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A Consolidation of the Neoclassical Macroeconomic Competitive General Equilibrium Theory via Keynesianism (Part 1 and Part 2)

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Angelos Angelopoulos[1]

Abstract

The unique existence of a Keynes (1936) [- Harrod (1939) - Domar (1946)] - Sollow (1956) - Swan (1956) - Phelps (1961, 1965) dynamic Walrasian (1874) [or (perfectly) competitive general equilibrium in the philosophy of Arrow-Debreu (1954) and McKenzie (1954, 1959)] is proven. The same is afterwards accomplished for the Ramsey (1928) - Cass (1965) - Koopmans (1965) case. Both projects are executed in an atypical (to macroeconomics) way. It is finally shown that when the first one of the previous two Walrasian general equilibrium notions is specified (and collapses) into to its Keynesian (1936) general equilibrium illustration (and only), the two competitive general equilibrium concepts coincide.

Key words: Growth, Competitive General Equilibrium, Existence, Uniqueness, Equivalence.

Classification: D5, O40

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0. Preface

This paper consists of 2 parts, Part 1 and Part 2. Part 1 contains section 1 and section 2 of the paper. Part 2 is comprised of sections 3, 4 and 5 of the paper. This is Part 1.

1. Introduction

The truth is that Economics gain public recognition through macroeconomics. In a world of (socio-political) economic pluralism, for the sake of accuracy, macroeconomics build and study macro-economies.

There are, in point of fact, two genres of solidly justifiable reasons to instigate a macro-economy, one of applied and one of theoretical ones.

In practice, at least in the modern economic world where efficacious macro-economic policy is not a serendipity, macro-economies are foremostly advertised by their creators as being evidence based and, because of this, destined to correctly inform the policy maker.

In theory howbeit, when looking behind the scenes, macro-economies are simplified maquettes of perplex humanitarian societies, which means that macro-economies are basically nothing else than pure theoretical and elementarily instructive constructions.

Yet, when macro-economies (as utter notional establishments) come into their analytical climax, they are eventually found to be designed with quite fancy and, occasionally, overwhelming actuarial techniques, as a means of acting as unbiased estimators of reality, by relying onto that fundamental principle of positivism which avers that by enhancing a macro-economy as a mathematical-economic model one increases the chances of its credibility.

In the end, however, regardless of how much strenuous a macroeconomic analysis can become, macroeconomic (as all economic) solutions are doomed to be approximately gauged and established, with simulation and perturbation mathematical economic methods.

Tersely and corporately speaking therefore, high-technically constructed macro-economies are, above all, virtual structures and remain, in essence, simplistically made up conceptual artefacts, which become, in the end, economic vessels for empirical testing and laboratorial fields for policy experiments, meaning (or, at least, aspiring) in this multiple identity-and-intent way to be good and reliable proxies of the real-life world's sovereign countries, nations, states, commonwealths and the like territorial regions, when the aforementioned peripheries are (specifically and exclusively) taken with their economic facet.

So, by putting aside (and irrespectively of) all the credences and discrepancies above, the ultimate focal point, scope and worth of any coherent macroeconomic investigation is to approximate and represent economic communities as totalities, in which the whole group's mutual, collective or public interest is to be eventually pursued and served (and not necessarily and unilaterally by a governmental agent).

Still, this is the concern and job of a general equilibrium theorist, who principally works along the spectrum of microeconomics.

Even so, by renouncing or undermining the value of microeconomics as being faceless, one could struggle to establish macroeconomics as self-standing in the discipline of Economics.

Clearly, then, this would have ended being a void and paradoxic attempt, simply because to examine the whole the unit is needed.

Thence, the nexus of macro-economies with micro-economies is vigorous and indispensable. The two of them are structurally connected. The first are viable only through and because of the second.

When, particularly, one moves behaviourally (thus, progressively in the field of humanitarian sciences) from the individual to the aggregate level, (macro) economies become plexuses of (fictional) agents and markets. To this end, by mimicking the microeconomic paradigm, they get populated by artificial economic entities, while they get partitioned into fabricated commercial venues. To put it differently, (macro) economies get elevated into (mathematically polished) economic networks of interacting agents, inside interacting markets (see also in **Comment 0**, in the Auxiliary Text of the Appendix).

Ideally then, as in the small-scale case of microeconomics, a complete (formal) macroeconomic exploration should be conducted across time, across (national and international, with localities being taken into account) geographical space and under uncertainty (hence, upon randomness and private and imperfect knowledge as well).

Time rolls throughout this paper, but the economy's spatial dimension is frozen, while a veil of certainty underpins it. This is because aspatial and deterministic dynamic macroeconomic systems of agents and markets can be handled more easily.

Their manipulation facilitates the first-hand and neat derivation of the core results of this theory. Once these systems are upgraded and enriched with more intricacies concerning vicinity and luck, the analysis does become more generous and powerful, but its fundamental results become blurry, harder to be extracted.

Now, an economy's aggregate product, equivalently, its aggregate income, becomes just about the major issue of concern in macroeconomics, and especially in growth theory, which leads the way in macroeconomics (see also in **Comment 1**, in the Auxiliary Text of the Appendix[2]).

Aggregate production-to-supply side macroeconomics, in particular, highlight the role of the factors of production in an economy's growth (i.e., its output expansion), with capital-accumulated, labour and technology being the three most prominent sources (and accelerators) of production, to wit, the three salient inputs that are combined (and feasibly technologically constrained) into/by an exogenous atemporal production function (equivalently, production technology)[3].

Unequivocally, although all schools of macroeconomics have (here and there) mentioned and (in one way or another) recorded the chief role of production (and supply) in the economy's size, such macroeconomics appear most usually, most bluntly and most systematically with the (behavioural) neoclassical dressing.

In neoclassical growth theory, as a general conclusive overview after what the largest samples of the relevant literature's data, history, information and knowledge have documented and descriptively inferred, one may induce that an economy's output grows with time as, preponderantly, capital-accumulated grows with time, but with the (desirably enlarging) capital-stock being eroded and slowed-down according to an exogenous time-rule and via a given depreciation factor, i.e., through a time invariant coefficient applied to capital-accumulation along time.

This paper remains silent as far as neoclassical technology and its advancement (i.e., the technological progress or, more generally, the technical change) is concerned when a production function is put into force, so, in this production mechanism, capital-accumulated (and labour) are reported as quantitative production factors exclusively, and the quality of their quantities is not taken into account for

[2]See in the Auxiliary Text of the Appendix.
[3]See in the Auxiliary Text of the Appendix.

growth purposes (only their quantities, volumes or magnitudes are, and do matter for growth). As usually, in contrast to the economy's (illiquid) asset, labour (either measured in manpower or in time devoted to work) does not decay over time from being employed into the production practice, neither gets accumulated.

In all the mainstream macroeconomic theories, at each time instant, when the macroeconomic system of agents and markets is in balance, there is no unemployment or underemployment of the economy's factors of production (beyond the unaccountable temporarily, circumstantially and naturally existing one, for either Keynesian or neoclassical reasons). That being the case, *all* the available (that are actually offered) productive resources are (with some criterion) *efficiently* employed (and, symmetrically, *all* the available supplied resources are *efficiently* demanded). In the end, macroeconomics always predict the attainment of the potential output.

Time grows continuously or discreetly in dynamic macroeconomics, either up to an upper time-bound (in the case of short-run analysis) or unboundedly (in the case of long-run analysis). Under the unstoppable cultivation, flourishing and maturation of dynamic macroeconomics in the literature by the profession of Economics, discrete time analysis, although mathematically tedious, has ended being far more preferable for computational, empirical, monitoring and measurement of macroeconomic performance and testing the applicability of macroeconomic policy matters, overshadowing the alternative employment of a continuous time-horizon, which, although articulate, is intangible, hence, less transparent, tractable and interpretative. At the same time, given the two options as far as the economy's time-scale is concerned (the small and the large one), on the proviso that steady-states are attained so that the required intuition and normativeness is gained, long term dynamic analysis becomes more fruitful, not only because economic theory is better served with it, but also because its short term analogue is short sighted, failing to capture the big (stretched into the time-line) economic picture.

Inescapably, as it was explained in the prequel, pertinent markets, agents and tradable commodities are entangled into (behavioural) neoclassical growth theory scenarios.

Accordingly, one popularly admissible contextualisation by the proponents of behavioural macroeconomics suggests that the involved forward looking, immortal and, whenever the occasion calls for it, imperfectly altruistic (that is, caring for both themselves and their offspring) and imperfectly patient (that is, keeping income to be spent for joy and satisfaction both today and tomorrow) agents, the households in particular, are dynastic, all-knowing, non-myopic and perfect foresighted decision makers.

In the generic absence of any exterior disturbances, the households decide into a prefect world of determinism and of possession (and usage) of full and public information. They are, in fact, seen in sequential generations, which are replicated in time endlessly. Typically, their (quantitive, i.e., able to be quantified and measured) welfare is utilitarian and is pulled by consumption (and, wherever applicable, leisure) exclusively[4], when consumption and leisure are not measured (and, thereby, not depicted in the utility function) qualitatively. To make (more precisely, to duplicate) their instantaneous optimal choices, therefore, the households are (ultimately) exogenously endowed with preferences (equivalently, a personal atemporal utility function) and (whenever appropriate) a certain degree of (im)patience or altruism (equivalently, a subjective stationary discount factor)[5], which weighs and distributes their original (purchasing) preferences over time[6].

Clearly, to prevent ay prematurely arising misinterpretations, this is a trustworthy conceptual agreement if and only if the following terms and conditions hold. Although each trans-generational household is (truly) based and decides once and for all at $t = 0$, which is the unique (actually existing) decisional point for the decision maker, thus, the date at which all the exogenous priors are attached to the household, the household imaginatively projects itself forwardly and discretely, placing itself at each tangible fragment of the endless life ahead of it, therewith, seen as if making copycat instantaneous decisions at each $t \in \mathbb{N}$ and then compiling and bringing all these multiple (pseudo) decisions into/as a unique consolidated decision back at $t = 0$. In the end, therefore, the household's optimal decisions are (dynamic programming) plans, which are practically visualised (at $t = 0$ and along/for all $\mathbb{N}$) as time-indexed (convergent) sequences of real numbers. At $t = 0$, any (in steady

[4]See in the Auxiliary Text of the Appendix.
[5]See in the Auxiliary Text of the Appendix.
[6]See in the Auxiliary Text of the Appendix.

states) general equilibrium occurs for every single $t \in \mathbb{N}$, i.e., concerns each and every date of $\mathbb{N}$ (with particular attention being afterwards placed at the limiting case of $t = \infty$). A general equilibrium is pragmatically attained (in steady states) at $t = 0$ and, by being artificially attained at every $t \in \mathbb{N}$, spans and encapsulates at once all $\mathbb{N}$. So, having these arrangements in mind, no confusion shall be arising in the aftermath.

Typically, all (private) agents, households or firms, are (fully) rational when being in pursue of their lucrative activities. They are, for the sake of argument, masterminds. By functioning as decisional machines, they display the traditional neoclassical behaviour of the *homo economicus* and they (manage to fully, not incompletely or boundedly) maximise their objectives (subject to market constraints exclusively). So far as possible, agents' rationality to begin with, and their total rationality afterwards, are above suspicion and inextinguishable neoclassical stipulations.

Additionally, all agents are benevolent (innocuous and benign). They are always well meant (well intentioned and well minded) and in so being the economy does not shelter unethical characters. To a certain extent, when prosperous (macro) economies are modelled, the population of which is (sensibly) normalised to morality, and which are (pragmatically) institutionally fortified, the likelihood to find a (seizable) coercive interface for the mean agent is extremely low, if not negligible.

Nevertheless, the fabric of the strotyline goes deeper and, in the end, any (degree of) possible imperfect side of an agent's behaviour and personality has been perfectified, so that even the (definitely more substantial) odds of having bipolar (sub-populations of) agents with hybrid behaviour (rational or irrational, fully rational or boundedly rational, good or bad, depending on the decision) are ruled out.

In the standard decentralised backdrop, moreover, *both* the households' planing is private *and* the households have private ownership of the production's inputs and output (as well as of the economy's productive organisations, that is, the economy's production facilities, equipment and plants; and since households own the firms as well, the firms are passive agents, used as facilitating devices). The all-powerful households themselves and alone, consequently, manage the business of the free-market economy, orchestrating its operation and diverting its outcomes in their own interests. Without a (governmental) public sector, households are left absolutely alone and responsible for themselves, and (as a result) all the economic benefits freely and immediately flow for the sake of them (and only).

In the follow up, under all these circumstances, the apposite issue inevitably boils down to the optimal intertemporal distribution of the (optimally) produced resources of a market-structured economy among the households, whenever of course the intention is to trace allocations of the composite product that are brought about as natural economic phenomena as time elapses, without having the government (a public supervisor and administrator, a central authority and institution and, eventually, a household-targeting economic policy maker and dispenser) outwardly participating into the economy's distributive (and re-distributive) affairs across time, by intervening between the privates as a separate agent and by meddling with the private trade in the economy's markets, thereby, by putting the economy into a state of bilateral (both private and public) economic governance.

Neoclassical competitive (or Walrasian) general equilibrium theory naturally arises then as the mainstream (and as the epitome of all relevant theories) answer, on the plain basis that it is that universal architecture which *self-fulfils* and *stably preserves* all the normative neoclassical laws, principles and properties of economic nature. This is, in turn, because this vehement calibrating theory is built upon the timeless Smith's (1776) invisible hand and the vintage Walras' (1874) tâtonnement process, respectively, which two (retro, but still eloquent) conceptions constitute the backbone of the (neo)classical general equilibrium theory.

Allocatively speaking, the beauty and preponderance of this quiescently automated (in its leadership, guidance and conduction) theory of economic liberalism is inspired by its sheer flexibility and generality in humanitarian terms.

Indeed, among a multitude of other reasons, this laissez-faire theory is gifted with the following traits: each agent, who is identically and unconditionally behaving with any other (same type) agent's behaviour, cares solely for herself and pursues her own interests and only, without being concerned with what any other agent (of any sort) does; attains in this spirit an (not necessarily unique) optimal allocation on her behalf as a unit, but then miraculously, this (independently and identically distributed to any other) unit allocation is found to be harmless (in any sense) uponto any other unitary or group allocation; hence, when eventually all the units are assembled together and get pair-wisely contrasted or compared with respect to their distributional atomic privileges, this allocation is found to be (grand coalitionally) optimal as well (on top of being normative in any other humanitarian aspect).

Ultimately, provided that a macro-economy ends up simply being a large-scaled micro-economy, the state of the art Arrow-Debreu (1954) - McKenzie (1954, 1959) competitive general equilibrium is to be tracked down (or engineered) into a growing macro-economy.

This means that a neoclassical (here, decentralised, discrete-dynamic, long-run, steady-state, aspatial and deterministic) macroeconomic competitive general equilibrium shall be comprised of

1. (time) processes of market-clearing prices, that uniquely go together with
2. (both technologically or productively constrained and) budget-constrained (both unit optimal and group optimal) households' plans or, equivalently, dynamic feasible allocations of the quantities of the economy's aggregate output among the economy's households.

Given any (exogenous) priors that are intertwined with the economy's mains, the above two shall be the economy's (endogenous) posteriors.

And to align with the previous humanitarian ascriptions to this notion, apart form socially efficient (i.e., Pareto optimal), the competitive general equilibrium allocations shall be declared 'equitable' (iff 'fair and impartial' iff 'just and neutral') as well, at least in the anonymity (i.e., objectivity or impersonality) sense of satisfying the equal treatment property[7]

In the accustomed *perfectly* competitive macroeconomic regimes now, *per se* homogenisation of agents (of indifferently-indefinite number) of the same class and of markets-and-their-commodities (also of indifferently-indefinite number) of the same family is legitimately applicable. Under a *status quo* of unlimited contestability, the zero-profits (that is, the no-arbitrage or no-spread) condition is a valid stipulation as well, since it is an imputation that can be always and somehow formally verified.

In short, all households are symmetric, all firms are tantamount one to the other and all the markets accommodating and transacting the same type of commodity are uniform, admitting all of them, in turn, a representative consideration.

[7]See in the Auxiliary Text of the Appendix.

Of course, to keep things straight, perfect competition is not an impervious conceptualisation in (Walrasian general equilibrium tabularised) macroeconomics.

For example, the neoclassical macroeconomic story has it that extraneous (random) shocks (which here are excluded) are able to revert agents of some category into heterogeneity and/or their markets into incompleteness. Stochasticity would thence be attached to the herein storyline and the economy's competitive general equilibrium would be derailed out of its normal operation and canonical stability, having not only its normativeness compromised, but its ordinary theoretical implementation disrupted as well.

Or another (new Keynesian neoclassical) scenario has it that market prices are inelastic to excess demands and excess supplies, thus, inflexible, sticky or rigid and, practically, sluggish and unable to quickly adjust to their competitive general equilibrium levels, so the fulfilment of general equilibrium itself is smacked and disequilibrium prevails into the economy's markets (at least contemporaneously, while the prices' resistance holds).

To stick to the smoothly-linear (i.e., non-costate and unbendable) scenario of no markets' failure, neither of agents' failure, and, thereupon, avoid economic entropy, this paper does not allow any outer imperfections (noises, frictions, ridges, predicaments, glitches, pitfalls and so forth) ruin the perfection of the competitive general equilibrium.

As a result, in the above macroeconomic plot of hitting and conserving economy-wide balance with 'ideal equality' and 'perfect imperfections', the germane competitive general equilibrium issue has been straightforwardly resolved within every single generation of decision makers.

In general equilibrium, more accurately,

1. the representative (representing the whole society) household is (directly socio-widely) optimally-efficiently allocated with the economy's aggregate product or income [or, equivalently, every single homogenous household ends up with the per capita (or per head) product or income], while

2. there is a (unique, according to the neoclassical law of one price) average price-level spanning all the homogenous markets of some class, i.e., there is a (unique) representative price for each representative market.

In essence, what happens is that the competitive general equilibrium allocative theme has been simplified to equal (equivalently, equitable) divisions-distributions, so as to be transferred to its (second and concurrent) among-generations cake cutting (and sharing) problem. *A posteriori*, that is to say, the proposed dynamic (macroeconomic) competitive general equilibrium allocative script is twofold, because by adding the time-dimension, things are getting complicated by the involved inter-generational dependencies and conflicts of interests.

So now, in an equally sophisticated version, there is one representative household which unfolds in time in multiply-alternate heterogenous (hence, differentially allocated) generations. And eventually, along with the endogenous generation of representative general equilibrium prices, the neoclassical dynamic (macroeconomic) perfectly competitive general equilibrium seeks for optimal intertemporal rules for the aggregate output distribution to the representative household, which household does not any more solely consume (and settles down by drawing dis-utility as well).

The reason being that if all the pie, when shared out to its entitled (i.e., its authorised and qualified) owners and recipients, gets eaten up presently and none of it is kept, stored and piled-up for tomorrow's productive purposes, then there will be no more pie created (so as to be eaten) in the future. When today (where today is any time-point they find themselves being currently based) the decision-makers see prospective time and offspring (or other generations) in front of (or with) them, they do not live ephemerally, just for the moment, and parsimoniously, only for themselves. So beyond the short run, and even if they actually dis-like it and receive disutility from the in reference action, they save-invest (for extra-precautional reasons), and not just consume. And they do this perpetually, repeating the same activity at each time-instant ahead of them and within any forthcoming clone economy.

Ergo, instantaneously thinking, Pareto optimality is non-negligibly fractured because consumption is leaky, so the dampening effects to the properties of competitive general equilibrium coming from this macroeconomic archetype are, transiently, undeniable. But this is a temporary circumstance. Holistically and hyper-contemporaneously viewing the same situation, nonetheless, the gentrification of the (candidate) competitive general equilibrium allocations is conceptually materialised, by means of 'rationalising' the act of capital-accumulation (but as long as over/massive accumulation of capital does not occur; technically, capital accumulated is not allowed to go to infinity).

Indeed, at some/any time instant, the representative household that plans ahead does momentarily and voluntarily (i.e., consciously and intentionally) suffer a reduction in consumption (hence, a loss in utility and Pareto optimality), because some asset has to be (competitively to consumption and cumulatively to the already existing asset) created and kept as a reserve. And this is just because in the right-next time instant, this asset-accumulated (equivalently, this sacrificed level of consumption) becomes the source that supplies again the household with optimal consumption (hence, optimal-maximum utility and, in turn, Pareto optimality).

In (a competitive) general equilibrium as a matter of fact, and in marginal utilitarian terms, the degree of the optimal utility or efficiency that is abandoned and gone shall be equal to the degree of the new optimal utility or efficiency that is accrued and added. It is, more accurately, the famous dynamic competitive general equilibrium condition, dubbed 'Euler Equation', which encapsulates and describes this conceptualisation[8].

