q=0* and consumption would tend to infinity. To rule this out, we set *ζ=0*.

Solving for human capital
A partial solution of *(H)* is easily evaluated from (18) as

---

[11] Dixit and Pindyck (1994, p 163) report only the first term in the rhs of (34b). Though at the end the second kind CHF is removed from the solution, its inclusion is important in order to clarify the behavior of consumption in the zero income threshold.
[12] For the derivatives w.r.t. to parameters (α,β) see Ancarani and Casaneo (2008).

$$H_t^P = \frac{\theta\mu - \lambda k}{(r+\theta)r} + \frac{y_t}{r+\theta} \quad (36a)$$

Thus the complete solution to the SDE takes the form:

$$H = \frac{\theta\mu - \lambda k}{r(r+\theta)} + \frac{y_t}{r+\theta} + \xi \cdot F[\alpha; \beta; q(y)] \quad (36b)$$

Constant *(ξ)* is determined so that the consumption function is well-behaved everywhere, i.e. the Euler equation is satisfied even when income constraints become binding. This requires that the 'smooth pasting' condition $c'(y)|_{y=\mu\Omega} = 0$ holds. Substituting (36b) into (8) and differentiating w.r.t. *(y)* we obtain

$$1/\xi = -q'(\mu\Omega) \cdot (r+\theta) \cdot F'[\alpha; \beta; q(\mu\Omega)] \quad (37)$$

Putting this into (36b) and using (36a) and (19a,b,c), expressions (20) and (21) for consumption and precautionary savings respectively are readily obtained.

A2. A log-linear approximation of precautionary savings

First, a simple log-linear approximation of the CHF around the income upper bound *(μΩ)* is obtained:

$$\ln F[\alpha; \beta; q(y)] \approx \ln F[\alpha; \beta; q(\mu\Omega)] + \psi \cdot (y - \mu\Omega) \quad (38a)$$

where

$$\psi = \frac{\partial[lnF]}{\partial y}\bigg|_{y=\mu\Omega} = \frac{1}{F} \cdot \frac{\partial F}{\partial y} = q'(\mu\Omega) \cdot \frac{\alpha}{\beta} \cdot \frac{F[\alpha+1; \beta+1; q(\mu\Omega)]}{F[\alpha; \beta; q(\mu\Omega)]} = \frac{2r}{\sigma^2} \cdot \Phi \quad (38b)$$

and

$$\Phi := \frac{F[\alpha+1; \beta+1; q(\mu\Omega)]}{\beta \cdot F[\alpha; \beta; q(\mu\Omega)]} \quad (38c)$$

Next step is to look for a further approximation of expression *(Φ).* Applying the recurrent formula as in Daalhuis (2010, equation 13.3.3), expression (38c) is rewritten as:

$$\Phi\ = \frac{1}{\alpha} - \left(\frac{1}{\alpha} - \frac{1}{\beta}\right) \cdot \frac{F[\alpha;\beta+1;q(\mu\Omega)]}{\beta \cdot F[\alpha;\beta;q(\mu\Omega)]} \tag{38d}$$

The fractional form in the r.h.s. can now be approximated in terms of the deviation *ε=[q(Ω)/β-1]*, as defined in (25). Omitting functional arguments for simplicity and assuming that *(ε)* is relatively small so that $ln(1+\varepsilon) \approx \varepsilon - \varepsilon^2/2$, the approximation formula displayed in Daalhuis (2010, equation 13.8.4) is reduced to

$$F(\alpha;\beta;q) \approx (\beta)^{\frac{\alpha-1}{2}} \cdot \exp\left[\frac{1}{4}\varepsilon^2\beta\right] \cdot \left\{(1+\varepsilon)\sqrt{\beta} \cdot \Delta_1 + \Delta_2\right\} \tag{39a}$$

Expressions $(\Delta_j, j=1,2)$ are defined as

$$\Delta_1 = G\left(\alpha - \frac{1}{2};\ -\varepsilon\sqrt{\beta}\right) \ \text{and}\ \ \Delta_2 = G\left(\alpha - \frac{3}{2};\ -\varepsilon\sqrt{\beta}\right) \tag{39b}$$

where *G(.;.)* is the so-called parabolic cylinder function (PCF). An analogous expression is obtained for *F(α;β+1;q)* in terms of deviation *ε΄=[q(μΩ)/(β+1)-1]*. For a relatively large (*β*), it is assumed that

$$\sqrt{\beta/(1+\beta)} \approx 1 \tag{40a}$$

$$\varepsilon\sqrt{\beta} \approx \varepsilon'\sqrt{\beta+1} \tag{40b}$$

$$(1+\varepsilon)\sqrt{\beta} \approx (1+\varepsilon')\sqrt{\beta+1} \tag{40c}$$

The above assertions are plotted in Fig. 7 and look justified. This makes the terms with the parabolic cylinder functions roughly equal when applied for arguments (*β*) and (*β+1*).

