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PROFIT MAXIMIZATION UNDER POINT AND QUANTITY RATIONING: WHY DO SENSITIVITY ANALYSIS AND LE CHATELIER PRINCIPLE STILL FACE PROBLEMS IN THE PRESENCE OF CONSTRAINTS

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PROFIT MAXIMIZATION UNDER POINT AND QUANTITY RATIONING: WHY DO SENSITIVITY ANALYSIS AND LE CHATELIER PRINCIPLE STILL FACE PROBLEMS IN THE PRESENCE OF CONSTRAINTS

by

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Abstract

The paper considers a simple example of unconstrained maximization, i.e., that of unrestricted profit maximization by a firm facing constant input prices, as compared to two restricted (constrained) profit problems under "point" or "quantity" rationing of some inputs.

In the former a single constraint is imposed, indicating specific point prices and total allowable expenditure on rationed inputs, while in the latter rationed input quantities are fixed.

Local sensitivity and Le Chatelier effects in every optimization problem are now obtained, in matrix theory terms, via either a primal or a dual method. A difficulty, however, appears in constrained optimization models, whose s-o-c are expressed in the form of a matrix that must be semi definite or definite in the tangent subspace of its constraints' hyper surface and, thus, cannot be used directly for either purpose. Economists have not exploited fully all the existing mathematical analysis: they have only succeeded in performing sensitivity analysis via the primal method, by the use of "bordered Hessians". Otherwise the difficulty still exists and, in fact, appears not to have been recognized.

The profit maximization problem, even under point or quantity constraints, is so simple that the above difficulty becomes as transparent as possible, while the steps required for resolving it are close at hand. Finally, a diagrammatic illustration of profit maximization under quantity rationing is possible, if there are only two inputs: then, we can show global sensitivity and Le Chatelier effects and also specify the conditions under which they may be upset.

1. Introduction

In this paper we consider a firm that produces its output by using n inputs. Its technology is given by the production function $f(x) = y$.

$f(x): R^n_{++} \to R$ is **well behaved** if

(i) $f(0_n) = 0$, $f(x)$ is finite for every finite $x \in R^n_+$. For every $y > 0$ there exist $x \in R^n_+$ with $f(x) = y$.

(ii) $f \in C^2$ on R^n_{++}, with first and second partial derivatives $f_i(x)$ and $f_{ij}(x)$, $i, j = 1, \ldots, n$.

(iii) for any $y > 0$ there exist x with positive gradient vectors $f_x(x)$, i.e., $x \in S \equiv \{x \in R^n_{++} \mid f(x) = y, f_x(x) \geq 0_n\}$. For any $x \in S$, $f(x)$ is strongly concave, i.e., f is a a strictly concave function with a negative definite Hessian matrix, $F(x) \equiv [f_{ij}(x)]$.

The firm is competitive in all markets, facing constant input prices $w > 0_n$ and output price $p > 0$. Throughout the paper, except in section 5, p is set equal to one.

Our analysis relies on classical optimization techniques in matrix theory terms. All vectors are treated as column vectors, unless they are enclosed within parentheses or appear as function arguments, while matrices are denoted by capital letters: thus e.g. 0, 0_n or O_{nm} denote, respectively, the zero scalar, a vector of n zeros, or a matrix of zeros. A prime after a vector or a matrix denotes transposition.

The paper is organized as follows. **Section 2** examines the unrestricted profit maximization problem, which is compared to two

restricted (constrained) profit maxima, namely, those under “point” or “quantity” rationing of some inputs. Since the original problem is an unconstrained one, “point” rationing can be dealt with quite smoothly. **Section 3** introduces the dual method of comparative statics via the Envelope theorem. Again, having an unconstrained original profit maximum facilitates sensitivity analysis immensely: indeed the appropriate Envelope problems, under “point” and “quantity” rationing, appear both in an unconstrained and in a constraint form! This felicitous feature forces the researcher to recognize the difference of the second-order-conditions of the two forms and understand why the s-o-c of the latter form cannot be used directly for sensitivity analysis.
Section 4 considers all possible interrelations that can be obtained between unrestricted and restricted profit maxima and examines various manifestations of distinct local Le Chatelier effects. On the other hand, **Section 5** is the epitome of simplicity, offering a diagrammatic illustration of global comparative static and Le Chatelier effects and their upsets, when the firm uses only two inputs, one of which may be fixed in quantity. Finally, **Section 6** concludes with a historical survey of the relevant economic literature and the specific mathematical analysis that has to be taken into account so as to permit sensitivity analysis and Le Chatelier Principle in the presence of constraints in more complex optimization problems.

2. Unrestricted and restricted profit maximization under point and quantity rationing

The unrestricted, or first – best, profit maximization problem is given by

$$\pi(w) = \max_x \{f(x) - w'x\} \quad , \qquad (P^f)$$

if $x^f \in S^o$ satisfies both

$$x^f \in foc \equiv \{x : f_x(x) - w = 0_n\} \qquad (1)$$

and

$$x^f \in soc \equiv \{x : F(x) \text{ is negative definite}\}, \qquad (2)$$

then x^f attains a strict local maximum of (P^f). Using the Implicit function theorem we can find, in principle, $x^f \equiv x(w)$ by solving the identities $f_x(x(w)) \equiv w$ in (1), with $x(w) \in C^1$ in a neighborhood of any $w > 0_n$.

(P^f) will be contrasted with two restricted, or second best, profit maximization problems, namely, those under "point" or "quantity" rationing of some inputs. Thus our former x' bundle will be given by (x, z), with $x \geq 0_p, z \geq 0_m, 1 \leq m < n$ and $p = n - m$, while $w > 0_p$ and $r > 0_m$ denote the respective input prices.

In **point rationing** an equality constraint , $a'z = b$, is imposed on the choice of rationed inputs, where $a > 0_m$ denote point prices and b the allowable expenditure on rationed inputs. The profit maximization problem is now given by

$$\pi(w, r, a, b) = \max_{x,z} \{f(x,z) - w'x - r'z \mid a'z = b\}. \qquad (P^p)$$

If x^p , z^p and λ^p satisfy both

$$\text{foc} \equiv \{f_x(x^p,z^p) - w = 0_m,\; f_z(x^p,z^p) - r - \lambda^p a = 0_\rho \text{ and } a'z^p = b\} \tag{3}$$

and

$$\text{s-o-s c} \equiv \left\{ \begin{array}{l} F(x^p,z^p) \text{ is negative definite on the} \\ \text{tangent subspace } T = \{(\zeta,\eta)' \in R^n \mid \\ (0'_\rho, a')(\zeta,\eta)'' = 0\} \text{ and } (\zeta,\eta) \neq 0_n \end{array} \right\}, \tag{4}$$

then the implicit function theorem works and $x^p \equiv x(w,r,a,b)$, $z^p \equiv z(w,r,a,b)$ and $\lambda^p \equiv \lambda(w,r,a,b)$ attain a strict local maximum of (P^p), with x^p, z^p and $\lambda^p \in C^1$ in a neighbourhood of any $(w, r, a, b) > 0'_{n+m+1}$. We also note that the Jacobian matrix of (3), in x^p, z^p and $-\lambda^p$ i.e., the Bordered Hessian of (P^p)

$$\begin{bmatrix} F(x^p,z^p), & c \\ c', & 0 \end{bmatrix}, \text{ with } c' \equiv (0'_\rho, a'), \tag{5}$$

in an invertible matrix at $(w, r, a, b) > 0'_{n+m+1}$.[1] Finally, a simple inspection of (3) verifies that $x(w, r, a, b)$, $z(w, r, a, b)$ and $\lambda(w, r, a, b)$ are homogeneous functions of degrees zero and (- one), respectively, in (a, b).

In quantity (or straight) rationing of some inputs we may, first, consider **gross** profit maximization, namely,

$$\begin{aligned} \pi(w,\bar{z}) &\equiv \max_x \{f(x,\bar{z}) - w'x\} \\ &\equiv \max_{x,z} \{f(x,z) - w'x \mid z = \bar{z}\}, \end{aligned} \tag{P^g}$$

or, second, **net** profit maximization, namely.

$$\begin{aligned} \pi(w,r,\bar{z}) &\equiv \max_x \{f(x,\bar{z}) - w'x - r'\bar{z}\} \\ &\equiv \max_{x,z} \{f(x,z) - w'x - r'z \mid z = \bar{z}\}. \end{aligned} \tag{P^q}$$

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[1] For a proof see e.g. **Drandakis** (2003), Lemma 1.

In the second version of (P^q) and (P^g) the m constraints appear explicitly, while in the first version of (P^q) and in (P^g) it is clear that if $x^g = x^g(w, \bar{z})$ satisfy both

$$foc \equiv \{f_x(x, \bar{z}) - w = 0_\rho\} \qquad (6)$$

and

$$soc \equiv \{F_{xx}(x, \bar{z}) \text{ is negative definite}\}, \qquad (7)$$

then x^g attains a strict local maximum, with $x^g(w, \bar{z})$ depending on w and $\bar{z}$, but not on r. On the other hand, the second versions of (P^q) and (P^g) lead to $x^g = x^g(w, \bar{z}), z = \bar{z}$ and the m langrangean multipliers $\mu_j^g \equiv \mu_j^g(w, \bar{z})$ and $\mu_j^q \equiv \mu_j^q(w, r, \bar{z})$ $j = 1, \dots, m$, satisfying, respectively, both

$$foc \equiv \{f_x(x, z) - w = 0_\rho, \; f_{z_m}(x^g, z) - \mu^g = 0, \; z = \bar{z}\} \qquad (8^g)$$

and

$$foc \equiv \{f_x(x, z) - w = 0_\rho, \; f_{z_m}(x^q, z) - r - \mu^q = 0, \; z = \bar{z}\}, \qquad (8^q)$$

as well as

$$soc \equiv \left\{ F(x^q, z) \text{ is negative definite on } \{(\zeta, \eta) \in R^n \mid [O_{m\ell}, I_{mm}](\zeta, \eta) = 0_m \} \text{ and } (\zeta, \eta) \neq 0'_n \right\} \qquad (9)$$

and attaining a strict local maximum, with x^q, μ^g and $\mu^q \in C^1$ in a neighborhood of any $(w, \bar{z})$ or $(w, r, \bar{z})$. It is clear that $\mu^q(w, r, \bar{z}) = \mu^g(w, \bar{z}) - r$. It is also evident that (9) reduce to (7), since in the tangent subspace ζ is unrestricted while $\eta = 0_m$. Again the gradient matrix $[O_{m\ell}, I_{mm}]$ of the m constraints in (x, z) has rank equal to m and so the Bordered Hessian of (P^q) and (P^g)

$$\begin{bmatrix} F(x^q, z) & C \\ C' & O_{mm} \end{bmatrix}, \text{ with } C = \begin{bmatrix} O_{\rho r} \\ I_{mr} \end{bmatrix}, \qquad (10)$$

is an invertible matrix.

Finally, let us note that, while any solution of (P^f) for $w > 0_n$ generates positive profits, nothing definite can be said about $\pi(w, r, a, b)$ or $\pi(w,r,\bar z)$.

Indeed, for any $x^o \in S$ and $x^1 \in S$, $f(x^1) < f(x^o) + f_x(x^o)'(x^1 - x^o)$, because of strict concavity of $f(x)$; so if $x^1 = 0_n$ we are led to $f(x^o) - f_x(x^o)' x^o > 0$, or to $\pi(w) > 0$ for the corresponding input prices.
However, in (P^p) or (P^q) we can easily see that

$$\pi(w,r,a,b) = \{f(x^p,z^p) - f_x(x^p,z^p)'x^p - f_z(x^p,z^p)'z^p\} + \lambda^p b$$

and

$$\pi(w,r,\bar z) = \{f(x^q,\bar z) - f_x(x^q,\bar z)'x^q - f_z(x^q,\bar z)'\bar z\} + \mu^{q\prime} \bar z$$

and so, if b is much bigger than $a' z(w, r)$, or some $\bar z_j$ are much bigger than $z_j(w, r)$, then $\lambda(w, r, a, b)$ or the corresponding $\mu_j(w,r,\bar z)$ may become so negative that $\pi(w, r, a, b) < 0$ or $\pi(w,r,\bar z) < 0$.
To avoid any complication from having inequality constraints in (P^p) or (P^q)[2], we will assume that b or $\bar z_j$ are not very big, so that $f_z(x^p, z^p) =$ $= r + \lambda^p a > 0_m$ and $f_z(x^q,\bar z) = r + \mu^q > 0_m$. Thus the firm operates within S despite the constraints, with

$$\pi(w,r,a,b) = f(x^p,z^p) - w'x^p - r'z^p > 0$$

and

$$\pi(w,r,\bar z) = f(x^q,\bar z) - w'x^q - r'\bar z > 0$$

and $\lambda(w, r, a, b)$ positive, zero, or negative depending on how big b is relative to $a'z(w, r)$ and similarly for $\mu(w,r,\bar z)$.

