Standard Stochastic Dominance

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Abstract: We propose a new stochastic dominance (SD) criterion based on Kimball's (1993) notion of standard risk aversion, which assumes decreasing absolute risk aversion (DARA) and decreasing absolute prudence (DAP). To implement the proposed criterion, we develop linear systems of optimality conditions for a given prospect relative to a discrete or polyhedral choice set in a general state-space model. An empirical application to historical stock market data shows that small-loser stocks are more appealing to standard risk averters than the existing mean-variance and SD criteria suggest, due to the positive skewness of their return distribution. Depending on the assumed trading strategy and evaluation horizon, accounting for standardness increases abnormal returns by about one to seven percentage points per annum relative to the second-order SD criterion and 50 to 200 basis points per annum relative to mean-variance and higher-order SD criteria. An analysis of the mean-variance tangency portfolio shows that the opportunity cost of the mean-variance approximation to direct utility maximization can be substantial.

Key words: Stochastic dominance, utility theory, standardness, prudence, temperance, linear programming, bootstrapping, portfolio efficiency, asset pricing, skewness, kurtosis, momentum, reversal.

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1. Introduction

Stochastic Dominance (SD) criteria impose a set of general regularity conditions for risky choice without assuming a particular functional form for the utility function. The most common regularity conditions are non-satiation and risk aversion, but additional assumptions about higher-order risk aversion are often required to exclude pathological functions and achieve an acceptable level of discriminating power. For example, third-order SD imposes the additional assumption of prudence (or skewness love) and fourth-order SD adds temperance (or kurtosis aversion). Levy (2006) provides an excellent survey of this literature.

This study develops and implements a new SD criterion based on Kimball's (1993) notion of Standard Risk Aversion. We refer to the resulting choice criterion as 'Standard Stochastic Dominance' (StSD). We will derive a system of linear equations for testing StSD relations in the spirit of the revealed preference approach (Samuelson (1938), Afriat (1967)). Linear Programming (LP) can identify feasible solutions or detect and quantify possible violations of the linear system. Using this approach, we will show that imposing standardness helps to improve the economic meaning and discriminating power of SD tests in a financial investment application.

Standardness combines decreasing absolute risk aversion (DARA) and decreasing absolute prudence (DAP). These higher-order risk aversion properties are required for consistency with important stylized facts about the relation between wealth, risk exposure and asset allocation. Specifically, standardness ensures that introducing a zero-mean background risk makes people less willing to accept another independent risk and, in addition, an increase in the risk level of an asset reduces the demand for that asset.

Despite the appeal of DARA and DAP, the common N-th order SD (NSD) rules do not impose these regularity conditions. As a case in point, SD of any order allows for a quadratic utility function (provided that the bliss point lies outside the relevant range of outcomes), despite the well-known undesirable economic properties of the quadratic form. The NSD rules therefore seem too permissive for analyzing prospects with a skewed outcome distribution. For example, Basso and Pianca (1997) show that NSD allows for financial option prices that are inconsistent with DARA, and Post, Fang and Kopa (PFK14) show that NSD underestimates the pricing errors to small-cap stocks for DARA investors.

Following Shalit and Yitzhaki (1994), Post (2003) and Kuosmanen (2004), we focus on stochastic dominance for a choice set of all convex combinations of a finite number of base prospects, for example, mixtures of financial securities, agricultural cropping programs or health care treatments. The Appendix briefly outlines how to derive solutions for the simpler case of a discrete choice set of a finite number of prospects (mixtures not allowed), including the simplest case of pairwise comparison of two given prospects.

Rather than imposing a particular shape for the joint probability distribution, we focus on a general state-space model. To arrive at finite optimization problems, we assume a finite number of states. This assumption is not restrictive because empirical studies generally use discrete sample distributions and experimental studies generally use a discrete population distribution. In addition, many continuous population distributions can be approximated accurately with some discrete distribution, for example, using a finite number of random draws from the population distribution.

The empirical part of our study applies the proposed StSD rule and several other decision criteria to analyze the efficiency of a broad stock market portfolio using historical return data.

The application can be viewed as an empirical test for a rational representative-investor model of capital market equilibrium or, alternatively, as a revealed-preference test for the observed behavior of investors who adopt a passive strategy of broad diversification. Empirical findings about which market segments and active strategies outperform a passive strategy may also be useful for active money managers. Apart from analyzing a market portfolio, we also use our StSD efficiency test to examine the mean-variance (MV) efficient tangency portfolio in order to estimate the opportunity cost of the MV approximation to direct utility maximization.

2. Standard Stochastic Dominance (StSD)

We consider M distinct prospects with risky outcomes, $X_1, \cdots, X_M \in \mathbb{R}_{++} \coloneqq (0, +\infty)$, that are treated as random variables with a discrete, state-dependent, joint probability distribution characterized by R mutually exclusive and exhaustive scenarios with probabilities $p_r > 0$ and realizations $x_{1,r}, \cdots, x_{M,r}$, $r = 1, \cdots, R$. Let $\mathcal{X}_0 := \{X_1, \cdots, X_M\}$. We assume that convex combinations of the individual prospects are feasible and the choice set is given by the convex hull $\mathcal{X} := Conv(\mathcal{X}_0) = \left\{\sum_{j=1}^{M} \lambda_j X_j : \sum_{j=1}^{M} \lambda_j = 1;\ \lambda_j \geq 0,\ j = 1, \dots, M\right\}$.

Our M prospects may be mixtures of more basic choice alternatives. To allow for general linear restrictions, the mixtures may represent the vertices of a general polyhedral choice set. To represent or approximate dynamic intertemporal choice problems, the prospects could be dynamic mixtures, such as actively managed investment portfolios with mixing weights that depend on a conditioning information set.

We use $Y \in \mathcal{X}$ for the evaluated prospect and y_r, $r = 1, \cdots, R$, for its realizations. The ordering of the scenarios is inconsequential for our analysis and we are free to label the scenarios by their ranking with respect to the evaluated prospect: $y_1 \leq \cdots \leq y_R$. In practice, this means that the scenarios may have to be re-ordered if we change the scenarios (for example, in statistical resampling procedures) or the evaluated prospect. The ranked outcomes $\{y_1, \cdots, y_R\}$ represent a partition of the interval $[y_1, y_R]$. We will use y_m for the median outcome, that is, $m \coloneqq \min_r\{r: \mathbb{P}(Y \leq y_r) \geq 0.5\}$.

Decision makers' preferences are described by four times continuously differentiable utility functions $u(x) \in \mathcal{C}^4: \mathbb{R}_{++} \to \mathbb{R}$. Let

$$a_u(x) := -\left(\frac{u''(x)}{u'(x)}\right); \tag{1.1}$$

$$p_u(x) \coloneqq -\left(\frac{u'''(x)}{u''(x)}\right), \tag{1.2}$$

the absolute risk aversion (ARA) quotient and the absolute prudence (AP) quotient, respectively. The Standard SD (StSD) utility functions follow:

$$\mathcal{U}_4^* := \{u \in \mathcal{U}_4 : u'(x) > 0; u''(x) < 0;\ a'_u(x) \leq 0; p'_u(x) \leq 0 \quad \forall x \in \mathbb{R}_{++}\}; \tag{2.1}$$

$$\mathcal{U}_4 := \{u \in \mathcal{C}^4 : u'(x) \geq 0;\ u''(x) \leq 0; u'''(x) \geq 0; u''''(x) \leq 0 \qquad \forall x \in \mathbb{R}_{++}\}. \tag{2.2}$$

The StSD functions $\mathcal{U}_4^*$ are a subset of the fourth-order stochastic dominance (FOSD) functions $\mathcal{U}_4$. The restrictions on the signs of the derivatives have the economic meaning of non-satiation ($u'(x) \geq 0$), risk aversion ($u''(x) \leq 0$), prudence ($u'''(x) \geq 0$) and temperance ($u''''(x) \leq 0$).

Standardness requires the additional restrictions of decreasing absolute risk aversion (DARA; $a'_u(x) \leq 0$) and decreasing absolute prudence (DAP; $p'_u(x) \leq 0$); we allow the ARA and AP quotients to be constant. DAP excludes, for example, discontinuous drops in the curvature $-u''(x)$ and the ARA quotient. It is well-known that global DAP implies global DARA. However, our analysis imposes DAP for the range $\mathcal{D} := [y_1, y_R]$ but not for its complement $\mathcal{D}^C := (0, y_1) \cup (y_R, +\infty)$. Since local DAP does not imply local DARA for the same range, we explicitly impose DAP in addition to DARA. Standardness also requires strict monotonicity ($u'(x) > 0$) and strict risk aversion ($u''(x) < 0$), because $u'(x)$ and $u''(x)$ enter as the divisor in $a_u(x)$ and $p_u(x)$, respectively.

Importantly, the above regularity conditions are robust to aggregation across individual utility functions, for example, to the level of a family or an economy, and $\mathcal{U}_4^*$ is a convex set. Convexity ensures that an aggregate decision maker has the same general risk aversion properties as the individual decision makers. In addition, convexity allows for representing all admissible utility functions as mixtures of basic utility functions that form the extreme rays of $\mathcal{U}_4^*$. We will return to these topics after introducing some additional notation and results.

We can reformulate StSD in terms of the functions

$$\mathcal{A}_u(x) := \ln(u'(x)); \tag{3.1}$$

$$\mathcal{P}_u(x) := \ln(-u''(x)), \tag{3.2}$$

which are the negative anti-derivatives of the ARA and AP quotients, respectively: $-\mathcal{A}_u{}'(x) = a_u(x)$ and $-\mathcal{P}_u{}'(x) = p_u(x)$.

LEMMA 1 (ARA-AP LINKAGES)

$$a_u{}'(x) \leq 0 \Leftrightarrow \mathcal{A}_u{}''(x) \geq 0; \tag{4.1}$$

$$a_u{}'(x) \leq 0 \Leftrightarrow p_u(x) \geq a_u(x); \tag{4.2}$$

$$p_u{}'(x) \leq 0 \Leftrightarrow \mathcal{P}_u{}''(x) \geq 0. \tag{4.3}$$

The analysis of DARA SD by PFK14 is based on the log convexity condition (4.1). Our analysis of StSD will also make use of the link between the ARA and AP quotients (4.2) and the log convexity condition (4.3).