[Fig. 7, here]

Considering the remaining parts, we obtain the approximation:

$$\Phi\ \approx \frac{1}{\alpha} - \left(\frac{1}{\alpha} - \frac{1}{\beta}\right) \cdot \left(\frac{\beta+1}{\beta}\right)^{\frac{\alpha}{2}} \cdot \exp\left[\frac{1}{4}(\varepsilon')^2(\beta+1) - \frac{1}{4}\varepsilon^2\beta\right] \tag{41a}$$

For the income mean normalized at unity *(μ=1)*, the transformed variable (*β*) is likely to be large and well above the level of *α=r/θ*, for a wide range of parameter values. Hence, the following simplifications might be further considered:

$$(\varepsilon')^2(\beta+1)-\varepsilon^2\beta=-\frac{q^2}{(\beta+1)\beta}+1\approx-\frac{q^2}{\beta^2}+1=-\varepsilon(\varepsilon+2) \qquad (41b)$$

With *β>> α:*

$$\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\cdot\left(1+\frac{1}{\beta}\right)^{\frac{\alpha}{2}}\approx\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\left(1+\frac{\alpha}{2\beta}\right)=\frac{1}{\alpha}-\frac{1}{2\beta}-\frac{\alpha}{\beta^2}\approx\frac{1}{\alpha} \qquad (41c)$$

Thus, expression *(Φ)* is simplified to

$$\Phi\approx\frac{1}{\alpha}-\frac{1}{\alpha}\cdot\exp\left[-\frac{1}{4}\varepsilon\cdot(\varepsilon+2)\right]\approx\frac{1}{4\alpha}\varepsilon\cdot(\varepsilon+2) \qquad (42a)$$

Substituting the above into (38b) and recalling (19a), it becomes

$$\psi=\frac{2r\theta}{\sigma^2}\cdot\left(\frac{\varepsilon}{2}+2\right) \qquad (42b)$$

Thus the log-linear form (24) is finally obtained.

A3. Properties

The square-root process

Consider the square-root process as in (3), with $k=0$

$$dy=\theta(\mu-y)\cdot dt+\left[\sigma\sqrt{y}\,\right]\cdot dz \qquad (43)$$

The probability of hitting $y=0$ is zero if $\sigma^2<2\theta\mu$.

Taking expectations

$$E[y_t|y_0]=y_0e^{-\theta t}+\mu\left(1-e^{-\theta t}\right) \qquad (44a)$$

The variance is given by the expression

$$var[y_t|y_0]=y_0\frac{\sigma^2}{\theta}\left(e^{-\theta t}-e^{-2\theta t}\right)+\frac{\mu\sigma^2}{2\theta}\left(1-e^{-\theta t}\right)^2 \qquad (44b)$$

The long-term mean of the process is (μ) and the long-term variance is

$$\tau^2=\mu\,\sigma^2/(2\theta). \qquad (44c)$$

Thus, condition (2) for a zero probability of hitting *y=0,* is simplified to $\tau<\mu$, i.e. long-term standard deviation is below the long-term mean.

To get an impression of the square root process, five random realizations are plotted in Fig. 8.

[Fig. 8 here]

Optimal tax rule

From (25) it is easy to obtain

$$\frac{\partial\varphi}{\partial\varepsilon}=\frac{1}{2}(\varepsilon+1)>0 \quad \text{and} \quad \frac{\partial^2\varphi}{\partial\varepsilon^2}=\frac{1}{2}>0 \tag{45a}$$

From (30a)

$$\frac{\partial\varepsilon}{\partial\rho}=[1+\rho]^{-2}>0 \quad \text{and} \quad \frac{\partial^2\varepsilon}{\partial\rho^2}=-2[1+\rho]^{-3}<0 \tag{45b}$$

Setting $\partial\rho/\partial k=0$ and recalling that $\Omega>1$, we obtain from (30b)

$$\left.\frac{\partial^2\rho}{\partial k^2}\right|_{k*}=-\frac{2\sigma^2}{\theta}\cdot\left[k^2+\frac{\sigma^2\mu}{\lambda}\Omega\right]^{-2}\cdot\left[k+\frac{\theta\mu}{\lambda}(\Omega-1)\right]<0 \tag{46}$$

By Leibnitz's rule

$$\left.\frac{\partial^2\varepsilon}{\partial k^2}\right|_{k*}=\frac{\partial^2\varepsilon}{\partial\rho^2}\left[\frac{\partial\rho}{\partial k}\right]^2+\frac{\partial\varepsilon}{\partial\rho}\cdot\frac{\partial^2\rho}{\partial k^2}=(-)(0)+(+)(-)<0 \tag{47a}$$

$$\left.\frac{\partial^2\varphi}{\partial k^2}\right|_{k*}=\frac{\partial^2\varphi}{\partial\varepsilon^2}\left[\frac{\partial\varepsilon}{\partial k}\right]^2+\frac{\partial\varphi}{\partial\varepsilon}\cdot\frac{\partial^2\varepsilon}{\partial k^2}=(+)(0)+(+)(-)<0 \tag{47b}$$

Thus $\varepsilon(k)$ and $\varphi(k)$ are concave at $k=k^*$.

Proof of Proposition 4

Differentiating (24) w.r.t. φ, we get easily

$$\frac{\partial s}{\partial\varphi}=-\left[\frac{1}{\varphi}+(\mu\Omega-y)\right]\cdot s<0 \tag{48a}$$

$$\frac{\partial^2 s}{\partial \varphi^2} = \frac{1}{\varphi^2} s - \left[\frac{1}{\varphi} + (\mu\Omega - y)\right] \cdot \frac{\partial s}{\partial \varphi} > 0 \qquad (48b)$$

$$\frac{\partial^2 s}{\partial k^2} = \frac{\partial^2 s}{\partial \varphi^2}\left[\frac{\partial \varphi}{\partial k}\right]^2 + \frac{\partial s}{\partial \varphi} \cdot \frac{\partial^2 \varphi}{\partial k^2} = (+)(+) + (-)(-) > 0 \qquad (48c)$$

Hence, precautionary savings are convex w.r.t. the tax hike *(k)*.

