Caratheodory's Theorem
on
Constrained Optimization
and
Comparative Statics

by

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Abstract

The paper considers Caratheodory's Theorem on the properties of the Inverse of the Bordered Hessian of an Optimization Problem. After a new proof of the complete theorem, using matrix theory methods, the paper considers the sensitivity of the optimal solution in parameters appearing either in the objective or constraint functions. We also prove a second theorem that compares the symmetric submatrices on the main diagonal of the Inverse Matrix, before and after new constraints are introduced to the problem : such comparisons are essential for Le Chatelier Principle. Our "primal method" of comparative statics is as simple and elegant as any "dual method" that uses the Envelope Theorem.

Keywords : Caratheodory's Theorem in Constrained Optimization, Comparative Statics, Le Chatelier Principle.

JEL Classification : C61

1. Introduction

Comparative Statics examine the impact of changes in the parameters of a behavioral system, by comparing equilibrium positions before and after the change. The methodology developed by *Hicks* (1939) and especially *Samuelson* (1947) has derived important results whenever the equilibrium values of the choice variables of the system can be regarded as the solution of an extremum problem.

Today, however, *Samuelson's* (1947) method of proof is considered as unduly complicated and is usually avoided in favor of "dual" methods, like those of *Silberberg* (1974), (1990) and *Hatta* (1980). While *Samuelson's* "primal" method examines directly the system of the rates of change of the choice variables in the parameters – but relies on cumbersome properties of determinants – the latter employ the Envelope Theorem and consider the parameters of the original problem as choice variables of appropriate "dual" problems. Their advantage is simplicity, which is conducive to generality, and elegance of proofs.

This paper offers a primal type of proof for comparative statics in optimization problems, using *Caratheodory's* (1935) *Theorem* on the properties of the inverse of the bordered hessian matrix A new proof of the theorem *in matrix theory terms* is given, a proof that is remarkably simple and transparent and leads easily to further important results.

But why has such a proof not been given until now? The reason must be attributed to the fact that the relevant literature followed two independent and separate paths : matrix theory methods were introduced by "the economatricians", while *Caratheodory's* theorem did not attract any substantial attention and was left unexplored.

Caratheodory (1935) devoted a chapter of his classic book on ordinary maxima and minima, in which he introduced the examination of the properties of the inverse matrix. The theorem was stated in the closing paragraphs of *Samuelson's* (1947) Mathematical Appendix A, but nearly escaped notice by later researchers. Except for a single reference in *Silberberg* (1971), it was as late as 1994 that *Takayama* gave it a prominent place and called it the *Caratheodory-Samuelson Theorem* ; see *Takayama* (1994, p.130) and (1985, pp. 163, 167).

On the other hand, matrix theory methods were introduced into sensitivity analysis of parameter variation in the mid-sixties by *Barten*, *Dhrymes* and *Goldberger*. *Dhrymes* (1967) used convenient matrix operations in his analysis of consumption and production theory, so as to avoid complicated determinant manipulations and obtain "a unity of formal apparatus" between econometric and economic theory. *Barten* was probably the first who used matrix notation and examined comparative statics in consumer theory in terms of "the fundamental matrix equation" ; his analysis forms the basis of *Goldberger*'s essay (1967). Finally *Barten*, *Kloek* and *Lempers* (1969) extended *Dhrymes*' comparative static analysis, by showing that it could be accomplished without assuming that the hessian of the optimization problem was negative definite.

The paper is divided in two parts, the first of which examines all aspects of *Cartheodory*'s theorem while the second deals with comparative statics and the *Le Chatelier Principle*. Section 2 considers the bordered hessian of a constrained optimization problem and derives the properties of its inverse, without attempting its computation. The inverse is composed of four submatrices, about which a lot can be said. Using only matrix theory methods we derive their properties and prove Caratheodory's theorem, both when the number of constraints is less than the number of choice variables and in the limiting case of equal numbers of constraints and variables. Section 3 compares two such optimization problems ; the original one and another in which additional constraints are present. Under conditions guaranteeing that the hessians of both problems are identical, the inverses of the two bordered hessians are compared. An important result can be shown which forms the core of the *Le Chatelier Principle* ; namely, the differences of comparable submatrices on their main diagonal are all (semi)definite matrices.

Building on the theorems of Part I we can easily introduce, in Section 4, parameters appearing in the objective or constraint functions and examine alternative specifications of the behavioral system. Our matrix notation save time and space and leads effortlessly to comparative static results in each model. Finally, section 5 utilizes the results of section 3 and establishes all *Le Chatelier* effects derivable in each alternative system.

2. Properties of the Inverse of the Bordered Hessian Matrix

We consider the constrained optimization problem

$$\max_x \{ f(x) \ | h^1(x)=0,\dots,h^m(x)=0 \} = \max_x \{ f(x) \ | h(x)=0'_m \} \tag{I}$$

where the objective and constraint functions are real valued on R^n and f, h^j ε C^2. The gradient vectors of f, h^j are $f_x(x)$ and $h_x^j(x)$, with $H_x(x)=[h_x^1(x),...,h_x^m(x)]$ the gradient matrix of the constraints. Feasible points $x \in R^n$ exist for $m \leq n$ and are **regular** if

$$r\,(H_x(x)) = m \quad . \tag{R}$$

All vectors are treated as column vectors, while matrices are denoted by capital letters. A prime (') denotes transportation. In addition to the choice variables, x, various parameters may appear in the objective and constraint functions. Until §4 we treat all such parameters as constants and do not consider them explicitly.

(I) is well behaved having interior solutions. If x^0 is regular and a solution of (I), there exist lagrangean multipliers, $\lambda^0 = (\lambda_1^0,...,\lambda_m^0)'$, and we have the necessary conditions :

$$\text{f-o-c} \quad \left\{ \begin{array}{c} f_x(x^0) = \sum_{j=1}^{m} \lambda_j^0 h_x^j(x^0) \equiv H_x(x^0)\lambda^0 \\ h(x^0) = 0'_m \end{array} \right\} \tag{1}$$

and

$$\text{s-o-n-c} \quad \left\{ \begin{array}{l} \textit{The hessian matrix of (I) at } x^0,\ \lambda^0 \textit{ is} \\ \textit{negative semidefinite on the tangent plane} \end{array} \right\} \tag{2}$$

The hessian matrix is given by $A\ (x, \lambda) = F_{xx}\ (x) - H_{xx}\ (x,\lambda)$, where $H_{xx}(x,\lambda) \equiv \sum_{j=1}^{m} \lambda_j H_{xx}^j(x)$ is the matrix of the second-order derivatives of h^j [1]. The tangent plane at a regular x^0 is given by $T = \left\{ \eta \in R^n \,\middle|\, H_x(x^0)'\eta = 0_m \right\}$.

[1] Note the difference between the gradient matrix, $H_x(x)$, and $H_{xx}\ (x,\lambda)$ which is the sum of all $H^j_{xx}\ (x)$ weighted by the corresponding λ_j, $j = 1, ..., m$.

