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Threshold Endogeneity in Threshold VARs: An Application to Monetary State Dependence *

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Abstract

We contribute a new method for dealing with the problem of endogeneity of the threshold variable in threshold vector auto-regression (TVAR) models. Drawing on copula theory, enables us to capture the dependence structure between the threshold variable and the vector of TVAR innovations, independently of the marginal distribution of the threshold variable. A Monte Carlo demonstrates that our method works well, and that ignoring threshold endogeneity leads to biased estimates of the threshold parameter and the variance-covariance error structure, thus invalidating dynamic analysis. As an application, we assess the effects of interest-rate shocks on output and inflation: when 'expected' inflation exceeds 3.6%, the effects of monetary policy are faster and stronger than otherwise.

Keywords: VAR; State Dependency; Copula; Monte Carlo; Monetary Policy; Impulse Response; Davig-Leeper.

JEL Codes: E40; E50; C32.

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1 Introduction

Vector Auto-Regressive (VAR) models have become a standard tool for studying multivariate relationships in macro and financial economics. Although most commonly used in a linear setting, there can be instances where a non-linear VAR representation would be more appropriate. For example, to allow for asymmetry between size, sign, and state dependence of shocks.

As a consequence, VARs have been extended to the class of threshold models (henceforth *TVAR*), allowing for shifts and asymmetric parameter responses to changes in the state of a threshold variable (e.g., Tsay, 1998). The threshold variable chosen is often taken to be an observed time series (reflecting some particular hypothesis) or an index summarizing information from many variables. Typical channels to justify threshold outcomes include the presence of real and nominal rigidities, financial frictions, tipping points, business-cycle phases, ceilings/floors, adjustment costs, asymmetric policy preferences, etc.

Given this rich backdrop, TVARs have been applied in many contexts. A particularly common application has been to examine the transmission of stabilization policy. For example, Balke (2000), Peersman and Smets (2002), and Tenreyro and Thwaites (2016) examined monetary policy transmission conditional on financial and real conditions.[1] Likewise, Auerbach and Gorodnichenk (2012), Ramey and Zubairy (2018), Ramey, Zeev and Zubairy (2023) and Ghassibe and Zanetti (2022) look at threshold issues relating to the transmission of fiscal policy (see also Kourtellos et al, 2013).

Notwithstanding, a salient but overlooked issue in the estimation of TVAR models – and principal contribution of our paper – is the potential *endogeneity of the threshold variable*. The literature assumes that the threshold variable is exogenous. In some

[1]VAR studies of the monetary transmission mechanism are common in the literature, see the comprehensive survey of Ramey (2016). For a survey of linear and nonlinear VARs, see Kilian and Lütkepohl (2017).

contexts, this might be considered contentious or at the very least worth testing.[2] Endogeneity will arise if the threshold variable is contemporaneously correlated with the error (innovation) terms. For instance, if there are unobserved or omitted variables which confound both threshold and dependent variables. Specific examples include where the threshold variable (a) is meant to capture changes in business-cycle conditions (e.g., Sims and Zha, 2006; Kim et al., 2008); (b) is an index combining information from different economic sources (i.e., a financial index capturing financial stress conditions, e.g., Floro and van Roye, 2017); (c) is used as a state variable in forward-looking models to endogenously capture regime shifts on policy variables (see, e.g., Davig and Leeper, 2007, 2008) and agents' reactions to realizations of state variables (for example, see the equity returns model of Turner et al, 1989).

Although a long-standing issue in quantitative economics, endogeneity bias has not yet been addressed in the context of threshold VARs. This is problematic since ignoring true threshold endogeneity will in general lead to biased estimates of not only the threshold parameter, but also the dynamic and variance-covariance parameters. This in turn would impair impulse response function (IRF) analysis.

In the *single equation* framework (see Kourtelos et al., 2016; Kourtelos et al., 2021; Christopoulos et al., 2021, Chen et al., 2023), threshold endogeneity bias has been recently addressed using control variables in the rhs (right-hand side) of the threshold regression to adjust the conditional mean for the bias. This approach is in the spirit of binary-choice selectivity models that allow for endogenous regime switching (e.g., Maddala, 1983; Lee, 1982, 1983; Vella, 1998). It relies on a limited information estimation framework to adjust for the endogeneity of the threshold variable.

[2]To illustrate, when discussing the Reinhart and Rogoff (2010) threshold debt-growth controversy, Kourtellos et al. (2013) argue that their assumption of the exogeneity of public debt is implausible, and that once threshold endogeneity is allowed for, there is no threshold effect from debt on growth. If this is correct, any inference and policy conclusions based on such an assumption could be highly misleading. The same may be said for other variables on which the threshold literature has often focused – such as democracy indices, trade intensity, institutional quality (in addition to those discussed above), etc.

To deal with the threshold variable endogeneity problem in a *multiequation* context, we extend the above approach to the TVAR and structural TVAR frameworks. Specifically, we draw upon **copula theory**. Copulas are a means of accounting for the dependence structure between random variables: Sklar's theorem states that any multivariate joint distribution can be written in terms of univariate marginal distribution functions, and a copula. Copulas have been widely applied in economics and finance (e.g., Patton, 2012), as well as for forecasting, climate research, reliability analysis, and so on. The method was also suggested by Park and Gupta (2012) to tackle the endogeneity problem of a single regressor.

In that spirit, we first employ a Gaussian *multivariate* copula. We demonstrate that this choice yields a particularly neat, precise and interpretable form,[3] and enables us to include a transformation of the threshold variable based on the Gaussian copula in the rhs of the TVAR equations across the different regimes. In this way, we orthogonalize the relationship between the threshold variable and the entire vector of the VAR reduced form (or structural) error terms. For completeness, our Appendices consider copulas other than the Gaussian case.

The method has several attractive features: (1) It is a free-of-instruments approach and can allow for different dependence structures of the threshold and innovation terms across the model regimes. Such different dependencies may be a feature in practice. (2) It allows the distribution of the threshold variable to be of unknown form. Indeed, there is no parametric restriction on the threshold variable other than it must be continuous. Thus, whilst the copula choice may be Gaussian, the threshold variable need not be. (3) It can work satisfactorily even in cases where the dependence structure between the innovations and the threshold variable is nonlinear, for instance,

[3] See, e.g., Patton (2006, 2012), Growiec (2013), and Tófoli et al. (2017) for discussions and applications of copula theory to economics and econometrics.

as captured by Archimedean (as opposed to elliptical) copulas.[4] This tail-dependence aspect is attractive since in empirical applications the threshold value is often located at some extreme value, where there is some mass of distribution. (4) It is computationally very easy to apply since it only requires us to include, in the rhs of the equations, copula transformations of the marginal distributions of the threshold variable over its distinct regimes, truncated at the threshold value. These transformations can be obtained by the quantile function of the normal distribution.

To verify the efficiency of our method to control for threshold endogeneity bias, we perform a Monte Carlo (MC). There, we also appraise the consequences of (instead) ignoring threshold endogeneity and examine robustness to cases where the dependence between the TVAR innovations and the threshold variable is nonlinear, and of varying signs and strengths. Simulation results demonstrate the severity of ignoring endogeneity both in the estimates of the variance-covariance matrix of innovations and the estimates of the IRFs of the TVAR variables with respect to structural errors. We find that our method controls for this problem efficiently. Its benefits are more obvious in cases where the correlation structure and the error variances change markedly across the regimes, i.e., the piece-wise structural form of the model is more profound.

Empirical illustration We implement the method to asses the efficiency and state dependence of monetary policy. In our baseline case, we find that when a measure of expected inflation exceeds 3.6%, the impact of monetary policy changes. Moreover, partitioning the historical inflation data into periods of high and low inflation states on the basis of this threshold value, reveals a good match with established views on economic developments. Ignoring threshold endogeneity, moreover, tends to bias

[4] See, e.g., see Park and Gupta (2012). These authors also showed the robustness of the method against deviations from the normality assumption of the regressor error term.

upward estimates of the threshold value and generally make the economy appear more sluggish in response to monetary shocks.

Organization Section 2 presents our method to control for threshold endogeneity in the TVAR. Section 3 discusses the estimation method. When the threshold value is unknown, we suggest a concentrated least-squares procedure (e.g., Kourtellos et al., 2016), based on a grid search to estimate the threshold and other parameters. Section 4 discusses the rationale for threshold and state-dependent effects of monetary policy shocks on the economy. Section 5 implements the endogenous TVAR framework to examine the case for monetary inflation state dependence with expected inflation as the threshold variable, and makes comparison to a linear and an exogenous threshold VAR . Section 6 concludes. Additional material is in the appendices.

2 An endogenous threshold VAR

Consider the following two ($i = 1, 2$) state TVAR allowing for p lags:[5]

$$y_t = \begin{cases} \Gamma_0^{(1)} + \Gamma_1^{(1)} y_{t-1} + \Gamma_2^{(1)} y_{t-2} + .,.. + \Gamma_p^{(1)} y_{t-p} + u_t^{(1)}, & \text{if } z_t \leq \delta \\ \\ \Gamma_0^{(2)} + \Gamma_1^{(2)} y_{t-1} + \Gamma_2^{(2)} y_{t-2} + ... + \Gamma_p^{(2)} y_{t-p} + u_t^{(2)}, & \text{if } z_t > \delta \end{cases} \tag{1}$$

where y_t is a $K \times 1$ vector of variables, $\Gamma_0^{(i)}$ is a $K \times 1$ vector of intercepts, $\Gamma_l^{(i)}$ ($l = 1, 2, ..., p$) are $K \times K$ matrices of transition coefficients, $u_t^{(i)}$ is a $K \times 1$ vector of the TVAR innovation terms, i.e., $u_t^{(i)} = (u_{1t}^{(i)}, u_{2t}^{(i)}, ..., u_{Kt}^{(i)})'$,, with mean $\mathbb{E}(u_t^{(i)}) = 0$ and variance-covariance matrix $\mathbb{VAR}(u_t^{(i)}) = \Sigma_{uu}^{(i)}$ which is different across i: $\Sigma_{uu}^{(1)} \neq \Sigma_{uu}^{(i)}$.

Variable z_t is a threshold variable with threshold value δ. Given δ, the support

[5]In the remainder of our treatment, we shall assume two states, and accordingly $i = 1, 2$.

of z_t can be partitioned, at any point t, into the following intervals (regions): $\mathcal{Z}_t^{(1)} = \{-\infty < z_t \leq \delta\}$ and $\mathcal{Z}_t^{(2)} = \{\delta < z_t < \infty\}$, which correspond to the two distinct states of the TVAR model (1), i.e., $\mathcal{Z}_t^{(1)} \cap \mathcal{Z}_t^{(2)} = \varnothing$, implied that the model is segmented over the values of z_t, i.e. $\mathcal{Z}_t^{(1)}$ and $\mathcal{Z}_t^{(2)}$.

For a sample of $t = 1, 2, ..., T$ observations of y_t, model (1) can be expressed more compactly:

$$y_t = \begin{cases} \Gamma^{(1)} \mathbf{Y}_{t-1} + u_t^{(1)}, & \text{if } z_t \leq \delta \\ \\ \Gamma^{(2)} \mathbf{Y}_{t-1} + u_t^{(2)}, & \text{if } z_t > \delta \end{cases} \tag{2}$$

for $t = p + 1, ..., T$, where $\mathbf{Y}_{t-1} = (1, y_{t-1}, ..., y_{t-p})$ and $\Gamma^{(i)} = (\Gamma_0^{(i)}, \Gamma_1^{(i)}, ..., \Gamma_p^{(i)})$. The threshold variable endogeneity problem in the above model arises from the contemporaneous correlation between the threshold variable z_t and the vector of innovation terms $u_t^{(i)}$, implying that $\mathbb{E}(u_t^{(i)} | \mathcal{Z}_t^{(i)}) \neq 0$. Even if $u_t^{(i)}$ is uncorrelated with $\mathbf{Y}_{t-1}$, i.e., $\mathbb{E}(u_t^{(i)} | \mathbf{Y}_{t-1}) = 0$, as often assumed in VARs, in practice it is likely that $u_t^{(i)}$ is correlated with z_t across the two states of the model, i.e., $\mathbb{E}(u_t^{(i)} | \mathcal{Z}_t^{(i)}) \neq 0$, due to the influence of common economic factors. Ignoring this correlation will lead to biased predictions and parameter estimates. These parameters also include those of the variance-covariance matrix of $u_t^{(i)}$, denoted $\Sigma_{uu}^{(i)}$, used to identify structural shocks from the innovation terms $u_t^{(i)}$ and perform IRF analysis of y_t to structural shocks, one of the main objectives of VAR analysis.

The use of copulas allows us to capture the relationship between $u_t^{(i)}$ and the threshold variable z_t, causing the threshold endogeneity problem, by adjusting the conditional mean of y_t with the expectation terms $\mathbb{E}(u_t^{(i)} | \mathbf{Y}_{t-1}, \mathcal{Z}_t^{(i)})$. Accordingly, we rely on the following assumption:

Assumption 1 *Consider the following block of assumptions:*

1. $u_t^{(i)} \sim IID(0, \Sigma_{uu}^{(i)})$, *with* $\mathbb{E}\left(u_t^{(1)} u_s^{(2)'}\right) = 0$ *for* $t \neq s$

2. z_t *has a continuous and strictly monotonically increasing probability distribution, and* $\mathbb{E}(u_t^{(i)} | \mathcal{Z}_t^{(i)}) \neq 0$.

3. $\mathbb{E}(u_t^{(i)} | \mathbf{Y}_{t-1}, \mathcal{Z}_t^{(i)}) = \mathbb{E}(u_t^{(i)} | \mathcal{Z}_t^{(i)})$.

Assumption 1 is standard for endogenous regime switching models. It is an extension of that made by Christopoulos et (2021) to the multivariate framework. It implies that z_t is endogenous – i.e., correlated with $u_t^{(i)}$ – and that $u_t^{(i)}$ and $\mathbf{Y}_{t-1}$ are uncorrelated, implying $\mathbb{E}(u_t^{(i)} | \mathbf{Y}_{t-1}) = 0$ which is a standard assumption of VAR models. Under this assumption, we can derive an analytic relationship for $\mathbb{E}(u_t^{(i)} | \mathcal{Z}_t^{(i)})$, which can capture the dependence between $u_t^{(i)}$ and z_t. This will be done under the additional assumption that $u_t^{(i)}$ is normally distributed. But, it can also be done under the assumption that $u_t^{(i)}$ follows a Student's t distribution (both of which belong to the family of elliptical distributions).

To implement the copula theory in this context, we first define the truncated distributions of z_t over each of its interval supports $\mathcal{Z}_t^{(i)}$:

$$
\begin{aligned}
F_{\mathcal{Z}^{(1)}}(\mathcal{Z}_t^{(1)}) &= \frac{F_z(z_t)}{p_\delta} \qquad \text{if } 0 \leq F_z(z_t) \leq p_\delta \;\; (z_t \leq \delta) \\
F_{\mathcal{Z}^{(2)}}(\mathcal{Z}_t^{(2)}) &= \frac{F_z(z_t) - p_\delta}{1 - p_\delta} \quad \text{if } p_\delta < F_z(z_t) \leq 1 \;\; (z_t > \delta)
\end{aligned}
\tag{3}
$$

where $p_\delta = \Pr(z_t \leq \delta)$. These distributions are obtained by scaling $F_z(z_t)$, at p_δ, so that they integrate to one and constitute proper distribution functions. Next, we define copula functions to represent the conditional distribution of vector $u_t^{(i)}$ on

$\mathcal{Z}_t^{(i)}$, denoted by $F_{u^{(i)}|\mathcal{Z}_t^{(i)}}(u_{1t}^{(i)}, .., u_{Kt}^{(i)}|\mathcal{Z}_t^{(i)})$, needed to obtain analytic formulas for $\mathbb{E}(u_t^{(i)}|\mathcal{Z}_t^{(i)})$.

Definition 1 *Consider two $K+1$ dimension copulas*

$C^{(i)} : [0,1]^{K+1} \to [0,1]$, $F_{u^{(i)}|\mathcal{Z}_t^{(i)}}(u_{1t}^{(i)}, .., u_{Kt}^{(i)}|\mathcal{Z}_t^{(i)})$ *which can be written as*

$$F_{u^{(i)}|\mathcal{Z}_t^{(i)}}(u_{1t}^{(i)}, .., u_{Kt}^{(i)}|\mathcal{Z}_t^{(i)}) = \frac{\partial}{\partial F_{\mathcal{Z}^{(i)}}} C^{(i)}\left(F_{u_1^{(i)}}(u_{1t}^{(i)}), ..., F_{u_K^{(i)}}(u_{Kt}^{(i)}), F_{\mathcal{Z}^{(i)}}(\mathcal{Z}_t^{(i)})\right) \tag{4}$$

where $F_{u_k^{(i)}}(u_{kt}^{(i)})$, $k = 1, 2, ..., K$, are the marginal distributions of error terms $u_{kt}^{(i)}$.[a]

[a] See, e.g., Erderly (2017). The copula functions C_i, defined by (4), can be thought of as scaled functions of gluing copulas along the support interval of threshold variable z_t, for $z_t \leq \delta$ and $z_t > \delta$. See also Siburg and Stoimenov (2008).

Based on (4), we can write the conditional density of vector $u_t^{(i)}$ on $\mathcal{Z}_t^{(i)}$ as follows:

$$f_{u^{(i)}|\mathcal{Z}_t^{(i)}}(u_{1t}^{(i)}, .., u_{Kt}^{(i)}|\mathcal{Z}_t^{(i)}) = c^{(i)}\left(F_{u_1^{(i)}}(u_{1t}^{(i)}), ..., F_{u_K^{(i)}}(u_{Kt}^{(i)}), F_{\mathcal{Z}^{(i)}}(\mathcal{Z}_t^{(i)})\right) f_{u_1^{(i)}}(u_{1t}^{(i)})...f_{u_K^{(i)}}(u_{Nt}^{(i)}) \tag{5}$$

where

$$c^{(i)}\left(F_{u_1^{(i)}}(u_{1t}^{(i)}), ..., F_{u_K^{(i)}}(u_{Kt}^{(i)}), F_{\mathcal{Z}^{(i)}}(\mathcal{Z}_t^{(i)})\right) = \frac{\partial^{K+1} C^{(i)}\left(F_{u_1^{(i)}}(u_{1t}^{(i)}), ..., F_{u_K^{(i)}}(u_{Kt}^{(i)}), F_{\mathcal{Z}^{(i)}}(\mathcal{Z}_t^{(i)})\right)}{\partial F_{u_1^{(i)}} ... \partial F_{u_K^{(i)}} \partial F_{\mathcal{Z}^{(i)}}}$$

is the copula density of $u_t^{(i)}$ and $\mathcal{Z}_t^{(i)}$, and $f_{u_k^{(i)}}(u_{kt}^{(i)})$ in (5) are the marginal densities of error terms $u_{kt}^{(i)}$, for $k = 1, 2, .., K$.

The representation of the conditional density $f_{u^{(i)}|\mathcal{Z}_t^{(i)}}(u_{1t}^{(i)}, .., u_{Kt}^{(i)}|\mathcal{Z}_t^{(i)})$, given by (5), enables us to derive an analytic expression for it under a known copula distributions $C^{(i)}$ (or densities $c^{(i)}$) and the marginal distributions of error terms $u_{kt}^{(i)}$. Then, we can obtain analytic formulas for $\mathbb{E}(u_t^{(i)}|\mathcal{Z}_t^{(i)})$. To this end, in the below proposition, we assume that the innovation terms $u_{kt}^{(i)}$ are normally distributed and $C^{(i)}$ is Gaussian.