The marginal propensity of income to precautionary save is defined as

$$MPPS = \frac{\partial s}{\partial y} = \frac{r}{r+\theta} \cdot \exp\left[-\frac{2\theta\varphi}{\sigma^2}(\mu\Omega - y)\right] \qquad (49a)$$

Its partial derivatives are

$$\frac{\partial MPPS}{\partial \varphi} = -\frac{2\theta}{\sigma^2} \cdot MPPS < 0 \qquad (49b)$$

$$\frac{\partial^2 MPPS}{\partial \varphi^2} = \left[\frac{2\theta}{\sigma^2}\right]^2 \cdot MPPS > 0 \qquad (49c)$$

Recalling (47b), it is obvious that

$$\frac{\partial^2 MPPS}{\partial k^2} = \frac{\partial^2 MPPS}{\partial \varphi^2}\left[\frac{\partial \varphi}{\partial k}\right]^2 + \frac{\partial MPPS}{\partial \varphi} \cdot \frac{\partial^2 \varphi}{\partial k^2} = (+)(+) + (-)(-) > 0 \qquad (49d)$$

Hence, MPPS is convex w.r.t. *(k).* Differentiating (24) w.r.t. *(y)* and applying Ito's Lemma, the time-varying volatility of precautionary savings is obtained as

$$\sigma_s = \sigma_y \frac{\partial s}{\partial y} = \sigma\sqrt{y} \cdot MPPS \qquad (50)$$

Thus the volatility of savings is convex w.r.t. *(k)*.

Appendix B

B1. Calibration

To calibrate persistence and variance parameters for the mean-reverting process in (3), per capita GDP for the period 1999-2013 was detrended by an HP filter to obtain the transitory components of income, ($\tilde{y}(j,t),\ j = 1 \dots 12$). Then, percentage deviations are cross-section estimated as:

$$\tilde{y}(j,t) = \rho\tilde{y}(j,t-1) + C(j) + T(t) \tag{51}$$

where *C(j)* and *T(t)* denote county-specific and time effects respectively. Results are displayed in Table 1, together with similar results obtained by Sandri (2011) for the US economy during 1980-2008 for comparison.

Three sub-periods are considered for estimation. The first period, 1999-2005, refers to the pre-crisis period and yields a speed of reversion $\theta=1-\rho=0.34$ with an income volatility $\sigma=1.292\%$. To capture the effects of the global crisis the estimation was repeated for the period 2006-2013 yielding a more persisting process (i.e. lower $\theta=0.25$) and volatility $\sigma=2.292\%$, more than two times higher than before the crisis.

However, as noted by Blanchard (1993), the uncertainty prevailing after a large shock is not just in anticipation of slower growth but may encompass a magnifying factor of the 'animal spirits' kind. To better capture the uncertainty in the immediate aftermath of the 2008 crisis, the mean-reverting process is estimated again over the sub-period 2006-2009 and parameters now become $\theta=0.13$ and $\sigma=4.18\%$. As the crisis was expected to last for many years, it seems more appropriate that the long run variance of the process should be considered in explaining the behavior of households on the eve of the consolidation programs. This is obtained through (44c) and is found to be equal to *8.20%*, four times larger than before the slump.

The rest of the parameters are set so as to conform with annual frequency: real interest rates are set at $r=3\%$, the upper income margin is set at $\Omega=1.10$ to denote the deterioration in credit facilitation and frequency is set equal to $\lambda=1/2$, implying that austerity measures were assumed to be taken within the first two years of the crisis. Income is scaled with a mean equal to one *(μ=1)*.

B2. Data sources

1. AMECO Database, European Commission.

2.KPMG Tax rates in the Euro Area.
The total of tax changes is obtained by summing changes in the VAT rate and the income tax rate effected during 2009-2013 over the average in 2006-2008. To avoid country specific effects, a simple unweighted sum was taken.
Data are available at
http://www.kpmg.com/GLOBAL/EN/SERVICES/TAX/TAX-TOOLS-AND-RESOURCES/Pages/tax-rates-online.aspx

3.Eurostat: Quarterly Savings rates for Euro area 17
The gross saving rate of households is defined as gross saving (ESA95 code: B8G) divided by gross disposable income (B6G), with the latter being adjusted for the change in the net equity of households in pension funds reserves (D8net). Gross saving is the part of the gross disposable income which is not spent as final consumption expenditure. Detailed data and methodology on site http://ec.europa.eu/eurostat/sectoraccounts

4.ECB: Quarterly data for Government debt as percent of GDP
Euro area 17 (fixed composition) - Maastricht assets/liabilities - General government (ESA95)-NCBs - All sectors without general government (consolidation) (ESA95) - NCBs - Financial stocks at nominal value - Percentage points, ser(t)/ sum(GDP(t), GDP(t-1), GDP(t-2), GDP(t-3)). Neither seasonally nor working day adjusted.

Table 1. Estimating the persistence of per capita GDP in the EA 12

	US Sandri (2011)	EA 12	EA 12	EA 12
Period	1980-2008	1999-2005	2006-2013	2006-2009
Std of Δlny		1.561%	2.904%	4.042%
Nobs. (annual)		72	96	48
Persistence *(ρ)*	0.73	0.66	0.75	0.87
SE (t-stat)		0.0995 (6.63)	0.0867 (8.63)	0.218 (3.95)
R2		0.613	0.783	0.824
DW		1.867	1.453	1.603
Std (σ)	1.20%	1.292%	2.922%	4.181%
Implied speed of reversion θ=1-ρ	*0.27*	*0.34*	*0.25*	*0.13*
Implied long run Std $\tau=\sigma/\sqrt{2\theta}$	*1.63%*	*1.57%*	*4.13%*	*8.20%*

Notes: Initial data of per capita GDP in constant 2005 prices for the 12 members of the Euro Area are detrended by HP filter with smoothing parameter set to 100. Deviations are obtained as percent of trend and found to be stationary as in Table 2. Estimation is obtained by OLS with fixed country and period effects. The implied long-run variability is obtained as in (41d). For comparison, a similar estimate is shown for the pre-crisis US economy; see Sandri (2011).
Source: AMECO.

