Towards a general theory of real capital

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Abstract

We present a unified theoretical framework capable to address the main real capital decision problems. Initially we specify a continuous-time, terminal-horizon real capital model and use it to trace the properties of optimal *utilisation*, *maintenance*, and *expansionary investment* policies, as well as their interactions. From this analysis we find that these policies are uniquely determined by the property that certain substitution rates be equal to the own price of capital stock. Prominent among the optimal policies are the *extremal policies* and those for which these rates exhibit discontinuities. Thus, under certain conditions related to the efficiency of *utilisation* and *maintenance*, the capital owner is advised to apply *extremal policies* like, *stopping and idling*, *stopping and discarding*, and *downgrading and depleting*. Additionally we examine the possibility of *overhauling investment* or *stripping disinvestment*, if the own price of capital stock becomes higher (lower) than the marginal market price of such investment. Also we extend the model to an infinite-horizon equidistant-sequence of replacement investments and show, among other things, that the *optimal service life* is uniquely determined and negatively related to the uncertainty due to major technological breakthroughs. Finally we solve the model for a general class of operating functions.

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I. Introduction

A household, a firm or a whole economy owns an assortment of durable goods that are somehow aggregated into what we call capital stock. Apart from its *utilisation*, the owner of this capital stock may alter its state through several decisions. Referring to them by the terms employed to describe the respective activities, these decisions comprise : *expansionary investment*,[1] *replacement investment*, *maintenance,*[2] *overhauling investment, stripping disinvestment*, *discarding or abandoning*, and *idling*. So far the study of their determinants has been conducted under three fundamental premises. The first is that only a few of these activities carry significant macroeconomic implications to deserve attention. This has led researchers to focus on the analysis mainly of *expansionary* and *replacement investments* and to a much lesser extent on *maintenance* expenditures.[3] The second premise is that each decision is taken at a different point in time during the useful lives of the durables that make up the capital stock, so each can be studied in isolation from the others. With the exception of a few studies stressing the contemporaneous nature of some of these activities,[4] this has allowed researchers to adopt partial equilibrium approaches and single equation estimating techniques. Finally, the third premise is that *investment* is completely reversible (irreversible) in the sense that once put in place it can (cannot) be resold at no (any) economic loss to the investor.

At the time of their introduction these premises were motivated partly by the prevailing stylised facts and partly by the state of economic theory. For example, when Jorgenson (1963) proposed his renowned theory of *expansionary investment*, he was justified in downplaying the importance of *replacement investment* because, even though the latter accounted for over 50% of gross investment in the United States, it appeared to be a stable proportion of the capital stock. But a few years later Feldstein and Foot (1971), Eisner (1972), and others, were equally justified in criticising his conceptualisations because more recent data from McGraw-Hill surveys and other sources showed that the ratio of replacement investment to capital stock varied significantly, particularly in the short run. Our view in this paper is that both the stylised facts and the state of economic theory have changed significantly so that a major overhaul of the established theory of real capital is warranted.

To highlight the nature of contemplated changes, consider first the premise of reversibility (irreversibility). Jorgenson (1963) assumed that the non-depreciated part of an investment could be resold at the same price that it had been purchased initially (complete reversibility). This was quite natural at the time because economic theory was being developed under the presumption that economic agents operated in complete markets. Then Arrow (1968) came along and dispensed with the assumption of complete reversibility by postulating that once undertaken investment could not be resold (complete irreversibility). In turn, this meant that investors operated in perfectly incomplete markets. But whatever the state of second hand markets might have been in the late 1960's,[5] their breath and depth in recent decades has increased significantly under the advancing wave of globalisation. Hence, given that investors may have now the option to resell their investment at some cost,[6] one necessary generalisation is to account for the implications of this possibility.[7]

Nor is it adequate any more to treat each of the aforementioned decisions as if they are taken in isolation from the others. For this might have been a reasonable approximation when the average useful life of capital stock was 20 years and planning of investment in a capital budgeting framework was little practised. But now the average age of capital stock in all industrial nations has declined dramatically under the influence of rapid technological advances and ignoring the time profiles of one policy while deciding on the time profiles of the others may easily turn out to be suboptimal, and hence costly. Therefore, a general theory of real capital should allow for all possible interactions among these policies.

Moreover, aside from the generalisations just suggested, the present paper investigates the consequences of several other extensions. One of them is the incorporation into the theory of explicit policies for *utilisation* and *maintenance*. Another is the provision for the appearance of minor and major technological changes or breakthroughs. And still another is the provision for *safety* in the *operation* and the *disposal* of durables at the end of their useful lives. Owing to all these revisions and extensions the results that emerge are quite illuminating. With or without discontinuities, the policies of *utilisation*, *maintenance*, and *expansionary investment* turn out to be uniquely determined. More specifically, in the absence of linearities, i.e. in the presence of adjustment costs, the capital owner is advised to apply the policies so as to equate certain substitution rates to the own price of capital stock. But in the presence of linearities he is told to resort to the *extreme policies* indicated by the solution, prominent among which are the policies of *stopping and idling*, *stopping and discarding or abandoning*, and *downgrading and depleting*. Additionally we show that, if at any intermediate period the own price of capital stock becomes higher (lower) than the marginal market price of investment, the owner should undertake *overhauling investment* or *stripping disinvestment,* respectively. Last, but not least, it is shown that the *optimal service life* is uniquely determined and negatively related to the uncertainty due to minor and major technological breakthroughs.

The paper is organised as follows. In Section II we specify a continuous-time, terminal-horizon real capital problem and derive necessary and almost sufficient conditions for the existence of optimal policies. The specification is identical to the model we analysed in Bitros and Flytzanis (2000) with two important differences. These are first that the capital owner is allowed to undertake *expansionary investment* and, second, that the capital stock enters into the operating functions. In Section III we characterise the properties of the optimal policies and trace their interactions. Additionally, in the same section, we extend the model to an infinite-horizon equidistant sequence of replacement investments and investigate the consequences of this generalisation on the optimal service life of the capital stock. Then, in Section IV, we obtain a general solution by applying our analysis to a class of operating functions whose elasticities with respect to the policy variables are independent of the capital stock and a specific solution for an indicative but common specification of these functions. Finally, in Section V, we summarise our conclusions.

II. The model

Consider a capital owner whose operations are designed and implemented on the basis of the following blocks of information:

Capital Stock

$\mathbf{S} = \mathbf{S(t)}$ **:** *Resale value of capital stock,* expressing the size and state of equipment, structures and generally all real assets owned and used by the capital owner with $\mathbf{S} \geq \mathbf{J}$, where $\mathbf{J} \geq \mathbf{0}$ is some lowest operating capital stock level which we will call *safety level.*[8]

Utilisation & Maintenance

$\mathbf{u} = \mathbf{u(t)}$ **:** *Utilisation intensity*, relative to some maximum, with $\mathbf{0} \leq \mathbf{u} \leq \mathbf{1}$.

$\mathbf{m} = \mathbf{m(t)}$ **:** *Maintenance intensity,* expressed as effort or expense relative to some maximum, with $\mathbf{0} \leq \mathbf{m} \leq \mathbf{1}$. We will be referring to either *utilisation* or *maintenance* as *operating policies* and denote their combination by the vector variable:

$\vec{u}:(u,m)$: *Operating policy pair.*

$r(u,m,S)$: *Current revenue flow.* Strictly increasing in u, strictly decreasing in m, concave in (u,m). It can have either sign, usually positive.