Proposition 1 *Let $u_t^{(i)} \sim \mathcal{N}IID(0, \Sigma_{uu}^{(i)})$ and $C^{(i)}$ is a multivariate Gaussian copula, then the vector of the TVAR innovations $u_t^{(i)} = (u_{1t}^{(i)}, u_{2t}^{(i)}, ..., u_{Kt}^{(i)})'$ has the following single factor representation:*

$$u_t^{(i)} = \Lambda^{(i)} z_t^{(i)*} + \varepsilon_t^{(i)}, \tag{6}$$

where $\Lambda^{(i)} = \Sigma_u^{(i)1/2} R_{u^{(i)*} z_t^{(i)*}} R_{z_t^{(i)*} u^{(i)*}}$; $\Sigma_u^{(i)} \equiv diag[\sigma_{u_k}^{(i)2}]$, $k = 1, 2, ..., K$, and $R_{u^{(i)*} z_t^{(i)*}} = R'_{z_t^{(i)*} u^{(i)*}}$ is a $K \times 1$ vector of Pearson correlation coefficients between $u_t^{(i)}$ and $\mathcal{Z}_t^{(i)}$
$z_t^{(i)*} = \Phi^{-1}(F_{\mathcal{Z}^{(i)}}(\mathcal{Z}_t^{(i)}))$ which constitutes a copula transformation of $\mathcal{Z}_t^{(i)}$, where $\Phi^{-1}(\cdot)$ is the quantile function of the normal distribution, and $\varepsilon_t^{(i)}$ is a $K \times 1$ vector of error terms, defined as $\varepsilon_t^{(i)} = \Omega_\varepsilon^{(i)1/2} e_t^{(i)}$, where $\Omega_\varepsilon^{(i)}$ is the variance-covariance matrix of $\varepsilon_t^{(i)}$ given as
$\Omega_\varepsilon^{(i)} = \left(\Sigma_u^{(i)1/2} R_{u^{(i)*} u^{(i)*}} \Sigma_u^{(i)1/2} - \Sigma_u^{(i)1/2} R_{u^{(i)*} z^{(i)*}} R_{z^{(i)*} u^{(i)*}} \Sigma_u^{(i)1/2}\right)$,
and $e_t^{(i)} = (e_{1t}^{(i)}, e_{2t}^{(i)}, ..., e_{Kt}^{(i)})'$ is a $K \times 1$ vector of $\mathcal{N}IID(0, 1)$ variables which are independent of $\mathcal{Z}_t^{(i)}$ and its copula transformation $z_t^{(i)*}$.

Proof: Appendix A

The relationship between $u_t^{(i)}$ and the copula-transformed variable $z_t^{(i)*}$, given by (6), is due to the central result of copula theory (Sklar's Theorem) decomposing the joint distribution of vector $u_t^{(i)}$ and $Z_t^{(i)}$, into a part that captures the dependence structure between them based on copula $C^{(i)}$ and the marginal distributions of the error terms $u_{kt}^{(i)}$.

The linear feature of this relationship is due to the Gaussian assumption made.

As noted before, a linear relationship between $u_t^{(i)}$ and $z_t^{(i)*}$ can also be obtained under the assumption that $C^{(i)}$ is a multivariate Student's t-copula and $u_{kt}^{(i)}$ follows the Student's t-distribution, for all k (see Christopoulos et al. 2021). In Appendix B, however, we check the robustness of this relationship under *non-Gaussian* copulas. We demonstrate that the method works well in a non-Gaussian environment.

It is important to note that, to obtain condition (6), no assumption is made about the marginal distribution of the truncated variable $z_t^{(i)*}$. This variable constitutes a copula transformation of threshold variable z_t over its interval support $\mathcal{Z}_t^{(i)}$. It can be obtained from the marginal distribution $F_{\mathcal{Z}^{(i)}}(\mathcal{Z}_t^{(i)})$ based on the quantile function $\Phi^{-1}(\cdot)$, without making any particular assumption about the true formula of $F_{\mathcal{Z}^{(i)}}(\mathcal{Z}_t^{(i)})$. The distribution $F_{\mathcal{Z}^{(i)}}(\mathcal{Z}_t^{(i)})$ can be efficiently estimated from the data, in a first step, by a non-parametric method (e.g., Silverman, 1986), or by using the empirical cumulative distribution function, see Rice (2007). The Glivenko-Cantelli theorem[6] establishes that these estimates of $F_{\mathcal{Z}^{(i)}}(\mathcal{Z}_t^{(i)})$ almost surely converge to the true cumulative distribution function, implying that the sample estimates of $z_t^{(i)*}$ are consistently estimated.

That there is no need to know the distribution of z_t constitutes an interesting feature of the copula method. Furthermore, in many applications of threshold models in economics, one might expect z_t to follow different distributions across the regions $\{-\infty < z_t \leq \delta\}$ and $\mathcal{Z}_t^{(2)} = \{\delta < z_t < \infty\}$, which may be characterized by different degree of asymmetry and fat-tails.[7] The assumption that $u_{kt}^{(i)}$, for all k, are normally distributed is often made in economics both in single or multivariate regression models for convenience (e.g., Spanos, 2018). It assumes the following two attractive properties of error term distributions: symmetry and a reasonable degree of fat-tails that one would expect to be satisfied from well-specified econometric models. In Ap-

[6]See Glivenko (1933), Cantelli (1993).

[7]To recall the transformed variable $z_t^{(i)*}$ is normal, but not necessarily the original vartiable z_t.

pendix B we verify the robustness of this assumption against different distributional choices for $u_{kt}^{(i)}$.

Condition (6), provided by Proposition 1, implies that the expectation term $\mathbb{E}(u_t^{(i)}|\mathcal{Z}_t^{(i)})$ is given by the following linear relationship: $\mathbb{E}(u_t^{(i)}|\mathcal{Z}_t^{(i)}) = \Lambda^{(i)} z_t^{(i)*}$. Given this, we can then adjust TVAR model (2) to control for the endogeneity of the threshold variable z_t as follows:

$$y_t = \begin{cases} \Gamma^{(1)}\mathbf{Y}_{t-1} + \Lambda^{(1)} z_t^{(1)*} + \varepsilon_t^{(1)}, & \text{if } z_t \leq \delta \\ \\ \Gamma^{(2)}\mathbf{Y}_{t-1} + \Lambda^{(2)} z_t^{(2)*} + \varepsilon_t^{(2)}, & \text{if } z_t > \delta \end{cases} \tag{7}$$

where $\mathbb{E}(z_t^{(i)*}\varepsilon_t^{(i)}) = 0$. The conditional mean of y_t given $\mathbf{Y}_{t-1}$ and $\mathcal{Z}_t^{(i)}$ is then given by,[8]

$$\mathbb{E}(y_t|\mathbf{Y}_{t-1}, \mathcal{Z}_t^{(i)}) = \Gamma^{(i)}\mathbf{Y}_{t-1} + \Lambda^{(i)} z_t^{(i)*} \tag{8}$$

where the copula-transformation terms $z_t^{(i)*}$ work as a control function to capture the correlation structure between the threshold variable and the TVAR vector of innovations $u_t^{(i)}$. The new vector of innovation terms $\varepsilon_t^{(i)}$ of model (7) are orthogonal to the threshold variable z_t over the two intervals $\mathcal{Z}_t^{(1)}$ and $\mathcal{Z}_t^{(2)}$, by the copula theory (see Proposition 1).

The estimates of $\varepsilon_t^{(i)}$ can be employed to obtain the structural errors of a structural TVAR model net of the influence of threshold variable z_t. We can then use these structural errors to perform IRF analysis of y_t with respect to them based on a triangular, or a structural, factorization of the variance-covariance of $\varepsilon_t^{(i)}$, denoted $\Omega_\varepsilon^{(i)}$.

Consequently, it should be clear that ignoring threshold endogeneity will lead to biased estimates of vector $\varepsilon_t^{(i)}$ and its variance-covariance matrix, thus leading to biased estimates of the variance-covariance matrix of the structural errors, and the IRFs implied by a structural representation of the TVAR model. This problem might

[8] The exogenous TVAR is a special case of (8) with $\Lambda^i = 0 \; \forall i$.

be more severe in TVAR models using composite indices (as threshold variables) consisting of a large number of variables, which are likely to be correlated with the VAR innovation terms.

The severity of this problem is amply demonstrated in Appendix B, where we discuss the Monte Carlo properties relating to the endogenous and exogenous threshold VARs, respectively labeled *TVAR(N)* and *TVAR(X)*. In that analysis, we examine different dependence structures between the threshold variable and the structural error terms via linear and nonlinear dependencies, and of different signs and strengths across the states. Results indicate that our method can efficiently control for the endogeneity effects of the threshold variable on the parameter estimates, including the threshold parameter. This is true even where the correlation structure between the threshold variable and the TVAR innovations changes considerably across the states. Ignoring threshold endogeneity, by contrast, leads to severely biased estimates of the TVAR parameters, especially the variance-covariance matrix, and, hence, the implied IRFs.

Figure 1, reproduced from that Appendix, graphically highlights these biases using a bivariate VAR. The true DGP and that generated by the copula method are indistinguishable, whereas the *TVAR(X)* case tends to overshoot markedly and is characterized by an excessively smooth reversion pattern (compared to the true dynamic). We shall see a largely similar picture in our empirical application: since the misspecified *TVAR(X)* model systematically overestimates the threshold parameter and leads to more sluggish dynamics responses compared to the *TVAR(N)* specification.

3 Estimation of the endogenous threshold VAR

If the threshold value δ is known, then estimation of the augmented TVAR model (7) is straightforward, based on least squares. Otherwise, δ can be estimated by the concentrated least squares (e.g., Kourtellos et al., 2016). According to this, δ is estimated based on a grid search procedure over possible values of the threshold variable z_t, by solving the optimization problem:

$$\widehat{\delta} = \underset{\delta \in Q_\delta}{\arg\min}\, RSS(\delta) \tag{9}$$

where the objective function to be minimized, is the sum of squared residuals function for a value of $\delta \in Q_\delta$, where Q_δ is the sample set of all possible values of z_t:[9]

$$RSS(\delta) = \sum\nolimits_{t=1}^{T} \left(\varepsilon_t^{(1)\prime}\varepsilon_t^{(1)} + \varepsilon_t^{(2)\prime}\varepsilon_t^{(2)}\right),$$

Given $\widehat{\delta}$, the LS estimates of the remaining slope coefficients of model (7), denoted

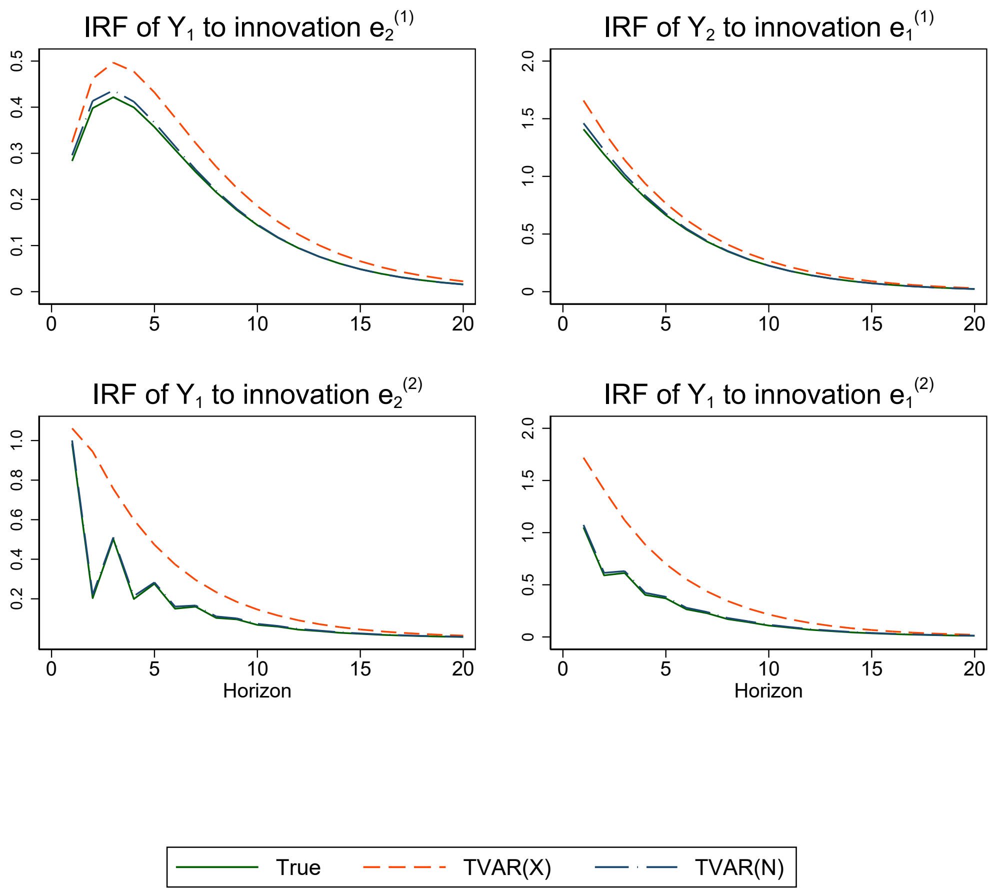
$$\theta(\widehat{\delta}) = (\Gamma^{(1)}(\widehat{\delta}),\ \Gamma^{(2)}(\widehat{\delta}),\ \Lambda^{(1)}(\widehat{\delta}),\ \Lambda^{(2)}(\widehat{\delta})),$$

and those of the variance-covariance matrix $\Omega_\varepsilon^{(i)}$, denoted as $\widehat{\Omega}_\varepsilon^{(i)}(\widehat{\delta})$, can be obtained in a second step. The asymptotic results of Chan (1993), Samia and Chan (2011), and Tsay (1998) imply that, under stationarity conditions for y_t and z_t, as well as distinct (strong) threshold effects (i.e., $\Gamma^{(1)} \neq \Gamma^{(2)}$), the estimator $\widehat{\delta}$ is strongly consistent, while $\theta(\widehat{\delta})$ and $\widehat{\Omega}_\varepsilon^{(i)}(\widehat{\delta})$ are $\sqrt{T}$-consistent. In Appendix B, we perform a Monte Carlo which demonstrates that the estimator performs well even in small samples.[10]

[9]We report the $RSS(\delta)$ metric in the later estimation tables. Note also that Q_δ can be trimmed at a lower and upper bound to ensure a sufficient number of data points for the estimation of the model across the two states.

[10]Note that, since $z_t^{(i)*}$ is estimated with an error, inference about the estimates of $\theta(\widehat{\delta})$ can be improved by employing a bootstrap method to calculate standard errors of the estimates. Note, however, that in our MC exercise, we found that this estimation error is negligible. This is true even for small T.

Figure 1: Impulse Response Functions from Monte Carlo Exercises



Notes: The figure graphically presents estimates of IRFs of y_{1t+h} (or y_{2t+h}) to structural shocks $e_{2t}^{(i)}$ (or $e_{1t}^{(i)}$), for $h = 1, 2, ..., 20$ periods ahead and across the two states of the model, obtained by our method (copula approach), including the threshold variable as an additional regressor in the TVAR model and the case ignoring the threshold variable endogeneity. This is done against the true (theoretical) values of the IRFs, generated by the appendix TVAR model (B.1) under the error-term structure (B.2). The reported estimates of the IRFs in the graphs constitute average values of them based on estimates of (B.1) and (B.2) over 1000 bootstrap iterations. We assume $T = 500$ observations.

4 Application: The Dynamic Effects of Monetary Policy

A positive monetary shock is typically predicted to lower prices and output. However, even across that broad picture, empirical evidence is mixed. Often the impacts on output are mild (and/or insignificant), while prices can take an extended horizon to fall or decline (almost two years or more after the shock, e.g., Bernanke et al. 2005, Eichenbaum et al. 2011; Coibion, 2012; Ramey, 2016; Doh and Foerster, 2022). This diverse behavior may be attributable to the assumption of linearity considered in such studies.

Indeed, the hypothesis that monetary policy impacts are nonlinear and state dependent, moreover, is a long-standing one (recall the *pushing on a string* analogy attributed to Keynes[11]). Broadly speaking, though, we might categorize the arguments for asymmetry into *economic mechanisms* and *policy preferences*.

Economic Mechanisms High inflation can clearly impact policy effectiveness and transmission. Inflation reduces real interest rates, which most noticeably impacts demand for investment, durable goods, and housing. Inflation also redistributes resources across agents (savers, lenders) affecting for instance mortgage refinancing (e.g., Eichenbaum et al. 2022), as well as deepening uncertainty (Vavra, 2014). The existence of state-dependent pricing would also change and enhance the role and transmission of monetary policy. In 'high' inflation states, there may be a faster, and less sluggish, response of inflation and output to monetary shocks. The reason being that state-dependent price setters can freely reset prices when conditions change, with less attendant fallout on demand. This is consistent with models of state-dependent pricing (Álvarez et al., 2019, Ascari and Haber, 2021).

[11]The case for monetary asymmetry based on business cycle thresholds was analyzed by Tenreyro and Thwaites, 2016, see also Manea, 2022).

Policy Preferences There is growing empirical evidence that monetary policy rules may be nonlinear and, in particular, respond asymmetrically to inflation (e.g., Kim et al., 2005; Boivin et al. 2011; Barakchian and Crowe, 2012; Kazanas and Tzavalis, 2015). Davig and Leeper (2007, 2008) – see also Aksoy et al. (2006) and Liu et al. (2009) – attribute this behavior to preemptive movements by the central-bank to slow down inflation expectations by triggering a more aggressive monetary policy (when inflation rates persistently stray above particular levels). These movements can increase the efficiency of monetary policy in states of high and rising current and expected inflation.

4.1 TVAR Specification

Like many empirical studies (e.g., Sims, 1992; Gerlach and Smets, 1995; Peersman and Smets, 2002; Mavroeidis, 2021), we rely on the canonical three-equation New Keynesian (NK) monetary VAR comprising of a measure of real activity, inflation and a monetary instrument: specifically, the annualized growth rate of the index of real industrial production (g_t), the annualized inflation rate (π_t) and, the monetary policy instrument (r_t). For the latter, we alternatively consider the effective Federal Funds rate and the yield on 2-year constant maturity government bonds.[12] This prototypical model is sometimes augmented with a financial premium to capture financial interactions, which we shall also examine. The data are discussed in Appendix C.

We are not of course the first to assess monetary shocks using threshold methods (see, for example, Jääskelä, 2007; Galvao and Marcellino, 2014; Tenreyro and Thwaites, 2016; Zakipour-Saber, 2019; Ascari and Haber, 2021). Compared to such studies we

[12]We do so given that the FFR could be considered problematic during lower bound episodes (see, e.g., Swanson and Williams, 2014). The two-year rate is a natural substitute for the FFR, because it is closely related to the expected path of future short rates (e.g., Bernanke et al. 2004; McGough et al. 2005). Furthermore, to influence market beliefs about the expected path of short rates, the Fed relies on forward guidance to influence the expected rate path, which often operates with roughly two-years horizon.

make four innovations. First, these studies have been performed in the context of *exogenous* threshold models. This in itself is our key methodological contribution. However it also aligns with Ramey (2016) who, while discussing high-frequency identification, notes that shocks retain some predictability and thus are not necessarily exogenous and serially uncorrelated. This would suggest that the monetary VAR advocated in the literature may still be subject to non-trivial endogeneity. Second, those studies also used samples which did not encompass the Global Financial Crisis (GFC) and its aftermath of highly accommodative policy. The efficacy and possible state dependency of policy in these later, very distinct periods is therefore of interest.

In fact, our full sample is 1970-2021. This traverses the Great Inflation of the late 70s/early 80s, the Great Moderation thereafter, and the period around the GFC and Covid pandemic, where the FFR approached the lower bound and a package of 'unconventional' monetary policies were adopted to ease financial conditionss. Third, we use a variety of formulations for robustness. Finally, we make a detailed comparison of our results to linear and exogenous threshold VARs.

To identify the monetary policy shocks, we assume a standard recursive ordering of the variables: real economic activity, inflation, then the policy rate. This ordering implies that the effects of the monetary policy shocks on inflation and output are reached after one period. To see if results remain robust to alternative identification strategies, we also draw on the narrative approach of identifying monetary policy shocks – in particular, using Miranda-Agrippino and Ricco (2021).[13] In this approach, we also include the narrative monetary shock (denoted m) as a first variable in the VAR system, in addition to the other three variables (see Plagborg-Møller and Wolf, 2021, 2022)

Threshold variables are usually constructed as a moving average. Here we use a

[13]This identification method combines narrative and high-frequency approaches (respectively, Romer and Romer, 2004; Gertler and Karadi, 2015), and is robust to information asymmetries of either the monetary authorities or market participants.

20-month moving average of inflation rates, akin to a measure of inflation expectations (see Kim et al., 2005; Davig and Leeper, 2007). Note that this threshold variable also depends on *current* inflation, which can cause threshold bias endogeneity problems (and in turn validate our approach).

5 Empirical analysis

Now we discuss our results. We demonstrate clear evidence of threshold effects, and that the endogenous case constitutes the best specification of the data. For our baseline *TVAR(N)* specification, we find that when our measure of expected inflation exceeds 3.6% percent, the effect of monetary policy changes considerably: the reaction of the economy to a monetary shock in that high-inflation state is stronger and more efficient. It also emerges that interest rates must be held to a higher level and for longer to return inflation to base, relative to the low-inflation state. Thereafter, we show the impulse and cumulative response functions associated with our different cases.

5.1 TVAR estimates

Table 1 provides a snapshot of some system metrics of our alternative VAR specifications considered for the FFR and the 2 year rate (denoted as $2y$) as the monetary instrument, as well as including narrative monetary shock m.[14] The lower panel of the table contains results based on the VAR specification using the excess bond premium (Gilchrist and Zakrajšek, 2012; Gertler and Karadi, 2015).