Table 2. Unit root tests for the deviations of per capita GDP in EA 12

Sample: 1999 2013				
Method	Statistic	Prob.	Cross-sections	Obs.
Null: Unit root (assumes common unit root process)				
Levin, Lin & Chu t*	-4.272	0.00	9	121
Null: Unit root (assumes individual unit root process)				
Im, Pesaran and Shin W-stat	-2.602		7	94
ADF - Fisher Chi-square	26.66	0.0086	6	81
PP - Fisher Chi-square	11.9274	0.1545	4	56

Notes: The hypothesis of a common unit root is strongly rejected. Rejection for each particular country is also strong by the W-stat and the ADF test, but weaker by the PP-Fisher test.

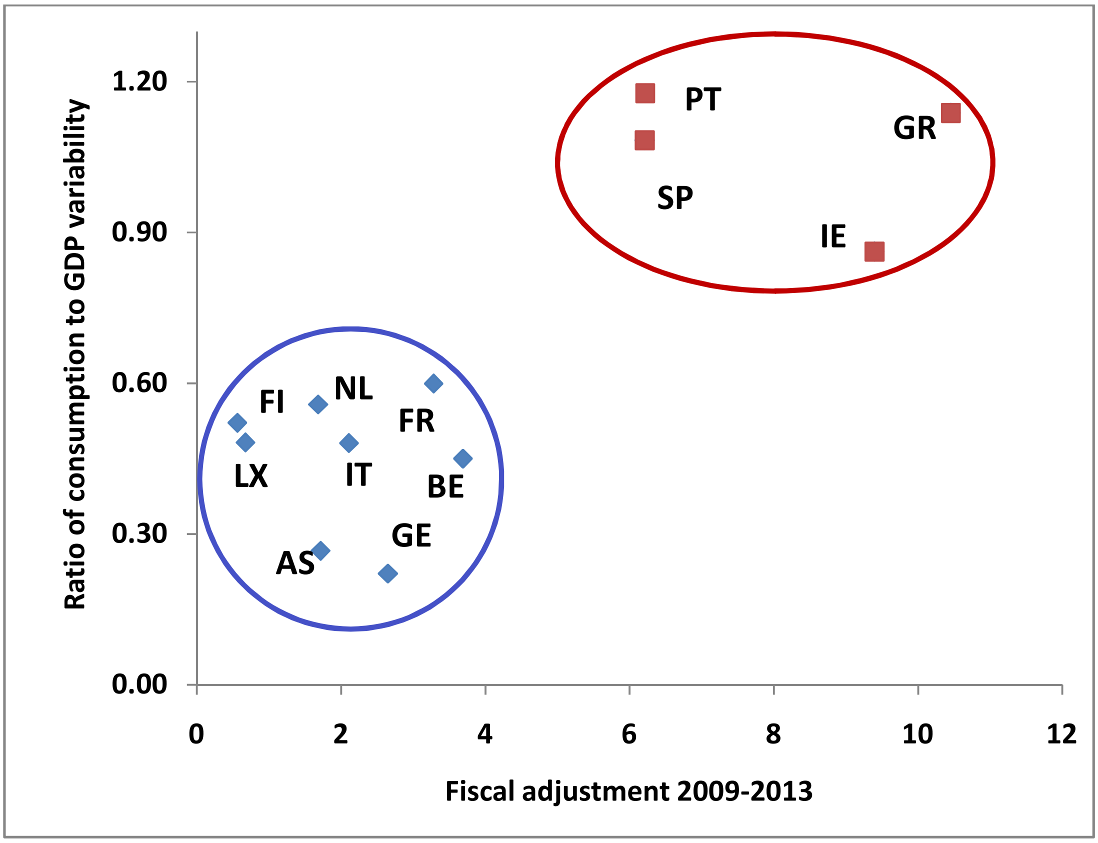

FIG.1: Post-crisis excess consumption volatility vs. the size of fiscal adjustment 2009-2013 as a ratio to GDP.

Note: Excess volatility is the ratio of average consumption variability over that of GDP per capita. Variability in each period is obtained as a 3-year moving average of standard deviations. Variable is per capita consumption and per capita GDP in constant 2005 prices.

Source: AMECO .