On the other hand if x^0, λ^0 satisfy (1) and if

$$\text{s-o-s-c} \quad \left\{ \begin{matrix} A(x^0, \lambda^0) \text{ is negative definite} \\ \text{on } T \text{ for all } \eta \neq 0_n \end{matrix} \right\} \tag{2'}$$

then x^0 achieves a strict local maximum of (I) .

Let $A = A(x^0, \lambda^0)$ and $B = H_x(x^0)$. Then the bordered Hessian of (I) is given by the $(n+m) \times (n+m)$ symmetric matrix

$$\begin{bmatrix} A, & B \\ B', & O_{mm} \end{bmatrix} \tag{3}$$

and we have [2] :

Lemma 1 : If (R) and (2') are satisfied, the bordered hessian of (I) is invertible. If (2) hold and the bordered hessian is invertible, then (2') are satisfied.

We will assume that (R), (1) and (2') are satisfied at x^0 , λ^0. Then the inverse of the bordered hessian exists, is symmetric and nonsingular. Before turning to examine its properties, we will consider the following general result :

Lemma 2 : Let $\begin{bmatrix} U, & V \\ V', & W \end{bmatrix}$ be a nonsingular $(n+m) \times (n+m)$ matrix with U an $n \times n$ submatrix and $m \leqq n$ If $r(V) = m - i$, $i = 0, 1, \ldots, m$, then $r(U) \geqq n - m + i$ and $r(W) \geqq i$.

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[2] The proofs of all Lemmas in this section are in the Appendix

We conclude immediately that $r\,(A)$ must be at least $n - m$; the hessian of (I) is thus not required to be negative definite, as we would have done for the hessian of an unconstrained problem.

Let us then define the inverse of the bordered hessian of (I) by

$$\begin{bmatrix} U\ , & V \\ V'\ , & W \end{bmatrix} = \begin{bmatrix} A\ , & B \\ B'\ , & O \end{bmatrix}^{-1} \tag{4}$$

Because of its symmetry, U and W are both symmetric submatrices. By definition we have

$$\begin{bmatrix} U\ , & V \\ V'\ , & W \end{bmatrix} \begin{bmatrix} A\ , & B \\ B'\ , & O \end{bmatrix} = I_{(n+m)(n+m)}\ , \tag{5}$$

namely,

$$UA + VB' = I_{nn}\ , \tag{5a}$$

$$UB = O_{nm}\ , \tag{5b}$$

$$V'A + WB' = O_{mn} \quad \text{and} \tag{5c}$$

$$V'B = I_{mm}\ . \tag{5d}$$

Without computing the inverse of the bordered hessian, we have sufficient information for establishing its general properties.

First, from (5d) we see that $r\,(V'B) = m$. Since the rank of the product of two matrices is less than or equal to the minimum of the ranks of the two matrices, we conclude that

$$r\,(V') = r\,(V) = m \tag{6}$$

Second, Lemma 2 and (6) imply that $r\,(U) \geqq n - m$. However (5b) shows that n (U) $\geqq$ m . Thus it is clear that

$$r\,(U) = n - m. \tag{7}$$

Third, we see from (5a) that $UAU + VB'U = U$ which, using (5b), reduces to

$$UAU = U \tag{8}$$

Similarly, from (5c) and (5d) we get

$$V'AV = -W. \tag{9}$$

We can then prove, using (8) and (9), the following :

<u>Lemma 3</u> : For any m with $1 \leqq m < n$

(i) Matrix U is negative semidefinite and

(ii) When the hessian A is negative (semi) definite, W is positive (semi) definite.

We also see from (5a), (5d) and (5c), (5b) that

$$UAV = O_{nm} \quad and \quad V'AU = O_{mn}. \tag{10}$$

Finally, we note an interesting inverse-type relationship between $f_x(x^0)$ and λ^0 : for any m , $1 \leqq m < n$, not only $f_x(x^0) = B\,\lambda^0$, but also $\lambda^0 = V'B\lambda^0 = V'f_x(x^0)$. Indeed we have the following :

<u>Lemma 4</u> : V' is a pseudoinverse of B. In addition to $V'B = I_{mm}$ it satisfies (i) $BV'B = B$ and (ii) $V'BV' = V'$. However, the $n \times n$ matrix $BV' = I_{nn} - AU$ is not even symmetric.

This completes the proof of Caratheodory's Theorem. For any m , $1 \leqq m < n$, we have :

<u>Theorem 1 (Caratheodory)</u>

The inverse of the bordered hessian of (I) exists, is symmetric and

(i) $r(V) = m$, $r(U) = n\text{-}m$,

(ii) U is negative semidefinite,

(iii) when A is negative (semi)definite, W is positive (semi)definite, and

(iv) V' is a pseudoinverse of B.

We should not complete this section before examining the inverse of the bordered hessian when we have the maximum number of independent constraints in (I). With n constraints it is clear that x^0 is determined solely by $h(x) = 0_n'$, since $r(B) = n$ and B is invertible. λ^0 is now determined from $f\text{-}o\text{-}c$, $\lambda^0 = B^{-1} f_x(x^0)$, and (5d) shows that now V' is the inverse of B , $V' = B^{-1}$. Finally (5d) and (9) show that $U = O_{nn}$ and $W = - B^{-1} AB^{-1'}$. Thus we have :

<u>Theorem 2</u>

For $m = n$ and $r(B) = n$

(i) V' is the inverse of B ,

(ii) $U = O_{mn}$ and

(iii) $W = -B^{-1} AB^{-1}{}'$

The inverse matrix is

$$\begin{bmatrix} O_{nn} \ , & B^{-1'} \\ B^{-1} \ , & -B^{-1}AB^{-1'} \end{bmatrix} = \begin{bmatrix} A \ , & B \\ B', & O_{nn} \end{bmatrix}^{-1} , \qquad (11)$$

as can be checked directly. It presents an interesting interchange of the O_{nn} submatrix of the bordered hessian into the O_{nn} submatrix of the inverse and shows clearly that no requirement on A need be imposed.

3. The Inverse of the Bordered Hessian as the number of constraints increases

Suppose that f and h^j , $j=1, ...,m$, remain the same, but new constraints are added so that their number reaches m', with $m + m^+ = m' < n$. We assume that the nxm' submatrix $\widetilde{B}=[B,B^+]$, the augmented gradient matrix of the m' constraints, is of full rank and, also, that (1) and (2') continue to hold properly modified. Let x^1 , λ^1 denote the new solution with m^+ new multipliers. In general $x^1 \neq x^0$ and so the new bordered hessian has no relation with the old one.