PROPOSITION 1 (EXPONENTIATION OF ANTI-DERIVATIVES)

$$\mathcal{U}_4^* = \{u \in \mathcal{U}_4 : u'(x) = \exp(\mathcal{A}_u(x)); u''(x) = -\exp(\mathcal{P}_u(x)); -\mathcal{A}_u, -\mathcal{P}_u \in \mathcal{U}_2\}; \tag{5.1}$$

$$\mathcal{U}_2 := \{u \in \mathcal{C}^2 : u'(x) \geq 0;\ u''(x) \leq 0 \quad \forall x \in \mathbb{R}_{++}\}. \tag{5.2}$$

PROOF Clearly, (5.1) obeys all properties of (2.1): (i) $u'(x) = \exp(\mathcal{A}_u(x)) > 0$; (ii) $u''(x) = -\exp(\mathcal{P}_u(x)) < 0$; (iii) $-\mathcal{A}_u''(x) \leq 0 \Leftrightarrow a'_u(x) \leq 0$; (iv) $-\mathcal{P}_u''(x) \leq 0 \Leftrightarrow p'_u(x) \leq 0$. In addition, (2.1) obeys all properties of (5.1): (i) $u'(x) = \exp(\ln(u'(x))) = \exp(\mathcal{A}_u(x))$; (ii) $u''(x) = -\exp(\ln(-u''(x))) = -\exp(\mathcal{P}_u(x))$; (iii) $(u'(x) > 0) \wedge (u''(x) < 0) \Rightarrow 0 \leq a_u(x) = -\mathcal{A}_u'(x)$; (iv) $a'_u(x) \leq 0 \Leftrightarrow -\mathcal{A}_u'(x) \leq 0$; (v) $(u''(x) < 0) \wedge (a'_u(x) \leq 0 \Rightarrow u'''(x) > 0) \Rightarrow 0 \leq p_u(x) = -\mathcal{P}_u'(x)$; (vi) $p'_u(x) \leq 0 \Leftrightarrow -\mathcal{P}_u'(x) \leq 0$.■

PROPOSITION 2 (CONVEXITY) *The standard utility functions* $\mathcal{U}_4^*$ *form a convex cone in the function space:* $u, v \in \mathcal{U}_4^* \Rightarrow au + bv \in \mathcal{U}_4^*\ \forall a, b \geq 0: a + b > 0$.

PROOF Clearly, $u, v \in \mathcal{U}_4^* \Rightarrow au, bv \in \mathcal{U}_4^*\ \ \forall a, b > 0$, and therefore we need to prove only $\mathcal{U}_4^* \ni u + v =: w$. It also seems obvious that w is strictly increasing and $-w''$ is positive and decreasing as required, and what remains to be proven is that $-w''$ it is log-convex. Using (3), this condition is equivalent to convexity of $\mathcal{P}_w$, or $\mathcal{P}_w(y) \geq \mathcal{P}_w(x) - p_w(x)(y - x)$ for all $x, y \in \mathbb{R}_{++}$. Our strategy is to derive this inequality from the properties of u and v. Since $-u''$ and $-v''$ are log-convex, $\mathcal{P}_u$ and $\mathcal{P}_v$ are convex, or

$$\begin{cases} \mathcal{P}_u(y) \geq \mathcal{P}_u(x) - p_u(x)(y - x) \\ \mathcal{P}_v(y) \geq \mathcal{P}_v(x) - p_v(x)(y - x) \end{cases}.$$

Exponentiating both sides of both equations gives

$$\begin{cases} -u''(y) \geq -u''(x)\exp\big(-p_u(x)(y - x)\big) \\ -v''(y) \geq -v''(x)\exp\big(-p_v(x)(y - x)\big) \end{cases}.$$

Adding the two equations and dividing both sides by $-w''(x)$ gives

$$\frac{-w''(y)}{-w''(x)} \geq \left(\frac{-u''(x)}{-w''(x)}\right)\exp\big(-p_u(x)(y - x)\big) + \left(\frac{-v''(x)}{-w''(x)}\right)\exp\big(-p_v(x)(y - x)\big).$$

Taking the logarithm of both sides, and using the strict concavity of the logarithmic function, yields

$$\ln(-w''(y)) - \ln(-w''(x)) > \left(\frac{-p_u(x)u''(x) - p_v(x)v''(x)}{-w''(x)}\right)(y - x)$$

$$\Leftrightarrow \mathcal{P}_w(y) > \mathcal{P}_w(x) - p_w(x)(y - x). \blacksquare$$

DEFINITION 1 (STSD EFFICIENCY) *A given prospect* $Y \in \mathcal{X}$, *is StSD efficient if it is the optimum for some admissible utility function* $u \in \mathcal{U}_4^*$, or $Y = \operatorname{argmax}_{X \in \mathcal{X}} \sum_{r=1}^{R} p_r u(x_r)$.

LEMMA 2 (KKT CONDITIONS) *A given prospect* $Y \in \mathcal{X}$ *is StSD efficient if and only if it obeys the Karush-Kuhn-Tucker first-order optimality conditions for some admissible utility function* $u \in \mathcal{U}_4^*$*:*

$$\sum_{r=1}^{R} p_r u'(y_r)\big(y_r - x_{j,r}\big) \geq 0, \qquad j = 1, \cdots, M. \tag{6}$$

Although the StSD functions are of infinite dimension, the optimality conditions (6) evaluate marginal utility only at the discrete outcome levels $y_1, \cdots, y_R$, allowing for a formulation of finite dimension. We can use a piecewise-linear representation of $\mathcal{P}_u(x)$ (or piecewise-constant AP quotient $p_u(x)$), the details of which are given in the next section. Repeated integration by parts of $u''(x) = -\exp\big(\mathcal{P}_u(x)\big)$ yields a composite utility function $u(x) = u_1(x) + u_2(x)$ that consists of a piecewise-exponential component $u_1(x) = -p_u(x)^{-2}\exp\big(\mathcal{P}_u(x)\big)$ and a piecewise-linear component $u_2(x)$ that ensures continuity of $u'(x)$ and $u(x)$. Unfortunately, optimization over such functions is a non-linear and non-convex problem, which makes this approach unpractical. The next section therefore introduces a useful linearization.

3. Linear StSD Conditions

Our strategy is to linearize the conditions $-\mathcal{A}_u \in \mathcal{U}_2$, $-\mathcal{P}_u \in \mathcal{U}_2$, $u \in \mathcal{U}_4$ and the exponentiations $u'(x) = \exp(\mathcal{A}_u(x))$ and $u''(x) = -\exp(\mathcal{P}_u(x))$. We use an exact linearization for $-\mathcal{A}_u \in \mathcal{U}_2$, $-\mathcal{P}_u \in \mathcal{U}_2$, and $u \in \mathcal{U}_4$ using particular Riemann sums, and a (tight) local linear approximation for the exponential function. We will also use a local linear approximation to the logarithmic function in order to normalize the levels of log marginal utility.

PROPOSITION 3 (LINEARIZED LOG DERIVATIVES) *For any utility function $u \in \mathcal{U}_4^*$, we can represent $\mathcal{A}_u(y_r)$ and $\mathcal{P}_u(y_r)$, $r = 1, \cdots, R$, using decreasing and convex piecewise-linear functions that are linear in a finite number of parameters and obey a set of linear restrictions:*

$$\mathcal{A}_u(y_r) = \sum_{k=r}^{R-1} \alpha_k (y_{k+1} - y_r) + \alpha_R, \qquad r = 1, \cdots, R; \tag{7.1}$$

$$\alpha_r \geq 0, \qquad r = 1, \cdots, R-1; \tag{7.2}$$

$$\mathcal{P}_u(y_r) = \sum_{k=r}^{R-1} \pi_k (y_{k+1} - y_r) + \pi_R, \qquad r = 1, \cdots, R; \tag{8.1}$$

$$\sum_{k=r}^{R-1} \pi_k \geq \sum_{k=r+1}^{R-1} \alpha_k, \qquad r = 1, \cdots, R-2; \tag{8.2}$$

$$\pi_r \geq 0, \qquad r = 1, \cdots, R-1. \tag{8.3}$$

PROOF The mean-value theorem for integration implies that the values of the anti-derivative can be represented using a Riemann sum: $\mathcal{A}_u(y_r) = \sum_{k=r}^{R-1} a_u\left(z_k^{(a)}\right)(y_{k+1} - y_k) + \ln\left(u'(y_R)\right)$, $1, \cdots, R$, where the sampling points $z_k^{(a)} \in [y_r, y_{r+1}]$ are selected such that $a_u\left(z_k^{(a)}\right) = \ln\left(u'(y_r)/u'(y_{r+1})\right)/(y_{r+1} - y_r)$, $r = 1, \cdots, R-1$. We can further decompose the ARA coefficients as $a_u\left(z_r^{(a)}\right) = \sum_{k=r}^{R-1} \alpha_k$, using the ARA decrements $\alpha_r = a_u\left(z_r^{(a)}\right) - a_u\left(z_{r+1}^{(a)}\right)$, $r = 1, \cdots, R-2$, and the ARA level $\alpha_{R-1} = a_u\left(z_{R-1}^{(a)}\right)$. Consequently, the values of the anti-derivative can be built via summation by parts using (7.1), where $\alpha_R = \ln\left(u'(y_R)\right)$. The non-negativity conditions (7.2) reflect risk aversion ($\alpha_{R-1} \geq 0$) and DARA ($\alpha_r \geq 0$, $r = 1, \cdots, R-2$). Similarly, the levels of $\mathcal{P}_u(y_r)$ in (8.1) are built from the AP decrements $\pi_t = p_u\left(z_r^{(p)}\right) - p_u\left(z_{r+1}^{(p)}\right)$, $r = 1, \cdots, R-2$, and the ARA level $\pi_{R-1} = p_u\left(z_{R-1}^{(p)}\right)$, for sampling points $z_r^{(p)} \in [y_r, y_{r+1}]$ such that $z_r^{(p)} = \ln(u''(y_r)/u''(y_{r+1}))/(y_{r+1} - y_r)$, $r = 1, \cdots, R-1$, and $\pi_R = \ln\left(-u''(y_R)\right)$. Inequalities (8.2) follow from (4.2), $a_u\left(z_{r+1}^{(a)}\right) = \sum_{k=r+1}^{R-1} \alpha_k$, $p_u\left(z_r^{(p)}\right) = \sum_{k=r}^{R-1} \pi_k$, and $z_{r+1}^{(a)} \geq z_r^{(p)}$. The non-negativity conditions (8.3) reflect prudence ($\pi_{R-1} \geq 0$) and DAP ($\pi_r \geq 0$, $r = 1, \cdots, R-2$). ■