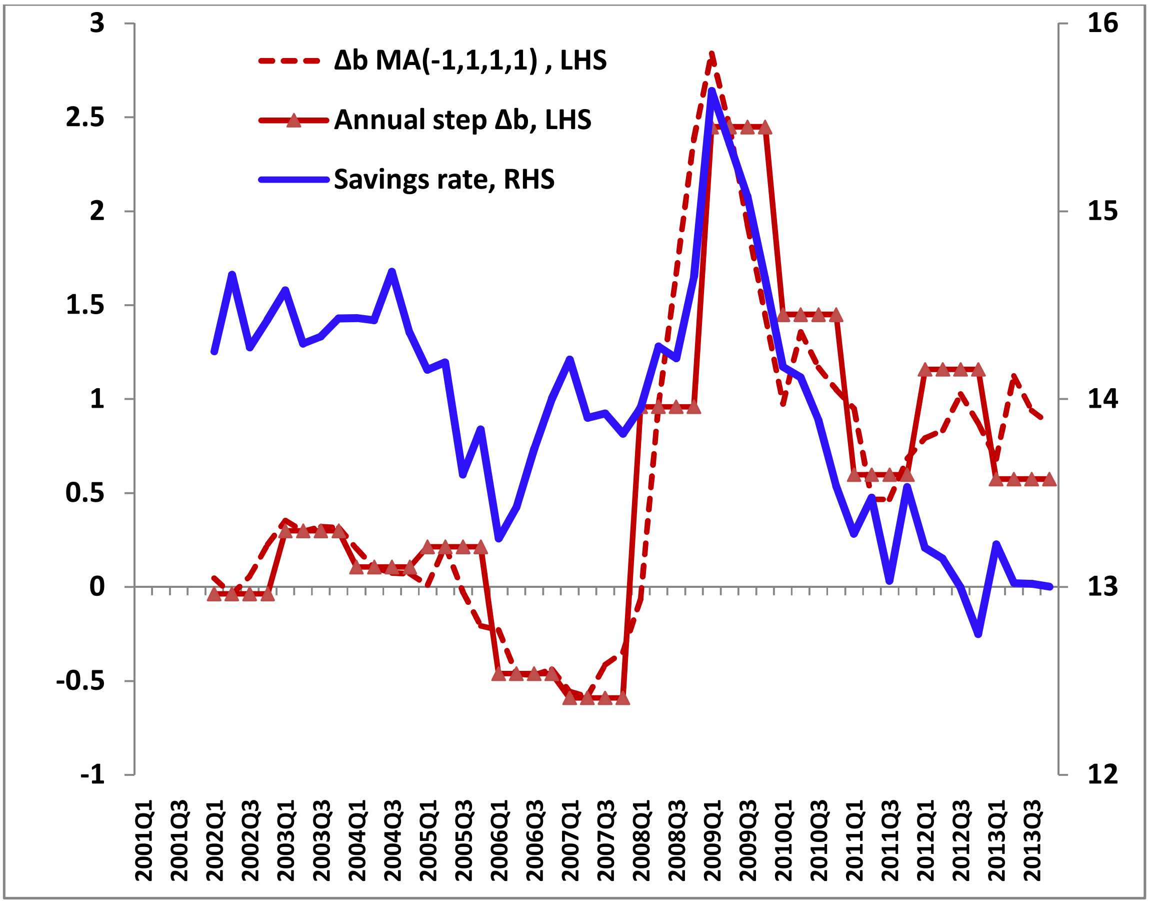

FIG.2 Quarterly savings rates of households and changes in the debt to GDP ratio in the Euro Area (EA 17).

Note: Step changes in debt are obtained from annual figures and are evenly distributed per quarter. A smooth trajectory is plotted as a moving average of (t-1,t,t+1,t+2,t+3) for the quarterly debt data. It is tilted forward to reflect incoming news about debt developments before they are finalized at the end of each year.

Source: Eurostat, seasonally adjusted households saving rates 2002Q1-2013Q4.

Annual debt data from AMECO. Quarterly debt data from ECB, not seasonally adjusted.

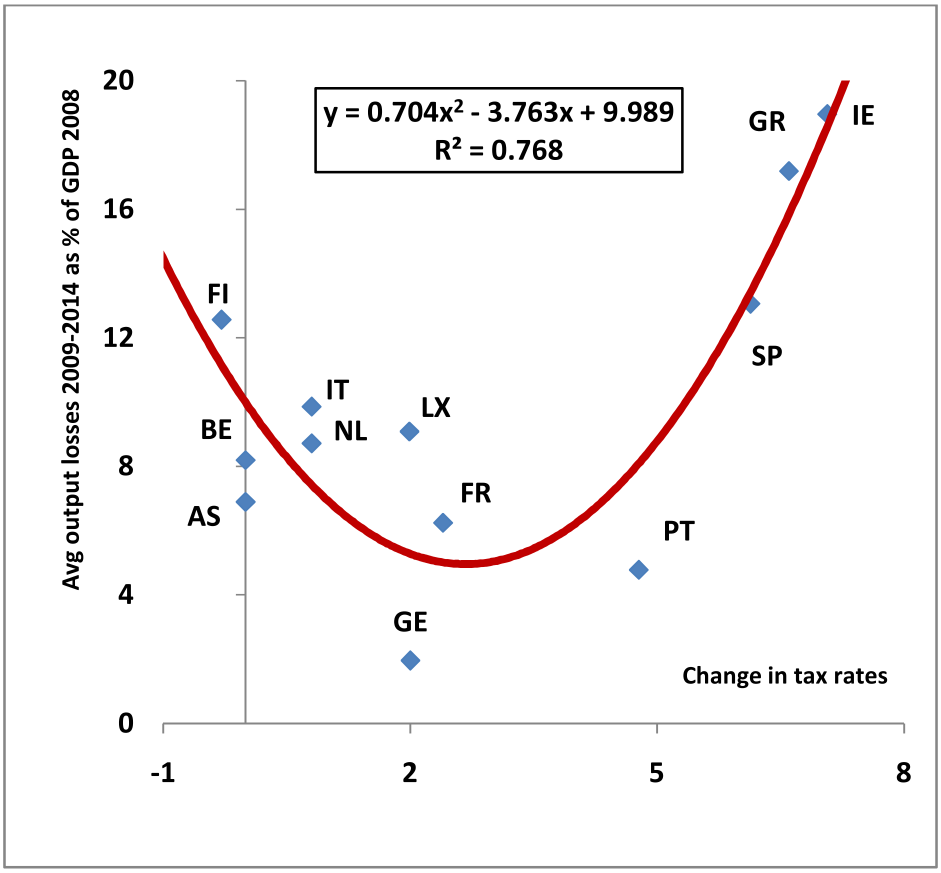

FIG.3 Average output losses during 2009-2014 vs. total changes in the tax rates 2009-2013. Losses are expressed as % of GDP in 2008.
Source: AMECO; tax rates from KPMG.