To test for the presence of significant threshold effects, we report a likelihood ratio (LR) statistic testing the null of a VAR (i.e., without threshold effects) against the alternative of the TVAR specification (with and without controlling for threshold

[14]The full set of individual results can be found in Table 1A-Table 4 in Appendix D.

endogeneity). Since the two threshold states cannot be identified under the null, bootstrapped p-values of the LR statistic are reported, calculated based on a non-overlapping block-bootstrap of fixed-length using 1000 iterations.

In terms of this test statistic, the linear *VAR* is decisively rejected against both TVARs. However, the *TVAR(N)* fits the data better than the *TVAR(X)* on the objective function $RSS(\delta)$, from equation (9), and the bic. Further supporting evidence comes from the estimates of the copula transformations of the threshold variable, i.e., $\Lambda^{(i)} z_t^{(i)*}$, which are significant across both states (results available). Our results hold for all TVAR specifications considered, i.e., using FFR or the two year rate, and including the unexpected monetary shocks m in the model.[15]

For the *TVAR(N)*-FFR model, the estimated threshold value of expected inflation is 3.6% (unless otherwise stated, we can consider this the baseline case). For the other two *TVAR(N)* cases (with the 2y and the unexpected monetary shocks m), we find a comparable range of $3.6\% - 4.2\%$. This value, note, is close to but above the official 2% target rate.[16] To confirm this statistically, we performed a t-ratio test testing $\delta = 2$ against $\delta > 2$. Based on the bootstrap distribution of this test statistic, we find that the null is always rejected against its alternative. This suggests that the impact of monetary policy is ostensibly the same as long as expected inflation is within a roughly $0 - 4\%$ range.

[15]For the TVAR specification with the monetary shocks variable m included, note that the endogeneity of the threshold can arise from the correlation between the threshold variable and the innovations of the remaining variables: u_g, u_π and u_r.

[16]The 2% inflation target was explicitly adopted by the FOMC in January 2012. Moreover, in 2016 the FOMC (FOMC, 2016) clarified that its inflation target is symmetric: "... The Committee would be concerned if inflation was persistently running above or below this objective." Bullard (2018), however, has argued that the Federal Reserve has used an implicit inflation target of 2 percent since 1995. By contrast, Blanchard et al. (2010) recommend a higher inflation rate during 'good' times to leave more room for nominal rate cutting during 'bad' times. See also the discussion in Mumtaz and Theodoridis (2023).

Table 1: Cross-Model Evaluation Metrics

	FFR			2 year			FFR + Narrative shock m		
	TVAR(N)	TVAR(X)	VAR	**TVAR(N)**	TVAR(X)	VAR	**TVAR(N)**	TVAR(X)	VAR
	Estimates of Monetary VAR								
LR	91.90	86.53		73.88	95.71		167.51	115.89	
vs. VAR	[0.000]	[0.000]		[0.000]	[0.000]		[0.000]	[0.000]	
bic	-8465.28	-8412.42	-8403.43	-7718.629	-7698.93	-7694.554	-10110.58	-10100.67	-10083.44
$RSS(\delta)$	0.0567	0.0582		0.0550	0.0563		0.0394	0.0397	
$\widehat{\delta}$	0.0359	0.0436		0.0359	0.0402		0.0418	0.0463	
	$\{0.0180, 0.0460\}$	$\{0.0190, 0.0550\}$		$\{0.0226, 0.0490\}$	$\{0.0270, 0.0535\}$		$\{0.0290, 0.0472\}$	$\{0.0190, 0.0550\}$	
$\%\triangle$		21.4			12.0			11.0	
$\widehat{\delta} = 0.02$	[0.020]			[0.025]			[0.031]		
	Estimates of Monetary VAR Augmented with Finance Premium								
LR	229.51	157.88		148.02	131.02		217.37	205.58	
vs. VAR	[0.000]	[0.000]		[0.000]	[0.000]		0[0.000]	[0.000]	
bic	-7844.84	-7830.59	-7881.87	-12772.67	-12778.38		-13030.68	-9181.08	
$RSS(\delta)$									
$\hat{\delta}$	0.0371	0.0405		0.0376	0.0390		0.0389	0.0389	
	$\{0.0190, 0.0570\}$	$\{0.002384, 0.0572\}$		$\{0.0209, 0.0543\}$	$\{0.0189, 0.05904\}$		$\{0.0189, 0.0589\}$	$\{0.0222, 0.0556\}$	
$\%\triangle$		9.2			3.8			0.0	
$\hat{\delta} = 0.02$	[0.020]			[0.018]			[0.023]		

Notes: Numbers in parentheses, braces and squared brackets respectively indicate standard errors, 68% confidence intervals and p-values. LR stands for the likelihood ratio, and the p-value below tests the respective model against the linear VAR. bic is the Bayesian Information Criterion. $RSS(\delta)$ refers to objective function (9). The final row tests the null that the threshold value is 2% versus an alternative that it exceeds 2% for the *TVAR(N)* case. An empty cell means not applicable. We use a VAR lag length of 2.

Another interesting conclusion is that ignoring threshold endogeneity tends to overstate the threshold estimates. For instance, the *TVAR(X)*-FFR estimates the threshold at 4.5% (almost 80bp, or over 20%, above the baseline). Similar differences are found for the other two TVAR*(N)* cases considered.[17]

Figure 2 plots the two inflation series (i.e., actual and expected) overlaid with the endogenously determined threshold scalar, and with associated state-dependent shading (pertaining to the high inflation state). Around 46% of the time, inflation exceeds its threshold (most of the high inflation state is dated prior to the early 1990s), compared with 37% in the *TVAR(X)* case.[18] These states match well with established views on contemporary US monetary and inflation histories (e.g., Rudebusch, 2002; Bianchi, 2013; Davig and Doh, 2014):

- the Great Inflation of the 1970s (associated with the accommodative Burns regime) with inherited rates of expected inflation exacerbated by the Nixon Shock and the two oil shocks bookmarking the decade;
- the subsequent Volcker disinflation of money targeting and high policy rates, and its two attendant deep recessions;
- the Great Moderation of macroeconomic stability from the mid-1980s;
- the GFC from 2007/9 followed by highly accommodative monetary (and fiscal) policy;
- the Covid shock and global supply-chain disruption, the subsequent high inflation, and the upward revision of nominal policy rates.

[17]We also tested for differences in the bootstrapped distributions of the δ estimators across the models controlling for and ignoring threshold endogeneity based on Kolmogorov-Smirnov test and found that the null of equality of the two distributions is decisively rejected. This is what one might expect if the estimator that ignores threshold endogeneity is biased.

[18]From a statistical viewpoint, the lower $\widehat{\delta}$ implies a larger number of observations in the high inflation state which may sharpen model inference in this state.

5.2 Impulse and cumulative impulse response analysis

We now study the dynamic effects of monetary policy shocks on activity and inflation. To this end, we present IRFs and cumulative IRFs (CIRFs) obtained by a Cholesky decomposition of the TVAR innovations to identify the structural shocks. All such structural shocks have a unit variance. Their impact on y, π and r, though, differs from variable to variable, across high and low inflation states and between models reflecting the different coefficients and variance-covariance estimates (recall Proposition 1).

Figure 3 and 4 respectively compare the *TVAR(N)* with the *VAR* and the *TVAR(X)*, for the FFR 3-equation case (i.e., the baseline case in Table 1).[19] Unlike the VAR, the threshold models have distinct results for the different inflation states. All models predict similar outcomes regarding the long-run effects of a positive monetary shock: output and inflation tend to fall after the monetary contraction then revert to base. However, there are some stark differences among the models with respect to the dynamic paths, as well as their size and degree of persistence.

In the linear VAR, a one standard deviation positive monetary shock has an outsize and highly persistent absolute effect on output and inflation. Price behavior in the linear case for instance appears excessively sticky, with the result that output must be held markedly below base for a long period. Comparing the TVARs, Figure 4, shows again at least in the high inflation state, that the exogenous TVAR makes output look more sluggish and prices stickier relative to the endogenous case. The standard deviation of the monetary shocks is lower in both regimes for the *TVAR(N)* case, suggesting that monetary policy can be more efficient in high-inflation states.

Consider also the CIRFs (in which we multiply the lower triangle matrix by 100), see Figure 5, inflation and output continue to increase for up to a year after the shock in the low-inflation state, thus confirming the puzzling price behavior following the

[19] In the appendix we also plot the 2 year case.

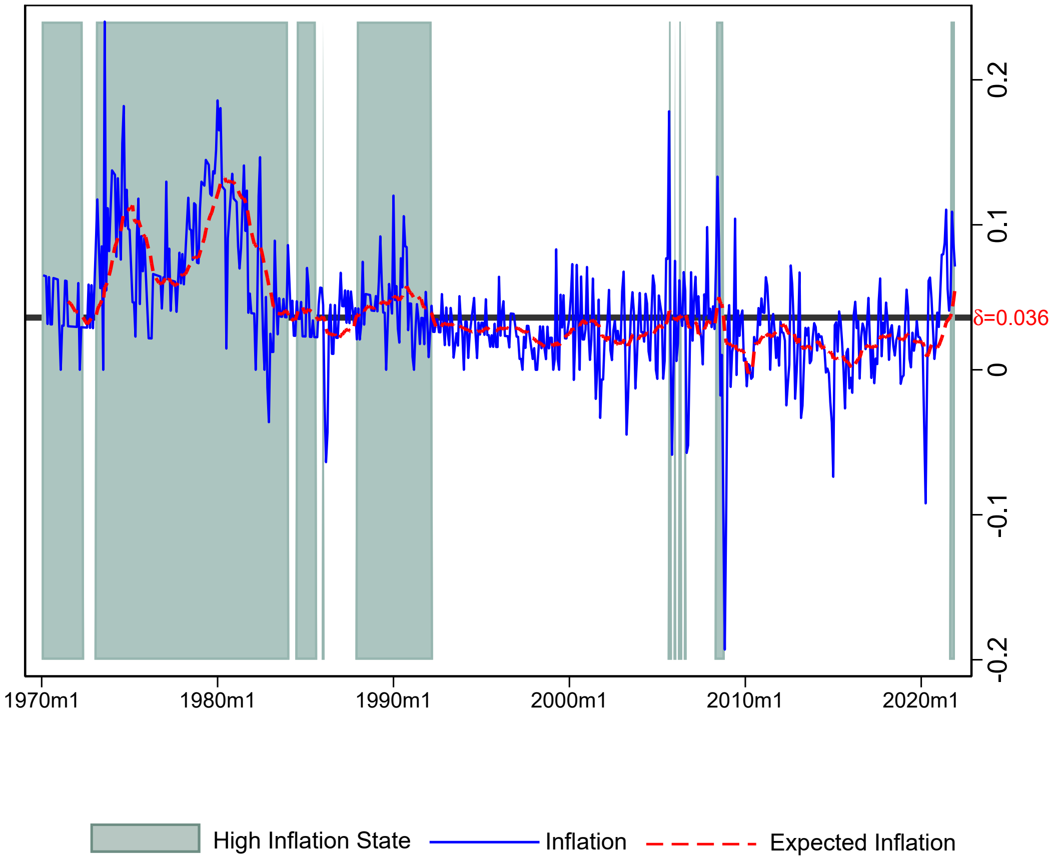
monetary policy shock noted in the literature. This puzzling behavior does not seem to appear in the TVARs, especially the *TVAR(N)*, in the high inflation state. The panels indicate that, although the size and persistence of effects of the monetary shocks on inflation and output, in general, look similar in the high inflation states across the two TVAR models, the *TVAR(X)* model predicts that inflation is stickier than under the endogenous threshold case and the output costs are much larger and longer lasting. The estimates of the IRFs and CIRFs of the *TVAR(N)* model for inflation and output fall rapidly below zero (within 4, 5 months) after the shock occurs and quickly return to the base, compared to the *TVAR(X)*. This is in line with Davig and Leeper's (2008) thesis that monetary policy works more efficiently in the high inflation state.

In contrast to the high inflation state, the results of the figures indicate that the pattern of the IRFs and CIRFs is both qualitatively and quantitatively more similar across the two threshold models in the low inflation state, and is closely related to the linear *VAR*. The degree of stickiness of inflation and output to a contractionary monetary shocks is higher than under the high inflation state for both the *TVAR(N)* and *TVAR(X)* models, and it takes a longer period (almost a year) for a monetary shock to have negative effects on inflation and output. These results imply that the price puzzling behavior of the monetary policy is a low-inflation phenomenon. This is consistent with Karadi and Reiff (2019) theory model where firms are not willing to change prices as frequently as in the high inflation state to avoid adjustment, while they change prices more frequently when inflation is higher (e.g., Alvarez et al., 2019).

The above results hold also for the 2y interest rate and the case that the unexpected monetary policy shock m is used as structural shock (see Table 1). In summary, these results clearly indicate that monetary tightening is contractionary, and this is more evident in the high inflation state where monetary policy needs to more strongly anchor inflation expectations.

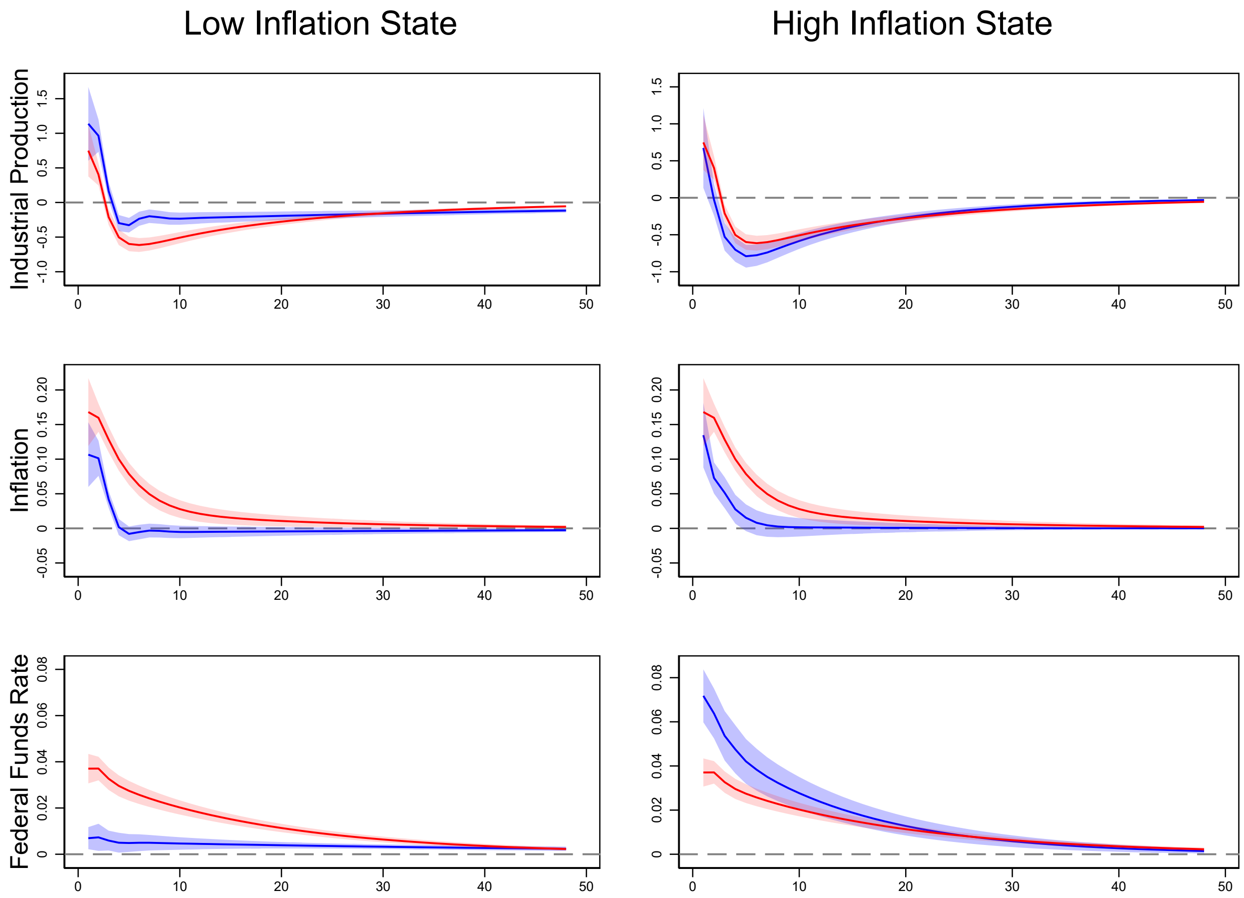
Figure 2: Inflation and Expected Inflation

(overlaid with high-inflation, state-dependent shading)



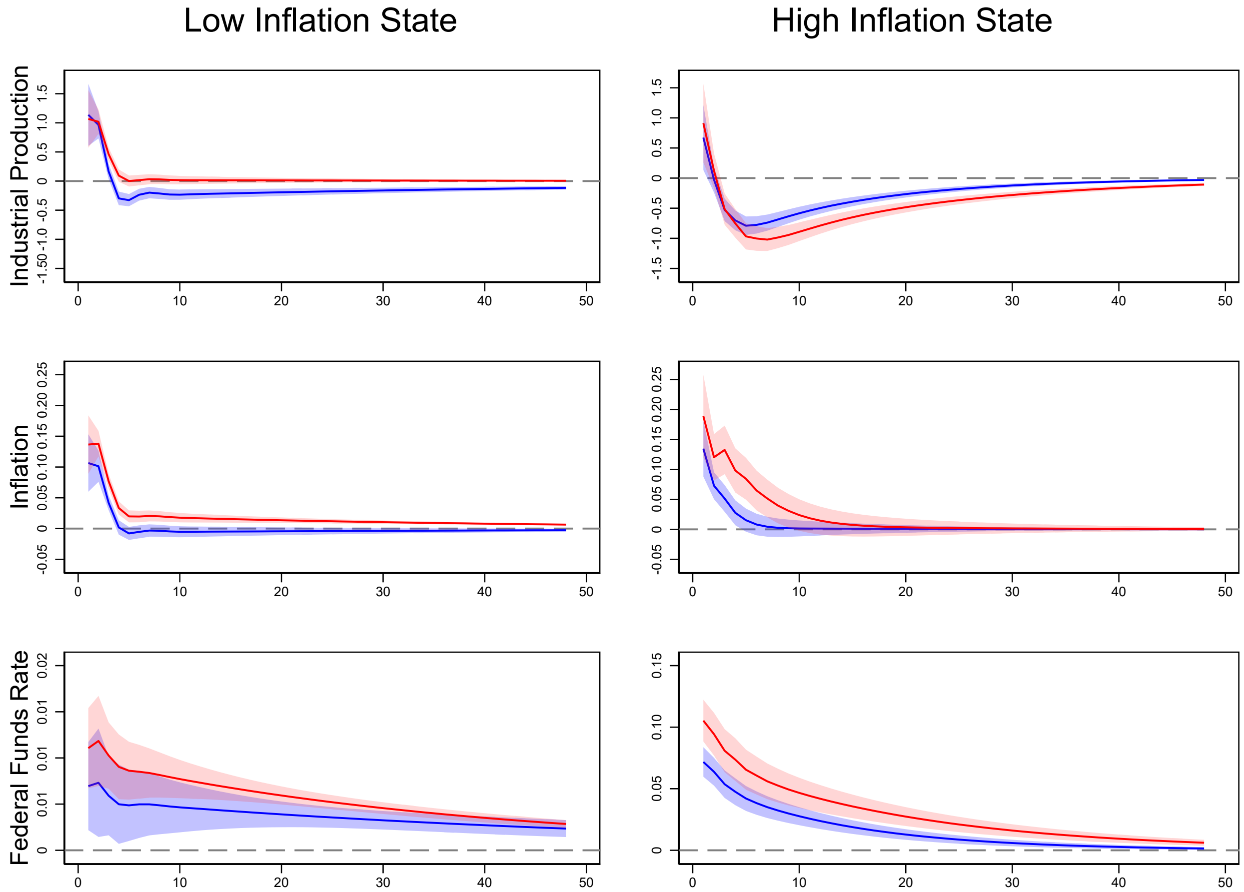
Notes: This figure plots the monthly inflation series in blue and a measure of expected inflation (a 20 month moving average) in red dash. The figure also overlays shaded green vertical bars which indicate the high inflation state, i.e., where expected inflation exceeds the empirically determined threshold value of 0.036 (indicated by the horizontal black line).