PROPOSITION 4 (LINEARIZED DERIVATIVES) *For any utility function $u \in \mathcal{U}_4$, we can represent $u'(y_r)$ and $-u''(y_r)$, $r = 1, \cdots, R$, using decreasing and convex piecewise-linear functions that are linear in a finite number of parameters and obey a set of linear restrictions:*

$$u'(y_r) = \sum_{k=r}^{R-1} \beta_k (y_{k+1} - y_r) + \beta_R, \qquad r = 1, \cdots, R; \tag{9.1}$$

$$\beta_r \geq 0, \qquad r = 1, \cdots, R; \tag{9.2}$$

$$-u''(y_r) = \sum_{k=r}^{R-1} \gamma_k (y_{k+1} - y_r) + \gamma_R, \qquad r = 1, \cdots, R; \tag{10.1}$$

$$\sum_{k=r}^{R-1} \gamma_k (y_{k+1} - y_r) + \gamma_R \leq \sum_{k=r-1}^{R-1} \beta_r, \qquad r = 2, \cdots, R-1; \tag{10.2}$$

$$\sum_{k=r}^{R-1} \gamma_k (y_{k+1} - y_r) + \gamma_R \geq \sum_{k=r+1}^{R-1} \beta_r, \qquad r = 1, \cdots, R-2; \tag{10.3}$$

$$\gamma_r \geq 0, \qquad r = 1, \cdots, R. \tag{10.4}$$

PROOF By analogy to Proposition 3, the derivatives $u'(y_r)$ in (9.1) are built via summation by parts from the increments of the second-order derivative $\beta_r = u''\left(z_{r+1}^{(2)}\right) - u''(z_r^{(2)})$, $r = 1, \cdots, R-2$, $\beta_{R-1} = -u''(z_{R-1}^{(2)})$, for sampling points $z_r^{(2)} \in [y_r, y_{r+1}]$ such that $u''\left(z_r^{(2)}\right) = (u'(y_{r+1}) - u'(y_r))/(y_{r+1} - y_r)$, $r = 1, \cdots, R-1$, and $\beta_R = u'(y_R)$. The non-negativity conditions (9.2) reflect non-satiation ($\beta_R \geq 0$), risk aversion ($\beta_{R-1} \geq 0$), and prudence ($\beta_r \geq 0 \;\; r = 1, \cdots, R-2$). Similarly, the second-order derivatives $u''(y_r)$ in (10.1) are built from the decrements of the third-order derivative $\gamma_r = u'''(z_r^{(3)}) - u'''(z_{r+1}^{(3)})$, $r = 1, \cdots, R-2$, $\gamma_{R-1} = u'''(z_{R-1}^{(3)})$, for sampling points $z_r^{(3)} \in [y_r, y_{r+1}]$ such that $u'''\left(z_r^{(3)}\right) = (u''(y_{r+1}) - u''(y_r))/(y_{r+1} - y_r)$, $r = 1, \cdots, R-1$, and $\gamma_R = -u''(y_R)$. Inequalities (10.2) follow from $u'''(x) \geq 0$, $-u''\left(z_{r-1}^{(2)}\right) = \sum_{k=r-1}^{R-1} \beta_r$, and $z_{r-1}^{(2)} \leq y_r$; similarly, (10.3) follow from $u'''(x) \geq 0$, $-u''\left(z_r^{(2)}\right) = \sum_{k=r}^{R-1} \beta_r$, and $y_r \leq z_r^{(2)}$. The non-negativity conditions (10.4) follow from risk aversion ($\gamma_R \geq 0$), prudence ($\gamma_{R-1} \geq 0$), and temperance ($\gamma_r \geq 0 \;\; r = 1, \cdots, R-2$). ■

Propositions 3 and 4 give an exact linearization rather than an approximation. When using fixed sampling points (for example, the midpoints $z_r = 0.5(y_r + y_{r+1})$, $r = 1, \cdots, R-1$), a Riemann sum gives an approximation for the definite integral in question, the accuracy of which generally depends on the coarseness of the partition. By contrast, our sampling points $z_r = z_r^{(a)}$, $z_r^{(p)}$, $z_r^{(2)}$, $z_r^{(3)}$, $r = 1, \cdots, R-1$, are derived endogenously as the points where the tangent of $h(y) = \mathcal{A}_u(y)$, $\mathcal{P}_u(y)$, $u'(y)$, $u''(y)$ is parallel to the secant that joins $(y_r, h(y_r))$ and $(y_{r+1}, h(y_{r+1}))$. The mean-value theorem implies that these tangency points exist and that the Riemann sum equals the definite integral for this specification regardless of the coarseness of the partition.

Proposition 4 deviates in an important way from Post and Kopa (2013, Theorem 1). For FOSD, they model $u'(y_r)$ ($u''(y_r)$) with a piecewise-quadratic (piecewise-linear) function that is obtained by integrating over a piecewise-constant third-order derivative $u'''(y_r)$. This approach would be inconsistent with standardness, because a piecewise-quadratic function implies locally increasing ARA and AP. For this reason, we use two separate piecewise-linear functions, based on two different sets of sampling points, $z_r^{(2)}$ for $u'(y_r)$ and $z_r^{(3)}$ for $u''(y_r)$, rather than a single piecewise quadratic function. The two approaches converge as the partition is refined ($(w_{r+1} - w_r) \to 0$), but only our formulation allows for a coarse partition.

We have now derived two sets of linear conditions: (7) and (8) express $\mathcal{A}_u(y_r)$ and $\mathcal{P}_u(y_r)$ as linear functions of the levels and differences of the ARA and AP quotients; (9) and (10) express $u'(y_r)$ and $u''(y_r)$ as linear functions of the levels and differences of the second-order and third-order derivatives. The relation between these sets of restrictions is non-linear: $u'(y_r) = \exp\left(\mathcal{A}_u(y_r)\right)$ and $u''(y_r) = -\exp\left(\mathcal{P}_u(y_r)\right)$. We will develop a set of linear conditions based on a (tight) local first-order approximation to the exponentiation.

Let $f_s \in \mathcal{U}_4^*$, $s = 1, \cdots, S$, be 'frame functions', or prior parametric utility functions that are fitted to the decision problem in question. Since marginal utility $u'(y)$ is decreasing, it seems natural to assume that at least one of the frame functions obeys the following minimal goodness criterion:

$$u'(y_m) \geq \min_{\substack{s=1,\cdots,S \\ r=1,\cdots,R}} f_s{}'(y_r) = \min_{s=1,\cdots,S} f_s{}'(y_R). \tag{11}$$

Our default choice is $S = 4$ CRRA power functions $f_s(x) := \frac{\eta_s}{1-\theta_s} x^{1-\theta_s}$, $s = 1, \cdots, 4$, with risk aversion coefficients $\theta_1 = 0.5$, $\theta_2 = 1$, $\theta_3 = 2$ and $\theta_4 = 4$ (a plausible range in many applications), and η_1, η_2, η_3, and η_4 selected to obtain the desired normalization (see below). Additional or alternative frame functions may be used depending on the application in question.

Each frame function generates a set of plausible starting values for the derivatives, $f_s{}'(y_r)$ and $f_s{}''(y_r)$, $r = 1, \cdots, R$, $s = 1, \cdots, S$. We may apply a local first-order Taylor series approximation of the exponential function $g(x) := \exp(x)$ at point $\mathcal{A}_u(y_r)$ around point $\mathcal{A}_{f_s}(y_r)$, and a similar approximation at point $\mathcal{P}_u(y_r)$ around point $\mathcal{P}_{f_s}(y_r)$:

LEMMA 3 (EXPONENTIATION) *For any utility function $u \in \mathcal{U}_4^*$ and set of frame functions $f_s \in \mathcal{U}_4^*$, $s = 1, \cdots, S$,*

$$u'(y_r) = g(\mathcal{A}_u(y_r)) \geq f_s{}'(y_r)\left(1 + \mathcal{A}_u(y_r) - \mathcal{A}_{f_s}(y_r)\right), \qquad r = 1, \cdots, R; \tag{12.1}$$

$$-u''(y_r) = g(\mathcal{P}_u(y_r)) \geq -f_s{}''(y_r)\left(1 + \mathcal{P}_u(y_r) - \mathcal{P}_{f_s}(y_r)\right), \qquad r = 1, \cdots, R. \tag{12.2}$$

PROOF Since $g(x)$ is convex and $g(x) = g'(x)$, we find $g(x_2) = g(x_1) + g'(x_1)(x_2 - x_1) = g(x_1)(1 + x_2 - x_1)$. Setting $x_1 = \mathcal{A}_{f_s}(y_r)$ and $x_2 = \mathcal{A}_u(y_r)$ gives (12.1); $x_1 = \mathcal{P}_{f_s}(y_r)$ and $x_2 = \mathcal{P}_u(y_r)$ gives (12.2). ■

Our frame functions are reminiscent of Meyer's (1977a,b) Stochastic Dominance With Respect to a Function but differs from that approach in an important way. SDWRF uses given functions as exogenous lower and upper bounds for the ARA quotient $a_u(x)$. By contrast, our approach uses the frame function to generate sampling points for a local linear approximation to the exponentiation in order to build *endogenous* lower bounds for the values of $u'(y_r)$ and $-u''(y_r)$, $r = 1, \cdots, R$, that are consistent with the values of $\mathcal{A}_u(y_r)$ and $\mathcal{P}_u(y_r)$. StSD can however be used in combination with SDWRF.†

† For example, it seems appropriate in many cases to assume that the relative risk aversion (RRA) quotient $r_u(x) = a_u(x)x$ is not smaller than 1 at the highest outcome level, or $r_u(y_R) \geq 1$. A value of 1 seems relatively low compared with empirical estimates in many applications. In addition, Arrow observed that

To avoid numerical instability, we normalize utility such that marginal utility has an average value of unity, or

$$1 = \sum_{r=1}^{R} p_r \, u'(y_r) \coloneqq \overline{u'(Y)}. \tag{13.1}$$

We deviate from the median-based normalization of PFK14 (Eq. (8.3)) in order to allow for a better comparison across different choice criteria (for a given data set) and to follow the convention in the asset pricing literature (our application area). The median-based normalization inflates the average level of marginal utility and common inefficiency measures if the marginal utility function is skewed (assigns a relatively high value to the worst outcomes). Hence, this normalization can induce a bias towards utility functions with a relatively low absolute prudence, such as the negative exponential ($p_u(x) = a_u(x)$). By contrast, our mean-based normalization allows for a skewed marginal utility function without inflating the levels of marginal utility and inefficiency. This alternative normalization requires a non-trivial adjustment to the normalization of log marginal utility of PFK14 (Eq. (7.2)).