That being the case, i.e., by making allowances for the household's providence, which is expressed by the fact that the representative household keeps steadily forestalling (and postponing) some consumption so as to (safely and reliably) keep enjoying optimal (i.e., maximum) utility forever, dynamic competitive general equilibrium allotments [i.e., streams of periodical bundles (of endogenous assignments) of consumption and/versus non-consumption] are found to be (Pareto optimal) efficient in the following sense: since the dynasty's lifetime (the one taken in some sense of being collated in time; for example, it can be infinitely summable when discounted back at $t = 0$) utility is maximised, for each one of these trans-generational allocations, there does not exist an alternative inter-temporal arrangement (i.e., a dynamic schedule, scheme, program, profile of repeatedly instantaneous competitive or substitutional choices between consumption and savings) that would leave the household better off in terms of its lifetime utility (which is, once more, the household's utility when viewed, as a totality, in one and only snapshot, that is, all at once)[9].

[8] See in the Auxiliary Text of the Appendix.
[9] See in the Auxiliary Text of the Appendix.

So gradually the spoil in Pareto optimality (by having the household waiving its right to consume everything instantaneously) is mitigated, until it completely sinks and fades away with time.

Under the tenet of utilitarian welfarism and, in the macroeconomic style of modelling at least, the doctrine of (classification and subsequent) representation of agents and markets of the same group, one massive stream of relevant to this discourse models in the literature are the Overlapping Generations (OLG) ones, which are ubiquitous in macroeconomics. In them, two generations (young versus old) always coexist and interact within a single (*ex ante*-*ex post*) time-period (of two time-points), so practically, and irrespectively of its longevity, the focal household of the focal problem finitely lives and dies.

Samuelson (1958) pioneered the rigorous formulation of these models without production[10]. Diamond (1965) popularised these models in macroeconomics and growth theory. Galor and Ryder (1989) and Galor (1992) gave concrete theoretical foundations to the OLG models with production. For a systematic overview of these models see in Blanchard and Fisher (1989), chapter 3. Uniqueness and dynamic (Paretian or not) efficiency of this sort of competitive general equilibrium are intrinsically elusive properties. However, despite their pathogenicities, i.e., their irreparable inbuilt theoretical-economic inconsistencies and abnormalities, the extra-ordinary advantage of models of that ilk is that they, in effect, collapse analytically to merely static (multiple and identical single-shot) settings. Thereupon, they were reasonably initially coined and designed for pure exchange economies, that is, they were devised so as to innately resemble to the genuine (atemporal) competitive general equilibrium framework, before becoming spuriously dynamic. See, as a specimen, into the works of Gale (1973), Balasko and Shell (1980) and Balasko et al. (1980). For a condensed, concise and rigorous review of the OLG competitive general equilibrium concept (and its associated theory) see in Aliprantis et al. (1989), chapter 5.

This paper, in the other hand, goes in quest of (and fosters) a macroeconomic competitive general equilibrium idea which retains the pure dynamic picture. It stresses (adheres to and works on) that template where the meaningful tactic is to pass to the competitive general equilibrium via growth theory (not the unusual and cumbersome

[10]See in the Auxiliary Text of the Appendix.

converse). On this account, it relies up-onto the emblematic (neo-Keynesian) neoclassical (synthetic with Keynes, 1936[11]) Sollow (1956) - Swan (1956) growth model, which was optimally-endogenised both by the influential work of Phelps (1961, 1965) without utility, and by the celebrated texts of Cass (1965) and Koopmans (1965) (which were based on the seminal and ingenious paper of Ramsey, 1928) with utility[12]. Those models (and all the refinements upon them) are the building blocks of neoclassical growth theory, occupying the lion's share in the related (bulky) literature.

From there on, the analysis of this paper is carried out as follows.

First things first, the paper marks its puissance by constructing (and proving the unique existence of) both a Keynes (Harrod-Domar)-Sollow-Swan-Phelps [K(HD)SSP] competitive general equilibrium and of a Ramsey-Cass-Koopmans (RCK) competitive general equilibrium (in sections 2 and 3, respectively). Each one of these two (entirely differently stylised and customised) concepts is autonomous, independent and self-sufficient in terms of its optimal endogeneity.

In effect, within the neoclassical macroeconomic growth - extrapolated to perfectly competitive general equilibrium - agenda, and as long as agents' personas and markets are both frictionless and homogenous, this paper prompts two separate and self-sustainable strata of dynamic competitive general equilibrium set-ups.

More concretely, the paper furnishes (this domain of) neoclassical macroeconomics both with a portfolio of models with (immediately) grand optimality (the utilitarian one) and with a collection of models containing (initially) non-utilitarian optimality (in which, nevertheless, welfare is still utilitarian, which is again optimised).

Even by making an intuitive fact-based inference, after this propitious grouping, sorting or permutation (or, more accurately speaking, suitable stratification) of the neoclassical macroeconomic models of that ilk, the variation, dispersion, scattering or spreading between them is minimised, all of them become elastic, responsive or sensitive in the (substitutional) rough idea of expressing one to the other and, eventually, all of them become adjacent and equally spaced or distant around a central point, concentrating into a single uniform concept.

[11]See in the Auxiliary Text of the Appendix.
[12]See in the Auxiliary Text of the Appendix.

So, the terminal (and naturally emerging) question this paper asks (and goes on answering formally) is whether these two conceptions (which capture and occupy all the relevant neoclassical macroeconomic plenary) can be properly blended together so as to somehow overlap and create a uniform analytical environment.

To make inroads towards exploring this prospect or opportunity (or, at least, inquiry), the ground work of the two previous sections is summoned and by putting (in section 4) into direct contrast the two landmark competitive general equilibria that have been generated by this paper's project, sufficient conditions are elicited under which their (desirable) coincidence is allowed. To be more precise, it is shown that if the KSSP model is downgraded and manifested regressively, into its Keynesian ingredient solely, then the two prequel notions coincide into a single analytical body.

The paper (in section 5) critically summarises its findings and concludes that what was deduced in section 4 is a firm indication that the majestic work of Keynes (1936) is the indisputable point of origin in macroeconomics (but see also in **Comment 2** of the Auxiliary Text of the Appendix).

Finally, whilst the foundations of the analysis have been set (and digested) linguistically, en route to (and in preparation of) the main (mathematical) text of this paper, an album of several bonus and subsidiary (generalised at this point) mathematical-economic assumptions is critically needed in advance, so as to codify the two competitive general equilibria of the paper.

One is that the representative household possesses strictly-positive initial (endowment of) capital-stock, while the household does not perform borrowing (i.e., negative savings-investment) at any time point, thus, the household accumulates wealth instantaneously, not debt. Another is that aside from sections' 3 and 4 state of affairs, where labour contrivances make brief appearances, there is no labour production factor, no labour (unit or marginal) price (aka wage) and no labour market (and no 'labour versus leisure problem') to appear (and to be solved) in the rest of the paper('s two parts). One more is that all markets clear with the (strict) 'aggregate supply *equals* the aggregate demand' condition[13]. In conjunction with the former impositions comes, as a collateral, the general axiom of strict positiveness that has to be satisfied everywhere, jointly with the postulation that both the utility and the

[13]See in the Auxiliary Text of the Appendix.

production (functions) satisfy the strictest possible version of the usual normative properties. The surrounding neoclassical macroeconomic axiom that none macroeconomic (aggregate) quantity is to be addressed and tackled pecuniarily is also implementable (while, finally, the reader is referred to **Comment 3** of the Auxiliary Text of the Appendix).

Important to be mentioned is also the fact that the analysis is fundamentally based on time sequences (the terms of) which are embedded into recurrence relationships. Thence, one last general suggestion to be held is that all the (deterministic) recurrence relations of the text carrying time sequences, the straightforwardly arisen ones, the underlying ones which do eventually constructively emerge, or even those that exist latently given a time sequence, are to be deemed, named and treated as (deterministic) difference relations at the end of the day. This compulsive ascription becomes a legitimate connotation, as at least the explicit recurrence equations of the paper are time sequences that can be defined and solved primitively iteratively (or recursively)[14].

In the sequel main text of this paper of course, and contingently on each section's context, scores of other simplifying (situational and more specialised, but equally crucial) presumptions, which are based onto common-knowledge macroeconomic normalisations and conventionalities, are also to be held.

Though the mathematical facade of all these hypotheses is non-negotiable, there is, with respect to their economic compartment, a common rule accompanying the employment of each and every one of them. This is that although the familiar logic of pursuing the greater possible exhaustiveness and density in the economic layout may be being violated to the extent of potentially weakening the interpretative ability of the model, none loss or damage in mathematical generality occurs with conscripting them. Hence, the analysis' results and merits are robust, that is, impeccable and immune as far as their mathematical validity is concerned, irrespectively of any posterior extension that could come, in an effort to make the model seem more persuasive, from the economic side. To put it simply: any (seemingly irregular) lack or shortage in the model's (seemingly overlooked) economic parameters does not cause any deficiency in the mathematical affluence and potency of the model.

[14]See in the Auxiliary Text of the Appendix.

2. Keynes [- Harrod - Domar] - Sollow - Swan - Phelps competitive (or Walrasian) general equilibrium

Let the (macro) economy $\mathcal{E}_t^{KSSP}$, $t = 0, 1, 2, ..$, which is portrayed as follows

$$\{\mathcal{E}_t^{KSSP}\}_{t=0}^{\infty} = \{\mathbb{R}_+; C_t, S_t = I_t, K_t, r_t, \pi_t, K_0, \delta, F\}_{t=0}^{\infty}$$

$$\text{[or, equi-notationally, as } \{\mathcal{E}_t^{KSSP}\}_{t\in\mathbb{N}} = \{\mathbb{R}_+; K_t, r_t, \Pi, C, S = I, K_0, \delta, F\}_{t\in\mathbb{N}}],$$

and which (in the long run and under any general equilibrium totality in the universe of general equilibria) should (indispensably normatively) satisfy the limiting (across time) existential property

$$\lim_{t\to\infty} \mathcal{E}_t^{KSSP} = \mathcal{E}_{\infty}^{KSSP}(< \infty) =$$

$$\{\mathbb{R}_+; \lim_{t\to\infty} C_t(\bullet) < \infty, \lim_{t\to\infty} [S_t(\bullet) = I_t(\bullet)] < \infty, \lim_{t\to\infty} K_t(\bullet) < \infty, \lim_{t\to\infty} r_t(\bullet) < \infty, \lim_{t\to\infty} \pi_t(\bullet) < \infty, K_0, \delta, F\}.$$

The symbolic equivalence between these two compactified illustrations of the economy in reference will be fluently yielded from the subsequent text, which performs an analytical anatomy of the economy, dismantling it into its foundational parts.

Let us start with probing the basics.

$\mathbb{R}_+$ is the extended decision (action or choice) set of $\{\mathcal{E}_t^{KSSP}\}_{t\in\mathbb{N}}$, $\delta \in (0, 1)$ is the depreciation factor involved into the firm's production, $K_0 > 0$ is the household's initial endowment of asset (accumulated), and the bijection

$$F : \mathbb{R}_{++} \to \mathbb{R}_{++}$$

is the economy's well-behaved production function satisfying the following (widely acknowledged in the literature) properties: it is (sufficiently twice and, practically, infinitely many times) continuously differentiable over $\mathbb{R}_{++}$, strictly increasing and strictly concave on its unique input (i.e., its domain), while its first derivative tends to zero (respectively, infinity) as its argument approaches arbitrarily close to infinity (respectively, zero). Note that the asymptotic properties supra do not imply that F has an upper bound (attaining, thereby, a supremum). Note also that, technically, F fails to satisfy the conditions of Inada (1963), since zero is not included in its values, i.e., the interior of the cookie-cutter domain $\mathbb{R}_+$ is directly reckoned[15].

An exemplar algebraic expression of F is the Decreasing Returns to Scale-DRS (or homogeneous of degree strictly less than 1) Cobb-Douglas (1928) production (power) function, for example the

$$F(\star) = A\sqrt{\star} = A\star^{\frac{1}{2}} = Y_t(> 0),\ A > 0,$$

but other functional formats are also credible and welcomed [especially when the domain $\mathbb{R}_+$ is restricted more than just excluding zero; consider, for instance, the also ergonomic logarithmic function on $(1, \infty)$].

Cobb-Douglas or not, such an algebraic relationship, by summarising the alternative values of F, represents the economy's instantaneous (private) aggregate output [or (private) aggregate income] values [when the economy's product (or income) is demonstrated in real (i.e., demonetised) terms]. Here, for the time being, $\star$ denotes the instantaneous capital stock ('s alternative quantities or values) in any such formulation, hence, all the credit is given to capital accumulation through-in the economy's productive operational mode. The effect of labour, the second pillar of production, has been partialled out. Technical change (of neoclassical type) is also obliterated, having null influence.

Note that instantaneously and in theory, given the above specifications of production, there can be unlimited supplies of capital-accumulated, hence, unlimited supplies of produced output. Any credible general equilibrium notion outside the (Edgeworth, 1881) box of a pure exchange (of finite initial endowments) economy, however, capitally seeks for (endogenous-optimal) finite values for these two quantities at each time instant. Then, instant allocations (of the finite output) contain

[15] See in the Auxiliary Text of the Appendix.

finite quantities as well. This is an indispensable normative asymptotic (within time) existential property for $\{\mathcal{E}_t^{KSSP}\}_{t\in\mathbb{N}}$.

$\{\mathcal{E}_t^{KSSP}\}_{t\in\mathbb{N}}$ is a government-less closed economy and reserves for its own business two (domestic or national) markets:

(1) a financial market for the economy's illiquid asset-accumulated

and

(2) a market for the economy's product (comprising the sales of goods and services, which are both durables and non-durables, or equivalently, reflecting the expenditure on savings-investment and consumption, respectively).

Using the standard classical axioms, techniques and tricks, both money (the domestic currency, or equivalently, the economy's liquid asset and price-less commodity) and the price of the second market supra are ditched [for details, see in section 3; but see also in **Comment 4**, in the Auxiliary Text of the Appendix, for the potential implications when the public sector is interfering (and eventually gets blended) into the economy's markets].

Having (in the background) performed these analytical manoeuvres, all the (rest of the) economy's prices become real (or equivalently, relative), while the values of all the economy's (aggregatively) demanded and supplied quantities also become real (or equivalently, deflated); and all of them are (in the foreground) analytically treated as such.

Let us now turn into surveying the intricacies.

The functions r_t and π_t, $t = 0, 1, 2, ...$, will be the last to be elucidated, when the household's budget constraint is put under the analytical microscope.

As $t \in \mathbb{N}$, lastly, $C_t, S_t = I_t, K_t$ (symbolising the economy's aggregate consumption, aggregate savings=investment and aggregate capital accumulated, respectively) are the (ultimately) bivariate and univariate (i.e., with two independent variables and one explanatory variable, respectively) functions

$$C_t, S_t = I_t : \mathbb{R}_{++} \times (0, 1) \to \mathbb{R}_{++},\ C_t(Y_t, s), S_t(Y_t, s) = I_t(Y_t, s) > 0$$

and

$$K_t : (0,1) \to \mathbb{R}_{++},\ K_t(s) > 0.$$

To procure their properties, their particularised conceptual framing, which leads (whenever this is possible) to some specified functional formula for them, is first required.

To get down to such an accomplishment, a series of logical deductions [reached by following the footsteps of, historically, some of the primest figures (and distinctively revolutionary contributors) into the macroeconomics' industry] are needed.

De facto, for starters, for every $t \in \mathbb{N}$, C_t and $I_t = S_t$ linearly-positively depend on Y_t, since for a Keynesian (non-volatile in time by original conceptualisation[16], but exogenous for the time being and, for sure, not yet capturing optimality) savings rate $s \in (0,1)$, one has that

$$C_t(Y_t) = (1-s)Y_t \text{ and } I_t(Y_t) = S_t(Y_t) = sY_t,\ t = 0, 1, 2, ...,$$

where, by dint of Keynes (1936), the (100%) potential(ly achieved) income is split-up (and spent) into two separate portions or fractions (ultimately, percentages), with

$$s(= \tfrac{dI_t(Y_t)}{dY_t} = \tfrac{dS_t(Y_t)}{dY_t})$$

being the marginal propensity to save, as opposed to

$$(1-s)(= \tfrac{dC_t(Y_t)}{dY_t}),$$

which is the marginal propensity to consume [when, of course, investment becomes savings, not the other (neoclassical) way around].

[16]See in the Auxiliary Text of the Appendix.

Above, autonomous consumption has been dropped off the analytical radar screen without any loss in mathematical generality, while, to fill procedurally the gap between savings and investment without having to worry about the direction of their correspondence, investment is not autonomous to begin with, given the forthcoming inclination of this aggregate macroeconomic measure to be, eventually, infused with the neoclassical interpretation.

However, autonomous investment, i.e., the fact that $I_t = \overline{I}_t = S_t(Y_t) = sY_t$, $t = 0, 1, 2, ...$, will be kept as a background analytical pledge for the time being. It will naturally appear and drastically operate in section 4 when Keynes (1936) will be left to be solo acting in the KSSP model (see also in **Comment 5** of the Auxiliary Text of the Appendix).

Seen, at this juncture, as functions of (just) income, C_t and $I_t = S_t$, $t \in \mathbb{N}$, whose (common) infimum is zero and common range is $(0, \infty)$, are infinitely many times continuously differentiable, strictly increasing and above unbounded (on their domain, $\mathbb{R}_{++}$). These functions end up communicating the (rock solid in Keynes' era) classical Say's (1803) law, which in turn summarises the simple (but superb) idea that supply determines (and creates its own) demand.

So the real novelty of Keynes (1936) is elsewhere. To wit, Keynes (1936) makes a breakthrough and inverts the former law, challenging, in this way, the classical monopoly in Economics. More precisely, since C_t and $I_t = S_t$, $t \in \mathbb{N}$, are surjective, they are eventually (given that they are injective as well) bijective, so they can be inverted, so that finally there exists the inverse function

$$Y_t : \mathbb{R}_{++} \to \mathbb{R}_{++}, \text{ defined by } Y_t(I_t = S_t) = \tfrac{1}{s} I_t = \tfrac{1}{s} S_t,\ t \in \mathbb{N},$$

which agrees with original (inverted) function as far as its mathematical properties are concerned, and in which

$$\tfrac{dY_t(I_t)}{dI_t} = \tfrac{dY_t(S_t)}{dS_t}) = \tfrac{1}{s} > 1,\ t = 0, 1, 2, ...$$

is the Keynesian (1936) output-multiplier, capturing the (supreme in contextualisation) Keynesian multiplying (amplifying) effect that investment (hence, capital) has into the economy's product, thus, growth. Ultimately, given that there also exists the twin (to the previous one) function

$$Y_t : \mathbb{R}_{++} \to \mathbb{R}_{++}, \text{ defined by } Y_t(Ct) = \tfrac{1}{1-s}C_t,\ t \in \mathbb{N},$$

in which

$$\tfrac{dY_t(C_t)}{dC_t} = \tfrac{1}{1-s} > 1,\ t = 0, 1, 2, ...,$$

Keynes (1936) updates (and, in his mind, swaps) Say's law with the transpose of this law, that is to say, invents one new law of his own, brilliantly claiming that demand determines (and creates its own) supply.

Although this (timelessly innovative) fact is usually withheld from any neoclassical growth dynamic analysis, demand side macroeconomics continue to have dramatic repercussions to the expansion of output at every single (statically considered) instant of the time-line, especially as long as s (and, in particular, s^{-1}) gets to be interpreted behaviourally and becomes a choice output (hence, growth) multiplying variable (see also in section 4).

With production (driven fundamentally by capital-accumulated) being explicitly modelled, Harrod (1939) and Domar (1946) add the dimension of time into an upgraded Keynesian framework, reproducing with loyalty the Keynesian ideas on the one hand (by refusing to comprehend the Keynesian analysis as of being a protectorate of neoclassicism), but by managing to surpass the original (and problematic) Keynesian-like short-term analytics on the other hand. Neoclassicists maintain that the growth solution this model predicts is not (and cannot be sensibly restrained so as to be) stable in the limit, neither favourable for an economy, so they defy the value-added of this model.

So finally, with clear neoclassical supplements, the (neo-Keynesian neoclassicists) Sollow (1956) and Swan (1956) supersede all their severely Keynesian-inclined predecessors, by offering long-run balanced growth solutions, confined into the idea of steady-states. From that point and on, neoclassicism (and its associated mathematical machinery) rules in macroeconomics as well, becoming the main intellectual apparatus in dynamic, specifically, macroeconomic calculus.

In fact, by going even further-and-beyond Keynesianism, but by still keeping faithfully (as a neoclassical-synthetic analytical leverage) the Keynesian conceptual wrapping (and, of course, the analogous decision-theoretic covering), for $t = 0, 1, 2, ...$, one may non-incidentally and non-superficially define the (double-domain) functions

$$C_t, S_t = I_t : \mathbb{R}_{++} \times (0, 1) \to \mathbb{R}_{++},$$

where

$$C_t(Y_t, s) = (1 - s)Y_t \text{ and } S_t(Y_t, s) = I_t(Y_t, s) = sY_t.$$

Although Von Neumann and Allais had previously touched (in their texts) this issue, this analytical culmination is, effectively, realisable by virtue of Phelps (1961, 1965) (and several other authors who independently worked on, propagated and escalated the analysis of this topic around that time), who (contrary to all his/their precursors) broke a taboo issue and described steady states of maximum (in sustainability) capital-accumulated versus maximum (in efficiency) consumption, reached by a intergenerational reciprocality-promoting savings-investment rate. Ever after, the consumption-maximising $s \in (0, 1)$ legitimately becomes an optimal-endogenous (implicitly, choice) variable for optimally-refurbished growth (to, implicitly, competitive general equilibrium) purposes (see, for instance, in Acemoglu, 2009, chapter 2).