$w(u,m,S)$: *Capital stock deterioration or wear and tear flow,* expressing current deterioration of S due to *usage.* It includes *ageing* and *current upkeep* due to *maintenance*. Increasing in u, decreasing in m, convex in (u,m). It can have either sign, usually positive. We will be referring to r and w as *operating flow functions.* Of special interest are also the elasticities of these functions with respect to the operating policies.

Expansionary Investment

$i = i(t)$: *Expansionary investment,* with $i(t) \geq 0$.[9]

$c(i)$: *Cost of expansionary investment flow,* convex and strictly increasing, with $c(0)=0$ and $c(i) \geq i$.

$\dot{S} = -w(u,m,S) + i$: *Current overall rate of capital stock adjustment.*

$q = r(u,m,S) - c(i)$: *Current overall revenue flow* net of *expansionary investment* cost.

Overhauling investment or Stripping disinvestment

$I = \Delta S = S^+ - S$: A finite jump in the capital stock level S, of either sign, usually positive acting like *overhauling investment,* or negative acting like *stripping disinvestment.* Its cost will be represented by

$C(S,I,t)$: *Current cost of overhauling* or *revenue from stripping*, at time t, with current capital stock S. Convex, strictly increasing in I, with $C(S,0,t)=0$ and $C \geq I$. Thus, it is positive when I is positive, expressing the cost of *overhauling investment*, and it is negative when I is negative, expressing the revenue from *stripping disinvestment*.[10] Taking into account transaction costs and costly reversibility, we assume that the marginal market cost $C_I(S,0^+,t)$ and revenue $C_I(S,0^-,t)$ are strictly bigger and strictly smaller than 1, respectively. Concerning its monotonicity with respect to S, it could be of either sign, usually decreasing: $C_S \leq 0$.

Major technological breakthroughs

$\mathbf{F(t)}$: Probability of technological obsolescence by time $\mathbf{t}$, with $\mathbf{F(0)=0}$ and $\mathbf{F(t)<1}$. We will examine mainly the usual exponential case: $\mathbf{F(t)=1-e^{-\theta t}}$.

Discounting

Assuming *discount rate* $\boldsymbol{\rho}$, we will have the *discount factor* $\mathbf{e^{-\rho t}}$. Actually, to express *effective current values* where we take also into account the technological obsolescence effect, we will use the terms:

$\boldsymbol{\varphi}\mathbf{(t)=[1-F(t)]e^{-\rho t}}$: *Effective discount factor.*

$\boldsymbol{\sigma}\mathbf{(t)}=-\boldsymbol{\varphi}\mathbf{'(t)}/\boldsymbol{\varphi}\mathbf{(t)}$: *Effective discount rate.*

In particular, the exponential case $\mathbf{F(t)=1-e^{-\theta t}}$ gives $\boldsymbol{\varphi}\mathbf{(t)=e^{-\sigma t}}$, where $\boldsymbol{\sigma\equiv\theta+\rho}$ is the *constant effective discount rate.*

Objective

The objective adopted by the capital owner is to maximise the discounted expected total profit from using the stock for a time period $\mathbf{T}$ and *selling, discarding* or *idling* what is left from it at the end of this period.[11] Moreover, while pursuing the above, the capital owner is expected to allow for the possibility of *overhauling* or *stripping* at some intermediate time τ.[12]

The problem

$$\underset{\{\mathbf{u,\,m,\,i,\,I,\,T},\tau\}}{\boldsymbol{Max}} \left\{ \mathbf{B=Q+R(T,S_T)-E(S,I,\tau)} \right\}, \tag{1}$$

where:

$\mathbf{Q}=\int_0^T \mathbf{[r(u,m,S)-c(i)]}\boldsymbol{\varphi}\mathbf{(t)dt}$: *Expected total net revenue* from operation.

$\mathbf{R}=\phi\mathbf{(T)S_T}$: *Expected revenue from resale*, where $\mathbf{S_T=S(T)}$.

$\mathbf{E}=\phi(\tau)\mathbf{C(S,I,}\tau\mathbf{)}$: *Expected cost (revenue) of overhauling (stripping).*

$\dot{\mathbf{S}}=-\mathbf{w(u,m,S)+i}$: *Stock adjustment,* with $\mathbf{S(0)=S_0}$.

$\mathbf{0\le u\le 1,\ 0\le m\le 1,\ S\ge J}$: *Operating and stock constraints*.

Remark 1. For the operating policies $\{u, m\}$ we have the natural bounds $0 \le u \le 1$ and $m \ge 0$. In addition, we assumed an upper bound on maintenance, which we normalised to 1. In the absence of such an upper bound, e.g. if m represents actual expense flow, we would have the usual limiting conditions on the derivatives of the operating functions:

$$r_m \to -\infty \quad \text{or} \quad w_m \to 0 \text{ as } m \to +\infty$$

Similarly, concerning the investment policy variables $\{i, I\}$ and their cost functions. We note that in particular cases one or both could be absent.

Optimality conditions

In the setting of optimal control theory we consider the *Hamiltonian* expression:

$$H = \varphi(t)[r(u,m,S) - c(i)] + \lambda(t)[-w(u,m,S) + i] \tag{2}$$

where $\lambda(t)$ is a Co-state variable, representing the *present own price* of capital stock S. Using *effective current* rather than *present values*, we divide by the strictly positive quantity $\varphi(t)$, and set

$$\mu(t) = \frac{\lambda(t)}{\varphi(t)} : \textit{Current effective own price of } S \Rightarrow \frac{\dot{\mu}}{\mu} = \frac{\dot{\lambda}}{\lambda} + \sigma(t). \tag{3}$$

Without overhauling or stripping

As usually we will consider first the case without state discontinuities, i.e. without *overhauling* or *stripping*. According to Seierstad & Sydsaeter (1986, p.335), the assumed convexities give rise to the following almost necessary and sufficient conditions for the optimal solution:

$$\max_{u,m}\{H = \varphi(t)[r(u,m,S) - c(i)] + \lambda[-w(u,m,S) + i] \,\|\, 0 \le u \le 1,\ 0 \le m \le 1\} \tag{4.1}$$

$$\Rightarrow \max_{u,m}\{H = \varphi(t)[r(u,m,S) - \mu w(u,m,S) - c(i) + \mu i] \,\|\, 0 \le u \le 1,\ 0 \le m \le 1\}$$

$$\Rightarrow \max_{u,m}\{r(u,m,S) - \mu w(u,m,S) \,\|\, 0 \le u \le 1,\ 0 \le m \le 1\}, \qquad \text{(i)}$$

$$\max_{i}\{-c(i) + \mu i\}, \qquad \text{(ii)}$$

i.e., given S and μ, the operating and investment policies are decided independently. This is a consequence of the assumed separable forms for the effects of the corresponding policies on the revenue and on the capital stock.

Concerning the time development of capital stock $\mathbf{S}$, we have the state equation:

$$\dot{\mathbf{S}} = -\mathbf{w(u,m,S)} + \mathbf{i}, \text{ with } \mathbf{S(0)} = \mathbf{S_0}. \tag{4.2}$$

For the time development of the own price of capital stock $\boldsymbol{\mu}$ we have the co-state equation:

Assuming the safety level $\mathbf{J}$ is not reached (4.3)

$\dot{\boldsymbol{\lambda}} = -\partial \mathbf{H}/\partial \mathbf{S} = -\boldsymbol{\varphi}\mathbf{r_S} + \boldsymbol{\lambda}\mathbf{w_S}$, with $\boldsymbol{\lambda}\mathbf{(T)} = \partial \mathbf{R}/\partial \mathbf{S_T} = \boldsymbol{\varphi}\mathbf{(T)}$
and $\boldsymbol{\lambda}\mathbf{(t)}$ continuous

$\Rightarrow$ $\dot{\boldsymbol{\mu}} = -\mathbf{r_S(u,m,S)} + \boldsymbol{\mu}\mathbf{w_S(u,m,S)} + \boldsymbol{\mu\sigma}\mathbf{(t)}$, with $\boldsymbol{\mu}\mathbf{(T)} = \mathbf{1}$

and $\boldsymbol{\mu}\mathbf{(t)}$ continuous.