If it happens, however, that $x^1 = x^0$ despite the addition of the new constraints, then from the f-o-c of the two problems we see that

$$f_x(x^0)=\sum_{j=1}^{m}\lambda_j^0 h_x^j(x^0)=\sum_{j=1}^{m'}\lambda_j^1 h_x^k(x^0)$$

and thus

$$\sum_{j=1}^{m}(\lambda_j^1-\lambda_j^o)h_x^j(x^o)+\sum_{k=m+1}^{m'}\lambda_k^1 h_x^k(x^0)=0_n$$

If any term of this sum is nonzero, a violation of the rank condition, $r(\widetilde{B})=m'$, results. Thus we must necessarily have

$$\begin{aligned}&\lambda_j^1=\lambda_j^0,\ j\in\{1,...,m\} \quad and\\&\lambda_j^1=0,\ j\in\{m+1,...,m'\}.\end{aligned} \qquad (12)$$

In other words, the new constraints must be "just binding" at x^0. But this means that the hessian of the new problem remains exactly the same as before. As a consequence, the bordered hessian of the new problem has its borders augmented, $\widetilde{B}=\{B, B^+\}$, its upper left corner unchanged and it is still an invertible matrix [3].

An intriguing question immediately arises about the inverses of the bordered hessians of the two problems : is it possible to establish some relationships between their comparable submatrices? The odds of doing so seem rather remote, since all comparable elements of the two matrices are different from one another. However, Theorem 1 holds for the new inverse as well and Theorem 2 is quite instructive : thus, some important findings can be deduced.

Let us denote the new bordered hessian and its inverse by

$$\begin{bmatrix} A\,, & B\,, & B^+ \\ B'\,, & & O_{m'm'} \\ B^{+'} & & \end{bmatrix} \quad and \quad \begin{bmatrix} \widetilde{U} & \widetilde{V} & \widetilde{V}^+ \\ \widetilde{V}' & \widetilde{W} & \widetilde{W}_c \\ \widetilde{V}^{+'} & \widetilde{W}_c' & \widetilde{W}^+ \end{bmatrix}, \qquad (13, 14)$$

with $B, \widetilde{V}$ nxm and $B^+, \widetilde{V}^+$ nxm^+ matrices, while $\widetilde{U}$ is still an nxn matrix.

Corresponding to (5) we have the following relations among the submatrices of the new bordered hessian and its inverse :

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[3] In fact (2'), for the new problem, is weaker than before since the tangent plane $\widetilde{B}'\eta = 0_{m'}$ has its dimensions reduced.

$$\widetilde{U}A+\widetilde{V}B'+\widetilde{V}^{+}B^{+}{}'=I_{nn} \tag{15a}$$

$$\widetilde{U}B=O_{nm}\ ,\quad \widetilde{U}B^{+}=O_{nm^{+}} \tag{15b,b'}$$

$$\widetilde{V}'A+\widetilde{W}B'+\widetilde{W}_{c}B^{+}{}'=O_{mn} \tag{15c}$$

$$\widetilde{V}'B=I_{mm}\ ,\quad \widetilde{V}'B^{+}=O_{mm^{+}} \tag{15d,d'}$$

$$\widetilde{V}^{+}{}'A+\widetilde{W}_{c}'B'+\widetilde{W}^{+}B^{+}{}'=O_{m^{+}n} \tag{15e}$$

$$\widetilde{V}^{+}{}'B=O_{m^{+}m}\ ,\quad \widetilde{V}^{+}{}'B^{+}=I_{m^{+}m^{+}}\qquad , \tag{15f,f''}$$

while we also have $r(\widetilde{V})=m$, $r(\widetilde{V}^{+})=m^{+}$, $r(\widetilde{U})=n-m'$, $\widetilde{U}A\widetilde{U}=\widetilde{U}$, $\widetilde{V}A\widetilde{V}=-\widetilde{W}$, $\widetilde{V}^{+}{}'A\widetilde{V}^{+}=-\widetilde{W}^{+}$, etc.

All these relations can be utilized to show a remarkable property that characterizes all the symmetric submatrices on the main diagonal of the two inverses : indeed we can prove [4] that $U-\widetilde{U}$, $W-\widetilde{W}$ and $-\widetilde{W}^{+}$ are themselves negative (semi)definite matrices !

First, we see that

$U-\widetilde{U}=UAU-\widetilde{U}A\widetilde{U}=(U-\widetilde{U})A(U-\widetilde{U})+\widetilde{U}A(U-\widetilde{U})+(U-\widetilde{U})A\widetilde{U}$

and thus

$B^{+'}(U-\widetilde{U})B^{+}=B^{+'}(U-\widetilde{U})A(U-\widetilde{U})B^{+}+B^{+'}\widetilde{U}A(U-\widetilde{U})B^{+}+B^{+'}(U-\widetilde{U})A\widetilde{U}B^{+}=$
$=B^{+'}(U-\widetilde{U})A(U-\widetilde{U})B^{+}.$

Then for any $k\neq 0_k$ $B^{+}k\equiv\theta\neq 0_n$ and $(U-\widetilde{U})\theta\equiv\eta$ we deduce that $\eta\neq 0_n$. Otherwise $\theta=Bc$ for some $c\in R^{m}$ and thus $\theta=B^{+}k=Bc$ implies $(B,B^{+})\begin{pmatrix} c\\ -k\end{pmatrix}=0_n$ contradicting the rank condition. In addition

[4] In the proof that follows we use (15) repeatedly, without explicit references.

$B'\eta = B'(U-\tilde{U})\theta = 0_m$. Thus $\theta'(U-\tilde{U})\theta = \eta' A\eta$ and (2') holds. We conclude therefore that

$B^{+'}(U-\tilde{U})B^{+}$ is a negative definite $m^+ \; x \; m^+$ matrix. (16)

Of course, there may be $\theta \in R^n, \theta \neq 0_n$ and $\theta \neq B^+k$, for which $\eta = (U-\tilde{U})\theta = 0_n$. We reach therefore our second conclusion, namely, that $U-\tilde{U}$ is a negative semidefinite matrix.

Quite similarly we see that

$$(V-\tilde{V})' A(V-\tilde{V}) = V'AV - V'A\tilde{V} - \tilde{V}'AV + \tilde{V}'A\tilde{V} = -WB'V + WB'\tilde{V} + \tilde{V}'BW - \tilde{W}B'\tilde{V} - \\ - \tilde{W}_c B^{+}{}'\tilde{V} = W - \tilde{W}.$$

Thus for any $k \in R^m, k \neq 0_m, \eta \equiv (V-\tilde{V})k \neq 0_n$ and we reach the conclusion that $\eta' A\eta = k'(W-\tilde{W})k$. But $B'\eta = B'(V-\tilde{V})k = 0_m$ and thus (2') shows again that

$W-\tilde{W}$ is negative definite. (17)

Finally, $-\tilde{W}^+ = \tilde{V}^{+'}A\tilde{V}^+$ and so for all nonzero $k \; \varepsilon \; R^{m+}$ and $\tilde{V}^+k \equiv \eta \neq 0_n$ we have $-k'\tilde{W}^+k = \eta' A\eta$. Again $B'\eta = B'\tilde{V}^+k = 0_m$ and thus

$-\tilde{W}^+$ is negative definite. (18)

We have completed the proof of the following theorem which is at the heart of *Le Chatelier Principle*, as we will see below.