Figure 3: IMPULSE RESPONSES: *TVAR(N)* (BLUE) AND LINEAR *VAR* (RED)



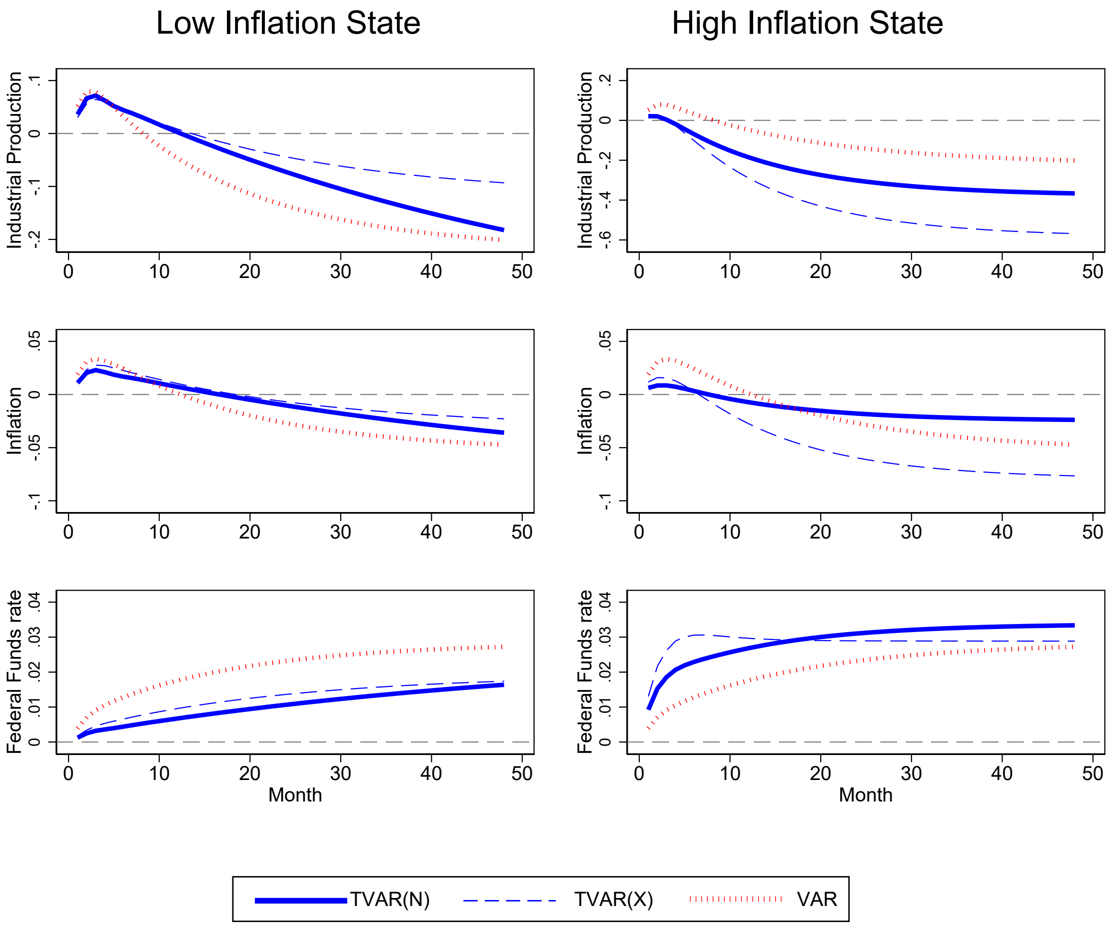
Notes: This panel of figures show the responses to a one standard deviation shock of the monetary innovation for the endogenously determined TVAR model (in blue) and the linear model (in red) across the two inflation states. 68% confidence intervals are shown in shaded bands. The high inflation state occurs when expected inflation (defined as a 20 month moving average) exceeds the threshold: $z_t > 0.0359$. In the linear case there is by definition no state dependency for the impulse responses. All values have been scaled by 100 for visual convenience. The horizontal axis measures months.

Figure 4: Impulse Responses: *TVAR(N)* (Blue) and *TVAR(X)* (Red)



Notes: This panel of figures show the responses to a one standard deviation shock of the monetary innovation for the endogenously determined TVAR model (in blue) and the exogenously determined *TVAR(X)* model (in red) across the two inflation states. 68% confidence intervals are shown in shaded bands. The high inflation state occurs when expected inflation (a 20 month moving average) exceeds the threshold: $z_t > 0.0359$. In the linear case there is by definition no state dependency for the impulse responses. All values have been scaled by 100 for visual convenience. The horizontal axis measures months.

Figure 5: Cumulative IRFs



Notes: Cumulative IRFs for Industrial Production growth and Inflation across low and high inflation states. Solid blue lines indicate the case of allowing for threshold endogeneity, the dashed blue lines represent the exogenous threshold estimates, and the dotted red indicates the linear VAR.

6 Conclusions

The contribution of this paper has been to put forward a new method to control for threshold endogeneity in threshold VARs. As far as we are aware, our paper is the first to tackle this. We do so by appealing to copula theory, which is a simple and instruments-free method of capturing threshold endogeneity.

To study the consequences of ignoring threshold endogeneity on the VAR estimates and impulse response functions (IRFs), and to evaluate the performance of the suggested method, we perform a Monte Carlo study. That study also examined the robustness of our method to nonlinear dependence structures between the threshold variable and the TVAR innovations implied by copulas of the Archimedean family. Our results demonstrate that our method can be successfully implemented to estimate the model parameters and uncover the true IRFs.

As an application, we examined state dependency in candidate monetary VARs. For a baseline case, we find that when expected inflation exceeds 3.6%, the effects of monetary policy on both inflation and output are faster and stronger than otherwise. Partitioning the historical data into high and low-inflation states matches economic developments. Ignoring threshold endogeneity tends to overstate the threshold parameter and the effects of monetary shocks on output and inflation in the high-inflation state and understates them in the low-inflation state. Compared to the endogenous threshold VAR, both the linear and the exogenous threshold VAR, suggest that prices are sticky and output sluggish in response to monetary shocks in the high-inflation state. Our results, by contrast, are more in line with state-dependent pricing models (in which inflation reactions are faster and output reactions more muted) and consistent with models which suggest that monetary policy is more efficient in high-inflation states.

Accordingly, we have established that endogenous threshold VARs overcome the biases and impaired dynamic analysis of exogenous threshold VARs. Given the huge popularity of VARs, and the burgeoning interest in nonlinearity and state dependence, our method should prove useful in the wide variety of applications addressed by the the threshold literature.

References

Aksoy, Y., A. Orphanides, D. Small, V. Wieland, and D. Wilcox (2006) A quantitative exploration of the opportunistic approach to disinflation. *Journal of Monetary Economics* 53, 1877-93.

Barakchian, S. M. and C. Crowe (2012) Monetary policy matters: Evidence from new shocks data, *Journal of Monetary Economics*, 60, 8, 950–966.

Álvarez, F., M. Beraja, Gonzalez-Rozada, M. and A. Neumeyer (2019). From Hyperinflation to Stable Prices: Argentina's Evidence on Menu Cost Models. *Quarterly Journal of Economics*, 134, 1, 451–505.

Ascari, G. and T. Haber (2021). Non-linearities, state-dependent prices and the transmission mechanism of monetary policy. *Economic Journal*, 110, 465, 664–686.

Auerbach, A. J. and Y. Gorodnichenko (2012). Measuring the Output Responses to Fiscal Policy. *American Economic Journal: Economic Policy*, 4(2), 1-27.

Balke, N. (2000). Credit and economic activity: Credit regimes and nonlinear propagation of shocks. *Review of Economics and Statistics*, 82, 344–349.

Bauer, M. D. and E. T. Swanson (2022). A Reassessment of Monetary Policy Surprises and High-Frequency Identification, NBER Chapters, in: NBER *Macroeconomics Annual* 2022, 37, National Bureau of Economic Research, Inc.

Bernanke, B. S., J. Boivin, and P. Eliasz (2005) Measuring the Effects of Monetary Policy: A Factor-Augmented Vector Autoregressive (FAVAR) Approach. *Quarterly Journal of Economics*, 120, 1, 387–422.

Bernanke, B. S., V. R. Reinhart, and B. P. Sack (2004). Monetary Policy Alternatives at the Zero Bound: An Empirical Assessment. *Brookings Papers on Economic Activity* 34 (2): 1–100.

Bianchi, F. (2013). Regime switches, Agents' Beliefs, and Post-World War II U.S. Macroeconomic Dynamics. *Review of Economic Studies*, 80(2), 463-490.

Blanchard, O. J., G. Dell'Ariccia and P. Mauro (2010). Rethinking Macroeconomic Policy. *Journal of Money, Credit and Banking*, 42(s1), 199-215.

Boivin, J., M. T. Kiley, and F. S. Mishkin (2011) How has the Monetary Transmission Mechanism Evolved Over Time? In *Handbook of Monetary Economics*, edited by B. Friedman and M. Woodford, 3A, 369–422.

Bullard, J. (2018). What Is the Best Strategy for Extending the U.S. Economy's Expansion? Speech at CFA Society Chicago—Distinguished Speaker Series Breakfast Sept. 12, 2018 Chicago, Ill.

Cantelli, F. P. (1933). Sulla determinazione empirica delle leggidi probabilita. *Giornale dell'Istituto Italiano degli Attuari*, 4:421–424

Chen, C., T. Stengos and Y. Sun (2023) Endogeneity in semiparametric threshold regression models with two threshold variables. *Econometric Reviews* (forthcoming).

Christopoulos, D. K., P. McAdam and E. Tzavalis (2021). Dealing With Endogeneity in Threshold Models Using Copulas. *Journal of Business & Economics Statistics*, 39, 166-178. Stability: Evidence and Some Theory. *Quarterly Journal of Economics*, 115 (1), 147–180.

Coibion, O. (2012). Are the Effects of Monetary Policy Shocks Big or Small? *American Economic Journal: Macroeconomics*, 4 (2), 1–32

Davig, T. and Doh, T. (2014). Monetary Policy Regime Shifts and Inflation Persistence. *Review of Economics and Statistics*, 96(5): 862–875.

Davig, T. and Leeper, E. M. (2007). Generalizing the Taylor principle. *American Economic Review*, 97(3), 607–35.

Davig, T. and Leeper, E. M. (2008). Endogenous monetary policy regime change. In NBER *International Seminar on Macroeconomics 2006*. Cambridge, MA: National Bureau of Economic Research, 345–91.

Doh, T. and A. Foerster (2022) Have Lags in Monetary Policy Transmission Shortened? *Economic Bulletin*, Federal Reserve Bank of Kansas City, December, 1-3.

Eichenbaum, M., N. Jaimovich and S. Rebelo (2011). Reference prices, costs, and nominal rigidities. *American Economic Review*, 101, 234–262.

Eichenbaum, M., S. Rebelo and A. Wong (2022) State-Dependent Effects of Monetary Policy: The Refinancing Channel. *American Economic Review*, 112(3), 721–761.

Erdely, A. (2017). Copula-based piecewise regression, in Ubeda F. M., de Amo Artero E., Durante F., and Fernandez S. J. (eds). *Copulas and Dependence Models with Applications*, Springer, New York, pp. 69–81.

Fernandez, C. and Steel, M. F. J. (2012). On Bayesian modeling of fat tails and skewness. *Journal of American Statistal Association*, 93, 359–371.

Floro, D. and van Roye, B. (2017). Threshold effects of financial stress on monetary policy rules: A panel data analysis. *International Review of Economics & Finance*, 51(C), 599-620.

FOMC (2016). Statement on Longer-Run Goals and Monetary Policy Strategy Adopted effective January 24, 2012; as amended effective January 26, 2016. Available at www.federalreserve.gov/monetarypolicy/files/fomc_longerrungoals_20160126.pdf.

Galvao, A. B. and M. Marcellino (2014). The effects of the monetary policy stance on the transmission mechanism. *Studies in Nonlinear Dynamics & Econometrics*, 18, 3, 217-236.

Gerlach, S. and F. Smets (1995). The monetary transmission mechanism: Evidence from the G-7 countries, BIS Working Papers 26, Bank for International Settlements.

Gertler, M. and P. Karadi (2015). Monetary Policy Surprises, Credit Costs, and Economic Activity. *American Economic Journal: Macroeconomics*, 7 (1), 44–76.

Ghassibe, M. and F. Zanetti (2022) State dependence of fiscal multipliers: the source of fluctuations matters. *Journal of Monetary Economics*, 132(C), pages 1-23.

Glivenko, V. I. (1933). Sulla determinazione empirica delle leggi di probabilita. *Giornale dell'Istituto Italiano degli Attuari*, 4, 92–99

Gilchrist, S. and E. Zakrajšek (2012) Credit spreads and business cycle fluctuations. *American Economic Review* 102 (4): 1692–1720.

Growiec, J. (2013). A microfoundation for normalized CES production functions with factor-augmenting technical change and copulas. *Journal of Economic Dynamics and Control*, 37(11), 2336-2350.

Harvey, C. R. and A. Siddique (2000). Conditional Skewness in Asset Pricing Tests. *Journal of Finance*, 55, 1263-1295.

Jääskelä, J. (2007). More Potent Monetary Policy? Insights from a Threshold Model, RBA Research Discussion Papers rdp2007-07, Reserve Bank of Australia.

Kazanas, T. and E. Tzavalis (2015). Unveiling ECB's monetary policy behavior under different inflation regimes. *Economica*, 82, 912-937.

Kilian, L. and H. Lütkepohl (2017). *Structural Vector Autoregressive Analysis*, Cambridge University Press.

Kim, C.-J., J. Piger and R. Startz (2008). Estimation of markov regime-switching regressions with endogenous switching. *Journal of Econometrics* 143 (2), 263–273.

Kim, D. H., D. R. Osborn, and M. Sensier (2005). Nonlinearity in the Fed's monetary policy rule. *Journal of Applied Econometrics*, 20, 621–39

Kourtellos, A., T. Stengos and C. M. Tan (2013). The effect of public debt on growth in multiple regimes. *Journal of Macroeconomics*, 38(PA), 35-43.

Kourtellos, A., T. Stengos and C. M. Tan (2016). Structural threshold regression. *Econometric Theory*, 32, 827–860.

Kourtellos, A., T. Stengos and Y. Sun (2022). Endogeneity in semiparametric threshold regression. *Econometric Theory*, 38, 3, 562–595.

Lee, L. F. (1982). Some Approaches to the Correction of Selectivity Bias. *Review of Economic Studies*, XLIX, 355–372.

Lee, L. F. (1983). Generalized Econometric Models With Selectivity. *Econometrica*, 51, 507–512.

Liu, Z., D. F. Waggner and T. Zha (2009). Asymmetric expectation effects of regime shifts in monetary policy. *Review of Economic Dynamics*, 12, 284–303.

Maddala, G. S. (1983). *Limited-Dependent and Qualitative Variables in Econometrics*, Cambridge: Cambridge University Press.

Manea, C. (2022). Monetary Policy with Financially-Constrained and Unconstrained Firms, available at https://cristina-manea-econ.com/wp-content/uploads/2020/08/Heterogenous_firms_monetary_policy_cmanea.pdf.

Mavroeidis, S. (2021) Identification at the Zero Lower Bound. *Econometrica*, 89(6), 2855-2885.

McGough B., G. D. Rudebusch and J.C. Williams (2005). Using a long-term interest rate as the monetary policy instrument. *Journal of Monetary Economics*, 52, 855-879.

Miranda-Agrippino, S. and G. Ricco (2021). The Transmission of Monetary Policy Shocks. *American Economic Journal: Macroeconomics*, 13(3), 74-107.

Mumtaz, H. and K. Theodoridis (2023) The Federal Reserve's Implicit Inflation Target and Macroeconomic Dynamics. A SVAR analysis. *Internation Economic Review* (forthcoming).

Papantonis, I, L. Rompolis and E. Tzavalis (2023). Improving variance forecasts: The role of Realized Variance features. *International Journal of Forecasting* (forthcoming).

Park, S. and S. Gupta (2012), Handling endogenous regressors by joint estimations using copulas. *Marketing Science*, 31, 567-586.

Patton, A. J. (2006). Modeling Asymmetric Exchange Rate Dependence. *International Economic Review*, 47, 527–556.

Patton, A. J. (2012). A Review of Copula Models for Economic Time Series. *Journal of Multivariate Analysis*, 110, 4–18.

Peersman, G. and F. Smets (2002). Are the effects of monetary policy in the euro area greater in recessions than in booms?, in Mahadeva and Sinclair (eds). *Monetary transmission in Diverse Economies*, Cambridge University Press, 28-48.

Plagborg-Møller, M. and C. K. Wolf (2022). Instrumental Variable Identification of Dynamic Variance Decompositions. *Journal of Political Economy*, 130 (8), 2164–2202.

Plagborg-Møller, M. and C. K. Wolf (2021) Local projections and VARs estimate the same impulse responses. *Econometrica*, 89, 2, 955–980.

Ramey, V. A. (2016). Macroeconomic shocks and their propagation. In *Handbook of Macroeconomics*, volume 2, 71–162.

Ramey, V. A. and S. Zubairy (2018) Government Spending Multipliers in Good Times and in Bad: Evidence from US Historical Data. *Journal of Political Economy*, 126(2), 850–901.

Ramey, V. A., N. B. Zeev and S. Zubairy (2023) Do Government Spending Multipliers Depend on the Sign of the Shock? *American Economic Association Papers & Proceedings*, March.

Reinhart, C. M. and K. S. Rogoff (2010) Growth in time of debt. *American Economic Review Papers and Proceedings*, 100, 573-578.

Rice, J. A. (2007). *Mathematical Statistics and Data Analysis*, 3rd ed. Cengage Learning.

Romer, C. D. and D. H. Romer (2004). A New Measure of Monetary Shocks: Derivation and Implications. *American Economic Review*, 94, 4, 1055–1084.

Rudebusch, G. D. (2002) Term Structure Evidence on Interest Rate Smoothing and Monetary Policy Inertia. *Journal of Monetary Economics* 49, 1161–1187.

Samia, N. I. and Chan, K.-S. (2011). Maximum likelihood estimation of a generalized threshold stochastic regression model. *Biometrika*, 98(2):433–448.

Siburg, K. F. and Stoimenov, P. A. (2008). Gluing copulas. *Communications in Statistics, Theory and Methods*, 37, 3124–3134.

Silverman, B. W. (1986). *Density Estimation for Statistics and Data Analysis*. Chapman & Hall, London

Sims, C. A. (1992). Interpreting the macroeconomic time series facts: The effects of monetary policy. *European Economic Review*, 36, 5, 975-1000.

Sims, C. A. and T. Zha (2006). Were there regime switches in US monetary policy? *American Economic Review* 96, 54–81.

Spanos, A. (2018). Mis-specification Testing in Retrospect. *Journal of Economic Surveys*, 32, 541–577.

Swanson, E. T., and J. C. Williams (2014). Measuring the Effect of the Zero Lower Bound on Medium- and Longer-Term Interest Rates. *American Economic Review* 104 (10): 3154–85.

Tenreyro, S. and G. Thwaites (2016). Pushing on a String: US Monetary Policy Is Less Powerful in Recessions. *American Economic Journal: Macroeconomics*, 8(4), 43-74.

Tófoli, P., Ziegelmann, F. A. and O. Candido (2017). A Comparison Study of Copula Models for European Financial Index Returns. *International Journal of Economics and Finance*, 9(10):155-177.

Tsay, R. S. (1998). Testing and modeling multivariate threshold models. *Journal of the American Statistical Association*, 93, 1188–1202.

Turner, C. M., R. Startz, and C. R. Nelson (1989). A Markov model of heteroskedasticity, risk, and learning in the stock market. *Journal of Financial Economics*, 25, 3–22.

Vavra, J. (2014) Inflation Dynamics and Time-Varying Volatility: New Evidence and an Ss Interpretation. *Quarterly Journal of Economics*, 129, 1, 215–258.

Vella, F. (1998). Estimating Models With Sample Selection Bias: A Survey. *Journal of Human Resources*, 33, 127–169.

Zakipour-Saber, S. (2019). State-dependent Monetary Policy Regimes, Research Technical Papers 4/RT/19. Central Bank of Ireland.

A Proof of Proposition 1

If C_i follows a multivariate Gaussian copula, then the copula density function $c^{(i)}\left(F_{u_1^{(i)}}(u_{1t}^{(i)}),...,F_{u_K^{(i)}}(u_{Kt}^{(i)}),F_{\mathcal{Z}^{(i)}}(\mathcal{Z}_t^{(i)})\right)$ can be written as

$$c^{(i)}\left(F_{u_1^{(i)}}(u_{1t}^{(i)}),...,F_{u_K^{(i)}}(u_{Kt}^{(i)}),F_{\mathcal{Z}^{(i)}}(\mathcal{Z}_t^{(i)})\right) = \phi^{(i)}(u_{1t}^{(i)*},...,u_{Kt}^{(i)*},z_t^{(i)*})$$
$$= \frac{1}{|R^{(i)}|^{1/2}}\exp\left(-\frac{1}{2}U_t^{*(i)\prime}(R^{(i)-1}-I)U_t^{*(i)}\right), \tag{A.1}$$

where $u_{kt}^{(i)*} = \Phi^{-1}(F_{u_k^{(i)}}(u_{kt}^{(i)}))$ and $z_t^{(i)*} = \Phi^{-1}(F_{\mathcal{Z}^{(i)}}(\mathcal{Z}_t^{(i)})$, $U_t^{(i)*\prime} = (u_t^{(i)*\prime}\vdots z_t^{(i)*})$ is a $(K+1)\times 1$ vector stacking the standardized random variables $u_{kt}^{(i)*}$ and $z_t^{(i)*}$ and $R^{(i)}$ is Pearson's correlation coefficients matrix for vector $U_t^{*(i)}$.

