The Pricing of Multiple Options with Default Risk

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Abstract

This paper is on the pricing of multiple options written by a financial firm having additional liabilities such as debt. The existence of fixed or contingent obligations of the firm, such as debt and options, reduces the value of options written. The correlation among prices of assets underlying call options has a negative effect on option value. It is also shown that options can have a significant impact on debt value and bond spreads.

1. Introduction

An important class of derivatives consists of options with default risk on the part of the seller. The counterparty with a long position in these options is not guaranteed the usual payoff upon exercise, but rather a smaller amount, depending on the salvageable assets of the firm and on other competing claims.

Examples of this class include over-the-counter (OTC) options, bond and mortgage insurance, and guarantees in project financing. However, the range of applications of these "vulnerable options" is much broader. Consider for instance, off-balance sheet (OBS) activities, which involve future, contingent payoffs. Many OBS items such as loan commitments and forward contracts either have option characteristics or can be decomposed in options for valuation purposes. The size of OBS activities has increased dramatically in the last two decades, with their nominal value exceeding assets on bank balance sheets several times (Sinkey, 1998). Thus, vulnerable options represent a substantial part of value in financial institutions, and it is important to see how credit risk affects that value.

Exercising a single option is unlikely to have an effect on the writing firm's ability to cover its liabilities, due to the small size of such a transaction. However, the simultaneous exercise of several options can have an impact, especially in the presence of large liabilities such as debt, particularly deposits. This impact would be most severe when the correlation among the assets underlying the options increases, say, during a market downturn characterized by simultaneous "unwinding" of positions. In addition, an associated and still unanswered question is the effect of vulnerable options on other liabilities of the firm.

The literature on options with credit risk is traced to Johnson and Stulz (1987), who provided distribution-free results for vulnerable options and found that these options have unique features. For instance, it is possible for the value of an option to decline with the volatility of the assets of the writer. Hull and White (1995), Jarrow and Turnbull (1995), Klein (1996) and Klein and Inglis (2001) extended

Johnson and Stulz (1987) to include the valuation of options in the presence of fixed liabilities of equal seniority to the option written.

This paper pursues the line of research by Johnson and Stulz (1987, hereafter JS), Klein (1996), and Klein and Inglis (2001). It extends the above papers by studying the pricing of multiple options in the presence of other liabilities, using a numerical integration approach. A special feature of this paper is that it illustrates the effect of options on the value of debt, thus extending the Merton (1974) model. Numerical examples show that credit risk reduces the value of options written by a financial institution. The volatility of the assets of the firm is negatively related to the value of the options, while, increasing the correlation between the asset underlying the call option and the assets of the firm increases the value of the option. By contrast, the correlation between underlying assets in two call options written by the firm has a negative effect on option value. Finally, the existence of contingent liabilities, such as options, has a negative effect on the market value of debt issued by the firm.

The rest of the paper is organized as follows: Section 2 presents the one-option case. Section 3 shows how generalizing to several options can be implemented and provides numerical examples for the case of two call options. Section 4 presents an application to debt pricing and section 5 concludes.

2. The Basic Model: One Option

Let us begin with the description of the model set up and some assumptions. A firm with asset value U has written a European, OTC call option on an asset S_1 expiring at time T with an exercise price X_1. The two assets follow a joint lognormal diffusion. According to the risk-neutrality approach to pricing derivatives of Cox and Ross (1976) and Harrison and Pliska (1981), we can write the dynamics of these assets as follows:

$$dS_i = S_i r dt + S_i \sigma_i dW_i \tag{1}$$

where r is the constant risk free rate of interest, dW_i is an increment of the standard Wiener process, $i = \{1,2\}$, and S_2 denotes U. The volatility of each asset is σ_1 and σ_U, respectively, and the correlation coefficient between the two assets is $\rho_{1,U}$. Without loss of generality, we assume that the asset underlying the option is a non-dividend paying stock.