Assuming the safety level $\mathbf{J}$ is reached (4.3′)

$\dot{\boldsymbol{\lambda}} \le -\partial \mathbf{H}/\partial \mathbf{S}$ and $\mathbf{S} \ge \mathbf{J}$ with *Complementary Slackness (CS),*

$\boldsymbol{\lambda}\mathbf{(T)} \ge \partial \mathbf{R}/\partial \mathbf{S_T}$ and $\mathbf{\min S} \ge \mathbf{J}$ with *CS,*

and with *downward discontinuities* for $\boldsymbol{\lambda}$ at the times of hitting and leaving the *safety level.*

$\Rightarrow$ $\dot{\boldsymbol{\mu}} \le -\mathbf{r_S(u,m,S)} + \boldsymbol{\mu}\mathbf{w_S(u,m,S)} + \boldsymbol{\mu\sigma}\mathbf{(t)}$ and $\mathbf{S} \ge \mathbf{J}$ *with CS,*

$\boldsymbol{\mu}\mathbf{(T)} \ge \mathbf{1}$ and $\mathbf{\min S} \ge \mathbf{J}$ with *CS ,*

and *downward discontinuities* for $\boldsymbol{\mu}$ at the times of hitting and leaving the *safety level* $\mathbf{J}$.

Finally, the total operating period $\mathbf{T}$ is determined by the terminal condition:

$$\mathbf{H} + \partial \mathbf{R}/\partial \mathbf{T} \,\|_{\mathbf{T}} = \mathbf{0} \Rightarrow \boldsymbol{\varphi}\mathbf{(T)[r_T - c_T]} + \boldsymbol{\lambda}_\mathbf{T}\mathbf{[-w_T + i_T]} + \boldsymbol{\varphi}\mathbf{'(T)S_T} \Rightarrow \tag{4.4}$$

$\mathbf{r_T - c_T - \boldsymbol{\mu}_T w_T + \boldsymbol{\mu}_T i_T - \boldsymbol{\sigma}(T)S_T = 0}$, provided that $\mathbf{T} > \mathbf{0}$,
($\le \mathbf{0}$ if $\mathbf{T} = \mathbf{0}$), where the $\mathbf{T}$ subscript refers to values at the terminal time.

With overhauling or stripping

Finally, considering the possibility of spiked investment causing a jump discontinuity in $\mathbf{S}$ at time τ, we note that it will be advantageous if the *current effective own price* of capital $\boldsymbol{\mu}(\tau)$ equals the *marginal market price of investment* $\mathbf{C_I(S,I,}\tau\mathbf{)}$. This means that we will have *overhauling*: $\mathbf{I} > \mathbf{0}$, if $\boldsymbol{\mu}(\tau)$ rises above $\mathbf{C_I(S,0^+,\tau)}$, and *stripping*: $\mathbf{I} < \mathbf{0}$, if $\boldsymbol{\mu}(\tau)$ falls below $\mathbf{C_I(S,0^-,\tau)}$. Then, according to Seierstad and Sydsaeter (1987, pp.196 and 207), $\boldsymbol{\mu}(\tau)$ will exhibit a discontinuity determined by the condition:

$$\mathbf{C_S(S,I,\tau)} = \boldsymbol{\Delta\mu} \tag{5.1}$$

The amount of *overhauling /stripping* is determined by the condition that the *marginal market price* of $\mathbf{I}$ before be equal to the *current effective own price of capital stock* $\boldsymbol{\mu}$ after.[13] That is:

$$\mathbf{C_I(S,I,\tau)} = \boldsymbol{\mu}^+ , \tag{5.2}$$

assuming, in the case of *stripping,* that this does not drive the stock below its safety level $\mathbf{J}$. Finally, we note the following remarks:

Remark 2. We have assumed that the cost function of *overhauling/stripping* is known in advance. Actually it is only known at the zero time of planning. In the absence of any other information we can assume for convenience that it is constant in time. Also, on general grounds, we can assume that *overhauling/stripping* is not advantageous at the beginning or it would have been incorporated into the initial capital stock level. The above remarks are expressed by the condition:

$$\mathbf{C_I(S,0^-)} < \boldsymbol{\mu_0} < \mathbf{C_I(S,0^+)}, \text{ where } \boldsymbol{\mu_0} = \boldsymbol{\mu}(\mathbf{0}). \tag{5.3}$$

Remark 3. Since *overhauling/stripping* changes the values of $\{\mathbf{S},\boldsymbol{\mu}\}$, it necessitates also the modification of the optimal policies as determined above.

III. Optimal policies

In comparing policies $\{\mathbf{u,m,i}\}$ for given $\mathbf{S}$, we will call *backward* the direction of decreasing revenue flow $\mathbf{r}$, decreasing deterioration flow $\mathbf{w}$, and increasing investment flow $\mathbf{i}$. We will call *forward* the reverse direction. Thus, in the space of operating policies $\vec{\mathbf{u}} : (\mathbf{u,m})$, we distinguish the extremal policies:

$$\vec{\mathbf{u}}_\mathbf{o} : (\mathbf{u}=\mathbf{0},\mathbf{m}=\mathbf{1}) \text{, } \textit{Backward most extremal.} \tag{6.1}$$

$$\vec{\mathbf{u}}_\mathbf{1} : (\mathbf{u}=\mathbf{1},\mathbf{m}=\mathbf{0}) \text{, } \textit{Forward most extremal.} \tag{6.2}$$

Also for given $\mathbf{S}$, we can characterise policy triplets $(\mathbf{u,m,i})$ according to their effect on $\mathbf{S}$ as follows:

$$\dot{\mathbf{S}} = -\mathbf{w}(\mathbf{u},\mathbf{m},\mathbf{S}) + \mathbf{i} = \mathbf{0} \qquad \textit{Stock equilibrium}, \tag{7.1}$$

$$\dot{\mathbf{S}} = -\mathbf{w}(\mathbf{u},\mathbf{m},\mathbf{S}) + \mathbf{i} > \mathbf{0} \qquad \textit{Upgrading}, \tag{7.2}$$

$$\dot{\mathbf{S}} = -\mathbf{w}(\mathbf{u},\mathbf{m},\mathbf{S}) + \mathbf{i} < \mathbf{0} \qquad \textit{Downgrading}, \tag{7.3}$$

Moreover, by considering their effect on the net revenue, we will characterise policies as follows:

$$\mathbf{q} = \mathbf{r}(\mathbf{u},\mathbf{m},\mathbf{S}) - \mathbf{c}(\mathbf{i}) = \mathbf{0} \qquad \textit{Revenue equilibrium}, \tag{8.1}$$

$$\mathbf{q} = \mathbf{r}(\mathbf{u},\mathbf{m},\mathbf{S}) - \mathbf{c}(\mathbf{i}) > \mathbf{0} \qquad \textit{Profit making}, \tag{8.2}$$

$$\mathbf{q} = \mathbf{r}(\mathbf{u},\mathbf{m},\mathbf{S}) - \mathbf{c}(\mathbf{i}) < \mathbf{0} \qquad \textit{Loss making.} \tag{8.3}$$

All above notions could be restricted to apply only to the *operating policy pairs* $\vec{\mathbf{u}}:(\mathbf{u},\mathbf{m})$, by considering their effect on the operating functions $\mathbf{r}(\mathbf{u},\mathbf{m},\mathbf{S})$ and $\mathbf{w}(\mathbf{u},\mathbf{m},\mathbf{S})$. Thus, an *operating policy pair* $\vec{\mathbf{u}}:(\mathbf{u},\mathbf{m})$ would be *upgrading* (*downgrading*) and *profit (loss) making* if $-\mathbf{w}(\mathbf{u},\mathbf{m},\mathbf{S})$ and $\mathbf{r}(\mathbf{u},\mathbf{m},\mathbf{S})$ are respectively greater (lower) than $\mathbf{0}$.