LEMMA 4 (NORMALIZING LOG MARGINAL UTILITY) *Normalization (13.1) implies the following joint condition for marginal utility and log marginal utility:*

$$u'(y_m) - \left(\frac{\min_{s=1,\cdots,S} f_s{}'(y_R) - 1}{\min_{s=1,\cdots,S} \mathcal{A}_{f_s}(y_R)} \right) \mathcal{A}_u(y_m) \leq 1. \tag{13.2}$$

PROOF Using convexity of $u'(x)$ and (11), we know that $\min_s f_s{}'(y_R) \leq u'(y_m) \leq \overline{u'(Y)} = 1$. Concavity of the logarithmic function therefore implies $\mathcal{A}_u(y_m) \geq \lambda \left(\min_{s=1,\cdots,S} \mathcal{A}_{f_s}(y_R) \right) + (1-\lambda)\ln(1)$, with $\lambda \coloneqq \frac{u'(y_m)-1}{\min_s f_s{}'(y_R)-1}$. It follows that $\mathcal{A}_u(y_m) \geq \left(\frac{u'(y_m)-1}{\min_s f_s{}'(y_R)-1} \right) \left(\min_{s=1,\cdots,S} \mathcal{A}_{f_s}(y_R) \right)$ or, rearranging terms, (13.2). ■

Whereas (12.1) bounds $u'(y_m)$ from *below* using linear *outer* approximations to the *exponential* function $\exp\left(\ln\left(u'(y_m)\right)\right)$, (13.2) bounds $u'(y_m)$ from *above* using linear *inner* approximations to the *logarithmic* function $\ln(u'(y_m))$. Since the median $u'(y_m)$ generally is not too far below the average value $\overline{u'(Y)} = 1$, the linear approximation to the logarithmic function works well and the inequalities become approximate equalities in our applications.

We can combine Lemma 1, 3 and 4 and Proposition 1, 3 and 4 to derive the following linear conditions for StSD efficiency:

THEOREM 1 (CONDITIONS FOR STSD) A *necessary condition for StSD efficiency of a given prospect $Y \in \mathcal{X}$ is that, for any given set of frame functions $f_s \in \mathcal{U}_4^*$, $s = 1, \cdots, S$, there exists a solution, $\alpha_r^*, \pi_r^*, \beta_r^*, \gamma_r^*$, $r = 1, \cdots, R$, to the following system of linear inequalities*:

$\lim_{x\to\infty} r_u(x) > 1$ is needed for utility to be bounded from above. To impose $r_u(y_R) \geq 1$, one could use the restriction $\alpha_{R-1} \geq 1/y_R$, based on $r(y_R) = a_u(y_R) y_R$ and $\alpha_{R-1} = a_u(z_{R-1}^{(a)}) \geq a_u(y_R)$ (due to DARA and $z_{R-1}^{(a)} \leq y_R$).

$$\sum_{r=1}^{R} p_r \left(\sum_{k=r}^{R-1} \beta_k (y_{k+1} - y_r) + \beta_R\right)(y_r - x_{j,r}) \geq 0, \qquad j = 1, \cdots, M; \tag{14.1}$$

$$\sum_{k=r}^{R-1} \beta_k (y_{k+1} - y_r) + \beta_R \geq f_s{}'(y_r)\left(1 + \sum_{k=r}^{R-1} \alpha_k (y_{k+1} - y_r) + \alpha_R - \mathcal{A}_{f_s}(y_R)\right), \quad r = 1, \cdots, R, s = 1, \cdots, S; \tag{14.2}$$

$$\sum_{k=r}^{R-1} \gamma_k (y_{k+1} - y_r) + \gamma_R \geq -f_s{}''(y_r)\left(1 + \sum_{k=r}^{R-1} \pi_k (y_{k+1} - y_r) + \pi_R - \mathcal{P}_{f_s}(y_R)\right), \quad r = 1, \cdots, R, s = 1, \cdots, S; \tag{14.3}$$

$$\sum_{r=1}^{R} p_r \left(\sum_{k=r}^{R-1} \beta_k (y_{k+1} - y_r) + \beta_R\right) = 1; \tag{14.4}$$

$$\left(\sum_{k=m}^{R-1} \beta_k (y_{k+1} - y_m) + \beta_R\right) - \left(\frac{\min_{s=1,\cdots,S} f_s{}'(y_R) - 1}{\min_{s=1,\cdots,S} \mathcal{A}_{f_s}(y_R)}\right)\left(\sum_{k=m}^{R-1} \alpha_k (y_{k+1} - y_m) + \alpha_R\right) \leq 1; \tag{14.5}$$

$$\sum_{k=r}^{R-1} \pi_k \geq \sum_{k=r+1}^{R-1} \alpha_k, \qquad r = 1, \cdots, R-2; \tag{8.2}$$

$$\sum_{k=r}^{R-1} \gamma_k (y_{k+1} - y_r) + \gamma_R \leq \sum_{k=r-1}^{R-1} \beta_r, \qquad r = 2, \cdots, R-1; \tag{10.2}$$

$$\sum_{k=r}^{R-1} \gamma_k (y_{k+1} - y_r) + \gamma_R \geq \sum_{k=r+1}^{R-1} \beta_r, \qquad r = 1, \cdots, R-2; \tag{10.3}$$

$$\alpha_r \geq 0, \qquad r = 1, \cdots, R-1; \tag{7.2}$$

$$\pi_r \geq 0, \qquad t = 1, \cdots, R-1; \tag{8.3}$$

$$\beta_r \geq 0, \qquad t = 1, \cdots, R; \tag{9.2}$$

$$\gamma_r \geq 0, \qquad t = 1, \cdots, R. \tag{10.4}$$

PROOF Restrictions (14.1) follow from (6) and (9.1); (14.2) from (12.1), (9.1) and (7.1); (14.3) from (12.2), (10.1) and (8.1); (14.4) from (13.1) and (9.1); (14.5) from (13.2), (9.1) and (7.1). To prove the necessary condition, suppose that the evaluated prospect is optimal for $u \in \mathcal{U}_4^*$, and hence also for the normalized function $v = u/\overline{u'(Y)} \in \mathcal{U}_4^*$. A solution to the linear system then is $\alpha_r = a_v\left(z_r^{(a)}\right) - a_v\left(z_{r+1}^{(a)}\right), r = 1, \cdots, R-2$, $\alpha_{R-1} = a_v\left(z_{R-1}^{(a)}\right)$, and $\alpha_R = \ln(v'(y_R))$; $\pi_r = p_v\left(z_r^{(p)}\right) - p_v\left(z_{r+1}^{(p)}\right), r = 1, \cdots, R-2$, $\pi_{R-1} = p_v\left(z_{R-1}^{(p)}\right)$, and $\pi_R = \ln(-v''(y_R))$; $\beta_r = v''\left(z_{r+1}^{(2)}\right) - v''\left(z_r^{(2)}\right)$, $r = 1, \cdots, R-2$, $\beta_{R-1} = -v''(z_{R-1}^{(2)})$, and $\beta_R = v'(y_R)$; $\gamma_r = v'''\left(z_r^{(3)}\right) - v'''\left(z_{r+1}^{(3)}\right)$, $r = 1, \cdots, R-2$, $\gamma_{R-1} = v'''\left(z_{R-1}^{(3)}\right)$, and $\gamma_R = -v''(y_R)$. ■

The linear system gives a necessary but not sufficient condition. The system cannot falsely classify an efficient prospect as inefficient but it may falsely classify an inefficient prospect as efficient. Some feasible solutions may not obey all regularity conditions, because the four sets of sampling points $z_r = z_r^{(a)}$, $z_r^{(p)}$, $z_r^{(2)}$, $z_r^{(3)}$, $r = 1, \cdots, R-1$, may be different and, in addition, the local linear approximation for the exponential and logarithmic functions in (14.2), (14.3) and (14.5) may be imperfect.

The use of a necessary condition in Theorem 1 is consistent with the general hierarchical structure of the SD rules. Non-dominance by lower-order SD rules (FSD and SSD) is a necessary condition for non-dominance by higher-order rules (TSD, FOSD), DARA SD and StSD; and the latter is a necessary condition for optimality for any given admissible utility function.

The strength of the necessary condition increases with the accuracy of the partition $\{y_1, \cdots, y_R\}$ and with the number of frame functions, $f_s \in \mathcal{U}_4^*$, $s = 1, \cdots, S$. As the partition is refined $((y_{r+1} - y_r) \to 0)$, the four sets of sampling points converge, as $z_r \in [y_r, y_{r+1}]$ for all $z_r = z_r^{(a)}$, $z_r^{(p)}$, $z_r^{(2)}$, $z_r^{(3)}$, $r = 1, \cdots, R-1$. Similarly, the approximation error for the exponential and logarithmic functions shrink as we increase the number of frame functions, $f_s \in \mathcal{U}_4^*$, $s = 1, \cdots, S$.

We can diagnose solutions *a posteriori*, that is, after testing the necessary condition, to detect possible false efficiency classification and determine the need for refinements. If the optimal marginal utility levels $u'(y_r) = \sum_{k=r}^{R-1} \beta_k^* (y_{k+1} - y_r) + \beta_R^*$, $r = 1, \cdots, R$, and curvature levels $-u''(y_r) = \sum_{k=r}^{R-1} \gamma_k^* (y_{k+1} - y_r) + \gamma_R^*$, $r = 1, \cdots, R$, do not show a log-convex pattern, then an approximation error must have occurred. It is straightforward to check for log-convexity, either by visual inspection of a log-level plot or with a small linear program.

