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Stochastic Arbitrage Opportunities: Set Estimation and Statistical Testing

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Stelios Arvanitis*and Thierry Post†

Abstract

A Stochastic Arbitrage Opportunity is a zero-cost investment portfolio that enhances every feasible host portfolio for all admissible utility functions. This concept generalizes simpler concepts based on dominating or enhancing a single benchmark portfolio. The present study provides a formal theory of consistent estimation of the set of all arbitrage portfolios that meet the description of being a Stochastic Arbitrage Opportunity. Two Empirical Likelihood Ratio tests are developed: one for the null that a given arbitrage portfolio is qualified and another one for the alternative that the portfolio is not qualified. Apart from considering generalized concepts and hypotheses based on multiple host portfolios, the statistical assumption framework is also more general than in earlier studies that focus on special cases with a single benchmark portfolio. Various extensions and generalizations of the theory are discussed.

Key words: Portfolio analysis; Arbitrage portfolios; Asset pricing; Asymptotic statistics; Empirical Likelihood.

1 Introduction

Stochastic Dominance (SD) is a time-honored maxim for investment decision making that avoids a parametric specification of investor risk preferences by relying on a general class of utility functions (Hadar and Russell (1969); Hanoch and Levy (1969); Rothschild and Stiglitz (1970); Bawa (1975)). This principle is particularly appealing for asset classes and investment strategies with asymmetric risk profiles that defy the standard mean-variance approach, for example, small-cap stocks, momentum strategies, and option combinations.

The application of SD was traditionally focused on pairwise comparison between two alternatives (e.g., portfolios, strategies or funds) but over time theoretical concepts and analytical methods were developed also for multivariate analysis of sets of alternatives for asset pricing and portfolio analysis.

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A milestone on this path was the modeling of non-smooth restrictions on the Expected Shortfall or Lower Partial Moment of a portfolio relative to a given benchmark, which facilitated the development of various problem formulations and optimization algorithms for building a portfolio that stochastically dominates a given benchmark (Shalit and Yitzhaki (1994); Rockafellar and Uryasev (2000); Dentcheva and Ruszczynski (2003); Kuosmanen (2004); Roman, Darby-Dowman and Mitra (2006)). Post, Karabati and Arvanitis (2018) develop a statistical theory based on the Empirical Likelihood (EL) method for the associated efficient set and the optimal portfolio.

Arvanitis and Post (2023), henceforth AP23, further generalize the concepts and methods by accounting for uncertainty about the relevant benchmark portfolio or host portfolio. This type of uncertainty arises in many applications in empirical asset pricing and delegated money management. To account for it, AP23 introduce the general concept of a Stochastic Arbitrage Opportunity (SAO): a zero-cost investment portfolio that enhances all host portfolios that can be constructed from a set of base assets (instead of a single benchmark portfolio), for all elements of a general set of utility functions.

For empirical estimation and statistical inference, AP23 sketch a theory of consistent estimation of the set of SAOs and consistent testing whether or not a given arbitrage portfolio is a SAO. The present study provides a formalization of that statistical theory and additional discussion about extensions and generalizations.

We demonstrate how the analysis by Post, Karabatı and Arvanitis (2018) of optimization with SD constraints can be generalized to SAO choice for a general host portfolio set (K) and utility class ($\mathcal{U}$). We utilize a generalization of the Weak Independence assumption of Post, Karabatı and Arvanitis (2018) in order to obtain versions of set consistency for the empirical SAOs, notably avoiding the use of (arbitrary) tolerance parameters. The derivations are novel; they are among other things based on the construction of a set of approximate solutions of the empirical optimization problem that satisfy restricted versions of the empirical dominance inequalities.

We also show how statistical inference about membership of the SAO set can be performed using Blockwise Empirical Likelihood Ratio (BELR) tests, in the spirit of the tests for the simpler concepts of non-dominance, efficiency and optimality by Davidson and Duclos (2014), Post and Poti (2017) and Post (2017). Two Empirical Likelihood Ratio tests are developed: one for the null that a given arbitrage portfolio is a true SAO and another one for the alternative that the portfolio is not a true SAO. We furthermore introduce new robust versions of the testing procedures that allow for the possibility that the set of binding dominance inequalities is infinite.

Although the focus in this study is on a set of host portfolios and a set of arbitrage portfolios, the results naturally also apply to simpler cases in which one of the two sets is a singleton. The simplest special case arises for the pairwise analysis between two given standard portfolios or, equivalently, the analysis of one arbitrage portfolio and one host portfolio. Another special case is the efficiency analysis of a given standard portfolio and portfolio optimization with SD constraints relative to a given standard benchmark portfolio.

Section 2 summarizes the relevant economic concepts. Section 3 provides the statistical theory. Section 4 concludes by discussing various extensions and generalizations of the theory.

2 Stochastic Arbitrage Opportunities

2.1 Standard portfolios, arbitrage portfolios and combined portfolios

The focus is on a single-period investment problem with N base assets with cashflows at the investment horizon, $\boldsymbol{x} := (x_1 \cdots x_N)^{\mathrm{T}} \in \mathbb{R}^N$, $N < \infty$. The prices of the base assets are taken as given, and are collected in the price vector $\boldsymbol{p} \in \mathbb{R}^N$. Cash flows may be converted to gross returns $\boldsymbol{r} := \mathrm{diag}^{-1}(\boldsymbol{p})\,\boldsymbol{x}$.

The joint probability distribution of the cash flows is denoted by $\mathbb{P}$; its support is included in $\mathcal{X}^N$, where $\mathcal{X} := [a, b]$, $-\infty < a < b < \infty$. The distribution is generally conditional on the business cycle and/or financial market conditions. It is assumed below that expectations and probabilities are evaluated with respect to $\mathbb{P}$, unless an empirical counterpart of $\mathbb{P}$ is explicitly specified.

Investment alternatives are evaluated based on the expected utility of the cash flows, using utility functions $u : \mathbb{R} \to \mathbb{R}$. Instead of specifying a particular functional form, a uniformly bounded and convex set of functions, $\mathcal{U}$, is defined using general functional properties. The functions are assumed to be continuously differentiable, strictly increasing and strictly concave. To facilitate numerical analysis, it is assumed that $u(b) = 0$ and $u'(a) = 1$, which is harmless because utility rankings are invariant to positive linear transformation.

All aforementioned utility functions can be represented using strictly positive mixtures of elementary Russell and Seo (1989) functions $v_{2;\phi}(x) := -(\phi - x)_+$, $\phi \in \mathcal{X}$. A formal definition of the utility function class is $\mathcal{U}_2 := \left\{u(x) = \int_{\mathcal{X}} v_{2;\phi}(x)dW(\phi); W \in \mathcal{W}\right\}$, where $\mathcal{W}$ is the set of strictly increasing cumulative weighting functions $W : \mathcal{X} \to [0, 1]$.

$\mathcal{U}$ is endowed with the sup metric, to facilitate the asymptotic theory of the inferential procedures in Section 3. The asset pricing results in AP23 (Thm. 3.3.1) of AP23 rely on first-order optimality conditions that require a stronger topology like $W^{1,\infty}(\mathbb{R})$-see Adams and Fournier (2015).

The utility set $\mathcal{U}$ is not necessarily closed because singular utilities obtained as limits of strictly increasing and concave functions are inconsequential for the analysis. The closure of the set of utilities, $\mathrm{cl}(\mathcal{U})$, is however useful for the numerical analysis and the analysis of the limiting behavior of empirical constructs used in Section 3.

The portfolio possibilities are described by two distinct portfolio sets: a set of host portfolios or benchmarks, $\mathrm{K} \subseteq \mathbb{R}^N$, and a set of arbitrage portfolios, $\Delta \subseteq \mathbb{R}^N$. These portfolios sets are described in more detail below.

Host portfolios are standard portfolios that require a strictly positive net investment. The asset position vector $\boldsymbol{\kappa} \in \mathrm{K}$ must obey the budget restriction $\boldsymbol{p}^{\mathrm{T}}\boldsymbol{\kappa} = c$, for budget $c > 0$. Asset positions may be converted to portfolio weights $\boldsymbol{\lambda} := c^{-1}(\boldsymbol{p} \odot \boldsymbol{\kappa})$.

K is assumed to be closed and bounded but it need not be convex. If it is not convex, then $\mathrm{conv}(\mathrm{K}^{(0)})$ is used for its convex hull; if it is convex, then $\mathrm{K}^{(0)}$ is used for its set of extreme elements. If K takes a polyhedral shape, then the number of extreme elements is finite; in this case, the continuous set $\mathrm{K} = \mathrm{conv}(\mathrm{K}^{(0)})$ can be replaced by $\mathrm{K}^{(0)}$, for numerical purposes.

Arbitrage portfolios $\boldsymbol{\delta} \in \Delta$ are combinations of short and long positions with a net investment of zero, or $\boldsymbol{p}^{\mathrm{T}}\boldsymbol{\delta} = 0$. They can be added to a host portfolio $\boldsymbol{\kappa} \in \mathrm{K}$ to form a combined portfolio $\boldsymbol{\lambda} = (\boldsymbol{\kappa} + \boldsymbol{\delta})$ that is a standard portfolio ($\boldsymbol{p}^{\mathrm{T}}\boldsymbol{\lambda} = c$). The set of all feasible combined portfolios is given by the vector subspace sum $\Lambda_0 := \mathrm{K} + \Delta$. The asset positions may equivalently be expressed using weights in the combined portfolio, or $\boldsymbol{\gamma} := c^{-1}(\boldsymbol{p} \odot \boldsymbol{\delta})$.

It is assumed that the arbitrage portfolio set is a bounded and convex polytope that is characterized by R linear restrictions: $\Delta := \left\{\boldsymbol{\delta} \in \mathbb{R}^N : \mathbf{A}\boldsymbol{\delta} \leq \boldsymbol{a}\right\}$; here, $\mathbf{A}$ is $(R \times N)$, and $\boldsymbol{a}$ is $(R \times 1)$. It is assumed that the 'passive' solution $\mathbf{0}_N \in \Delta$ is the default choice if SAOs do not exist.