In addition to the above, let us make some further assumptions: The firm has issued zero-coupon, non-callable debt in the amount of B, which matures at time T. The value of the firm is independent of its capital structure and financial distress does not affect the joint distribution of asset returns. In addition, all liabilities have equal seniority to claims on the final assets of the firm at time T. Thus, in the event of default by the firm, option holders receive either the claim due to them or a proportion of the firm's assets. The current value of a call option at time T is given by:

$$
\begin{aligned}
c &= e^{-rT} E_{1,U} \left\{ \begin{array}{lll} S_{1T} - X_1 & S_{1T} > X_1, & U_T \geq (S_{1T} - X)/a_1 \\ a_1 U_T & S_{1T} > X_1, & U_T < (S_{1T} - X)/a_1 \\ 0 & & \text{otherwise} \end{array} \right\} \\
&= e^{-rT} E_{1,U} \{\min(a_1 U_T, \max(S_{1T} - X_1, 0))\}
\end{aligned}
\tag{2}
$$

where $E_{1,U}$ is the expectations operator under the risk-neutral probability measure over the two assets, and the claim proportion is given by:

$$
a_1 = \frac{\max(S_{1T} - X_1, 0)}{(\max(S_{1T} - X_1, 0) + B)} \tag{2a}
$$

The top line of Equation (2) in the large brackets gives the call payoff if the option is exercised and if credit risk is absent. This is a standard European call payoff. The second line gives the payoff if the option is exercised but the option holder receives only a share of the assets of the firm, proportional to the total liabilities of the firm outstanding on the expiration date. The max function can be omitted in both cases, because the option is exercised. A write-down factor in the value of the options as in Klein and Inglis (2001) is ignored for simplicity. In addition, option buyers usually require some form of collateral to be posted by the option writer. Such collateral can be easily incorporated in the model. The second equality in Equation (2) gives a comprehensive formula for the value of the option.

Because of the nonlinear nature of Equation (2), Klein and Inglis (2001) solve for the option price numerically by using three-dimensional binomial trees. However, the problem structure accepts a straightforward calculation by integration. If lower case symbols denote asset values at time T, the value of the call is:

$$c = \frac{e^{-rT}}{\sigma_1 \sigma_U T}\left[\int_0^\infty \int_0^\infty \frac{1}{s_1 u} \min\left(a_1 u,\ \max\left(s_1 - X_1, 0\right)\right) f_2 \, ds_1 \, du\right] \tag{3}$$

where

$$f_2 = f_2(d) = (2\pi)^{-1} |\Sigma|^{-1/2} \exp\left[-\frac{1}{2} d' \Sigma^{-1} d\right] \tag{3a}$$

is the density function of the standard bivariate normal distribution with mean (0,0) and correlation matrix

$$\Sigma = \begin{bmatrix} 1 & \rho_{1,U} \\ \rho_{1,U} & 1 \end{bmatrix} \tag{3b}$$

while the elements of vector $d = (d_1, d_2)$ are given by

$$d_i = \frac{\ln\left(s_i / S_i\right) - (r - \sigma_i^2 T / 2)}{\sigma_i \sqrt{T}}. \tag{3c}$$

and a_1 is given by Equation (3) with $S_{1,T}$ replaced by s_1.

JS assume that the firm has no other liabilities except the option. Thus, the JS model is a special case of the above if we set $B = 0$ in Equation (2). Notice that default occurs if the terminal value of the assets of the firm is insufficient to meet the combined payoff to the option and debt holders.

Some numerical examples illustrate the difference between any of the three models (Black-Scholes, JS, and vulnerable call with debt) as well as the effect of parameter changes on the value of these options. From the last three columns of Table 1, we observe that the value of the vulnerable calls, c_{JS} or c, is lower than the value of the standard option c_{BS}. This confirms the results of JS and makes financial sense. However, c is lower than both the standard and the JS options. This is natural because the option in our model takes into account the debt obligation, which is taken to be 80% of firm value in the standard case. Changing the relative size of debt compared to options does not alter the qualitative results presented.

Further observation of Table 1, reveals that increases in volatility of the underlying asset, σ_1, lead to increases in the value of all options. The volatility of the assets of the firm writing the option has a negative effect not only on c_{JS}, as JS had observed, but also on c. This result is interesting and can be explained intuitively by the fact that the value of the firm is like collateral to option holders. If that collateral cannot be trusted to be available at the expiration of the option, the option becomes less valuable.