The linear system is relatively small with $\mathcal{O}(R)$ variables and constraints. We can test the linear system using mathematical programming. The specific formulation of the program would of course depend on the specific application area and decision problem. Our empirical section will develop a linear program for testing StSD efficiency of a stock market portfolio relative to all portfolios formed from a set of base assets.

4. Empirical application

A. Data Sets

The market portfolio in our analysis is a value-weighted average of all non-financial common stocks that are listed on the NYSE, AMEX or NASDAQ exchange and covered by the CRSP and COMPUSTAT databases. It is compared with 25 actively managed stock portfolios that are formed, and periodically rebalanced, based on the market capitalization and lagged return of individual stocks. These portfolios are of particular interest because a wealth of empirical research suggests that contrarian and trend-following strategies earn a return premium over passive investment that defies rational explanation. Our analysis also includes a riskless asset with return equal to the time-series average of the T-bill yield in the relevant sample period.

We consider three different sets of portfolios based on the lagged return in three different estimation windows: (i) the past month (window '1-1'), (ii) the eleven months before the past month (window '2-12'), (iii) the four years before the past year (window '13-60'). These three data sets are designed to capture the known short-term reversal effect (Jegadeesh (1990)), momentum effect (Jegadeesh and Titman (1993)) and long-term reversal effect (De Bondt and Thaler (1985)), respectively.

We use gross value-weighted portfolio returns for the entire sample period covered by the data library of Kenneth French. Gross return seems the relevant return definition, because it is positive ($X \in \mathbb{R}_{++}$) and proportional to invested wealth. We analyze evaluation horizons of one month ($H = 1$), one quarter ($H = 3$) and one year ($H = 12$). The sample periods are July 1926–December 2013, 1926Q2–2013Q4 and 1927–2013 for window '1-1'; January 1927–December 2013, 1927Q1–2013Q4 and 1927–2013 for window '2-12'; January 1931–December 2013, 1931Q1–2013Q4 and 1931–2013 for window '13-60'. The time-series are relatively long and provide an accurate partition of the observed range of market returns.

Table I shows descriptive statistics for the nine data sets (three estimation windows times three evaluation horizons). For the sake of brevity, we tabulate only the statistics for the five portfolios in the first size quintile, where the reversal and momentum effects are strongest. Not surprisingly, the return distribution of small-cap stocks appears far from normal, witness the high levels of skewness and kurtosis. A distinct pattern is that the return distribution of small-loser (S/L) stocks tends to have high upside potential compared with small-winner (S/W) stocks. For example, for estimation window '1-1' and evaluation horizon $H = 12$, skewness is as high as 5.35 for S/L stocks and only 0.21 for S/W stocks. Presumably, many S/L stocks feature high levels of financial leverage and behave as 'out-of-the-money' call options after large price drops. This pattern suggests that SD criteria may diverge from the basic mean-variance (MV) criterion.

[Insert Table I about here]

B. Empirical Methodology

To test whether the market portfolio is StSD efficient, we design a linear program for the linear system of Theorem 1. In this application, the individual prospects are the returns to 25 risky stock portfolios ($X_1, \cdots, X_{25}$) and the T-bill yield (X_F), and the evaluated prospect is the market portfolio return (Y). The R time-series observations are interpreted as scenarios with equal probabilities $p_r = R^{-1}, r = 1, \cdots, R$. Recall that we normalize the average marginal utility to unity (13.1). Following another convention in the asset pricing literature, we require the 'pricing kernel' ($u'(Y)$) to be consistent with the equity premium by imposing a zero 'pricing error' for the T-bill ($\sum_{r=1}^{R} p_r u'(y_r)(y_r - x_{F,r}) = 0$). We are therefore left with the task of evaluating the pricing errors for the $M = 25$ stock portfolios.

Our objective is to minimize (across all admissible utility functions) the maximum of the 25 pricing errors:

$$\xi(Y) = \min_{u \in \mathcal{U}_4^*} \max_{j=1,\cdots,N} \sum_{r=1}^{R} p_r u'(y_r)\big(y_r - x_{j,r}\big). \tag{15}$$

We can implement this mini-max criterion by minimizing a model variable ξ subject to the restriction $\xi \geq \sum_{r=1}^{R} p_r u'(y_r)\big(y_r - x_{j,r}\big)$ and the linear system of Theorem 1. A value of $\xi^* = 0$ is required to classify the market portfolio as efficient; $\xi^* > 0$ implies inefficiency.

Our results are robust to the specification of the objective function. We prefer the mini-max criterion because it allows for a straightforward economic interpretation of the objective as the largest abnormal return that can be achieved without leverage or short selling. In addition, the linear formulation considerably reduces the computer time for our bootstrap procedure (see below). A follow-up analysis suggests that minimizing a weighted sum of squared errors in the spirit of the Generalized Method of Moments using Quadratic Programming leads to similar results and conclusions.

Apart from the StSD efficiency test, we also apply efficiency tests based on DARA SD (dropping temperance and DAP), FOSD (dropping DARA and DAP), TSD (dropping DARA, temperance and DAP) and SSD (dropping prudence, DARA, temperance and DAP). Finally, we apply a MV efficiency test that assumes a linear marginal utility function with coefficients based on the equity premium and an average value of unity, so that the pricing errors amount to Jensen's alphas.

For statistical inference, we will use a re-centered IID bootstrap approach that repeatedly applies the LP test to random pseudo-samples that are based on the original sample. To re-center the bootstrap process, we subtract the estimated pricing error $\hat{\alpha}_j := R^{-1}\sum_{r=1}^{R} u'(y_r)\left(y_r - x_{j,r}\right)$ from every return observation: $\hat{x}_{j,r} := x_{j,r} - \hat{\alpha}_j \quad r = 1, \cdots, R$. We implement the bootstrap by generating 10,000 pseudo-samples of the same size as the original sample through random draws with replacement from the re-centered original sample, and test portfolio efficiency in every pseudo-sample. Finally, we compute the critical values for the original test statistics from the percentiles of the bootstrap distribution. Similar results were obtained by using an empirical-likelihood bootstrap that re-centers the bootstrap distribution by weighting the time-series observations with their empirical likelihood.

C. Empirical Results

Table II summarizes the test results (test statistic, bootstrap p-value, pricing errors) for the various decision criteria. Since the results for the MV, TSD and FOSD criteria show no material differences, we omit the MV and FOSD results. In addition, we focus on the small-cap errors and omit the pricing errors for large-cap stocks for the sake of brevity.

Figure 1 graphically illustrates the properties of the relevant pricing kernels for the data set based on evaluation window '1-1' and evaluation horizon $H = 12$. The graphs for the other eight data sets look similar and are omitted for brevity. The figure also includes the unrestricted kernel with the lowest variance among all kernels that price all assets correctly in this sample. This kernel is constructed without imposing any of the utility-based regularity conditions and it is completely fitted to the data to serve as benchmark for the restricted kernels. Clearly, it does not resemble a well-behaved marginal utility function, as it is not smooth, tends to convexity in the center of the distribution and takes large negative values in the right tail.

Naturally, the SSD efficiency test (which does not impose higher-order risk aversion restrictions) yields the smallest pricing errors and the highest p-values. For the estimation window '13-60', the p-value falls the range of conventional significance levels (0.01 to 0.10) and the violations of portfolio efficiency are only marginally significant. However, the SSD pricing kernel (in the center left panel of Figure 1) is a far cry from a well-behaved marginal utility function. The kernel is flat for the bulk of the observations and drops abruptly to zero for large gains. This pattern is not consistent with standardness and casts doubt on the economic meaning of the SSD results. Like the unrestricted kernel, the SSD kernel seems over-fitted to the data and assigns an artificially low weight to the scenarios with the highest market returns.

The TSD criterion imposes prudence and avoids concave segments of the kernel. The pricing errors increase and the p-values decrease considerably for every sample. Particularly, the

S/L pricing errors increase by more than two percentage points per annum in every sample. By contrast, the errors for the S/W stocks are substantially less affected by the assumption of prudence. This explains why the test statistic for the estimation window '2-12', which is based on the drift of S/W stocks, is more robust to the decision criterion than for the estimation windows '1-1' and '13-60', which are based on the reversal of S/L stocks.

The TSD kernel closely resembles the linear MV kernel and shows large violations of DARA on the wide return interval in our samples. The MV/TSD kernel avoids penalizing S/L stocks for their positive skewness, but it doesn't reward these stocks either. Almost identical results are obtained using the FOSD criterion. This finding is not surprising, given that restrictions on the fourth-order derivative generally have a minimal effect on the shape of the utility function.

The DARA SD criterion leads to a substantial increase in the pricing errors. Depending on the assumed trading strategy and evaluation horizon, the abnormal returns to S/L stocks are some 40 to 180 basis point per annum larger than for the MV/TSD/FOSD criterion. By contrast, the pricing error of S/W stocks is not materially affected by imposing DARA, leading to an increase of the error for the average small-cap stock. Although the kernel is convex, it exhibits large linear segments in violation of DARA (log-convexity). These violations reflect approximation errors in the local linear approximation of the exponentiation. Similar deviations did not surface in PFK14 because their median-based normalization biases the optimal kernel towards a negative exponential shape.

The StSD criterion imposes the additional assumption of DAP. The incremental effect of imposing DAP in addition to DARA is smaller than the effect of imposing DARA in addition to risk aversion and/or prudence. Nevertheless, imposing DAP increases the pricing errors for S/L stocks by some 10 to 20 basis point per annum in the typical sample. In addition, the optimal StSD kernel is better behaved than the DARA kernel. The optimal StSD kernels closely resemble the negative exponential shape that obeys all regularity conditions. We now arrive at this standard shape with an (unbiased) means-based normalization and by explicitly imposing standardness.