Thenceforth, the portal to a competitive general equilibrium notion akin to the previously narrated tale of (neo-Keynesian) neoclassical growth is a (from scratch) makeover of the Keynes to Harrod-Domar, to Sollow-Sawn, to Phelps economic (growth-to-competitive-general-equilibrium) organism. The cells of this organism have to be the known ones, but its nucleolus will have to be reconstructed.

So, as a first crack to the story, what is fundamentally required (as the properties of the previous two functions are still pending) is the extraction of the mathematical behaviour of the function K_t, $t \in \mathbb{N}$.

Consider first $K_0 > 0$, which is exogenously given at $t = 0$ and, consequently, gets invalidated as a variable, so for $t = 0$ all the statements that follow infra will hold trivially. Next, as usual, capital-accumulated (the values of which are notated with $\bullet$, in the absence of the formal acquisition of its parametrisation thus far) evolves in time according to the cliché (dated back to Ramsey, 1912, and untouchable up to today, since it is heavily assimilated and supported by neoclassicists, with disproportionate and mild post-Keynesian criticism or objection) neoclassical law of motion of capital-accumulated, which is the first order (with one time lag) difference equation

$$\{K_t(\bullet) = I_{t-1}(Y_{t-1}, s) + (1-\delta)K_{t-1}(\bullet)\}_{t=1,2,..}, \text{ for every } s \in (0,1) \text{ and } Y_t > 0,\ t \in \mathbb{N},$$

where, for $t = 1, 2, ...$, $K_{t-1}(\bullet)$ is already attained and given. From the above expression it becomes evident that, for $t = 1, 2, ...$, K_t is a positive function of (the function) I_{t-1}, which is, in turn, conditional on $s \in (0,1)$ and $Y_{t-1} > 0$, thus, K_t is a function of the savings-investment ratio (and trivially of income) as well[17]. Conclusively,

$$K_t : (0,1) \to \mathbb{R}_{++},\ K_t(s) > 0,\ t \in \mathbb{N}\ (K_0 > 0 \text{ given}).$$

This function is not exogenously strictly positive. Upon the validity of the previous difference equation, **Claim 1** (find its proof in the Appendix) suggests that when the time-sequence of investment values contains strictly positive terms, then the associated time-sequence of the capital-accumulated values is also strictly positively termed.

Note also at this point that in contrast with the economy's production function (or technology), F, which is atemporal, K_t, $t = 0, 1, 2, ..$, is not a (time) stationary

[17]See in the Auxiliary Text of the Appendix.

function, meaning that it does not (necessarily) retain the same functional form at every time instant. Concurrently, remark in advance that the functional formulae of C_t and $S_t = I_t$, $t \in \{0, 1, 2, ..\}$, are to be kept, eventually, fixed across time and in this sense the former are time invariant functions. This means that, had one desired to exhaust mathematical accuracy, the values $c_t = C(.,.)$ and $s_t = S(.,.) = I(.,.) = i_t$, $t \in \mathbb{N}$, of the constant in time functions C and $S = I$, respectively, should have been manipulated in the first place. This practice is not undertaken here so as to annihilate notational proliferation and, thereby, avert presentational disarray. This explains, however, the two alternative ways to equip the economy with, which were laid out in the preamble of this section.

Thereupon, for every $t \in \mathbb{N}$, C_t and $S_t = I_t$ admit further parametrisation with respect to $s \in (0, 1)$ by being defined as below:

$$C_t(\bullet, s) = (1 - s)F(K_t(s)) \text{ and } S_t(\bullet, s) = I_t(\bullet, s) = sF(K_t(s)),$$

while their mathematical properties are still awaited.

First of all however, as $t \in \mathbb{N}$, note that once s is optimally chosen, K_t (along with $S_t = I_t$ and C_t) is also optimally chosen and, subsequently, Y_t is optimally chosen as well. This (*inter alia*; see soon afterwards as the relevant competitive general equilibrium notion, along with its characterisation, will be finally introduced) means that, for any $t \in \mathbb{N}$, Y_t is eventually neutralised as a choice variable and everything above (indifferently) holds for any value of Y_t. So eventually, all attention is to be placed onto the $C_t(\bullet, s) > 0$ and $S_t(\bullet, s) = I_t(\bullet, s) > 0$, $t = 0, 1, 2, ..$, values of the previous two functions (having the aggregate product's values running in the background).

Now, before getting into this, for sure C_t and $S_t = I_t$, $t \in \mathbb{N}$, are (at least twice) jointly continuously differentiable (thus, continuously differentiable on each one of their arguments separately). Further, on impulse, as $t \in \mathbb{N}$, consumption is inversely (negatively) related with the economy's savings-investment rate and savings-investment is directly (positively) connected with the same ratio, but apparently, to formally validate the monotonicity (with respect to the argument s) properties of C_t and $S_t = I_t$ the mathematical properties of K_t (which, intuitively, should also be tied to the same direction with the savings-investment rate) need to be unlocked beforehand. Additionally, in the absence the previous fundamental knowledge, the two functions C_t and $S_t = I_t$, $t \in \mathbb{N}$, are not (necessarily) jointly injective, neither (necessarily) jointly surjective [since they are not definitely such functions on their partial domain, $(0, 1)$].

So say, first of all, that K_t, $t \in \mathbb{N}$, is bijective and (at least twice) continuously differentiable over $(0, 1)$, and define (by keeping track with the inner flow and natural interpretation of the model) the strictly positive 'marginal capital accumulation with respect to the savings-investment rate',

$$\frac{dK_t(s)}{ds} > 0, \ \forall s \in (0,1), \text{ as } t \in \mathbb{N},$$

so that, K_t, $t \in \mathbb{N}$, is strictly increasing on $s \in (0, 1)$ (equivalently, on its domain).

Then, for $t = 0, 1, 2, ...$, the viability of the condition

$$\frac{d^2K_t(s)}{ds^2} < 0, \ \forall s \in (0,1),$$

equivalently, the fact that K_t is strictly concave on its domain, is reasonably invoked as well, on the following basis: by the well behaviour of F, for $t \in \mathbb{N}$, we already (and extraneously) have in hand that

$$\frac{d^2F(K_t(s))}{dK_t^2(s)} < 0, \ \forall K_t(s) > 0, \ \forall s \in (0,1),$$

with the assumption that, at each time instant, the (strictly positive) 'marginal capital-accumulation with respect to the savings-investment rate' drops as the savings-investment rate rises being on par with the condition supra, because this circumstance is actually the one that causes the (strictly positive) 'marginal production with respect to capital accumulation' climb down as capital agglomeration climbs up (as the savings-investment rate scales up)[18]; in other words, at each time point, as the savings-investment rate increases, the capital accumulated (hence, the output)

[18]See in the Auxiliary Text of the Appendix.

increases as well, but its cumulativity (i.e., its technical ability to get accumulated) diminishes, thus, its productivity (i.e., its technical ability to produce) sinks as well (up until it tends to entirely clog).

Finally, as $t \in \mathbb{N}$, while K_t does not attain a supremum, $\inf\limits_{s\in(0,1)} K_t(s) = 0$.

For $t \in \mathbb{N}$ now, consider the two (known to be at least twice) jointly continuously differentiable [over $\mathbb{R}_{++} \times (0, 1)$] functions

$$\{C_t(\bullet, s) = (1-s)F(K_t(s)) : s \in (0,1)\}$$

and

$$\{S_t(\bullet, s) = I_t(\bullet, s) = sF(K_t(s)) : s \in (0,1)\}.$$

Then, for $t = 0, 1, 2, ...,$

$$\tfrac{\theta S_t(\bullet,s)=\theta I_t(\bullet,s)}{\theta s} = F(K_t(s)) + sF'(K_t(s)),\ s \in (0,1),$$

where $F(K_t(s)) > 0$, $\forall K_t(s) > 0$, $\forall s \in (0,1)$, by construction, while one has that

$$F'(K_t(s)) = \tfrac{\theta F(K_t(s))}{\theta s} = \tfrac{dF(K_t(s))}{dK_t(s)} \tfrac{dK_t(s)}{ds} > 0,\ \forall K_t(s) > 0,\ \forall s \in (0,1),$$

where the above follows by simply applying the chain rule of differentiation, since

$$\tfrac{dF(K_t(s))}{dK_t(s)} > 0,\ \forall K_t(s) > 0,\ \forall s \in (0,1),$$

by the well behaviour of F, and

$$\tfrac{dK_t(s)}{ds} > 0,\ \forall s \in (0,1),$$

as explained in the prequel, so that finally

$$\frac{\theta S_t(\bullet,s)=\theta I_t(\bullet,s)}{\theta s} > 0,\ [\forall K_t(s) > 0,]\ \forall s \in (0,1),$$

so that indeed (without any further scepticism) $S_t = I_t$ is a strictly increasing function of $s \in (0,1)$. Also, instantaneously, while the infimum of this function is zero, this function rests unbounded from above, so that (with respect to this argument) this function is eventually both injective and surjective (hence, invertible).

However, although the properties of the bivariate function $S_t = I_t$, $t \in \mathbb{N}$, have been entirely cleared out, some fundamental aspects of C_t, $t \in \mathbb{N}$, will be still remaining a mystery. The reason for this being that, when considered globally, monotonicity with respect to $s \in (0,1)$ is a fuzzy property for C_t, $t \in \mathbb{N}$. Indeed, when taking

$$\frac{\theta C_t(\bullet,s)}{\theta s} = -F(K_t(s)) + (1-s)F'(K_t(s)),\ (1-s) \in (0,1),\ t = 0,1,2,...,$$

then (by appealing to exactly the same reasoning as before) it holds that

$$-F(K_t(s)) < 0 \text{ and } (1-s)F'(K_t(s)) > 0,\ \forall K_t(s) > 0,\ \forall s \in (0,1),$$

so, globally, the sign of this derivative is indeterminate.

It is conjectured that this happens because there are two opposite shadow forces, i.e., two unobservable stimulants, which operate simultaneously and outwardly to the system, when consumption is parametrised with respect to the savings-investment rate across time.

First, the Keynesian (and pure statically interpreted) one, according to which consumption moves to the opposite direction from the one that the savings-investment ratio shifts, as time evolves (see also in section 4).

Second, the neoclassical (and pure dynamically interpreted) one, according to which, as explained in footnote 17 supra, (current) consumption depends positively on savings-investment (of the previous fragment of time), which in turn, as deduced previously, depends positively on the savings-investment rate, so consumption (seen as a non-ephemeral action) is mobilised into the same direction that the savings-investment rate variates, as time progresses.

Heuristically concluding so far, along time, instantaneous consumption has to be increasing for/on some $s \in (0, 1)$, while it has to be decreasing for/on some other $s \in (0, 1)$ (the remaining ones). Intuitively thinking as matter of fact, at each time instant, as s fluctuates onwardly going from zero to one (both of them being values not taken by s) two distinct subsequent stages have to be crossed.

Phase 1. For the first (relatively small) values of s, as s is going upwards (following its predetermined course from 0 to 1), the neoclassical effect prevails, so consumption increases as well, as the corresponding (to the moderate rise in s) increase in savings-investment (transformed into capital accumulated) works in favour of consumption, because, in its initial values and as it was already formally claimed, s feeds at an increasing rate (the cumulativity of) the capital accumulated[19], which (being, in fact, the unique source of output) fuels affluently (that is, with high productivity) production, which, in turn reaches adequately the position to trigger consumption diffusion and induce consumption spillovers inside the economy; the negative effect of non-consuming (i.e., of relinquishing and loosing consumption) is cancelled out by the generous boost up in consumption in the follow up; the consumer (decision maker) is well compensated for saving-investing; overall, consumption lifts up as s does (within limits) the same; s leaves the household better-off (or does not harm it whatsoever).

Phase 2. Phase 1 takes place only up to reaching a certain level or threshold value of s, beyond which the Keynesian effect kicks in, because as s keeps enlarging (taking now relatively large values) the exactly opposite (to previously) circumstances come forward by being provoked in the same spirit, and consumption scales down; in its ending values, s breeds at a decreasing rate (of cumulativity) capital accumulated[20], which spurs with low (now) productivity production; the (due to the high stimulus in s) magnification of savings-investment (converted into capital-accumulated)

[19]See in the Auxiliary Text of the Appendix.
[20]See in the Auxiliary Text of the Appendix.

works, ultimately, in expense of (and blocks) consumption, as the positive effect of the forthcoming (due to explosion in s) building-up in consumption cannot offset the original (incurred by the excessive quantity of s) high abandonment of consumption; altogether, consumption declines as s surges; when s takes exuberant values and capital gets (over) accumulated massively, the act of forfeiting consumption does not payback, leaving the household worse-off because consumption peters out.

So phase 2 turns upside down phase 1, but the two phases are essentially the two opposite sides of the same coin.

Spontaneously thinking, if this is the case, an optimally chosen endogenous general equilibrium value for s should be one that, at whichever date of the time horizon, leaves the decision maker indifferent between the two phases, that is, an $s^\star \in (0, 1)$ that does leave either a net consumption payoff or a net consumption loss when keeping (saving-investing) 'some' of the pie, instead of eating (consuming) all of it. Such an $s^\star$, which would be (or give) the critical (turning) point of alternation (i.e., of switch or flip) between the two consecutive stages, shall not be stimulating consumption over savings-investment, neither will be inciting saving-investing over consuming.

In a competitive general equilibrium, further on, any such attained $s^\star$ would (globally) maximise the objectives of both the household and the firm. That is to say, would indeed be (or give) a turning (and not an inflection or saddle) stationary point for both consumption and profits related functions. Ideally, there should be only one attained $s^\star$ of such idiosyncrasy, i.e., the involved global maxima shall be strict.

This $s^\star$ shall be called the *golden* (*savings-investment rate*) *decision rule*, or *golden rule* for short, while the competitive general equilibrium that this $s^\star$ designates shall be referred to as the *golden rule competitive general equilibrium*.

Ultimately, all the informal standpoints that have been expressed heretofore will be formally discovered in the proof of Theorem 1 that closes this section. For now, formality is prolonged and if such phenomena do occur in the first place, they are evolving irrelevantly to the economy's mechanics.

Finally, even if being (at some of its parts) draft, deficient and inconclusive, all the information that has been gathered so far is dense enough to put one in the position to assemble the economy's parts back into a single and succinct analytical shape.

In short, up to this point, both by taking for granted all the specifications concerning the economy's structural priors that have been acknowledged insofar and by introducing a few more protagonists and procedures of the model, as $t = 0, 1, 2, ...$ and for any $(s, Y_t) \in (0, 1) \times \mathbb{R}_{++}$, $\{\mathcal{E}_t^{KSSP}\}_{t \in \mathbb{N}}$ is completely characterised by the following (dynamic) system of equations:

$$\begin{cases} C_t(Y_t, s) + I_t(Y_t, s) = F(K_t(s)) = Y_t = r_t(s)K_t(s) + \pi_t(K_t(s)), \pi_t(K_t(s)) \in \mathbb{R} \\ I_t(Y_t, s) = K_{t+1}(s) - (1 - \delta)K_t(s)[\iff K_{t+1}(s) = I_t(Y_t, s) + (1 - \delta)K_t(s)] \\ K_{t+1}(s) = r_t(s)K_t(s) + (1 - \delta)K_t(s) \\ [S_t(Y_t, s) =]I_t(Y_t, s) = sF(K_t(s)) \\ C_t(Y_t, s) = (1 - s)F(K_t(s)), \end{cases}$$

where (when, from now on, 'well' means 'appropriately', i.e., as in the prequel text) well-defined δ and K_0 are given, well-behaved K_t, $t \in \mathbb{N}$, and F are also given,

while

$$K_{t+1}(s) = r_t(s)K_t(s) + (1 - \delta)K_t(s), \ t = 0, 1, 2, ...$$

is the [existing independently to (and in parallel with) the neoclassical one] fundamental financial time-law of motion of capital, when capital does flow cumulatively and discretely along the time-line, but when, additionally, capital-stock gets instantaneously depreciated as well; this is also a first order (with one time lapse) difference equation with respect to capital-accumulated; its interpretation is that the (conglomerated) capital first yields income (i.e., interest) for its owner and then gets depreciated (i.e., trimmed) in production, before being returned back (for replenishment) to its owner, so as to be added to the current instant's interest (i.e., growth of the original capital-stock) and contribute into the creation of the next instant's capital-accumulated;

and

$$\{C_t(\bullet, s) + I_t(\bullet, s)(= Y_t) = r_t(s)K_t(s) + \pi_t(K_t(s)), \pi_t(K_t(s)) \in \mathbb{R}\}_{t\in\mathbb{N}}$$

is the household's budget constraint, when the household realises income at every date twofoldly, from speculation and from entrepreneurship; that is to say, the household raises financial benefits by being rewarded with the (representative) return (or interest) to (saving-investing and) asset accumulating and by gaining the (representative) proprietor's dividend out of the firm's profits[21];

three more critical (clarifying) comments as far as the household's budget line and its consolidation with the two time-laws of motion of capital-accumulated (the neoclassical and the financial one) are concerned are in order:

1. At every $t \in \mathbb{N}$, in a realistic neoclassical market-based economy, K_t gets parametrised with respect to one more variable as well, which was deliberately wiped off (since it had nothing to offer) so far, the economy's (real) rate of interest, $r_t > 0$, so that

$$K_t : \bullet \times \mathbb{R}_{++} \to \mathbb{R}_{++}, \text{ such that } K_t(\bullet, r_t) > 0,$$

is the household's (neoclassical) 'aggregate supply of capital-accumulated function' (realised from the household and towards the firm, within the relevant market), which is (at least twice) continuously differentiable over $\mathbb{R}_{++}$, strictly increasing and bijective on (its partial domain concerning) r_t[22], thus, invertible when framed into this sub-domain, which means that

$$r_t : \mathbb{R}_{++} \to \mathbb{R}_{++}, \text{ such that } r_t(K_t(s)) > 0,\ s \in (0, 1),$$

[21]See in the Auxiliary Text of the Appendix.
[22]See in the Auxiliary Text of the Appendix.

and ultimately

$$r_t : (0, 1) \to \mathbb{R}_{++}, \text{ such that } r_t(s) > 0,$$

is the household's (traditionally taken) inverse (aggregate) supply (of capital accumulated) function, carrying by construction (independently on whether it is set in motion with respect to capital accumulated or, eventually, directly with respect to the savings-investment quota) exactly the same properties as the aforesaid concerning K_t; an asymptotic particularity is put into force however: while, as $t \in \mathbb{N}$, an appropriate inference makes one realise that $\inf_{s \in (0,1)} r_t(s) = 0$ and no supremum (with respect to $s \in (0, 1)$) is actually attained for r_t, for any $s \in (0, 1)$, by relying onto Proposition 1 right after, it could be (reasonably, mathematically thinking, and realistically, economically speaking) assumed that $r_t(s) \in (0, 1)$, $t \in \mathbb{N}$, by means of the (implied or inherited from this Proposition) fact that $1 > \overline{r}(s)(\leftarrow \{r_t(s)\})$ as $t \to \infty$ (see also in Remark 3 that postscripts this section), so under the logic of the anticipation of a definite steady-state outcome, to reinforce economic (and, specifically, real-life financial market's) credibility and without any loss in mathematical generality, one could methodically (and purposefully) put $(0, 1)$ in lieu of $\mathbb{R}_{++}$ in the previous two functional representations of r_t, $t \in \mathbb{N}$, wherever appropriate; hence, it could be (and is actually from now on) claimed that r_t attains a (least) upper bound (uniformly the value 1) as $t \in \mathbb{N}$.