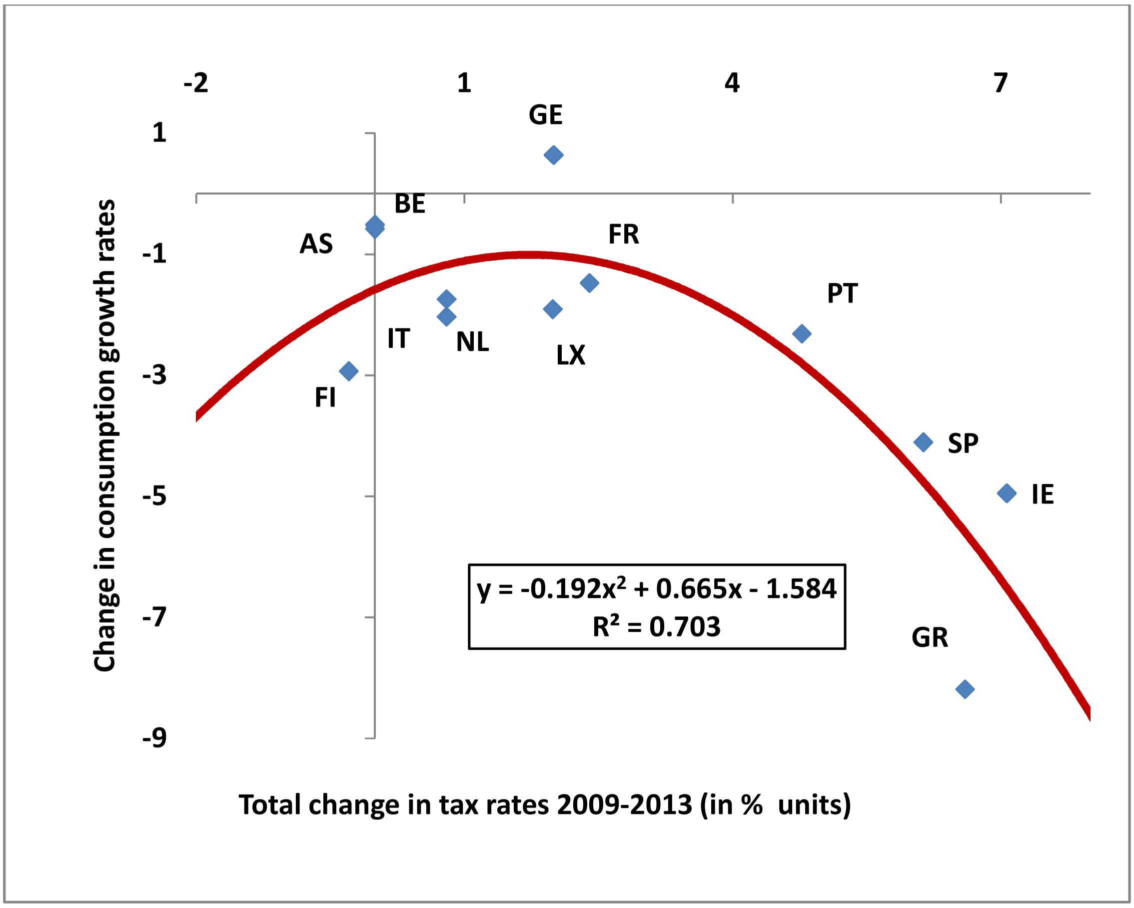

FIG.4 Changes in per capita consumption growth rates 2009-2013 over the pre-crisis period vs. total changes in tax rates 2009-2013.
Source: AMECO; tax rates from KPMG.