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[2] It must be noted, for example, that (P^p) and (P^q) incorporate only the profits (or losses) generated in the production of the firm's output. If the firm's technology permits the consideration of inequality constraints in (P^p) or (P^q), the possibility that the firm may "rent out" its unused "capacity" has to be taken into account.

3. Comparative static analysis via the Envelope Theorem

Comparative static analysis in (P^f), (P^p) or (P^q), examines the rates of change of their solutions as the parameters of each problem vary. This is now done in matrix theory terms, via two methods: either a **primal method**, through differentiation of f-o-c with respect to parameters and evaluation of the properties of the resulting matrix equation system, or a **dual method** that starts from the maximal value function of each problem and their derivative properties, through the solution of appropriately specified Envelope problems. Each envelope problem compares the profit secured by the firm under two alternative policies: a specific feasible, but passive, policy of input use is compared to the corresponding optimal policy.

In (P^f) both approaches are quite simple. First, from (1) we get

$$F(x(w))\, X_w(w) = I_{mm}, \qquad (11)$$

with $X_w(w) \equiv [\partial x_i(w) / \partial w_1]$. Since the Hessian of $f(x(w)$ is invertible, we see that

$$X_w(w) = F(x(w))^{-1} \qquad (11')$$

is a symmetric and negative definite matrix.

On the other hand, $\pi(w) \equiv f(x(w)) - w' x(w)$ has the derivative properties

$$\pi_w(w) = X_w(w)\, f_x(x(w)) - X_w(w)\, w - x(w) = -\, x(w) \qquad (12)$$

and

$$\Pi_{ww}(w) = -\, X_w(w), \qquad (13)$$

which is a symmetric matrix. For any $w^o > 0_m$ we denote $x^o \equiv x(w^o)$ and consider the **Envelope problem**

$$0 = \max_w \{ f(x^o) - w' x^o - \pi(w) \}, \qquad (E_P^f)$$

where **parameters have become the choice variables and the former choice variables are treated as parameters**. It is evident that the maximum of (E^f_P) cannot possibly be positive but is at most equal to zero, since the

$$\text{f-o-c} \;\; \{ -x^o - \pi_w(w) = 0_m \} \qquad (14)$$

are satisfied at w^o, as we know from (12). If we also have the

$$\text{s-o-s-c} \;\; \{ -\Pi_{ww}(w^o) \;\text{ is negative definite} \}, \qquad (15)$$

then we attain a strict local maximum of zero.

We thus see that

$$X_w(w^o) = -\Pi_{ww}(w^o) \qquad (11'')$$

is a negative definite matrix.

Both approaches become more involved in (P^p) or (P^q). Thus only the dual method is presented here, with the primal method briefly sketched in **Appendix A**.

In **point rationing**, the derivative properties of π (w, r, a, b) are

$$\pi^p_w = -x^p, \quad \pi^p_r = z^p, \quad \pi^p_a = -\lambda^p z^p, \quad \text{and} \quad \pi^p_b = \lambda^p \qquad (16)$$

and the symmetric matrix Π (w, r, a, b) =

$$\begin{bmatrix} \pi^p_{ww} & \pi^p_{wr} & \pi^p_{wa} & \pi^p_{wb} \\ \pi^p_{rw} & \pi^p_{rr} & \pi^p_{ra} & \pi^p_{rb} \\ \pi^p_{aw} & \pi^p_{ar} & \pi^p_{aa} & \pi^p_{ab} \\ \pi^p_{bw} & \pi^p_{br} & \pi^p_{ba} & \pi^p_{bb} \end{bmatrix} = \begin{bmatrix} -X^p_w & -X^p_r & -X^p_a & -x^p_b \\ Z^p_w & Z^p_r & Z^p_a & z^p_b \\ -\lambda^p Z^p_w - z^p\lambda^{p\prime}_w & -\lambda^p Z^p_r - z^p\lambda^{p\prime}_r & -\lambda^p Z^p_a - z^p\lambda^{p\prime}_a & -\lambda^p z^p_{b} - z^p\lambda^p_{b} \\ -\lambda^{p\prime}_w & -\lambda^{p\prime}_r & -\lambda^{p\prime}_a & -\lambda^p_b \end{bmatrix} \qquad (17)$$

where function arguments are suppressed and superscripts denote problem (P^u). We note that

(i) $\lambda^p \gtrless 0$ and $\pi^p_b \gtrless 0$ when b is smaller or bigger than a' z (w, r), while $\lambda^p = 0$ implies π (w, r) = π (w, r, a, b)

(ii) the symmetry of Π^p implies that of X^p_{wr}, Z^p and Z'^p $\lambda_{aa} + \lambda'$, while we also see that

$X^p_{rw} = X^p_{wa} ... Z^p ... \lambda_b$ or $X^p_{ar} = X^p_{b} ...$ and,

similarly, $Z^p_{ar} = Z^p_{b} ...$ as well as $-\lambda^p_a - \lambda^p_b z' z'' = \lambda^p_b$.

The Envelope problem in (P^μ) appears in two forms: for any $(w^o, r^o$, a^o, $b^o)$ and $x^o \equiv x\,(w^o, r^o, a^o, b^o)$, $z^o \equiv (w^o, r^o, a^o, b^o)$ and $\lambda^o = (w^o, r^o, a^o$, $b^o)$, we may consider a **constrained envelope problem**, namely,

$$0 = \max_{w,r,a,b}\{f(x,z) - w x^o - r z^o (w, r, a, b) \mid a' z = b\} \qquad (E^c_{P^p})$$

or, due to the linearity of the constraint in b, we may consider an **unconstrained envelope problem** for any (w^o, r^o, a^o), z^o and $x^o \equiv x\,(w^o$, $r^o, a^o, a^{o\prime} z^o)$, namely,

$$0 = \max_{w,r,a}\{f(x,z) - w x^o - r z^o (w, r, a, a' z)\} . \qquad (E_{P^p})$$

The latter is simpler and will be examined first. However the former is quite instructive since it shows what has to be done so that the s-o-s-c of a constrained optimization problem can be turned into envelope curvature conditions suitable for sensitivity analysis. On top of that, we can immediately verify here that these curvature conditions are non other than the s-o-s-c of the unconstrained optimization problem (E_{P^p}).

(E_{P^p}) is characterized by

$$\text{f-o-c} \;\; \{ - x^o - \pi^p_w = 0,\; - z^o - \pi^p_r = 0,\; -\pi^p_a - \pi^p_b z^o = 0 \} \qquad (18)$$

which, as we know from (16), are satisfied at (w^o, r^o, a^o) and $a^{o\prime} z^o$. Also the matrix of partial derivatives of (18) with respect to (w, r, a), namely,

$$-\Pi^p_{w,r,a} = -\begin{bmatrix} \Pi^p_{ww} + \pi_{wb} z' & \Pi^p_{wr} & \Pi^p_{wa} + \pi_{wb} z' \\ \Pi^p_{rw} & \Pi^p_{rr} + \pi_{rb} z' & \Pi^p_{ra} - \pi_{rb} \\ \Pi^p_{aw} + \pi_{bw} z & \Pi^p_{ar} + \pi_{br} z & \Pi^p_{aa} + \pi_{ab} z' + z\pi_{ba} + z\pi_{bb} z' \end{bmatrix} =$$

$$= \begin{bmatrix} X^{ppppo}_{wra} & X, & Xxz + b' \\ Z^{ppppo}_{wra} & Z, & Zzz\ b' \\ \lambda Z^{pppppo}_{wra} & Z, & (Zzz)\ b \end{bmatrix}, \qquad (19)$$

as we can easily see from (17), satisfies at (w^o, r^o, a^o) the

$$\text{sosc} \begin{cases} \text{For any } (\zeta,\eta,\theta) \in R^{n+m} \\ -(\zeta,\eta,\theta)\,\Pi^{p0}_{w,r,a}\,(\zeta,\eta,\theta)' < 0 \\ \text{if } (\zeta,\eta,\theta) \neq 0'_{n+m} \neq t(0,0_m,a^o) \\ \text{for any } t > 0 \end{cases}. \qquad (20)$$

On the other hand in $(E^c_{P^p})$, we have

$$= f(xz^o) + \xi - wx'^o \quad rz''^o \quad (w,r,a,b) \quad (b \quad az^o)$$

and so the

$$\text{foc}\{x - \pi = -\pi^{op} = -\pi - \xi_{wrmamp}\,z^{op} = 0, \quad \pi^{po}\,z0, \quad -\pi^{po}_b + \xi = 0, \ azb\} = \qquad (21)$$

are satisfied at (w^o, r^o, a^o, b^o) with $\xi^{oo} = \lambda$, as we know from (16). Since the (n + m +1) x (n + m + 1) matrix $-\Pi^p$ is the matrix of the partial derivatives of the first n + m -1 equations in (21), we also have at (w^o, r^o, a^o, b^o) the

$$\text{sosc} \begin{cases} \text{For any } (\zeta,\eta,\theta,\kappa) \in R^{n+m+1} \quad -(\zeta,\eta,\theta,\kappa)\,\Pi^{po}\,(\zeta,\eta,\theta,\kappa)' < 0 \\ \text{on the tangent subspace} \\ T\{(0,0_m,z^o,1)(\zeta,\eta,\theta,\kappa)' = 0\} \\ \text{if } (\zeta,\eta,\theta,\kappa) \neq 0'_{n+m+1} \neq t(0,0_m,a^o,b^o) \\ \text{for any } t > 0. \end{cases} \qquad (22)$$

When (21) and (22) are satisfied at (w^o, r^o, a^o, b^o), a strict local maximum of $(E^c_{P^p})$ is attained.

It must be emphasized that (22) cannot be used directly for comparative static analysis because we do not have complete information about the properties of the (n+m+1) x (n+m+1) matrix Π^{po} of second partial derivatives of π (w, r, a, b). We only know that its representation in the tangent subspace, which is of dimensions (n+m) x (n+m), must be positive definite for $(\zeta, \eta, \theta) \neq 0'_{n+m} \neq t(0,0,a^0)$ for any $t > 0$. But to ascertain the implications of the above property we must, first, find a representation of Π^{po} in its tangent subspace and, second, specify the submatrices appearing in it and explain their meaning.

Fortunately this can be done quite easily[3]. Indeed a matrix E^0, whose first n+m rows and columns form an identity matrix and its last row is given by $(0_n, z^0)$, can do the job! E^0 is an (n+m+1) x (n + m) matrix with $r(E^0)$ = n + m and, thus, it provides a basis for all (n + m + 1) vectors in the tangent subspace of $-\Pi^0(w^0, \alpha^0, b^0)$, since $(0_n, z^0, -1)$ $E^0 = (0_n, z^0 - z^0) = (0_n, 0_m)$.

We see therefore that the product matrix, $-E^{0\prime} \Pi^{p0} E^0$, is a representation of $-\Pi^{po}$ restricted to its tangent subspace and, so, must be negative definite for all $(\zeta, \eta, \theta) \neq 0'_{n+m} \neq t(0_n, a^0)$ for any $t > 0$.

Our final task, then, is already at hand: we can see quite easily that $-E^{o\prime} \Pi^{po} E^o = -\Pi^{po}_{w,r,a}$ in (19), which also gives us its submatrices expressed in terms of the rates of change of the solution of (P^p) and, finally, leads to the s-o-s-c in (20).

We conclude, therefore that the **Envelope curvative conditions** of (E_{P^p}) and $(E^c_{P^p})$ are the following :

$$
(e\text{-}c\quad c)\begin{cases}\text{Matrix } -E^{o\prime}\Pi^{po}E^{o}, \text{ as given in (19),}\\ \text{is negative definite for } (\zeta,\eta,\theta)\\ \neq 0'_{n+m} \neq t(0_n, a^o) \text{ and } t > 0\end{cases} \tag{23}
$$

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[3] See e.g. **Luenberger** (1973), chapter 10 on constrained optimization.