Correlation between the underlying asset and the assets of the firm needs special mention. As the correlation coefficient declines from 0.9 to –0.9, this reduces the value of both vulnerable calls dramatically. A negative correlation means that, when the underlying asset price increases, the value of the assets of the firm tends to decline. Although the reverse could have some positive effect due to an increase in collateral value, that benefit is not as strong to overcome the loss of value due to the drop in the underlying asset price. A decline in correlation appears to have a more pronounced effect on c than c_{JS}. The interest rate and the time to expiration have a positive effect on all options in this example.

The underlying stock price and the value of the firm are positively related to option values, a fact that makes sense. Increasing the "quasi"-debt ratio $\kappa = B/U$, leaves less assets for the option holders and reduces the value of the call. Notice that the JS option is not affected because that model does not account for debt. Finally, from Table 1 we find that options under credit risk are less sensitive to common parameters than standard options are. Noted first by JS, this applies for the extended model also and has important implications for the hedging of vulnerable options.

3. Multiple Options

Let us now extend the model by assuming that the firm has written two call options on assets S_1 and S_2 expiring at time T with exercise prices X_1 and X_2, respectively. All assets follow the stochastic diffusion of Equation (1), and all claimants have equal priority in the event of default by the firm. The value of the first call is given by the following:

$$c = e^{-rT} E_{1,2,U} \left\{ \min\left(a_2 U_T, \max\left(S_{1T} - X_1, 0\right)\right)\right\} \tag{4}$$

where

$$a_2 = \frac{\max\left(S_{1T} - X_1, 0\right)}{\max\left(S_{1T} - X_1, 0\right) + \max\left(S_{2T} - X_2, 0\right) + B} \tag{4a}$$

and $E_{1,2,U}$ is the risk neutrality expectations operator over the three assets. The brackets of Equation (4) contain the payoff to the counterparty in the first option contract, if the assets of the firm are sufficient to cover the option or when the firm defaults. In the latter case, the option holder receives just a proportion of his claim.

Translating Equation (2) into integrals is straightforward:

$$c = \frac{e^{-rT}}{\sigma_1 \sigma_2 \sigma_U \left(T\right)^{3/2}} \left[\int_0^\infty \int_0^\infty \int_0^\infty \frac{1}{s_1 s_2 u} \min\left(a_2 u,\ \max\left(s_1 - X_1, 0\right)\right) f_3 \, ds_1 ds_2 du \right] \tag{5}$$

where f_3 is the density function of the standard trivariate normal distribution with mean (0, 0, 0) and correlation matrix

$$\sum = \begin{bmatrix} 1 & \rho_{1,2} & \rho_{1U} \\ \rho_{1,2} & 1 & \rho_{2,U} \\ \rho_{1,U} & \rho_{2,U} & 1 \end{bmatrix} \tag{5a}$$

The elements of the vector $d = (d_1, d_2, d_3)$ are analogous to Equation (3c), with the third asset denoting the assets of the firm, and a_2 is given by Equation (3) with $S_{i,T}$ replaced by s_i. The value of a put is computed in a similar way.

Table 2 provides a numerical illustration. The JS model is computed in its original form, without accounting for the second option liability. Increasing the volatility of the underlying asset price leads to increases in all options in the last three columns. Other things equal, the volatility of the second asset has a negative effect on the first call, because the second option gets a higher proportion of the available assets in the event of default. The volatility of the assets of the firm has a negative effect on the option, as expected from the analysis of the previous section.

The correlation between the two optioned asset prices has a negative effect on the value of the option. Of course, the reason is that, as the asset prices move together, they tend to be exercised together, thereby increasing the total claim on the available assets in case of default. The correlation of the asset underlying the first option and the assets of the firm has a positive effect on option value, as shown in the previous section. The correlation between the second asset and the assets of the firm has an unclear effect. The interest rate, time to expiration, current stock price and assets of the firm have a positive effect as in the one-option case. When the price of the stock underlying the second option increases, it reduces the value of the first option, apparently because the chances of exercising (and receiving a share of the total claim) become better for the second option. The quasi-debt ratio has a negative effect on the first option, as expected. Finally, it is confirmed that vulnerable options are less sensitive than standard options to changes in common parameters, as mentioned earlier.