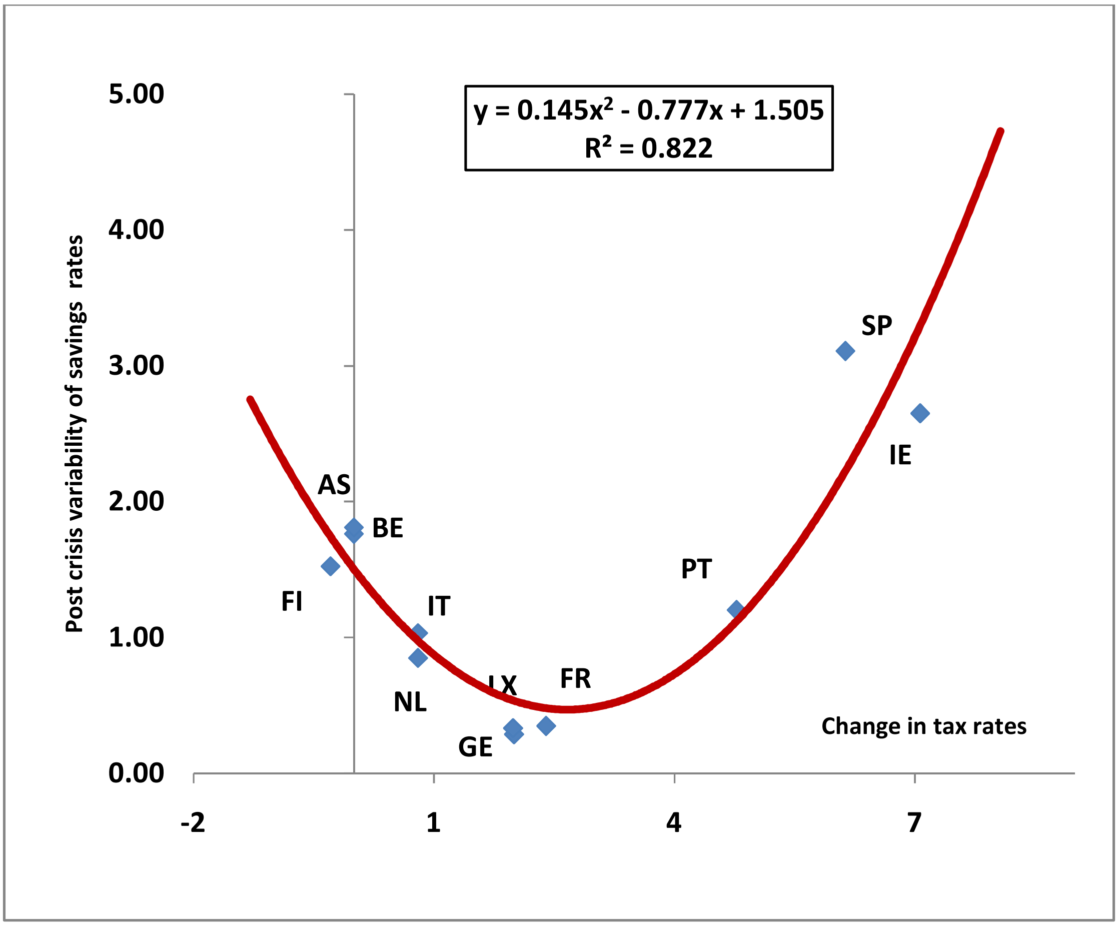

FIG.5 Savings rate volatility 2009-2013 vs. total changes in the tax rates.

Note: Volatility is measured by the standard deviation of percentage rates. Data for Greece not reported.

Source: Eurostat; tax rates from KPMG.

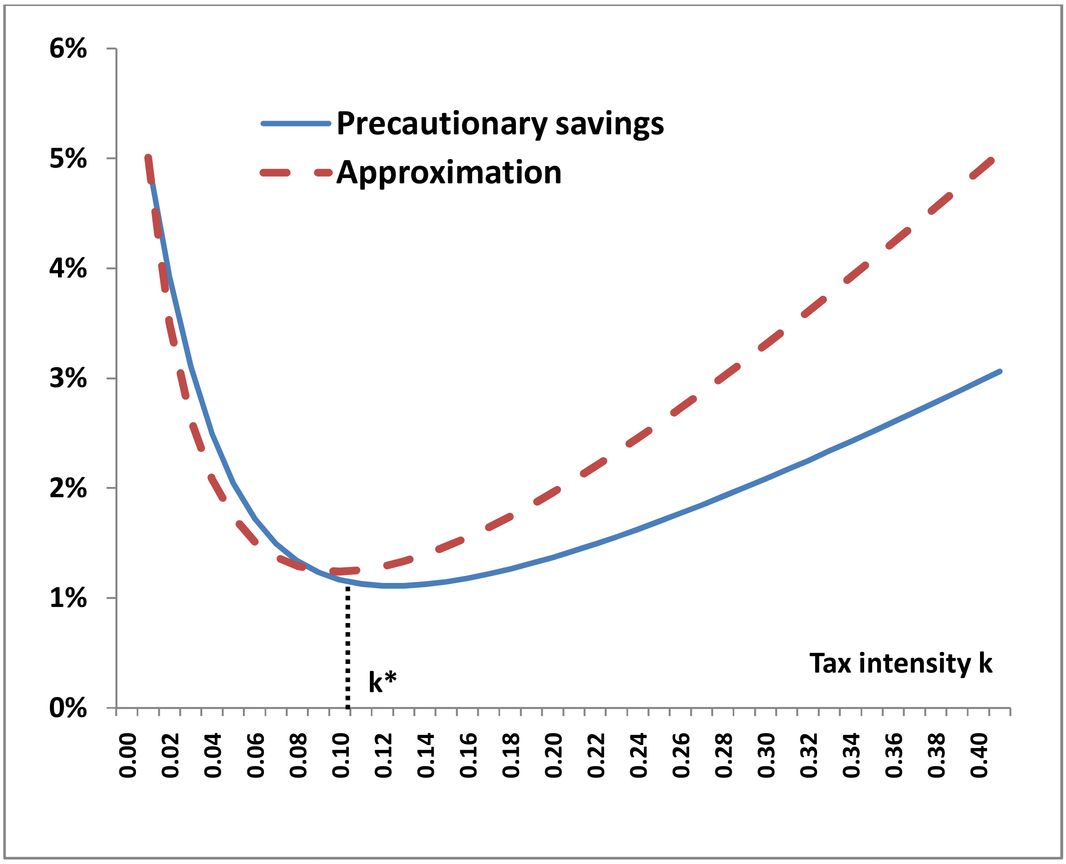

FIG.6 Precautionary savings and the log-linear approximation.
Parameter values set at $\theta=0.13$, $\mu=1$, $\sigma=0.082$, $\lambda=1/2$, $r=3\%$, $\Omega=1.10$.