<u>Theorem 3</u>

As new just binding constraints are added in (I), all symmetric submatrices $B^{+'}(U-\tilde{U})B^+$, $W-\tilde{W}$ *and* $-\tilde{W}^+$ are negative definite. $U-\tilde{U}$ is only negative semidefinite.

Before leaving this section, let us consider the effects of introducing new independent and just binding constraints, one by one. Starting with the unconstrained problem

$$\max_{x} \; f(x) \,, \tag{Iu}$$

$A = F_{xx}(x^0)$ is negative definite and thus we have [5]

$${}^{0}U = A^{-1} \text{ negative definite.}$$

With the first constraint, ${}^{1}b = h^{1}_{x}(x^0)$, the submatrices on the main diagonal of the inverse matrix are ${}^{1}U$ and the scalar ${}^{1}w$. Thus we have :

$${}^{0}U - {}^{1}U \text{ negative definite and } -{}^{1}w < 0 \,.$$

With the $i+1$ constraint, $i = 1, \ldots, n-1$, ${}^{i}B = [\mathrm{h}^{1}_{\mathrm{x}}(\mathrm{x}^{0}), \ldots, \mathrm{h}^{\mathrm{i}}_{\mathrm{x}}(\mathrm{x}^{0})]$ and ${}^{i+1}b = \mathrm{h}^{\mathrm{i+1}}_{\mathrm{x}}(\mathrm{x}^{0})$. The submatrices on the main diagonal of the inverse matrix are ${}^{i+1}U$, the $i \times i$ matrix ${}^{i+1}W_{ii}$ and the scalar ${}^{i+1}w$.

Thus we have :

$${}^{i+1}b' \, ({}^{i}U - {}^{i+1}U) \, {}^{i+1}b \,, \quad {}^{i}W - {}^{i+1}W$$

$$\text{negative definite, while}$$

$${}^{i}U - {}^{i+1}U \text{ negative semidefinite} \tag{19}$$

$$\text{and } -{}^{i+1}w < 0 \,.$$

Finally, let us note that with A invertible we could compute the inverse matrix, $U = A^{-1} - A^{-1} B \, (B' A^{-1} B)^{-1} B' A^{-1}$, $V = A^{-1} B \, (B'A^{-1} B)^{-1}$ and $W = -(B' A^{-1} B)^{-1}$, for any number of constraints. Could this help us in proving Theorems 1 and 3 more easily or, perhaps, in deriving more specific results ? The answer is an emphatic no.

[5] Superscripts indicate the number of constraints.

4. Comparative Static Analysis

We consider explicitly parameters that appear in (I) and examine the effects of their variation on the solution of the problem. Several alternative cases are of interest and will be presented one by one : the first three are quite simple and show what the various submatrices of the inverse matrix stand for.

(i) constraint levels are nonzero and may vary

Suppose that we have the problem

$$\max_x \{ f(x) \;|h^1(x)=\gamma_{1,\dots,} h^m(x)=\gamma_m \} \;, \tag{I$_\mathrm{i}$}$$

with everything else as above. The solution, $x(\gamma)$, $\lambda(\gamma)$, is a function of the constraint levels. Letting $X_\gamma(\gamma)=\left[\dfrac{\partial x_i(\gamma)}{\partial \gamma_j}\right]$ and $\Lambda_\gamma(\gamma)=\left[\dfrac{\partial \lambda_i(\gamma)}{\partial \gamma_j}\right]$, we can differentiate the first – order conditions.

$$\left\{ \begin{matrix} f_x(x(\gamma)) \equiv H_x(x(\gamma))\lambda(\gamma) \\ h(x(\gamma)) \equiv \gamma' \end{matrix} \right\} \tag{1$_\mathrm{i}$}$$

and get *the fundamental matrix equation of comparative statics*

$$\begin{bmatrix} A \;, & B \\ B' \;, & O_{mm} \end{bmatrix} \begin{bmatrix} X_\gamma \\ -\Lambda_\gamma \end{bmatrix} = \begin{bmatrix} O_{nm} \\ I_{mm} \end{bmatrix} \tag{20$_\mathrm{i}$}$$

when parameters γ vary. But the bordered hessian is invertible and thus

$$\begin{bmatrix} X_\gamma \\ -\Lambda_\gamma \end{bmatrix} = \begin{bmatrix} U \;, & V \\ V', & W \end{bmatrix} \begin{bmatrix} O_{nm} \\ I_{mm} \end{bmatrix} = \begin{bmatrix} V \\ W \end{bmatrix} \;. \tag{21$_\mathrm{i}$}$$

We see that $V = X_\gamma$ and $W = -\Lambda_\gamma$ provide us with the rates of change of $x(\gamma)$ and $\lambda(\gamma)$ in γ .

We cannot say much about the signs of $X_\gamma(\gamma)$ and $\Lambda_\gamma(\gamma)$. We only see that

$$X'_\gamma B = I_{mm} \quad , \; X'_\gamma f_x(x(\gamma)) = X'_\gamma B\lambda(\gamma) = \lambda(\gamma) \quad , \tag{22}$$

while

$$\Lambda_\gamma = -W = X'_\gamma A X_\gamma \tag{23}$$

is a symmetric matrix which, when Lemma 3 (ii) applies, is negative (semi) definite.

(ii) parameters appear in the objective function and may vary

Here we have the problem

$$\max_x \{ f(x,a) \; | h(x) = 0'_m \} \tag{I$_{ii}$}$$

with k parameters α_i appearing in f. The solution of (I$_{ii}$) is $x(\alpha)$, $\lambda(\alpha)$ and if we differentiate the first – order conditions we obtain

$$\begin{bmatrix} X_\alpha \\ -\Lambda_\alpha \end{bmatrix} = \begin{bmatrix} U\;, & V \\ V'\,, & W \end{bmatrix} \begin{bmatrix} -F_{x\alpha} \\ O_{mk} \end{bmatrix} = \begin{bmatrix} -UF_{x\alpha} \\ -V'F_{x\alpha} \end{bmatrix} \quad , \tag{21$_{ii}$}$$

We see that the $k\,x\,k$ matrix

$$F_{ax} X_a = -F_{ax} UF_{xa} \tag{24}$$

is symmetric and positive semidefinite, since U is symmetric and negative semidefinite. (24) is the first of the main comparative static results about the rates of change of $x(\alpha)$. It is not as satisfactory as we would wish it to be, since it does not specify something about X_α itself : but this is not possible in a model as general as (I$_{ii}$) [6].

We also see that

$$X'_a B = O_{km} \; , \quad X'_a f_x(x(a),a) = 0_k \quad , \tag{25}$$

[6] In simple models, where parameters enter f linearly, both $F_{x\alpha}$ and X_α are $n\;x\;n$ matrices and the former becomes a matrix of constant elements or even an identity matrix.

while nothing more can be said about $\Lambda_a = V' F_{x\alpha}$ except its comparison to $\lambda(\alpha) = V' f_x (x(\alpha), \alpha)$.