These conditions lead to the following comparative static results for (P^p)
It is clear that we have :

(i) $\zeta' X_w (w^o, r^o, a^o, b^o)\, \zeta < 0$ for $\zeta \neq 0$,

(ii) $\eta' Z_r (w^o, r^o, a^o, b^o)\, \eta < 0$ for $\eta \neq 0_m \neq t\, a^0$ for any $t > 0$,

since differentiating the constraint $a' z (w, r, a, b) \equiv b$ w/r r we get $a' Z_r (w, r, a, b) = 0_m$,

(iii) if $\lambda (w^o, r^o, a^o, b^o) > 0$ (< 0), the $Z^{po}_a + z^{po} {z^{po}_b}'$ is negative (positive) semi-definite of rank $m - 1$

and, finally,

(iv) if $\lambda (w^o, r^o, a^o, b^o) = 0$, then the last m rows and columns of $P' E^{po}$ E^o become zeros[4]

On the other hand, in **quantity rationing** the profit functions $\pi^q(w,\bar z)$ and $\pi^{qr}(w,r,\bar z)$ have derivative properties

$$\pi^q_w = \pi^{qr}_w = x(w,\bar z), \quad \pi^{qr}_r = -\bar z,$$
$$\pi^q_{\bar z} = f_z(x(w,\bar z),\bar z) = \mu(w,\bar z) \quad \text{and} \qquad (24)$$
$$\pi^{qr}_{\bar z} = f_z(x(w,\bar z),\bar z) - r = \mu^r(w,r,\bar z)$$

and the symmetric matrices

$$\Pi^q = \begin{bmatrix} \Pi^q_{ww} & \Pi^q_{w\bar z} \\ \Pi^q_{\bar z w} & \Pi^q_{\bar z\bar z} \end{bmatrix} = \begin{bmatrix} X_w & X_{\bar z} \\ M_w & M_{\bar z} \end{bmatrix}$$

and (25)

$$\Pi^{qr} = \begin{bmatrix} \Pi^{qr}_{ww} & \Pi^{qr}_{wr} & \Pi^{qr}_{w\bar z} \\ \Pi^{qr}_{rw} & \Pi^{qr}_{rr} & \Pi^{qr}_{r\bar z} \\ \Pi^{qr}_{\bar z w} & \Pi^{qr}_{\bar z r} & \Pi^{qr}_{\bar z\bar z} \end{bmatrix} = \begin{bmatrix} X_w & O_{nm} & X_{\bar z} \\ O_{mn} & O_{mm} & -I_m \\ M^r_w & -I_m & M^r_{\bar z} \end{bmatrix},$$

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[4] Ignoring the last zero rows and columns of the matrix in (19), we get the $n \times n$ matrix $\begin{bmatrix} X^p_w & X^{po}_r \\ Z^{po}_w & Z^{po}_r \end{bmatrix}$ whose interesting relationship to $\begin{bmatrix} X^{fo}_w & X^{f}_r \\ Z^{fo}_w & Z^{fo}_r \end{bmatrix}$ will be considered in the next section.

respectively.[5]

The **Envelope problem** in (P^q) appears also in two forms: for any specific parameter values (w^o, z^o) and $x^o \equiv x(w^o, z^o)$, $\mu^o \equiv \mu(w^o, r^o, z^o)$ also fixed we may consider a **constrained envelope problem**, namely,

$$0 = \max_{w,r,z}\{f(x^o, z) - w'x^o - r'z(w,r,z) \mid z = \bar{z}\} \,. \qquad (E^c_{P^q})$$

Due to the linearity of the m constraints, we may also consider an **unconstrained envelope problem**, for specific (w^o, r^o) and $x^o \equiv x(w^o, \bar{z})$, namely,

$$0 = \max_{w,r}\{f(x^o, \bar{z}) - w'x^o - r'z(w,r,\bar{z})\} \,. \qquad (E_{P^q})$$

Again we examine (E_{P^q}), first, which is characterized by

$$f \quad o \quad c\{x^o - \pi_{wm}(w,r,\bar{z}) = 0_p;\quad \bar{z} - \pi(w,r,\bar{z}) = 0\} \qquad (26)$$

which, as we know from (24), are satisfied at (w^o, r^o) and attain a strict local maximum of zero, if for the symmetric n x m matrix

$$-\begin{bmatrix} \Pi_{ww}(w^o, r^o, \bar{z}), & \Pi_{wr}(w^o, r^o, \bar{z}) \\ \Pi_{rw}(w^o, r^o, \bar{z}), & \Pi_{rr}(w^o, r^o, \bar{z}) \end{bmatrix} = \begin{bmatrix} X_{wm}(w,z), & O \\ O, & O_{mm} \end{bmatrix}, \qquad (27)$$

we also have the

$$sosc\ \left\{ -(\zeta', \eta') \begin{bmatrix} \Pi^{qo}_{ww}, & \Pi^{qo}_{wr} \\ \Pi^{qo}_{rw}, & \Pi^{qo}_{rr} \end{bmatrix} \begin{pmatrix} \zeta \\ \eta \end{pmatrix} < 0 \quad \text{for } \zeta \neq 0_p \right\} . \qquad (28)$$

On the other hand in $(E^c_{P^q})$ we have, using

$$= f(x^o, z^o) - w'x^o - r'z(w,r,\bar{z}) + \xi'(\bar{z} - z^o)$$

with ξ the vector of the m lagrangean multipliers,

$$foc\{x^o - \pi^q_{wm} = 0_p,\ z^o - \pi^q = 0,\ -\pi^q + \xi = 0,\} \quad \text{and } z^o = \bar{z}\} \,, \qquad (29)$$

---

[5] It is obvious, from $x^g \equiv x^q$, $\mu^g \equiv r + \mu^q$ and (24) – (25), that $M^{qq}_{ww} = M_{mm}$ and $M^{qg}_{zz} = M$.

which are satisfied at (w^o, r^o, z^o) and $\xi^o = \mu^o$.

Since the matrix of the partial derivatives of the first $\ell + m + m$ equations in (29) is $-\Pi^q$ as given in equation (25) and since the gradient matrix of the m constraints in $(w, r, \bar{z})$ is given by $[O_{m\ell}, O_{mm}, -I_{mm}]$, we also have at (w^o, r^o, z^o) the

$$\text{s-o-s c}\begin{cases}\text{For any } (\zeta,\eta,\theta) \in R^{\ell+m+m},\ (\zeta,\eta,\theta)' \neq 0 & -(\zeta,\eta,\theta)\Pi^q(\zeta,\eta,\theta)' < 0\\ \text{on the tangent subspace } T = \{(O_{m\ell}, O_{mm}, -I_{mm})(\zeta,\eta,\theta)' = 0\} & \\ & \end{cases}. \quad (30)$$

Again these s-o-s-c cannot be used directly for comparative static analysis. To find a matrix that represents $-\Pi^{qo}$ when it is restricted in its tangent subspace, we use the $(\ell + m + m) \times (\ell + m)$ matrix E^o whose first $\ell + m$ rows and columns form an identity matrix and its last m rows consist of zeros. E^o is an $(\ell + m + m) \times (\ell + m)$ matrix with $r(E^o) = \ell + m$ and can, thus, provide a basis for all $(\ell + m + m)$ vectors in the tangent subspace of $-\Pi(w^o, r^o, z^o)$, since $(O_{m\ell}, O_{mm}, -I_{mm}) E^o = (O_{m\ell}, O_{mm})$. We see therefore that a representation of $-\Pi^{qo}$ restricted to its tangent subspace is given by $-E^{o\prime}\Pi^{qo}E^o$ and it is simply the $(\ell + m) \times (\ell + m)$ matrix in (27).[6]

We conclude then that the **Envelope curvature conditions** of (E_{p^q}) and $(E^c_{p^q})$ are given by

$$(\text{e-c c})\begin{cases}\text{Matrix } -E^{o\prime}\Pi^{qo}E^o \text{ is negative}\\ \text{definite for } (\zeta,\eta)' \neq 0_n\end{cases}. \quad (31)$$

It is clear from (31) that the only comparative static result of (P^q) we have obtained, so far, is that $X_w(w, \bar{z})^o$ is a negative definite $\ell \times \ell$ matrix.

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[6] It must be admitted that the last paragraph could have been avoided, if we had noted that in the tangent subspace of (30) $\theta_m = 0$, while (ζ, η) is unrestricted; thus (30) would immediately coincide with (28). This was done on penpose so that $(E^c_{p^q})$ would be specified for (P^q).

We cannot end this section without a comparison of the two alternative methods for doing comparative static analysis. As the reader has seen in **Appendix A**, **the primal method in constrained optimization problems** examines the bordered Hessian of the problem, a matrix having additional rows and columns than the Hesssian, depending on the number of constraints imposed. Correspondingly however, **the primal method produces comparative static results for all choice variables, including the lagrangean multipliers**. On the other hand, **the dual method in constrained optimization problems** focuses on a reduced matrix of the Hessian of the optimal value function, a matrix restricted in the tangent subspace of the Hessian and with a smaller number of rows and columns depending on the number of constraints imposed. Consequently, however, the comparative static results produced, so far, by the dual method are limited to the rates of change of choice variables **minus** those of the lagrangean multipliers. It is obvious from the s-o-s-c of the unconstrained envelope problems (E_p) and (E_q) as given in (20) and (28), respectively, that no restrictions on the signs of π^{qq}_{bb} and $^{qo}_{zz}$ can be established. Does this difference point to a structural deficiency of the dual method? Not at all, as we will see in the next section.

4. Interrelations between unrestricted and restricted profit maxima; the various manifestations of distinct local Le Chaterlier Effects.

It is evident that for any (w, r) and (a, b) or $\bar{z}$ we must have

$$\pi(w,r) \geq \pi(w,r,a,b) \text{ and } \pi(w,r) \geq \pi(w,r,\bar{z}) \,. \tag{32}$$

Equalities may appear in (32) only when –by chance or design– rationing constraints happen to be "just binding", with either $a' z(w, r) = b$ or $z(w, r) = \bar{z}$. Otherwise, it is impossible to relate their solutions and compare their rates of change as parameters vary.

In some cases, however, it is possible to establish interrelations between (P^f) and (P^p), or (P^f) and (P^q) or of all three, by appropriate choices of alternative subsets of parameters so that maximum value functions are brought into contact with one another, thereby creating tangencies and producing proper curvature conditions on the rates of change of their solutions. The **Envelope theorem** is not only involved in all such cases, but appropriate **Envelope problem** can also be designed so as to bring about such results. In this more general setting, in which one of the profit functions depends on actual parameter values while the other depends also on properly chosen "shadow" values of some parameters, there are for greater opportunities for such tangencies between π^f, π^p, or π^q to occur.

In our **first comparison**, (P^f) is assumed to have been solved when the **point rationing** constraint $a'z = b$, $a >_m 0$, $b > 0$ is imposed. Since $a' z(w, r) \neq b$, in general, we can reach an envelope tangency at the first best optimum quite simply: we only have to select b so that

$a'z(w,r) = b$. The feasibility of z (w, r) under this point rationing constraint implies that (P^p) has the same solution as (P^f), i.e., that

(i) x (w, r, a, b) ≡ x (w, r) , (ii) z (w, r, a, b) ≡ z (w, r) and

(iii) λ (w, r, a, b) = 0

Thus from $b \equiv a'z(w,r) \equiv b(w,r,a)$ we get the derivative properties $b'_w \equiv a'Z^f_w$, $b'_r \equiv a'Z^f_r$ and $b_a \equiv z$ and, so, we can compare the rates of change of the solutions of (P^f) and (P^p) at the first best optimum. As shown in **Appendix B** we get

$$\begin{aligned} &\text{(i)}\ \ X^p_w + (1/a'Z^f_r a)X^f_r a a' X^{f\prime}_r = X^f_w\,, \\ &\text{(ii)}\ \ Z^p_r + (1/a'Z^f_r a)Z^f_r a a' Z^f_r = Z^f_r \quad \text{and} \qquad\qquad (33) \\ &\text{(iii)}\ \ \lambda^p_b = (1/a'Z^f_r a)\,, \end{aligned}$$

where function arguments are suppressed, while the presence of some "shadow' parameter values is indicated by a superscript. Even before looking at the proof of (33) in **Appendix B**, it must be noted that $\lambda_b(w,r,a,b) = (1/a'Z^f_r a) < 0$ and that the rates of change of the solutions of (P^p) can be and are indeed expressed in term of those of (P^f) and the known a_j `s , j = 1, … , m. The important finding is that all matrices in the second terms of the ℓ - h – sides of (33) are negative semi definite of rank 1, since $a'Z^f_r a < 0$ and matrix aa′ is positive semi definite of rank 1 but with positive main diagonal elements. It is obvious that an Envelope tangency is attained at the first best optimum, with $\pi(w,r) = \pi(w,r,a,b)$ and π (w, r) more convex than $\pi(w,r,a,b)$ there. The envelope curvature conditions (at the first best optimum) are given by

$$\begin{cases} 0 > x^{ii}_{w}(w,r,a,b) > x^{ii}_{w}(w,r), & \text{all } i \\ 0 > z^{jj}_{r}(w,r,a,b) > z^{jj}_{r}(w,r), & \text{all } j \qquad (34) \\ -\lambda_b(w,r,a,b) > 0 & \end{cases}$$

On the other hand, if **quantity rationing** constraints $z = \bar{z}$, $\bar{z} = z(w,\bar{r})$, are imposed, then by choosing $\bar{z} = z(w,r)$ a tangency between (P^f) and (P^q) is produced at the first best optimum and

$$\text{(i)}\quad x(w,\bar{z}) \equiv x(w,r) \quad \text{and} \quad \mu(w,r,\bar{z}) = 0_m .$$

Then we get, since $X^{qf}_{zr} Z = X$, or $X^{qf1}_{zr} = X Z^{-1}$,

$$\text{(i)}\quad X^{q}_{ww} = X^{f}_{ww} + X^{qf}_{wz} Z^{f}_{w} = X - X Z^{-1} X' \ldots X \qquad (35)$$

$$\text{and} \quad \text{(iii)}\; M^{qf1}_{zr} = -Z^{-1} = O ,$$

since $M^{qf}_{zr} Z = O$ and $M I^{q}$ from (25). Thus the envelope curvature conditions (at the first best optimum) are given by

$$\begin{cases} 0 > x^{ii}_{ww}(w,\bar{z}) > x^{ii}_{ww}(w,r), & \text{all } i \\ -\mu^{j}_{z}(w,r,\bar{z}) > 0, & \text{all } j \end{cases} \qquad (36)$$

as shown in **Appendix B**.