S-optimal policies

The *Maximality Principle* (4.1) determines the optimal policy triplet $\{\mathbf{u},\mathbf{m},\mathbf{i}\}$ for given $\{\mathbf{S},\boldsymbol{\mu}\}$. Using the convexities of the operating functions and applying the *theory of convex programming,* we find that for each given $\mathbf{S}$ the allowed *optimal operating policies* for various $\boldsymbol{\mu}$ values are also solutions of the constraint optimisation problem:

$$\max_{\mathbf{u},\mathbf{m}} \{\mathbf{r}(\mathbf{u},\mathbf{m},\mathbf{S}) \| \mathbf{w}(\mathbf{u},\mathbf{m},\mathbf{S}) \le \mathbf{w},\ \ \mathbf{0} \le \mathbf{u} \le \mathbf{1},\ \ \mathbf{0} \le \mathbf{m} \le \mathbf{1}\}, \tag{9}$$

By the strict monotonicity properties of function $\mathbf{r}$ we can replace the inequality constraint by the corresponding equality.[14]

Considering the constraint optimisation problem above, we note that for each $\mathbf{S}$ the policy pairs $\vec{\mathbf{u}}:(\mathbf{u},\mathbf{m})$ maximising revenue for given deterioration, define an upper semicontinuous path of *operating policy pairs* parameterised by $\mathbf{w}$. In practice this means piecewise continuous. We will call these policies $\boldsymbol{S}-$***0ptimal policies***. The path will be continuous if the convexity condition is strict for one of the operating flow functions. In this case for each $\mathbf{S}$ the optimal policies of *utilisation* and *maintenance* are tied together continuously in a $\mathbf{1}-\mathbf{1}$ fashion, forming a continuous path of *operating policy pairs* represented by:

$$\vec{u} = \vec{u}(w,S) : u = u(w,S), m = m(w,S) . \tag{10}$$

The maximal value function involved measures the maximal operating revenue r for given deterioration w. It is increasing concave in w. The Lagrange multiplier

$$z(\vec{u},S) = \frac{dr}{dw} , \tag{11}$$

given by the derivative of this maximal function, is the *marginal revenue of capital stock deterioration.* It measures the *substitution rate* between revenue and capital stock deterioration caused by $S - 0ptimal\ \ policies$. Moreover, for reasons that will become obvious shortly, we define:

$$z_u = \frac{r_u(u,m,S)}{w_u(u,m,S)} , \ z_m = \frac{r_m(u,m,S)}{w_m(u,m,S)} , \tag{12}$$

where the subscripts denote indices for z and partial derivatives for $\{r,w\}$. Clearly, for each given S these ratios measure the *substitution rate* between revenue and stock deterioration caused by each operating policy separately. By the convexity properties, z_u is decreasing in u, z_m is increasing in m, and the path of $S - 0ptimal\ \ policies$ separates the operating policies space into two regions:

$R^+(S) : z_u \le z_m$ *High intensity operating policies,*

$R^-(S) : z_u \ge z_m$ *Low intensity operating policies.*

Using the above definitions, we arrive at the following conclusions:

Conclusion 1: For each given S the *marginal revenue of capital stock deterioration* caused by $S - 0ptimal\ \ policies$ is positive. Also it is decreasing in the *forward direction* of increasing r and w . If the *operating policies* act independently:

$$\begin{aligned} r(u,m,S) &= \alpha(u,S) - \beta(m,S) \\ w(u,m,S) &= \gamma(u,S) - \delta(m,S), \end{aligned} \tag{13}$$

then the *forward direction* in the $S - 0ptimal\ \ policies$ coincides with the direction of increasing *utilisation* and decreasing *maintenance*.

Conclusion 2: The $S - 0ptimal\ \ policies$ in the interior of the control space are those for which: $z = z_u = z_m$. By implication, in the *high intensity region* we are *over-*

maintaining for given *utilisation* or, equivalently, we are *over-utilising* for given *maintenance*, and conversely in the *low intensity region*.

Additionally, for each given $\mathbf{S}$, we define the *substitution equilibrium operating policies*:

$$\vec{\mathbf{u}} = \vec{\mathbf{u}}^*(\mathbf{S}) : \mathbf{u} = \mathbf{u}^*(\mathbf{S}), \mathbf{m} = \mathbf{m}^*(\mathbf{S}), \tag{14}$$

as the ***S* – *0ptimal policies*** for which it holds that: $\mathbf{z}^*(\mathbf{S}) = \mathbf{1}$. Otherwise, they are the most extremal ones:

$$\vec{\mathbf{u}}_0 : (\mathbf{u} = \mathbf{0}, \mathbf{m} = \mathbf{1}), \quad \text{if} \quad \mathbf{z}_0(\mathbf{S}) \le \mathbf{1}$$

$$\vec{\mathbf{u}}_1 : (\mathbf{u} = \mathbf{1}, \mathbf{m} = \mathbf{0}), \quad \text{if} \quad \mathbf{z}_1(\mathbf{S}) \ge \mathbf{1},$$

where $\mathbf{z}_0(\mathbf{S})$ and $\mathbf{z}_1(\mathbf{S})$ are the *maximal* and the *minimal substitution rates,* respectively.

Next, we define similar notions for *expansionary investment policy***.** The *marginal cost of expansionary investment* is given by the derivative of the cost function and depends only on $\mathbf{i}$:

$$\boldsymbol{\xi}(\mathbf{i}) = \frac{\mathbf{dc}}{\mathbf{di}} \tag{15}$$

The *equilibrium expansionary investment policy* is the investment policy for which we have $\mathbf{c'(i^*)} = \mathbf{1}$. Otherwise it is the corresponding *most extremal expansionary investment policy*. It follows from our assumptions regarding $\mathbf{c(i)}$ that the *equilibrium expansionary investment policy* is always the *extremal*: $\mathbf{i}^* = \mathbf{0}$.

Collecting the above, we obtain the following:

Proposition 1*.* For given $\mathbf{S}$ and $\boldsymbol{\mu}$:

(i) The *optimal operating policy* pairs $\vec{\mathbf{u}} : (\mathbf{u}, \mathbf{m})$ are the ***S* – *0ptimal policies*** for which the *marginal revenue of capital stock deterioration* $\mathbf{z}$ equals the *current effective own price of capital stock* $\boldsymbol{\mu}$:

$$\mathbf{z}(\vec{\mathbf{u}}, \mathbf{S}) = \boldsymbol{\mu} \;\Rightarrow\; \vec{\mathbf{u}} = \vec{\mathbf{u}}(\mathbf{S}, \boldsymbol{\mu}) \tag{16}$$

if this rate is attained. Otherwise it is the corresponding *most extremal operating policy* pair.

(ii) The *optimal expansionary investment policy* $\mathbf{i}$ is determined by the condition that the *marginal cost of expansionary investment* equals the *current effective own price of capital stock* $\boldsymbol{\mu}$:

$$\boldsymbol{\xi}(\mathbf{i}) = \boldsymbol{\mu} \;\Rightarrow\; \mathbf{i} = \mathbf{i}(\boldsymbol{\mu}), \tag{17}$$

if this rate is attained. Otherwise it is the corresponding *most extremal expansionary investment policy*.