Let us partition $R^{(i)}$, as[1]

$$R^{(i)} = \begin{bmatrix} R_{u^{(i)*}u^{(i)*}} & R_{u^{(i)*}z_t^{(i)*}} \\ R_{z^{*\mathcal{H}}u^{(i)*}} & R_{z_t^{(i)*}z_t^{(i)*}} \end{bmatrix},$$

where $R_{u^{(i)*}u^{(i)*}}$ is the $K\times K$ matrix of correlation coefficients between the elements of $u_t^{(i)*}$, $R_{u^*z_t^{(i)*}}$ is a $K\times 1$ vector of the correlation coefficients between $u_t^{(i)*}$ and $z_t^{(i)*}$ ($R_{z^{*\mathcal{H}}u^{(i)*}}$ is the transform of $R_{u^{(i)*}z_t^{(i)*}}$,$R_{z^{*\mathcal{H}}u^{(i)*}} = R'_{u^*z_t^{(i)*}}$), and

[1] $R_{\varepsilon^*\varepsilon^*}$ is the correlation matrix of vector ε_t^*, $R_{\varepsilon^*z^{*\mathcal{H}}}$ (or $R'_{z^{*\mathcal{H}}\varepsilon^*}$) is the correlation matrix between vectors ε_t^* and $z_t^{*\mathcal{H}}$, and $R_{z^{*\mathcal{H}}z^{*\mathcal{H}}}$ is the correlation matrix of vector $z_t^{*\mathcal{H}}$.

$R_{z_t^{(i)*} z_t^{(i)*}} = 1.$

Using the copula density (A.1) and assuming that $u_{kt}^{(i)} \sim \mathcal{N}IID(0, \sigma_{u_k}^2)$, the conditional density function $f_{u^{(i)}|\mathcal{Z}_t^{(i)}}(u_{1t}^{(i)}, .., u_{Kt}^{(i)}|\mathcal{Z}_t^{(i)})$ (see equation (5)) can be derived as follows:

$$\begin{aligned}
& f_{u^{(i)}|\mathcal{Z}_t^{(i)}}(u_{1t}^{(i)}, .., u_{Kt}^{(i)}|\mathcal{Z}_t^{(i)}) \\
&= \phi^{(i)}(u_{1t}^{(i)*}, ..., u_{Kt}^{(i)*}, z_t^{(i)*}) \prod_{k=1}^{K} f(u_{kt}) \\
&= \frac{1}{|R^{(i)}|^{1/2}} \exp\left(-\frac{1}{2} U_t^{(i)*\prime}(R^{(i)-1} - I) U_t^{*(i)}\right) \prod_{k=1}^{K} \frac{1}{\sigma_{u_k}\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\frac{u_{kt}^{(i)2}}{\sigma_{u_k}^2}\right\} \\
&= \frac{1}{|R^{(i)}|^{1/2}\left(\sqrt{2\pi}\right)^K \prod\limits_{k=1}^{K}\sigma_{u_i}} \exp\left\{-\frac{1}{2} U_t^{(i)*\prime}(R^{(i)-1} - I) U_t^{*(i)}\right\} \prod_{k=1}^{K} \exp\left\{-\frac{1}{2}\frac{u_{kt}^{(i)2}}{\sigma_{u_k}^2}\right\} \\
&= \frac{1}{|R^{(i)}|^{1/2}\left(\sqrt{2\pi}\right)^K |\Sigma_u^{(i)}|^{1/2}} \exp\left\{-\frac{1}{2} U_t^{(i)*\prime}(R^{(i)-1} - I) U_t^{*(i)}\right\} \exp\left\{-\frac{1}{2} u_t^{(i)\prime}\Sigma_u^{(i)-1} u_t^{(i)}\right\} \\
&= \frac{1}{\left(\sqrt{2\pi}\right)^K |R^{(i)}|^{1/2}|\Sigma_u^{(i)}|^{1/2}} \exp\left\{-\frac{1}{2}\left(U_t^{(i)*\prime} R^{(i)-1} U_t^{*(i)} - U_t^{(i)*\prime} U_t^{*(i)} + u_t^{(i)\prime}\Sigma_u^{(i)-1} u_t^{(i)}\right)\right\}
\end{aligned} \tag{A.2}$$

where $\Sigma_u^{(i)} \equiv diag[\sigma_{u_k}^2]$. Using $u_t^{(i)} = \Sigma_u^{(i)1/2} u_t^{(i)*}$ from the definition of multivariate copula and $u_{kt}^{(i)} \sim \mathcal{N}IID(0, \sigma_{u_k}^2)$,

$k=1,2,...K$, the quadratic form entering (A.2) can be written as follows:

$$
\begin{aligned}
&U_t^{(i)*\prime}R^{(i)-1}U_t^{(i)*}-U_t^{(i)*\prime}U_t^{(i)*}+u_t^{\prime}\Sigma_u^{(i)-1}u_t\\
&=u_t^{(i)*\prime}R_{u^{(i)*}u^{(i)*}}u_t^{(i)*}+u_t^{(i)*\prime}R_{u^{(i)*}z^{(i)*}}z_t^{(i)*}+z_t^{(i)*}R_{z^{(i)*}u^{(i)*}}u_t^{(i)*}+z_t^{(i)*\prime}R_{z^{(i)*}z^{(i)*)}}z_t^{(i)*}-u_t^{(i)*\prime}u_t^{(i)*}-z_t^{(i)*\prime}z_t^{(i)*}+u_t^{(i)\prime}\Sigma_u^{(i)-1}u_t^{(i)}\\
&=u_t^{(i)*\prime}R_{u^{(i)*}u^{(i)*}}u_t^{(i)*}+u_t^{(i)*\prime}R_{u^{(i)*}z^{(i)*}}z_t^{(i)*}+z_t^{(i)*}R_{z^{(i)*}u^{(i)*}}u_t^{(i)*},
\end{aligned}
\tag{A.3}
$$

since $z_t^{(i)*\prime}R_{z^{(i)*}z^{(i)*)}}z_t^{(i)*}=z_t^{(i)*\prime}z_t^{(i)*}=(z_t^{(i)*})^2$ and $u_t^{(i)*\prime}u_t^{(i)*}=u_t^{(i)\prime}\,\Sigma_u^{(i)-1}\Sigma_u^{(i)-1}u_t^{(i)}=u_t^{(i)\prime}\Sigma_u^{(i)-1}u_t^{(i)}$.

Using results on quadratic forms for symmetric matrices, like matrix $R^{(i)}$, relationship (A.3) can be written as

$$
\begin{aligned}
&u_t^{(i)*\prime}R_{u^{(i)*}u^{(i)*}}u_t^{(i)*}+u_t^{(i)*\prime}R_{u^{(i)*}z^{(i)*}}z_t^{(i)*}+z_t^{(i)*}R_{z^{(i)*}u^{(i)*}}u_t^{(i)*}\\
&=\left(u_t^{(i)*}-R_{u^{(i)*}z^{(i)*}}R^{-1}_{z^{(i)*}z^{(i)*}}R_{z^{(i)*}u^{(i)*}}z_t^{(i)*}\right)'\left(R_{u^{(i)*}u^{(i)*}}-R_{u^{(i)*}z^{(i)*}}R_{z^{(i)*}z^{(i)*)}}R_{z^{(i)*}u^{(i)*}}\right)^{-1}\left(u_t^{(i)*}-R_{u^{(i)*}z^{(i)*}}R^{-1}_{z^{(i)*}z^{(i)*}}R_{z^{(i)*}u^{(i)*}}z_t^{(i)*}\right)\\
&=\left(u_t^{(i)*}-R_{u^{(i)*}z^{(i)*}}R_{z^{(i)*}u^{(i)*}}z_t^{(i)*}\right)'\left(R_{u^{(i)*}u^{(i)*}}-R_{u^{(i)*}z^{(i)*}}R_{z^{(i)*}u^{(i)*}}\right)^{-1}\left(u_t^{(i)*}-R_{u^{(i)*}z^{(i)*}}R_{z^{(i)*}u^{(i)*}}z_t^{(i)*}\right)\\
&=\left(u_t^{(i)*}-R_{u^{(i)*}z^{(i)*}}R_{z^{(i)*}u^{(i)*}}z_t^{(i)*}\right)'\Sigma_u^{(i)1/2}\Sigma_u^{(i)-1}\left(R_{u^{(i)*}u^{(i)*}}-R_{u^{(i)*}z^{(i)*}}R_{z^{(i)*}u^{(i)*}}\right)^{-1}\Sigma_u^{(i)-1}\Sigma_u^{(i)1/2}\left(u_t^{(i)*}-R_{u^{(i)*}z^{(i)*}}R_{z^{(i)*}u^{(i)*}}z_t^{(i)*}\right)\\
&=\left(\Sigma_u^{(i)1/2}u_t^{(i)*}-\Sigma_u^{(i)1/2}R_{u^{(i)*}z^{(i)*}}R_{z^{(i)*}u^{(i)*}}z_t^{(i)*}\right)'\left(\Sigma_u^{(i)1/2}R_{u^{(i)*}u^{(i)*}}\Sigma_u^{(i)1/2}-\Sigma_u^{(i)1/2}R_{u^{(i)*}z^{(i)*}}R_{z^{(i)*}u^{(i)*}}\Sigma_u^{(i)1/2}\right)^{-1}\left(\Sigma_u^{(i)1/2}u_t^{(i)*}-\Sigma_u^{(i)1/2}R_{u^{(i)*}z^{(i)*}}R_{z^{(i)*}u^{(i)*}}z_t^{(i)*}\right)\\
&=\varepsilon_t^{(i)\prime}\left(\Sigma_u^{(i)1/2}R_{u^{(i)*}u^{(i)*}}\Sigma_u^{(i)1/2}-\Sigma_u^{(i)1/2}R_{u^{(i)*}z^{(i)*}}R_{z^{(i)*}u^{(i)*}}\Sigma_u^{(i)1/2}\right)^{-1}\varepsilon_t^{(i)}
\end{aligned}
\tag{A.4}
$$

where $\varepsilon_t^{(i)}=u_t^{(i)}-\Sigma_u^{(i)1/2}R_{u^{(i)*}z^{(i)*}}R_{z^{(i)*}u^{(i)*}}z_t^{(i)*}$, $R_{z^{(i)*}z^{(i)*)}}=1$ and $u_t^{(i)}=\Sigma_u^{(i)1/2}u_t^{(i)*}$.

Also note that the quantity $|R^{(i)}|^{1/2}|\Sigma_u^{(i)}|^{1/2}$, entering the denominator of (A.2), can be written as follows:

$$
\begin{aligned}
|R^{(i)}|^{1/2}|\Sigma_u^{(i)}|^{1/2} &= \left(|\Sigma_u^{(i)1/2}||R^{(i)}||\Sigma_u^{(i)1/2}|\right)^{1/2} \\
&= \left|\Sigma_u^{(i)1/2} R_{u^{(i)*}u^{(i)*}} \Sigma_u^{(i)1/2} - \Sigma_u^{(i)1/2} R_{u^{(i)*}z^{(i)*}} R_{z^{(i)*}u^{(i)*}} \Sigma_u^{(i)1/2}\right|^{1/2}.
\end{aligned}
\tag{A.5}
$$

This result follows immediately by noticing that $\left|\Sigma_u^{(i)}\right| = |\Sigma_u^{(i)1/2}||\Sigma_u^{(i)1/2}|$ since $\Sigma_u^{(i)} = diag[\sigma^2_{u_k}]$, and using

$$
|R^{(i)}| = |R_{z^{(i)*}z^{(i)*})}| \left|R_{u^{(i)*}u^{(i)*}} - R_{u^{(i)*}z^{(i)*}} R_{z^{(i)*}u^{(i)*}}\right| = \left|R_{u^{(i)*}u^{(i)*}} - R_{u^{(i)*}z^{(i)*}} R_{z^{(i)*}u^{(i)*}}\right|,
$$

since $R_{z^{(i)*}z^{(i)*})} = 1$.

The result of the proposition follows, directly, by substituting relationships (A.4) and (A.5) into (A.2), and noticing that $|R^{(i)}|^{1/2}|\Sigma_u^{(i)}|^{1/2}$ is determinant of $\Omega_\varepsilon^{(i)} = \Sigma_u^{(i)1/2} R_{u^{(i)*}u^{(i)*}} \Sigma_u^{(i)1/2} - \Sigma_u^{(i)1/2} R_{u^{(i)*}z^{(i)*}} R_{z^{(i)*}u^{(i)*}} \Sigma_u^{(i)1/2}$, which is the variance-covariance matrix of the vector of error terms $\varepsilon_t^{(i)} = u_t^{(i)} - \Sigma_u^{(i)1/2} R_{u^{(i)*}z^{(i)*}} R_{z^{(i)*}u^{(i)*}} z_t^{(i)*}$. ■

B Monte Carlo

The true data generating process assumed in our MC study is the following bivariate TVAR model:[2]

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{cases} \begin{pmatrix} 0.7 & 0.1 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} y_{1t-1} \\ y_{2t-1} \end{pmatrix} + \begin{pmatrix} u_{1t}^{(1)} \\ u_{2t}^{(1)} \end{pmatrix}, & \text{if } z_t \leq \delta \\ \begin{pmatrix} 0.1 & 0.7 \\ 0.7 & 0.1 \end{pmatrix} \begin{pmatrix} y_{1t-1} \\ y_{2t-1} \end{pmatrix} + \begin{pmatrix} u_{1t}^{(2)} \\ u_{2t}^{(2)} \end{pmatrix}, & \text{if } z_t > \delta \end{cases} \tag{B.1}$$

where the value of the threshold parameter δ is set to the 3rd quantile of their assumed distributions (see below). The initial values of y_{1t} and y_{2t} are set to zero, and the vector of innovation terms $u_t^{(i)} = (u_{1t}^{(i)}, u_{2t}^{(i)})'$ has the following lower-triangular recursive structure, across the two states:

$$\begin{pmatrix} u_{1t}^{(i)} \\ u_{2t}^{(i)} \end{pmatrix} = \begin{cases} \begin{pmatrix} 1.0 & 0.0 \\ 0.8 & 1.0 \end{pmatrix} \begin{pmatrix} e_{1t}^{(1)} \\ e_{2t}^{(1)} \end{pmatrix}, & \text{if } z_t \leq \delta \\ \begin{pmatrix} 1.0 & 0.0 \\ 0.8 & 1.0 \end{pmatrix} \begin{pmatrix} e_{1t}^{(2)} \\ e_{2t}^{(2)} \end{pmatrix}, & \text{if } z_t > \delta \end{cases} \tag{B.2}$$

where $e_{jt}^{(i)}$, $j = 1, 2$, are two independent zero-mean normally distributed error terms with unit variance which stand for the structural errors of the model for the two

[2] Without loss of generality, we abstract from a vector of intercepts.

equations $j = 1, 2$. The above structural error causation scheme is often assumed in economic studies and can be identified by the Cholesky decomposition of the variance-covariance matrix of the TVAR innovation terms $u_{1t}^{(i)}$ and $u_{2t}^{(i)}$, i.e., $\Sigma_{uu}^{(i)}$. For threshold variable z_t, we consider the following two distributions:

$$z_t \sim U(-4, 3) \quad \text{and} \quad z_t \sim \mathcal{N}(0, 4.08)$$

these are chosen to have the same variance for valid comparisons.

We consider different dependence structures between the structural error terms $e_{jt}^{(i)}$, $j = 1, 2$, and the threshold variable z_t, in the following tabular order:

1. Linear for both equations with the common correlation

$$Corr(e_{jt}^{(1)}, z_t) = Corr(e_{jt}^{(2)}, z_t) = -0.8$$

2. Also linear, but with different degrees of correlation across the states:

$$Corr(e_{jt}^{(1)}, z_t) = 0.8 \neq Corr(e_{jt}^{(2)}, z_t) = -0.5$$

   for both equations $j = 1, 2$.

3. Next, we consider the case that the VAR innovations are generated by Student's t-distribution, with five degrees of freedom. This distribution can generate severe excess kurtosis of the innovations. In addition, we considered the skewed t-distribution of Fernandez and Steel (2012) with five degrees of freedom, which, in addition to excess kurtosis, can generate skewness. Both of these additional features are often encountered in financial data (see, e.g., Harvey and Siddique, 2000 or Papantonis et al. 2023, for a survey).

4. Finally, we assume a *non-linear* dependence structure between $e_{jt}^{(i)}$ and z_t. This

is assumed to be the same across the two equations j and the two states i, for simplicity. In this case, we generate z_t based on bivariate copulas between z_t and $e_{jt}^{(i)}$.

For that fourth case, we consider the following three alternative copulas: the Clayton, Joe, and Frank copulas, which belong to the Archimedean family of copulas. These assume different dependence structures at the tails of the joint distribution of z_t and $e_{jt}^{(i)}$. The Frank copula is a symmetric copula which assumes stronger dependence around the center of the distribution and weaker at its lower and upper tails. On the other hand, the Clayton and Joe copulas are asymmetric and assume stronger lower and upper tail dependence, respectively.

The dependence parameter values considered in our Monte Carlo exercises are set to: $\psi = 2.50$ for Clayton; $\psi = 2.50$ for Joe, and $\psi = 5.2$ for Frank. These values imply a Spearman correlation coefficient between z_t and $e_{jt}^{(i)}$ which is about 0.70.

We perform 1000 MC iterations using sizes of T ={200, 500} observations. In each iteration, we estimate the threshold parameter δ and the remaining parameters of TVAR model (B.1), based on the estimation procedure described before. The distributions $F_{\mathcal{Z}^{(i)}}(\mathcal{Z}_t^{(i)})$, are estimated non-parametrically using a Gaussian kernel. For the estimation of δ, we trim out the 10% of the observations of z_t from each end of its sample to increase the degrees of freedom in the estimation of the model for both states. In Tables B.1–B.5, we present the average values of the bias (denoted as BIAS) and mean square error (MSE) of the estimates of threshold parameter δ and the variance-covariance parameters of the TVAR innovation terms $u_{1t}^{(i)}$ and $u_{2t}^{(i)}$, denoted $\sigma_{u_1}^{(i)2}$, $\sigma_{u_2}^{(i)2}$ and $\sigma_{u_1u_2}^{(i)}$. The values of these metrics are calculated based on 1000 iterations. For reasons of space, we have omitted the results for remaining parameters of the TVAR model.

Tables B.1 and B.2 present results for the cases that the dependence structure between z_t and $e_{jt}^{(i)}$ is linear, and $T = 200$ and $T = 500$, respectively. Table B.3 presents results for the case that the VAR innovations follow the Student's t and skewed-t distributions. Finally, Tables B.4 and B.5 present results where that dependence structure is nonlinear (i.e., it is obtained by the three aformenioned Archimedean copulas) and again for $T = 200$ and $T = 500$, respectively. We present results from our method controlling for the endogeneity of the threshold variable using the copula transformations $z_t^{(i)*}$, and for the case that we ignore the endogeneity problem of z_t.

Inspection of the results leads to several interesting conclusions. First, ignoring the issue of the threshold variable endogeneity leads to biased estimates and large MSE values of the threshold parameter δ and the variance-covariance terms of innovation terms $u_{1t}^{(i)}$ and $u_{2t}^{(i)}$ (mainly, the variances $\sigma_{u_1}^{(i)2}$ and $\sigma_{u_2}^{(i)2}$). This is true for both states of the model. These problems appear more severe in cases where the correlation structure between z_t and $e_{jt}^{(i)}$ changes across the two states, z_t is *not* normally distributed, i.e., $z_t \sim U(-4, 3)$, or z_t is generated by the three alternative Archimedean copulas, and the size of T is smaller. For these cases, the bias of the above parameters remains substantial even where $T = 500$.