2. By combining the two (simultaneously viable) difference equations of capital-accumulated, as $t \in \mathbb{N}$ and $s \in (0, 1)$, and given that instantaneous consumption is strictly positive, one gradually concludes that

$$\text{(a) } I_t(\bullet, s) = r_t(s)K_t(s) \iff \pi_t(K_t(s)) > 0,$$

as opposed to

$$\text{(b) } I_t(\bullet, s) < r_t(s)K_t(s) \iff \pi_t(K_t(s)) \leq 0,$$

so that, instantaneously and as long as profits remain strictly positive, the return from the asset-accumulated fully and exactly finances the undertaken investment (which a financially orthodox recycling condition), with the residual expenditure on consumption being, therefore, exhaustively financed by the profits (as it was to be expected, in an economy where no labour income is earned, that is, the entrepreneurial income fully substitutes the labour income, meaning that the household is an employer or self-employed, not an employee); so (by having long ago bypassed the idea of attaining strictly positive profits, something that occurs only in non-competitive general equilibrium market set ups where competition is impeded and squashed from the firm's side), if the competitive firm settles with (the worst case) scenario that the production-revenues break even with the production-costs and no profits are materialised, then the financial market's payoffs (have to) cover for consumption as well (apart from definitely shielding the economy's savings-investment); notwithstanding, if case (a) is valid, several needful results are obtained:
(i) the two sequences (of investment and capital-accumulation) are co-monotone,
(ii) instantaneously, in a *ceteris paribus* argumentation styled comparative statics analysis, investment (which is a percentage of capital-accumulation) directly-proportionally determines capital accumulation and *vice versa*, when the cause and effect direction is changed, capital-accumulated straightly-proportionally shapes investment, and
(iii) instantaneously, for every $t \in \mathbb{N}$ and $s \in (0, 1)$, one could start with the equalities

$$I_t(\bullet, s) = r_t(s)K_t(s)(> 0) \text{ and } C_t(\bullet, s) = \pi_t(K_t(s))(> 0)$$

and construct the household's synthesised budget equation (by adding the two equating expressions by members), while inversely, given the household's (consolidated) budget line as a composition of spendings and incomes, one could decompose it into its two (just above presented) autonomous pieces (or equations); concluding, the specifications of the model regarding the co-validity of the two independent laws that describe the mobility of capital-accumulated in time emits, as $t \in \mathbb{N}$ and $s \in (0, 1)$, the model's canonical condition

$$r_t(s)K_t(s) \geq I_t(\bullet, s), \pi_t(K_t(s)) \in \mathbb{R}, \text{ with } \begin{cases} r_t(s)K_t(s) > I_t(\bullet, s), \pi_t(K_t(s)) \leq 0 \\ r_t(s)K_t(s) = I_t(\bullet, s), \pi_t(K_t(s)) > 0; \end{cases}$$

Comment 6 in the Auxiliary Text of the Appendix embellishes further this topic.

3. At every $t \in \mathbb{N}$,

$$\pi_t : \mathbb{R}_{++} \rightarrow \mathbb{R}, \text{ such that } \pi_t(K_t(s)) \in \mathbb{R},\ s \in (0, 1),$$

defined by

$$\pi_t(K_t(s)) = F(K_t(s)) - r_t(K_t(s))K_t(s)$$

and ultimately (when non-beneficiary parametrisation is skipped)

$$\pi_t : (0, 1) \rightarrow \mathbb{R}, \text{ such that } \pi_t(s) \in \mathbb{R},$$

is the firm's (real) profits function (but as with C and $S = I$, one could more exhaustively define the atemporal function Π); it can be easily shown that this function inherits differentiability and strict concavity from F globally, but global monotonicity cannot be specified for it (specifically, the profits function is definitely strictly decreasing at the ending values of capital accumulated, because when the latter tends to infinity, the first derivative of the profit function tends to the negative value of the economy's interest rate); upon the maximisation of profits at every consecutive date profits are destroyed; this happens when, as time ascends to infinity, the limits of all the involved time-sequences are attained; as a testimony, the following proposition is offered:

Proposition 1. Assume that the time-sequences $\{K_t(s)\}_{t\in\mathbb{N}}$ and $\{r_t(s)\}_{t\in\mathbb{N}}$ are convergent as $t \to \infty$ and $s \in (0,1)$. Then, when profits are maximised over $s \in (0,1)$ at every $t \in \mathbb{N}$, the following equivalence is true (for whichever arisen optimal s, at whichever $t \in \mathbb{N}$): the firm accrues zero (maximum) profits instantaneously $[\pi_t(K_t(s)) = 0 \iff \pi_t(s) = 0,\ \forall\ t \in \mathbb{N}]$ iff instantaneously, F becomes (reduces to) the Constant Returns to Scale-CRS (homogenous of degree 1) linear (Cobb-Douglas, power) function $[F(K_t(s)) = r_t(K_t(s))K_t(s) \iff (F \circ K_t)(s) = r_t(s)K_t(s), \forall\ t \in \mathbb{N}]$.

Proof of Proposition 1. *See in the Appendix.*

Continuing, for $t = 0, 1, 2, ...$, the previous (dynamic) system of equations that completely characterises $\{\mathcal{E}_t^{KSSP}\}_{t=0}^{\infty}$ may be (in a bifurcational or piece-wise sense):

reduced to the following pair of difference equations

$$\begin{cases} K_{t+1}(s) = (1-\delta)K_t(s) + sF(K_t(s)) \iff K_{t+1}(s) = (1-\delta)K_t(s) + I_t(Y_t, s), \\ K_{t+1}(s) = r_t(s)K_t(s) + (1-\delta)K_t(s), \end{cases}$$

when, in general, $\pi_t(s) \in \mathbb{R}$,

while may be specifically brought down to the difference equation

$$K_{t+1}(s) = (1-\delta)K_t(s) + r_t(s)K_t(s), \text{ when, in particular, } \pi_t(s) > 0,$$

when, in both alternative cases, the difference equation is viable subject to the household's budget constraint, for any $(s, Y_t) \in (0,1) \times \mathbb{R}_{++}$, for any well-defined δ and K_0 and for any well-behaved $\{K_t, r_t\}_{t\in\mathbb{N}}$ and F.

The couple of the difference equations in the (weaker) first unwinding of the economy (in difference equations) supra, which case contains as a special case the second (stricter, and in difference equations as well) release of the economy, and which will be always meant to be also coupled by the household's budget line, shall be (from now on and in the remainder of this paper) referred to as the KSSP-Difference Equation(s) [KSSP-DE(s)], which shall be also (and more compendiously) completely characterising $\{\mathcal{E}_t^{KSSP}\}_{t=0}^{\infty}$.

Note, as a means of saving analytical energy, that the second version (and bifurcation) of the (consolidated into a single) KSSP-DE above is directly viable upon the revealed profits' strictly-positive discretion. Oppositely, as long as it is imminently known that $\pi_t(s) \leq 0$, $s \in (0,1)$ and $t \in \mathbb{N}$, the first bifurcation (the one of the dual KSSP-DE) is straightforwardly viable, without having to worry whether to resort into the second bifurcation or not. If, nevertheless, none profits-related information is in-advance available, the split-up (into two KSSP-DEs) case [which is, eitherways, both cases inclusive] is to be ultimately held analytically liable for consistent reasoning, so no such analytical dilemmas exist. Observe also that when the consolidated KSSP-DE is satisfied, the non-consolidated one is satisfied as well, but the reverse is not true. Finally, while both the consolidated KSSP-DE and the second KSSP-DE of the first (doubly sustainable) bifurcation capture (and may examine and support) the joint convergence of the time-sequences of capital-accumulated and interest rate (in the sense of both of them being convergent simultaneously: one is convergent when/and/given that the other is convergent as well), the first KSSP-DE of the first (twofoldly manifested) bifurcation embodies (and can talk about the opportunity of) the independent or autonomous convergence of the time-sequence of capital-accumulated.

We are now ready to give a precise definition for the (intergenerational) competitive general equilibrium of $\{\mathcal{E}_t^{KSSP}\}_{t\in\mathbb{N}}$.

Definition 1. Let the well-defined δ and K_0, the well-behaved F and $\{K_t, r_t\}_{t=0}^{\infty}$, and $\{C_t, S_t = I_t, \pi_t\}_{t=0}^{\infty}$ being (constructively) defined as above. Then, at $t = 0$ and for every $t \in \mathbb{N}$, an *(in steady states) golden rule competitive general equilibrium* of $\{\mathcal{E}_t^{KSSP}\}_{t\in\mathbb{N}}$ (or, equivalently, for the KSSP-DE subject to the household's budget constraint) is:

(comprised of)

(i) the quadruplet of convergent (to their steady-states) time-sequences

$$[\{C_t^*(Y_t^*, s^*)\}_{t=0}^{\infty}, \{S_t^*(Y_t^*, s^*) = I_t^*(Y_t^*, s^*)\}_{t=0}^{\infty}, \{K_t^*(s^*)\}_{t=1}^{\infty}, \{Y_t^* = F(K_t^*(s^*))\}_{t=0}^{\infty}]$$

(ii) together with the convergent (to its steady-state) time-sequence $\{r_t^*(s^*)\}_{t=0}^{\infty}$,

(iii) and along with the (unique) $s^* \in (0,1)$ being involved above, for which, for every $t \in \mathbb{N}$:

(it simultaneously holds that)

$$C_t^*(\bullet, s^*) = \max_{s\in(0,1)} C_t(\bullet, s)$$

(the representative household globally maximises its consumption, subject to its budget constraint)

and

$$\frac{dF(K_t(s))}{dK_t(s)}|_{s=s^*} = r_t^*(s^*)$$

(the representative firm globally maximises its profits, subject to its production-technology constraint)

To make a start on working productively with the residual theoretical messages that Definition 1 transmits, two clarifying remarks are first needed for its (mathematical-economic) refinement and disambiguation.

Remark 1. As a corollary to Proposition 1, upon the attainment of a golden rule competitive general equilibrium for $\{\mathcal{E}_t^{KSSP}\}_{t\in\mathbb{N}}$, the depicted profits of $\{\mathcal{E}_t^{KSSP}\}_{t\in\mathbb{N}}$ are erased. This practically means that the endogenous production (by a golden rule competitive general equilibrium) of a time sequence of profits need not be incorporated inside Definition 1, since such a sequence is stationary convergent to zero, thus, exists vacuously. Also, using the posterior knowledge of Proposition 1 again, inside Definition 1 one may replace

$$\{Y_t^* = F(K_t^*(s^*))\}_{t\in\mathbb{N}} \text{ with } \{r_t^*(s^*)K_t^*(s^*)[> I_t^*(Y_t^*, s^*)]\}_{t\in\mathbb{N}},$$

but for the sake of preserving (by relying onto the prior information of the model) definitional totality for this competitive general equilibrium concept, this action is not undertaken. Lastly, Proposition 1 makes it clear that for any golden rule competitive general equilibrium one should be always working with the first bifurcation of the KSSP-DE.

Remark 2. *A priori* (when going *for* a golden rule competitive general equilibrium) it holds that, instantaneously, the economy's (real) interest rate value is endogenously generated by the (supply equals demand) clearance condition of the economy's capital-accumulated market, but is actually endogenously computed by the firm's profits (first order) maximisation condition, i.e., by setting the first derivative of the profits function equal to zero, by which condition the interest rate (i.e., the marginal or unit price of capital-accumulated) is equalised with the marginal product of capital-accumulated[23]. *A posteriori* (when being *in* a golden rule competitive general equilibrium) it holds that, instantaneously, the market's (real) interest rate value is (also) given by the average product of capital-accumulated (this is due to

[23]See in the Auxiliary Text of the Appendix.

Remark 1, by which it follows that

$$\{r_t^*(s^*) = \tfrac{F(K_t^*(s^*))}{K_t^*(s^*)} = \tfrac{Y_t^*}{K_t^*(s^*)}\}_{t=0}^{\infty},$$

but there is none inconsistency to be found around in the ambience, since the posterior production function collapses to a linear function without a constant, in which the marginal product and the average product of capital-accumulated coincide).

Next, partially in an attempt to hedge away the risk of misunderstandings and dodge interpretational turbulences as far as Definition 1 is concerned, comes a list of diversified characterisations accompanying the golden rule competitive general equilibrium of $\{\mathcal{E}_t^{KSSP}\}_{t\in\mathbb{N}}$. In the main, the motivation behind these characterisations is to stress that this competitive general equilibrium conception should (desirably) uniquely exist and should be loaded with all the normative humanitarian properties.

Characterisation 1. Each replica economy $\mathcal{E}_t^{KSSP}$, $t = 0, 1, 2, ...$, admits an instantaneous golden rule competitive general equilibrium. At $t = \infty$ specifically, a golden rule competitive general equilibrium for $\mathcal{E}_\infty^{KSSP}(= \lim\limits_{t\to\infty} \mathcal{E}_t^{KSSP})$ is the six-tuplet

$$[\overline{C}(\overline{Y}, s^*), \overline{S}(\overline{Y}, s^*) = \overline{I}(\overline{Y}, s^*), \overline{K}(s^*), \overline{Y} = F(\overline{K}(s^*)), \overline{r}(s^*), s^*] \in \mathbb{R}_{++}^4 \times (0, 1)^2.$$

Characterisation 2. The pair or (more generally) bundle (of convergent time sequences)

$$\{C_t^*(Y_t^*, s^*)\}_{t=0}^{\infty}, [\{S_t^*(Y_t^*, s^*) = I_t^*(Y_t^*, s^*)\}_{t=0}^{\infty} \Rightarrow] \{K_t^*(s^*)\}_{t=1}^{\infty}\}$$

is proclaimed a *golden rule competitive general equilibrium allocation* (pro-allocation and anti-allocation, respectively) of a *golden rule competitive general equilibrium income, product* or *output*, $\{Y_t^* = F(K_t^*(s^*))\}_{t=0}^{\infty}$, of $\{\mathcal{E}_t^{KSSP}\}_{t\in\mathbb{N}}$, while (the convergent time sequence)

$\{r_t^*(s^*)\}_{t=0}^{\infty}$ is declared a *golden rule competitive general equilibrium price* (*real interest rate*) of $\{\mathcal{E}_t^{KSSP}\}_{t\in\mathbb{N}}$ (that is uniquely associated with a golden rule competitive general equilibrium allocation)

and

$s^\star$ is announced as the (unique) *golden* (*decision*) *rule* for which the golden rule competitive general equilibrium is achieved.

Note 1: Analogous characterisation is to be made specifically for the golden rule competitive general equilibrium of $\mathcal{E}_\infty^{KSSP}$ using only the steady-states. Before getting into this, observe that a (dynamic) allocation (which is identified with a pair of time-sequences) contains (countable infinitely many) instantaneous allocative pairs (or bundles). A finite number of instantaneous pairs of allotments (time-ordered and listed, or not), or a pair of subsequences of the original allocation's sequences, constitute a partial allocation (or a sub-allocation) of $\mathcal{E}_t^{KSSP}$, $t \in \mathbb{N}$. The same terminology, however, is to be used for the pro-allocation or the anti-allocation of the allocation.

Characterisation 3. A golden rule competitive general equilibrium for $\{\mathcal{E}_t^{KSSP}\}_{t\in\mathbb{N}}$ exists if and only if

there exists a (unique) s^* such that:

1. the representative household (globally) maximises its consumption at every $t \in \mathbb{N}$,

2. the representative firm (globally) maximises its profits at every $t \in \mathbb{N}$,

and for this s^*:

3. there exists a K_0 such that $\lim_{t\to\infty} K_t^*(s^*) = \overline{K}(s^*, K_0)$,

4. there exists a (unique) $\overline{r}(s^*)$ [the limit of the sequence $\{r_t^*(s^*)\}_{t\in\mathbb{N}}$].

Characterisation 4. The (unique) golden rule competitive general equilibrium of $\{\mathcal{E}_t^{KSSP}\}_{t\in\mathbb{N}}$ exists if and only if

there exists a (unique) s^* such that:

1. the representative household (globally) maximises its consumption at every $t \in \mathbb{N}$,

2. the representative firm (globally) maximises its profits at every $t \in \mathbb{N}$,

and for this s^*:

3. for every K_0, it holds that $\lim_{t\to\infty} K_t^*(s^*) = \overline{K}(s^*)$,

4. there exists a (unique) $\overline{r}(s^*)$ [the limit of the sequence $\{r_t^*(s^*)\}_{t\in\mathbb{N}}$].

Note 2: Given condition 4 of Characterisation 4, Condition 2 of Characterisation 4 allows for the existence of a unique golden rule competitive general equilibrium price (of Characterisation 2), while condition 3 of Characterisation 4 packages on its own (and implies) the unique existence of a (Characterisation's 2) golden rule competitive general equilibrium allocation (of a golden rule competitive general equilibrium product) that couples the previous price.

Note 3: Characterisations 1-4 will be the base (in particular, the logical infrastructure) of the proof of Theorem 1 that follows.

For the next two characterisations a utility function needs to make its debut.

Characterisation 5. Define (at $t = 0$ and for every $t \in \mathbb{N}$) an atemporal (instantaneous) utility (providing) function

$$u : \mathbb{R}_{++} \to \mathbb{R}_{++},\ u(C_t(Y_t, s)) = u(C_t(\bullet, s)) > 0,\ (s, Y_t) \in (0, 1) \times \mathbb{R}_{++}$$

$$[\text{and eventually } (u \circ C_t) : (0, 1) \to \mathbb{R}_{++},\ (u \circ C_t)(s) > 0,\ s \in (0, 1)],$$

for the household. For every $t \in \mathbb{N}$, the function $(u \circ C_t)$ is, now, defined by the same algebraic expression. To host the utility function, the economy is then outlined as

$$\{\mathcal{E}_t^{KSSP}\}_{t=0}^{\infty} = \{\mathbb{R}_+; C_t, S_t = I_t, K_t, r_t, \pi_t, K_0, \delta, F, u\}_{t=0}^{\infty},$$

whilst its new guest-element, u, has to satisfy certain properties. With respect to instantaneous consumption (which, in turn, ends up critically-depending on the stationary savings-investment ratio), this function is injective but not surjective, below bounded by (its infimum) zero, (at least twice) continuously (compositely) differentiable, strictly increasing and strictly concave, while the function's first derivative tends to infinity as instantaneous consumption approaches arbitrarily close to zero (because s does the same). The household's lifetime or trans-generational utility is a (free of properties in this model) mechanism that [apprehends, somehow-cumulatively (i.e., abstractly-additively) combines and] maps derivative-sequences of instantaneous utilities (which are convergent when their primitive-sequences of instantaneous consumptions are convergent) onto $(0, +\infty)$, so it may be denoted formally as

$$v : \mathbb{N} \to \mathbb{R}_{++},\ v(\{u(C_t(\bullet, s))\}_{t \in \mathbb{N}}) > 0$$

$$[\text{and eventually } v : \mathbb{N} \to \mathbb{R}_{++},\ v(\{C_t(\bullet, s)\}_{t=0}^{\infty}) > 0].$$

Since, in a golden rule competitive general equilibrium, the household's instantaneous consumption [versus savings, investment and capital-accumulation, and over

$s \in (0,1)]$ is maximised, its instantaneous utility (function) is (given the properties of this function) maximised as well, so that the household's lifetime utility (map) is definitely (also) maximised (irrespectively of its actual definition, that is to say, irrelevantly onto the instantaneous weights the household assigns to the instantaneous utilities, as utility is mounded along time). Then, by construction, any golden rule competitive general equilibrium allocation for $\{\mathcal{E}_t^{KSSP}\}_{t\in\mathbb{N}}$ is socially efficient (in any utilitarian sense), when, in a regime where consumption (hence, current welfare) races against savings-investment (hence, capital accumulated and future welfare) as time elapses, social optimality is looked through somewhat differentiated (inter-and-trans generational) lenses (for this matter, the reader is also referred to **Comment 7** of the Auxiliary Text of the Appendix).

Characterisation 6. Any golden rule competitive general equilibrium allocation for $\{\mathcal{E}_t^{KSSP}\}_{t\in\mathbb{N}}$ fulfils trivially the equal treatment property. At the same time (**Claim 2**:), none generation envies the (with this general equilibrium conception) allocated consumption of any other generation, to wit, any golden rule competitive general equilibrium allocation for $\{\mathcal{E}_t^{KSSP}\}_{t\in\mathbb{N}}$ is envy-less or envy-free (look for the proof of **Claim 2** in the Appendix). Concluding, on the consumption side, any golden rule competitive general equilibrium pro-allocation for $\{\mathcal{E}_t^{KSSP}\}_{t\in\mathbb{N}}$ is (trans and inter generational) equitable, i.e., fair and impartial (in any utilitarian sense). Additionally, every generation of a golden rule competitive general equilibrium of $\{\mathcal{E}_t^{KSSP}\}_{t\in\mathbb{N}}$ is rewarded (when acting as a factor of production) according to the contribution this generation makes into this generation's product. So this general equilibrium notion is equitable, i.e., just and neutral, on the production side as well.

Note 4: Characterisations 5 and 6 guarantee that a golden rule competitive general equilibrium (which is stable at $t = \infty$) is instantaneously stable as well, so that it cannot be shattered within $T = \{0, 1, 2, ...\}$.

Finally, this section winds up by proving that a competitive general equilibrium notion of Definition 1 uniquely exits and, in this way, reaching the first milestone of the paper.

Theorem 1. The (unique) golden rule competitive general equilibrium of $\{\mathcal{E}_t^{KSSP}\}_{t\in\mathbb{N}}$ exists.