Particularly large is the increase of the pricing error of S/L stocks from 12.14 percent for the SSD criterion to 18.86 percent for the StSD criterion in the data set with estimation window '1-1' and evaluation horizon $H = 12$. The increase of 6.72 percentage points clearly shows the discriminating power of standardness. Assuming that the second-order derivative is only non-positive (SSD) leads to much weaker results than assuming that the second-order derivative is negative, non-decreasing and log-concave (StSD). Imposing standardness seems an effective remedy against over-fitting the kernel to the data.

[Insert Table II about here]

[Insert Figure 1 about here]

As a robustness test, we repeated our analysis after excluding the first size quintile (S/L, S/2, S/3, S/4 and S/W) and the early sub-period 1927–1962, common robustness tests in the empirical asset pricing literature. The original results and conclusions are robust to these exclusions. The general momentum and reversal patterns weaken by several percentage points, but the effect of imposing standardness is comparable with that in the full sample, making the incremental effect even more pronounced. For example, the long-term reversal effect ('13-60' window) becomes insignificant for $H = 1, 3$ when using the SSD test, but it remains significant when standardness is assumed. The robustness analysis also reveals additional examples of the incremental effect of imposing DAP (in addition to prudence and DARA).

D. Evaluating the Tangency Portfolio

The above analysis evaluates a passive market portfolio. One may expect larger differences between the various decision criteria for active portfolios that use short selling and/or derivative securities. To illustrate this point, we will now evaluate the SD efficiency of the MV *tangency portfolio*, or the combination of the benchmark portfolios that maximizes the Sharpe ratio without weight restrictions other than a budget constraint.

The tangency portfolio plays a pivotal role in asset pricing theory and tests. In the Capital Asset Pricing Model, all investors mix the tangency portfolio with a riskless asset (two-fund separation) and the market portfolio of risky assets equals the tangency portfolio. More generally, the tangency portfolio defines the unique pricing kernel in complete markets and the minimum-variance kernel in incomplete markets. The tangency portfolio also plays a pivotal role in empirical tests to gauge the MV efficiency of a given portfolio and the goodness of a given pricing kernel (Gibbons, Ross and Shanken (1989) and Hansen and Jagannathan (1997)).

In contrast to the market portfolio, the tangency portfolio generally features short positions. Naturally, unlimited short selling is not realistic, as it introduces a potential solvency risk ($x < 0$) to the investor (and to her clients and brokers). To avoid extreme positions and solvency risk, we scale the tangency portfolio by mixing it with the T-bill to obtain the same standard deviation as the passive market portfolio. Importantly, this approach affects neither the relative weights of the risky assets nor the portfolio's Sharpe ratio.

Whereas the (scaled) tangency portfolio is MV efficient by construction, it need not be optimal for utility maximizers because it ignores the utility-based regularity conditions. To analyze sub-optimality, we construct the tangency portfolio in each of our nine samples, mix it with the T-bill to adjust the risk level, and subject it to our SD efficiency tests. Since the tangency portfolio is optimized for a given return distribution, we now do not use the bootstrap; the original empirical distribution now serves as the population distribution. Table III summarizes the results. These results cannot be interpreted in terms of pricing kernels and pricing errors, as a portfolio with short positions cannot represent the aggregate market portfolio.

Panel A shows the optimal weights assigned to the five small-cap portfolios and the T-bill; the (smaller) weights for remaining 20 portfolios are omitted for the sake of brevity. Not surprisingly, the unrestricted optimization longs S/L stocks and shorts S/W stocks for estimation windows '1-1' and '13-60', and takes the opposite positions for '2-12'. Panel B shows that the tangency portfolio has a very high mean compared with its (fixed) standard deviation. Unfortunately, MV optimization often trades upside potential for downside risk, reducing the meaning of standard deviation as a risk measure. Indeed, the tangency portfolio doesn't exhibit the high positive skewness of a typical concentrated portfolio of small-caps. For the estimation window '2-12', the tangency portfolio even introduces negative skewness by shorting the positively skewed S/L stocks.

The StSD test results in Panel C demonstrate that the tangency portfolio is sub-optimal for every standard risk averter in each of our nine samples. The degree of sub-optimality decreases with the length of the estimation window and increases with the evaluation horizon.

For estimation window '1-1' and evaluation horizon $H = 12$, the optimization error for S/L stocks is as high as 21.52 percent per annum. Compared with the variance averter, the standard risk averter is *less* inclined to short low-yielding (but positively skewed) small-cap stocks (S/L or S/W, depending on H) and *more* inclined to long high-yielding small-cap stocks and short large-cap stocks, in order to preserve the positive skewness of long position.

The SSD criterion struggles to detect the sub-optimality of the tangency portfolio by allowing for implausible patterns of higher-order risk aversion. As before, imposing standardness substantially increases the test statistic. The incremental effect of imposing DAP (in addition to DARA) is now much larger than for the analysis of the market portfolio. Notably, for window '2-12' and horizon $H = 12$, the StSD test statistic is 13.10 percent per annum, some 6.45 percent points above the value for DARA SD. Due to the high reward-to-variability ratio, a high level of general risk aversion is needed to rationalize the tangency portfolio and the effects of higher-order risk aversion become very pronounced.

[Insert Table III about here]

5. Concluding Remarks

StSD involves more discriminating power than the existing SD criteria by requiring DARA and DAP (in addition to non-satiation and risk aversion). Given the universal appeal of these regularity conditions, we see little risk of specification error and hence the additional power seems a 'free lunch'.

Our analysis does not impose complete monotonicity, or mixed risk aversion (Caballe and Pomansky (1996)), which requires additional restrictions on the signs of the derivatives of orders five and higher. We subscribe to the theoretical motivation of mixed risk aversion, and, in addition, our analysis (notably Proposition 4) can be extended in a straightforward manner to include additional model variables and linear constraints for the higher-order derivatives. Such higher-order restrictions however have no material effect on the shape of the optimal utility function in our experience.

We also do not impose increasing relative risk aversion (IRRA). There exist compelling theoretical and empirical arguments to support the notion that the relative risk aversion (RRA) quotient $r_u(x) := a_u(x)x$ is constant or increasing slightly for individual utility functions of wealth. However, these arguments do not carry over directly to the utility of income or consumption. In addition, IRRA is not robust to aggregation across utility functions. For example, the combined utility of multiple decision makers with power utility functions with different RRA levels exhibits *decreasing* RRA.

Our empirical application illustrates the benefits of StSD by means of a substantial increase in the pricing errors of small-loser stocks and an economically more meaningful shape for the pricing kernel compared with the traditional MV and higher-order SD criteria. Depending on the assumed trading strategy and evaluation horizon, we find that the abnormal returns to small-loser stocks for standard risk averters are about one to seven percentage points per annum larger than for the SSD criterion and about 50 to 200 basis points larger than for MV and higher-order SD criteria. Whereas the optimal kernel for SSD seems pathological and the kernel for the higher-order SD criteria resembles the questionable linear MV kernel, the StSD kernel approximates a negative exponential shape. The StSD criterion also shows that the MV tangency portfolio is substantially sub-optimal for standard risk averters in our samples.

Compared with the analysis of DARA SD by PFK14, we introduced additional variables for the second-order derivative and the AP quotient and additional restrictions for temperance and DAP. The empirical application illustrates the goodness of the resulting linear system. In our analysis of the market index, the incremental effect on the test statistic of imposing DAP (in addition to DARA) is some 10 to 20 basis point per annum in the typical sample, modest in comparison with the effect of imposing DARA. Nevertheless, the optimal StSD kernel is clearly better behaved than the DARA SD kernel. In addition, in our analysis of the MV tangency portfolio (rather than the passive portfolio), the incremental effect of DAP becomes as large as multiple percentage points, and StSD substantially increases the estimated opportunity cost of the MV approximation.

We also introduced several other methodological refinements (besides imposing DAP). First, by using different sets of *sampling points* for the derivatives of different orders, we allow for a coarse partition of the outcomes range, which is useful, for example, for low-frequency returns. By comparison, the analysis of DARA SD optimality (but not that of DARA SD efficiency) in PFK14 implicitly assumes that the utility function is a piece-wise cubic function, an assumption that becomes restrictive if the partition is coarse. Second, our *mean-based normalization* avoids 'punishing' skewed marginal utility functions that assign a high value to the worst outcomes. The median-based normalization of PFK14 artificially increases the level of the errors for such functions by pushing the average value of marginal utility above unity (in order to equate the median value with unity). Third, using *multiple frame functions* (our default choice is a set of CRRA functions with different RRA levels) has the advantage of reducing error for the local linear approximation to the exponentiation.

Our empirical results challenge the notions that (i) the MV criterion gives a close approximation to expected utility for analyzing stock returns, (ii) higher-order SD rules properly accounts for higher-order risk aversion and (iii) accounting for higher-order moments *reduces* the abnormal (risk-corrected) returns to active investment strategies. Small-cap stocks seem *more* appealing to standard risk averters than the classical decision criteria suggest. In particular, small-loser stocks seem to entail a relatively high probability of future price reversals, introducing a high upside potential compared with their downside risk. A standard risk averter will be more inclined to buy, and less inclined to short, these stocks than a variance averter.

Appendix: StSD Optimality

We now analyze the case of the discrete choice set $\mathcal{X}_0 := \{X_1, \cdots, X_M\}$ rather than its convex hull, $\mathcal{X} = Conv(\mathcal{X}_0)$. We will need some additional notation. Collect all possible outcomes across prospects and states in $\{w \in \mathbb{R}_{++} : w = x_{i,r} \;\; i = 1, \cdots, M; r = 1, \cdots, R\}$; rank the elements in ascending order, $w_1 \leq \cdots \leq w_S$; and use $q_{i,s} := \mathbb{P}[X_i = w_s] = \sum_{r: x_{i,r} = w_s} p_r$, $i = 1, \cdots, M$, $s = 1, \cdots, S$.

DEFINITION 2 (STSD OPTIMALITY) *A given prospect X_i, $i \in \{1, \cdots, M\}$, is StSD optimal if it is the optimum for some admissible utility function $u \in \mathcal{U}_4^*$:*

$$\sum_{s=1}^{S} u(w_s)\left(q_{i,s} - q_{j,s}\right) \geq 0, \qquad j = 1, \cdots, M. \tag{A.1}$$

Optimality cannot be analyzed using the KKT optimality conditions (6). Our approach is to introduce variables and constraints for the utility levels $u(w_s)$, $s = 1, \cdots, S$, in addition to the variables and constraints for the derivatives and their logs in Propositions 3 and 4.