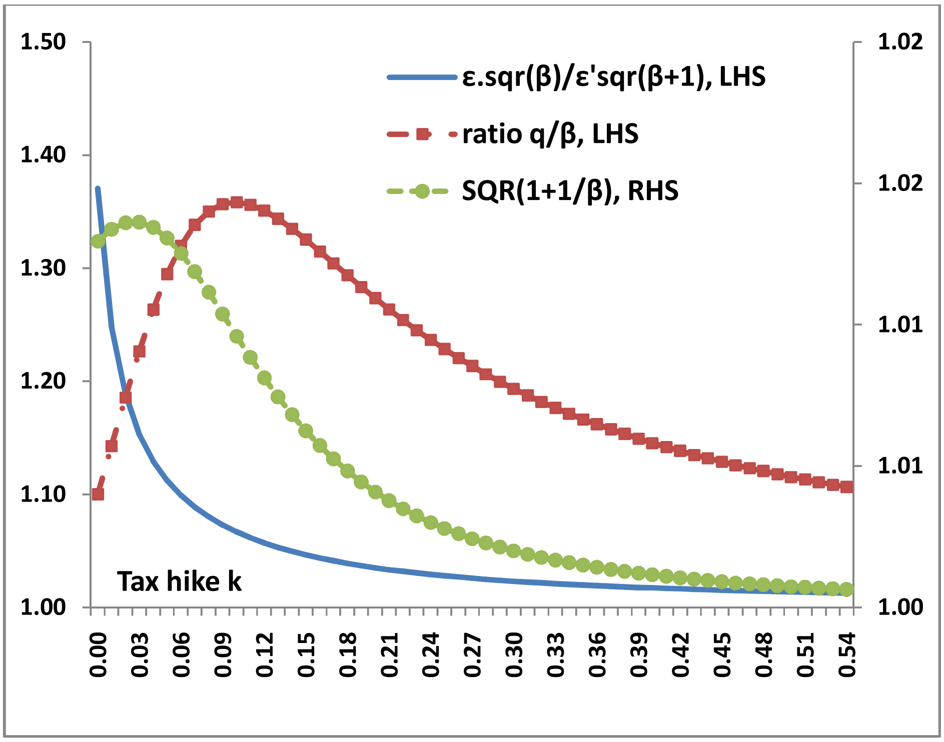

FIG. 7: The variables used in the log-linear approximation as functions of fiscal intensity *(k)*.

Parameter values set at $\theta=0.13$, $\mu=1$, $\sigma=0.082$, $\lambda=1/2$, $r=3\%$, $\Omega=1.10$.

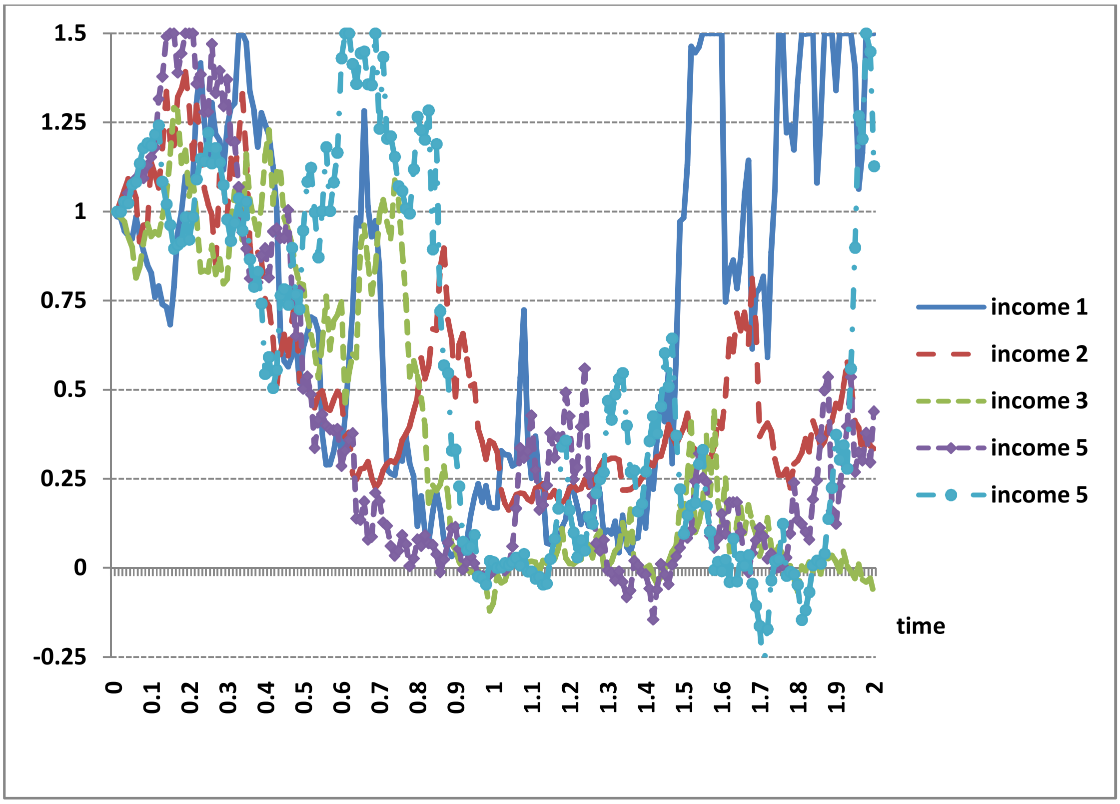

FIG. 8: Random realizations of a stochastic square process with parameters *θ=0.5, μ=1, σ=0.2, Ω=1.50*. Time unit *dt=0.01*.