The first set of ℓ inequalities in (36) are the Le Chatelier effects established by **Samuelson** (1947, pp. 36-38) as he introduced the **Le Chatelier Principle** in the economic literature.

The **second comparison** starts with the solution of (P^p) or (P^q) and considers the possibility of attaining an envelope tangency there if λ (w, r, a, b) $\neq 0$ or if $\mu(w,r,\bar{z}) \neq 0_m$, respectively.

With **point rationing** we can select the "shadow" prices of rationed inputs, r, by

$$r \equiv r + \lambda(w,r,a,b)a = f_{zm}(x(w,r,a,b),z(w,r,a,b)) \equiv r(w,r,a,b) > 0 , \qquad (37)$$

with derivative properties $R_{a} = \lambda I_{mm} + a\lambda'_{a}$ ... $\lambda^{p\prime}$

and $r_b = a\lambda_b$.

It is easily shown that such a choice of r (w, r, a, b) transforms the f-o-c of (P^p) into those of (P^f) and, thus, leads to $\pi(w,r) \equiv \pi(w,r,a,b)$ and to

$$\text{(i)}\quad x(w,r) \equiv x(w,r,a,b) \quad \text{and} \quad \text{(ii)}\quad z(w,r) \equiv z(w,r,a,b) .$$

We show in **Appendix B** that

$$\text{(i)}\quad X^{f}_{ww} = X^{p}_{ww} - (1+\lambda^{p}_{b}/b)\, x^{p}_{b} x^{p}_{b}{}' \quad X$$

and (38)

$$\text{(ii)}\quad Z^{f}_{rr} = Z^{p}_{rr} - (1+\lambda^{p}_{b}/b)\, z^{p}_{b} z^{p}_{b}{}' \quad Z$$

and also that

$$\lambda_{b}^{p}(w,r,a,b) \equiv -(1/a\, z' Z^{f} z) \quad 0$$

since $Z^{f}_{rr}(w,r)$ is a negative definite matrix for $(w,r) > 0_{n}'$.

The envelope curvature conditions at the second best optimum are

$$\begin{cases} x^{ii}_{ww}(w,r) \lessgtr x^{ii}_{ww}(w,r,a,b)\ 0, \text{all } i \\ z^{jj}_{rr}(w,r) \lessgtr z^{jj}_{rr}(w,r,a,b)\ 0, \text{all } j \end{cases} . \qquad (39)$$

On the other hand, in **quantity rationing**, the solution of (P^q) can be transformed into that of (P^f) if we select r by

$$r \equiv r + \mu(w,r,\bar z) = f_z(x(w,\bar z),\bar z) \equiv r(w,r,\bar z) > 0 . \qquad (37')$$

Then (i) $x(w,r) \equiv x(w,\bar z)$ and (ii) $z(w,r) = \bar z$

and so

$$\text{(i)}\quad X^{f}_{ww} = X^{q}_{ww} - M_{wz} M^{-1}_{zz} M_{wz}{}' \quad M \quad X$$

and (40)

$$\text{(ii)}\quad Z^{f}_{rr} M_{zz} = I \qquad ,$$

since $R_w M_{rm} = R M^{q}_{zz}$ ..., and from $Z^{f}_{rr} M_{zz} =$

we see that $M_{zz}(w,r,\bar z) = Z(w,r)^{-1}$ is the Inverse of a negative definite matrix for $w > 0$, r, $r > 0_m$ and $\bar z > 0_m$, as shown in **Appendix B**.

The envelope curvature conditions at the second best optimum are

$$\begin{cases} x_i^f(w,r) - x_i^p(w,z) \le 0, \text{ all } i \\ z_j^f(w,r) \le 0, \text{ all } j \end{cases} \quad (41)$$

Additional interrelations between fist and second best profit maxima are obtained if, having the solution of either one, an appropriate **Envelope problem** is solved to produce tangencies and curvature conditions at the **other** profit maximum.

Thus the **third comparison** starts with the solution of (P^p) in **point rationing** and the envelope problem

$$0 \equiv \max_b \{\pi^p(w,r,a,b) - \pi^f(w,r)\}$$
$$\text{or} \quad \pi^f(w,r) \equiv \max_b \{\pi^p(w,r,a,b)\}, \qquad (E_b)$$

without knowing anything about (P^f).

(E_b) is characterized by

$$f-oc\{\pi_b(w,r,a,b) = 0\} \qquad (42)$$

and

$$sosc\{\pi_{bb}(w,r,a,b) < 0\}, \qquad (43)$$

with b determines implicitly by solving

$$\lambda(w,r,a,b) = 0, \qquad (42')$$

since $\lambda_b = \lambda_b(w,r,a,b) < 0$ from (43).

The $b \equiv b(w,r,a)$ has derivative properties $b_w = -(1/\lambda_b^p)x_b^p$, $b_r = -(1/\lambda_b^p)z_b^p$

and $b_a = z^p$ since $\lambda_a^p = 0$, as we can see from (17) .

With the help of (E_b) we get

$$\text{(i)} \; x^f(w,r) \equiv x^p(w,r,a,b), \quad \text{(ii)} \; z^f(w,r) \equiv z^p(w,r,a,b)$$

and thus

$$\text{(i)} \quad X_{ww}^f = X_{ww}^p - \lambda_b^p (1/\lambda_b^p)\, x_b x_b'$$

and (44)

$$\text{(ii)} \quad Z_{rr}^f = Z_{rr}^p - \lambda_b^p (1/\lambda_b^p)\, z_b z_b'$$

We see therefore that the envelope curvature conditions at the first best profit maximum are given by

$$\begin{cases} x^{i}_{w_i}(w,r) - x^{i}_{w_i}(w,r,a,b) \le 0, \text{ all } i \\ z^{j}_{r_j}(w,r) - z^{j}_{r_j}(w,r,a,b) \le 0, \text{ all } j \end{cases} \tag{45}$$

On the other hand, in **quantity rationing**, if (P^q) has been solved we can consider

$$0 = \max_{\bar z}\{\pi(w,r,z) - \bar\pi(w,r)\}$$

or $$\bar\pi(w,r) \equiv \max_{\bar z}\{\pi(w,r,z)\}, \tag{$E_{\bar z}$}$$

with

$$\text{foc}\{\pi_{\bar z}(w,r,z) = 0_m\} \tag{46}$$

and

$$\text{sosc}\begin{cases} \Pi_{\bar z\bar z}(w,r,z) \text{ is a negative} \\ \text{definite matrix} \end{cases}. \tag{47}$$

z is determined implicitly from

$$\pi_{\bar z}(w,r,z) \equiv f_z(x(w,z),z) - r \equiv \Phi(w,r,z) = 0_m. \tag{46′}$$

With the help of $(E_{\bar z})$ and the derivative properties of $z = \bar z(w,r)$, or $Z^{fq}_{w} = M^{-1} X^{q\prime}_{wz}$ and $Z^{fq}_{r} = M^{-1}$, we get

$$\text{(i)}\ \ x(w,r) \equiv \bar x(w,z), \text{ and (ii)}\ \ z(w,r) \equiv z$$

and thus

$$\begin{aligned} &\text{(i)}\ \ X^{f}_{ww} - X^{q}_{ww} = X^{q}_{wz} M^{-1} X^{q\prime}_{wz}, \\ &\text{(ii)}\ \ Z^{fq}_{rz} = M^{-1} \end{aligned} \tag{48}$$

and the envelope curvature conditions at the first best profit maximum

$$\begin{cases} x^{i}_{w_i}(w,r) - x^{i}_{w_i}(w,z) \le 0, \text{ all } i \\ z^{j}_{r_j}(w,r) \le 0, \text{ all } j \end{cases} \tag{49}$$

Our **fourth comparison** starts with the solution of (P^f) and produces an Envelope tangency at (P^q) or (P^p).[7]

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[7] The reason for considering (P^q) first will become apparent below.

With the **quantity rationing** constraints, $z = \bar z$, we consider the **Envelope problem**

$$0 = \max_r \{\pi(w,r,\bar z) - \pi(w,r)\}$$

$$\equiv \max_r \{\pi(w,\bar z) - r'\bar z - \pi(w,r)\}. \qquad (E_r^q)$$

or $\quad \pi(w,\bar z) \equiv -\max_r \{-r'\bar z - \pi(w,r)\} \equiv \min_r \{r'\bar z + \pi(w,r)\}.$

No prior knowledge of $\pi(w,\bar z)$ is necessary: the second form of (P^q) follows directly from the first and $\bar z$ is the known rector of fixed inputs. (E_r^q) is characterized by

$$\text{foc} \{\bar z + \pi_r(w,r) = 0_m\} \qquad (50)$$

and

$$\text{s-o-c} \{\pi_{rr}(w,r) \text{ is positive definite}\}. \qquad (51)$$

$r \equiv r(w,\bar z)$ is determined implicitly by (50) or $\bar z \equiv -\pi_r(w,r) \equiv z(w,r)$ and has the derivative properties $R_w = -Z_r^{f-1} Z_w$ and $R_{\bar z} = Z_r^{f-1}$. Thus we get

(i) $x(w,\bar z) \equiv x(w,r)$, (iii) $\mu(w,r,\bar z) \equiv r - r = 0_m$

and we can derive their rates of change

$$\text{(i)} \quad X_w^q = X_w^f - X_r^f Z_r^{f-1} X_r^{f\prime}$$
$$\text{and (ii)} \quad M_{zr}^q = Z_r^{f-1} - \bar Z_r^{-1} = O_{mm}. \qquad (52)$$

The envelope curvature conditions at the second best profit maximum are

$$\begin{cases} 0 > x_{ww}^{ii}(w,\bar z) > x_{ww}^{ii}(w,r), & \text{all } i \\ -\mu_{z_j}^j(w,r,\bar z) > 0, & \text{all } j \end{cases} \qquad (53)$$

When, however, (P^f) has been solved and a **point rationing** constraint is imposed, then an envelope tangency at (P^p) can only be obtained if $x(w, r, a, b)$, $z(w, r, a, b)$ and $\lambda(w, r, a, b)$ are already known. Even if we treat (x^p, z^p, λ^p) as given and consider an Envelope problem (E_r^p) analogous to (E_r^q), we cannot proceed and determine r implicitly from the first - order identities, $z^p \equiv -\pi_r(w,r)$ if we do not know how $z^p = z(w, r, a, b)$ vary with their parameters. But if (P^p) has also to be solved, do we need (E^p_r) for attaining a tangency at (P^p)? Is it not easier to rely on our second comparison and, having the solution of (P^p), simply define $\bar r$ using (37) ?