All the above integrals were computed using the *Mathematica* software and can be easily generalized to more than two options. For instance, in Episcopos (2002), multiple puts were used in deposit insurance. Approximating multiple integrals in higher dimensions is conducted efficiently with the quasi-Monte Carlo (QMC) technique. The typical estimation error in QMC increases with the integral dimension but it is mainly dependent on the sample size (Evans and Swartz, 2000). The only impediment is computation time, which can be prohibitive after some point.

4. An Application: Pricing Debt

Merton (1974) has shown that the value of debt equals the value of risk free debt minus a put option on the assets of the firm and exercise price the terminal value of debt. Debtholders write this put to stockholders, who, in turn, exercise it when the assets of the firm recede below the firm's debt obligation.

It would be interesting to see how contingent liabilities can affect the value of debt. Let us assume that the firm has written just one call option, and that creditors have equal priority with option holders in the event of default by the firm. By analogy with Equation (2), the current value of debt, F, is given by

$$F = e^{-rT} E_{1,U} \{\min(b_1 U_T, B)\} \tag{6}$$

where

$$b_1 = \frac{B}{\left(\max(S_{1T} - X_1, 0) + B\right)} \tag{6a}$$

Equation (6) gives F as the present value of the expected payoff to bondholders, who receive either their full claim or a proportion of the assets of the firm given by Equation (6a), when the firm is in default. The Merton (1974) model is a special case of Equation (6) if the firm writes no option, that is, $b_1 = 1$. The value of debt is given by:

$$F = \frac{e^{-rT}}{\sigma_1 \sigma_U T}\left[\int_0^\infty \int_0^\infty \frac{1}{s_1 u} \min\left(b_1 U_T, B\right) f_2 ds_1 du\right] \quad (7)$$

Although we can look at the sensitivity of F to various parameters, it is more important to see how F changes with the current value of the firm. Table 3 compares the value of Merton debt and F with the risk free debt for a range of values of the firm. Credit risk disappears for high values of U. Thus, both the Merton debt and the debt of this model behave like risk free debt. However, as U declines, the value of debt declines, but in a manner that cannot be explained by the usual default put of the equity holders. This is due to the call option liability that becomes more binding as asset value falls. As a matter of fact, the difference in the two debt values can be attributed exactly to the additional liability created by the option. In other words, the shareholders' put option is more valuable in the presence of an option liability.

In practice, it is often convenient to work with spreads of bonds, if these bonds are traded, with Treasury bonds. The spread between the yield of such bonds and the risk free rate is:

$$s^* = \left(-1/T\right)\ln(F^*/B) - r \quad (8)$$

where F^* is the value of either F, as provided by Equation (6) or the usual Merton (1974) debt and can be very large, especially for low values of U, exactly when the risk for default increases further, due to the option liability. Furthermore, accounting for the existence of an option should provide a warning signal in the form of larger spread much earlier than the Merton (1974) model does.

The last result has important implications: The empirical applicability of the Merton (1974) model and its extensions, known also as contingent claims models, has been the subject of much research. While some papers are partly supportive of the contingent claims model (for instance, Anderson and Sundaresan (2000) on endogenous bankruptcy), others have found contradictory evidence. As an indicative example, Jones, *et al* (1984) report that that the spreads predicted by options models are consistently lower than observed market spreads, particularly in investment grade bonds. With regard to financial firms writing options, this is consistent with the vulnerable debt approach presented here. Thus, Table 3 could explain empirical results such as those of Jones, *et al* (1984), at least in the case of financial institutions issuing debt.

5. Conclusions

This paper is on the joint valuation of multiple options written by a financial firm, when the firm has other liabilities such as debt. The paper extends the received theory in the direction of multiple options and examines the sensitivity of option values to various parameters. The existence of fixed or contingent obligations, such as debt and options, reduces the value of options written. As an associated application, debt is priced under the assumption that the financial firm has an option liability, partly extending the contingent claims model of credit risk. Options written by the firm can have a significant effect on the market value of debt.