Proof of theorem 1. *See in the Appendix.*
[To stage the proof articulately, the reader should become aware of its preparatory essentials and preliminaries, which are as follow. The proof is broken into 3 distinct and sequentially (strictly with the specific order) occurring parts. Upon the conclusion of the third part the proof is completed and the (unique) golden rule competitive general equilibrium of Definition 1 (that is, for $\{\mathcal{E}_t^{KSSP}\}_{t\in\mathbb{N}}$ or, equivalently, for the KSSP-DE subject to the household's budget constraint) exists. Throughout the whole proof the following are given as $t \in \mathbb{N}$: exogenous (atemporal) $\delta \in (0,1)$ and $K_0 > 0$, exogenous well-behaved (time invariant) F and (time variant) K_t, r_t, and exogenously appropriately constructed (time stationary) $C_t, S_t = I_t, \pi_t$.]

Upon the completion of the proof of Theorem 1 (and when this proof is paired with the proof of Proposition 1), Remark 3 ties up any loose ends. It offers several terminal (and of paramount importance) refinements to the theory that has been proposed so far. It characterises further the golden rule competitive general equilibrium by binding together the information that has been separately produced by the two proofs. It also makes it clear that this theory concedes but does not imitate the results and conclusions of the other relevant theories.

Remark 3. In between the lines of the third part of the proof of Theorem 1 it can be read that irrespectively of whichever and somehow generated $s^\star$ [which can indifferently be (or not be) a golden (decision) rule of a golden rule competitive general equilibrium], the following equivalence is true:

'for some given $K_0 > 0$, the (strictly positively termed) sequence $\{K_t^\star(s^\star)\}_{t=0}^{\infty}$ (which could hypothetically be a golden rule competitive general equilibrium partial anti-allocation, but practically it can never be such as we are about to discover) is convergent to $\overline{K}(s^\star, K_0)$

if and only if

for every $\geq t \in \mathbb{N}$ (see in Part 3 of the proof of Theorem 1 for what this notation stands for), given a well behaved F and a $\delta \in (0,1)$, $K_t^\star(s^\star) > 0$ is the unique value for which is holds that

$$\frac{F(K_t^\star(s^*))}{K_t^\star(s^*)} = \frac{\delta}{s^\star} \iff s^* = \frac{\delta K_t^\star(s^\star)}{F(K_t^\star(s^\star))} \iff s^* F(K_t^\star(s^\star)) = I_t^*(\bullet, s^*) = \delta K_t^\star(s^\star) \iff$$

$$F(K_t^\star(s^\star)) = \frac{\delta}{s^\star} K_t^\star(s^\star) \iff F(K_t^\star(s^\star)) = \frac{\delta}{s^*} K_t^\star(s^\star) \iff K_t^\star(s^\star) = \frac{s^*}{\delta} F(K_t^\star(s^\star))$$

[where the last equality is (from some time-point and on) interpreted as: instantaneously, the average product of capital-accumulated is equal to the fraction of the depreciation ratio over the savings-investment ratio; or equivalently as: instantaneously, the amount of capital-accumulated that is depreciated (gone or lost) is equal to the new investment that is created; or equivalently as: instantaneously, the capital-accumulated to output ratio (aka the share or ratio of capital-accumulated in the output) is the inverse of the previous fraction; or equivalently as: instantaneously, F becomes (collapses to) a Constant Returns to Scale-CRS (homogenous of degree 1) linear (Cobb-Douglas, power) function; and finally, provided that $(0 <)s^* < \delta(< 1)$, equivalently as: instantaneously, capital-accumulated becomes the product's (hence, the growth's) accelerator, where in fact, the inequality $s^* < \delta$ reflects the proper scenario, since it mirrors the interpretation that, across $\mathbb{N}$, the melt down in capital-accumulation is faster than the creation of investment, so capital-accumulated (and the household's coffer) cannot ever hit infinity by fleeing from its steady state]';
this last (balance purporting) condition in this equivalence, apart from being a desirable (and naturally or endogenously arising, either being given the cue from the inequality $s^* < \delta$, or not) restraint on capital-accumulated since (as the equivalence itself puts it) this condition holds and pulls back the accumulated asset's expansion

by ascribing to wealth a (terminal) limit-value, it ultimately enables one to characterise (out of the instantaneous optimal-endogenous output's values) the instantaneous optimal-endogenous values of capital-accumulated (apart from the optimal-endogenous ones of consumption and savings-investment) as below

$$\{S_t^\star(\bullet, s^\star) = I_t^\star(\bullet, s^\star) = s^* F(K_t^\star(s^\star)) = s^* Y_t^*\}_{t=0}^{\infty},$$

$$\{C_t^\star(\bullet, s^\star) = (1 - s^*) F(K_t^\star(s^\star)) = (1 - s^*) Y_t^*\}_{t=0}^{\infty},$$

$$\{K_t^\star(s^\star) = \tfrac{s^*}{\delta} F(K_t^\star(s^\star)) = \tfrac{s^*}{\delta} Y_t^*\}_{t=0}^{\infty},$$

so that not only an optimal-endogenous (consumption-maximising, for example) s^*, which acquires a total endogenous characterisation of itself, fully (and on its own) endogenously generates (and computes) a (golden rule, for example) growth in steady-states, but all the associated to this growth notion (s^*-dependent and endogenously produced) time-sequences evolve in known time laws, i.e., are termed under specified time paths and not under arbitrary time patterns, hence, this growth concept is completely characterised and publicised, and (upon the optimal-endogenous generation of s^*) immediately computed according to this (its) total characterisation; note in advance that the collection of the above (s^*-dependent) time-sequences of numbers cannot refer to the partial allocations of a golden rule (growth to) competitive general equilibrium (see why right after); note further that in the limit (i.e., at $t = \infty$ specifically), this critical condition inside the initially stated equivalence, taking now the particular (in steady-states) expression

$$\frac{\overline{K}(s^*)}{\overline{Y} = F(\overline{K}(s^*))} = \frac{s^\star}{\delta},$$

could have been also obtained from the fundamental neoclassical law (or difference equation) of mobility in time of capital-accumulated, which expression (with the above specifications operating across time and for $t = 0, 1, 2, ..$) could have been rewritten as

$$K^{\star}_{t+1}(s^*) - K^*_t(s^*) = I^*_t(\bullet, s^*) - \delta K^*_t(s^*) \iff \Delta K^{\star}(s^*) = I^*_t(\bullet, s^*) - \delta K^*_t(s^*),$$

and since at $t = \infty$ it holds that $\Delta K^{\star}(s^*) = 0$, it would have been (indeed) implied that in the limit

$$\overline{I}(\bullet, s^*) = s^*\overline{Y} = \delta\overline{K}(s^*) \iff \tfrac{\overline{K}(s^*)}{\overline{Y}} = \tfrac{s^{\star}}{\delta}.$$

Things, however, do not turn up being so much attentive for the golden rule (growth to) competitive general equilibrium (in steady-states). By deeper looking into and combinatorially grasping what is independently communicated and suggested by the proof of Proposition 1 and by the proof of Theorem 1, the following certitude leaks. As attractive it may be according to the previous discussion, while the key limiting growth-characterising condition that is being involved into the previous equivalence is (assuming that it accommodates an s^* that maximises the steady-state of consumption) the necessary and sufficient condition for having achieved a K(HD)SSP type growth in steady-states, the same condition (on its own accounted) is irrelevant for the (unique) limiting existence of a K(HD)SSP style (growth to) competitive general equilibrium (in steady-states). Indeed, there is, first of all, a hidden catch if one translates this limiting condition as a golden rule competitive general equilibrium characterising condition. To find the latent inconsistency, say that this is a self-sustained condition of that ilk and therefore end up with the result

$$\tfrac{s^{\star}}{\delta} = \tfrac{\overline{K}(s^*)}{\overline{Y}} \iff s^{\star} = \tfrac{\delta\overline{K}(s^*)}{F(\overline{K}(s^*))} = \tfrac{\delta\overline{K}(s^*)}{\overline{r}(s^{\star})\overline{K}(s^*)} = \tfrac{\delta}{\overline{r}(s^{\star})} > 1, \text{ since } \overline{r}(s^{\star}) < \delta,$$

which is a contradiction [upon, of course, the embracing of the argumentative agenda of the proof of Proposition 1, since then the argumentative grid of the third part of the proof of Theorem 1 designates the limit (that is to say, the steady-state) of the convergent sequence of the (real) interest rates('s values) that specifically dwells inside Definition 1 to have been squeezed into $(0, \delta)$ (thus, to eventually belong in $(0, 1)$)].

At the same time, one may arrive to the same or to a similar contradiction without necessarily manipulating explicitly the limiting version of this camouflaged as a competitive general equilibrium condition. Indeed, for instance, if the instantaneous consumption is invoked and this condition instantaneously (from some time-instant and onwards) holds, then, by letting $K_t^\star(\bullet) > 0$, $t \in \mathbb{N}$, variating for irrelevant values of $s^\star$, it follows that

$$C_t^*(\bullet, \bullet) = F(K_t^*(\bullet)) - s^* F(K_t^*(\bullet)) = F(K_t^*(\bullet)) - \delta K_t^*(\bullet), \geq t \in \mathbb{N},$$

which function is instantaneously maximised (now with respect to $K_t^\star(\bullet)$, $t \in \mathbb{N}$) within a golden rule competitive general equilibrium, so by the associated first order maximisation condition one acquires that

$$(C_t^*)'(\bullet, \bullet) = 0 \iff F'(K_t^*(\bullet)) = \delta \iff r_t^\star(K_t^*(\bullet)) = \delta, \geq t \in \mathbb{N},$$

i.e., to begin with, one ends up with a contradiction, since the overall conclusion is that $\overline{r}(\overline{K}(\bullet)) = \delta$, when it holds that $\overline{r}(\overline{K}(\bullet)) < \delta$, while, to continue with, one ends up with

$$s^* = \tfrac{\delta K_t^*(\bullet)}{F(K_t^*(\bullet))} = \tfrac{\delta K_t^*(\bullet)}{r_t^\star(K_t^*(\bullet)) K_t^*(\bullet)} = \tfrac{\delta}{r_t^\star(K_t^*(\bullet))} = \tfrac{\delta}{\delta} = 1, \geq t \in \mathbb{N},$$

inside a golden rule competitive general equilibrium, which is an obvious and fundamental violation of the model's harmony in more than one respects. To rephrase this situation, in a (consumption-maximising) K(HD)SSP type growth in steady-states where labour has been paused as a production factor [so the real labour income (or, symmetrically, the real labour cost) cannot boost the denominator of the previous fraction so as to drop the fraction to a value strictly less than 1], and in which instantaneous market-clearing for/in general equilibrium with a profit maximising firm also occurs (so that $F'(K_t^*(\bullet)) = r_t^\star(K_t^*(\bullet))$, $t \in \mathbb{N}$), which general equilibrium is in particular competitive or Walrasian and the firm attains zero maximum profits (so that $F(K_t^*(\bullet)) = r_t^\star(K_t^*(\bullet)) K_t^*(\bullet)$, $t \in \mathbb{N}$), to accept the previous condition as a *passe-partout* general equilibrium law is a tentative conclusion[24]. Now, in this section's model, this fallacious outcome is obtained because (less than) half of the truth is told (and has nothing to do with the absence of labour). When (in the third part of the proof of Theorem 1, given the proof of Proposition 1) all the truth is told, according to which s^* becomes eventually the one that maximises instantaneously both the household's consumption and the firm's profits, while, for this optimal-endogenous s^*, the sequence of the optimal-endogenous capital-accumulated values is jointly convergent with the sequence of the endogenous interest rate values, then the savings-investment rate, the depreciation rate and the interest rate do not ever synchronise in time inside a relationship which equates their values. Instead, the procured (by these two proofs) condition

$$\overline{r}(s^\star) < \delta \iff \overline{r}(s^\star)\overline{K}(s^*) < \delta\overline{K}(s^*) \iff [\overline{r}(s^\star) - \delta]\overline{K}(s^*)$$

ultimately becomes the limiting golden rule competitive general equilibrium characterising condition. This condition states that, in the limit, the marginal supplement (i.e., the marginal interest) of the steady-state value of the capital-accumulated lies strictly below the marginal depletion (i.e., the marginal devaluation) of the same quantity. Hence, along $\mathbb{N}$, the capital-accumulated has been getting worn out faster than getting amplified, and, on this account, has not flown to infinity. The coefficient $|\overline{r}(s^\star) - \delta| \in (0,1)$ applied to $\overline{K}(s^*)$ turns back the net (after depreciation) steady-state augmentation (or accumulation) of the capital in the long run. And as a corollary to the limiting conclusions supra, an (*a posteriori*) endogenous framing (hence, characterisation) of δ inside a golden rule competitive general equilibrium can be also retrieved. To wit,

$$\overline{r}(s^\star)\overline{K}(s^*) < \delta\overline{K}(s^*) \iff F(\overline{K}(s^*)) = \overline{Y} < \delta\overline{K}(s^*) \iff$$

$$\delta > \frac{\overline{Y}}{\overline{K}(s^*)}(=\text{the limiting average product of capital-accumulated}),$$

[24]See in the Auxiliary Text of the Appendix.

so that eventually $\delta \in (\frac{\overline{Y}}{\overline{K}(s^*)}, 1)$. This action is obviously legitimate because

$\frac{\overline{Y}}{\overline{K}(s^*)} < 1 \iff \overline{Y} < \overline{K}(s^*) \iff \overline{r}(s^\star)\overline{K}(s^*) < \overline{K}(s^*) \iff \overline{r}(s^\star) < 1$, which is true.

To carry on the verification of what is recommended by this line of argumentation, the in-reference limiting condition that characterises a golden rule competitive general equilibrium can be extended and written (according to the prequel construction of the model) as

$$\overline{I}(\bullet, s^*) < \overline{r}(s^\star)\overline{K}(s^*) < \delta\overline{K}(s^*) \iff s^*\overline{Y} < \overline{r}(s^\star)\overline{K}(s^*) < \delta\overline{K}(s^*), \text{ so that}$$

1. $s^* < \frac{\overline{r}(s^\star)\overline{K}(s^*)}{\overline{Y}} = \frac{\overline{r}(s^\star)\overline{K}(s^*)}{\overline{r}(s^\star)\overline{K}(s^*)} = 1$, which is true, and

2. $s^* < \frac{\delta\overline{K}(s^*)}{\overline{Y}} = \frac{\delta(s^\star)\overline{K}(s^*)}{\overline{r}(s^\star)\overline{K}(s^*)} = \frac{\delta}{\overline{r}(s^\star)}(> 1)$, which is again true because $s^* < 1$.

Finally, by keeping track with the herein specifications, to familiarise ourselves further and tighter with this competitive general equilibrium notion and gain better conceptual insights with respect to its validity, observe that within a golden rule competitive general equilibrium the instantaneous elasticity of the production function (or of the output) with respect to (its original independent or explanatory variable, or input) capital-accumulated is constant and, in fact, equal to 1. Indeed, as $t \in \mathbb{N}$ and irrelevantly to any attained s^*:

$$\epsilon_{t}{}^{F}_{K_t^*(\bullet)}(K_t^*(\bullet)) = \frac{dlnF(K_t^*(\bullet))}{dlnK_t^*(\bullet)} = \frac{\frac{dF(K_t^*(\bullet))}{F(K_t^*(\bullet))}}{\frac{dK_t^*(\bullet)}{K_t^*(\bullet)}} = \frac{dF(K_t^*(\bullet))}{dK_t^*(\bullet)}\frac{K_t^*(\bullet)}{F(K_t^*(\bullet))} = r_t^*(\bullet)\frac{K_t^*(\bullet)}{r_t^*(\bullet)K_t^*(\bullet)} = 1.$$

At the same time, the instantaneous contribution (share, ratio, fraction, part, portion or percentage) of the capital accumulated's income onto the economy's income is 1 (or 100%). Indeed, as $t \in \mathbb{N}$ and irrelevantly to any attained s^*:

$$Y_t^* = r_t^*(\bullet)K_t^*(\bullet) \iff \frac{r_t^*(\bullet)K_t^*(\bullet)}{Y_t^*} = 1.$$

All things considered, inside a golden rule competitive general equilibrium, the production function is a CRS (linearly homogenous) Cobb-Douglas (power) function (as it actually is).

Appendix

1. List of Proofs

Proof of Claim 1

Let $s \in (0,1)$ for the whole proof. Since, by exogenous assumption, $K_0 > 0$ and $(1-\delta) > 0$, it follows that $(1-\delta)K_0 > 0$ and then, since $I_0(Y_0, s) > 0$ by construction, $K_1(\bullet) = [I_0(Y_0, s) + (1-\delta)K_0] > 0$, which then secures (for $I_1(Y_1, s) > 0$) that $K_2(\bullet) = [I_1(Y_1, s) + (1-\delta)K_1(\bullet)] > 0$, which in turn ensures (for $I_2(Y_2, s)) > 0$) that $K_3(\bullet) = [I_2(Y_2, s) + (1-\delta)K_2(\bullet)] > 0$, so that one may iteratively conclude that (for $I_{t-1}(Y_{t-1}, s) > 0$) $K_t(\bullet) = [I_{t-1}(Y_{t-1}, s) + (1-\delta)K_{t-1}(\bullet)] > 0$ in general, for $t = 1, 2, ...$ and $K_0 > 0$ and $\delta \in (0,1)$ given.

PS_0: For $K_0 > 0$, the same conclusion can be derived (recursively again, but straightforwardly) by using the second (independent) difference equation of the capital-accumulated, which will be defined in the sequel main text.

Proof of Claim 2

Pick a time instant $t \in \mathbb{N}$. Within a golden rule competitive general equilibrium of $\{\mathcal{E}_t^{KSSP}\}_{t\in\mathbb{N}}$ the utility of date's t generation is $(u^* \circ C_t^*)(s^*) > 0$. Pick then a time point $(t \neq)q \in \mathbb{N}$. Within the same golden rule competitive general equilibrium of $\{\mathcal{E}_t^{KSSP}\}_{t\in\mathbb{N}}$, the utility of date's t generation accrued by date's q consumption is $(u^* \circ C_q^*)(s^*) > 0$. Generation t envies generation's q instantly allocative consumption quantity within a golden rule competitive general equilibrium of $\{\mathcal{E}_t^{KSSP}\}_{t\in\mathbb{N}}$ if and only if $(u^* \circ C_t^*)(s^*) < (u^* \circ C_q^*)(s^*)$. Since $(u \circ C_t)$ has the same functional format across $\mathbb{N}$, it follows that $(u^* \circ C_t^*)(s^*) = (u^* \circ C_q^*)(s^*)$. Therefore, generation t is not jealous of generation's q consumption. In general, upon a golden rule competitive general equilibrium's (partial) pro-allocation, there does not exist a generation that becomes instantaneously strictly better off (in terms of utilitarian welfare) with some other's generation consumption.

Proof of Proposition 1

Let the stationary consumption, savings-investment and profits functions defined as in the main text of section 2. For given functions $K_t, r_t > 0$, as $t \in \mathbb{N}$, and for every $(s, Y_t) \in (0,1) \times \mathbb{R}_{++}$ it holds that

$$Y_t = C_t(Y_t, s) + I_t(Y_t, s) = r_t(s)K_t(s) + \pi_t(s), \pi_t(s) \in \mathbb{R},\ t \in \mathbb{N}.$$

For $s \in (0,1)$, suppose now that $\pi_t(s) > 0$, at some $t \in \mathbb{N}$. Then, for this $t \in \mathbb{N}$, as $s \in (0,1)$, it holds that

$$C_t(\bullet, s) + I_t(\bullet, s) > r_t(s)K_t(s) \iff C_t(\bullet, s) > r_t(s)K_t(s) - I_t(\bullet, s).$$

Since the range of (the not necessarily surjective with respect to s) C_t is $\mathbb{R}_{++}$, it follows that[25]

$$r_t(s)K_t(s) - I_t(\bullet, s) \geq 0 \iff r_t(s)K_t(s) \geq I_t(\bullet, s) = K_{t+1}(s) - (1-\delta)K_t(s),$$

or equivalently

$$[r_t(s) + (1-\delta)]K_t(s) \geq K_{t+1}(s) \iff r_t(s) + (1-\delta) \geq \tfrac{K_{t+1}(s)}{K_t(s)}\ [1],$$

where (to backup the validity of [1]) $[r_t(s) + (1-\delta)] > 0$, at this t, as $s \in (0,1)$ and for a given $\delta \in (0,1)$. At the same time, it is independently (and simultaneously) true that

$$(1-\delta)K_t(s) + r_t(s)K_t(s) = K_{t+1}(s) \iff [r_t(s) + (1-\delta)]K_t(s) = K_{t+1}(s)[2],$$

[25]For this argument see also in the comments of PS_1 at the end of the proof of Proposition 1.

at this t, as $s \in (0,1)$ and for a given $\delta \in (0,1)$. Next, since, for $s \in (0,1)$, we know in advance that

$$\lim_{t\to\infty} r_t(s) = \overline{r}(s),$$

we can consider, without loss of generality, the representative $\overline{r}(s)$ along $\mathbb{N}$, in which case [for whichever specified $s, \delta \in (0,1)$] equation [2] becomes the first oder (i.e., with one time gap) and homogenous (i.e., with zero constant) linear difference equation with a constant coefficient

$$K_{t+1}(s) = [\overline{r}(s) + (1-\delta)]K_t(s),\ t = 0, 1, 2, ...,$$

the (dependant on $K_0 > 0$) 'general solution form' of which (effectively, the one conveying all the possible K_0 particular solutions) turns out (from the relevant theory) to be

$$K_{t+1}(s) = [\overline{r}(s) + (1-\delta)]^t K_0,\ t = 0, 1, 2, ...$$

This difference equation has an asymptotically stable equilibrium solution-orbit (and attains equilibrium values)[26] if and only if

$$\overline{r}(s) + 1 - \delta < 1 \iff \overline{r}(s) < \delta \text{ [hence, } \overline{r}(s) \in (0,1)\text{, since } \delta \in (0,1)],$$

for some determined $s, \delta \in (0,1)$. Now, for some given $K_0 > 0$ and as $s \in (0,1)$), we know beforehand that

$$\{K_t(s)\}_{t=0}^{\infty} \to \overline{K}(s, K_0),$$

[26]This is the normative characterisation concerning this difference equation. See its meaning in PS_2 at the end of the proof of Proposition 1.

so that the condition $\overline{r}(s) < \delta$ is for sure valid, while also, as $t \to \infty$, for some given $K_0 > 0$ and as $s \in (0,1)$, it follows that

$$K_{t+1}(s, K_0) = K_t(s, K_0) = \overline{K}(s, K_0),$$

so that (upon this last conclusion) [1] becomes $\overline{r}(s) \geq \delta$, which is a contradiction (to the already valid opposite condition). So finally, as $s \in (0,1)$, it holds that $\pi_t(s) \leq 0$ at this (randomly) picked $t \in \mathbb{N}$. This guarantees that the maximum profit at every time point is zero.