PROPOSITION 5 (LINEARIZED UTILITY AND DERIVATIVES) *For any utility function $u \in \mathcal{U}_4$, we can represent $-u(w_s)$, $u'(w_s)$ and $-u''(w_s)$, $s = 1, \cdots, S$, using decreasing and convex piecewise-linear functions that are linear in a finite number of parameters and obey a set of linear restrictions:*

$$-u(w_s) = \sum_{k=s}^{S-1} \upsilon_k (w_{k+1} - w_s) + \upsilon_S, \qquad s = 1, \cdots, S; \tag{A.2.1}$$

$$\upsilon_s \geq 0, \qquad s = 1, \cdots, S-1; \tag{A.2.2}$$

$$u'(y_s) = \sum_{k=s}^{S-1} \beta_k (w_{k+1} - w_s) + \beta_S, \qquad s = 1, \cdots, S; \tag{A.3.1}$$

$$\sum_{k=s}^{S-1} \beta_k (w_{k+1} - w_s) + \beta_S \leq \sum_{k=s-1}^{S-1} \upsilon_s, \qquad s = 2, \cdots, S-1; \tag{A.3.2}$$

$$\sum_{k=s}^{S-1} \beta_k (w_{k+1} - w_s) + \beta_S \geq \sum_{k=s+1}^{S-1} \upsilon_s, \qquad s = 1, \cdots, S-2; \tag{A.3.3}$$

$$\beta_s \geq 0, \qquad s = 1, \cdots, S; \tag{A.3.4}$$

$$-u''(w_s) = \sum_{k=s}^{S-1} \gamma_k (w_{k+1} - w_s) + \gamma_S, \qquad s = 1, \cdots, S; \tag{A.4.1}$$

$$\sum_{k=s}^{S-1} \gamma_k (w_{k+1} - w_s) + \gamma_S \leq \sum_{k=s-1}^{S-1} \beta_s, \qquad s = 2, \cdots, S-1; \tag{A.4.2}$$

$$\sum_{k=s}^{S-1} \gamma_k (w_{k+1} - w_r) + \gamma_S \geq \sum_{k=s+1}^{S-1} \beta_s, \qquad r = 1, \cdots, S-2; \tag{A.4.3}$$

$$\gamma_s \geq 0, \qquad s = 1, \cdots, S. \tag{A.4.4}$$

PROOF The utility levels $u(w_S)$ in (A.2.1) are built from the decrements of the first-order derivative $v_s = u'(z_s^{(1)}) - u'(z_{s+1}^{(1)})$, $s = 1,\cdots,S-2$, $v_{S-1} = u'(z_{S-1}^{(1)})$, for sampling points $z_s^{(1)} \in [w_s, w_{s+1}]$ such that $u'\left(z_s^{(1)}\right) = (u(w_{s+1}) - u(w_s))/(w_{s+1} - w_s)$, $s = 1,\cdots,S-1$, and $v_S = -u(w_S)$. The non-negativity conditions (A.2.2) follow from non-satiation. Inequalities (A.3.2) follow from $u'(x) \geq 0$, $-u'\left(z_{s-1}^{(1)}\right) = \sum_{k=s-1}^{S-1} v_s$, and $z_{s-1}^{(1)} \leq w_s$; similarly, inequalities (A.3.3) follow from $u'(x) \geq 0$, $-u'\left(z_s^{(1)}\right) = \sum_{k=s}^{S-1} v_s$, and $w_s \leq z_s^{(1)}$. The remaining variables and constraints are as in Proposition 3, *mutatis mutandis*. ■

Our Proposition 5 deviates from Proposition 4 in PFK14 by accounting for the second-order derivative and by using two separate piece-wise linear functions, with two different sets of sampling points, for $u(w_s)$ and $u'(w_s)$. The two functions are not directly linked through integration but through the inequality constraints (A.3.2) and (A.3.3). The two approaches converge as the partition is refined ($(w_{r+1} - w_r) \rightarrow 0$), but only our formulation allows for a coarse partition.

The other lemma's and propositions do not require adjustment other than an obvious change in notation from y_r, $r = 1,\cdots,R$, to w_s, $s = 1,\cdots,S$. We can then combine Proposition 5 with Lemmas 3 and 4 and Propositions 3 and 4 to derive a linear system for (A.1) by analogy to Theorem 1.

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Table I Descriptive Statistics

Shown are descriptives for gross returns for evaluation period of $H = 1,3,12$ months to the CRSP all-share index, one-month T-bill and stock portfolios formed on return in the 1 month, 2-12 months or 13-60 months prior to the formation date. For the sake of brevity, we report only the statistics for the small-cap segment (the first size quintiles), where the reversal and momentum effects are strongest. 'S/L' denotes small-loser stocks; 'S/W' small-winner stocks; 'S/2'; 'S/3' and 'S/4' are the remaining three past-return quintiles in the small-cap stock segment. We use data on gross value-weighted portfolio returns. The sample periods are July 1926–December 2013, 1926Q2–2013Q4 and 1927–2013 for window '1-1'; January 1927–December 2013, 1927Q1–2013Q4 and 1927–2013 for window '2-12'; January 1931–December 2013, 1931Q1–2013Q4 and 1931–2013 for window '13-60'. The original data are from the data library of Kenneth French.

1-1	*H*=1				*H*=3				*H*=12			
	Mean	StDev	Skew	Kurt	Mean	StDev	Skew	Kurt	Mean	StDev	Skew	Kurt
Market	100.94	5.40	0.12	7.42	102.81	10.67	0.73	9.36	111.89	20.35	-0.44	-0.01
T-bill	100.29	0	0	0	100.86	0	0	0	103.54	0	0	0
S/L	102.47	10.96	2.80	22.51	107.40	23.52	3.79	31.99	139.83	88.07	5.35	38.94
S/2	101.56	9.74	2.94	30.80	104.67	21.20	3.62	33.29	122.18	45.31	1.83	7.74
S/3	101.37	9.36	4.01	46.75	104.12	20.65	4.48	46.48	119.06	41.79	2.31	12.34
S/4	101.06	9.11	3.63	46.13	103.18	19.96	3.71	37.90	114.32	34.28	0.09	-0.38
S/W	100.19	9.14	2.09	20.58	100.58	19.16	1.59	12.85	103.71	33.28	0.21	0.17
2-12	*H*=1				*H*=3				*H*=12			
	Mean	StDev	Skew	Kurt	Mean	StDev	Skew	Kurt	Mean	StDev	Skew	Kurt
Market	100.93	5.41	0.13	7.38	102.80	10.70	0.73	9.30	111.89	20.35	-0.44	-0.01
T-bill	100.29	0	0	0	100.86	0	0	0	103.54	0	0	0
S/L	100.96	11.00	3.00	23.91	102.89	23.51	2.84	19.64	113.78	45.35	1.09	1.80
S/2	101.43	9.54	3.61	37.60	104.29	20.50	4.06	37.65	119.85	43.73	2.28	11.50
S/3	101.63	8.87	3.62	42.36	104.90	18.95	3.71	34.09	122.34	40.11	1.89	8.68
S/4	101.69	9.06	3.67	43.43	105.07	19.97	5.79	73.38	124.79	52.86	4.21	29.15
S/W	101.91	8.90	1.51	16.66	105.73	17.55	2.03	17.98	126.19	41.69	1.42	3.91
13-60	*H*=1				*H*=3				*H*=12			
	Mean	StDev	Skew	Kurt	Mean	StDev	Skew	Kurt	Mean	StDev	Skew	Kurt
Market	100.96	5.35	0.23	7.91	102.87	10.56	0.91	10.11	112.13	19.78	-0.44	0.22
T-bill	100.29	0	0	0	100.86	0	0	0	103.55	0	0	0
S/L	101.91	10.72	3.26	28.01	105.73	22.61	3.50	28.00	125.33	44.63	1.66	5.87
S/2	101.65	9.16	3.59	38.99	104.96	19.29	4.35	42.99	122.77	46.82	4.01	26.83
S/3	101.71	8.44	2.31	19.55	105.12	17.98	3.63	31.20	121.83	31.30	0.63	0.67
S/4	101.57	9.11	3.07	28.32	104.70	17.58	2.44	16.96	121.54	39.84	2.04	8.71
S/W	100.99	8.69	0.33	7.94	102.96	16.70	0.52	4.43	113.10	31.81	0.13	0.13

Table II Efficiency of the Market Portfolio

Shown are results for testing efficiency of the value-weighted market portfolio relative to 25 stock portfolios formed on size and lagged return and the one-month T-bill for evaluation periods of $H = 1{,}3{,}12$ months and estimation periods of 1-1, 2-12 and 13-60 months. For interpretation and comparability, the test statistics and pricing errors for an interval of H months are 'annualized' by multiplication with $(12/H)$. For brevity, we report only the errors for the small-cap segment (the first size quintiles), where the reversal and momentum effects are strongest. 'S/L' denotes small-loser stocks; 'S/W' small-winner stocks; 'S/2'; 'S/3' and 'S/4' are the remaining three past-return quintiles in the small-cap stock segment. We use data on gross value-weighted portfolio returns. The sample periods are July 1926–December 2013, 1926Q2–2013Q4 and 1927–2013 for window '1-1'; January 1927–December 2013, 1927Q1–2013Q4 and 1927–2013 for window '2-12'; January 1931–December 2013, 1931Q1–2013Q4 and 1931–2013 for window '13-60'. The original data are from the data library of Kenneth French. Bootstrap p-values are based on 10,000 pseudo-samples from the original sample after re-centering the means.