Although the focus here is on portfolio sets, the analysis applies to simpler cases in which K and/or Δ is singleton. To make a pairwise comparison of two given standard portfolios $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$, we may simply apply our framework to $\mathrm{K} = \{\boldsymbol{\kappa}_1\}$ and $\Delta = \{\boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_1\}$; to analyze the efficiency of a given standard portfolio $\boldsymbol{\kappa}_1$ relative to standard portfolio set Λ and find an alternative portfolio $\lambda \in \Lambda$ that stochastically dominates $\boldsymbol{\kappa}_1$, we may simply set $\mathrm{K} = \{\boldsymbol{\kappa}_1\}$ and $\Delta = \Lambda - \boldsymbol{\kappa}_1$.

2.2 Stochastic Arbitrage Opportunities

AP23 introduce two versions of the concepts of stochastic enhancement and SAO: a strict version for asset pricing theory and a weak version for numerical purposes.

To allow for compact notation, the expected utility increment is denoted by $D(u, \boldsymbol{\kappa}, \boldsymbol{\delta}, \mathbb{P}) := \mathbb{E}\left[u(\boldsymbol{x}^{\mathrm{T}}(\boldsymbol{\kappa} + \boldsymbol{\delta}))\right] - \mathbb{E}\left[u(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\kappa})\right]$, for $(u, \boldsymbol{\kappa}, \boldsymbol{\delta}) \in \mathcal{U} \times \mathrm{K} \times \Delta$.

Definition 2.2.1 (Strict SAO). *An arbitrage portfolio $\boldsymbol{\delta} \in \Delta$ (strictly) stochastically enhances a given benchmark $\boldsymbol{\kappa} \in \mathrm{K}$, or $(\boldsymbol{\kappa} + \boldsymbol{\delta}) \succ_{(\mathcal{U},\mathbb{P})} \boldsymbol{\kappa}$, if $D(u, \boldsymbol{\kappa}, \boldsymbol{\delta}, \mathbb{P}) > 0$ for all $u \in \mathcal{U}$. It is a (strict) SAO if such enhancement is achieved for all $\boldsymbol{\kappa} \in \mathrm{K}$. The set of all feasible (strict) SAOs is given by*

$$\Delta^{\mathrm{SAO}}_{(\mathcal{U},\mathrm{K},\mathbb{P})} := \{\boldsymbol{\delta} \in \Delta : D(u, \boldsymbol{\kappa}, \boldsymbol{\delta}, \mathbb{P}) > 0 \; \forall \, (u, \boldsymbol{\kappa}) \in \mathcal{U} \times \mathrm{K}\}. \tag{1}$$

Definition 2.2.2 (Weak SAO). *An arbitrage portfolio $\boldsymbol{\delta} \in \Delta$ (weakly) stochastically enhances a given benchmark $\boldsymbol{\kappa} \in \mathrm{K}$, or $(\boldsymbol{\kappa} + \boldsymbol{\delta}) \gtrsim_{(\mathcal{U},\mathbb{P})} \boldsymbol{\kappa}$, if $D(u, \boldsymbol{\kappa}, \boldsymbol{\delta}, \mathbb{P}) \geq 0$ for all $u \in \mathrm{cl}\,(\mathcal{U})$. It is a (weak) SAO if such enhancement is achieved for all $\boldsymbol{\kappa} \in \mathrm{K}$. The set of all feasible (weak) SAOs is given by*

$$\Delta^{\mathrm{WAO}}_{(\mathcal{U},\mathrm{K},\mathbb{P})} := \{\boldsymbol{\delta} \in \Delta : D(u, \boldsymbol{\kappa}, \boldsymbol{\delta}, \mathbb{P}) \geq 0 \; \forall \, (u, \boldsymbol{\kappa}) \in \mathrm{cl}\,(\mathcal{U}) \times \mathrm{K}\}. \tag{2}$$

Thus, SAOs are solutions to the following semi-infinite inequality system:

$$\begin{gathered} D(u, \boldsymbol{\kappa}, \boldsymbol{\delta}, \mathbb{P}) \geq 0 \; \forall \, (u, \boldsymbol{\kappa}) \in \mathrm{cl}\,(\mathcal{U}) \times \mathrm{K}; \\ \boldsymbol{\delta} \in \Delta. \end{gathered} \tag{3}$$

Finite discretizations of $\mathbb{P}$, $\mathrm{cl}\,(\mathcal{U})$ and K allow for numerical analysis using finite mathematical programming problems. A number of common and general approaches are reviewed below.

Continuous distributions $\mathbb{P}$ can be discretized using, e.g., Monte-Carlo simulation methods or lattice models. These approaches are not needed in empirical applications with discrete estimators, such as the empirical distribution $\mathbb{P}_T$ that was used by AP23 or the probability distributions implied by Generalized Method of Moments and Generalized Empirical Likelihood that were used by Post, Karabati and Arvanitis (2018).

If the benchmark set K is continuous, then it may be replaced without harm by its set of extreme elements $\mathrm{K}^{(0)}$, e.g., the vertices of a polytope. This replacement is allowed due to the convexity property of the stochastic enhancement relation with respect to the host portfolio positions.

Since expected utility is a linear function of $u \in \mathrm{cl}(\mathcal{U})$, and $\mathrm{cl}(\mathcal{U})$ is a convex set, $\mathrm{cl}(\mathcal{U})$ may be replaced with the set of its extreme elements, or $\mathcal{U}^{(0)} : \mathrm{cl}(\mathcal{U}) = \mathrm{conv}\left(\mathcal{U}^{(0)}\right)$, without loss of generality. For many relevant specifications of $\mathcal{U}$, the extreme elements $u \in \mathcal{U}^{(0)}$ are low-dimensional functions, which reduces the numerical complexity of searching over $\mathrm{cl}(\mathcal{U})$. Notably, for n-th degree SD, $\mathcal{U}_n^{(0)}$ consists of the one-parameter functions $v_{n;\phi}(x) = -(\phi - x)_+^{n-1}$.

Total boundedness of the support, the Lipschitz continuity property of the positive part operator $(\cdot)_+$, and the compactness of Δ and K imply that the one-dimensional parameter space can be represented or approximated by a finite set with arbitrary level of precision. For any $\epsilon > 0$, there exists a finite $\mathcal{U}_{n,\epsilon}^{(0)} \subseteq \mathcal{U}_n^{(0)}$ such that $\sup_{\kappa,\delta} \sup_{\phi \in \mathcal{U}^{(0)}} \inf_{\phi^\star \in \mathcal{U}_{n,\epsilon}^{(0)}} |D(u_{n,\phi}, \boldsymbol{\kappa}, \boldsymbol{\delta}, \mathbb{P}) - D(u_{n,\phi^\star}, \boldsymbol{\kappa}, \boldsymbol{\delta}, \mathbb{P})| < \epsilon$; $\mathcal{U}_{n,\epsilon}^{(0)}$ approximates $\mathcal{U}^{(0)}$ with precision ϵ. Such a finite discretization is inconsequential for portfolio optimization for small enough values of the parameter.

Using the above replacements, system (3) can be reduced to:

$$D(u, \boldsymbol{\kappa}, \boldsymbol{\delta}, \mathbb{P}) \geq 0 \; \forall \, (u, \boldsymbol{\kappa}) \in \mathcal{U}^{(0)} \times \mathrm{K}^{(0)}; \tag{4}$$
$$\boldsymbol{\delta} \in \Delta.$$

In what follows, it is assumed that $\mathrm{K}^{(0)}$ is finite; this assumption is justified by the fact that K is polyhedral in many applications. It is also assumed that for any $\epsilon > 0$, there exists a finite $\mathcal{U}_\epsilon^{(0)}$ that approximates $\mathcal{U}^{(0)}$ with precision ϵ. Since $\mathcal{U} \subseteq \mathcal{U}_2$, and $\mathcal{U}_2$ is the convex hull of $\mathcal{U}_2^{(0)}$, this assumption is harmless.

Under these assumptions, the semi-infinite system of inequalities (3) can be represented or approximated by the following finite system; given some choice of ϵ:

$$D(u,\boldsymbol{\kappa},\boldsymbol{\delta},\mathbb{P}) \geq 0\ \forall\,(u,\boldsymbol{\kappa}) \in \mathcal{U}_{\epsilon}^{(0)} \times \mathrm{K}^{(0)}; \tag{5}$$

$$\boldsymbol{\delta} \in \Delta.$$

Maximizing a concave objective function $G(\boldsymbol{\delta},\mathbb{P})$ subject to system (5) yields a finite convex optimization problem. The choice of the objective function naturally is application specific, just as the choice of $\mathcal{U}$ and K.

3 Empirical counterparts

3.1 Consistency properties

It is assumed that the latent $\mathbb{P}$ is estimated by the empirical measure $\mathbb{P}_T$ from a time series sample $(\boldsymbol{x}_t)_{t=1,\dots,T}$. Since cashflows may include trend components, they generally are not stationary random variables. To eliminate or mitigate this econometric problem, cash flows may be transformed to returns $\boldsymbol{r} := \mathrm{diag}^{-1}(\boldsymbol{p})\,\boldsymbol{x}$, and positions may be transformed to weights $\boldsymbol{\lambda} := c^{-1}(\boldsymbol{p} \odot \boldsymbol{\kappa})$ and $\boldsymbol{\gamma} := c^{-1}(\boldsymbol{p} \odot \boldsymbol{\delta})$, without altering the substance of the economic choice problem. In order to focus on the effect of sampling error on the stochastic enhancement constraints, it is furthermore assumed that the constraints in K and Δ are deterministic.

The following empirical weak SAO set is constructed:

$$\Delta^{\mathrm{WAO}}_{(\mathcal{U},\mathrm{K},\mathbb{P}_T)} := \left\{\boldsymbol{\delta} \in \Delta : (\boldsymbol{\kappa}+\boldsymbol{\delta}) \gtrsim_{(\mathcal{U},\mathbb{P}_T)} \boldsymbol{\kappa},\ \forall \boldsymbol{\kappa} \in \mathrm{K}\right\} \tag{6}$$

where the dominance relation $\gtrsim_{(\mathcal{U},\mathbb{P}_T)}$ is interpreted in (2) with $\mathbb{P}_T$ in place of $\mathbb{P}$.