PS_1: Strictly (and more accurately) speaking, since profits are strictly positive, the implied relationship $r_t(s)K_t(s) \geq I_t(\bullet, s)$, $t \in \mathbb{N}$, $s \in (0,1)$, holds with the strict equality (see in the main text of section 2), so that the veracity that consumption is, instantaneously, strictly positive simply follows. Indeed, the proof that follows (up until its terminal contradiction) continues to be valid if just the equality is summoned. Here, given that this action is permitted by the autonomous context of the proof since we do not know yet that the instantaneous consumption starts from being arbitrarily close to zero, the weaker inequality is employed for reasons of generality. And the general conclusion by combining, herein, the (main text's) case of the (strict) inequality with the (main text's) case of the (strict) equality is that, simultaneously, the (real) income from asset-accumulated should be always (i.e., in any case) definitely enough to cover (and should be the one that actually pays) for the (real) spending for (savings-)investment.

PS_2: This (desirable) condition means that the difference equation is solved with convergent K_0-initiated time-sequences. In particular, we can have two cases. First, for any (all) K_0, the generated time-sequence of capital-accumulated converges to some (unique) K_0-dependent limit. Second, and more strictly, for whichever K_0, the generated time-sequence of capital-accumulated (uniformly) converges to the same/uniform (unique) K_0-independent limit. In the second (uniform convergence) case, the difference equation attains (more prudently) a unique general equilibrium value. (Undesirably,) the difference equation has an asymptotically unstable equilibrium solution-trajectory (and does not attain equilibrium values) if and only if for some/any K_0 the produced time-sequence of capital accumulated either diverges to infinity, or may be terminating to some unknown limiting value.

PS_3: In this postscript, the procedure of another way to prove this proposition is also described. If one assumes that at some picked $t \in \mathbb{N}$ the profits are strictly positive, then (when the two involved sequences are convergent) directly from the (stricter) consolidated KSSP-DE one comes up with the limiting condition $\overline{r}(s) = \delta$, $s \in (0,1)$. At the same time however, as an implication, the (weaker, non-consolidated) split up into two different KSSP-DEs should be satisfied (as well), because this general case for (real) profits being any real number contains the particular case where the (real) profits are a strictly positive number. But then, the proof shows that $\overline{r}(s) < \delta$, $s \in (0,1)$, which inequality violates the (having originally assumed to be true) equation between the two terms.

Proof of Theorem 1

Part 1. In this part of the proof it is shown that, for a golden rule competitive general equilibrium, there exists a unique s^* which (uniformly in time) globally maximises the aggregate consumption at each time instant. Consider $C_t : \mathbb{R}_{++} \times (0,1) \to \mathbb{R}_{++}$, defined by $C_t(\bullet, s) = (1-s)F(K_t(s))$, which is the (appropriately constructed) representative household's consumption function at every $t \in \mathbb{N}$. Clearly, at every $t \in \mathbb{N}$, as $s \to 1$, $C_t(\bullet, s) \to 0$. At the same time, for $t = 0, 1, 2, ...$, as $s \to 0$ [equivalently, as $C_t(\bullet, s) \to F(K_t(s))$], $K_t(s) \to 0$ (by construction) and, subsequently, $F(K_t(s)) \to 0$ (again by construction), so that, eventually, $C_t(\bullet, s) \to 0$. The first partial derivative of $C_t(\bullet, s)$ is

$$\tfrac{\theta C_t(\bullet, s)}{\theta s} = -F(K_t(s)) + (1-s)F'(K_t(s)),\ t = 0, 1, 2, ...,$$

which is of indeterminate sign, thus, C_t, $t = 0, 1, 2, ..$, is of indeterminate monotonicity on its partial (with respect to s only) domain [see also in the main text (in section

2) of the paper]. For the graphical depiction of this function (see right ahead) useful becomes the following property. While it will be shown in this part of the proof that

$$\lim_{s \to s^*} C_t'(\bullet, s) = 0,\ t \in \mathbb{N},$$

for a unique s^*, it simultaneously holds that

$$\lim_{s \downarrow 0} C_t'(\bullet, s) = +\infty \text{ and } \lim_{s \uparrow 1} C_t'(\bullet, s) = -\infty, \text{ for every } t \in \mathbb{N}.$$

The validity of the first limit above relies onto the $F'(0) = \infty$ property of the well-behaved production function with respect to capital-accumulation. The validity of the second one is based onto the hypothesis of the main text that the instantaneous capital-accumulated function is above unbounded with respect to $s \in (0,1)$, while additionally, on the fact that the well-behaved F has no upper bound with respect to capital-accumulation. Subsequently, the second partial derivative of $C_t(\bullet, s)$, for $t = 0, 1, 2, ...,$ is

$$\tfrac{\theta C_t^2(\bullet, s)}{\theta s^2} = [-F(K_t(s)) + (1-s)F'(K_t(s))]' =$$

$$= -F'(K_t(s)) + (1-s)F''(K_t(s)) - F'(K_t(s)) = -2F'(K_t(s)) + (1-s)F''(K_t(s)),$$

where $F'(K_t(s)) > 0$, $\forall K_t(s) > 0$, $\forall s \in (0,1)$ [by applying the chain rule of differentiation; see also in the main text (in section 2) of the paper], $(1-s) \in (0,1)$ and by appealing to the Faà di Bruno's formula we get that

$$F''(K_t(s)) = \tfrac{\theta F^2(K_t(s))}{\theta s^2} = \tfrac{d^2F(K_t(s))}{dK_t^2(s)}[\tfrac{dK_t(s)}{ds}]^2 + \tfrac{dF(K_t(s))}{dK_t(s)}\tfrac{d^2K_t(s)}{ds^2},$$

in which $\frac{d^2F(K_t(s))}{dK_t^2(s)} < 0$, $\forall K_t(s) > 0$, $\forall s \in (0,1)$ (by construction), $\frac{dF(K_t(s))}{dK_t(s)} > 0$, $\forall K_t(s) > 0$, $\forall s \in (0,1)$ (by construction as well) and $\frac{d^2K_t(s)}{ds^2} < 0$, $\forall s \in (0,1)$ (also by construction), so that, eventually, $F''(K_t(s)) < 0$, $\forall K_t(s) > 0$, $\forall s \in (0,1)$, which finally implies that

$$\frac{\theta C_t^2(\bullet,s)}{\theta s^2} < 0,\ \forall s \in (0,1),$$

equivalently, C_t, $t = 0, 1, 2, ..,$ is (globally) strictly concave with respect to its argument s. By combining all the results obtained above regarding the properties of C_t, $t \in \mathbb{N}$, it is clear that the strictly concave C_t, $t = 0, 1, 2, ..,$ when being visualised as variating with respect to $s \in (0, 1)$ exclusively, starts from being arbitrarily close to zero, then, strictly increases on $(0, s^*)$ and up to attaining its global maximum (which is strict, i.e., is attained for a unique s^*), and, finally, strictly decreases on $(s^*, 1)$ and up until approaching zero arbitrarily close again. For more clarity, see in graph 1 of the List of Figures below in the Appendix (which is plotted for some $t \in \mathbb{N}$). Since the consumption function is (time) stationary by construction, $s^\star$ has to be such. To conclude, for a golden rule competitive general equilibrium, there exists a unique (and uniform along $\mathbb{N}$) s^* such that

$$C_t^*(\bullet, s^*) = \max_{s\in(0,1)} C_t(\bullet, s),\ \forall t \in \mathbb{N},$$

or equivalently, the set

$$\arg\max_{s\in(0,1)} C_t(\bullet, s)$$

is a (the same) singleton, $\forall t \in \mathbb{N}$.

Part 2. In this part of the proof it is shown that, for a golden rule competitive general equilibrium, the attained s^* in Part 1 of the proof (which, uniformly in time, uniquely globally-maximises the representative household's budget-constrained consumption instantaneously) also maximises (uniformly in time and globally) the representative firm's (constrained by production or technology) profits instantaneously (which maximum profits, when eventually Proposition 1 comes into effect after the completion of the third part of this proof, are constantly zero). Initially, autonomously and outside the hunt of a golden rule competitive general equilibrium, it is proven that the profits' maximisation condition (or equality) of Definition 1 is satisfied for exactly one $s \in (0,1)$. For every $t \in \mathbb{N}$, by composing $K_t : (0,1) \to \mathbb{R}_{++}$ with $F : \mathbb{R}_{++} \to \mathbb{R}_{++}$ and obtaining $(F \circ K_t) : (0,1) \to \mathbb{R}_{++}$, where $(F \circ K_t)(s) > 0$, $s \in (0,1)$, one has that

$$(F \circ K_t)'(s) = F'(K_t(s)) = \frac{\theta F(K_t(s))}{\theta s} = \frac{dF(K_t(s))}{dK_t(s)} \frac{dK_t(s)}{ds} = \frac{d(F \circ K_t)(s)}{dK_t(s)} K_t'(s) = \frac{d(F \circ K_t)}{dK_t}(s) K_t'(s),$$

where by construction: $\frac{d(F \circ K_t)}{dK_t}(s) > 0$ and $K_t'(s) > 0$, $\forall s \in (0,1)$.

But since (also by construction and at every time-instant) F is strictly concave with respect to capital-accumulation, so that F' is strictly decreasing with respect to capital-accumulation, one concludes that $\frac{d(F \circ K_t)}{dK_t}$, $t = 0, 1, 2, ...$, which is a function of s, is a (strictly positive and continuously differentiable, for sure, and) strictly decreasing function with respect to s [indeed, by definition and at every time-point, capital-accumulation increases (decreases) if and only if the savings-investment rate increases (decreases)]. At the same time, $r_t > 0$, $t = 0, 1, 2, ...$, which is a (bijective) strictly positive function also of $s \in (0,1)$, is (by construction) continuously differentiable and strictly increasing on s, while it is above bounded by 1 (its supremum) and below bounded by zero (its infimum). Of course, there is not a solid guarantee that the two functions intersect (necessarily at one point, if the case of intersection is valid), since the derivative function of F is not necessarily a surjection. So, (again) by co-employing the visual-schematic aid via plotting the one function versus the other at the same plane for any $t \in \mathbb{N}$ so that graph 2 of the List of Figures below in the Appendix arises for some $t \in \mathbb{N}$[27], it needs to be formally shown that the two functions intersect at exactly one point. This is done as follows. Fix a $t \in \mathbb{N}$ and assume initially that $\frac{d(F \circ K_t)}{dK_t}$ starts by being strictly above r_t and remains (while strictly plunges with respect to s) strictly above it as s evolves, that is, as $r_t(s), \frac{d(F \circ K_t)}{dK_t}(s) > 0$ for $s \in (0,1)$, it holds that

[27] In this figure, for instructive purposes, (the important parts of) both $\frac{d(F \circ K_t)}{dK_t}$ and r_t are depicted as straight lines for some fixed $t = 0, 1, 2, ...$, Nothing, of course, changes if they are pictured as (strictly decreasing and strictly increasing, respectively) smooth curves.

$$\frac{d(F \circ K_t)}{dK_t}(s) > r_t(s) \iff \frac{\frac{d(F \circ K_t)}{dK_t}(s)}{r_t(s)} > 1,$$

both for every $s \in (0, 1)$ and as $s \to 0, 1$. Since (by construction) K_t with respect to s has no upper bound, it holds that $K_t(s) \to \infty$ as $s \to 1$, so that the following condition is then true

$$\lim_{s \to 1 \iff K_t(s) \to \infty} \left[\frac{d(F \circ K_t)}{dK_t}(s) = \frac{dF(K_t(s))}{dK_t(s)}\right] = 0$$

by the well-behaviour of F with respect to capital-accumulation, so that finally

$$\lim_{s \to 1} \frac{\frac{d(F \circ K_t)}{dK_t}(s)}{r_t(s)} = \frac{0}{1} = 0,$$

with which we arrive to the contradiction, because we have initially assumed that the former limit is strictly greater than one. Pick now a $t \in \mathbb{N}$ and consider the other case where $\frac{d(F \circ K_t)}{dK_t}$ starts by being strictly below r_t and remains (while strictly descends with respect to s) strictly below it as s evolves. Then, by mimicking the prior argumentation but with the employment of $s \to 0$ this time, one arrives again to a contradiction, by having begun with the same as before limit being strictly smaller than one, while ending up with this limit tending to (plus) infinity. Concluding, at every $t \in \mathbb{N}$, the two functions necessarily have a point of intersection, which is exactly one given their graphical shapes and/or given their mathematical properties. Finally, for a golden rule competitive general equilibrium, it is proven that the s-coordinate of this unique crossing-point is not arbitrary, but coincides at every time instant with the (time-stationary) $s^\star \in (0, 1)$ of Part 1 of the proof. Once more, select a (whichever) $t \in \mathbb{N}$ and say that $s' \in (0, 1)$ with $s' \neq s^\star$ is the unique s that maximises (subject to production or technology constraints) the firm's profits at this t, in which (already) the following equality (i.e., the household's budget constraint) is certainly true for an s^* (consumption maximising) golden rule competitive general equilibrium:

$$C_t^*(\bullet, s^*) + I_t^\star(\bullet, s^\star) = r_t^\star(s^\star)K_t^\star(s^\star) + \pi_t^\star(s^\star),$$

at which equality, apart from the maximum consumption, all the rest of the golden rule competitive general equilibrium values follow immediately from the consumption maximising $s^\star$ as well. Then, at this $t \in \mathbb{N}$, $s' \neq s^\star$ does not maximise C_t, because $s^\star$ uniquely does so, so for s' it holds that

$$C_t(\bullet, s') < C_t^*(\bullet, s^*) = \max_{s\in(0,1)} C_t(\bullet, s),$$

and consequently for an s' (profits maximising) and ($\neq$) an s^* (consumption maximising) golden rule competitive general equilibrium it holds that

$$C_t^*(\bullet, s') + I_t^\star(\bullet, s^\star) < r_t^\star(s^\star)K_t^\star(s^\star) + \pi_t^\star(s^\star),$$

which is a direct violation of the household's linearity in its budget constraint. By means of contradiction, this means that $s' = s^\star$ for a golden rule competitive general equilibrium (that is, the same $s \in (0, 1)$ which maximises consumption maximises profits as well).

Part 3. Part 3 of the proof works its way through by not explicitly taking for granted the entrenchment of the previous 2 parts of the proof. In particular, the proof at this part proceeds according to the following methodology. It shows that the sequences of 'some' optimal values of capital accumulation and 'some' general equilibrium values of interest rates are *independently convergent* [by reaching uniform (i.e., the same always) steady states] on any (i.e., irrespectively of 'some') optimally-endogenously arisen s^*, hence, specifically on the s^* of the previous two parts of the proof. Then, it is implied that the two sequences are *jointly convergent* as well on the (uniquely and uniformly in time attained) s^* which (globally) maximises both consumption and profits (and then, finally, postscript 5 is implied). To embark upon the execution of this project, let us commence by deducing (by invoking the Bolzano-Weierstrass Theorem) that there exists a subsequence of the (uniformly) bounded (from above, by the sequence's supremum value 1) sequence (of strictly positive terms) $\{r_t^\star(\bullet\}_{t=0}^\infty$ which converges, which truncated sequence sufficiently characterises the original sequence, since it is (again) definitely uniformly (above) bounded by 1. To conclude the proof, it needs to be proven that the (optimally termed) sequence $\{K_t^\star(\bullet)\}_{t=0}^\infty$ (uniformly) converges (to its unique optimal limit) for any (whichever) given $K_0 > 0$. To engage into this task, we first need to show that the particular sequence is convergent for some (particularly given) K_0. Consider, for this purpose, the (subject to the household's budget line) partial KSSP-DE of the first relevant bifurcation of the economy:

$$\text{‘}\{K_{t+1}^*(\bullet) = (1-\delta)K_t^*(\bullet) + s^*F(K_t^*(\bullet))\}_{t=0}^\infty,$$
$$\text{for some (particularly) given capital-stock (initial condition) } K_0^\star(\bullet) := K_0 > 0\text{’},$$

in which, upon the preceding argumentation, $K_t^\star(\bullet) > 0$ is left to be variating at every $t \in \mathbb{N}$ for irrelevant values of $s^\star$, so that $K_t^\star(\bullet) > 0$, $t \in \mathbb{N}$, becomes the instantaneous instrumental variable in production. Then, for some (particularly) given $K_0 > 0$, it holds that:

$$\{K_t^*(\bullet)\}_{t=0}^\infty \text{ is convergent to (its uniquely existing-attained limit) } \overline{K}(\bullet, K_0),$$

or equivalently

$$\lim_{t\to\infty} K_t^*(\bullet) = \overline{K}(\bullet, K_0) \iff \lim_{t\to\infty} [K_{t+1}^*(\bullet) - K_t^*(\bullet)] = 0,$$

or equivalently

$$\text{for every } q \in \mathbb{N} \text{ (after } q \geq t^{28}\text{, for some } t \in \mathbb{N})^{29}\text{ , } K_{t+1}^*(\bullet) = K_t^*(\bullet)[= \overline{K}(\bullet, K_0)],$$

[28]That is, after q having exceeded t.

[29]Form now on, this condition will be notated as: $\forall\ (\geq t) \in \mathbb{N}$. But independently, t will be still indexing the time-sequences along the whole $\mathbb{N}$, maintaining the general characterisation $\mathbb{N} \ni t$. With these arrangements, no confusion shall be arising.

or

equivalently (by performing the appropriate replacements inside the KSSP-DE)

$$\text{for every} \geq t,\ \delta K_t^*(\bullet) = s^* F(K_t^*(\bullet))[= I_t^*(\bullet, \bullet)]$$

$$\text{[and, specifically, } \delta \overline{K}(\bullet, K_0) = s^* F(\overline{K}(\bullet, K_0)) = \overline{I}(\overline{Y}, \bullet), \text{ at } t = \infty].$$

Now let the stationary (that is, with constant in time functional representation) functions $f, g : \mathbb{R}_{++} \to \mathbb{R}_{++}$, which are defined by

$$f(K_t^*(\bullet)) = s^* F(K_t^*(\bullet))$$

and

$$g(K_t^*(\bullet)) = \delta K_t^*(\bullet),$$

respectively, for every $t \in \mathbb{N}$. Then, for some (particularly) given $K_0 > 0$:

the (particularly) K_0-initiated sequence $\{K_t^\star(\bullet)\}_{t=0}^\infty$ converges

[to its unique steady state $\overline{K}(\bullet, K_0) > 0$]

if and only if

at every $\geq t \in \mathbb{N}$, the two functions f, g have a unique point of intersection

[namely, the (unique) point $\{\overline{K}(\bullet, K_0), (g[\overline{K}(\bullet, K_0)] = f[\overline{K}(\bullet, K_0)])\}$, where $\overline{K}(\bullet, K_0) > 0$ is the unique limit of the (particularly) K_0-initiated sequence $\{K_t^\star(\bullet)\}_{t=0}^\infty$].