(iii) parameters appear in the constraint functions and may vary

When k parameters are present in $h^j (x, \alpha) = 0$, our problem is

$$\max_x \{ f(x) \mid h(x,a) = 0'_m \} \tag{I$_{iii}$}$$

and has the solution $x(\alpha)$, $\lambda(\alpha)$.

Differentiating the first – order conditions and solving for X_α and Λ_α we get

$$\begin{bmatrix} X_a \\ -\Lambda_a \end{bmatrix} = \begin{bmatrix} U\,, & V \\ V'\,, & W \end{bmatrix} \begin{bmatrix} H_{xa} \\ -H'_a \end{bmatrix} = \begin{bmatrix} UH_{xa} & - & VH'_a \\ V'H_{xa} & - & WH'_a \end{bmatrix} \quad , \tag{21$_{iii}$}$$

where $H_\alpha (x, \alpha)$ is the $k \times m$ gradient matrix in α of $h (x (\alpha), \alpha)$ and $H_{xa}(x(a),a,\lambda(a)) \equiv \sum_{j=1}^{m} \lambda_j(a) H^j_{xa}(x(a),a)$ is an $n \times k$ matrix.

Are there any conclusions that can be reached here ? Even if we consider

$$H_{ax} X_a = H_{ax} U H_{xa} - H_{ax} V H'_a \tag{26}$$

and

$$H_a \Lambda_a = - H_a V' H_{xa} + H_a W H'_a \quad , \tag{27}$$

we can say something about only one of the two terms : $H_{\alpha x}\, UH_{x\alpha}$ is symmetric and negative semidefinite, while $H_\alpha W H'_\alpha$ is symmetric and may be positive (semi)definite. On appropriate subspaces, however, things may be different. If $k > m$ and $r\,(H_a) = m$, then there are $k - m$ linearly independent $\eta \in R^k ,\ \ \eta \neq 0_k$, with $H'_a \eta = 0_m$ and thus we observe that $H_{\alpha x} X_\alpha$ is symmetric and that

$$\eta' H_{ax} X_a \eta = \eta' H_{ax} U H_{xa} \eta \leqq 0 \quad , \tag{28}$$

while

$$\eta' H_a \Lambda_a \eta = 0 \tag{29}$$

If however, $0<k\leq m$ and $r\,(H_\alpha) = k$, no such nonzero η can be found and we are left with (26) and (27).

We conclude therefore that with parameters in the constraint functions we have some comparative static results only when the number of parameters exceeds that of constraints.

Finally we also have

$$X'_a B = -H_a \quad , \quad X'_a f_x(x(a)) = -H_a \lambda(a) \ . \tag{30}$$

(iv) constraints are given by *h (x, β) = γ′* and both *β* and *γ* may vary

If we consider problem

$$\max_x \{ f(x) \mid h(x,\beta) = \gamma' \} \quad , \qquad (\mathrm{I}_{iv})$$

with ℓ parameters in the constraint functions in addition to the m constraint levels, the solution is $x\,(\beta, \gamma)$ and $\lambda\,(\beta, \gamma)$ [7].

As β, γ vary we get

$$\begin{bmatrix} X_\beta \ , & X_\gamma \\ -\Lambda_\beta \ , & -\Lambda_\gamma \end{bmatrix} = \begin{bmatrix} U \ , & V \\ V' , & W \end{bmatrix} \begin{bmatrix} H_{x\beta} \ , & O_{nm} \\ -H'_\beta \ , & I_{mm} \end{bmatrix} = \begin{bmatrix} UH_{x\beta} - VH'_\beta \ , & V \\ V' H_{x\beta} - WH'_\beta \ , & W \end{bmatrix}, \qquad (21_{iv})$$

which combine both (21_{iii}) and (21_{i}) and offer the same comparative static results as in **(i)** and **(iii)**.

[7](Iiv) may be thought of as having been derived from (Iiii), if $k > m$ and $r\,(H_\gamma\,(x, \alpha)) = m$ for some m $\alpha_i\,'s$. Letting $\alpha = (\beta, \gamma)$, with $\ell = k - m$, $h\,(x, \beta, \gamma) = 0'_m$ may in principle be solved for γ in terms of (x, β), or, $\gamma' = g\,(x, \beta)$.

Suppose however, that we consider $X_\beta + X_\gamma H'_\beta$ and $\Lambda_\beta + \Lambda_\gamma H'_\beta$. Then it is obvious that

$$X_\beta + X_\gamma H'_\beta = UH_{x\beta} \tag{31}$$

and

$$\Lambda_\beta + \Lambda_\gamma H'_\beta = -V'H_{x\beta} , \tag{32}$$

which lead to

$$H_{\beta x} \{X_\beta + X_\gamma H'_\beta\} = H_{\beta x}UH_{x\beta} \tag{33}$$

being a symmetric and negative semidefinite matrix in complete analogy with $F_{ax} X_\alpha$ in (24)

Naturally the question arises about why the difficulties in deriving comparative static results in **(i)** and **(iii)** are overcome in (33). Or, really, about which type of combined variations in β and γ is captured in (31) and (32). Are $X_\beta + X_\gamma H'_\beta$ and $\Lambda_\beta + \Lambda_\gamma H'_\beta$ the rates of change of the solution to some identifiable optimization problem ?

We economists have a pretty good idea about what is involved here. Our experience from solving optimization problems in the theory of consumption, embodied in the seminal paper of *McKenzie* (1957), shows what must be done.

(v) the compensated version of (I_{iv})

Indeed let us consider the problem

$$\max_x \{ f(x) \mid h(x,\beta)=h(z,\beta) \} \quad , \tag{I_c}$$

in which z is a constant vector in the domain of definition of the constraints. Each constraint here does not have to respect any fixed level, as in **(iv)**, but is required to "pass" through some fixed $z \in R^n$. In other words when parameters β vary, constraint levels respond appropriately so as to "compensate" for these variations in β and make possible the uninterrupted attainment of z.

Since z does not vary the solution of (I_o) is a function of $(\beta; z)$. Let us use the symbols $s(\beta; z)$ and $\mu(\beta; z)$ for the solution of (I_c). Then

$$\text{f-o-c} \quad \left\{ \begin{matrix} f_x(s(\beta;z)) \equiv H_x(s(\beta;z),\beta)\mu(\beta;z) \\ h(s(\beta;z),\beta) \equiv h(z,\beta) \end{matrix} \right\}, \tag{1_c}$$

while (R) and (s-o-c) are as in (I_{iv}) , but A, B and H_{xx} are given by
$A = F_{xx}(s(\beta;z)) - H_{xx}(s(\beta;z),\beta,\mu(\beta;z)),\quad B = H_x(s(\beta;z),\beta)$ and

$$H_{xx} \equiv \sum_{j=1}^{m} \mu_j(\beta;z) H_{xx}^j(s(\beta;z),\beta) \quad .$$

As β vary we get the fundamental equation for (I_c)

$$\begin{bmatrix} A, & B \\ B', & O_{mm} \end{bmatrix} \begin{bmatrix} S_\beta(\beta;z) \\ -M_\beta(\beta;z) \end{bmatrix} = \begin{bmatrix} H_{x\beta} \\ H_\beta(z,\beta)' - H_\beta(s(\beta;z),\beta)' \end{bmatrix} . \tag{20_c}$$

Comparing (I_c) with (I_{iv}) it is obvious that $x(\beta,h(z,\beta)) \equiv s(\beta;z)$ and $\lambda(\beta,h(z,\beta)) \equiv \mu(\beta;z)$. We see that as β vary, with z constant, γ respond and

$$X_\beta(\beta,h(z,\beta)) + X_\gamma(\beta,h(z,\beta)) H_\beta(z,\beta)' = S_\beta(\beta;z) \tag{34}$$

and

$$\Lambda_\beta(\beta,h(z,\beta)) + \Lambda_\gamma(\beta,h(z,\beta)) H_\beta(z,\beta)' = M_\beta(\beta;z) \ . \tag{35}$$

These equations may be called *the generalized Slutsky-Hicks equations* and hold for any z.