We see therefore, from our examination of all interrelations of any two of (P^f), (P^p) and (P^q), that under **quantity rationing** conditions (36), (41), (49) and (53) exhibit four distinct manifestations of the **Le Chatelier Principle**, while under **point rationing** three such manifestations appear in (34), (39) and (45). Apparently the complexity of the single point rationing constraint, $a' z(w, r, a, b) \equiv b$ is enough to preclude an efficient utilization of (E^p_r) for reaching an envelope tangency at the second best profit maximum.[8]

We must not also forget the possibility of establishing interrelations between all three profit maxima. For example, let us assume that (P^p) has been solved. Then, selecting $\bar{\mathbf{r}} = \mathbf{r} + \lambda(w,r,a,b)a$, we obtain (P^f), as we have seen in **second comparison (a)**. But suppose that we also want to ascertain the repercussions of imposing quantity rationing constraints, chosen so as to have $\bar z \equiv z(w,r,a,b) \equiv z(w,\bar r)$. Then the f – o – c of (P^p) are transformed into $\{f_x(x^p,\bar z) - w = 0,\ f_{z_m}(x^p,\bar z) - r - \lambda(w,r,a,b)a = 0,\ z = \bar z\}$ or equivalently into the f – o – c $\{f_x(x(w,\bar z),\bar z) - w = 0,\ f_{z_m}(x(w,\bar z),\bar z) - r - \mu(w,r,\bar z) = 0,\ z = \bar z\}$, of (P^q) with (i) $x(w,z) \equiv x(w,r,a,b)$ and (iii) $\mu(w,r,z) \equiv \lambda(w,r,a,b)a$.
Differentiating (i) w/r to w and r we get

$X^q_w + X^q_z Z^p_w = X^p_w$ and $X^q_z Z^p_r = X^p_r$. Thus we get

$X^q_w = X^p_w - X^q_z Z^p_w$,

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[8] **Samuelson** (1947), pp. 163-71, and **Graaff** (1947-48) have independently examined **point rationing** in the Theory of consumer choice: in that context further problems appear.

from which we see that $X^{q}_{z} Z^{p}_{r} X^{q\prime}_{z}$ is a negative semi definite matrix, even if we do not know anything about X^{q}_{z} .

Similarly

$M^{q}_{zz} = \lambda^{p}_{b}\, b\, b'$ or $z' M^{q}_{zz} z = \lambda^{p}_{b} < 0$.

We thus see that we get the envelope curvature conditions at (P^p)

$$\begin{cases} x^{i}_{w_i}(w,r) \le x^{i}_{w_i}(w,r,a,b) \le x^{i}_{w_i}(w,\bar{z}) \le 0, \text{ all } i \\ z^{j}_{r_j}(w,r) \le z^{j}_{r_j}(w,r,a,b) \le 0, \text{ all } j \end{cases} \tag{54}$$

Finally, it is evident, from our **second comparison** in **point** or **quantity rationing**, as well as, from the **fourth comparison** in **quantity rationing**, that we have

$\lambda_b (w, r, a, b) = (1/a' Z_r(w, r)^{-1} a) < 0$, although $\lambda (w, r, a, b) \gtrless 0$ and $M_{zz}(w,r,\bar{z}) = Z_r(w,r)^{-1}$ is negative definite, although $\mu(w,r,\bar{z})$ may not be a zero vector. Our proof is based on the negative definiteness of $Z_r(w,r)$ for all $(w,r)' > 0_n$ and $(x(w,r),z(w,r))' \in \overset{o}{S}$. If $\lambda_b (w, r, a, b)$ were not a negative scalar and if $M_{zz}(w,r,\bar{z})$ were not a negative definite matrix, for all admissible parameter values, then no envelope tangency could emerge between (P^p) or (P^q) and (P^f) or between (P^f), (E^q) and (P^q) at the second best profit maximum.

5. Global Comparative Static and Le Chatelier effects and their upsets: a diagrammatic analysis with two inputs

We consider a competitive firm that produces its output with two inputs, facing positive input and output prices, (w, p). Thus

$$\pi(w_1, w_2, p) = \max_x \{pf(x) - w'x\} \,, \qquad (P^f)$$

with f strongly concave and $|F(x)| = f_{11}(x) f_{22}(x) - f_{12}(x)^2 > 0$. With input demand functions $x^i(w, p)$, $i = 1, 2$, we get the comparative static results

$$X_w(w,p) = \frac{1}{p} F(x(w,p))^{-1} = \frac{1}{p\{f_{11}f_{22} - f_{12}^2\}} \begin{bmatrix} f_{22} & -f_{12} \\ -f_{12} & f_{11} \end{bmatrix} \qquad (53)$$

and

$$x_p(w,p) = -\frac{1}{p} F(x(w,p))^{-1} f_x(x(w,p)) = \frac{1}{p\{f_{11}f_{22} - f_{12}^2\}} \begin{pmatrix} f_{12}f_2 - f_{22}f_1 \\ f_{12}f_1 - f_{11}f_2 \end{pmatrix} . \qquad (54)$$

The firm's **expansion path**, $x_2(x_1)|_{w_1/w_2}$, with w_1 / w_2 = a constant, is determined implicitly by

$$f_1(x_1, x_2(x_1)) / f_2(x_1, x_2(x_1)) = w_1 / w_2 \qquad (55)$$

and has a slope

$$x_2'(x_1)\big|_{w_1/w_2} = \frac{f_{12}f_1 - f_{11}f_2}{f_{12}f_2 - f_{22}f_1} = \frac{x_p^2(w,p)}{x_p^1(w,p)} \,, \qquad (56)$$

while its **constant – marginal – product curves**, $x_2(x_1)|_{w_1/p}$ and $x_2(x_1)|_{w_2/p}$ are determined implicitly by

$$f_i(x_1, x_2(x_1)) = w_i / p \,, \quad i = 1, 2 \,, \qquad (57)$$

having slopes

$$x_2'(x_1)\big|_{w_i/p} = -\frac{f_{i1}(x_1, x_2(x_1))}{f_{i2}(x_1, x_2(x_1))} \qquad (58)$$

and

$$x_2'(x_1)\Big|_{w_2/p} = -\frac{f_{21}(x_1, x_2(x_1))}{f_{22}(x_1, x_2(x_1))} \quad . \qquad (59)$$

We see that the EP has a positive slope when both inputs are **normal**, with $x_{ip}(w,p) > 0$, has a zero slope when $x_2(w, p)$ is a **neutral** input and a negative slope when $x_2(w, p)$ is an **inferior** input, with EP turning towards the x_1, axis; similarly the EP has an infinite or negative slope when x_1 (w, p) is neutral or inferior, with EP turning towards the x_2 axis.
We also see that both CMP curves have positive slopes when $f_{12}(x) > 0$ and negative slopes when $f_{12}(x) < 0$, since $f_{11}(x)$, $f_{22}(x) < 0$.
If $f_{12}(x) = 0$ then the slope in (58) is infinite, while that in (59) is zero. Of course, the sign of $f_{12}(x)$ depends, in general, on $x \in S$ and may be positive or negative, within limits, in one region of S or in another.
In Economic terms two inputs are called **complements** (co operant) when $f_{12}(x) > 0$, **substitutes** when $f_{12}(x) < 0$ and **independent** when $f_{12}(x) = 0$.

When any two of $x_2(x_1)\big|_{w_1/w_2}$, $x_2(x_1)\big|_{w_1/p}$ or $x_2(x_1)\big|_{w_2/p}$

intersect at some $x \in S$, for the same input and output prices, then the third one also passes through the same point. We also establish the relations among the slopes of the EP and the CMP curves at a point where all three intersect. Thus we can easily show that

(i) when $f_{12}(x) > 0$ then

$$x_2'(x_1)\Big|_{w_1/p} > x_2'(x_1)\Big|_{w_1/w_2} > x_2'(x_1)\Big|_{w_2/p} > 0 \ , \qquad (60)$$

(ii) when $f_{12}(x) = 0$ then

$$+\infty = x_2'(x_1)\Big|_{w_1/p} > x_2'(x_1)\Big|_{w_1/w_2} > x_2'(x_1)\Big|_{w_2/p} = 0 \ , \qquad (61)$$

while (iii) when $f_{12}(x) < 0$, but both inputs are normal, then

$$x''_{21}(x_1)\Big|_{w_2/p} < x_{21}(x_1)\Big|_{w/p} < 0 \qquad (62)$$

while $x'_{21}(x_1)\Big|_{w_1/w_2}$ is still positive.[9]

Let us finally note from (53) that

$x^{12}_{w_2}(w,p) = x^{21}_{w_1}(w,p) \lessgtr 0$ when $f_{12}(x(w,p)) \gtrless 0$. Thus when the two inputs are complements, they are also **gross complements**, with < 0 in the first and > 0 in the second inequality. On the contrary substitute inputs are also **gross substitute**, with > 0 in the first and < 0 in the second inequality.

Those properties hold, not only for two inputs, but for any number of inputs: when **all** inputs are complements they are also gross complements, while when all inputs are substitutes they are also gross substitutes[10].

We can then turn to the diagrams in Figure 1-3, which illustrate the interrelations of EP and the CMP curves, when $f_{12}(x)$ is either positive or negative or changes sign over S, separated by an $f_{12}(x) = 0$ locus[11].

In **Figure 1(a)** $f_{12}(x)$ is everywhere positive. Point A denotes a profit maximum, since both CMP curves have the appropriate shapes and if x_1 or x_2 rise from A then $f_1(x)$ or $f_2(x)$ fall. The figure also shows that, for a finite increase in w_1, the profit maximum moves to point A′ on the original $x_{21}(x)\Big|_{w_2/p}$ curve with both x_1 and x_2 smaller. In **Figure 1(b)** however, with $f_{12}(x)$ everywhere negative, the profit maximum at A

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[9] In the interest of brevity we do not offer a complete analysis of the above interrelations, when one of the inputs becomes inferior: the interested reader can easily do it, taking into account that a strongly concave f(x) is also strongly quasi-concave. These two conditions set the upper, positive, and lower, negative, limits within which $f_{12}(x)$ may range.

[10] See **Rader** (1968), who first examined the implications of $f_{ij}(x) > 0$, $j \neq i$ for $x^i(w, p)$. See also **Takayama** (1985), ch. 4, for a complete proof.

[11] Such an $f_{12}(x) = 0$ locus does not, in general, coincide with a particular production isoquant.

moves to A′, for a finite decrease in w_1: we thus see that x_1 increases while x_2 decreases. Both figures lead to an important **global** comparative static result: when both inputs are complements (substitutes) of one another, they are also gross complements (gross substitutes) of one another.[12,13]

In **Figures 2(a) and (b)** the $f_{12}(x) > 0$ region is followed by an $f_{12}(x) < 0$ one as output increases, or the opposite. In 2(a) we first observe how the positively sloped CMP curves become negatively inclined as soon as they enter the $f_{12}(x) < 0$ region. With A the original maximum profit point, greater finite decrease in w_1 move A to A′, A′′ and finally to A′′′. But although $f_{12}(x) < 0$ near A′′ and the two inputs are substitutes, they are also gross complements, since both increase as w_1 decreases: indeed, not only $x_1(w_1'', w_2, p) - x_1(w_1, w_2, p) > 0$ but also $x_2(w_1'', w_2, p) - x_2(w_1, w_2, p) > 0$ for $w_1'' < w_1' < w_1$. Thus x_2 is a gross complement of x_1 although as we approach A′′ $f_{12}(x)$ is negative. Of course at A′′′, which is reached when the decrease of w_1 is greater, we observe again a restoration of the implication that "substitute inputs" imply "gross substitute inputs". In **Figure 2(b)** an $f_{12}(x) < 0$ region is succeeded by an $f_{12}(x) > 0$ one. Again decreases in w_1 lead A to move to A′, A′′ and A′′′ with $x_2(w_1'', w_2, p) < x_2(w_1, w_2, p)$. Thus, although A′′

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[12] To keep all diagrams as simple as possible, we have not drawn the new EP that passes through A′. Neither have we considered the effects of all possible changes in w_1 and w_2, which we leave for the interested reader.

[13] We also note that of the well known examples of production functions, the Cobb-Douglas function - $f(x) \equiv A x_1^{a_1} x_2^{a_2}$, $A, a_i > 0$, $a_1 + a_2 < 1$ - has $f_{12}(x) > 0$ while the CES - $f(x) \equiv A[a_1 x_1^{\tau} + a_2 x_2^{\tau}]^{\frac{s}{\tau}}$, $A, a_1, a_2 > 0$, $\tau < 1$, $1 > s > 0$ - has $f_{12}(x) > 0$ when $1 > s > \tau$, $f_{12}(x) < 0$ when $1 > \tau > s > 0$ and $f_{12} = 0$ when $1 > \tau = s > 0$. In the latter case f(x) is an additive function, like $A[a_1 x_1^{\tau_1} + a_2 x_2^{\tau_2}]$, $A, a_i > 0$ and $1 > \tau_1, \tau_2 > 0$. Both types of production functions lead to diagrams similar to Figure 1.

is in the $f_{12}(x) > 0$ region and x_1 and x_2 are complements, x_1 and x_2 are gross substitutes.