This empirical set can be shown to possess a number of favorable consistency properties under three general assumptions:

Assumption 3.1 (Stationarity and Mixing). $(\boldsymbol{x}_t)$ *is stationary and absolutely regular with mixing coefficients* $(m_k)_{k\in\mathbb{N}}$ *that satisfy* $m_k = O\left(k^{-r}\right)$ *for* $r > 1$.

Those regularity conditions are compatible with stationary versions of ARMA-GARCH or Stochastic Volatility models, or more generally, processes that satisfy a general class of stochastic recurrence equations, see for example Mikosch and Straumann (2006, Section 4).

Assumption 3.2 (Lipschitz Continuity). *The population goal* $G(\cdot,\mathbb{P})$ *is Lipschitz continuous in* $\boldsymbol{\delta}$ *with Lipschitz coefficient that is continuous at* $\mathbb{P}$ *w.r.t. weak convergence.*

Given the boundedness of the support of $\boldsymbol{x}_0$, Lipschitz continuity is easy to establish for standard goals such as expected return and expected utility. A uniform integrability condition implied by the boundedness of the support ensures the required continuity of the coefficient in these cases. The objective function of AP23 (the weighted average of individual Appraisal Ratios) also satisfies the assumption, due to the CMT and the aforementioned uniform integrability; the Lipschitz coefficient is in this case a continuous function of moments, as long as the variance involved is strictly positive.

The final assumption generalizes the Weak Independence assumption of Post, Karabatı and Arvanitis (2018). It takes care of consistency problems that may arise for the empirical solutions due to the binding inequalities (equivalences), or $D(u,\boldsymbol{\kappa},\boldsymbol{\delta},\mathcal{F})=0$, for some non-constant utility functions $u\in \mathrm{cl}\,(\mathcal{U})$.

Assumption 3.3 (Joint Enhancement). *For any non-constant* $v\in cl(\mathcal{U})$*, there exists some* $\boldsymbol{\delta}\,(v)$ *that solves* (3)*, such that* $D(v,\boldsymbol{\kappa},\boldsymbol{\delta}\,(v)\,,\mathbb{P})>0$ *for all* $\boldsymbol{\kappa}\in K$*.*

The Joint Enhancement (henceforth JE) assumption essentially ensures the existence of some neighborhood of every weak SAO $\boldsymbol{\delta}_1\in\Delta^{\mathrm{WAO}}_{(\mathcal{U},\mathrm{K},\mathbb{P})}$ that contains strict SAOs $\boldsymbol{\delta}_2\in\Delta^{\mathrm{SAO}}_{(\mathcal{U},\mathrm{K},\mathbb{P})}$, if strict SAOs exist. JE allows for different $\boldsymbol{\delta}$ for different utility functions v instead of a single $\boldsymbol{\delta}$ for all such utilities. It mitigates the adverse effect of the binding inequalities on consistency; given any weak SAO $\boldsymbol{\delta}$ for which $D(v,\boldsymbol{\kappa},\boldsymbol{\delta},\mathbb{P})=0$ for some $(v,\boldsymbol{\kappa})$, consider the portfolio defined using the Lebesgue integral $\boldsymbol{\gamma}:=\int_{\mathcal{U}-\mathcal{U}^{=}}\boldsymbol{\delta}\,(u)\,d\left((1-c)\,w_v+cw\,(u)\right),\ w\in\mathcal{W}$, where $\mathcal{W}$ is the set of strictly monotone Borel measures defined on $\mathcal{U}$, and w_v the degenerate measure at v, $c\in(0,1)$. JE implies that $D(u,\boldsymbol{\kappa},\boldsymbol{\gamma},\mathbb{P})>0$ for all $(u,\boldsymbol{\kappa})\in\mathcal{U}-\mathcal{U}^{=}\times\mathrm{K}$, and by choosing c appropriately small, it can be chosen to lie as close to $\boldsymbol{\delta}$ as desired.

Various consistency results are obtained based on the above assumptions. The limit theory evolves as $T\to\infty$. The squiggly arrow $\rightsquigarrow$ denotes convergence in distribution. The analysis also uses Painleve-Kuratowski (PK) convergence $\left(\overset{\mathrm{PK}}{\rightsquigarrow}\right)$ for sequences of compact subsets of Δ. With high probability (w.h.p.) refers to probability converging to one. $\boldsymbol{\delta}^{*}(\mathbb{P})$ denotes the set of optimal weak SAOs, i.e., $\arg\max_{\Delta^{\mathrm{WAO}}_{(\mathcal{U},\mathrm{K},\mathbb{P})}}\left(G(\boldsymbol{\delta},\mathbb{P})\right)$. Given $\delta>0$, $\Delta^{(\delta)}(\mathbb{P})$ denotes the set of δ-optimal weak

SAOs, i.e. $\left\{\boldsymbol{\gamma} \in \Delta^{\text{WAO}}_{(\mathcal{U},\text{K},\mathbb{P})} : \max_{\Delta^{\text{WAO}}_{(\mathcal{U},\text{K},\mathbb{P})}} G(\boldsymbol{\delta},\mathbb{P}) - G(\boldsymbol{\gamma},\mathbb{P}) \leq \delta\right\}$.

Theorem 3.4 . *Under Assumptions 3.1-3 the following results are obtained: (i) If $\Delta^{\text{WAO}}_{(\mathcal{U},\text{K},\mathbb{P})} - \Delta^{=}_{\mathbb{P}} \neq \emptyset$, and JE occurs, then for any $v \in \mathcal{U} - \mathcal{U}^{=}$, and for any $c_T \to 0$ such that $m_T c_T \to \infty$, we have that $\Delta^{\text{WAO}}_{(\mathcal{U},\text{K},\mathbb{P}_T)} \cap \Delta^{(c_T)}(\mathbb{P}) \overset{PK}{\rightsquigarrow} \boldsymbol{\delta}^*(\mathbb{P})$; (ii) If $\Delta^{\text{WAO}}_{(\mathcal{U},\text{K},\mathbb{P})} - \Delta^{=}_{\mathbb{P}} = \emptyset$, then $\Delta^{\text{WAO}}_{(\mathcal{U},\text{K},\mathbb{P}_T)} \overset{PK}{\rightsquigarrow} \Delta^{=}_{\mathbb{P}}$.*

Proof of Theorem 3.4. First notice that any element of $\Delta^{=}_{\mathbb{P}}$ belongs to $\Delta^{\text{WAO}}_{(\mathcal{U},\text{K},\mathbb{P})}$. Section 2.4 shows that when non-trivial weak SAOs do not exist, then $\Delta^{\text{WAO}}_{(\mathcal{U},\text{K},\mathbb{P})} = \Delta^{=}_{\mathbb{P}}$. Hence, if $\boldsymbol{\delta} \notin \Delta^{=}_{\mathbb{P}}$, then $\exists u \in \mathcal{U}, \boldsymbol{\kappa} \in K$ for which $D(u,\boldsymbol{\kappa},\boldsymbol{\delta},\mathbb{P}) < 0$. Assumption 3.1 and the compactness of $\mathcal{X}$, along with the FCLT of Rio (2017, Cor 4.1), imply that $\limsup_{T\to\infty} \mathbb{P}\left[D(u,\boldsymbol{\kappa},\boldsymbol{\delta},\mathbb{P}_T) \geq 0\right] = 0$, which establishes Theorem 3.4.(ii). Suppose next that non-trivial weak SAOs do not exist. Let $c_T = o(1)$ and such that $m_T c_T \to \infty$. Stationarity and mixing of $(\boldsymbol{x}_t)$ and the compactness of $\mathcal{X}$, with Corollary 2.E of Salinetti and Wets (1977) imply that we need to consider only fixed $u \in \mathcal{U}$ in our derivations. For any $\boldsymbol{\delta} \in \boldsymbol{\delta}^*(\mathbb{P})$ that does not have any trivial equivalencies, we have that $\min_K D(u,\boldsymbol{\kappa},\boldsymbol{\delta},\mathbb{P}) > 0$ for all $u \in \mathcal{U} - \mathcal{U}^{=}$, thus $\liminf_{T\to\infty} \mathbb{P}\left[\min_K D(u,\boldsymbol{\kappa},\boldsymbol{\delta},\mathbb{P}_T) \geq 0\right] = 1$ for all $u \in \mathcal{U}$. Suppose then that $\boldsymbol{\delta}$ has non-trivial equivalences. Then for any $u \in \mathcal{U} - \mathcal{U}^{=}$ that does not correspond to some equivalence or to some $u \in \mathcal{U}^{=}$, the previous analysis holds. If $u \in \mathcal{U} - \mathcal{U}^{=}$ and u corresponds to some non-trivial equivalence, then for large enough T, consider the strong SAO $\boldsymbol{\gamma}_T := \int_{\mathcal{U}-\mathcal{U}^{=}} \boldsymbol{\delta}(u)\, d\left((1-c^{\star}_T)\, w_v + c^{\star}_T w(u)\right)$ for L the Lipschitz coefficient of u. Here, $c^{\star}_T = \frac{c_T}{L\text{diam}(\Delta)} = O(c_T)$. Notice that since i) $\min_K D(u,\boldsymbol{\kappa},\cdot,\mathbb{P})$ is concave, ii) by construction $\min_K D(u,\boldsymbol{\kappa},\boldsymbol{\delta},\mathbb{P}) = 0$, and iii) $D\left(u,\boldsymbol{\kappa},\int_{\mathcal{U}-\mathcal{U}^{=}} \boldsymbol{\delta}(u)dw(u),\mathbb{P}\right) > 0$ due to Joint Enhancement, we have that

$$-m_T \inf_K D(u,\boldsymbol{\kappa},\boldsymbol{\gamma}_T,\mathbb{P}) \leq -m_T c^{\star}_T \inf_K D\left(u,\boldsymbol{\kappa},\int_{\mathcal{U}-\mathcal{U}^{=}} \boldsymbol{\delta}(u)dw(u),\mathbb{P}\right) < 0,$$