We are not interested however in any odd z but in $x(\beta,\gamma)$; besides $s(\beta;z) \neq x(\beta,\gamma)$ unless $z = x(\beta,\gamma)$. Thus we insist that all constraints continue to pass through $x(\beta,\gamma)$, even after β vary and γ respond. Thus we get the answer to our question,

$$S_\beta(\beta;x(\beta,\gamma))=X_\beta(\beta,\gamma)+X_\gamma(\beta,\gamma)H_\beta(x(\beta,\gamma),\beta)' \quad , \tag{34'}$$

$$M_\beta(\beta;x(\beta,\gamma))=\Lambda_\beta(\beta,\gamma)+\Lambda_\gamma(\beta,\gamma)H_\beta(x(\beta,\gamma),\beta)' \tag{35'}$$

and solving (20_c) we get

$$\begin{bmatrix} S_\beta \\ -M_\beta \end{bmatrix}=\begin{bmatrix} U\ , & V \\ V', & W \end{bmatrix}\begin{bmatrix} H_{x\beta} \\ O_{m\ell} \end{bmatrix}=\begin{bmatrix} UH_{x\beta} \\ V'H_{x\beta} \end{bmatrix} \tag{21_c}$$

Thus

$$H_{\beta x}S_\beta(\beta;x(\beta,\gamma))=H_{\beta x}UH_{\beta x} \tag{36}$$

is a symmetric and negative semidefinite $\ell x \ell$ matrix.

5. Le Chatelier Principle

Building on our results in §§ 3 and 4 , we can now compare the rates of change of the solution of problems (I_i) – (I_v) , with respect to any parameter, to those when new just binding constraints are added to the old ones.

Such comparisons, in simple models, result in showing that the difference in the rates of change of the choice variables, $X_\delta - \widetilde{X}_\delta$, where δ is any set of parameters, is a negative (or positive) semidefinite matrix. This *is Samuelson's Le Chatelier Principle* , introduced into Economics in 1947.

The Principle has been extended and generalized by *Pollak* (1969), *Silberberg* (1971), (1974) and *Hatta* (1980), (1987), with considerable simplification in the proof methods. While *Samuelson* (1947) and *Silberberg* (1971) relied on primal methods, using however cumbersome properties of determinants and *Jacobi's* Theorem, *Silberberg* (1974) and *Hatta* (1980) employ dual methods and the envelope theorem.

Here we are able to present a simple primal proof of *Le Chatelier Principle* based on Theorem 3, a proof which is equally elegant in form and general in scope as any dual method of proof.

We proceed by examining (I_i) , (I_{ii}) and (I_v) first. When the new m^+ constraints are added, we will use the inverse of the new bordered hessian, given in (14) above, repeatedly : so let us denote it by C for easiness of notation.

Then in case **(i)** of § 4 we have the new problem

$$\max_x \{ f(x) \; | h(x) = \gamma' \;\; , \;\; h^+(x) = \gamma^{+'} \} \qquad (\widetilde{I}_i)$$

and thus we get

$$\begin{bmatrix} \widetilde{X}_\gamma \\ -\widetilde{\Lambda}_\gamma \\ -\widetilde{\Lambda}^+_\gamma \end{bmatrix} = C \begin{bmatrix} O_{nm} \\ I_{mm} \\ O_{m^+m} \end{bmatrix} = \begin{bmatrix} \widetilde{V} \\ \widetilde{W} \\ \widetilde{W}_c' \end{bmatrix} \qquad (37)$$

if we differentiate the f-o-c with respect to γ and

$$\begin{bmatrix} \widetilde{X}_{\gamma^+} \\ -\widetilde{\Lambda}_{\gamma^+} \\ -\widetilde{\Lambda}^+_{\gamma^+} \end{bmatrix} = C \begin{bmatrix} O_{nm^+} \\ O_{mm^+} \\ I_{m^+m^+} \end{bmatrix} = \begin{bmatrix} \widetilde{V}^+ \\ \widetilde{W}_c \\ \widetilde{W}^+ \end{bmatrix} \qquad (38)$$

if we do the same with respect to γ^+ . Considering (21_i) and (37) we see that

$$X_\gamma - \widetilde{X}_\gamma = V - \widetilde{V} \; , \;\; \Lambda_\gamma - \widetilde{\Lambda}_\gamma = -(W - \widetilde{W}) \; , \quad -\widetilde{\Lambda}_{\gamma^+} = \widetilde{W}_c' \quad , \qquad (39)$$

from which we conclude that $\Lambda_{\gamma} - \widetilde{\Lambda}_{\gamma}$ and $-\widetilde{\Lambda}^{+}_{\gamma^{+}}$ are symmetric and positive definite matrices. Thus we derive the *Le Chatelier effect* for $(I_i) - (\widetilde{I}_i)$:

$$\{\ \Lambda_{\gamma} - \widetilde{\Lambda}_{\gamma} \text{ and } -\widetilde{\Lambda}^{+}_{\gamma^{+}} \text{ are positive definite matrices } \}\ . \qquad (\mathrm{LCE_i})$$

In case **(ii)** of § 4 the new problem is

$$\max_x \{\, f(x,a)\ |\widetilde{h}(x) = 0'_{m'}\ \} \qquad (\widetilde{I}_{ii})$$

and thus differentiation with respect to α gives us

$$\begin{bmatrix} \widetilde{X}_a \\ -\widetilde{\Lambda}_a \\ -\widetilde{\Lambda}^{+}_a \end{bmatrix} = C \begin{bmatrix} -F_{xa} \\ O_{mk} \\ O_{m^{+}k} \end{bmatrix} = \begin{bmatrix} -\widetilde{U}F_{xa} \\ -\widetilde{V}'F_{xa} \\ -\widetilde{V}^{+'}F_{xa} \end{bmatrix} \quad . \qquad (40)$$

We get then the *Le Chatelier effect* for $(\mathrm{I_{ii}})$ - $(\widetilde{I}_{ii})$:

$\{ F_{ax}(X_a - \widetilde{X}_a) = -F_{ax}(U - \widetilde{U})F_{xa}$ is a positive semidefinite matrix$\}$. $(\mathrm{LCE_{ii}})$

Similarly in **(v)** the new problem becomes

$$\max_x \{\, f(x)\ |\widetilde{h}(x,\beta) = \widetilde{h}(z,\beta)\ \} \qquad (\widetilde{I}_c)$$

and thus when $z = x(\beta, \gamma)$ we get

$$\begin{bmatrix} \widetilde{S}_{\beta} \\ -\widetilde{M}_{\beta} \\ -\widetilde{M}^{+}_{\beta} \end{bmatrix} = C \begin{bmatrix} H_{x\beta} \\ O_{m\ell} \\ O_{m^{+}\ell} \end{bmatrix} = \begin{bmatrix} \widetilde{U}H_{x\beta} \\ \widetilde{V}'H_{x\beta} \\ \widetilde{V}^{+'}H_{x\beta} \end{bmatrix} \quad , \qquad (41)$$

where $\widetilde{S}_{\beta} = \widetilde{X}_{\beta} + \widetilde{X}_{\gamma} H'_{\beta} + \widetilde{X}_{\gamma^{+}}\ H_{\beta}{}^{+'}$ and similarly for $\widetilde{M}_{\beta}$ and $\widetilde{M}^{+}_{\beta}$. Thus we derive the *Le Chatelier effect* for (I_c) - $(\widetilde{I}_c)$:

$\{ H_{\beta}(S_{\beta} - \widetilde{S}_{\beta}) = H_{\beta x}(U - \widetilde{U})H_{x\beta}$ is a negative semidefinite matrix$\}$. $(\mathrm{LCE_c})$

In addition to the above results, let us examine case **(iii)** of § 4 and the new problem

$$\max_x \{\, f(x)\ |\widetilde{h}(x,a) = 0_{m'} \qquad (\widetilde{I}_{iii})$$

Here we get

$$\begin{bmatrix} \widetilde{X}_a \\ -\widetilde{\Lambda}_a \\ -\widetilde{\Lambda}_a^{+} \end{bmatrix} = C \begin{bmatrix} H_{xa} \\ -H'_a \\ -H_a^{+'} \end{bmatrix} \tag{42}$$

and

$$H_{ax}\widetilde{X}_a = H_{ax}\widetilde{U}H_{xa} - H_{ax}\widetilde{V}H'_a - H_{ax}\widetilde{V}^{+}H_a^{+'} \tag{43}$$

Comparing (26) with (43) we see that $H_{\alpha x}(X_\alpha - \widetilde{X}_\alpha) =$ $= H_{ax}(U - \widetilde{U})H_{xa} - H_{ax}(V - \widetilde{V})H'_a + H_{ax}\widetilde{V}^{+}H_a^{+'}$ and we get the *Le Chatelier effect* for (I_{iii}) - $(\widetilde{I}_{iii})$:

$$\{H_{ax}(X_a - \widetilde{X}_a) \text{ is a negative semidefinite matrix on } H'_a\eta = 0_m \text{ and on } H_a^{+'}\eta = 0_m\} . \tag{LCE$_{iii}$}$$

For (LCE_{iii}) to be effective we must of course have $k > m'$.

6. Concluding Remarks

The paper exhibits the primal method of doing comparative static analysis and obtaining *Le Chatelier* effects in a manner as simple and elegant as any dual method. No computation of the inverse of the bordered hessian is necessary and no need of using any theorem on reciprocal determinants.

The impact of the optimization hypothesis is examined before we consider sensitivity analysis as such. This structural characteristic is responsible for two noteworthy features. First it makes possible the direct evaluation of the changes on the inverse matrix, brought about by the addition of new and just binding constraints. Second, it shows clearly that there is a common source of comparative static results for any behavioral system based on the same optimization problem. Of course, matrices *A* and *B* in the bordered hessian of the behavioral model will be different, depending on which parameters appear in which functions and in what way ; yet, it is obvious that none of our conclusions in Theorems 1-3 are thereby affected.

Thus let us consider the model of *Silberberg* (1974), (1990), or *Takayama* (1994), namely,

$$\max_x \{ f(x,\alpha) \mid h(x,\alpha) = 0_m' \} \ , \qquad (I_s)$$

which is a combination of (I_{ii}) and (I_{iii}) in § 4. *Caputo* (1999) considers it as the most general model, since the same parameters appear in the objective and constraint functions. Here the rates of change of the solution $x(\alpha)$, $\lambda(\alpha)$ are given by

$$\begin{bmatrix} X_\alpha \\ -\Lambda_\alpha \end{bmatrix} = \begin{bmatrix} U, & V \\ V', & W \end{bmatrix} \begin{bmatrix} -F_{x\alpha} + H_{x\alpha} \\ -H_\alpha' \end{bmatrix} = \ = \begin{bmatrix} -U\{F_{x\alpha} - H_{x\alpha}\} - VH_\alpha' \\ -V'\{F_{x\alpha} - H_{x\alpha}\} - WH_\alpha' \end{bmatrix} \qquad (21_s)$$

and thus $\{F_{\alpha x} - H_{\alpha x}\} X_\alpha\}$ is a symmetric and positive semidefinite matrix on the subspace $H_\alpha' \eta = 0_m$[8]. Similarly, as in § 5, we derive *the Le Chatelier* effect [9,10]

$$\{(F_{\alpha x} - H_{\alpha x})(X_\alpha - \tilde{X}_\alpha) \quad \text{is a positive semidefinite matrix on } H_\alpha' \eta = 0_m \text{ and on } H_\alpha^{+'} \eta = 0_{m^+}\} \ . \qquad (LCE_S)$$

If on the other hand we examine *Hatta*'s (1980) model, or

[8] See Silberberg (1974), pg. 162, and Caputo (1999), pp. 248-49.
[9] Both of these results require, of course, that the number of parameters exceeds the number of constraints.
[10] See Silberberg (1974), pg. 165.

$$\max_x \{ f(x,\alpha) \mid h(x,\alpha)=\gamma' \} \ , \qquad (I_h)$$

as well as its compensated version [11], than we can easily see again that the rates of change of the solution $s(\alpha\ ;\ x\ (\alpha,\ \gamma))$, $\mu\ (\alpha\ ;\ x(\alpha\ ,\ \gamma))$ are given by

$$\begin{bmatrix} S_\alpha \\ -M_\alpha \end{bmatrix} = \begin{bmatrix} U, & V \\ V', & W \end{bmatrix} \begin{bmatrix} -F_{x\alpha} + H_{x\alpha} \\ 0_{m\kappa} \end{bmatrix} = \begin{bmatrix} -U\{F_{x\alpha} - H_{x\alpha}\} \\ -V'\{F_{x\alpha} - H_{x\alpha\alpha}\} \end{bmatrix} \ . \qquad (21_h)$$

Thus $\{F_{\alpha x} - H_{\alpha x}\}\ S_\alpha$ is a symmetric and positive semidefinite matrix and, moreover, so is

$$\{F_{\alpha x} - H_{\alpha x}\}\{S_\alpha - \tilde{S}_\alpha\}\ ^{12}. \qquad (LCE_h)$$

It is clear from the two proceeding paragraphs that, no matter whether one prefers (I_s) or (I_h), all results of *Silberberg – Caputo* and *Hatta* can be obtained at least as easily via the primal method.