Finally, in **Figures 3(a) and (b)** we illustrate the effects of **quantity rationing** of input x_2 on the firm's demand response for x_1 when w_1 changes. We assume that $\bar{x}_2 = x_2(w_1, w_2, p)$ at the original profit maximum at point A. In both Figures 3(a) and (b) we see that A moves to A′, A′′ and finally to A′′′, with $w_1 > w_1' > w_1'' > w_1'''$. We see that at A′′ we get $x_1(w_1, \bar{x}_2, \bar{p})'' > x_1(w_1, w_2, p)$, namely, a greater increase in the demand for x_1 under quantity rationing relative to that when both inputs are variables. We thus observe that a global Le Chatelier Principle is upset at A′′ and restored again at A′′′.[14,15]

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[14] **Milgrom and Roberts** (1996) examine the global Le Chatelier Principle in a model quite more general than in the neoclassical theory of the firm. They show that in the latter case and with two inputs upsets of the global Principle are observed, when $f_{12}(x) > 0$ is followed by a $f_{12}(x) < 0$ region, as illustrated in Figure 3(a).

[15] Long ago **Samuelson** (1960a), page 372, pointed out that a global Le Chatelier effect can be upset if the firm's maximum profit, without and with quantity rationing, at a point like A in Figure 3(b) is "near the critical point where [the input] go from being substitutes to being complements, as measured by the sign of $[f_{12}(x)]$".

Figures

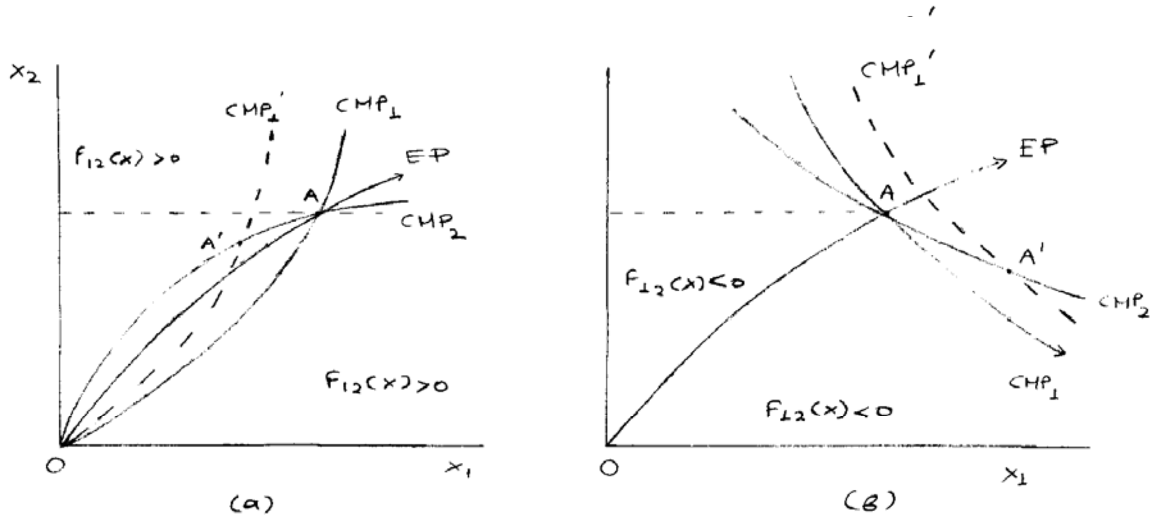

Figure 1

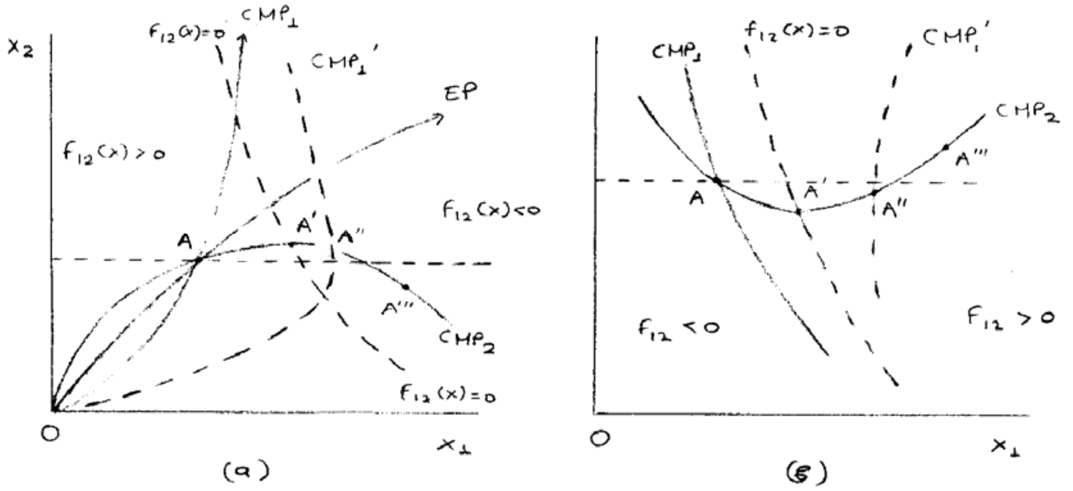

Figure 2

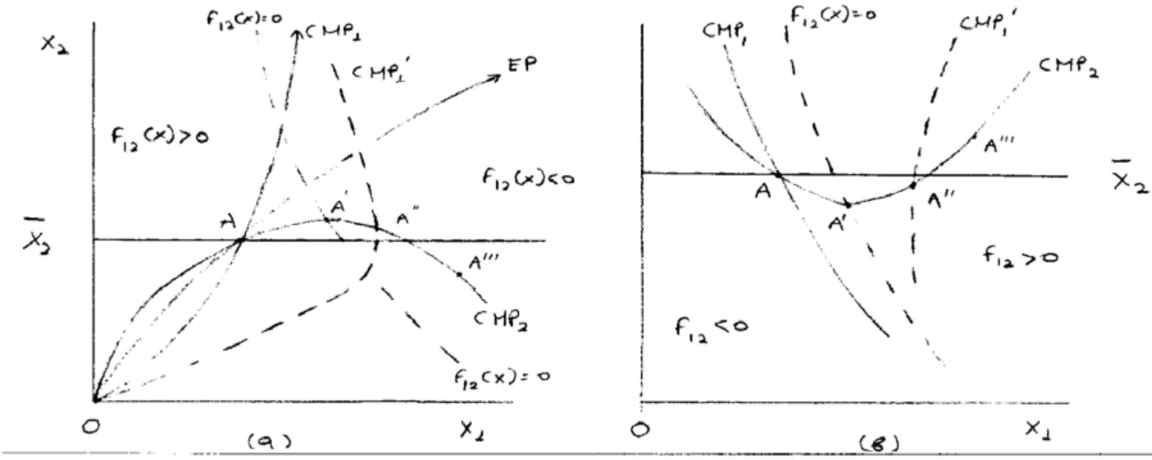

Figure 3

6. Concluding Remarks

Profit maximization is indeed a simple example of an optimization problem, even under quantity or point rationing. In compensation of its simplicity, it leads to a correspondingly smooth introduction to sensitivity analysis and Le Chatelier Principle that may prove quite helpful in more complex constrained optimization problems. This may explain the usefulness of considering it again at the present time.

Since the paper is concerned with the Envelope theorem and the specification of appropriate Envelope problems, it is natural to start with **Paul Samuelson**, who introduced both concepts into economic theory. **Samuelson** (1947, ch. 3) examines a regular unconstrained maximization problem, like

$$\phi(a) = \max_x \{f(x,a)\} \quad , \qquad \text{(U)}$$

for which he derives the basic comparative static result

$F_{xx}(x(a),a)X_a(a) = -F_{xa}(x(a),a)$, or

$F_{ax}(x(a),a)X_a(a) = -F_{ax}(x(a),a)F_{xx}(x(a),a)^{-1}F_{xa}(x(a),a) \quad ,$

as well as the derivative property of $\varphi(a)$, namely,

$\phi_a(a) = X_a(a)f'_x(x(a),a) + f_a(x(a),a)$

$\quad = 0 + f_a(x(a),a).$

The latter is no other but the "familiar relation of tangency between the envelope of a family of curves and the curves which it touches". He then considers the profit maximization problem and the impact of auxiliary constraints, or in our notation, problem

$$\max_x \left\{ f(x,z) - w'x - r'z \;\middle|\; C\begin{pmatrix} x - \bar{x} \\ z - \bar{z} \end{pmatrix} = 0_m \right\} \quad , \qquad \text{(P)}$$

where C′ is an n x m matrix of constant parameter values. Such a formulation of the auxiliary constraints is quite general. Indeed, if $C' = [O_{m\ell}, I_{mm}]$, then (P) is our (P^q), while if the constraint is $a'C' = (0_\ell, a)$, then we get the single constraint $a'z \equiv b$ of our (P^p)! However, he does not stop to consider the second-order conditions relating the envelope and the curves that it touches or to specify a suitable Envelope problem for comparative static analysis. As a matter of fact, **Samuelson** (1960b) and (1965) has specified appropriate Envelope problems for sensitivity analysis.[16]

Samuelson (1960b), examines problem (P^f) or $\pi(w) \equiv \max_x\{f(x) - w'x\}$ and the dual problem $\max_w\{-\pi(w) + x'w\} \equiv \min_w\{\pi(w) - x'w\}$. Then in section 6, he introduces a function n (x, w) ≡ f(x) – π (w) – w′ x and shows that $\max_w\{n(w;x)\} \equiv \max_w\{f(x) - x'w - \pi(w)\} \equiv 0$, for prescribed values of x and that $\max_x\{n(x;w)\} \equiv \max_x\{f(x) - w'x - \pi(w)\} \equiv 0$, for prescribed values of w. It is evident that $0 \equiv \max_w\{n(w;x)\}$ is non other but our (E_{P^f}). Is it not rather remarkable that no mention of an Envelope tangency is made?

Finally, in **Samuelsson** (1965) a constrained maximization problem

$$v(y) \equiv \max_x\{f(x) \mid y'x = 1\},$$

that is familiar in consumer theory, is examined. He then defines a "new fundamental function" n (x, y) ≡ f (x) – v(y) with $n(x;y) \le \max_x\{n(x;y) \mid y'x = 1\} = 0$, for prescribed y, and with $n(y;x) \le \max_y\{f(x) - v(y) \mid x'y = 1\} = 0$, for prescribed x. Clearly

[16] After many readings of **Samuelson** (1960b) I was able to see this, only after I knew the answer and I knew what to look for!

$0 = \max_{y}\{n(y;xy)\}$ is the appropriate specification of the envelope problem via which the dual method of comparative static analysis works in consumer theory.[17]

The next development in sensitivity analysis and Le Chatelier Principle was due to **Eugene Silberberg**.

First, **Silberberg** (1971) examined Le Chatelier Principle for the general unconstrained problem (U), as auxiliary and just binding constraints, $h^j(x(a)) = 0$, $j = 1, \ldots, m \le n$, are introduced into (U), namely,

$$\phi^m(a) \equiv \max_{x}\{f(x,a) \mid h(x) = 0\}_m \qquad (U^m)$$

and he proved the **generalized envelope theorem**

$$f(x(a),a) \equiv \phi^{n}(a) \equiv \ldots \equiv \phi(a),$$

$$f_a(x(a),a) = \phi_a^{n}(a) = \ldots = \phi_a(a) \quad \text{and}$$

$$f_{aa}(x(a),a) = \phi_{aa}^{n}(a) < \phi_{aa}^{n-1}(a) < \ldots < \phi_{aa}(a)$$

Then **Silberberg** (1974) introduced the dual method of sensitivity analysis via the Envelope theorem. For a general constrained maximization problem

$$\phi(a) \equiv \max_{x}\{f(x,a) \mid h(x,a) = 0\}_m , \qquad (P_S)$$

he introduced the dual problem

$$0 \equiv \min_{a}\{d(a;x^o) \equiv \phi(a) - f(x^o,a) \mid h(x^o,a) = 0\}_m \qquad (D_S)$$

where $x^o \equiv x(a^o)$.