(iii) For given $\mathbf{S}$ the optimal policies move in the forward direction of increasing revenue, increasing deterioration and decreasing *expansionary investment* when $\boldsymbol{\mu}$ decreases, and in the backward direction when $\boldsymbol{\mu}$ increases.

(iv) Allowing *overhauling/stripping*, and if the associated cost (revenue) is a decreasing function of the existing stock: $\mathbf{C_S} < \mathbf{0}$, then at the appropriate time τ we will observe a forward jump to higher operating flows $\{\mathbf{r}, \mathbf{w}\}$ and lower investment flow $\mathbf{i}$. The opposite will be observed if $\mathbf{C_S} > \mathbf{0}$.

(v) Assuming differentiability, the *interior optimal policies* $\{\mathbf{u}, \mathbf{m}, \mathbf{i}\}$ for given $\{\mathbf{S}, \boldsymbol{\mu}\}$ are determined by the equations:

$$\boldsymbol{\mu} = \mathbf{z} = \boldsymbol{\xi} \quad \Rightarrow \quad \boldsymbol{\mu} = \mathbf{z_u} = \mathbf{z_m} == \mathbf{c'(i)} \qquad \textbf{(18)}$$

Extremal S-optimal policies

Concerning the *extremal* ***S – 0ptimal policies*** $\mathbf{u} = \{\mathbf{0}, \mathbf{1}\}$ and $\mathbf{m} = \{\mathbf{0}\ , \mathbf{1}\}$, we mention the following special cases:

1. $\mathbf{z_u} \le \mathbf{z_m}$: *High intensity technology,* i.e. all operating policies are of high intensity. Then the optimal policies are *lower extremal*: $\mathbf{u} = \mathbf{0}$ or $\mathbf{m} = \mathbf{0}$.
2. $\mathbf{z_u} \ge \mathbf{z_m}$: *Low intensity technology,* i.e. all operating policies are of low intensity. Then the optimal policies are *upper extremal*: $\mathbf{u} = \mathbf{1}$ or $\mathbf{m} = \mathbf{1}$.

Finally we note that, because $\boldsymbol{\mu}$ develops continuously, if the substitution rate $\mathbf{z}$ is discontinuous at some *operating policy pair* for an interval of $\mathbf{S}$ levels, e.g. at extremal policies, then this policy will be *persistent,* if optimal, in the sense that it will be applied for a long period, if applied at all. Similarly, if $\boldsymbol{\xi}$ is discontinuous at some *expansionary investment policy*, e.g. at the extremal values, then these will be also *persistent*, if optimal. We mention now the following important cases where extremal policies are *persistent*:

1. $\vec{\mathbf{u}}_\mathbf{0} : (\mathbf{u} = \mathbf{0}, \mathbf{m} = \mathbf{1})$. The most backward operating policy pair is *persistent* if it defines finite substitution rate: $\mathbf{z_0} < +\infty$.
2. $\vec{\mathbf{u}}_\mathbf{1} : (\mathbf{u} = \mathbf{1}, \mathbf{m} = \mathbf{0})$. The most forward operating policy pair is *persistent*, if it defines nonzero substitution rate: $\mathbf{z_1} > \mathbf{0}$.
3. $\mathbf{i} = \mathbf{0}$: The zero investment policy is *persistent*, in general.
4. $\vec{\mathbf{u}}_- : (\mathbf{u} = \mathbf{0}, \mathbf{m} = \mathbf{0})$. The lowest intensity operating policy pair is *persistent* if all operating policies are strictly of high intensity: $\mathbf{z_u} < \mathbf{z_m}$.
5. $\vec{\mathbf{u}}_+ : (\mathbf{u} = \mathbf{1}, \mathbf{m} = \mathbf{1})$. The highest intensity operating policy pair is *persistent* if all operating policies are strictly of low intensity: $\mathbf{z_u} > \mathbf{z_m}$.

Safety level

The *Maximality Principle* expresses the *optimal policies* $\{\mathbf{u},\mathbf{m},\mathbf{i}\}$ as functions of $\mathbf{S}$ and the *current effective own price* of capital stock $\boldsymbol{\mu}$. Without *overhauling/stripping*, $\mathbf{S}$ develops continuously according to the state equation (4.2). Similarly, the *current effective own price of capital stock* $\boldsymbol{\mu}$ develops in time continuously, unless the stock $\mathbf{S}$ hits and is operated for a time period at its *safety level* $\mathbf{J}$. In the latter case $\boldsymbol{\mu}$ will exhibit a downward discontinuity at the times of hitting and leaving the *safety level*. However, this can not happen, because:

1. The time movement of $\mathbf{S}$ implies that when hitting and leaving the *safety level*, $\dot{\mathbf{S}}$ will be increasing going from negative to zero and then to positive values. Since the applied policies will be ***J-optimal***, they will be moving necessarily in the *backward direction* of decreasing $\mathbf{r}$, decreasing $\mathbf{w}$ and increasing $\mathbf{i}$.
2. The downward discontinuity of $\boldsymbol{\mu}$ when $\mathbf{S}$ hits and when it leaves the *safety level* implies that in both instances the optimal policies must move in the *forward direction* at capital stock $\mathbf{J}$.

Hence, $\mathbf{S}$ is not allowed to hit and be operated at the safety level. Instead we adjust the policies in the backward direction of lower operating rates $\{\mathbf{r},\mathbf{w}\}$ and higher investment rate $\mathbf{i}$, just enough so as to stay above the safety level. From these remarks we conclude that:

Proposition 2: Without *overhauling/stripping*, the *current effective own price of capital stock* $\boldsymbol{\mu}$ develops in time continuously according to the dynamic equation

$$\dot{\boldsymbol{\mu}} = -\mathbf{r}_{\mathbf{S}}(\mathbf{u},\mathbf{m},\mathbf{S}) + \boldsymbol{\mu}[\mathbf{w}_{\mathbf{S}}(\mathbf{u},\mathbf{m},\mathbf{S}) + \boldsymbol{\sigma}(\mathbf{t})]\,. \tag{19}$$

(i) If the *safety level* $\mathbf{J}$ is not binding: $\boldsymbol{\gamma} = \mathbf{0}$, we have $\boldsymbol{\mu}(\mathbf{T}) = \mathbf{1}$ and the terminal policies $\{\mathbf{u}_{\mathbf{T}},\mathbf{m}_{\mathbf{T}},\mathbf{i}_{\mathbf{T}}\}$ are the *substitution equilibrium policies* $\{\mathbf{u}^*,\mathbf{m}^*,\mathbf{i}^* = \mathbf{0}\}$ corresponding to the terminal stock $\mathbf{S}_{\mathbf{T}}$.

(ii) As the *safety level* $\mathbf{J}$ becomes binding: $\boldsymbol{\gamma} > \mathbf{0}$, we have $\mu(\mathbf{T}) = \mathbf{1} + \gamma$ and the *terminal policies* are displaced backward to lower operating rates and higher expansionary investment rate.