and thereby

$$\begin{gathered}\mathbb{P}\left[m_T \inf_K D(u,\boldsymbol{\kappa},\boldsymbol{\gamma}_T,\mathbb{P}_T-\mathbb{P}) \geq -m_T \inf_K D(u,\boldsymbol{\kappa},\boldsymbol{\gamma}_T,\mathbb{P})\right] \\ \geq \mathbb{P}\left[m_T \inf_K D(u,\boldsymbol{\kappa},\boldsymbol{\gamma}_T,\mathbb{P}_T-\mathbb{P}) \geq -m_T c^{\star}_T \inf_K D\left(u,\boldsymbol{\kappa},\int_{\mathcal{U}-\mathcal{U}^{=}} \boldsymbol{\delta}(u)dw(u),\mathbb{P}\right)\right],\end{gathered}$$

and due to Assumption 3.1.1 the CMT and the Portmanteau Theorem, the lim inf of the latter probability is greater than or equal to $\mathbb{P}\left[\min_K \mathcal{G}(u,\boldsymbol{\kappa},\boldsymbol{\delta}) \geq -\infty\right] = 1$. Hence, $\liminf_{T\to\infty} \mathbb{P}\left[\boldsymbol{\gamma}_T \in \Delta^{\text{WAO}}_{(\mathcal{U},\text{K},\mathbb{P}_T)}\right] =$

1 and $\boldsymbol{\gamma}_T \to \boldsymbol{\delta}$. The previous then imply that in all cases $\boldsymbol{\delta}$ lies in the lim inf of $\Delta^{\mathrm{WAO}}_{(\mathcal{U},\mathrm{K},\mathbb{P}_T)}$ w.h.p. Now, due to the definitions of $\boldsymbol{\delta}$, $\boldsymbol{\gamma}_T$, and the Lipschitz continuity of $G(\boldsymbol{\delta},\mathbb{P})$, we obtain

$$\begin{aligned} &0 \leq G(\boldsymbol{\delta},\mathbb{P}) - G(\boldsymbol{\gamma}_T,\mathbb{P}) \\ &\leq \max\nolimits_K Lc^{\star}_T \left\| \boldsymbol{\delta} - \textstyle\int_{\mathcal{U}-\mathcal{U}^{=}} \boldsymbol{\delta}(u) dw\,(u) \right\| \leq c^{\star}_T L \mathrm{diam}\,(\Delta) = c_T, \end{aligned} \tag{7}$$

where the first inequality follows from (7). This implies that $\boldsymbol{\gamma}_T \in \Delta^{(c_T)}_{(\mathcal{U},\mathrm{K})}(v,\mathbb{P})$. Obviously $\boldsymbol{\delta}^*(\mathbb{P}) \subseteq \Delta^{(c_T)}(\mathbb{P})$. Any other element of $\Delta^{(c_T)}(\mathbb{P})$ that lies in the empirical weak SAO set with asymptotically positive probability will necessarily converge to some element of $\boldsymbol{\delta}^*(\mathbb{P})$. The previous establish Theorem 3.4.(i), since there cannot exist accumulation points of sequences of elements of $\Delta^{(c_T)}(\mathbb{P})$ that lie outside $\boldsymbol{\delta}^*(\mathbb{P})$.□

The following result complements Theorem 3.4 by showing that the empirical optimal solutions also approximate the original as $T \to \infty$:

Theorem 3.5 (Empirical Solution Properties). *Under the premises of Theorem 3.4, the following results are obtained: i)*

$$\sup_{\boldsymbol{\delta}\in\Delta^{\mathrm{WAO}}_{(\mathcal{U},\mathrm{K},\mathbb{P}_T)}} G(\boldsymbol{\delta},\mathbb{P}_T) \rightsquigarrow \sup_{\boldsymbol{\delta}\in\Delta^{0}_{(\mathcal{U},\mathrm{K},\mathbb{P})}} G(\boldsymbol{\delta},\mathbb{P}), \tag{8}$$

where, $\Delta^0_{(\mathcal{U},\mathrm{K},\mathbb{P})} := \boldsymbol{\delta}^*(\mathbb{P})$ *if* $\Delta^{\mathrm{WAO}}_{(\mathcal{U},\mathrm{K},\mathbb{P})} - \Delta^{=}_{\mathbb{P}} \neq \emptyset$ *and Joint Enhancement (7) occurs, and* $\Delta^0_{(\mathcal{U},\mathrm{K},\mathbb{P})} := \Delta^{=}_{\mathbb{P}}$ *if* $\Delta^{\mathrm{WAO}}_{(\mathcal{U},\mathrm{K},\mathbb{P})} - \Delta^{=}_{\mathbb{P}} = \emptyset$*; ii) If* $\Delta^{\mathrm{WAO}}_{(\mathcal{U},\mathrm{K},\mathbb{P})} - \Delta^{=}_{\mathbb{P}} \neq \emptyset$ *and JE occurs, then every limit of any subsequence of elements of* $\boldsymbol{\delta}^*(\mathbb{P}_T)$ *lies in* $\boldsymbol{\delta}^*(\mathbb{P})$*; iii) If* $\Delta^{\mathrm{WAO}}_{(\mathcal{U},\mathrm{K},\mathbb{P})} - \Delta^{=}_{\mathbb{P}} = \emptyset$*, and , then* $\boldsymbol{\delta}^*(\mathbb{P}_T) \overset{PK}{\rightsquigarrow} \Delta^{=}_{\mathbb{P}}$*.*

Proof of Theorem 3.5. Assumption 3.1 the compactness of $\mathcal{X}$, along with the FCLT of Rio (2017, Cor 4.1), the concavity of $G(\boldsymbol{\delta},\mathcal{F}_T)$ and Skorokhod representations applicable due to Theorem 3.7.25 of Giné and Nickl (2016) imply the w.h.p. epi-convergence of the latter to $G(\boldsymbol{\delta},\mathcal{F})$, due to Corollary 2.E of Sallinetti and Wetts (1997). Then the results in i), ii), iii) follow via Theorem 3.4, employing Skorokhod representations applicable due to Theorem 3.7.25 of Giné and Nickl (2016), using Proposition 3.2, Ch. 5 of Molchanov (2006) and then reverting to the original probability space. Specifically for iii), and since, $\mathbb{E}_{\mathbb{P}_T}\left[u(\boldsymbol{x}^{\mathrm{T}}(\boldsymbol{\kappa}+\boldsymbol{\delta}))\right] - \mathbb{E}_{\mathbb{P}_T}\left[u(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\kappa})\right] \leq 0$ w.h.p., for any $\boldsymbol{\delta} \in \Delta$, $u \in \mathcal{U}, \boldsymbol{\kappa} \in K$, and that due to Assumption 3.2 $G(\boldsymbol{\delta},\mathbb{P}_T)$ is constant on $\Delta^{=}_{\mathbb{P}}$ w.h.p., it follows that: any $\boldsymbol{\delta} \notin \Delta^{=}_{\mathbb{P}}$ will not lie in $\Delta^{\mathrm{WAO}}_{(\mathcal{U},\mathrm{K},\mathbb{P}_T)}$ and thereby in $\boldsymbol{\delta}^*(\mathbb{P}_T)$ w.h.p., any $\boldsymbol{\delta} \in \Delta^{=}_{\mathbb{P}}$ lies in $\Delta^{\mathrm{WAO}}_{(\mathcal{U},\mathrm{K},\mathbb{P}_T)}$ a.s. for all T, and due to the constancy of G it also lies in $\boldsymbol{\delta}^*(\mathbb{P}_T)$ w.h.p.□

Among other things, these consistency results imply that the probability of two relevant types of decision errors vanishes asymptotically: (I) selecting an empirical SAO which is not a population SAO; (II) selecting a suboptimal SAO due to the false exclusion of better SAOs.

For Type I error, the reasoning is as follows. An arbitrage portfolio which is not a weak SAO, or $\boldsymbol{\delta} \notin \Delta^{\text{WAO}}_{(\mathcal{U},\text{K},\mathbb{P})}$, features at least one strict inequality $D(u,\boldsymbol{\kappa},\boldsymbol{\delta},\mathbb{P}) < 0$, for some $(u,\boldsymbol{\kappa}) \in \mathcal{U} \times \text{K}$. Since strict inequalities are robust to sampling error, the probability that such an arbitrage portfolio is falsely included in the empirical SAO set, or $\boldsymbol{\delta} \in \Delta^{\text{WAO}}_{(\mathcal{U},\text{K},\mathbb{P}_T)}$, is asymptotically negligible.

For analyzing Type II error, we face a non-vanishing probability of false exclusion of non-strict SAOs $\boldsymbol{\delta} \in \left(\Delta^{\text{WAO}}_{(\mathcal{U},\text{K},\mathbb{P})} - \Delta^{\text{SAO}}_{(\mathcal{U},\text{K},\mathbb{P})}\right)$, which feature equivalence $D(u,\boldsymbol{\kappa},\boldsymbol{\delta},\mathbb{P}) = 0$, for some $(u,\boldsymbol{\kappa}) \in \mathcal{U} \times \text{K}$. However, the neighborhood of these non-robust solutions includes strict SAOs $\boldsymbol{\delta}_T \in \Delta^{\text{SAO}}_{(\mathcal{U},\text{K},\mathbb{P})}$ which feature asymptotically strict (scaled) empirical inequalities $\sqrt{T} D(u,\boldsymbol{\kappa},\boldsymbol{\delta}_T,\mathbb{P}_T) > 0$ for all $(u,\boldsymbol{\kappa}) \in \mathcal{U} \times \text{K}$, and are therefore included in $\Delta^{\text{WAO}}_{(\mathcal{U},\text{K},\mathbb{P}_T)}$ with high probability.