The f-o-c of (D_S) give the envelope tangencies, while its s-o-c are subject to constraints and have to be transformed into the appropriate

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[17] See **Drandakis** (2007).

envelope curvature conditions so that they can be used for sensitivity analysis.[18,19]

With unusual candor **Silberberg** (1974) explains, in pp. 161-63 and footnotes 4 and 9, that he initially thought from **Samuelson**'s (1965) account about his "new fundamental function" $n(x,a) \equiv \{f(x,a) - \phi(a) \mid h(x,a) = 0'_m\}$, that he could use it in a "primal-dual problem", i.e.,

$$\min_{x,a}\{\phi(a) - f(x,a) \mid h(x,a) = 0'_m\}, \qquad (PD_S)$$

where minimization ranges over all $n + m$ independent variables (x, a). However, (PD_S) is not well behaved, since it is over-determined : some of the f-o-conditions are derived whenever the others are solved.

Difficulties of this kind are easy consequences of loose language and vague specifications. All subsequent researchers use the term "primal-dual method" while they refer to problem (D_S) and not to (PD_S).[20]

Reference must also be made to **Hatta** (1980), who cleverly designed a simpler constrained maximization problem,

$$\phi(a,\gamma) \equiv \max_{x}\{f(x,a) \mid h(x,a) = \gamma'\}, \qquad (P_H)$$

with γ the rector of constraint levels, which may also vary. (P_H) is sufficiently simple, that the dual method of sensitivity analysis can go through, using his "gain function method", or

[18] Those s-o-c are given in the matrix equation system (10) in **Silberberg** (1974), page 163. See also his footnote 9.

[19] It is clear that (D_S) is the negative of our $(E_p)^{cc}$ or $(E)_q$ in section 3 above. See also **Drandakis** (2009), s-o-c (7) in page 5.

[20] See e.g. **Caputo** (1999).

$$0 \equiv \phi \equiv \min_a \{g(a;x^o)\} = \min_a \{(a,h(x^o,a)) - f(x^o,a)\} \qquad (G_H)$$

which is an unconstrained problem leading directly to envelope tangencies and curvature conditions.[21]

We must conclude with a remark about an elementary, yet quite important, weakness that is still prevalent in the economic literature on constrained optimization problems. Such problems lead to s-o-c in which a Hessian matrix is semi definite or definite, but **subject to constraints**. The presence of such constraints precludes the possibility of using those s-o-c directly for comparative static analysis.

Of course, with the primal -or traditional- method of sensitivity analysis, such a difficulty is readily overcome by the use of the properties of the bordered Hessian of the problem and, in fact, it is not even mentioned. With the dual method, however, the presence of such constraints has to be dealt with, at least in general constrained optimization problems like that of **Silberberg** (1974). It is apparent that this difficulty has not attracted any attention in the economic literature, despite the fact that an appropriate mathematical analysis exists at least since **Luenberger** (1973).

In his chapter on Constrained Optimization problems, **David Luenberger** considers first the regularity conditions under which the tangent subspace can be expressed in terms of the gradient matrix of the constraints and, then, he prescribe a procedure under which the Hessian matrix appearing in the s-o-c can be reduced in dimensions so as to produce its representation in the tangent subspace. Having this reduced

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[21] Clearly, (G_H) is the negative of our (E_p) or (E_q) in section 3 above. See also **Drandakis** (2009).

matrix, we can examine the implications of definiteness or semi definiteness.

In our section 3, on the dual method via the Envelope theorem, we used **Luenberger**'s procedure to get the representations of Π^p or Π^q in their tangent subspaces, i.e., $E'\Pi^p E$ or $E'\Pi^q E$, respectively. The profit maximization problem is so simple that getting E, under point or quantity rationing, and completing the whole procedure becomes almost a triviality.[22]

[22] Of course E is not uniquely determined. As a matter of fact, **Luenberger**'s E is constructed differently so as to fit better his own purposes. Our E is simple and quite natural, given our interest –as economists- in sensitivity analysis and our exposure to the usefulness of "compensated parameter changes", in e.g. the theory of consumer behavior.

REFERENCES

Caputo M., (1999), "The relationship between two dual methods of comparative statics", *Journal of Economic Theory*, 243-250.

Caratheodory C., (1935), *Calculus of Variations and Partial Differentiable Equations of the First Order*, in German. English translation, 1967, Holden Day.

Drandakis E., (2003), "Caratheodory's theorem on constrained optimization and comparative statics", *Discussion Paper*, Athens University of Economics and Business.

__________, (2007), "The envelope theorem at work: utility (output) maximization under straight rationing", *Discussion Paper*, Athens University of Economics and Business.

__________, (2009), "The dual method via the envelope theorem for sensitivity analysis and Le Chatelier Principle in parametric optimization problem", *Discussion Paper*, Athens University of Economics and Business.

Graaff J. de, (1947-48), "Rothbarth's 'virtual prices' and the Slutsky equation", *Review of Economic Studies*, 91-95.

Hatta T., (1980), "The structure of the correspondence principle at an extremum point", *Review of Economic Studies*, 987-997.

Luenberger D., (1973), Introduction to Linear and Nonlinear Programming, Addison-Wesley.

Milgrom P. and J. Roberts, (1996), "The Le Chatelier Principle", *American Economic Review*, 173-79.

Samuelson P., (1947), *Foundations of Economic Analysis*, Harvard University Press.

_________, (1960a), "An Extention of the Le Chatelier Principle", *Econometrica*, 368-79.

_________, (1960b), “Structure of a Minimum Equilibrium System”, in Pfouts A. (ed.), *Essays in Economics and Econometrics*, Univ. of North Carolina press, 1-22.

_________, (1965), “Using full duality to show that simultaneously additive direct and indirect utility function implies unitary elasticity of demand”, *Econometrica*, 781-796.

Silberberg E., (1971), “The Le Chatelier Principle as a corollary to a generalized Envelope Theorem”, *Journal of Economic Theory*, 146-155.

___________, (1974), “A revision of comparative static methodology in economics”, *Journal of Economic Theory*, 159-172.

Takayama A., (1985), *Mathematical Economics*, 2nd edition, Cambridge University Press.

Appendix A

The **Primal method of comparative statics in constrained optimization problems** – like Profit maximization under point or quantity rationing – is based on two premises. The first is **Caratheodory's** (1935) **theorem** about the properties of the Inverse of the bordered Hessian matrix while the second is the evaluation of **Barten's** 1966 fundamental matrix equation system about the rates of change of the problems' solutions as their parameters vary.

In **point rationing**, the bordered Hessian is given in (5). Its Inverse exists and will be denoted by $\begin{bmatrix} U^p & v \\ v' & w \end{bmatrix}$, where $\begin{bmatrix} U^p_{xx} & U^p_{xz} \\ U^p_{zx} & U^p_{zz} \end{bmatrix}$ is an n x n symmetric matrix, v is an n vector and w is a scalar. Without computing the Inverse, **Caratheodory's** Theorem exploits its basic property that $\begin{bmatrix} U^p & v \\ v' & w \end{bmatrix}\begin{bmatrix} F^p & c \\ c' & 0 \end{bmatrix} = I_{(n+1)x(n+1)}$ and derives several results that we need, as shown e.g. in **Drandakis** (2003, section 2).

Thus we have:

(i) U^p is a negative semi definite matrix with $r(U^p) = n – 1$, since $U^p c = 0_n$. However the ℓ x ℓ submatrix U^p_{xx} is negative definite since, if $\zeta' U^p_{xx} \zeta = 0$ for any $\zeta \neq 0_\ell$, then $(\zeta', 0')_{mx}\begin{bmatrix} U^p_{xx} & U^p_{xz} \\ U^p_{zx} & U^p_{zz} \end{bmatrix}\begin{pmatrix} \zeta \\ 0_m \end{pmatrix} = \zeta' U^p \zeta = 0$

and, thus, $(\zeta, 0_m) \neq (0_\ell, a) \equiv c'$ would contradict the fact that $r(U^p) = n -1$.

(ii) From $v'F^p + w c' = 0'_n$ and $v'c = 1$ we get $v'F^p v + w c'v = v'F^p v + w = 0$,

or

$$w = - v' F^p v . \quad \text{(A.1)}$$

However, taking into account our assumption that $F(x, z)$ is a negative definite matrix on $\overset{o}{S}$, the same is true for $F^p_\lambda(x^p, z^p)$ and thus

$$w = -\, v' F^p v > 0\,. \qquad \text{(A.1′)}$$

If we then, differentiate the f-o-identities

$$\{f_x(x^p, z^p) - w \equiv 0_\rho,\;\; f_z(x^p, z^p) - r - \lambda^p a \equiv 0_n\,,\;\; a' z^p \equiv b\}$$

with respect to (w, r, a, b), it can be easily seen that we obtain the matrix equation system

$$\begin{bmatrix} F^p_{xx} & F^p_{xz} & 0 \\ F^p_{zx} & F^p_{zz} & a \\ 0'_\rho & a' & 0 \end{bmatrix} \begin{bmatrix} X^p_w & X^p_r & X^p_a & x^p_b \\ Z^p_w & Z^p_r & Z^p_a & z^p_b \\ -\lambda'^p_w & -\lambda'^p_r & -\lambda'^p_a & -\lambda^p_b \end{bmatrix} = \begin{bmatrix} I_n & O & O & 0_\rho \\ O & I_m & \lambda^p I_m & 0_m \\ 0'_\rho & 0'_m & -z^{p\prime} & 1 \end{bmatrix} \qquad \text{(A.2)}$$

and so

$$\begin{bmatrix} X^p_w & X^p_r & X^p_a & x^p_b \\ Z^p_w & Z^p_r & Z^p_a & z^p_b \\ -\lambda'^p_w & -\lambda'^p_r & -\lambda'^p_a & -\lambda^p_b \end{bmatrix} = \begin{bmatrix} U^p_{xx} & U^p_{xz} & \lambda^p U_{xz} - v_x z^{p\prime} & v_x \\ U^p_{zx} & U^p_{zz} & \lambda^p U_{zz} - v_z z^{p\prime} & v_z \\ v'_x & v'_z & \lambda^p v'_z - w z^{p\prime} & w \end{bmatrix} \qquad \text{(A.2′)}$$

Thus our comparative static results are:

(1) $X^p_w = U^p_{xx}$ is a negative definite matrix,

(2) $Z^p_r = U^p_{zz}$ is a negative semi definite matrix, as we can see from differentiating the constraint that $Z^p_r a = 0_m$,

(3) $\lambda^p_b = -w$ is a negative scalar from A.1 ,

as well as

(4) $X^p_a + x^p_b z^{p\prime} = \lambda^p U^p_{xz} = \lambda^p X^p_r$, (5) $Z^p_a + z^p_b z^{p\prime} = \lambda^p U^p_{zz} = \lambda^p Z^p_r$ and

(6) $-(\lambda'^p_a + w z^{p\prime}) = \lambda^p v'_z = \lambda^p \lambda'_b$,

which change signs if $\lambda^p > 0$ (< 0) and become zero matrices and vectors when $\lambda^p = 0$.

Finally we observe, if we compare (A.2′) and (17) in the text, that the Inverse of the bordered Hessian of (P^p) can be express in terms of submatrices of Π^p, namely,

$$\begin{bmatrix} U^p_{xx} & U^p_{xz} & V^p_x \\ U^p_{zx} & U^p_{zz} & V^p_z \\ V'^p_x & V'^p_z & W^p \end{bmatrix} = \begin{bmatrix} \Pi_{ww} & \Pi_{wr} & \Pi_{wb} \\ \Pi_{rw} & \Pi_{rr} & \Pi_{rb} \\ \Pi_{bw} & \Pi_{br} & \Pi_{bb} \end{bmatrix} \qquad \text{(A.3)}$$

In **quantity rationing** the bordered Hessian is given by (10), while its Inverse will be denoted by $\begin{bmatrix} U^q & V \\ V' & W \end{bmatrix}$, an (n + m) x (n + m) matrix. Again from its basic property we get:

(αα) $U^q_{xx}F_{xx} + U^q_{xz}F_{zx} = I$,

(αβ) $U^q_{xx}F_{xz} + U^q_{xz}F_{zz} + V_z = O_{m}$,

(αγ) $U_{xz} = O_{m}$,

(βα) $U^q_{zx}F_{xx} + U^q_{zz}F_{zx} = O_{m}$,

(ββ) $U^q_{zx}F_{xz} + U^q_{zz}F_{zz} + V_z = I_{mm}$,

(βγ) $U^q_{zz} = O_{mm}$,

(γα) $V'^q_x F_{xx} + V'^q_z F_{zx} = O_{m}$,

(γβ) $V'^q_x F_{xz} + V'^q_z F_{zz} = O_{mm}$ and

(γγ) $V'_{z} = I$.