For example in the standard model of consumer choice, which is either in (iv) or (v) of § 4 but with $h\ (x\ ,\ \beta) = \beta'\, x$, we have $A = F_{xx}(x^*)$, $B = \beta,\ H_\beta = x^*$, $H_{x\beta} = \lambda^*\, I_{nn}$, with $x^* = x\ (\beta,\ \gamma)$, $\lambda^* = \lambda\ (\beta,\ \gamma)$. Here $X_\beta = \lambda^* U - v x^{*\prime}$, $X_\gamma = v$, $\lambda_\beta = \lambda^*\, v - w x^*$, $\lambda_\gamma = -w$, and thus either X_β is symmetric and negative semidefinite on $x^{*\prime}\eta=0$ or S_β itself is symmetric and negative semidefinite[13].

If prices appear in the utility function[14], then $\max_x \{ f(x,\beta) \mid \beta' x = \gamma \}$ and

[11] (I_h) ia a combination of (I_{ii}) and (I_{iv}), while its compensated version combines (I_{ii}) and (I_c).
[12] See Hatta (1980), Theorems 6 and 8, respectively. Also see Hatta (1987), Proposition 4.
[13] The primal method is used by Barten and Bohm (1982) who build on the original formulation of Barten et al (1969).
[14] See Kalman (1968) where the traditional methodology of comparative statics is used.

$$\begin{bmatrix} X_\beta , & x_\gamma \\ -\lambda'_\beta , & -\lambda_\gamma \end{bmatrix} = \begin{bmatrix} U , & v \\ v' , & w \end{bmatrix} \begin{bmatrix} -F_{x\alpha} + \lambda * I_{nn} & , & 0_n \\ -x*' & , & 1 \end{bmatrix} = \begin{bmatrix} -UF_{x\alpha} + \lambda * U - vx*' & , & v \\ -v' F_{x\alpha} + \lambda * v' - wx*' & , & w \end{bmatrix} .$$

Here the first term, $- U F_{x\beta}$, expressing the price effect on the utility function, is not symmetric and thus either $\{F_{\beta x} - \lambda * I\}$ $X_\beta = -\{F_{x\beta} - \lambda * I\} U \{F_{x\beta} - \lambda * I\}$ is positive semidefinite on $x*'\eta = 0$, or $\{F_{\beta x} - \lambda * I\}\ S_\beta$ is positive semidefinite [15].

Let us finally note that the literature uses the dual method for comparative static analysis of cost or expenditure minimization, but neither of the two in the theory of consumer choice. Instead it relies on the properties of the "dual", or more exactly, the mirror image problem of expenditure minimization, which leads to beautifully simple results when parameters appear only in the constraint, but is completely silent when parameters appear in the objective function as well. Consequently, such a method cannot be extended even to slightly more general models and must be considered as an ad hoc methodology.

[15] The same results are obtained with the dual methods of Silberberg – Caputo and Hatta ; see Hatta (1980) pp. 995-6.

Appendix

Proof of Lemma 1.

Let $(\eta\ , k)' \in R^{n+m}$ and nonzero. If $\begin{bmatrix} A\ , & B \\ B'\ , & O \end{bmatrix}\begin{pmatrix} \eta \\ k \end{pmatrix} = 0_{n+m}$, then $A\eta + Bk = 0_n$ and $B'\eta = 0_m$. Hence $\eta' A\eta + \eta' Bk = \eta' A\eta = 0$ which would contradict (2') if $\eta \neq 0_n$. If $\eta = 0_n$, however, $Bk = 0_n$ and because of (R) $k = 0_m$. But this contradicts our assumption that $(\eta\ , k)'$ is nonzero. Thus the bordered hessian is invertrible.

On the other hand, if the bordered hessian is invertible and $(\eta, k)' \in R^{n+m}$ with $\eta \neq 0_n$ and $B'\eta = 0_m$, then

$$0 \neq (\eta , k)' \begin{bmatrix} A, & B \\ B', & O \end{bmatrix}\begin{pmatrix} \eta \\ k \end{pmatrix} = \eta' A\eta + \kappa' B'\eta + \eta' Bk = \eta' A\eta \quad .$$

Since (2) hold, (2') are satisfied.

Proof of Lemma 2.

Consider a composite matrix $[C, D]$ of rank q. If $r\,(C\,) = p$, then we will show that $r\,(D) \geqq q - p$.

But we can always find elementary matrices to multiply $[C, D]$ and bring it into a row echelon form[16] . Then in the place of C we will have a matrix with the first p rows nonzero, since $r\,(\,C) = p$, while in the place of D we will have a matrix with $q - p$ of the remaining rows nonzero. Thus the rank of D must be at least $q - p$.

Turning our attention to our nonsingular matrix, we examine first $[U\ , V]$ which must be of full rank, n . Since $r\,(V) = m - i$, $r\,(U) \geqq n - (m - i) = n - m + i$. On the other hand $[V'\ , W]$ is also of full rank , m ; since $r\,(V'\,) = r\,(V) = m - i$, we must have $r\,(W) \geqq i$ [17].

[16] See e.g. P. Dhrymes (1984), pp. 18-23.

[17] I owe this simple proof to my colleague Prof. E. Flynzanis.

Proof of Lemma 3.

(i) From (8) $k' UAU k = k' Uk$ for any nonzero $k \in R^n$. Letting $Uk \equiv \eta$ we have $\eta' A\eta = k' U k$. But from (5b) $B' U k = B' \eta = 0_m$. Thus, if $\eta \neq 0_n$, (2') implies $\eta' A\eta = k' Uk < 0$. But even through $k \neq 0_n$, $Uk = \eta$ may be zero. Thus $k' U k \leqq 0$.

(ii) From (9) $k' V' AV k = - k' W k$ for any nonzero $k \in R^m$. Letting $V k \equiv \eta \in R^n$ we get $\eta' A \eta = - k' W k$. Here, however, $B'\eta = B' V k = = k \neq 0_n$ and we cannot use (2') . If it happens that A itself is negative (semi)definite, then W is positive (semi)definite, since $\eta = V k \neq 0_n$.

Proof of Lemma 4.

(i) and (ii) are obvious from (5d). As for $B V' = I_{nn} - AU$, from (5a), it is true that

$$f_x(x^0) = B\lambda^0 = BV' f_x(x^0)$$

for any m , $1 \leqq m < n$. But this cannot hide the feet that $B V'$ is not in general a symmetric matrix [18].

[18] P. Dhrymes (1984), pp. 82-91, examines pseudoinverse matrices and the more stringent concept of a generalized inverse, which is unique. Unfortunately, simple examples show that V' is not a unique pseudoinverse of B.

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