Once more, figurative support is more than welcomed for the continuation of the proof. Fundamentally, in order to launch the graphs of f, g as $t \in \mathbb{N}$, notice that since f, g are (in therms of their algebraic expression) time invariant along $\mathbb{N}$, the graphical depictions of both of them also remain the same across $\mathbb{N}$. So, if a $\geq t \in \mathbb{N}$ is picked (which is also noted with t) and f versus g are plotted in the same plane (see in graph 3 of the List of Figures below in the Appendix), it has to formally deduced that f and g have a unique point of intersection at this time-instant. Towards proving that f and g intersect at exactly one point at this $\geq t \in \mathbb{N}$, assume first that g starts by being strictly below f and that f and g never cross as $K_t^\star(\bullet) \to \infty$, i.e., given that the inequality infra involves exclusively strictly positive elements,

$$\delta K_t^\star(\bullet) < s^\star F(K_t^\star(\bullet)) \iff \tfrac{F(K_t^\star(\bullet))}{K_t^\star(\bullet)} > \tfrac{\delta}{s^\star} > 0,$$

[for every $K_t^\star(\bullet) \in (0, +\infty)$ and] as $K_t^\star(\bullet) \to +\infty$,

or equivalently

$$\lim_{K_t^\star(\bullet) \to \infty} \tfrac{F(K_t^\star(\bullet))}{K_t^\star(\bullet)} > 0.$$

The last limit gives the indeterminate form $\frac{\infty}{\infty}$, since $\lim\limits_{K_t^\star(\bullet) \to \infty} F(K_t^\star(\bullet)) = \infty$,

given that F is above unbounded (with respect to capital-accumulation). Then, the L'Hospital's rule is applicable, from which it follows

(by noting that $\lim\limits_{K_t^\star(\bullet) \to \infty} F'(K_t^\star(\bullet)) = 0$, by appealing again to the well-behaviour of F with respect to capital-accumulation)

$$\text{that } \lim_{K_t^\star(\bullet) \to \infty} \tfrac{F(K_t^\star(\bullet))}{K_t^\star(\bullet)} = \lim_{K_t^\star(\bullet) \to \infty} \tfrac{F'(K_t^\star(\bullet))}{[K_t^\star(\bullet)]'=1} = \tfrac{0}{1} = 0,$$

which is a contradiction. Consider afterwards the second possible scenario at the originally picked $\geq t \in \mathbb{N}$, the one that g starts by being strictly above f and that

f and g never meet as $K_t^\star(\bullet) \to \infty$. Then, exactly analogously to before, one may arrive to a contradiction. Consequently, at the selected $\geq t \in \mathbb{N}$, f and g definitely intersect, while, by their graphical shapes and/or their mathematical properties, their point of intersection is at most one. Finally, we need to move towards securing that the sequence of 'some' (that we have in hand) $s^\star$-optimal capital accumulated values converges to the same (unique) steady state for (and irrespectively of) any/whichever given $K_0 > 0$. To pick up where we left off, re-write (equivalently) the partial KSSP-DE that is being employed as

$$\text{`}\{K_{t+1}^\star(\bullet) - K_t^\star(\bullet) = s^\star F(K_t^\star(\bullet)) - \delta K_t^\star(\bullet)\}_{t=0}^{\infty},$$

for some initially given $K_0 > 0$ (and subject to the household's budget constraint)',

while continue making reference to graph 3. Then understand that since graph 3 is perpetually (for $t = 0, 1, 2, ...$) replicated, any term of the sequence $\{K_t^\star(\bullet)\}_{t=0}^{\infty}$, in which some $K_0 > 0$ is given, can be captured by the (same) graph 3 that is picked for some (whichever) $t \in \mathbb{N}$. So (i) fix a $\geq t \in \mathbb{N}$, (ii) fix its graph 3 and (iii) fix the corresponding to it $\overline{K}(\bullet, K_0) > 0$ (which is generated by the previous unique intersection mechanism) to make reference into, and consider (exhaustively) all the following three alternative cases. Start with the case of some given $K_0 > 0$ with $K_0 > \overline{K}(\bullet, K_0) > 0$. Then, as $\{K_t^\star(\bullet)\}_{t=1}^{\infty}$ evolves (no matter how), it is clear (refer for example to graph 3) that it always holds that

$$\{s^\star F(K_t^\star(\bullet)) < \delta K_t^\star(\bullet) \iff s^\star F(K_t^\star(\bullet) - \delta K_t^\star(\bullet) < 0\}_{t=0}^{\infty},$$

or equivalently (from the preceding definition of the partial KSSP-DE) that

$$\{K_{t+1}^\star(\bullet) - K_t^\star(\bullet) < 0 \iff K_{t+1}^\star(\bullet) < K_t^\star(\bullet)\}_{t=0}^{\infty},$$

so that, equivalently, the sequence $\{K_t^\star(\bullet)\}_{t=0}^{\infty}$, in which the $K_0 > \overline{K}(\bullet, K_0)$ is given, is strictly decreasing. By referring (for clarity) to graph 3 again, it follows that $\overline{K}(\bullet, K_0)$ is the greater lower bound (and K_0 is the least upper bound) of that (strictly decreasing and bounded) sequence, so the sequence $\{K_t^\star(\bullet)\}_{t=0}^{\infty}$ [whenever we start with whichever $K_0 > \overline{K}(\bullet, K_0)$] converges to its infimum $\overline{K}(\bullet, K_0)$. Continue with the case of some given $K_0 > 0$ with $K_0 < \overline{K}(\bullet, K_0)$. Then, as $\{K_t^\star(\bullet)\}_{t=1}^{\infty}$ evolves (no matter how), it is clear (refer as a means of verification to graph 3 once more) that it necessarily holds that

$$\{s^\star F(K_t^\star(\bullet)) > \delta K_t^\star(\bullet) \iff s^\star F(K_t^\star(\bullet)) - \delta K_t^\star(\bullet) > 0\}_{t=0}^{\infty},$$

or equivalently (from the preceding definition of the partial KSSP-DE) that

$$\{K_{t+1}^\star(s^\star) - K_t^\star(s^\star) > 0 \iff K_{t+1}^\star(s^\star) > K_t^\star(s^\star)\}_{t=0}^{\infty},$$

so that, equivalently, the sequence $\{K_t^\star(\bullet\}_{t=0}^{\infty}$, in which the $K_0 < \overline{K}(\bullet, K_0)$ is given, is strictly increasing. By appealing (to acquire better understanding) to graph 3 again, it follows that $\overline{K}(\bullet, K_0)$ is the least upper bound (and K_0 is the greater lower bound) of that (strictly increasing and bounded) sequence, so the sequence $\{K_t^\star(\bullet)\}_{t=0}^{\infty}$ [whenever we start with whichever $0 < K_0 < \overline{K}(\bullet, K_0)$] converges to its supremum $\overline{K}(\bullet, K_0)$ (i.e., to the same value as in the first case). In the two alternative cases that were presented so far, the argumentative conclusions came as a combined consequence of the completeness property of the Real Numbers (the least upper bound property and the greatest lower bound property of any subset of $\mathbb{R}$) and the Monotone Convergence Theorem. Finish off with the last possible scenario where convergence is hit from the very beginning, i.e., for the given $K_0 > 0$ it holds that $K_0 = \overline{K}(\bullet, K_0)$. Then unstoppably, straight from the beginning of the sequence $\{K_t^\star(\bullet)\}_{t=0}^{\infty}$, its (unique) limit $\overline{K}(\bullet, K_0)$ (which is the same as with the two previous scenarios) is attained, i.e., we have the (trivial) case of stationary convergence. The proof is concluded.

PS_4: The obtained (strict) inequality at the end of Part 2 signifies to the forbidden (and in general puzzling) case of a golden rule competitive general equilibrium with free disposal (or general glut), according to which the household does not exhaust (into expenditure or spending) all its income at this t, so that there is (disliked) output left undisposed and, in the end, freely disposed as complimentary commodities (so long as the maintenance of strictly positive commodities' prices is a panacea).

PS_5: Given the companion (to Theorem 1) Proposition 1, inside a golden rule competitive general equilibrium, in which the sequence of capital accumulation is convergent, the steady state of the convergent sequence of interest rates belongs in $(0, \delta)[\subset (0, 1)]$. Regarding this phenomenon, which trails plenty of food for thought, a critical final point to be made is found in Remark 3 of the main text.

2. List of Figures

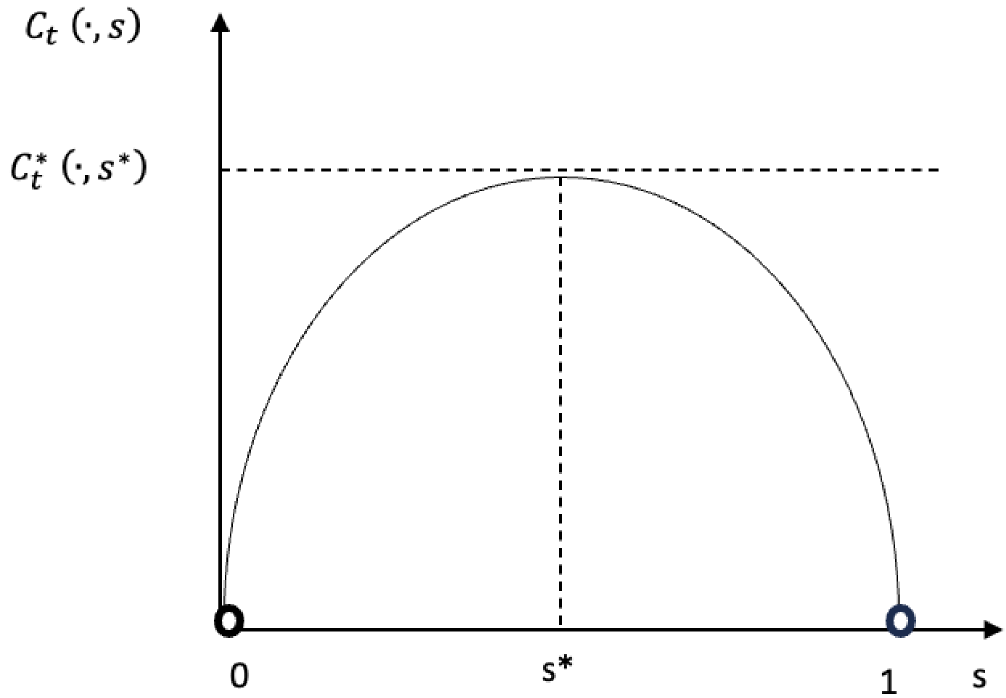

FIGURE 1. t=0,1,2,...

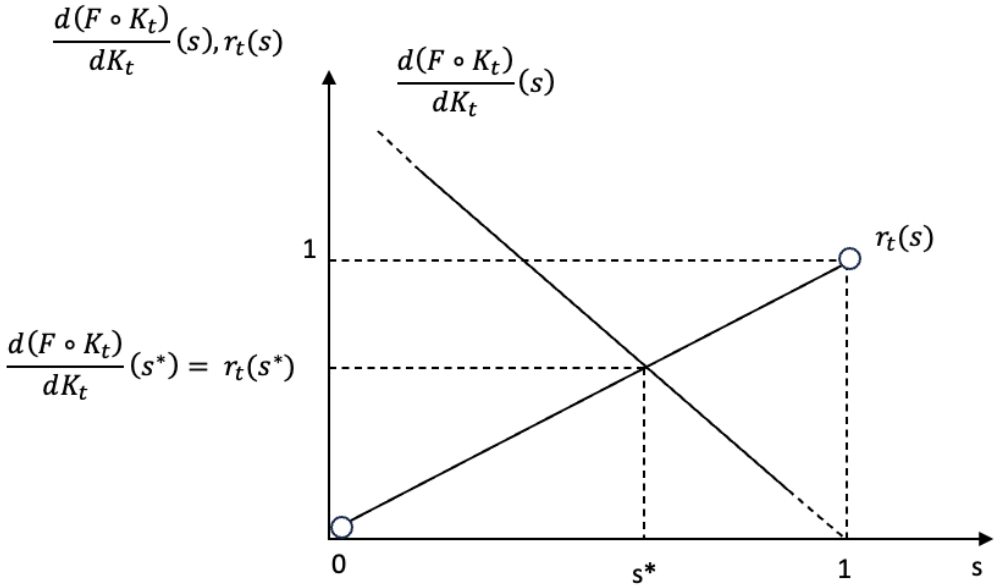

FIGURE 2. t=0,1,2,...

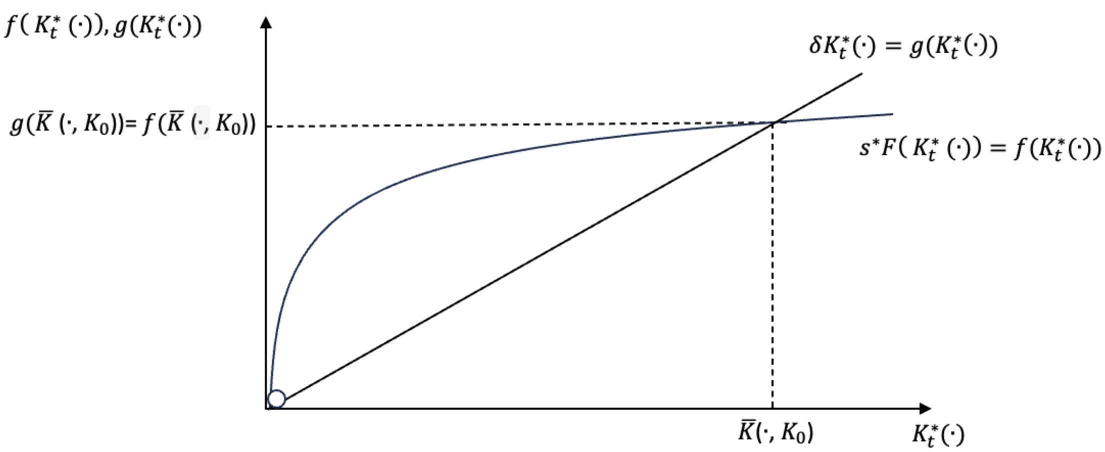

FIGURE 3. $\geq t$=0,1,2,...

3. Auxiliary Text

A. Comments

Comment 0: Macroeconomists indulge themselves into calling behavioural (micro-founded) macroeconomics modern (or contemporary) macroeconomics. In such macroeconomics, irrespectively of their denomination, aggregate (allocative) quantities arise from explicit summation (or integration) of the individual behaviours (i.e., optimal choices). All macroeconomics are micro founded, but modern macroeconomics have behavioural, specifically, micro-foundations.

Comment 1: This is self-evidently true. Quantitative macroeconomics, monetised or not, do additionally deal with such blastly impactful economic issues as: (i) the business cycle, (ii) finance, banking and entrepreneurship on the economy-wide level, (iii) the optimal labour unemployment ratio and the optimal inflation rate, (iv) the optimal fiscal, monetary, (international) trade and regulatory policies, reforms, measures, programs, schedules, laws, legislations, mandates and institutions, which are released both in the form of time rules (in which case they are uniform across time) and discretionarily (in which case they are circumstantial and time-based), being pro-cyclical, counter-cyclical, preemptive, remedial, group-based and/or place-based (as opposed to uniform across agents and/or space), (v) the optimal government budget deficit and debt and (vi) the optimal structure of the Balance of Payments. Albeit, all these (seemingly disparate) topics end up revolving around and being measured proportionally (i.e., in relation) to the economy's Gross Domestic Product (GDP), the optimisation of which, eventually, becomes the bliss point in macroeconomics.

Comment 2: By stepping upon this last argument one may briefly rotate the (stiff and opinion-less) analysis, by keeping a more prescriptive and advisory stance. In particular, one may arguably claim that no matter how much macroeconomic neoclassicism may be (improperly) attempting to refute, repress, hide, disembody, repudiate, isolate or seclude its Keynesian partial-identity, the dormant Keynesian philosophy pops out into the analytical foreground when the circumstances are right and the opportunities become favourable. All in all, to continue that kind of argumentation, the prototype Keynesian (1936) prestigious analysis is trans-macroeconomic, so (as it was to be expected) the Keynesian ambience encircles the neoclassical macroeconomic competitive general equilibrium, even marginally. Frequently, none the less, taking out Keynes from the analytical picture is just a super convenient (and the less analytical costly) thing to do, given that it is, candidly, tough to master (and know inside out) the Keynesian texts. This is mainly because the Keynesian readings (of the Keynesian writings) are subject to various (and occasionally conflicting) interpretations, so it is not unusual to come across with a Keynesian analyst in dire straights. To put it differently, decoding and interpreting the Keynesian features is a hard work. Indeed, aside from the majority of the freshly trained economists who frivolously (but reasonably) tend to disregard Keynes because they are nourished in the realm of neoclassicism, there is also a bunch of experienced neoclassicists-academics who show little appraisal and limited sympathy to the Keynesian attributes just because Keynes was using the 'wrong' (for them at least) words, even if he may (potentially) have had the 'right' ideas. This practice, however, puts the Keynesian alternative into an aggravating situation. So the disambiguation of the Keynesian ideas and jargon is of paramount importance, because upon it the miscellaneous (and more or less stealthy) ties and linkages between neoclassicism and Keynesianism are naturally coming into light. Before moving on, nevertheless, it is imperative to stress that this argument works the other way around as well. Indeed, it is expedient (and erroneous) mistaking Keynes as a substitution for, and not as an amelioration of, neoclassical economics, surpassing and suspending, thereby, whatever jumps out of the Keynesian jurisdiction. At the end of the day, and if this is not glaring enough, it needs to be anticipated (forecasted and foreseen) that Keynesianism and neoclassicism are intersected. They are neither mutually exclusive (i.e., disjoint), nor, certainly, collectively exhaustive subsets of macroeconomics (and Economics in general). That being so, so long as either of them keeps encoding and encrypting vocabulary or methods, the stakes are high, because the constant and thrusting flow of unprejudiced ideas between the two of them cannot be set in transit, so a measurable conceptual impoverishment of Economics is a credible threat (among other adverse effects this brings to the potential development of mathematical-economic analytical methods). To draw the line at this mental and philosophical argumentation, it is the opinion of the author (which is backed up by the findings) of this paper that the vanguard (overt and transpicuous) competitive general equilibrium analytical gates which secure the free (of analytical charge to either of them) transition form one discipline to the other should always be left wide open (hence, exploitable). And that, in order to hook up Keynes to the neoclassical macroeconomic circuit, there is no pressing need to use more underground and covert (i.e., non-lucid) market-based synthesising techniques (which, all of them, end up being rather analytically expensive when measuring the endured - with them - Keynesian losses), as, for instance, the colossal (neoclassical synthetic) Hicks (1937) and Hansen (1953) IS-LM (to, eventually, AD-AS) model does. Or even use, in a manner of speaking, clandestine 'smuggled through the back door' analytical shortcuts, as, for example, the gigantic (and incumbent in the literature) new Keynesian synthesis does, in which, conveniently, Keynesianism (is to be blamed for and) is invoked as an excuse of freezing the market prices and/or of shrinking the degree of competition in the markets. To put effort into encouraging and imparting a synthesis between the Keynesian analysis and the Walrasian general equilibrium is not something new. Although the last decades such attempts experience a characteristic deceleration and a persistent recession into the research industry, the literature has recored a number of competitive general equilibrium road-maps (thus, neoclassical guides) to Keynes. A nuanced (static) approach in a fiscal-monetary policy environment, for instance, is the striking paper of Dubey and Geanakoplos (2003), who are, nevertheless, based onto the IS-LM model's pseudo-markets.

Comment 3: And as usual, to make the system run flawlessly under its afore-assigned credentials, (at least) eight truism assertions (which are kept silently running in the background) have to be vitally made so as to avoid creating a conceptual maze;
• first, the most usual cast of a neoclassical macroeconomic act is as follows: (i) an agent (household, firm or the government) is a pre-specified economic role with a designated set of economic responsibilities and/or duties, while (ii) an economic unit (i.e., a participant of the economy) has or plays (i.e., is pre-assigned with more

than one) economic roles, thus, an economic unit may be more than one agents simultaneously, playing, thereby, multiple economic roles;

• second, an economic unit, which is identified with a family-enterprise (household agent/role) and/or a business-enterprise (firm agent/role) in an economy with a private sector exclusively, is, essentially, a pro-social coalition, that is, a fellowship of individuals, a company of persons, a partnership of decision makers who act all-together as a unit, motivated by a momentum of solidarity;

• third, (private) agents (that is, the already coalitional households and firms) decide autonomously or independently one to the other and, in particular, non-cooperatively, that is, without forming (further) coalitions (of coalitions);

• fourth, that there are no intermediate agents/roles of any kind to be placed in-between the household and the firm, which means that the same agent who (as a part of its role) produces does sell (supply, offer, provide or rent) as well, while the same agent who (as a part of its role) demands, does purchase or buy (or rent) as well;

• fifth, that there are no (quantitative) inventories of any type to be left at the agents' hands (hence, at the economy) immediately after the agents' actions, which means that, eventually, whatever is to be potentially supplied is actually supplied, while whatever is to be potentially demanded is actually demanded; equivalently, the planned economic actions and always identified with the realised economic actions;

• sixth, that households and firms cross-trade at the economy's markets under the familiar neoclassical (here closed and freed from government) macro-economic circuit (in which, however, all intermediate agents have been discarded), but households (or firms) do not trade among themselves in the same markets (for example, by negotiating and writing contractural agreements amongst them);

• seventh, that neither the household (role) nor the firm (role) possess unitary bargaining power [or any other kind of market (manipulative) power] when they cross-trade in the economy's markets, being unable therefore to affect or change with their actions the terms of trade, in particular, the markets' (quantities and/or) prices, at and post their endogenous generation in a competitive general equilibrium, being, in other words, utter (rational) price (and quantity) takers;

• eight, that the (benevolent) household (agent) and firm (agent) do not cooperate so as to form mal-intentioned or counter-social coalitions, which are otherwise known as tacit collusions (i.e., teams with anti-social team-spirit); that is, they do not join together (as co-members) into the same allied group (or exclusive club) so as to enhance (each one of them separately) their within-group positions, by increasing the market power of their lobby; and, in the end, by gaining [more accurately, by immorally (either licitly or illicitly) guaranteeing for themselves] extra-equilibrium privileges and other market-related advantages before, upon or after their trade, thereafter, causing mis-allocations (or even, bad or ill re-allocations) of the economy's resources.