Replacement

Taking into account the purchase price $\mathbf{P}_{\mathbf{0}}$ of the initial stock $\mathbf{S}_{\mathbf{0}}$, the total net profit to be maximised becomes:

$$\boldsymbol{\Pi}(\mathbf{T}) = \mathbf{B}(\mathbf{T}) - \mathbf{P}_{\mathbf{0}} = \mathbf{Q} + \mathbf{R}(\mathbf{T},\mathbf{S}_{\mathbf{T}}) - \mathbf{E}(\mathbf{S},\mathbf{I},\boldsymbol{\tau}) - \mathbf{P}_{\mathbf{0}} \tag{20}$$

The terminal condition remains the same as before:

$$\Pi'(T) = 0 \quad \Rightarrow \quad B'(T) = 0 \tag{21}$$

The above can be extended to a sequence of replacements. Assuming short replacement times relative to the planning period, the sequence can be extended to infinity. Considering equal replacement periods, constant purchase price and exponential probability of obsolescence, we face essentially a problem of constant future revenues with discount rate $\sigma = \theta + \rho$. The objective to be maximised becomes:

$$A(T) = \sum_v \Pi(T) e^{-\sigma v T} = \Pi(T) \frac{1}{1 - e^{-\sigma T}}, \tag{22}$$

and the terminal condition takes the form:

$$A'(T) = 0 \quad \Rightarrow \quad \Pi'(T) = \Pi(T) \frac{\sigma e^{-\sigma T}}{1 - e^{-\sigma T}}. \tag{23}$$

We note that since $\Pi(T)$ is assumed positive for profitability, the equation above gives $\Pi'(T) > 0$. This implies that service life in the replacement process becomes shorter as compared to the single operating period without replacement, because Π increases until it reaches $\Pi'(T) = 0$. The remaining optimality conditions remain the same as before. Concerning the new terminal condition, written explicitly it becomes:

$$r_T - c_T - w_T(1+\gamma) + i_T(1+\gamma) - \sigma S_T = [Q_T + e^{-\sigma T} S_T - e^{-\sigma T} C_T - P_0] \frac{\sigma}{1 - e^{-\sigma T}}. \tag{24}$$

When solved for T this gives:[15]

$$\kappa = \sigma K - \kappa e^{-\sigma T} \quad \Rightarrow \quad T = \frac{1}{\sigma} \ln \frac{\kappa}{\kappa - \sigma K} \tag{25}$$

where:

$\kappa = r_T - c_T - w_T(1+\gamma) + i_T(1+\gamma)$: *Terminal profit flow,*

$K = Q_T + S_T - e^{-\sigma T} C_T - P_0$: *Total profit.*

Assuming $\gamma = 0$, the terminal flows $\{r_T, w_T, i_T, c_T\}$ would be determined by the *substitution equilibrium policies* for the terminal stock $\{u^*, m^*, i^* = 0\}$. In fact as indicated bellow under some general conditions this is independent even of the capital stock, i.e. it depends only on the effectiveness of the existing technology. Then $\mathbf{T}$ becomes a function of the observable total quantities, increasing in the revenues $\{\mathbf{Q}_T, \mathbf{S}_T\}$, and decreasing in the costs $\{\mathbf{C}_T, \mathbf{P}_0\}$ and in the discount rate $\boldsymbol{\sigma}$. Of course $\mathbf{T}$ appears implicitly in these total quantities.

IV Applications

We will apply the results derived above so as to obtain the *optimal operating policies* for particular forms of *operating functions* that are of general interest. The existing technology of revenue $\mathbf{r}$, of deterioration $\mathbf{w}$ and of expansionary investment cost $\mathbf{c}$ allowed us to determine a path of ***S* − *0ptimal policies*** in the control space $\{\mathbf{u,m,i}\}$. At any time $\mathbf{t}$, if we know the $\mathbf{S}$ and $\boldsymbol{\mu}$, the *optimal operating policies* $\vec{\mathbf{u}}:(\mathbf{u,m})$ and $\mathbf{i}$ are determined by the condition (16)-(17), requiring that both the *marginal revenue of capital stock deterioration* and the *marginal cost of expansionary investment* be equal to the *current effective own price of capital stock*:

$$\mathbf{z}(\vec{\mathbf{u}},\mathbf{S}) = \boldsymbol{\xi}(\mathbf{i}) = \boldsymbol{\mu}\,. \tag{26}$$

In order to find the solution, i.e. the *optimal operating policies* as functions of time, we have to consider the time evolution of $\mathbf{S}$ and $\boldsymbol{\mu}$ as determined by the dynamical equations (4.2) and (19). We will do this for a general class of problems involving *separable operating functions*, and then we will solve for two specific but quite common analytic forms of these functions.

Solution for separable operating functions

We consider *operating functions* of the *separable* type:

$$\mathbf{r} = \mathbf{q}(\mathbf{u,m})\mathbf{f}(\mathbf{S}) \quad \text{and} \quad \mathbf{w} = \mathbf{s}(\mathbf{u,m})\mathbf{g}(\mathbf{S}) \tag{27}$$

i.e., we assume that the elasticities of the *operating functions* $\{\mathbf{r,w}\}$ with respect to the *operating policies* $\{\mathbf{u,m}\}$ are independent of the capital stock $\mathbf{S}$. Now we will call *forward* the direction of increasing operating rates $\{\mathbf{q,s}\}$, and *backward* the opposite direction. The *optimal operating policies* are obtained as solutions of the following equality constraint optimisation problem:

$$\max_{u,m}\{q(u,m)\|s(u,m)=s\} \tag{28}$$

The upper semi-continuous path of *optimal operating policy* pairs is now parameterised by $s:\vec{u}=\vec{u}(s)$. This path is continuous if one of the flow functions satisfies the convexity condition strictly. The Lagrange multiplier defined by the derivative of the maximal value function

$$\zeta(\vec{u})=\frac{dq}{ds} \quad \Rightarrow \quad \zeta(\vec{u})=z(\vec{u},S)g(S)/f(S) \tag{29}$$

is positive and decreasing in the forward direction. It expresses the ***substitution rate*** between revenue and capital stock deterioration rates q and s, with maximal value ζ_0 at the *backward most extremal policy* pair $\vec{u}_0$ and minimal value ζ_1 at the *forward most extremal policy* pair $\vec{u}_1$. For given S and μ the *optimal operating policies* $\vec{u}:(u,m)$ and i are determined now by the conditions:

$$z(\vec{u},S)=\mu \Rightarrow \zeta(\vec{u})=\mu g(S)/f(S),\ \xi(i)=\mu\,. \tag{30}$$

where $\xi(i)=c'(i)$ is always the *marginal cost of expansionary investment*. Also, as before, if any of the terms on the right side of these equations is outside the extremal values of ζ or ξ, respectively, then the *optimal operating policies* are the corresponding most extremal. Since we are considering first the case without *overhauling/stripping*, S and μ develop continuously, i.e., the right sides of the two equations above define continuous functions of time. So, we arrive at the following:

Conclusion 3: For operating functions of the chosen type, an *optimal operating policy* is:
(i) *Persistent* where the corresponding substitution rate has discontinuity.
(ii) *Skipped* in general where the corresponding substitution rate is constant, i.e. where the operating functions are linear.
Part (ii) implies that, if in time we pass through a region where the operating functions are linear, then we will witness sudden jumps in the policy variables. In particular, if the operating functions are linear everywhere then we will witness only extremal policy values, between *idling* and *full* operation, *discarding* and *full maintenance*, *zero* and *maximum* expansionary investment.

Concerning the time development of *optimal operating policies* we note that they are determined by the factor $\pi=\mu g(S)/f(S)$, while the *expansionary investment policy* by the factor μ. Thus, decreasing π means increasing operating rates $\{q,s\}$, while decreasing $\boldsymbol{\mu}$ means increasing operating flows $\{\mathbf{r = qf(S), w = sg(S)}\}$ and also decreasing investment flow $\mathbf{i}$.