When $\mathcal{U}^{(0)}_{\epsilon}$ and $\text{K}^{(0)}$ are used in place of $\mathcal{U}$ and K, i.e. $\Delta^{\text{WAO}}_{\left(\mathcal{U}^{(0)}_{n,\epsilon},\text{K}^{(0)},\mathbb{P}_T\right)}$ is the set of solutions to system (5) for $\mathbb{P}_T$ in place of $\mathbb{P}$, the results above still hold as long as $\epsilon \to 0$ as $T \to \infty$ and therefore $\mathcal{U}^{(0)}_{\epsilon}$ converges to $\mathcal{U}^{(0)}$ in the Painleve-Kuratowski topology. This is true even in the case where the latter convergence is in probability; e.g., whenever consistent estimators for a potentially latent upper bound of the support are used in the discretization.

3.2 Empirical Likelihood Ratio test

In their empirical analysis, AP23 construct arbitrage portfolios that are SAOs in a given sample, and test whether these are SAOs out of sample. This approach evaluates a single, optimized arbitrage portfolio with portfolio weights that are fixed and known out of sample, and thus avoids possible distortions of the test size that stem from evaluating multiple candidates or fitting portfolio weights to the test sample.

In this context, the natural choice of the null hypothesis seems $\mathbf{H}_0 : \boldsymbol{\delta} \in \Delta^{\text{WAO}}_{(\mathcal{U},\text{K},\mathbb{P})}$. After all, statistical theory predicts that the empirical optimal arbitrage portfolio converges to a true SAO, if pricing errors occur. Therefore, we first consider the testing of $\mathbf{H}_0$, and discuss the testing of the alternative $\mathbf{H}_1 : \boldsymbol{\delta} \notin \Delta^{\text{SAO}}_{(\mathcal{U},\text{K},\mathbb{P})}$, in the second part of this section.

The use of the hypothesis $\mathbf{H}_0$ is also consistent with common practice for pairwise dominance analysis. Standard statistical tests such as Linton, Maasoumi and Whang (2005) focus on the null of dominance, because the asymptotic distribution is degenerate under the alternative of non-dominance. This null is generalized here to multiple pairwise dominance relations that are based on multiple host portfolios.

Existing tests for pairwise dominance often rely on Kolmogorov-Smirnov or Cramer-von Mises test statistics that are straightforward empirical interpretations of the systems of dominance inequalities, and re-sampling methods that allow for asymptotically conservative and consistent inference under very general sampling schemes. These statistics are relatively inefficient for the data dimensions in our empirical application with non-overlapping low-frequency returns. In addition, re-sampling becomes computationally expensive if a large optimization problem needs to be solved for every pseudo-sample or sub-sample instead of the original sample only.

For these reasons, an alternative, computationally less demanding approach is adopted, based on a blockwise ELR statistic. A similar approach is used in the tests for non-dominance, efficiency and optimality by Davidson and Duclos (2014), Post and Poti (2017) and Post (2017). Our analysis is more general because we allow for multiple hosts portfolios, we test both the null and the alternative, and we allow for more general temporal data dependence.

A prominent role in the following analysis is played by the 'contact set', or set of the binding inequalities, $\mathrm{CS}_0 := \left\{(u, \boldsymbol{\kappa}) \in \mathcal{U}^{(0)} \times \mathrm{K}^{(0)} : D(u, \boldsymbol{\kappa}, \boldsymbol{\delta}, \mathbb{P}) = 0\right\}$. Its cardinality is denoted by $N_0(\boldsymbol{\delta}, \mathbb{P})$; this number is crucial for the approximation of the rejection region of the testing procedure. The analysis below describes sufficient conditions for finiteness of CS_0 for every member of $\Delta^{\mathrm{WAO}}_{(\mathcal{U},\mathrm{K},\mathbb{P})}$, as well as a modification of the procedure that accounts for the possibility of an infinite contact set.

To account for temporal dependence of investment returns, the time series is subdivided into $T^* := \lfloor {T-B}/{L} \rfloor + 1$ potentially overlapping blocks of B consecutive observations, $\mathcal{B}_s := \{\boldsymbol{x}_{(s-1)L+1}, \cdots, \boldsymbol{x}_{(s-1)L+B}\}$, $s = 1, \cdots, T^*$, with $L \leq B$. It is assumed that L is independent of T, and B diverges with rate slower than $\sqrt{T}$. The optimal block size depends on the context and involves a trade-off between the strength of the dynamic effects and the number of independent blocks.

Let $\mathcal{G}_T$ be the set of probability distributions on the discrete set of blocks, and $\mathbb{G}_T \in \mathcal{G}_T$ the empirical measure, or $\mathbb{G}_T(\mathcal{B}) := (T^*)^{-1} \sum_{s=1}^{T^*} \mathbb{I}\left[\mathcal{B}_s = \mathcal{B}\right]$, for any block $\mathcal{B}$. The ELR test is based on the smallest adjustments to the probability mass of $\mathbb{G}_T$ that suffice to qualify the evaluated arbitrage portfolio as a weak SAO, using the divergence from the adjusted distribution to the empirical measure of the blocks. The test statistic can be computed by solving the following Minimum Relative Entropy problem:

$$\min_{\mathbb{G}\in\mathcal{G}_T} \mathrm{KL}\left(\mathbb{G}_T \| \mathbb{G}\right) \tag{9}$$
$$\text{s.t. } \boldsymbol{\delta} \in \Delta^{\mathrm{WAO}}_{(\mathcal{U},\mathrm{K},\mathbb{G})}.$$

The Kullback-Leibler information criterion $\mathrm{KL}\left(\mathbb{G}_T \| \mathbb{G}\right) := \sum_{s=1}^{T^*} \mathbb{G}_T(\mathcal{B}_s)\ln(\mathbb{G}_T(\mathcal{B}_s)/\mathbb{G}(\mathcal{B}_s))$ is a well-known information-theoretic measure of the dissimilarity between two distributions with the same support (see Cover and Thomas (2006)). For $\epsilon > 0$, the finite system (5) is used so that (9) is approximated by an optimization problem that involves a finite system of moment inequalities:

$$\min_{\mathbb{G}\in\mathcal{G}_T} \mathrm{KL}\left(\mathbb{G}_T \| \mathbb{G}\right) \tag{10}$$
$$\text{s.t. } D(u,\boldsymbol{\kappa},\boldsymbol{\delta},\mathbb{G}) \geq 0 \; \forall (u,\boldsymbol{\kappa}) \in \mathcal{U}^{(0)}_{\epsilon} \times \mathrm{K}^{(0)},$$

where $D(u,\boldsymbol{\kappa},\boldsymbol{\delta},g) := \sum_{s=1}^{T^*} \mathbb{G}(s)\left(\frac{1}{B}\sum_{j=1}^{B} u(\boldsymbol{x}^{\mathrm{T}}_{(s-1)L+j}\left(\boldsymbol{\kappa}+\boldsymbol{\delta}\right)) - u(\boldsymbol{x}^{\mathrm{T}}_{(s-1)L+j}\boldsymbol{\kappa})\right)$.

To perform inference about $\mathbf{H}_0$, we use the blockwise ELR test statistic $\mathrm{ELR}_T = 2\frac{T}{T^*B} \cdot \mathrm{KL}\left(\mathbb{G}_T \| \mathbb{G}\left(\boldsymbol{\delta}\right)\right)$, with $\mathbb{G}\left(\boldsymbol{\delta}\right)$ the solution of (9). The exact limit null distribution of ELR is known to be a chi-bar-squared distribution, under the maintained assumptions about the data and the blocking structure. This null distribution cannot be directly implemented, because the mixing weights of the chi-bar-squared depend on the latent $\mathbb{P}$. However, we obtain an asymptotically conservative test using a majorizing distribution and statistical moment selection methods.

The chi-bar-squared is majorized by $\chi^2_{N_0(\boldsymbol{\delta},\mathbb{P})}$, a chi-squared with degrees of freedom equal to the number of binding inequalities whenever finite. The number of degrees of freedom can be estimated in a consistent way using the number of empirical moment conditions that are approximately binding, or $N_0(\boldsymbol{\delta},\mathbb{G}_T,c_T) := \mathrm{card}\left\{(u,\boldsymbol{\kappa}) \in \mathcal{U}^{(0)}_{\epsilon} \times \mathrm{K}^{(0)} : |D(u,\boldsymbol{\kappa},\boldsymbol{\delta},\mathbb{G}_T)| \leq c_T\right\}$, where $c_T > 0$ is a tolerance parameter that converges to zero at an appropriate rate.

Thus, asymptotically conservative p-values can be taken from $\chi^2_{N_0(\boldsymbol{\delta},\mathbb{G}_T,c_T)}$. This conservative approach guarantees that the estimated p-value is smaller than or equal to the nominal significance level, so that the test size, or the probability of false rejection of a SAO is under control, in large samples.

The following result derives the limit theory of the testing procedure defined by (10) and the rejection of $\mathbf{H}_0$ iff $\mathrm{ELR}_T \geq q\left(1-\alpha, \chi^2_{N_0(\boldsymbol{\delta},\mathbb{P}_T,c_T)}\right)$, where $q\left(1-\alpha, \chi^2_{N_0(\boldsymbol{\delta},\mathbb{P}_T,c_T)}\right)$ denotes the $1-\alpha$ quantile of the stochastic distribution $\chi^2_{N_0(\boldsymbol{\delta},\mathbb{P}_T,c_T)}$ for $\alpha \in (0,1)$. The result shows asymptotic conservatism and consistency.