Second, our method efficiently controls for the threshold endogeneity bias. The performance of the method is efficient even for cases where T is small (i.e., $T = 200$). Our method can substantially reduce both the BIAS and MSE of the threshold parameter δ and the parameters of the variance-covariance matrix of $u_{1t}^{(i)}$ and $u_{2t}^{(i)}$. Note that this is true even where the correlation between z_t and $e_{jt}^{(i)}$ changes across the two states. The performance of the method is also satisfactorily in cases in which (i) the TVAR innovations follow the Student's t and skewed-t distributions and (ii) the dependence between z_t and $e_{jt}^{(i)}$ is governed by the three Archimedean copulas, especially for the Clayton and Frank copulas.[3]

[3]Note that, for the Joe copula, our method does not substantially reduce the bias of the variance of error term $u_{1t}^{(i)}$ for the second state, i.e., $\sigma_{u_1}^{(2)2}$. This may be attributed to the high level of dependence of the

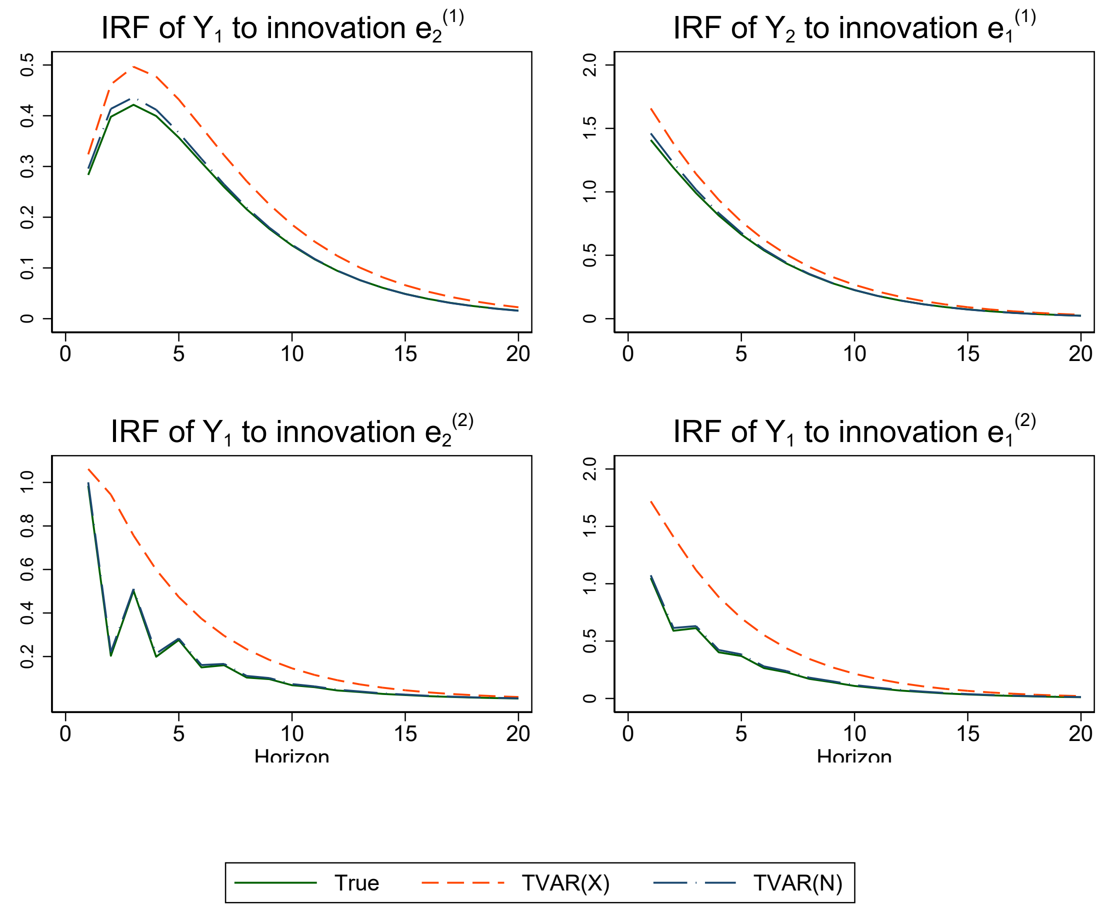
To better judge the consequences of ignoring threshold endogeneity, in Figure B.1, we graphically present the impulse response functions (IRFs) of two variables in the bivariate Monte Carlo VAR, Y_{1t} and Y_{2t}, to cross-equation structural errors (shocks) $e_{2t}^{(1)}$ and $e_{1t}^{(2)}$ for the two states. The plots of the figure present the true IRFs generated by the known artificial data (in green solid), the IRFs ignoring threshold endogeneity (red dash) and the IRFs estimated by our method controlling for the endogeneity in the threshold (blue dot-dash). Apart from the true ones, the IRFs reported are based on the average estimates of the model, over all iterations for the simulation scenario where $T = 500, Z_t \sim U(-4, 3)$ and $corr(e_{jt}^{(1)}, z_t) = 0.8$ and $corr(e_{jt}^{(2)}, z_t) = -0.5$ (see Panel B of Table B.2).

Inspection of the graphs clearly indicates that ignoring the threshold variable endogeneity leads to biased estimates of the IRFs. The magnitude of bias is substantial in both states of the model and lasts until the effects of the structural shocks die out (e.g., for about 15 periods ahead). Note that this happens even if the bias of the threshold parameter δ is not big enough. Our method of controlling for the threshold endogeneity bias can accurately estimate the true IRFs, for both states of the model. The IRFs estimated by our method follows closely the true ones generated by the data and the model.

In summary, the results of our MC exercise indicate that our method can efficiently control for the effects of threshold variable endogeneity on the estimates of the variance-covariance parameters of the TVAR innovations and their implied structural errors. This is true even where the correlation structure between the threshold variable and the TVAR innovations changes considerably across the states.

Joe copula at the upper tail of z_t and the lower number of observations available for the estimation of the parameters of the model in state. For the Archimedean copulas considered, our results indicate that the use of a Gaussian copula can substantially reduce the BIAS and MSE of the threshold parameter δ. Regarding the distribution of the TVAR's innovations $u_{jt}^{(i)}$, our method works much better in the case where $u_{jt}^{(i)}$ follows the Student's t distribution, which is symmetric.

Figure B.1: IMPULSE RESPONSE FROM MONTE CARLO, $T = 500$



Notes: The figure graphically presents estimates of IRFs of y_{1t+h} (or y_{2t+h}) to structural shocks $e_{2t}^{(i)}$ (or $e_{1t}^{(i)}$), for $h = 1, 2, ..., 20$ periods ahead and across the two states of the model, obtained by our method (copula approach), including the threshold variable as an additional regressor in the TVAR model and the case ignoring the threshold variable endogeneity. This is done against the true (theoretical) values of the IRFs, generated by the TVAR model (B.1) under the error-term structure (B.2). Apart from the IRFs based on theoretical values of the model, the reported estimates of the IRFs in the graphs constitute average values of them based on estimates of (B.1) and (B.2) over 1000 bootstrap iterations. We assume $T = 500$ observations.

Table B.1: Monte Carlo Results, $T = 200$

	$z_t \sim \mathcal{N}(0, 4.08)$							$z_t \sim U(-4, 3)$						
	δ	$(\sigma^{(1)}_{u_1})^2$	$(\sigma^{(1)}_{u_2})^2$	$\sigma^{(1)}_{u_1u_2}$	$(\sigma^{(2)}_{u_1})^2$	$(\sigma^{(2)}_{u_2})^2$	$\sigma^{(2)}_{u_1u_2}$	δ	$(\sigma^{(1)}_{u_1})^2$	$(\sigma^{(1)}_{u_2})^2$	$\sigma^{(1)}_{u_1u_2}$	$(\sigma^{(2)}_{u_1})^2$	$(\sigma^{(2)}_{u_2})^2$	$\sigma^{(2)}_{u_1u_2}$
Panel A: $\rho_{e^{(i)}_j z} = 0.80$ for $j = 1, 2$ equations and $i = 1, 2$ states														
Ignoring threshold endogeneity														
BIAS	-0.243	0.316	0.188	0.000	0.141	0.061	-0.060	-1.064	1.184	1.159	0.107	0.591	0.575	-0.056
MSE	0.398	0.011	0.065	0.010	0.033	0.063	0.053	1.644	1.619	1.633	0.011	0.465	0.515	0.003
Controlling for threshold endogeneity														
BIAS	0.024	0.008	-0.013	0.000	0.060	-0.018	-0.009	-0.015	0.067	-0.054	0.013	0.034	-0.000	0.006
MSE	0.049	0.002	0.021	0.010	0.012	0.058	0.054	0.025	0.019	0.042	0.000	0.050	0.119	0.000
Panel B: $\rho_{e^{(1)}_j z} = 0.80$ and $\rho_{e^{(2)}_j z} = -0.50$ for $j = 1, 2$ equations														
Ignoring threshold endogeneity														
BIAS	-1.524	0.821	0.825	0.097	0.193	0.163	-0.023	2.616	0.394	0.388	0.058	0.523	0.506	-0.144
MSE	8.900	0.838	0.885	0.016	0.107	0.157	0.022	7.105	0.226	0.271	0.015	0.323	0.343	0.031
Controlling for threshold endogeneity														
BIAS	0.000	0.028	0.029	0.015	0.008	-0.039	0.005	0.000	0.076	0.064	0.016	-0.021	-0.056	-0.001
MSE	0.024	0.016	0.037	0.006	0.048	0.107	0.022	0.009	0.021	0.043	0.006	0.048	0.121	0.023

Notes: The table presents the average values of the bias (denoted as BIAS) and mean square error (MSE) of the estimates of threshold parameter δ and the variance-covariance parameters of the TVAR innovations $u^{(i)}_{1t}$ and $u^{(i)}_{2t}$, denoted $\sigma^{(i)2}_{u_1}$, $\sigma^{(i)2}_{u_2}$ and $\sigma^{(i)}_{u_1u_2}$, across the two states $i = 1, 2$. This is done for the following cases: (i) ignoring the threshold endogeneity, (ii) controlling for it based on our suggested method (i.e., the copula approach). For all the above cases, the size of the sample is set to $T = 200$. We present results for the case that the threshold variables is distributed as $z_t \sim \mathcal{N}(0, 4.08)$ and the case that $z_t \sim U(-4, 3)$. Panel A presents results for the case that the correlation between z_t and structural TVAR errors $e^{(i)}_j$ is linear, given as $\rho_{e^{(i)}_j z} = 0.80$, for $j = 1, 2$ equations and $i = 1, 2$ states, while Panel B is for the case that the correlation structure changes across the two states, i.e., $\rho_{e^{(1)}_j z} = 0.80$and $\rho_{e^{(2)}_j z} = -0.50$ for $j = 1, 2$ equations.

Table B.2: Monte Carlo Results, $T = 500$

	$z_t \sim \mathcal{N}(0, 4.08)$							$z_t \sim U(-4, 3)$						
	δ	$(\sigma_{u_1}^{(1)})^2$	$(\sigma_{u_2}^{(1)})^2$	$\sigma_{u_1u_2}^{(1)}$	$(\sigma_{u_1}^{(2)})^2$	$(\sigma_{u_2}^{(2)})^2$	$\sigma_{u_1u_2}^{(2)}$	δ	$(\sigma_{u_1}^{(1)})^2$	$(\sigma_{u_2}^{(1)})^2$	$\sigma_{u_1u_2}^{(1)}$	$(\sigma_{u_1}^{(2)})^2$	$(\sigma_{u_2}^{(2)})^2$	$\sigma_{u_1u_2}^{(2)}$
Panel A: $\rho_{e_i^{(i)}z} = 0.80$ for $j = 1, 2$ equations and $i = 1, 2$ states														
Ignoring threshold endogeneity														
BIAS	-0.050	0.333	0.208	-0.002	0.148	0.083	-0.016	-1.017	1.266	1.263	0.111	0.647	0.629	-0.070
MSE	0.026	0.114	0.055	0.004	0.027	0.033	0.019	1.434	1.763	1.733	0.013	0.501	0.507	0.008
Controlling for threshold endogeneity														
BIAS	0.010	0.013	0.001	-0.006	0.037	0.001	-0.006	-0.001	0.080	-0.054	0.015	0.035	0.014	0.012
MSE	0.009	0.001	0.008	0.007	0.004	0.022	0.021	0.010	0.013	0.021	0.000	0.020	0.045	0.000
Panel B: $\rho_{e_i^{(1)}z} = 0.80$ and $\rho_{e_i^{(2)}z} = -0.50$ for $i = 1, 2$														
Ignoring threshold endogeneity														
BIAS	-0.603	0.981	0.977	0.099	0.129	0.127	0.003	-2.616	0.443	0.435	0.060	0.535	0.537	-0.145
MSE	3.462	1.028	1.047	0.011	0.048	0.075	0.127	6.980	0.227	0.244	0.007	0.302	0.321	0.025
Controlling for threshold endogeneity														
BIAS	-0.003	0.038	0.004	0.007	0.001	-0.001	0.006	-0.004	0.088	0.086	0.010	0.006	0.013	-0.005
MSE	0.005	0.008	0.017	0.003	0.017	0.046	0.008	0.002	0.014	0.023	0.003	0.023	0.048	0.008

Notes: The table presents the average values of the bias (denoted as BIAS) and mean square error (MSE) of the estimates of threshold parameter δ and the variance-covariance parameters of the TVAR innovations $u_{1t}^{(i)}$ and $u_{2t}^{(i)}$, denoted $\sigma_{u_1}^{(i)2}$, $\sigma_{u_2}^{(i)2}$ and $\sigma_{u_1u_2}^{(i)}$, across the two states $i = 1, 2$. This is done for the following cases: (i) ignoring the threshold endogeneity, (ii) controlling for it based on our suggested method (i.e., the copula approach). For all the above cases, the size of the sample is set to $T = 500$. We present results for the case that the threshold variables is distributed as $z_t \sim \mathcal{N}(0, 4.08)$ and the case that $z_t \sim U(-4, 3)$. Panel A presents results for the case that the correlation between z_t and structural TVAR errors $e_j^{(i)}$ is linear, given as $\rho_{e_j^{(i)}z} = 0.80$, for $j = 1, 2$ equations and $i = 1, 2$ states, while Panel B is for the case that the correlation structure changes across the two states, i.e., $\rho_{e_j^{(1)}z} = 0.80$ and $\rho_{e_j^{(2)}z} = -0.50$ for $j = 1, 2$ equations.

Table B.3: Monte Carlo Results – The Case of Non-Normal VAR Innovations

	δ	$\sigma^{(1)2}_{u_1}$	$\sigma^{(1)2}_{u_2}$	$\sigma^{(1)}_{u_1u_2}$	$\sigma^{(2)2}_{u_1}$	$\sigma^{(2)2}_{u_2}$	$\sigma^{(2)}_{u_1u_2}$	δ	$\sigma^{(1)2}_{u_1}$	$\sigma^{(1)2}_{u_2}$	$\sigma^{(1)}_{u_1u_2}$	$\sigma^{(2)2}_{u_1}$	$\sigma^{(2)2}_{u_2}$	$\sigma^{(2)}_{u_1u_2}$
	Ignoring threshold endogeneity							Controlling for threshold endogeneity						
Panel A: $T = 200$														
Student's-t innovations														
BIAS	−0.190	0.498	0.162	−0.020	0.376	0.079	−0.060	0.016	0.104	−0.094	−0.001	0.262	−0.031	0.000
MSE	0.200	0.242	0.054	0.010	0.138	0.113	0.043	0.022	0.005	0.094	0.019	0.069	0.121	0.045
skewed-t innovations														
BIAS	−0.146	0.427	0.267	0.120	1.110	0.71	0.010	0.047	0.089	0.036	0.004	0.910	0.529	0.003
MSE	0.145	0.193	0.107	0.014	1.240	0.730	0.025	0.049	0.010	0.034	0.024	0.857	0.535	0.030
Panel B: $T = 500$														
Student's-t innovations														
BIAS	−0.066	0.561	0.285	−0.001	0.471	0.216	−0.020	0.002	0.133	0.003	−0.001	0.284	0.077	0.000
MSE	0.028	0.301	0.083	0.004	0.206	0.095	0.016	0.009	0.022	0.038	0.004	0.073	0.000	0.056
skewed-t innovations														
BIAS	−0.061	0.493	0.333	0.030	1.333	0.878	0.020	0.006	0.105	0.116	0.004	0.962	0.688	0.003
MSE	0.029	0.237	0.114	0.050	1.769	0.861	0.090	0.007	0.010	0.019	0.09	0.927	0.458	0.010

Notes: The table presents the average values of the bias (denoted BIAS) and mean square error (MSE) of the estimates of threshold parameter δ and the variance-covariance parameters of the TVAR innovations $u^{(i)}_{1t}$ and $u^{(i)}_{2t}$, denoted $\sigma^{(i)2}_{u_1}$, $\sigma^{(i)2}_{u_2}$ and $\sigma^{(i)}_{u_1u_2}$, across the two states $i = 1, 2$, for cases that the distribution of the error terms $u^{(i)}_{1t}$ and $u^{(i)}_{2t}$ is Student's t with five degrees of freedom and skewed-t, with the same degrees of freedom. The threshold variable is $z_t \sim U(-4, 3)$. The table presents results for the following two cases: (i) ignoring the threshold endogeneity, (ii) controlling for it based on our suggested method relying on the Gaussian copula approach. The table consider the cases of $T = 200$ and $T = 500$, and assumes correlation coefficients $\rho_{e^{(i)}_i z} = 0.80$ for $j = 1, 2$ equations and $i = 1, 2$ states.

Table B.4: Monte Carlo Results, $T = 200$

	δ	$(\sigma^{(1)}_{u_1})^2$	$(\sigma^{(1)}_{u_2})^2$	$\sigma^{(1)}_{u_1u_2}$	$(\sigma^{(2)}_{u_1})^2$	$(\sigma^{(2)}_{u_2})^2$	$\sigma^{(2)}_{u_1u_2}$	δ	$(\sigma^{(1)}_{u_1})^2$	$(\sigma^{(1)}_{u_2})^2$	$\sigma^{(1)}_{u_1u_2}$	$(\sigma^{(2)}_{u_1})^2$	$(\sigma^{(2)}_{u_2})^2$	$\sigma^{(2)}_{u_1u_2}$
	$Clayton : \psi = 2.5$							$Joe : \psi = 2.5$						
	Ignoring threshold endogeneity													
BIAS	-1.537	0.404	0.113	0.302	0.684	0.309	-0.016	-1.009	0.843	0.732	-0.003	0.815	0.934	0.190
MSE	2.463	0.122	0.121	0.003	0.372	0.212	0.008	1.521	0.825	0.621	0.001	0.601	1.001	0.132
	Controlling for threshold endogeneity													
BIAS	-0.028	-0.159	-0.162	-0.018	0.031	-0.026	-0.008	-0.004	0.183	0.153	-0.019	0.719	0.056	0.011
MSE	0.022	0.013	0.031	0.009	0.096	0.046	0.000	0.003	0.041	0.056	0.000	0.512	0.068	0.001
	$Frank\ \psi = 5.2$													
	Ignoring threshold endogeneity													
BIAS	-0.772	0.623	0.532	-0.009	0.387	-0.185	-0.045							
MSE	1.052	0.420	0.415	0.000	0.216	0.098	0.026							
	Controlling for threshold endogeneity													
BIAS	0.005	0.142	0.101	-0.018	0.135	-0.033	0.000							
MSE	0.013	0.036	0.046	0.000	0.045	0.044	0.000							

Notes: The table presents the average values of the bias (denoted as BIAS) and mean square error (MSE) of the estimates of threshold parameter δ and the variance-covariance parameters of the TVAR innovations $u^{(i)}_{1t}$ and $u^{(i)}_{2t}$, denoted $\sigma^{(i)2}_{u_1}$, $\sigma^{(i)2}_{u_2}$ and $\sigma^{(i)}_{u_1u_2}$, across the two states $i = 1, 2$, when the dependence structure between z_t and $u^{(i)}_j$ is non-linear governed by the Clayton, Joe, and Frank copulas. The values of the copula parameters ψ considered are reported in parentheses. These imply a Spearman correlation coefficient about 0.70 between z_t and $e^{(i)}_{jt} \forall j, i$. This is done for the following cases: (i) ignoring the threshold endogeneity, (ii) controlling for it based on our suggested method (i.e., the copula approach). For all the above cases, the size of the sample is set to $T = 200$.