We thus see that

(i) $U^q = \begin{bmatrix} U^q_{xx} & O_m \\ O_m & O_{mm} \end{bmatrix}$ is a negative semidefinite matrix with $r(U^q) =$ n – m = ℓ, while U^q_{xx} is a negative definite matrix.

(ii) From $(V_x, V_z)\begin{bmatrix} F^q_{xx} & F^q_{xz} \\ F^q_{zx} & F^q_{zz} \end{bmatrix} + W(O_{mn}, I_{mm}) = O$ and

$(V_x, V_z)\begin{bmatrix} O_{nm} \\ I_{mm} \end{bmatrix} = I_{mm}$ we get

$V' F^q V + W(O, I_{mm}) V = V' F^q V + W = O_{mm}$,

or $$W = - V' F^q V \ , \qquad (A.4)$$

which means that the m x m matrix W is positive semi definite with positive main diagonal elements, since $F(x^a, \bar z)$ is a negative definite matrix.

Then if we differentiate the f–o–identities

$$\{f_\rho(x, z)^q - w = 0_\rho,\ f_{zm}(x, z)^q - r = 0,\ z = \bar z\}^q$$

w/r $(w, r, \bar z)$ we obtain the equation system

$$\begin{bmatrix} F^q_{xx} & F^q_{xz} & O \\ F^q_{zx} & F^q_{zz} & I_{nm} \\ O_{mn} & I_{mm} & O_{mm} \end{bmatrix} \begin{bmatrix} X^q_w & X^q_r & X^q_{\bar z} \\ Z^q_w & Z^q_r & Z^q_{\bar z} \\ M^q_w & M^q_r & M^q_{\bar z} \end{bmatrix} = \begin{bmatrix} I & O & O_{mm} \\ O & I & O_{mm} \\ O_m & O_{mm} & I_{mm} \end{bmatrix} \qquad (A.5)$$

From (A5) we conclude that the matrix of the rates of change of the solution of (P^q) is non other than the Inverse of the bordered Hessian. Thus the comparative static results are:

(1) $X^q_w = U_{xx}$ is a negative definite matrix

(2) $M^q_{\bar z} = -W$ and is a negative semi definite matrix with negative main diagonal elements, while

(3) $X^q_{\bar z} = O_m$ (4) $X^q_{r} = V_{zx}$

(5) $M^q_w = -V'_x$, (6) $M^q_r = -I_{mm}$.

Finally we observe, if we compare

$$\begin{bmatrix} X^q_w & X^q_r & X^q_{\bar z} \\ Z^q_w & Z^q_r & Z^q_{\bar z} \\ M^q_w & M^q_r & M^q_{\bar z} \end{bmatrix} = \begin{bmatrix} U & O & V \\ O & O & I \\ -V' & I & W \end{bmatrix} \qquad (A.5')$$

and (25) in the text, that the Inverse of the bordered Hessian of (2) can be expressed in terms of $-\Pi^{q}$. We thus see that $\Pi^{qq}_{ww} = -X_w$ is positive definite, while $\Pi^{qq}_{zz} = -M_z - V W V F V'$ is negative semi definite with negative main diagonal elements.

Indeed quantity rationing constraints are so simple, that we can show how the Inverse of the bordered Hessian is expressed in terms of the sub matrices of the bordered Hessian itself:

$$\begin{bmatrix} U^{q}_{xx} & O_{em} & V_x \\ O_{me} & O_{mn} & I_{mm} \\ V'_x & I_{mm} & W \end{bmatrix} = \begin{bmatrix} F^{q\,-1}_{xx} & O_{em} & -F^{-1}_{xx} F_{xz} \\ O_{me} & O_{nm} & I_{mn} \\ -F^{q}_{zx} F^{-1}_{xx} & I_{mn} & F^{q}_{zx} F^{q}_{xx} F_{xz} - F_{zz} \end{bmatrix} . \quad (A.6)$$

Just differentiate $\{ f_x(x,\bar{z})^{q} - w = 0_{\rho} ,\ f_z(x,\bar{z}) - r - \mu = 0 \}$ w/r $(w,r,\bar{z})$ to get

$$\begin{bmatrix} F^{qq}_{xx} & O_{m} \\ F^{qq}_{zx} & I_{mm} \end{bmatrix} \begin{bmatrix} X^{q}_{w} & X^{q}_{r} & X_{\bar{z}} \\ -M^{q}_{w} & M^{q}_{r} & M_{\bar{z}} \end{bmatrix} = \begin{bmatrix} I & O_{m} & -F_{xz} \\ O_{m} & I_{mm} & -F_{zz} \end{bmatrix} .$$

But the Inverse of the first matrix on the left is

$$\begin{bmatrix} F^{q\,-1}_{xx} & O_{m} \\ -F^{qq}_{zx} F^{-1}_{xx} & I_{mm} \end{bmatrix} \text{ and so } \begin{bmatrix} X^{q}_{w} & X^{q}_{r} & X_{\bar{z}} \\ -M^{q}_{w} & M^{q}_{r} & M_{\bar{z}} \end{bmatrix} =$$

$$= \begin{bmatrix} F^{qq\,-1}_{xx} & O_{m} & -F^{-1}_{xx} F_{xz} \\ -F^{qq}_{zx} F^{-1}_{xx} & I_{mm} & F^{qq}_{zx} F^{qq}_{xx} F_{xz} - F_{zz} \end{bmatrix} .$$

Appendix B

Several **interrelations** between (P^f), (P^p) and (P^g) are examined, in section 4, with the purpose of deriving various distinct manifestations of the **Le Chatelier Principle**.

First comparison. (a) (P^f) and (P^p) : with (P^f) solved and $b = a'z(w,r)$ we have seen that (P^p) is obtained with

(i) $x(w,r,a,b) \equiv x(w,r)$, (ii) $z(w,r,a,b) \equiv z(w,r)$ and

(iii) $\lambda(w,r,a,b) = 0$.

Differentiating **(ii)** w/r r and using the derivative properties of b we get: $Z^p_{rr} + Z^f_b a' Z^f_{rr} = Z^p_{rr}$. Since $a'Z^p_{rm} = 0$ from differentiating $a'z(w,r,a,b) = b$, we see that $Z^{pf}_{rr} + Z^f_b a' Z_a$, or $z^f_b = (1/a'Z_a)Z_{rr}$

and so we derive (33ii) in the text, i.e.,

$$Z^p_{rr} = (1/a'Z_a)[Z_a a'Z]Z^f_{rr} .$$

Differentiating **(i)** w/r w and r we get:

$$X^p_{ww} + x^f_b a'Z^f_{rw} = X^f_{ww} \text{ and } X^p_{rw} + x^f_b a'Z = X^f_{rw} .$$

From the second equation we get

$a'Z^p_{wr} + (a'Z_a)x^f_b = 0$ or $(a'Z_a)x^f_b = a'X^f_r$ or, $x^p_b = (1/a'Z_a)a'X^f_r$.

Thus we get (33i), namely,

$$X^p_{ww} = (1/a'Z_a)[X_a a'X]X^f_{ww} .$$

Finally, from **(iii)** we get

$\lambda^p_r + \lambda^f_b a'Z^f_{rr} = 0$ or $\lambda^p_b a'Z_a = -1$, $\lambda = -1/a'Z_a$. Thus we get (33iii), namely,

$$\lambda_b^{pf} = (1/a'Z^p a) < 0.$$

(b) (P^f) and (P^q) : With (P^f) solved and $z = z(w,r)$, we have seen that (P^q) is obtained. i.e., (i) $x(w,z) \equiv x(w,r)$ and (iii) $\mu(w,r,z) = 0_m$.

We have already shown in the text how (35i) and (35iii) are obtained. Obviously, $M_{zr}(w,r,z) = Z(w,r)^{-1}$ is a negative definite matrix while the second matrix in the ℓ - h - side of (35i) is negative semi-definite with negative main diagonal elements. So (36) are confirmed.

Second Comparison. (a) (P^p) and (P^f) : With (P^p) solved and $r \equiv r + \lambda(w,r,a,b)a$ we have seen that (P^f) is obtained. Thus we get (i) $x(w,r) \equiv x(w,r,a,b)$ and (ii) $z(w,r) \equiv z(w,r,a,b)$.

Differentiating (i) w/r w and b and (ii) w/r r and b we get

$X_{ww}^f = X_{ww}^p + x_b^p \lambda_w'$ and $X_b^p = x_b^f a x_b'$, or $X_{ww}^f = (1/\lambda) x_b x_b' X_{bb}$, as

well as $Z_{rr}^f = Z_{rr}^p + z_b \lambda_r'$ and $Z_{aa}^p = z_b a z_b' Z_{bb}$ or $Z_{rr}^f = (1/\lambda) z_b z_b' \lambda_{bb}$.

Having got (38), it remains to show that $\lambda_b(w, r, a, b) < 0$ in order to obtain the curvature conditions in (39). Indeed if we multiply

$Z_{rr}^f = Z_{rr}^p a z_b'$ by $z_b^p a'$ on the left, we get

$z_b^p a' Z_{rr}^f a z_b^p = z_b a' Z_{rr}^p a z_b' = z_b a' Z (a' Z a) z_b' z_b a' Z O$ and thus by symmetry we get

$$Z_{rr}^f (a' Z^p a) z_b z_b' Z_{bb} = .$$

But this shows that, in the envelope tangency at the second best optimum, we must have

$$\lambda_{br}(w,r,a,b) = (1/a' Z^f a) < 0.$$

(b) (P^q) and (P^f): With (P^q) solved and $r = r(w, \bar{z})$ we have seen that (P^f) is obtained. Thus we get

(i) $x(w,r) \equiv \bar{x}(w,\bar{z})$ and (ii) $z(w,r) \equiv \bar{z}$.

Differentiating (ii) w/r w and $\bar{z}$ we get

$Z^f_w + Z^f_r M^q_{rw} = O$ and $Z^f_r M^q_{r\bar{z}} = I$, while differentiating (i) w/r w we get

$X^f_w + X^f_r M^q_{rw} = X^q_w$ or, since $X^f_r = X^f_r$, $X^q_w = X^f_w - X^f_r (Z^f_r)^{-1} Z^f_w$. Thus we have derived (40) in the text, as well as that, in the envelope tangency at the second best optimum, $M_{\bar{z}}(w,r,\bar{z})$ is a negative definite matrix.

Third Comparison. (a) (P^p) and (E_b) for (P^f) : With (P^p) and (E_b) solved and b determined implicitly from (42′) we get

(i) $x(w,r) \equiv \bar{x}(w,r,a,\bar{b})$ and (ii) $z(w,r) \equiv z(w,r,a,\bar{b})$.

Differentiating (i) w/r w and (ii) w/r r we get equation (44) in the text,

$$X^f_w = X^p_w + (1/\lambda) x_b x_b'$$

and

$$Z^f_r = Z^p_r + (1/\lambda) z_b z_b'$$

and the envelope curvature conditions at the first best optimum in (45), since $\lambda_b(w,r,a,b) < 0$.

(b) (P^q) and (E_z) for (P^f) : With (P^q) and (E_z) solved and z determined implicitly by (46′) we get

(i) $x(w,r) \equiv \bar{x}(w,z)$ and (ii) $z(w,r) \equiv z$.

Differentiating (i) w/r w and (ii) w/r r , we get equation (48) in the text, XXXZXXMMXXMX fqqfqqqqqqqq wwzwwzzwwzzz − −$^{-1}$ =+ − −″ −

and

ZM fq rz = −$^{-1}$,

as well as the envelope curvature conditions at the first best optimum in (49) since $M_z(w,r,z)$ is a negative definite matrix.

Fourth Comparison. (a) (P^f) and (E_r^q) for (P^q) : With (P^f) and (E_r^q) solved and r determined implicitly from (50) we get

(i) $x(w,\bar{z}) \equiv x(w,r)$ and (iii) $(w,r,\bar{z})$ r r 0_m.

Differentiating (i) w/r w and (iii) w/r to $\bar{z}$, we get equation (52) in the text,

XXXRXXXZX qffffff wwrwrrr $^{-1}$ ′

and

MRMZO qqf zzzrwmn = − $^{-1}$,

as well as the envelope curvature conditions at the second best optimum in (53), since $M_{zr}(w,r,z)Z =$ f^{-1} is a negative definite matrix.