Solution for a specific type of rate functions

We will apply now the previous results to operating flow functions of the type:

$$\mathbf{r = q(u,m)S^{\varepsilon}} \quad \text{and} \quad \mathbf{w = s(u,m)S}. \tag{31}$$

We consider the new variable

$$\boldsymbol{\pi} = \boldsymbol{\mu}\frac{\mathbf{g(S)}}{\mathbf{f(S)}} = \mu \mathbf{S}^{1-\varepsilon} \quad \Rightarrow \quad \frac{\dot{\boldsymbol{\pi}}}{\boldsymbol{\pi}} = \frac{\dot{\boldsymbol{\mu}}}{\boldsymbol{\mu}} + (\mathbf{1}-\boldsymbol{\varepsilon})\frac{\dot{\mathbf{S}}}{\mathbf{S}}. \tag{32}$$

For the sequel we will consider only technological obsolescence probability of the exponential type, so that $\sigma \equiv \theta + \rho$ is constant so that the dynamical system defined by the optimality conditions becomes autonomous. The optimality conditions take the form:

$$\zeta(\vec{\mathbf{u}}) = \boldsymbol{\pi} \Rightarrow \frac{\mathbf{q_u(u, m)}}{\mathbf{s_u(u, m)}} = \frac{\mathbf{q_m(u, m)}}{\mathbf{s_m(u, m)}} = \boldsymbol{\pi} = \boldsymbol{\mu}\mathbf{S}^{1-\varepsilon} \qquad \text{(i)} \tag{33}$$

$$\boldsymbol{\xi}(\mathbf{i}) = \boldsymbol{\mu} \quad \Rightarrow \quad \mathbf{c'(i)} = \boldsymbol{\mu} \qquad \text{(ii)}$$

$$\dot{\mathbf{S}} = -\mathbf{sS} + \mathbf{i} \ \text{ with } \ \mathbf{S(0)} = \mathbf{S_0} \tag{34}$$

$$\dot{\boldsymbol{\mu}} = -\boldsymbol{\varepsilon}\mathbf{qS}^{\boldsymbol{\varepsilon}-\mathbf{1}} + \boldsymbol{\mu}(\mathbf{s}+\boldsymbol{\sigma}) \quad \text{with} \quad \boldsymbol{\mu}_{\mathbf{T}} = \mathbf{1}+\gamma \qquad \text{(i)} \tag{35}$$

$$\dot{\boldsymbol{\pi}} = -\boldsymbol{\varepsilon}\mathbf{q} + \boldsymbol{\pi}\mathbf{s} + \boldsymbol{\pi}\boldsymbol{\sigma} + \boldsymbol{\pi}(\mathbf{1}-\boldsymbol{\varepsilon})\frac{\mathbf{i}}{\mathbf{S}} \quad \text{with} \quad \boldsymbol{\pi}_{\mathbf{T}} = \boldsymbol{\mu}_{\mathbf{T}}\mathbf{S}_{\mathbf{T}}^{\mathbf{1}-\boldsymbol{\varepsilon}} \qquad \text{(ii)}$$

$$\mathbf{r_T} - \mathbf{c_T} - \boldsymbol{\mu}_{\mathbf{T}}\mathbf{w_T} + \boldsymbol{\mu}_{\mathbf{T}}\mathbf{i_T} - \boldsymbol{\sigma}\mathbf{S_T} = \mathbf{0} \tag{36}$$

From these conditions we conclude the following:

Conclusion 4:

(i) The *optimal expansionary investment policy* $\mathbf{i}$ is determined by the value of $\boldsymbol{\mu}$. It moves forward to lower rates when $\boldsymbol{\mu}$ decreases and conversely when it increases.

(ii) The *optimal operating policy* pair $\vec{\mathbf{u}}:(\mathbf{u,m})$ is determined by the value of $\boldsymbol{\pi} = \boldsymbol{\mu}\mathbf{S}^{1-\varepsilon}$. It moves forward to higher revenue rates $\mathbf{q}$ and deterioration rates $\mathbf{s}$ when $\boldsymbol{\pi}$ decreases, and conversely when it increases.

From equation 35(ii) we observe that, if the last term is zero, i.e. if $\boldsymbol{\varepsilon} = \mathbf{1}$ or $\mathbf{i} \equiv \mathbf{0}$, $\boldsymbol{\pi}$ does not involve $\mathbf{S}$. This means that the equation becomes autonomous and therefore $\boldsymbol{\pi}$ develops monotonously when continuous. We note also that the sign of the time derivative of $\boldsymbol{\pi}$ gives the monotonicity direction at any time, in particular at the terminal time as determined by conditions (35ii) and (36). Examining these two cases we obtain the following results:[16]

Proposition 3. Assuming constant returns to scale*:* $\boldsymbol{\varepsilon} = \mathbf{1}$*,* for $\mathbf{r} = \mathbf{q(u,m)S}$ and $\mathbf{w} = \mathbf{s(u,m)S}$, we have:

(i) Without *overhauling/stripping* the optimal policies develop in time as follows:

1. If the safety level is not binding: $\boldsymbol{\gamma} = \mathbf{0}$ then the *current effective price of capital stock* $\boldsymbol{\mu}$ is constant and the *optimal policies* are also constant, equal to the *substitution equilibrium policies*: $(\mathbf{u}^*, \mathbf{m}^*, \mathbf{i}^* = \mathbf{0})$.
2. If the safety level is binding: $\boldsymbol{\gamma} > \mathbf{0}$, then the *current effective price of capital stock* $\boldsymbol{\mu}$ is time increasing and the optimal policies move in time backward toward decreasing operating flows and increasing investment flow. In the presence of linearities we will witness sudden jumps to lower operating flows and higher investment flow.

(ii) A policy of *overhauling* will be applied provided the safety level is sufficiently high so that $\boldsymbol{\mu}_{\mathbf{T}} = \mathbf{1} + \boldsymbol{\gamma} > \mathbf{C}_{\mathbf{I}}(\mathbf{S}, \mathbf{0}^+)$.

(iii) A policy of *stripping* will be applied provided the safety level is sufficiently low so that $\mu_{\mathbf{T}} = \mathbf{1} + \gamma < \mathbf{C}_{\mathbf{I}}(\mathbf{S}, \mathbf{0}^-)$.[17]

Proposition 4. Without *expansionary investment*: $\mathbf{i} \equiv \mathbf{0}$, for $\mathbf{r} = \mathbf{q(u,m)S}^{\boldsymbol{\varepsilon}}$ and, $\mathbf{w} = \mathbf{s(u,m)S}$, the *optimal operating policies* move in time in the backward direction of increasing $\boldsymbol{\mu}$ and therefore decreasing operating flows $\{\mathbf{r}, \mathbf{w}\}$ if the safety level is sufficiently high, or if one of the following conditions holds: decreasing returns to scale: $\boldsymbol{\varepsilon} < \mathbf{1}$ and profit making terminal policies: $\mathbf{r}_{\mathbf{T}} > \mathbf{0}$, or increasing returns to scale: $\boldsymbol{\varepsilon} > \mathbf{1}$ and loss making terminal policies : $\mathbf{r}_{\mathbf{T}} < \mathbf{0}$.

In both cases considered we found that in general if the *safety level* is sufficiently restrictive, then in time *optimal operating policies* move in the backward direction of lower revenue and lower capital stock deterioration, with sudden jumps where we have linearities.