Table B.5: Monte Carlo Results, $T = 500$

	δ	$(\sigma_{u_1}^{(1)})^2$	$(\sigma_{u_2}^{(1)})^2$	$\sigma_{u_1u_2}^{(1)}$	$(\sigma_{u_1}^{(2)})^2$	$(\sigma_{u_2}^{(2)})^2$	$\sigma_{u_1u_2}^{(2)}$	δ	$(\sigma_{u_1}^{(1)})^2$	$(\sigma_{u_2}^{(1)})^2$	$\sigma_{u_1u_2}^{(1)}$	$(\sigma_{u_1}^{(2)})^2$	$(\sigma_{u_2}^{(2)})^2$	$\sigma_{u_1u_2}^{(2)}$
	$Clayton\ \psi = 2.5$							$Joe\ \psi = 2.5$						
	Ignoring threshold endogeneity													
BIAS	−1.250	0.302	0.035	−0.010	0.552	0.437	−0.015	−1.055	0.815	0.732	−0.008	0.815	0.934	0.206
MSE	2.011	0.092	0.079	0.000	0.333	0.171	0.000	1.586	0.848	0.662	0.004	0.673	1.203	0.066
	Controlling for threshold endogeneity													
BIAS	−0.011	0.118	−0.177	−0.010	0.262	−0.365	−0.010	0.000	0.204	0.173	−0.014	0.728	0.057	0.002
MSE	0.003	0.011	0.013	0.000	0.075	0.013	0.000	0.003	0.044	0.041	0.007	0.544	0.029	0.009
	$Frank\ \psi = 5.2$													
	Ignoring threshold endogeneity													
BIAS	−0.611	0.699	0.570	−0.008	0.365	0.596	−0.054							
MSE	0.792	0.621	0.506	0.004	0.212	0.076	0.016							
	Controlling for threshold endogeneity													
BIAS	−0.000	−0.126	−0.074	−0.010	0.133	−0.011	−0.015							
MSE	0.003	0.039	0.037	0.007	0.043	0.002	0.020							

Notes: The table presents the average values of the bias (denoted as BIAS) and mean square error (MSE) of the estimates of threshold parameter δ and the variance-covariance parameters of the TVAR innovations $u_{1t}^{(i)}$ and $u_{2t}^{(i)}$, denoted $\sigma_{u_1}^{(i)2}, \sigma_{u_2}^{(i)2}$ and $\sigma_{u_1u_2}^{(i)}$, across the two states $i = 1, 2$, when the dependence structure between z_t and $u_j^{(i)}$ is non-linear governed by the Clayton, Joe, and Frank copulas. The values of the copula parameters ψ considered are reported in parentheses. These imply a Spearman correlation coefficient about 0.70 between z_t and $e_{jt}^{(i)}$, for all j and i. This is done for the following cases: (i) ignoring the threshold endogeneity, (ii) controlling for it based on our suggested method (i.e., the copula approach). For all the above cases, the size of the sample is set to $T = 500$.

C Data

We use monthly data over the period 1970m1 to 2021m12 for (seasonally adjusted) real output, (seasonally adjusted) inflation rates, a measure of the monetary policy instrument, and a measure of the bond premium.

- For the real output measure, we used the Total Industrial Production Index ($2017 = 100$), FRED© mnemonic: `INDPRO`.
- The inflation rate is the growth rate of the Consumer Price index (1982-1940=100) for all urban Consumers, `CPIAUCSL`.
- The nominal monetary policy rate is the Federal Funds Rate, `FEDFUNDS`.
  - To address the concern over the zero lower bound we also use the two-year rate as the policy variable, `DGS2`. The sample size of this series is slightly shorter at 1976m1 to our final date of 2021m12.
- We augment the VAR with a fourth variable, namely, the excess bond premium taken from Gilchrist and Zakrajšek (2012), see also Gertler and Karadi (2015), Bauer and Swanson (2022).[4] The sample size of this series is slightly shorter at 1973m1 to our final date of 2021m12.

The summary statistics of these variables is given in Tble C.1 and plotted in Figure C.1.

Finally, the shock variable used in the analysis is based on the narrative analysis of Romer and Romer (2004) provided by Miranda-Agrippino and Ricco (2021).

Monthly data for the first two variables are annualized using $(Y_t/Y_{t-1})^{12} - 1$, where Y_t and Y_{t-1} are the values of the variables of interest in months t and $t-1$,

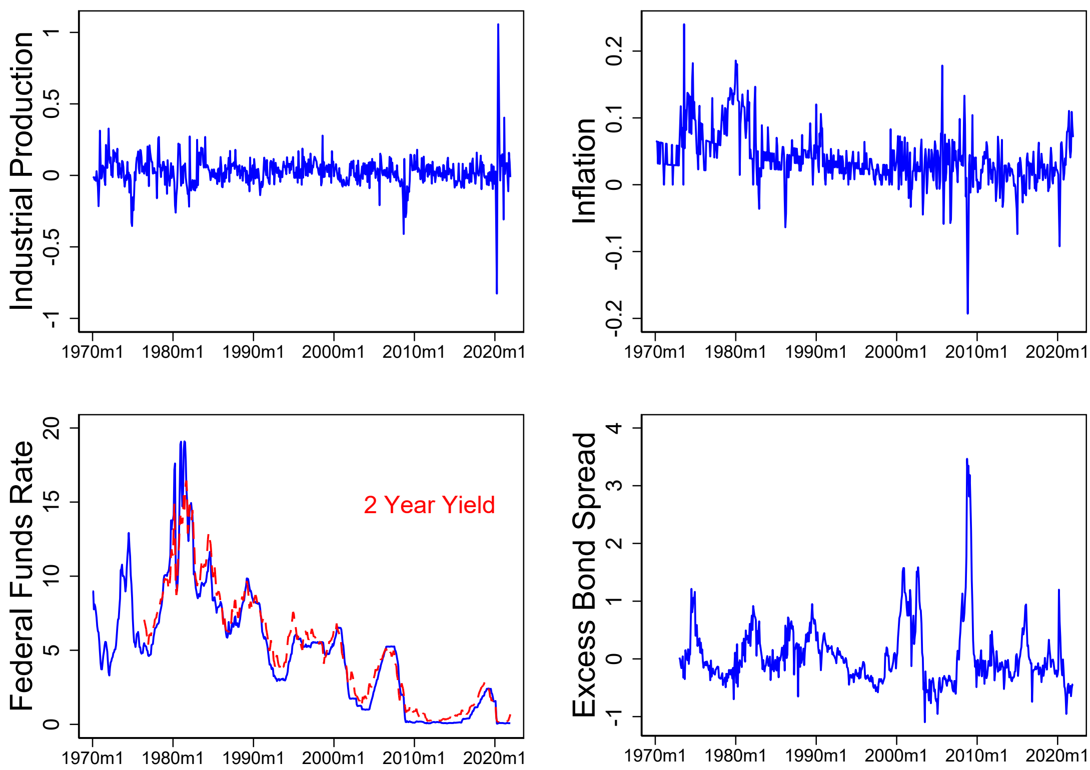
[4]For updated data see https://www.federalreserve.gov/econresdata/notes/feds-notes/2016/files/ebp_csv.csv

respectively. Both the growth rate of output and the inflation rate follow a stationary process, while the Federal Funds Rate is a trend stationary variable. We remove the trend by fitting a deterministic time trend model. The residuals from this model then follow a stationary stochastic process.

Table C.1: Summary Statistics: VAR Variables

Series	Mean	Median	Std. Dev	Max	Min
y	0.025	0.028	0.107	-0.827	1.057
π	0.040	0.033	0.041	-0.193	0.240
r	4.967	5.065	3.958	0.050	19.100
$2y$	5.062	4.903	3.814	0.115	16.458
ebp	0.061	-0.054	0.551	-1.096	3.466

Figure C.1: Data



Notes: Monthly series on Industrial Production Index, Inflation, the effective Federal Funds rate, the Market Yield on Treasury Securities at 2-Year Constant Maturity (shown in red dash), and the Excess Bond Premium. All series taken from FRED©, except for the final series which is taken from Gilchrist and Zakrajšek (2012).

D Full Set of Results

Tables 1A-1B presents results assuming the *FFR* is the policy instrument, while Table 2A-2B use the two-year government bond yield. Tables labeled with the letter **A** present results for the case controlling for endogeneity of the threshold variable (using our copula method), while those labeled **B** ignore threshold endogeneity. In addition, Tables 3A-3B present results for the case that our TVAR model also includes exogenous monetary shocks as a separate variable, denoted m. In that case, as the policy rate, r_t we continue to consider the FFR. As noted before, variable m is considered as exogenous (i.e., it is ordered first) to capture contemporaneous causation effects of monetary policy on g_t, π_t and r_t. Finally, Table 4 presents the benchmark linear VAR.

We report estimates of threshold parameter δ, the variance-covariance matrix of the innovations and the cumulative multipliers of the real growth and inflation rates to a structural monetary policy rate shock of one standard deviation, for horizons $h = \{1, 12, 24, 36, 48\}$ and across the states identified by the model estimates: $\mathcal{L}$ and $\mathcal{H}$ respectively denote the low- and high inflation state. The cumulative multipliers estimates are based on estimates of the IRFs of the alternative tabulated TVAR specifications.

All the TVAR specifications considered assume a lag structure of two (chosen based on the reported Bayesian information criterion, *bic*). The standard errors of the cumulative multipliers estimates and the confidence intervals of the IRFs estimates, respectively reported in the tables and figures, are constructed based on a non-overlapping block-bootstrap of fixed length chosen automatically via flat-top windows (e.g., Politis and White, 2004; Patton et al., 2009), based on 1000 iterations.

Finally, Figure D.1-Figure D.2 show the corresponding results for the 2 year rate as the monetary instrument.

Table 1A: Estimates of the TVAR model **controlling for** threshold endogeneity and Cumulative Responses

$$\delta = 0.0359\ [0.018, 0.046],\ LR = 91.895\ [0.0000],\ bic = -8465.28$$

$$\Sigma_{uu}^{\mathcal{L}} \times 100 = \begin{bmatrix} \sigma_{gg} = 0.9099 & & \\ \sigma_{\pi g} = 0.0213 & \sigma_{\pi\pi} = 0.0657 & \\ \sigma_{rg} = 0.0015 & \sigma_{r\pi} = 0.0005 & \sigma_{rr} = 0.0003 \end{bmatrix} \Sigma_{uu}^{\mathcal{H}} \times 100 = \begin{bmatrix} \sigma_{gg} = 0.8700 & & \\ \sigma_{\pi g} = 0.0092 & \sigma_{\pi\pi} = 0.0789 & \\ \sigma_{rg} = 0.0146 & \sigma_{r\pi} = -0.0010 & \sigma_{rr} = 0.0051 \end{bmatrix}$$

$horizon$	**1**	12	**24**	36	**48**
$g^{\mathcal{L}}$	0.035	0.002	-0.072	-0.133	-0.182
	(0.007)	(0.002)	(0.003)	(0.004)	(0.0034)
$g^{\mathcal{H}}$	0.021	-0.184	-0.301	-0.348	-0.366
	(0.0006)	(0.002)	(0.003)	(0.004)	(0.004)
$\pi^{\mathcal{L}}$	0.011	0.007	-0.010	-0.025	-0.035
	(0.001)	(0.002)	(0.003)	(0.004)	(0.004)
$\pi^{\mathcal{H}}$	0.006	-0.007	-0.020	-0.022	-0.025
	(0.004)	(0.003)	(0.005)	(0.006)	(0.007)

Notes: The table presents estimates of the cumulative responses of real industrial production growth and inflation rates to a monetary policy shock of one standard deviation. These are based on estimates of a TVAR model with two lags for the period 1970m1-2021m12, using as threshold variable a 20-month moving average of inflation and allowing for threshold endogeneity based on our copula approach of Section 2. In estimating threshold parameter δ, we have trimmed out the 10% of the observations of z_t from each end of its sample. Bootstrapped standard errors in parentheses, based on a wild parametric bootstrap method using one thousand iterations. LR is the likelihood ratio test statistic testing the null hypothesis that the VAR model is linear against its alternative hypothesis that is subject to threshold effects. The p-value of the LR statistic is calculated by the wild parametric method used to estimate the standard errors. bic is the Bayesian information criterion and $\Sigma_{uu}^{(i)}$, $i = \mathcal{L}, \mathcal{H}$, is the variance-covariance matrix of the vector of the TVAR innovations $u_t^{(i)}$. Accordingly, σ_{xx} and σ_{xy} indicate respectively the variance of x and the covariance of x and y.

Table 1B: Estimates of the TVAR model **ignoring** threshold endogeneity bias effects and Cumulative Responses

$$\delta = 0.0435\ [0.019, 0.055],\ LR = 86.526\ [0.0001],\ bic = -8412.42$$

$$\Sigma_{uu}^{\mathcal{L}} \times 100 = \begin{bmatrix} \sigma_{gg} = 0.9279 & & \\ \sigma_{\pi g} = 0.0291 & \sigma_{\pi\pi} = 0.0670 & \\ \sigma_{rg} = 0.0025 & \sigma_{r\pi} = 0.0004 & \sigma_{rr} = 0.0005 \end{bmatrix} \quad \Sigma_{uu}^{\mathcal{H}} \times 100 = \begin{bmatrix} \sigma_{gg} = 1.0277 & & \\ \sigma_{\pi g} = 0.0236 & \sigma_{\pi\pi} = 0.1015 & \\ \sigma_{rg} = 0.0189 & \sigma_{r\pi} = 0.0007 & \sigma_{rr} = 0.0065 \end{bmatrix}$$

$horizon$	**1**	12	**24**	36	**48**
$g^{\mathcal{L}}$	0.030	0.008	-0.004	-0.075	-0.093
	(0.0006)	(0.001)	(0.002)	(0.0025)	(0.003)
$g^{\mathcal{H}}$	0.026	-0.288	-0.479	-0.541	-0.568
	(0.0008)	(0.003)	(0.005)	(0.006)	(0.007)
$\pi^{\mathcal{L}}$	0.012	0.010	-0.007	-0.002	-0.007
	(0.001)	(0.002)	(0.004)	(0.004)	(0.005)
$\pi^{\mathcal{H}}$	0.012	-0.027	-0.059	-0.072	-0.075
	(0.001)	(0.009)	(0.017)	(0.022)	(0.025)

Notes: The table presents estimates of the cumulative responses of real industrial production growth and inflation rates to a monetary policy shock of one standard deviation. These are based on estimates of a TVAR model with two lags for the period 1970m1-2021m12, using as threshold variable the average inflation rate over the last J months, $z_t = \frac{1}{20}\sum_{j=0}^{19}\pi_{t-j}$ and ignoring threshold endogeneity bias. In estimating the threshold parameter δ, we have trimmed out the 10% of the observations of z_t from each end of its sample. Bootstrapped standard errors are in parentheses, based on a wild parametric bootstrap method using one thousand iterations. LR is the likelihood ratio test statistic testing the null hypothesis that the VAR model is linear against its alternative hypothesis that is subject to threshold effects. The p-value of the LR statistic is calculated by the wild parametric method used to estimate the standard errors. bic is the Bayesian information criterion and $\Sigma_{uu}^{(i)}$, $i = \mathcal{L}, \mathcal{H}$, is the variance-covariance matrix of the vector of the TVAR innovations $u_t^{(i)}$. Accordingly, σ_{xx} and σ_{xy} indicate respectively the variance of x and the covariance of x and y.

Table 2A: Estimates **controlling for** threshold endogeneity
Using the 2-year Government bond yield and Cumulative Responses

$$\delta = 0.035\ [0.019, 0.054],\ LR = 73.88\ [0.0001],\ bic = -7718.629$$

$$\Sigma_{uu}^{\mathcal{L}} \times 100 = \begin{bmatrix} \sigma_{gg} = 0.890 & & \\ \sigma_{\pi g} = 0.020 & \sigma_{\pi\pi} = 0.060 & \\ \sigma_{rg} = 0.001 & \sigma_{r\pi} = 0.0003 & \sigma_{rr} = 0.0005 \end{bmatrix} \quad \Sigma_{uu}^{\mathcal{H}} \times 100 = \begin{bmatrix} \sigma_{g} = 0.760 & & \\ \sigma_{\pi g} = 0.020 & \sigma_{\pi} = 0.060 & \\ \sigma_{rg} = 0.0006 & \sigma_{r\pi} = 0.0068 & \sigma_{rr} = 0.003 \end{bmatrix}$$

$horizon$	**1**	12	**24**	36	**48**
$g^{\mathcal{L}}$	0.051	0.042	-0.002	-0.003	-0.047
	(0.009)	(0.03)	0.003	(0.006)	(0.004)
$g^{\mathcal{H}}$	0.063	-0.155	-0.285	-0.339	-0.361
	(0.010)	(0.030)	(0.005)	(0.070)	(0.040)
$\pi^{\mathcal{L}}$	0.015	0.014	0.0021	-0.005	-0.010
	(0.001)	(0.002)	(0.005)	(0.004)	(0.005)
$\pi^{\mathcal{H}}$	0.0178	-0.031	-0.063	-0.076	-0.182
	(0.002)	(0.005)	(0.008)	(0.010)	(0.012)

Notes: The table presents estimates of the cumulative responses of real industrial production growth and inflation rates to a monetary policy shock of one standard deviation. These are based on estimates of a TVAR model with two lags for the period 1970m1-2021m12, using as threshold variable the average of inflation rate over the last twenty months, $z_t = \sum_{j=0}^{19} \pi_{t-j}$ and allowing for threshold endogeneity based on the copula approach, presented in Section 2. As the monetary policy instrument, we use the 2-year government bond yield. In estimating the threshold parameter δ, we have trimmed out the 10% of the observations of z_t from each end of its sample. Bootstrapped standard errors in parentheses, based on a wild parametric bootstrap method using one thousand iterations. LR is the likelihood ratio test statistic testing the null hypothesis that the VAR model is linear against its alternative hypothesis that is subject to threshold effects. The p-value of the LR statistic is calculated by the wild parametric method used to estimate the standard errors. bic is the Bayesian information criterion and $\Sigma_{uu}^{(i)}, i = \mathcal{L}, \mathcal{H}$, is the variance-covariance matrix of the vector of the TVAR innovations $u_t^{(i)}$.

Table 2B: Estimates **ignoring** threshold endogeneity
Using the 2-year Government bond yield and Cumulative Responses

$$\delta = 0.0402\ [0.019, 0.046],\ LR = 95.71[0.000],\ bic = -7698.93$$

$$\Sigma_{uu}^{\mathcal{L}} \times 100 = \begin{bmatrix} \sigma_{gg} = 0.900 & & \\ \sigma_{\pi g} = 0.020 & \sigma_{\pi\pi} = 0.070 & \\ \sigma_{rg} = 0.002 & \sigma_{r\pi} = 0.0001 & \sigma_{rr} = 0.0006 \end{bmatrix} \quad \Sigma_{uu}^{\mathcal{H}} \times 100 = \begin{bmatrix} \sigma_{gg} = 0.800 & & \\ \sigma_{\pi g} = 0.040 & \sigma_{\pi\pi} = 0.070 & \\ \sigma_{rg} = 0.008 & \sigma_{r\pi} = 0.0001 & \sigma_{rr} = 0.003 \end{bmatrix}$$

$horizon$	**1**	12	**24**	36	**48**
$g^{\mathcal{L}}$	0.049	0.036	-0.006	-0.024	-0.030
	(0.008)	(0.02)	(0.002)	(0.003)	(0.004)
$g^{\mathcal{H}}$	0.073	-0.178	-0.345	-0.429	-0.473
	(0.01)	(0.03)	(0.06)	(0.007)	(0.09)
$\pi^{\mathcal{L}}$	0.016	0.015	0.04	-0.0008	-0.003
	(0.02)	(0.03)	(0.05)	(0.06)	(0.05)
$\pi^{\mathcal{H}}$	0.041	-0.083	-0.174	-0.221	-0.247
	(0.003)	(0.09)	(0.015)	(0.02)	(0.023)

Notes: The table presents estimates of the cumulative responses of real industrial production growth and inflation rates to a monetary policy shock of one standard deviation. These are based on estimates of a TVAR model with two lags for the period 1970m1-2021m12, using as threshold variable the average of inflation rate over the last twenty months and ignoring threshold endogeneity bias. As the monetary policy instrument, we use the 2-year government bond yield. In estimating the threshold parameter δ, we have trimmed out the 10% of the observations of z_t from each end of its sample. Bootstrapped standard errors in parentheses, based on a wild parametric bootstrap method using 1000 iterations. LR is the likelihood ratio test statistic testing the null hypothesis that the VAR model is linear against its alternative hypothesis that is subject to threshold effects. The p-value of the LR statistic is calculated by the wild parametric method used to estimate the standard errors. bic is the Bayesian information criterion and $\Sigma_{uu}^{(i)}, i = \mathcal{L}, \mathcal{H}$ is the variance-covariance matrix of the vector of the TVAR innovations $u_t^{(i)}$.