Theorem 3.6 *Suppose that Assumption 3.1 holds, and, (a) CS is finite, (b) there exists some $\varepsilon > 0$ such that,*

$$\lambda_{\min}(\mathcal{V}) > \varepsilon,$$

where $\lambda_{\min}(\mathcal{V})$ denotes the minimum eigenvalue of

$$\mathcal{V} := \mathbb{E}\left[\left(u\left(\boldsymbol{x}_0^{\mathrm{T}}(\boldsymbol{\kappa}+\boldsymbol{\delta})\right) - u\left(\boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{\kappa}\right)\right)\left(u^\star\left(\boldsymbol{x}_0^{\mathrm{T}}(\boldsymbol{\kappa}^\star+\boldsymbol{\delta})\right) - u^\star\left(\boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{\kappa}^\star\right)\right)^{\mathrm{T}}\right]_{(u,\boldsymbol{\kappa}),(u^\star,\boldsymbol{\kappa}^\star)\in\mathrm{CS}_0},$$

(c) the tolerance parameter satisfies $c_T \to 0$, while, $\sqrt{T}c_T \to +\infty$ almost surely, (d) the block size satisfies $B \to +\infty$ and $B = O(T^\rho)$ for $0 < \rho < \frac{1}{2}$ and L is independent of T, and (e) $\epsilon \to 0$ with T, so that $\mathcal{U}_\epsilon^{(0)}$ converges to to a non-stochastic dense subset of $\mathcal{U}^{(0)}$ in probability, and card $\left(\mathcal{U}_\epsilon^{(0)}\right) = o_p(\frac{T}{B})$.
Then, the following results are obtained: (i)

$$\mathrm{ELR}_T \rightsquigarrow \begin{cases} \inf_{v\in\mathbb{R}_+^{N_0(\boldsymbol{\delta},\mathbb{P})}} (\mathbb{C}-v)^{\mathrm{T}}\mathcal{V}_{\mathbb{C}}^{-1}(\mathbb{C}-v), & \mathbf{H}_0 \wedge N_0(\boldsymbol{\delta},\mathbb{P}) \neq 0 \\ 0, & \mathbf{H}_0 \wedge N_0(\boldsymbol{\delta},\mathbb{P}) = 0 \\ +\infty, & \mathbf{H}_1 \end{cases}, \tag{11}$$

where $\mathbb{C}$ is a zero-mean Gaussian vector with covariance matrix

$$\mathcal{V}_{\mathbb{C}} := \mathcal{V} + 2\sum_{t=1}^{\infty}\mathbb{E}\left[\left(u\left(\boldsymbol{x}_0^{\mathrm{T}}(\boldsymbol{\kappa}+\boldsymbol{\delta})\right) - u\left(\boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{\kappa}\right)\right)\left(u^\star\left(\boldsymbol{x}_t^{\mathrm{T}}(\boldsymbol{\kappa}^\star+\boldsymbol{\delta})\right) - u^\star\left(\boldsymbol{x}_t^{\mathrm{T}}\boldsymbol{\kappa}^\star\right)\right)^{\mathrm{T}}\right]_{(u,\boldsymbol{\kappa}),(u^\star,\boldsymbol{\kappa}^\star)\in\mathrm{CS}};$$

(ii) Under $\mathbf{H}_0 \wedge N_0(\boldsymbol{\delta}, \mathbb{P}) \neq 0$,

$$\limsup_{T\to\infty} \mathbb{P}\left(\mathrm{ELR}_T \geq q\left(1-\alpha, \chi^2_{N_0(\boldsymbol{\delta},\mathbb{P}_T,c_T)}\right)\right) \leq \alpha, \tag{12}$$

while under $\mathbf{H}_0 \wedge N_0(\boldsymbol{\delta}, \mathbb{P}) = 0$,

$$\lim_{T\to\infty} \mathbb{P}\left(\mathrm{ELR}_T \geq q\left(1-\alpha, \chi^2_{N_0(\boldsymbol{\delta},\mathbb{P}_T,c_T)}\right)\right) = 0; \tag{13}$$

(iii) Under $\mathbf{H}_1$,

$$\lim_{T\to\infty} \mathbb{P}\left(\mathrm{ELR}_T \geq q\left(1-\alpha, \chi^2_{N_0(\boldsymbol{\delta},\mathbb{P}_T,c_T)}\right)\right) = 1. \tag{14}$$

Proof of Theorem 3.6. (11) follows as in the proof of Theorem 4.2.1 of Arvanitis, Post, Poti and Karabati (2021). Consider the case $\mathrm{H}_0 \wedge N_0(\boldsymbol{\delta}, \mathbb{P}) \neq 0$ Uniformly w.r.t. the $(i,j) \notin \mathrm{CS}_0$, it is found that, due to the definition of the tolerance parameter and the Birkhoff's ULLN, $\mathbb{E}_{\mathbb{G}_T}\left[u\left(\boldsymbol{x}_0^{\mathrm{T}}(\boldsymbol{\kappa}+\boldsymbol{\delta})\right) - u\left(\boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{\kappa}\right)\right] > c_T$, eventually, almost surely. Using Skorokhod representations, we also have that, since $\sqrt{T}c_T$ diverges to infinity almost surely, uniformly over the CS_0, $\left|\sqrt{T}\mathbb{E}_{\mathbb{G}_T}\left[u\left(\boldsymbol{x}_0^{\mathrm{T}}(\boldsymbol{\kappa}+\boldsymbol{\delta})\right) - u\left(\boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{\kappa}\right)\right]\right| \leq \sqrt{T}c_T$, eventually, almost surely. The previous along with (e) imply that $N_0(\boldsymbol{\delta}, \mathbb{P}_T, c_T) \rightsquigarrow N_0(\boldsymbol{\delta}, \mathbb{P})$, jointly with ELR_T, and thereby we obtain that

$$\mathrm{ELR}_T \rightsquigarrow \inf_{v\in\mathbb{R}_+^{N(\boldsymbol{\delta},\mathbb{P})}} (\mathbb{C}-v)^{\mathrm{T}} \mathcal{V}_{\mathbb{C}}^{-1} (\mathbb{C}-v)$$

$$= \inf_{v\in\mathbb{R}_+^{N(\boldsymbol{\delta},\mathcal{F})}} \begin{bmatrix} \mathbb{C}^{\mathrm{T}}\mathcal{V}_{\mathbb{C}}^{-1}\mathbb{C} \\ -\inf_{v\in C^o} (\mathbb{C}-v)^{\mathrm{T}} \mathcal{V}_{\mathbb{C}}^{-1} (\mathbb{C}-v) \end{bmatrix} \leq \mathbb{C}^{\mathrm{T}}\mathcal{V}_{\mathbb{C}}^{-1}\mathbb{C},$$

due to Proposition 3.4.1 of Silvapulle and Sen (2005), where C^o denotes the polar cone of $\mathbb{R}_+^{N_0(\boldsymbol{\delta},\mathbb{P})}$. Then the Portmanteau Theorem establishes (12). For the case $\mathbf{H}_0 \wedge N_0(\boldsymbol{\delta}, \mathbb{P}) = 0$, (11) implies that ELR_T is eventually zero w.h.p., hence (13) follows. Furthermore, under the alternative, the proof of Theorem 4.2.1 of Arvanitis, Post, Poti and Karabati (2021) implies that $\mathrm{ELR}_T \geq O_p(\frac{T}{B})$. Thereby the growth condition on the approximation of $\mathcal{U}$ along with (c), and via the use of Skorokhod representations, imply that the modified statistic diverges to infinity, while the quantile $q\left(1-\alpha, \chi^2_{N_0(\boldsymbol{\delta},\mathbb{P}_T,c_T)}\right)$ is almost surely bounded, hence (14) follows.□

For any $\boldsymbol{\delta} \in \Delta^{\text{WAO}}_{(\mathcal{U},\text{K},\mathbb{P})}$ that does not belong to the generic set of SAOs with finite contacts discussed above, the contact set finiteness condition (a) holds whenever $D(\cdot, \boldsymbol{\kappa}_j, \boldsymbol{\delta}, \mathbb{P})$ is analytic in the Russell-Seo threshold parameter for every extreme point $\boldsymbol{\kappa}_j$. Analyticity would be in turn obtained if $\mathbb{P}$ has an analytic density, due to our bounded support framework (see the relevant analysis after Assumption 2 in Arvanitis, Scaillet and Topaloglou (2023)).

Under the weaker assumption of a continuously differentiable density, another path for obtaining (a) is to ensure that the Hessian of $D(u_{n,\phi}, \boldsymbol{\kappa}_i, \boldsymbol{\delta}, \mathbb{P})$ w.r.t. ϕ has a finite number of zeros. For example, when $\mathcal{U} = \mathcal{U}_2$ and $\mathcal{U}^{(0)}$ is the set of Russell-Seo elementary utilities, following the analysis in Arvanitis, Scaillet and Topaloglou (2023), we obtain that the Hessian equals $\mathbb{P}(\{\phi = (\boldsymbol{\kappa}_i + \boldsymbol{\delta})\mathbf{x}\}) - \mathbb{P}(\{\phi = \boldsymbol{\kappa}_i\mathbf{x}\})$. Using then the diffeo-geometric analysis in Paragraph 5 of Kim and Pollard (1990), we obtain that the Hessian is not zero whenever $\mathbf{x}^{\mathrm{T}}\boldsymbol{\delta}$ is of the same sign for almost every value of $\mathbf{x}$ on the boundary of $\{\phi = \kappa_i\mathbf{x}\}$ locally uniformly in the null hypothesis.

Given (a), the minimum eigenvalue condition (b) holds whenever the random vector $\left(u\left(\boldsymbol{x}_0^{\mathrm{T}}\left(\boldsymbol{\kappa}, \boldsymbol{\delta}\right)\right) - u\left(\boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{\kappa}\right)\right)_{\text{CS}_0}$ consists of linearly independent random variables. The restriction bears similarity with forms of restricted Stochastic Dominance; utilities that would for example produce trivial contacts are excluded from the analysis, which is then carried out with a restricted subset of $\mathcal{U}$. Indicatively, whenever $\mathcal{U}$ is the set of Russell and Seo (1989) utilities, then the fact that the population support infimum a is finite along with a continuity argument imply that there exists some $\delta(\epsilon) > 0$ such that every utility corresponding to any threshold $z \in [a, a + \delta(\epsilon)]$ that also appears in CS_0, should be excluded from the analysis in order for Theorem 3.6 to hold. The Lipschitz continuity property of the utilities involved, along with that when ϵ exists, it can be chosen arbitrarily small, implies also that $\delta(\epsilon)$ can be chosen as small as desired. Practically, and under Assumption 3.1, such a restriction can be implemented when $\delta(\epsilon)$ is (a) chosen so that the random element $\left(\left(z - \boldsymbol{x}_0^{\mathrm{T}}\left(\boldsymbol{\kappa} + \boldsymbol{\delta}\right)\right)_+ - \left(z - \boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{\kappa}\right)_+\right)_{K^0,\, z \geq a_T + \delta}$, where a_T is the empirical support infimum, has a non singular empirical covariance matrix, and (b) the excluded Rusell-Seo utilities are immaterial for the analysis.