1-1	*H*=1				*H*=3				*H*=12			
	SSD	TSD	DSD	StSD	SSD	TSD	DSD	StSD	SSD	TSD	DSD	StSD
Test Stat	10.78	13.55	13.95	13.98	9.30	12.47	13.28	13.44	12.14	16.81	18.68	18.86
P-value	0.000	0.000	0.000	0.000	0.001	0.000	0.000	0.000	0.008	0.004	0.002	0.000
S/L	10.78	13.55	13.95	13.98	9.30	12.47	13.28	13.44	12.14	16.81	18.68	18.86
S/2	2.73	4.05	4.27	4.29	0.69	2.30	2.88	2.87	2.76	4.44	4.99	5.00
S/3	0.93	2.36	2.65	2.67	-0.34	0.62	1.21	1.18	0.87	2.63	3.08	3.11
S/4	-2.08	-0.94	-0.74	-0.71	-3.84	-2.75	-2.22	-2.23	-2.42	-0.97	-0.90	-0.85
S/W	-11.98	-11.53	-11.42	-11.39	-14.13	-12.94	-12.75	-12.71	-11.83	-10.91	-10.97	-10.94
2-12	*H*=1				*H*=3				*H*=12			
	SSD	TSD	DSD	StSD	SSD	TSD	DSD	StSD	SSD	TSD	DSD	StSD
Test Stat	8.61	9.51	9.51	9.52	7.82	9.06	9.18	9.25	7.27	9.95	10.15	10.34
P-value	0.000	0.000	0.000	0.000	0.001	0.001	0.000	0.000	0.004	0.002	0.002	0.001
S/L	-6.09	-1.95	-1.95	-2.40	-8.32	-5.80	-5.20	-5.18	-5.52	-3.86	-3.44	-3.34
S/2	0.92	5.07	5.07	4.73	0.28	1.54	2.20	2.11	2.01	3.12	3.76	3.71
S/3	4.42	7.40	7.40	7.14	4.05	4.83	5.39	5.32	5.25	6.44	7.07	7.05
S/4	5.02	9.51	9.51	9.52	4.31	5.44	6.02	5.97	5.53	6.72	7.73	7.68
S/W	8.61	9.51	9.51	9.52	7.82	9.06	9.18	9.25	7.27	9.95	10.15	10.34
13-60	*H*=1				*H*=3				*H*=12			
	SSD	TSD	DSD	StSD	SSD	TSD	DSD	StSD	SSD	TSD	DSD	StSD
Test Stat	4.76	7.02	7.39	7.44	4.24	5.86	6.32	6.38	6.13	7.85	8.16	8.27
P-value	0.014	0.003	0.001	0.000	0.059	0.021	0.013	0.019	0.040	0.011	0.008	0.006
S/L	4.76	7.02	7.39	7.44	3.09	5.68	6.32	6.32	5.13	7.47	8.14	8.19
S/2	3.23	5.56	5.81	5.86	2.71	4.46	5.04	5.06	3.82	5.31	6.46	6.33
S/3	4.76	6.58	6.83	6.88	4.24	5.86	6.32	6.38	6.13	7.85	8.16	8.27
S/4	3.61	5.21	5.37	5.41	3.15	4.81	5.23	5.22	6.13	7.81	8.15	8.06
S/W	-3.41	-1.92	-2.01	-1.98	-3.72	-2.64	-2.59	-2.61	-3.68	-2.13	-2.20	-2.13

Table III Sub-Optimality of the Tangency Portfolio

The table summarizes our SD efficiency analysis of the MV tangency portfolio formed from the 25 stock portfolios. Panel A shows the optimal weights assigned to the five small-cap portfolios and the T-bill. To avoid extreme positions and solvency risk, we scale the tangency portfolio with a position in the T-bill to obtain the same standard deviation as the passive market portfolio. Panel B displays descriptive statistics for the gross returns of the tangency portfolio. Panel C shows the SD efficiency test statistics for each of the nine samples. For interpretation and comparability, the test statistics for an interval of H months are 'annualized' by multiplication with the factor $12/H$. We use data on gross value-weighted portfolio returns. The sample periods are July 1926–December 2013, 1926Q2–2013Q4 and 1927–2013 for window '1-1'; January 1927–December 2013, 1927Q1–2013Q4 and 1927–2013 for window '2-12'; January 1931–December 2013, 1931Q1–2013Q4 and 1931–2013 for window '13-60'. The original data are from the data library of Kenneth French.

A. Optimal weights

	H=1			*H*=3			*H*=12		
	1-1	2-12	13-60	1-1	2-12	13-60	1-1	2-12	13-60
T-bill	0.79	0.35	0.47	0.58	0.28	0.42	0.76	0.50	0.03
S/L	1.06	-0.18	0.49	1.09	-0.39	0.70	0.09	0.60	0.97
S/2	-0.48	0.06	-0.36	-0.64	-0.05	-0.37	-0.63	-1.04	-0.51
S/3	-0.14	0.72	0.56	-0.33	0.93	0.42	0.09	0.72	0.47
S/4	-0.02	-0.13	-0.08	0.16	-0.35	-0.28	1.44	-0.19	-0.46
S/W	-0.71	0.28	-0.50	-0.82	0.33	-0.87	-1.44	0.17	-1.43

B. Descriptives

	H=1				*H*=3				*H*=12			
	Mean	StDev	Skew	Kurt	Mean	StDev	Skew	Kurt	Mean	StDev	Skew	Kurt
1-1	102.88	5.40	1.57	9.17	109.03	10.67	1.21	4.23	131.67	20.35	-0.03	0.21
2-12	102.18	5.41	-0.25	10.12	107.20	10.70	-0.62	4.83	126.37	20.35	-0.62	1.99
13-60	101.65	5.35	1.71	10.41	105.54	10.56	1.16	4.76	126.65	19.78	0.25	-0.05

C. Test statistics

	H=1				*H*=3				*H*=12			
	SSD	TSD	DSD	StSD	SSD	TSD	DSD	StSD	SSD	TSD	DSD	StSD
1-1	1.66	6.06	7.91	9.21	2.12	6.94	7.50	8.82	10.47	19.38	19.38	21.52
2-12	0	2.66	3.52	5.47	0	3.98	5.25	8.42	0.00	5.19	6.20	12.23
13-60	0	0.78	0.88	1.03	0	3.15	3.16	4.15	0.00	0.00	0.00	0.24

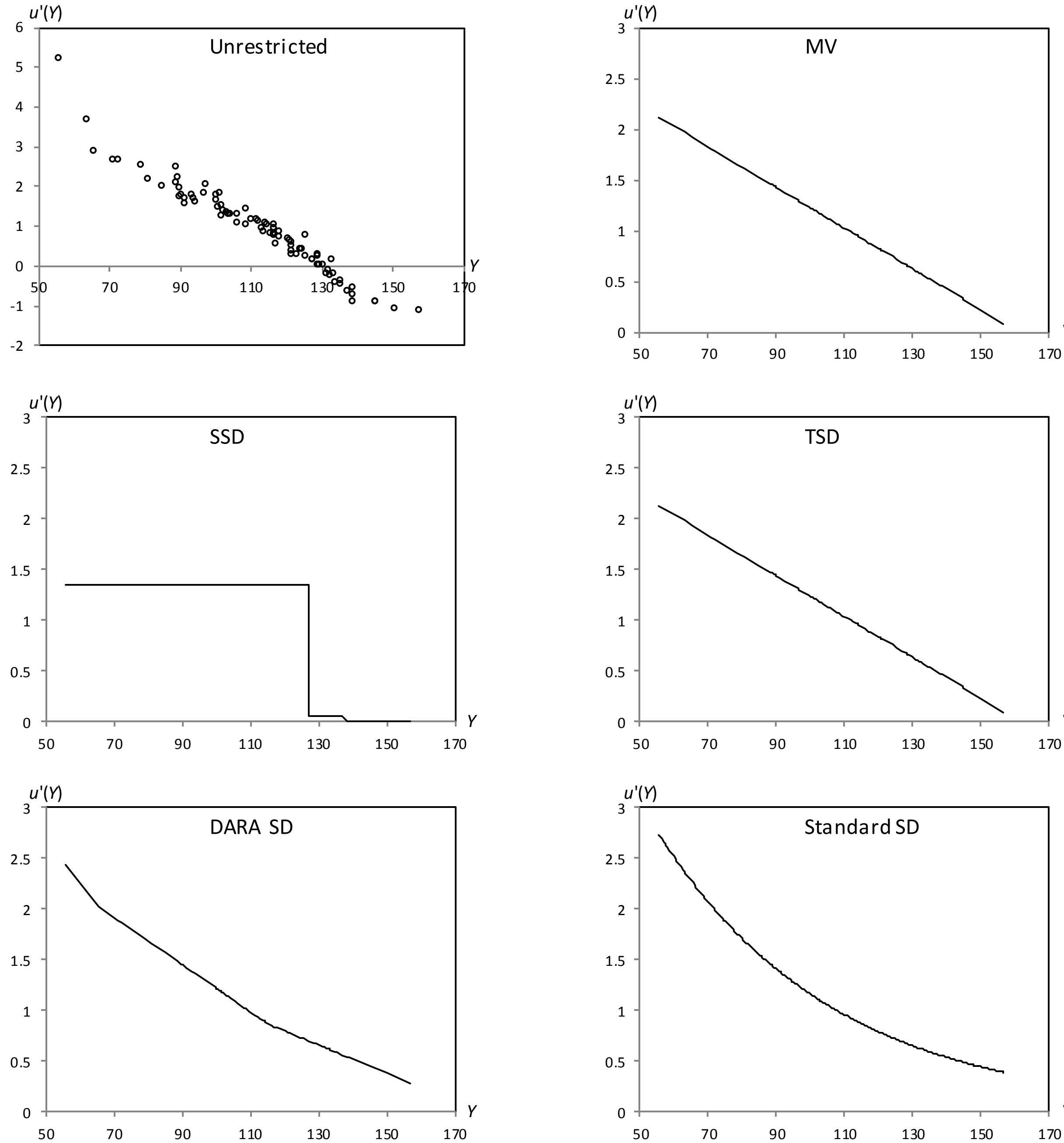

Figure 1: SD Pricing Kernels

The graphs show the pricing kernels generated by the tests for MV, SSD, TSD, DARA SD and StSD efficiency of the value-weighted market portfolio relative to the T-bill and 25 stock portfolios formed on size and lagged one-month return for an evaluation period of $H = 12$ months. A continuous function is obtain by linear interpolation between optimal marginal utility levels. The top left panel shows the values of the minimum-variance kernel that prices all assets correctly without imposing the regularity conditions for a well-behaved marginal utility function. We use data on gross value-weighted portfolio returns from 1927 to 2013 from the data library of Kenneth French.