Table 3A: Estimates controlling for threshold endogeneity bias effects including the unexpected monetary shocks variable m in the model and Cumulative Responses

$$\delta = 0.0412\ [0.029, 0.047],\ LR = 167.51\ [0.000],\ bic = -10110.58$$

$$\Sigma_{uu}^{\mathcal{L}} \times 100 = \begin{bmatrix} \sigma_{mm} = 0.0003 & & & \\ \sigma_{gm} = -0.0004 & \sigma_{gg} = 0.4110 & & \\ \sigma_{\pi m} = -0.00005 & \sigma_{\pi g} = 0.0012 & \sigma_{\pi\pi} = 0.047 & \\ \sigma_{rm} = 0.00003 & \sigma_{rg} = 0.0009 & \sigma_{r\pi} = 0.0003 & \sigma_{rr} = 0.0004 \end{bmatrix} \quad \Sigma_{uu}^{\mathcal{H}} \times 100 = \begin{bmatrix} \sigma_{mm} = 0.0021 & & & \\ \sigma_{gm} = -0.0016 & \sigma_{gg} = 0.7360 & & \\ \sigma_{\pi m} = -0.0010 & \sigma_{\pi g} = 0.0047 & \sigma_{\pi\pi} = 0.0089 & \\ \sigma_{rm} = 0.0001 & \sigma_{rg} = 0.00737 & \sigma_{r\pi} = -0.0012 & \sigma_{rr} = 0.0034 \end{bmatrix}$$

$horizon$	**1**	12	**24**	36	**48**
$g^{\mathcal{L}}$	-0.036	-0.0235	-0.077	-0.113	-0.136
	(0.005)	(0.049)	(0.042)	(0.039)	(0.038)
$g^{\mathcal{H}}$	0.198	0.032	-0.206	-0.320	-0.376
	(0.01)	(0.023)	(0.037)	(0.049)	(0.059)
$\pi^{\mathcal{L}}$	-0.010	-0.004	-0.0021	-0.034	-0.042
	(0.007)	(0.019)	(0.018)	(0.019)	(0.02)
$\pi^{\mathcal{H}}$	0.152	0.073	-0.089	-0.167	-0.205
	(0.04)	(0.010)	(0.022)	(0.029)	(0.035)

Notes: The table presents estimates of the cumulative responses of real industrial production growth and inflation rates to a monetary policy shock of one standard deviation. These are based on estimates of a TVAR model with two lags for the period 1970m1-2021m12, using as threshold variable the average of inflation rate over the last twenty months and allowing for threshold endogeneity based on the copula approach, presented in Section 2. As the monetary policy instrument, we use the 2-year government bond yield. In estimating the threshold parameter δ, we have trimmed out the 10% of the observations of z_t from each end of its sample. Fixed-length Block-bootstrap standard errors are in parentheses, using 1000 iterations. LR is the likelihood ratio test statistic testing the null hypothesis that the VAR model is linear against its alternative hypothesis that is subject to threshold effects. The p-value of the LR statistic is calculated by an analogous block-bootstrap method used to estimate the standard errors. bic is the Bayesian information criterion and $\Sigma_{uu}^{(i)}, i = \mathcal{L}, \mathcal{H}$ is the variance-covariance matrix of the vector of the TVAR innovations $u_t^{(i)} = (u_{mt}^{(i)}, u_{gt}^{(i)}, u_{\pi t}^{(i)}, u_{rt}^{(i)})'$.

Table 3B: Estimates **ignoring** threshold endogeneity bias effects including the unexpected monetary shocks variable m in the model and Cumulative Responses

$$\delta = 0.0464\ [0.030, 0.048],\ LR = 115.890\ [0.000],\ bic = -10100.67$$

$$\Sigma^{\mathcal{L}}_{uu}\times 100 = \begin{bmatrix} \sigma_{mm}= 0.0004 & & & \\ \sigma_{gm}= -0.00078 & \sigma_{gg}= 0.4866 & & \\ \sigma_{\pi m}= -0.00018 & \sigma_{\pi g}= 0.0075 & \sigma_{\pi\pi}= 0.0534 & \\ \sigma_{rm}= -0.000027 & \sigma_{rg}= 0.0009 & \sigma_{r\pi}= 0.00005 & \sigma_{rr}= 0.00055 \end{bmatrix} \quad \Sigma^{\mathcal{H}}_{uu}\times 100 = \begin{bmatrix} \sigma_{mm}= 0.00263 & & & \\ \sigma_{gm}= 0.00241 & \sigma_{gg}= 0.7190 & & \\ \sigma_{\pi m}= 0.00025 & \sigma_{\pi g}= -0.006 & \sigma_{\pi\pi}= 0.1967 & \\ \sigma_{rm}= -0.000067 & \sigma_{rg}= 0.0088 & \sigma_{r\pi}= -0.0016 & \sigma_{rr}= 0.00402 \end{bmatrix}$$

$horizon$	1	12	24	36	48
$g^{\mathcal{L}}$	-0.047	-0.046	-0.131	-0.177	-0.20
	(0.019)	(0.081)	(0.074)	(0.071)	(0.074)
$g^{\mathcal{H}}$	0.22	-0.091	-0.43	-0.61	-0.70
	(0.012)	(0.041)	(0.106)	(0.212)	(0.393)
$\pi^{\mathcal{L}}$	-0.056	-0.0009	-0.147	-0.212	-0.261
	(0.009)	(0.031)	(0.031)	(0.03)	(0.04)
$\pi^{\mathcal{H}}$	-0.165	0.192	0.521	0.695	0.788
	(0.014)	(0.029)	(0.078)	(0.138)	(0.227)

Notes: The table presents estimates of the cumulative responses of real industrial production growth and inflation rates to a monetary policy shock of one standard deviation. These are based on estimates of a TVAR model with two lags for the period 1970m1-2021m12, using as threshold variable the average of inflation rate over the last twenty months and ignoring threshold endogeneity bias. As monetary policy instrument, we use the 2-year government bond yield. In estimating the threshold parameter δ, we have trimmed out the 10% of the observations of z_t from each end of its sample. Fixed-length Block-bootstrap standard errors are in parentheses, based on 1000 iterations. LR is the likelihood ratio test statistic testing the null hypothesis that the VAR model is linear against its alternative hypothesis that is subject to threshold effects. The p-value of the LR statistic is calculated by an analogous block-bootstrap method used to estimate the standard errors. bic is the Bayesian information criterion and $\Sigma^{(i)}_{uu}, i = \mathcal{L}, \mathcal{H}$ is the variance-covariance matrix of the vector of the TVAR innovations $u^{(i)}_t = (u^{(i)}_{mt}, u^{(i)}_{gt}, u^{(i)}_{\pi t}, u^{(i)}_{rt})'$.

Table 4: ESTIMATES OF A LINEAR VAR MODEL OVER THE FULL SAMPLE AND CUMULATIVE RESPONSES

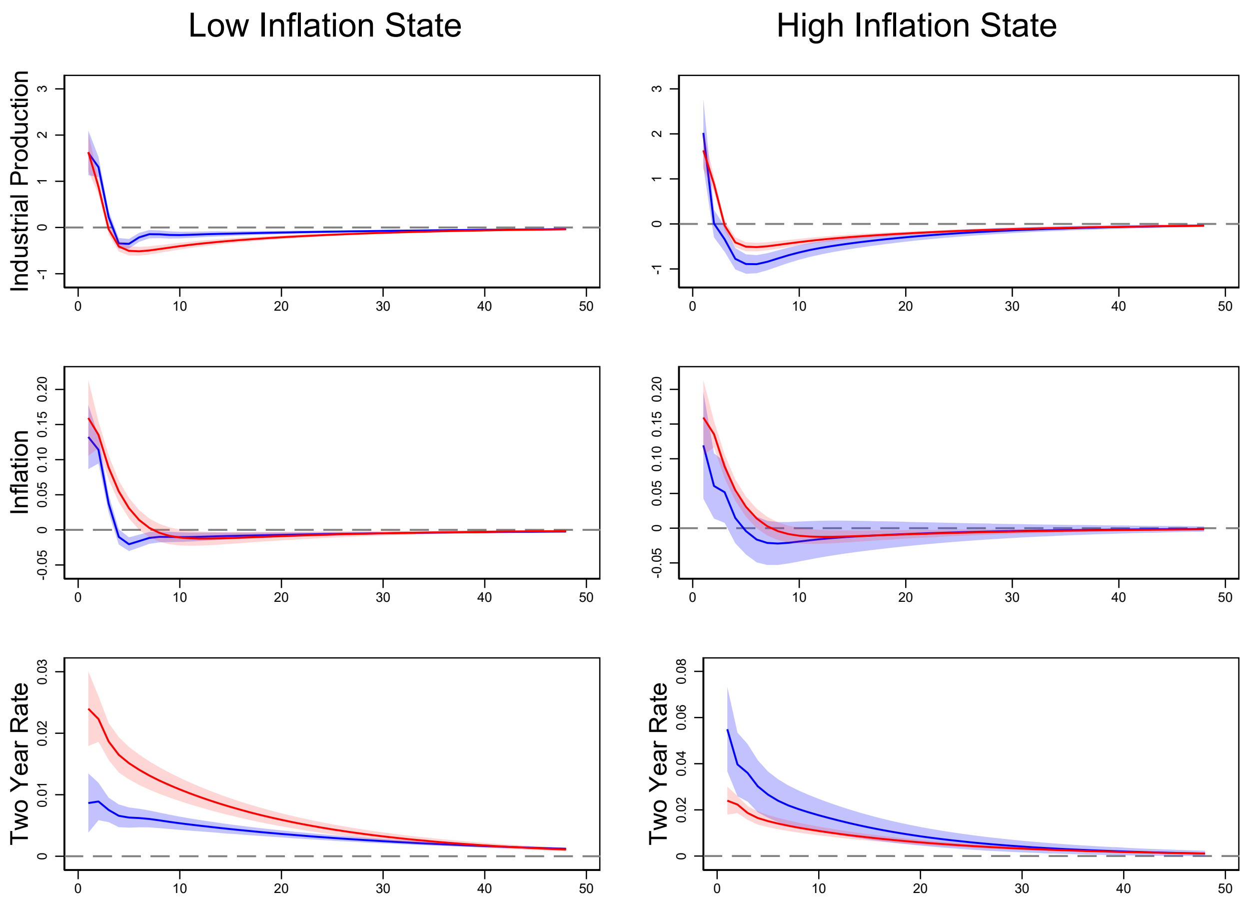
$$bic = -8403.43$$

$$\Sigma_{uu} \times 100 = \begin{bmatrix} \sigma_{gg} = 0.9650 & & \\ \sigma_{\pi g} = 0.0280 & \sigma_{\pi\pi} = 0.0950 & \\ \sigma_{rg} = 0.0073 & \sigma_{r\pi} = 0.0007 & \sigma_{rr} = 0.0023 \end{bmatrix}$$

$horizon$	**1**	12	**24**	36	**48**
g	0.024	-0.012	-0.241	-0.299	-0.328
	(0.003)	(0.011)	(0.018)	(0.023)	(0.025)
π	0.012	-0.004	-0.029	-0.043	-0.049
	(0.000)	(0.002)	(0.003)	(0.004)	(0.005)

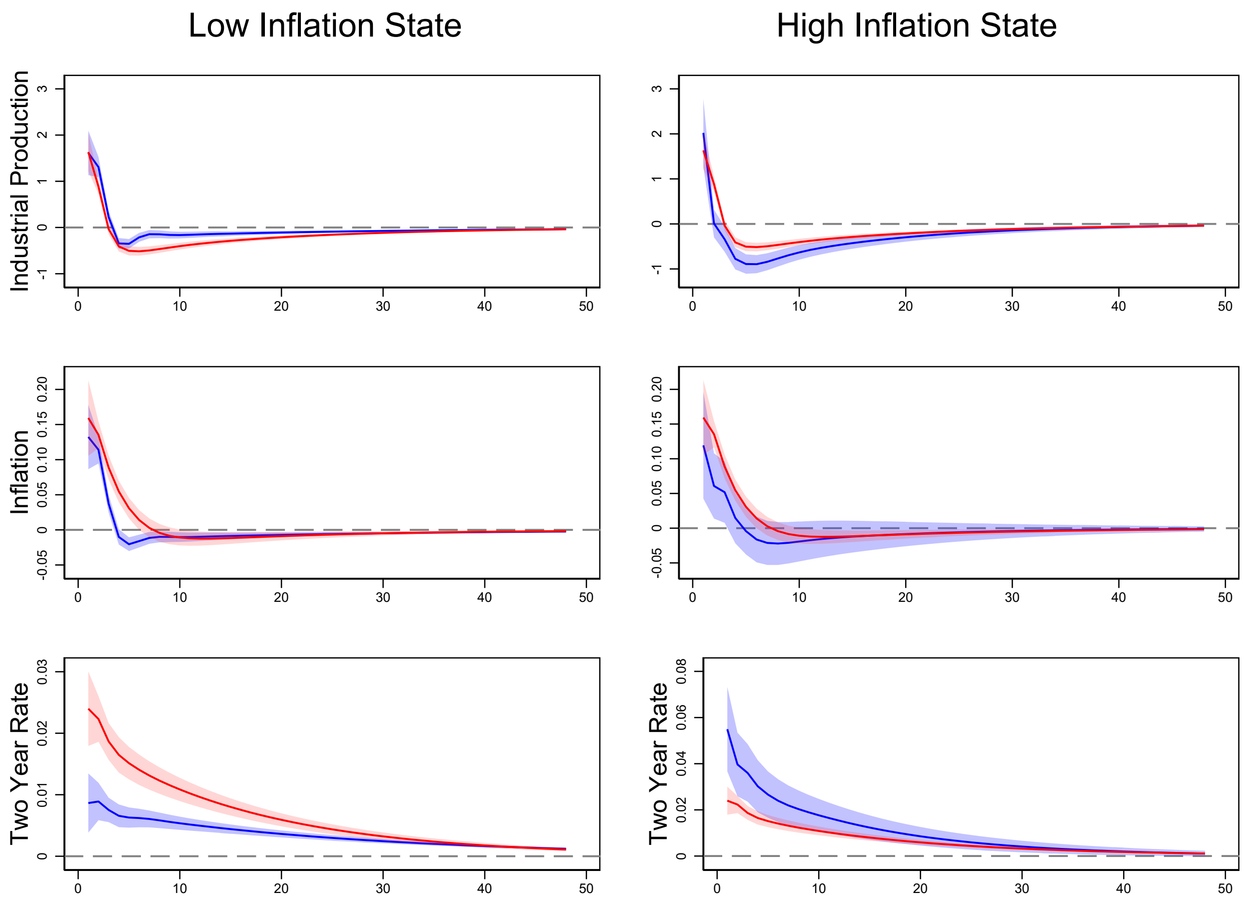
Notes: The table presents estimates of the cumulative responses of real industrial production growth and inflation rates to a monetary policy shock of one standard deviation. These are based on estimates of a linear VAR model for the full sample period. Bootstrapped standard errors in parentheses, based on a wild parametric bootstrap method using one thousand iterations. Σ_{uu} is the variance-covariance matrix of the vector of the VAR innovations u_t. Accordingly, σ_{xx} and σ_{xy} indicate respectively the variance of x and the covariance of x and y.

Figure D.1: Impulse Responses: *TVAR(N)* (Blue) and Linear *VAR* (Red)



Notes: See notes to Figure 3.

Figure D.2: Impulse Responses: *TVAR(N)* (Blue) and *TVAR(X)* (Red)



Notes: See notes to Figure 4.

References

Bauer, M. D. and E. T. Swanson (2022). A Reassessment of Monetary Policy Surprises and High-Frequency Identification, NBER Chapters, in: NBER *Macroeconomics Annual* 2022, 37, National Bureau of Economic Research, Inc.

Gertler, M. and P. Karadi (2015). Monetary Policy Surprises, Credit Costs, and Economic Activity. *American Economic Journal: Macroeconomics*, 7 (1), 44–76.

Miranda-Agrippino, S. and G. Ricco (2021) The Transmission of Monetary Policy Shocks. *American Economic Journal: Macroeconomics*, 13(3), 74-107.

Patton, A. J., D. N. Politis and H. White (2009). Correction to Automatic Block-Length Selection for the Dependent Bootstrap by D. Politis and H. White. *Econometric Review*, 28:4, 372-375.

Politis, D. and H. White (2004). Automatic Block-Length Selection for the Dependent Bootstrap. *Econometric Reviews*, 23(1), 53-70.

Romer, C. D. and D. H. Romer (2004). A New Measure of Monetary Shocks: Derivation and Implications. *American Economic Review*, 94, 4, 1055–1084.

E Additional Robustness Exercises

This section provides robustness tests of our results, aimed at gauging their sensitivity. The tests include estimating the TVAR model from the beginning of the sample until 2008m9, thus dropping the periods 2009-2015 after the GFC and the period 2020-2021 of the Covid-19 outbreak where the FFR approached the zero bound. In addition to this, we estimate a linear version of our VAR specification, without threshold effects, from 2008m10 onward until the end of the sample, since inflation during this period was below the threshold level 3.6%. The results of the model for this period should be consistent with those of the low-inflation states.

Tables E.1 and E.2 present the estimation results of the above robustness tests. Table E.1 presents estimates of the cumulative multipliers for the model allowing for endogeneity of the threshold variable over the period 1970m1-2009m9, while Table E.2 presents results for the linear VAR model over 2008m10-2021m21. The results of Table E.1 are consistent with those of Table 1A. The estimate of the threshold parameter δ is 3.7%, which is close to the aforementioned full-sample 3.6% value. A similar conclusion can be drawn for the estimates of the cumulative multipliers of real industrial production growth and inflation rates to a monetary shock of one standard deviation, across the two states. These are qualitatively similar to those obtained with the whole sample. Turning to the results of Table E.2, these are also consistent with those of low-inflation state obtained with the whole sample, both quantitatively and quantitatively.

Table E.1: Estimates controlling for threshold endogeneity bias effects (1970m1 - 2008m9) and Cumulative Responses

$$\delta = 0.0370\ 0.0203, 0.0537],\ LR = 64.200\ [0.000],\ bic = -6310.738$$

$$\Sigma_{uu}^{\mathcal{L}} \times 100 = \begin{bmatrix} \sigma_{gg} = 0.390 & & \\ \sigma_{\pi g} = 0.0015 & \sigma_{\pi\pi} = 0.041 & \\ \sigma_{rg} = 0.0008 & \sigma_{r\pi} = 0.0002 & \sigma_{rr} = 0.0003 \end{bmatrix} \Sigma_{uu}^{\mathcal{H}} \times 100 = \begin{bmatrix} \sigma_{gg} = 0.8100 & & \\ \sigma_{\pi g} = 0.0058 & \sigma_{\pi\pi} = 0.094 & \\ \sigma_{rg} = 0.013 & \sigma_{r\pi} = -0.0005 & \sigma_{rr} = 0.0053 \end{bmatrix}$$

$horizon$	**1**	12	**24**	36	**48**
$g^{\mathcal{L}}$	0.020	-0.004	-0.079	-0.148	-0.214
	(0.003)	(0.002)	(0.044)	(0.060)	(0.080)
$g^{\mathcal{H}}$	0.013	-0.223	-0.328	-0.367	-0.382
	(0.002)	(0.053)	(0.079)	(0.090)	(0.097)
$\pi^{\mathcal{L}}$	0.001	0.0002	-0.005	-0.009	-0.014
	(0.0002)	(0.0005)	(0.001)	(0.002)	(0.003)
$\pi^{\mathcal{H}}$	0.007	-0.006	-0.015	-0.019	-0.021
	(0.001)	(0.0009)	(0.002)	(0.002)	(0.002)

Notes: The table presents estimates of the cumulative responses of real industrial production growth and inflation rates to a monetary policy shock of one standard deviation. These are based on estimates of a TVAR model for the sample period 1970m1-2008m9, using as threshold variable the average of inflation rate over the last twenty months, $z_t = \sum_{j=0}^{19} \pi_{t-j}$ (akin to a measure of trend inflation), and allowing for threshold endogeneity based on the copula approach, presented in Section 2. In estimating the threshold parameter δ, we have trimmed out the 10% of the observations of z_t from each end of its sample. Bootstrapped standard errors in parentheses, based on a wild parametric bootstrap method using one thousand iterations. LR is the likelihood ratio test statistic testing the null hypothesis that the VAR model is linear against its alternative hypothesis that is subject to threshold effects. The p-value of the LR statistic is calculated by the wild parametric method used to estimate the standard errors. bic is the Bayesian information criterion and $\Sigma_{uu}^{(i)}$, $i = \mathcal{L}, \mathcal{H}$, is the variance-covariance matrix of the vector of the TVAR innovations $u_t^{(i)}$. Accordingly, σ_{xx} and σ_{xy} indicate respectively the variance of x and the covariance of x and y.

Table E.2: Estimates of a linear VAR model for period (2008m10 - 2021m12) and Cumulative Responses

$$bic = -2495.837$$

$$\Sigma_{uu} \times 100 = \begin{bmatrix} \sigma_{gg} = 0.0152 & & \\ \sigma_{\pi g} = 0.0730 & \sigma_{\pi\pi} = 0.060 & \\ \sigma_{rg} = 0.0041 & \sigma_{r\pi} = 0.0005 & \sigma_{rr} = 0.0001 \end{bmatrix}$$

$horizon$	**1**	12	**24**	36	**48**
g	0.023	-0.044	-0.110	-0.157	-0.189
	(0.990)	(0.0063)	(0.016)	(0.161)	(0.038)
π	0.0025	-0.010	-0.027	-0.039	-0.048
	(0.001)	(0.0015)	(0.004)	(0.008)	(0.011)

Notes: See notes to Table 4

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DEPARTMENT OF ECONOMICS

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