The restriction of the asymptotic behavior of the tolerance parameter (c) is usual in the econometric literature; see Andrews and Soares (2010) and references therein. The restriction on the block size divergence rate (d) is also standard (see, for example, Thm. 3 of Kitamura (1997)).

The optimal choice of block size depends crucially on the finer details of the temporal-dependence structure of the underlying time series processes (see, for example, Section 2 of Nordman, Bunzel and Lahiri (2013). The specification can be guided by theoretical as well as empirical characteristics of the underlying data and simulation analyses; e.g., the block size B can be chosen by some empirical variance minimization method such as the one in El Ghouch, Van Keilegom and McKeague (2011). When $\mathcal{U}^0 = \mathcal{U}^0_n$, condition (e) is implementable when for example, the empirical support is partitioned into $\lfloor ({}^{T}\!/_{B})^{\alpha} \rfloor$ sub-intervals, for some $\alpha < 1$, and the Russell-Seo thresholds are chosen as the boundaries of those sub-intervals.

The majorizing arguments in the construction of the critical values, as well as the restrictions on the limiting behavior of the tolerance parameter ensure that the test will be asymptotically conservative in both the degenerate and the non-degenerate cases.

The test is consistent under the alternative hypothesis, due to the divergence to infinity of the test statistic and the boundedness from above of the quantiles used as critical values.

Whenever (a) fails, a simple modification of the test statistic via the estimated number of contacts $N_0(\boldsymbol{\delta}, \mathbb{P}_T, c_T)$ would lead to a standard normal limiting null bound distribution. Specifically, performing the test based on the rejection rule

$$(\mathrm{ELR}_T - N_0(\boldsymbol{\delta},\mathbb{P}_T,c_T))\Big/\sqrt{2N_0(\boldsymbol{\delta},\mathbb{P}_T,c_T)} > \Big(q(1-\alpha,\chi^2_{N_0(\boldsymbol{\delta},\mathbb{P}_T,c_T)}) - N_0(\boldsymbol{\delta},\mathbb{P}_T,c_T)\Big)\Big/\sqrt{2N_0(\boldsymbol{\delta},\mathbb{P}_T,c_T)}$$

can be proven to satisfy (ii) and (iii) of the previous result, if V has a spectrum bounded away from zero, (c), (d), and (e) hold. This is obvious whenever CS_0 is finite from Theorem 3.6. Whenever CS_0 is infinite, then the spectrum condition and the fact that $\mathcal{U}^{(0)}_{\epsilon}$ is finite for any T, imply that under the null hypothesis the modified statistic will be bounded above by a standard normal distribution. The lhs of the rejection rule can be shown to converge in distribution to the $1-\alpha$ quantiles of the standard normal. Furthermore, under the alternative, the proof of Theorem 4.2.1 of Arvanitis, Post, Poti and Karabati (2021) implies that $\mathrm{ELR}_T \geq O_p(\frac{T}{B})$. Thereby the growth condition on the approximation of $\mathcal{U}$ along with (c), and via the use of Skorokhod representations, imply that the modified statistic diverges to infinity. Hence, under the particular restriction on the growth rate of $\mathcal{U}^{(0)}_{\epsilon}$ we obtain a robust modification of the original test that avoids (a).

The hypothesis $\mathbf{H}_0$ generally allows for more (locally) powerful tests than the alternative $\mathbf{H}_1 : \boldsymbol{\delta} \notin \Delta^{\mathrm{WAO}}_{(\mathcal{U},\mathrm{K},\mathbb{P})}$ or $\mathbf{H}_1 : \boldsymbol{\delta} \notin \Delta^{\mathrm{SAO}}_{(\mathcal{U},\mathrm{K},\mathbb{P})}$, because estimation error generally is more likely to yield false non-dominance classifications than false dominance relations, as the system of dominance inequalities is 'easier to break than to make'. Conservatism however could imply poor power performance on elements of the boundary between the two hypotheses; a possible local lack of power of the BELR test for the null could be mitigated using a second test for the alternative.

The analysis is hence completed with the design of an ELR test for the alternative hypothesis $\mathbf{H}_1 : \boldsymbol{\delta} \notin \Delta^{\text{SAO}}_{(\mathcal{U},\text{K},\mathbb{P})}$. For finite $\mathcal{U}^{(0)}_{\epsilon}$ and $\text{K}^{(0)}$, the condition $\boldsymbol{\delta} \in \Delta^{\text{SAO}}_{(\mathcal{U},\text{K},\mathbb{P})}$ amounts to a finite system: $D(u, \boldsymbol{\kappa}, \boldsymbol{\delta}, \mathbb{P}) > 0$, $\forall(u, \boldsymbol{\kappa}) \in \mathcal{U}^{(0)}_{\epsilon} \times \text{K}^{(0)}$. The condition $\boldsymbol{\delta} \notin \Delta^{\text{SAO}}_{(\mathcal{U},\text{K},\mathbb{P})}$ requires violations of this system. Those would imply degenerate asymptotic distributions for Kolmogorov-Smirnov or Cramer-von Mises type procedures-see for example Linton, Maasoumi and Whang (2005). Violations can however be additionally characterized as solutions to the alternative system $\sum_{u\in\mathcal{U}^{(0)}_{\epsilon}} \sum_{\boldsymbol{\kappa}\in\text{K}^{(0)}} \boldsymbol{\beta}_{u,\boldsymbol{\kappa}} D(u, \boldsymbol{\kappa}, \boldsymbol{\delta}, \mathbb{P}) \leq 0$; $\sum_{u\in\mathcal{U}^{(0)}_{\epsilon}} \sum_{\boldsymbol{\kappa}\in\text{K}^{(0)}} \boldsymbol{\beta}_{u,\boldsymbol{\kappa}} = 1$; $\boldsymbol{\beta}_{u,\boldsymbol{\kappa}} \geq 0$ $\forall(u, \boldsymbol{\kappa}) \in \mathcal{U}^{(0)}_{\epsilon} \times \text{K}^{(0)}$. The existence of binding inequalities for the alternative system could imply non-degeneracies.

Using the block structure above, the relevant Relative Entropy problem follows:

$$\begin{gathered}\min_{\mathbb{G}\in\mathcal{G}_T} \text{KL}\left(\mathbb{G}_T \| \mathbb{G}\right) \\ \text{s.t.} \sum_{u\in\mathcal{U}^{(0)}_{\epsilon}} \sum_{\boldsymbol{\kappa}\in\text{K}^{(0)}} \boldsymbol{\beta}_{u,\boldsymbol{\kappa}} D(u, \boldsymbol{\kappa}, \boldsymbol{\delta}, \mathbb{G}) \leq 0; \\ \sum_{u\in\mathcal{U}^{(0)}_{\epsilon}} \sum_{\boldsymbol{\kappa}\in\text{K}^{(0)}} \boldsymbol{\beta}_{u,\boldsymbol{\kappa}} = 1; \boldsymbol{\beta}_{u,\boldsymbol{\kappa}} \geq 0; \forall(u, \boldsymbol{\kappa}) \in \mathcal{U}^{(0)}_{\epsilon} \times \text{K}^{(0)}.\end{gathered} \tag{15}$$

This problem is bi-linear in $(\boldsymbol{\beta}, \mathbb{G})$ and hence bi-convex. It can be solved using an Alternating Direction Method of Moments (ADMM) algorithm that alternates between optimizing w.r.t. $\boldsymbol{\beta}$ and optimizing w.r.t. $\mathbb{G}$.

The contact set associated with testing the alternative hypothesis and is now defined as $\text{CS}_1 := \left\{\boldsymbol{\beta} : \sum_{u\in\mathcal{U}^{(0)}_{\epsilon}} \sum_{\boldsymbol{\kappa}\in\text{K}^{(0)}} \boldsymbol{\beta}_{u,\boldsymbol{\kappa}} D(u, \boldsymbol{\kappa}, \boldsymbol{\delta}, \mathbb{P}) = 0\right\}$, plays an analogous prominent role in the analysis. Here, non-finiteness of the contact set is more common than for testing the null. Thus, inference is performed by the modification of the empirical likelihood statistic via translation and scaling by the number of empirical contacts and moreover by a finite approximation of the simplex in which parameter $\boldsymbol{\beta}$ attains its values. The rejection rule used is:

$$\left(\mathrm{ELR}_T^{\star}-N_1(\boldsymbol{\delta},\mathbb{P}_T,c_T)\right)/\sqrt{2N_1(\boldsymbol{\delta},\mathbb{P}_T,c_T)} > \left(q(1-\alpha,\chi^2_{N_1(\boldsymbol{\delta},\mathbb{P}_T,c_T)})-N_1(\boldsymbol{\delta},\mathbb{P}_T,c_T)\right)/\sqrt{2N_1(\boldsymbol{\delta},\mathbb{P}_T,c_T)},$$

where $\mathrm{ELR}_T^{\star} = 2\frac{T}{T^*B}\cdot \mathrm{KL}\left(\mathbb{G}_T \| \mathbb{G}^{\star}\left(\boldsymbol{\delta}\right)\right)$, with $\mathbb{G}^{\star}\left(\boldsymbol{\delta}\right)$ the solution of (15), and, similarly to the previous, $N_1(\boldsymbol{\delta}, \mathbb{P}_T, c_T) := \mathrm{card}\left\{\boldsymbol{\beta} \in S_T : \left|\sum_{u\in\mathcal{U}_\epsilon^{(0)}} \sum_{\boldsymbol{\kappa}\in \mathrm{K}^{(0)}} \boldsymbol{\beta}_{u,\boldsymbol{\kappa}} D(u,\boldsymbol{\kappa},\boldsymbol{\delta},\mathbb{G}_T)\right| \leq c_T\right\}$ is the number of empirical contacts, and S_T a (potentially stochastic) finite discretization of the $(\#(\mathcal{U}_\epsilon^{(0)} \times K^0) - 1)$-simplex, that converges in probability to a dense subset of the $(\#(\mathcal{U}^{(0)} \times K^0) - 1)$-simplex as $T \to \infty$.