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Table 1. Numerical Values for a Single Call Option

σ_1	σ_U	$\rho_{1,U}$	R	T	S_1	U	$\kappa = B/U$	c_{BS}	c_{JS}	c
0.1								10.31	10.08	7.84
0.2								**13.27**	**11.03**	**7.97**
0.3								16.73	11.26	8.14
	0.1							13.27	11.15	8.05
	0.3							13.27	10.83	7.84
		-0.9						13.27	7.62	4.90
		0.9						13.27	11.68	8.80
			0.05					10.45	8.85	6.32
			0.15					16.36	13.30	9.72
				0.5				8.28	7.85	5.84
				1.5				17.67	13.02	9.52
					90			6.95	6.19	4.62
					110			21.25	16.13	11.45
						20		13.27	8.99	6.45
						40		13.27	12.13	9.05
							0.7	13.27	11.03	8.38
							0.9	13.27	11.03	7.58

Notes: c_{BS} = Black-Scholes (1973), c_{JS} = Johnson-Stulz (1987), c = Vulnerable call with debt. Except when otherwise indicated, parameter values are: $\sigma_1 = \sigma_U = 0.2$, $\rho_{1,U} = 0$, $R = 0.1$, $S_1 = 100$, $U = 30$, and the quasi-debt ratio, $\kappa = B/U = 0.80$. The base case is highlighted.

Table 2. Numerical Values for a Call Option: The Case of Two Calls

σ_1	σ_2	σ_U	$\rho_{1,2}$	$\rho_{1,U}$	$\rho_{2,U}$	R	T	S_1	S_2	U	$\kappa = B/U$	c_{BS}	c_{JS}	c
0.1												10.31	10.08	6.37
0.2												**13.27**	**11.03**	**6.62**
0.3												16.73	11.26	6.89
	0.1											10.31	10.08	6.49
	0.3											10.31	10.08	6.26
		0.1										10.31	10.15	6.43
		0.3										10.31	9.95	6.25
			-0.9									10.31	10.08	7.25
			0.9									10.31	10.08	5.51
				-0.9								10.31	9.62	5.62
				0.9								10.31	10.31	7.28
					-0.9							10.31	10.08	6.49
					0.9							10.31	10.08	6.21
						0.05						6.80	6.71	4.50
						0.15						14.20	13.69	8.12
							0.5					5.85	5.85	4.30
							1.5					14.53	13.51	7.91
								90				3.36	3.35	2.31
								110				19.61	18.13	10.58
									90			10.31	10.08	7.01
									110			10.31	10.08	5.66
										20		10.31	9.27	5.17
										40		10.31	10.26	7.19
											0.7	10.31	10.08	6.70
											0.9	10.31	10.08	6.05

Notes: c_{BS} = Black-Scholes (1973), c_{JS} = Johnson-Stulz (1987) assuming the second option liability is ignored, c = Vulnerable call with debt. Except when otherwise indicated, parameter values are: $\sigma_1 = \sigma_1 = \sigma_U = 0.2$, $\rho_{1,2} = \rho_{1,U}$ $\rho_{2,U} = 0$, $R = 0.1$, $S_1 = S_2 = 100$, $U = 30$, and $\kappa = B/U = 0.80$. The base case is highlighted.

Table 3. The Value of Debt

	Value of Firm										
	0	10	20	30	40	50	60	70	80	90	100
Risk Free Debt	27.15	27.15	27.15	27.15	27.15	27.15	27.15	27.15	27.15	27.15	27.15
Merton (1974) Debt	0.00	10.00	19.87	26.02	27.08	27.14	27.15	27.15	27.15	27.15	27.15
Debt with one call*	0.00	7.58	15.11	21.02	24.00	25.52	26.32	26.73	26.94	27.04	27.09

Parameter values are: $\sigma_1 = \sigma_U = 0.2$, $\rho_{1,U} = 0$, $R = 0.1$, $T = 1$, $S_1 = 100$, $X_1 = 100$, $B = 30$.