V Conclusions

This paper was motivated by three observations. The first of them has to do with the partial equilibrium approach that the bulk of research has adopted to study the decisions relating to real capital. Clearly, since it is founded on the presumption that each decision is taken in isolation from the others, this approach ignores the interactions among the policy options of capital owners and thus leads to sub-optimal policies. The second observation emanates from the realisation that, even though the more interesting case to study is that of *costly reversibility*, most of the research on real capital continues to be conducted as if investment were either *completely reversible* or completely *irreversible*. Finally, the third observation is that such important issues as uncertainty from minor and major technological breakthroughs, operating safety, friendly environmental disposal, etc. are not considered in a unified theoretical framework. Thus, what we set out to accomplish here was to present a model capable to address the problems of real capital policies as well as their interactions.

To this effect, initially we laid out a continuous-time, terminal-horizon real capital model and used it to trace the properties of *utilisation*, *maintenance*, and *investment* policies. From its analysis it turned out that these policies are uniquely determined. More specifically, the capital owner should apply the policies so as to equate the substitution rates $\mathbf{r_u / w_u}$, $\mathbf{r_m / w_m}$, and $\mathbf{c'}$ to the *current effective own price* of capital stock $\boldsymbol{\mu}$, whereas in the presence of linearities, i.e. in the absence of adjustment and disinvestment costs, he should apply extremal policies. Notable among the latter being the policies of *stopping and idling or mothballing* **(u=0,m=1,i=0)**, *stopping and discarding* (**u=0**,**m=0**,**i=0)**, and *downgrading and depleting* or *running down* (**u=1**,**m=0**,**i=0)**. Last, but not least, we examine the utilization and maintenance shifts resulting from the procedures of *overhauling* or *stripping* the capital stock.

Then, in order to cast the policy of *optimal service life* in its proper setting, we went on and extended the model to an infinite-horizon equidistant-sequence of replacement investments. From its analysis, there emerged several important results. For example, *optimal serve life* is uniquely determinate. By comparison to that obtained from the case of single operating period without replacement, the optimal service life is shorter. And the uncertainty due to minor and major technological breakthroughs reduces *optimal service life*, as does the interest rate.

Finally, to demonstrate the range of its applicability, we solved the model for a general class of *separable operating functions* and obtained particular solutions assuming either *constant returns to scale* or absence of *expansionary investment*.

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Endnotes

[1] We use the term *expansionary investment* to describe the activity of increasing capital stock through additions of new capital goods that do not affect the state of pre-existing ones. Investment activities that increase capital stock through modification of pre-existing capital goods are subsumed under the terms of *maintenance*, *overhauling investment* and *stripping disinvestment.*

[2] According to the definitions introduced in Bitros and Flytzanis (2000), maintenance may be distinguished into *regular* and *irregular*, with the former being applied at intervals recommended by the manufacturers' manuals that relate usually to the intensity with which durables are used. In turn, *irregular* maintenance may be further distinguished into maintenance proper or just *maintenance*, *upgrading* and *downgrading*. The difference among them being that *maintenance* leaves the resale value of the durable unchanged, whereas *upgrading* (*downgrading)* increases (decreases) it.

[3] This statement should not be interpreted to imply that the remaining decisions have not received some attention, albeit scanty. For example, see Rothwell and Rust (1993) on the optimal stopping of nuclear plants, Das (1991) on the idling of cement kilns, Arnott, Davidson and Pines (1983) on housing rehabilitation, and Bitros and Kelejian (1974) and Hahn (1995) on capital scrappage.

[4] The studies by Nashlund (1966), Thompson (1968), Kamien and Schwartz (1971) and Ye (1990) emphasised the simultaneous nature of optimal maintenance and service life decisions. Bitros (1976) highlighted the simultaneous nature of incremental investment, replacement investment and maintenance expenditures. Roll and Sachish (1978) studied the interrelationship between overhaul and replacement policies, and Bitros and Flytzanis (2000) developed a unified framework for the analysis of all real capital decisions with the exception of incremental investment.

[5] In his paper Arrow set out to investigate the implications of complete irreversibility for capital policies. So he was not required to justify his assumption by reference to the range of the then existing second hand markets. However, if one cared for the degree to which complete irreversibility was corroborated by reality, one would have concluded that it lacked support because in such large sectors as housing, shipping, aircraft, medical equipment, trucks, used industrial machinery, etc., the existence of second hand markets has been always quite robust. For evidence on this claim see, for example, Sen (1962), Waterson (1964), and Smith (1974).

[6] Abel and Eberly (1996) were the first to investigate the implications for optimal investment of "costly reversibility", i.e. of the possibility for an investor to purchase capital at a given price and sell it at a lower price.

[7] In addition, it should be observed that in the absence of some investment reversibility, the aggregation mentioned above for the derivation of capital stock might be extremely hard to carry out, if at all possible. To find out how demanding, and hence improbable, the conditions for such an aggregation would be, see, for example, Fisher (1982).

8 This *safety level* may be imposed by a regulatory agency for various reasons. One such reason is to oblige the owners to maintain certain technical standards for the *safe operation* of their durables. An example in this respect is the conventional requirement that motor vehicles pass a thorough technical examination every so many years or months before they can be issued valid circulation permits. Another reason is to prohibit owners from abandoning their durables and thus generating environmental externalities. Clearly, since due to this constraint the scrap value of durables is kept necessarily positive, for their owners to realise the corresponding revenues they must dispose them properly.

[9] Allowing $\mathbf{i(t)}$ to take on negative values does not present analytical difficulties. However, in this paper we decided not to do so for two reasons. First, because expansionary investment as we defined it above is either positive or zero, and, secondly, because reversibility is secured anyway through *stripping disinvestment*.

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[10] For example, in the case of complete irreversibility we would have $\mathbf{i} \geq \mathbf{0}$ and $\mathbf{I} \geq \mathbf{0}$.

[11] Later on we will examine also the option of replacing the capital stock and thus begin a new cycle in the optimal design and application of real capital policies. In fact, we will consider an infinite-horizon, equal service life, sequence of replacement investments.

[12] Dunne (1994), Caballero, Engel and Haltiwanger (1995), Cooper and Haltiwanger (1993), Cooper, Haltiwanger and Power (1999), and others have discovered in recent years "spiked" patterns of investment at the plant level. Our specification of the problem facing the capital owner is consistent with this literature since it allows for *overhauling, i.e.* spiked investment, at any $\boldsymbol{\tau}$ during the terminal horizon **T**. Moreover, notice that that we generalise even further by allowing for *stripping, i.e.* spiked disinvestment.

[13] Abel and Eberly (1996) established that under output demand uncertainty and costly reversibility the optimal investment policy of the firm would be to purchase (sell) capital if its *marginal revenue product* exceeded (fell short of) an upper (lower) value of its user cost of capital. By contrast, in our model uncertainty springs from technological change and optimal investment policy recommends that the firm purchase (sell) capital when the *marginal market price of investment* is lower (higher) than the *current effective own price of capital*.

[14] In fact this equivalence holds only for $\boldsymbol{\mu} \geq \mathbf{0}$. However if $\boldsymbol{\mu}$ is negative then the optimal policy triplet $\{\mathbf{u}, \mathbf{m}, \mathbf{i}\}$ would be necessarily the *forward most extremal*, $\vec{\mathbf{u}}_1 : (\mathbf{u} = \mathbf{1}, \mathbf{m} = \mathbf{0})$ and $\mathbf{i} = \mathbf{0}$, which is included above anyway for $\boldsymbol{\mu} = \mathbf{0}$.

[15] For comparison we note that the terminal condition without replacement is written: $\kappa = \sigma \mathbf{S}_{\mathbf{T}}$.

[16] The complete analysis underlying Propositions 3 and 4 is available on request from the authors.

[17] As mentioned earlier, the possibility of *overhauling/stripping* necessitates the recalculation of the *optimal operating policies,* using as final capital stock that one which results from the *marginal cost of overhauling/stripping.*