The limit theory of this testing procedure is derived in the following result.

Theorem 3.7 *Suppose that Assumption 3.1, and conditions (c) and (d) of Theorem 3.6 hold,* $\epsilon \to 0$ *with* T*, while,* $\mathcal{U}_\epsilon^{(0)}$ *converges in the Painleve-Kuratowski topology to a non-stochastic dense subset of* $\mathcal{U}^{(0)}$ *in probability,* S_T *converges in probability in the Painleve-Kuratowski topology to a dense non-stochastic subset of the* $(\#(\mathcal{U}_\infty^{(0)} \times K^0) - 1)$*-simplex, and the spectrum of* $\mathrm{Var}\left[\sum_{u\in\mathcal{U}_\epsilon^{(0)}} \sum_{\boldsymbol{\kappa}\in \mathrm{K}^{(0)}} \boldsymbol{\beta}_{u,\boldsymbol{\kappa}} \left(u\left(\boldsymbol{x}_0^{\mathrm{T}}\left(\boldsymbol{\kappa}+\boldsymbol{\delta}\right)\right) - u\left(\boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{\kappa}\right)\right)\right]_{\boldsymbol{\beta}}$, $\boldsymbol{\beta} \in \left\{\boldsymbol{\beta} : \sum_{u\in\mathcal{U}_\infty^{(0)}} \sum_{\boldsymbol{\kappa}\in \mathrm{K}^{(0)}} \boldsymbol{\beta}_{u,\boldsymbol{\kappa}} D(u_j,\boldsymbol{\kappa}_i,\boldsymbol{\delta},\mathbb{P}) = 0\right\}$ *is bounded away from zero, where* $\mathcal{U}_\infty^{(0)}$ *denotes the Painleve-Kuratowski limit in probability of* $\mathcal{U}_\epsilon^{(0)}$*. Then, the following results are obtained:*

(i) Under $\mathbf{H}_0$ *and if population contacts exist,*

$$\limsup_{T\to\infty} \mathbb{P}\left(\left(\mathrm{ELR}_T^{\star}-N_1(\boldsymbol{\delta},\mathbb{P}_T,c_T)\right)/\sqrt{2N_1(\boldsymbol{\delta},\mathbb{P}_T,c_T)} > \left(q(1-\alpha,\chi^2_{N_1(\boldsymbol{\delta},\mathbb{P}_T,c_T)})-N_1(\boldsymbol{\delta},\mathbb{P}_T,c_T)\right)/\sqrt{2N_1(\boldsymbol{\delta},\mathbb{P}_T,c_T)}\right) \leq \alpha, \tag{16}$$

while under $\mathbf{H}_0$ *and if population contacts do not exist,*

$$\lim_{T\to\infty} \mathbb{P}\left(\left(\mathrm{ELR}_T^{\star}-N_1(\boldsymbol{\delta},\mathbb{P}_T,c_T)\right)/\sqrt{2N_1(\boldsymbol{\delta},\mathbb{P}_T,c_T)} > \left(q(1-\alpha,\chi^2_{N_1(\boldsymbol{\delta},\mathbb{P}_T,c_T)})-N_1(\boldsymbol{\delta},\mathbb{P}_T,c_T)\right)/\sqrt{2N_1(\boldsymbol{\delta},\mathbb{P}_T,c_T)}\right) = 0; \tag{17}$$

(ii) Under $\mathbf{H}_1$*, and if moreover card* $\left(\mathcal{U}_\epsilon^{(0)}\right) = o_p(\frac{T}{B})$*,*

$$\lim_{T\to\infty} \mathbb{P}\left(\left(\mathrm{ELR}_T^{\star}-N_1(\boldsymbol{\delta},\mathbb{P}_T,c_T)\right)/\sqrt{2N_1(\boldsymbol{\delta},\mathbb{P}_T,c_T)} > \left(q(1-\alpha,\chi^2_{N_1(\boldsymbol{\delta},\mathbb{P}_T,c_T)})-N_1(\boldsymbol{\delta},\mathbb{P}_T,c_T)\right)/\sqrt{2N_1(\boldsymbol{\delta},\mathbb{P}_T,c_T)}\right) = 1. \tag{18}$$

Proof of Theorem 3.7. Analogous to the proof of Theorem 3.6, the distance between the number of contacts in S_t and $N^{\star}$ can be shown to converge to zero in probability. Using this, and arguments analogous to the ones concerning empirical process convergence, Skorokhod representations and bounds on infima of quadratic forms over cones in the proof of Theorem 4.2.1 of Arvanitis, Post, Poti and Karabati (2021), under the null, it can then be shown that the test statistic is asymptotically bounded from above by a random variable that weakly converges to a standard normal. Then (16) follows. (17) follows trivially when the contact set is empty. Under the alternative, the relevant part of the proof of Theorem 4.2.1 of Arvanitis, Post, Poti and Karabati (2021) implies that $\mathrm{ELR}_T^{\star} \geq O_p(\frac{T}{B})$. Thereby the growth condition on the approximation of $\mathcal{U}$ along with (c), imply that the test statistic diverges to infinity.□

The resulting test is also asymptotically conservative in both the degenerate and the non-degenerate cases, and consistent. The test extends the ELR test of Davidson and Duclos (2013) as it allows for non-singleton K and temporal dependence for the underlying stochastic processes.

4 Discussion

Generalizing the work of Post, Karabatı and Arvanitis (2018), we obtained versions of set consistency for the empirical SAOs. We avoided the use of (arbitrary) tolerance parameters in the empirical counterpart of the portfolio optimization problem. The derivations are novel; they used a set of approximate solutions of the empirical optimization problem that satisfy restricted versions of the empirical dominance inequalities.

We designed a procedure for statistical inference about membership of the SAO set via Blockwise Empirical Likelihood Ratio (BELR) tests. Two Empirical Likelihood Ratio tests were developed: one for the null that a given arbitrage portfolio is a SAO and another one for the alternative that the portfolio is not a SAO. The proposed conservative test strategy for the null is aimed at the validation of the theoretical prediction that the empirical optimal arbitrage portfolio is a true SAO if true SAOs exist. It controls the test size or the frequency of false rejection of a SAO. Conservatism however could imply poor performance on elements of the boundary between the two hypotheses; a possible local lack of power of the BELR test for the null could be mitigated using the second BELR test for the alternative. Refinements of the BELR based test strategies with better power under local alternatives are the subject of future research.

We also discussed new robust versions of the testing procedures that allow for the possibility that the set of binding dominance inequalities is infinite.

The focus has been on the empirical counterpart of the unconditional distribution. Extensions are possible to cases where $\mathbb{P}$ is a conditional distribution and/or estimated using alternative (and possibly more statistically efficient) non-parametric methods or (semi-)parametric models. The estimation may be based on a well-specified (semi-)parametric model or a fully non-parametric method, depending on the data properties and available side information. (Quasi) Maximum Likelihood or Nadaraya-Watson type estimators could be employed, respectively. Standard assumptions that involve smoothness of the associated likelihood function, or regularity properties of the unknown densities involved and the associated kernels, would suffice to obtain consistency. Parametric Likelihood Ratio tests and Smoothed Empirical Likelihood tests can replace the BELR test in these cases.

Although the motivation for this study was to handle multiple host portfolios, the analysis naturally also applies for the common special case in which K is a singleton and the ELR test becomes a test for pairwise dominance. Although the test is conservative, it may prove to have advantages compared to existing tests for pairwise dominance based on statistical resampling methods; e.g., the subsampling tests in Linton, Maasoumi and Whang (2005). For example, if the data are iid, the results of Canay (2010) (see Th. 3.2) and of Dembo and Zeitouni (2009) (see Th. 7.1.3) seem to imply large deviations advantages of our ELR test-at least for subsets of the alternative hypothesis that are sufficiently "far" from the null of being a SAO-compared to the existing tests.

Extending the comparison between our conservative BELR tests and resampling methods to non iid settings is of interest, given the data dependence in typical investment applications and the flexibility of the proposed BELR apparatus. In future work, we therefore hope to establish of a large-deviations property for blocks of stationary mixing data with a rate function (see, for example, Dembo and Zeitouni (2009)) that directly depends on the KL divergence between distributions on a space of limiting blocks. The form of the rate function (if it exists) could be connected to some sort of large-deviations optimality of the conservative BELR approach for appropriate elements of the alternative hypothesis.

Regardless of the relative efficiency of BELR tests with conservative chi-squared rejection regions compared to resampling methods, the BELR tests have a computational advantage for analyzing sets of portfolios, as the relevant optimization problem has to be solved only once instead of for a multitude of pseudo-samples or sub-samples. This seems an important advantage in large-scale applications and simulations.

The test for the alternative avoids in some cases degeneracy for the limiting alternative distribution by focusing on an alternative set of inequalities given by convex combinations of the original stochastic dominance system. It extends the ELR test for pairwise non-domimance of Davidson and Duclos (2013), since it allows for non-singleton hypotheses as well as stationarity and mixing for the stochastic processes involved.

The analysis also applies to alternative application areas where sets of risky alternatives are compared based on time-series estimates of the joint distribution functions. One such application area is the evaluation and combination of forecast models in Forecasting; see, e.g., Arvanitis et al. (2021). Encouragingly, the use of non-monotonic loss functions (outside $\mathcal{U}_2$) in forecasting does not affect the validity of our derivations and arguments.

Another interesting route for further research is to adjust the analysis to the case where the underlying probability distribution is estimated using cross-sectional data such as the comparison and combination of empirical well-being distributions in Well-being Analysis; see, e.g., Anderson, Post and Wang (2020).

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