Comment 4: Notably, in a tampered so as to be barter-looking economy without a foreign (or international) sector, if economic policy prescriptions and injections to the (private) system of agents and markets are to be allowed and the governmental agent gets introduced as a key economic player into the economy's market (trading or exchange) activity, but simultaneously one lets go of the governmental regulatory policy so that only (domestic) fiscal policy and (national) monetary policy stay within reach, monetary policy schemes remains viable only through the (direct or indirect) manipulation of the economy's interest rate, whose competitive general equilibrium value, then, would not have been pure (i.e., decentralised). Of course, in the presence of the public sector, competitive general equilibrium allocations would have been impure (i.e., centralised) in the first place.

Comment 5: Let us use a pedagogical compass so as to navigate (and point) towards the (hard to be figured) Keynesian direction and location. At some time instant, the autonomous investment, which does does not variate with the income, comes to be corresponded to the (and be converted into) savings, which have been formed conditionally on the income. Keynesians place their confidence that the autonomous investment, which does not itself fluctuate with the instantaneous income but gains this kind of dependency by the instantaneous savings, depends, instead, on the instantaneous rhythm of change of the output. Hence, a production that exhibits a positive (or growing) pace of alternation within some time-instant crowds in this instant's autonomous investment, and, in this manner, supports an economic boom (rather than an economic bust) instantaneously (this is the so called Keynesian acceleration effect, of the product towards the autonomous investment, happening statically, i.e., instantaneously). Thereby, in advance, the reader should be advised that in the world of Keynes (1936) one should start with the fact that the (aggregate) autonomous investment of $t = 0$ stays put along N, so that $\overline{I}_t = \overline{I}$, as $t \in \mathbb{N}$, because this quantity is neither affected by the output of one instant, nor it is affected by the growth of the product between any two instants. Then, the sequences of (aggregate) consumption, (aggregate) savings and (aggregate) income (or any other time-sequences of market-indices which affect the agents aggregatively, and which become vibrant upon the neoclassical additions to the Keynesian model) have to be stationary convergent as well, because the Keynesian model is essentially static and only $t = 0$ (or, equivalently, a/some/whichever single next time-instant) out of the whole $\mathbb{N}$ should be, effectively, accountable within it. In other words, when extracted from the KSSP totality so as to be independently considered and studied, the (left so as to be self-sufficiently operating) Keynes (K) model should be a special (here, trivial) case of the KSSP model and, in this way, should be included in (its superset) KSSP model.

Comment 6: Regarding this particular inwardly or innerly (i.e., by the model itself) arisen segment of the analysis, one could start alerting that the model is drifting away from reality, since one reasonably raised question would have been the following. Sure, but by completely abstaining from the two context-specific difference equations of capital accumulation that (exteriorly) co-furnish this general equilibrium model, does it then (autonomously) make sense to claim, as $t \in \mathbb{N}$ and $s \in (0, 1)$, that

$$\text{(i) } \pi_t(K_t(s)) > 0 \text{ and } I_t(\bullet, s) < r_t(s)K_t(s),$$

or even

$$\text{(ii) } \pi_t(K_t(s)) > 0 \text{ and } I_t(\bullet, s) > r_t(s)K_t(s)?$$

What is (in general) wrong with these two possibilities? More accurately, since these two previous cases are (in principle) mathematically-economic accessible, does the acceptance of these possibilities leave the other (context-independent) premises of the dynamic (non competitive in this case) general equilibrium model intact, or does this action lead (sideways and internally) to counterintuitive, baffling and conceptually self-contradicting phenomena to begin with, by breaching (or contaminating) the fundamental economic principles of dynamic macroeconomics? To see that a dead-end is hit in either cases, think first of prospect (i). Instantaneously, this scenario leads to the under-investment (or investment gap)

$$[I_t(\bullet, s) - r_t(s)K_t(s)] < 0,\ t \in \mathbb{N} \text{ and } s \in (0,1),$$

[in words: financial returns, apart form the profits, are additionally transported (from savings-investment) to consumption]

for the economy, in which situation more (than the normal) consumption (thus, efficiency) is attainable, but the potential output is not reached, hence, less (than the normal) production is realisable and, consequently, less consumption (and efficiency) is inflowed, and so on and on, which means that the economy is intrinsically (at its own responsibility) caught up into a vicious circle of having to deal with a continuous trade off between consumption and production. So, the model itself in case (i) is an imbroglio. Consider then the event (ii). This case is intimately compatible with the over-investment (or investment markup)

$$[I_t(\bullet, s) - r_t(s)K_t(s)] > 0,\ t \in \mathbb{N} \text{ and } s \in (0,1),$$

[in worlds: profits, apart from the financial returns, additionally travel (from consumption) to savings-investment]

for the economy, which plight also cannot qualify for an allocative efficiency improvement (or premium) since, similarly to before, more outflowed product (which soars up consumption and efficiency) can be obtained only by curtailing consumption and, thereby, lowering the allocative efficiency qualities, so the economy, once more and inherently (at its own fault), falls into an efficiency versus inefficiency (that is, uprisen efficiency versus shortened efficiency) trap from which it is difficult to escape. So, with hyper-investment of case (ii) as well, the model itself is a deadlock. To stay *vis-à-vis* with the neoclassical intertemporal macroeconomic sketch and evade disturbing analytical convolutions and ramifications, therefore, both these (bizarre or kinked) options have better to be (externally) obstructed.

Comment 7: Intuitively, unavoidably and certainly non-unprecedentedly in the province of neoclassical growth theory, for this competitive general equilibrium notion to end up carrying such a heavy tag, social efficiency needs to be (mandatorily) fenced and guarded (preferably by firewalls raised from the inside), by primitively protecting consumption over hazardous (non-steady) states of capital-accumulation (which are realisable through uncontrollable savings and investment), thus, via constantly leaving feasible the opportunity to mark up consumption whenever, momentarily, the intergenerational circumstance allows for it. Thence, this intergenerational-dynamic competitive general equilibrium model must accommodate at least one critical (innate or enforced, depending on the texture of the model) bridle on the stacked values of capital across time, aka a transversality condition, one that undercuts them and prevents them from taking arbitrarily large values, so that, technically speaking, capital-accumulated (through savings and investment) is never allowed to explode to infinity, having, thereby, its opponent, the household's consumption (which is instantaneously being traded off with the household's savings-investment) and, hence, the household's (utilitarian) welfare massively degenerated, up until to being entirely ruptured. Remark 3, which ends this section by marshalling its ideas, elaborates onto the recovery of such restraining mechanisms that are being effectuated upon to the usual culprit (that resides into the domicile of such models) for the dive in (and immersion of) social welfare: capital-accumulation. The simple (and common sense) idea behind all this prelude of Remark 3 is that having, instantaneously, the household consuming carefully, wisely and non-lavishly, does not mean that the household should consume frugally so as to (instead of making ends meet) unleash a savings-investment spree and accumulate a tsunami of wealth. The neoclassical growth (to competitive general equilibrium) theory does support the speculative-incentive based savings-investment (and financial holdings) of the household, but makes it clear that, in the end, the household always succumbs to its motive for (initially) survival and (subsequently) a decent (or moderately luxurious) well-being.

B. Footnotes

footnote 2: The same thing (and perhaps even more acutely) holds of course for (macro) economic development, which is simply a pro-social filtration of (macro) economic growth. When staying in the aggregate level, development pins down growth (hence, all the rest macroeconomic phenomena) when agents are imperfect, for example non-benevolent (immoral, corrupted, criminals or rent seekers), or when agents experience income-to-wealth inequality and, more generally, inequality in welfare (utilitarian or not), but whenever (quantitative) welfare is stimulated and scored by (quantifiable and measurable) qualitative factors as well (environmental, educational, healthcare and other such lifestyle related amenities which, all of them together, co-determine the quality of life). In fact, in the economics of development, where the qualitative spirit of the analysis is kept high, welfare is usually renamed to well-being. Thence, development comes to socially refine and, ultimately, socially sustain growth. And while there are, admittedly, various utensils available inside this field's toolbox than can be used for these purposes, ethics and egalitarianism, when found to be persistent within the peoples' communities, become the two most fundamental engines for the pro-social augmentation of the economy's output.

footnote 3: This not an analytical curfew. It is enough to consider only these three (physical capital and human capital, compounded with technology) productive sectors. Indeed, more or less, the economy's extant elemental natural resources (such as the land, the ground or the soil, the naturally existing forms of energy, the climate, etc) are (both in quantity and quality) external additions to the system, and (if not fixed) definitely in a continuous state of thriftiness, degeneration and extinction. So, provided that they cannot be substantially internally controlled, they bear infinitesimal importance in production. Simultaneously, all other manufactured sources of production (and any variation of them) can be seen as being materially-capitalised into machinery, mechanical and building equipment and other related physical productive capacity.

footnote 4: Which fact renders all the other households' economic actions (apart from their rest and consumption) unpleasant.

footnote 5: Note at this point that, by default, altruism and impatience of a dynasty are two notions inversely correlated: the more impatient a household is, the less altruistic it is (and *vice versa)*.

footnote 6: In other words, households can have both (primal) preferences concerning (quantities of) tradable (and purchasable) commodities (aka utility functions) and preferences concerning time (i.e., time-preferences, a term referring to the allocation of these utilities, that is, the primal preferences, across time).

footnote 7: Which amounts to the following principle: identical agents (initially assigned with the same endowments, priors or characteristics) get allocated identically in general equilibrium.

footnote 8: The (euphemistically labelled as) Euler equation is an iterative first-order maximisation condition (and a difference equation on capital accumulated) which characterises the household's inter-temporal optimal choice as an equality among the (expected, in principle) marginal cost-or-loss and the marginal benefit-or-gain of/from saving-investing. In other words, this condition reflects the idea that, since time evolves limitlessly and the household never dies, the household ultimately becomes indifferent between consuming (and receiving optimal welfare) either today (in the spot market) or tomorrow (in the future market). It spawns the usual competitive general equilibrium state of 'indifference' (and 'rightness and unbiasedness') when it comes to the mobility of consumption (and welfare) between the present and the future. It mirrors the interpretation that, in equilibrium, there is no intergenerational decisional cleavage. The marginal utility obtained from non-waiting and being impatient and egoistic is equal to the marginal utility derived from waiting and being patient and non-selfish. Impatience and altruism become synonymous concepts in equilibrium. And the household's age becomes an idle, nullified and ineffectual parameter. The Euler equation is the ultimate trans-generational and inter-generational equitability condition.

footnote 9: Pareto (1906) optimality-efficiency is the twin concept of the competitive (or Walrasian) general equilibrium notion. Upon the precedent of Debreu (1964), the two of them are innately related and form a totality. Remark, however, the following particularity in this case. Say that the goal is the optimisation-efficiency of the total (communal) utility. Then in principle, the Paretian social (i.e., grand coalitional) utilitarian welfare notion is weaker than (thus, superior to) any aggregate (that is, summable or, more generally, integrable) societal utilitarian welfare conceptualisation. This is because the first does not need to interrelate the agents' asymmetric preferences (equivalently, utility functions), whereas the second explicitly does so. As a result, the first one entails the broader ordinal utility functions, in contrast with the second one which only works with the narrower cardinal utility functions, since a common scale is needed when differential preferences are to be aggregated and get interpersonally comparable. In the herein case, nevertheless, the differentially initially endowed (with capital stock) cohorts have the same utility function, so none inter-generational common scale for their preferences is needed. The utilities trivially become comparable and transferable among generations. In this spirit, the dynasty's (additively time separable) lifetime utility function becomes also the dynamically aggregate social utility, which is again maximised. Consequently, the two social utilitarian welfare concepts overlap.

footnote 10: Which were already semi-formally invented (and circulating in the literature) by older works of Fisher and Allais.

footnote 11: And, more consistently, with its overlying dynamic analogue, which is the insightful Harrod (1939) - Domar (1946) model.

footnote 12: Where, as the history of Economics was unfurling and crude utility was being more and more rectified, utility excelled and eventually triumphed in (becoming, thereby, the unambiguous flag of) the land of neoclassicism. More accurately, utility emerged as providing (to decision makes with built-in acumen, both cognitive and emotional) custom-made numerical statuses of satisfaction. In this mode, utility was acknowledged as an arithmetic metric or index of pleasure or joy. Utility, in other words, has been tagged as a solidly quantifiable and actually measured personalised economic quantity (even if being heavily epistemically, physiologically and sentimentally skewed, thereby being a multifarious conception). And for the record, after the formatting (and crowning achievement) in economic theory of Debreu (1954, 1959), utility (functions) represent humanitarian (axiomatically behaving) preferences (at least in the neoclassical nomenclature). Upon this primitive and universal (axiomatic) generalisation, secondarily and specifically (i) habitual(ly figured, formed and arising) preferences, (ii) (generated) preferences based on explicit needs, tastes, affordability and availability, i.e., based on product variety, product substitutability, product characteristics and purchasing power or potential, (iii) (shaped) preferences based on the attitude towards risk, and so forth, may be, in the aftermath, creating specialised (formats of) more convincing utility functions. So other authors came then to highlight, for instance, the (due to all the practical limitations in the decision process) habitual or routine consumption rules (i.e., the habit formation parameter in a preferentially-constructed utility function), or other (quantifiable and measurable) subjective-qualitative and/or objective-quantitative elements of/in consumption that have produced preferentially-based utility (such as the tastes, the needs, the variety, the specifications, the substitutability, the affordability and the accessibility of products), or even the role of risk in an obtainable (return or) utility from consumption, when the former continues to be based and executed upon a preferential (ranking or ordering of commodities) system. In the course of time, Economists have realised and accepted that the manifold aspects of (preferential) utility, cardinal or ordinal, are numerous.

footnote 13: Equivalently, instantaneously, the household's budget set (or budget constraint) drops down to a budget line. Equivalently, instantaneously, there is an with-equality-feasible allocative relationship which simultaneously (and aggregatively) conjugates (and constrains) all the economy's resources. Equivalently, instantaneously, the titled as the *basic macroeconomic identity* is (indeed) an identity-equation. This admission does not leave room for a viable competitive general equilibrium with excess supply, that is, microeconomically speaking, with free disposal, while macroeconomically speaking, with general glut (neither, needles to say, with excess demand). This admission, furthermore, trivially guarantees that, instantaneously, the Walras (1874) law [which states that the budget constraint implies that the sum of the values of excess demands (or of excess supplies) of/in all the economy's markets is zero] is trivially satisfied.

footnote 14: There is a terminological convention that a number of authors naively rely upon to: they use the terms recurrence equation and difference equation as interchangeable, when the two of them are not (actually and always). To be more precise, an accurately defined (deterministic) difference equation, as a pair composed of a sequence of a (deterministic) real variable $\{a_n\}_{n=0}^{\infty}$ and its k^{th} differences, belongs in the broader class of (deterministic) recurrence relations, being a special case of a recurrence equation. There are, therefore, recurrence relations which are not difference relations. For example, the Ackermann numbers are not a difference equation. However, such cases are circumstantial and very rare, since the generic form of a recurrence equation is the difference equation. So, there is insubstantial peril to simply match the two of them (as this paper does).

footnote 15: However, F tends to zero as its argument tends to zero, i.e., zero is the infimum of F.

footnote 16: Since the importance of time as an infrastructural parameter is eliminated in the Keynesian practice.

footnote 17: Remark that, as $t = 1, 2, ...$ and $s \in (0, 1)$, since every $Y_t > 0$ is positively correlated with the alternative $K_t(s) > 0$ values via the production function, while every $K_t(s) > 0$ is, in it own turn, positively parametrised with the alternative $I_{t-1}(Y_{t-1}, s) > 0$ values through the fundamental law of motion (and accumulation) of capital across time, C_t, which positively depends on $Y_t > 0$ by the initial arrangements, ends up fluctuating positively with respect to the alternative $I_{t-1}(Y_{t-1}, s) > 0$ values and, of course, the alternative $K_t(s) > 0$ values [so that, eventually, the parametrisation of C_t is with respect to $(Y_t, s) \in (0, +\infty) \times (0, 1)$]; for $t = 0$, C_0 is simply (and neutrally) a positive function of $K_0 > 0$, which is a given constant; exactly analogous considerations are valid for the function $S_t = I_t$, $t \in \mathbb{N}$. The point to be made with this observation is the recovery of the formal explanation of the fact that the household has to save-invest in every time-point so as to become enabled to purchase the produced composite-product of the next time instant and consume a portion of it.

footnote 18: In this fashion, the hypothesis of the strict concavity of the production function is inwardly (from the model itself) self-gained and, therefore, permanently sealed. Besides, with flat (neoclassically styled) technology across time, the quality of the clustered capital remains constant over time as well, thus, it would have been insensible to substantiate, at each sequential time instant, the fall into the (strictly positive) marginal product of capital-accumulated (up to its ultimate tendency to completely evaporate) via the fall into its quality. In this model, the marginal product of capital-accumulated, hence, its productive capability, is not affected or influenced at all by the quality (i.e., the technical specification and qualification) of this capital-accumulated. In the presence of an optimal-endogenous savings-investment rate, to put it differently, the need to consider technological (or, more generally, technical) changes along the time line is extinguished, since there is a perfect conceptual substitutability among 'optimal-endogenous technology' and 'optimal-endogenous saving-investment ratio'. Aside all the aforementioned, it can be also detected that with the model's configurations the stationary (functional formula of the) production function (or, equivalently, of technology) with respect to capital-accumulation gets, in essence, shifted along time when taken with respect to s, because the (time-indexed) s-dependent capital-accumulation function changes functional format (to one, however, of exactly the same properties) from one time-instant to the other. In other words, this model contains a built-in vehicle of technology, the factor s.

footnote 19: Or equivalently, low levels of s are sufficient for a given increment in capital accumulation; s accelerates capital accumulation.

footnote 20: Or equivalently, high levels of s are needed for a given increment in capital accumulation; s retards capital accumulation.

footnote 21: But not, here, by receiving (the representative) earnings from labour, which proceeds depend positively both on the labour quantity and on the labour quality of the household (same thing as with the returns from the asset accumulated of the household). This means that the toil, effort, energy, talent, intelligence, creativity and innovativeness, experience, skills and capabilities of any humanitarian respect, learning and adaptiveness, technical knowledge and specification (or, more generally and all-inclusively, the productivity) that the household devotes on (respectively, with which the household conducts) its labouring in the firm should matter as well. To wit, there should be a positive correlation between the contribution the household-worker makes to the economy's production and the household's labour income as follows: (*ceteris paribus*) more labour quantity should give more labour income, but (*ceteris paribus*) more labour productivity should give more labour income as well. The same, again, applies to the household's income from capital-accumulated.

footnote 22: Also, while K_t remains unbounded from above with respect to r_t, the greater lower bound of K_t with respect to r_t is the zero value forever, i.e., for $t = 0, 1, 2, ...$

footnote 23: Of course, an analogous competitive general equilibrium condition follows through for labour, when labour gets inserted as a production factor, or, more generally, for any other production input whenever it is traded in a market as a commodity and starts being considered as an impetus for growth. By being placed directly oppositely to the elemental 'equal treatment property' allocative condition that should be fulfilled on the demand side, this 'marginal cost equals marginal gain' condition constitutes the fundamental equitability condition that any competitive general equilibrium should satisfy on the supply side. Indeed, it states that, in marginal terms, the factors of production are rewarded according to the contribution they make into the economy's output.

footnote 24: In accordance with this foray, remark also that in the overlapping generations competitive general equilibrium models with production and steady-states [say with discrete length of time again, and in which (dynamically) every economy is, as usual, a printed copy of the other] the golden rule of capital accumulation with maximum consumption along the time horizon, whenever the labour production factor is a constant (hence, a muted, as in section 2) quantity over time or, equivalently, the growth rate of the labour force is zero, is again (and among any two generations) identified with the equation between the marginal product of capital-accumulated (equivalently, the rate of interest) and the rate of depreciation, but there is not a rate of savings-investment in those models so as to create analogous problematic issues. If the work force grows in time at a strictly positive rate, then it is implied that (as in the KSSP model) the interest rate is strictly less than the depreciation rate. This alternatively indicates that labour is not a results-changing parameter in the KSSP model, so its absence is without any loss in mathematical generality.

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