THE ENVELOPE THEOREM

IN ITS PROPER PERSPECTIVE

by

Emmanuel Drandakis

Athens University of Economics and Business

February 1999

1. Introduction

The *Envelope Theorem* occupies today a central position among the basic tools of economic analysis. The theorem was introduced by *Samuelson* (1947), who used it to elucidate the proper relationship between long-run and short-run cost curves and to produce a general statement of the *Le Chatelier Principle*. Currently, the Envelope Theorem appears in many areas of economic theory: its main application is in providing simple and elegant proofs of comparative static results in optimization problems.

This paper attempts to demonstrate that the Envelope theorem is presented in the literature in a fragmented and rather elliptical way that distorts its very meaning. When placed in its proper perspective, *the Envelope Theorem is shown to involve comparisons of the performance of alternative policies*; namely, comparisons between the maximum (or minimum) value function of an optimization problem attained under the optimal policy and the value of the objective of the problem under any other economically relevant policy.

Let us consider a simple constrained maximization problem, like

$$\varphi(\alpha, \beta, \gamma) = \max_{x} \; \{f(x, \alpha) \mid h(x, \beta) = \gamma\} \qquad (1)$$

with f, h and the maximum value function $\varphi(\alpha, \beta, \gamma)$ sufficiently smooth and the optimal policy for the choice variables, $x(\alpha, \beta, \gamma)$, dependent on the values of parameters entering either the objective or the constraint functions[1].

As criterion of economic relevance of the alternative policies we will consider that of *feasibility*. In other words, we will examine policies or paths, $y(\alpha, \beta, \gamma)$, dependent on the values of the parametres, which satisfy the constraint of the problem

$$h(y(\alpha,\beta,\gamma),\beta)=\gamma \qquad (2)$$

for all (α,β,γ) and which are also sufficiently smooth.

[1] We could equally well examine a minimization problem and derive $x(\alpha, \beta, \gamma)$ and the minimum value function $\varphi(\alpha, \beta, \gamma)$.

Such a criterion is certainly the weakest possible one, since it encompasses all policies among which $x(\alpha, \beta, \gamma)$ is chosen. Can we have any meaningful comparisons between $\varphi(\alpha, \beta, \gamma)$ and $f(y(\alpha, \beta, \gamma), \alpha)$ for any odd $y(\alpha, \beta, \gamma)$? The answer is surely in the negative but, when the Envelope Theorem emerges, chaos is turned into light!

How do such feasible paths come about? There are many alternative possibilities, among which the following may be noted:

(i) These feasible paths may, in fact, be the optimal solutions of problems whose objective function is the same as in (1) and which are subject to various constraints, including *necessarily* those of (1). Or

(ii) they may be optimal solutions of problems having a different objective function, but whose constraints include *necessarily* the constraints of (1). Or

(iii) they may not be solutions of any optimization problem, but any paths satisfying the constrains of (1) along with other constraints.

Whatever the origin of $y(\alpha,\beta,\gamma)$, it is clear that $f(y(\alpha,\beta,\gamma),\alpha)$ may have any imaginable shape, but it must always lie below $\varphi(\alpha,\beta,\gamma)$. There may be cases, however, where it happens that at a particular configuration of parameters, $(\alpha^o, \beta^o, \gamma^o)$, we have

$$y(\alpha^o, \beta^o, \gamma^o) = x(\alpha^o, \beta^o, \gamma^o). \tag{3}$$

Then and only then we observe the equality

$$\varphi(\alpha^o, \beta^o, \gamma^o) = f(y(\alpha^o, \beta^o, \gamma^o), \alpha^o), \tag{4}$$

although all around $(\alpha^o, \beta^o, \gamma^o)$ nothing definite can be said about $\varphi(\alpha, \beta, \gamma) - f(y(\alpha, \beta, \gamma), \alpha)$ beyond its positivity. But this is enough! We will see that, at $(\alpha^o, \beta^o, \gamma^o)$, $\varphi(\alpha, \beta, \gamma)$ is tangential to $f(y(\alpha, \beta, \gamma), \alpha)$ and moreover, is more convex (or less concave) than $f(y(\alpha, \beta, \gamma), \alpha)$. Thus *the Envelope theorem springs into life*, despite the fact that $y(\alpha, \beta, \gamma)$ is completely arbitrary, or even grossly unreasonable in comparison to $x(\alpha, \beta, \gamma)$, having only (2) and (3) as its properties.

The paper is organized as follows. Section 2 considers the first and second-order derivative properties that the maximum value function inherits from those of the objective and constraint functions. Section 3 introduces feasible paths, examines the properties of the gain function

$$g(\alpha, \beta, \gamma) \equiv \varphi(\alpha, \beta, \gamma) - f(y(\alpha, \beta, \gamma), \alpha)$$

and derives the general form of the Envelope Theorem. Section 4 reviews briefly how the theorem is presented in the literature and underlines the differences with its general form developed in this paper. Finally, in section 5, we offer some suggestions for future research on extending the range of applications of the Theorem. It is pointed out that the scope of the Theorem, in its general form, is far wider than one may at first expect. If a feasible policy does not coincide with the optimal one at any point of a whole region of parameter values, then $\varphi(\alpha, \beta, \gamma) > f(y(\alpha, \beta, \gamma), \alpha)$ holds over that region; this seems to preclude any comparisons along the lines of the Envelope Theorem. Appearances, however, are deceptive. Indeed looking at the literature, from this point of view, we can find at least two areas of existing work, where this limitation of the Theorem has been overcome.

Three appendices complete the paper. The first includes proofs of the theorems appearing in the text; the second considers the "compensated" version of problem (1), so that variations in parameters β do not upset the feasibility of previously optimal or feasible policies; finally, the third examines Le Chatelier Principle and shows the simplifying effects of having such feasible policies on the curvature properties of the Envelope Theorem.

2. <u>Derivative Properties of the Maximum Value Function</u>

We consider a general constrained maximization problem

$$\varphi(\alpha, \beta, \gamma) \equiv \max_x \{ f(x, \alpha) \mid h(x, \beta) = \gamma \}, \qquad \text{(I)}$$

with n choice variables $x = (x_1, \ldots, x_n)'$, k parameters $\alpha = (\alpha_1, \ldots, \alpha_\kappa)'$ appearing in the objective function and two sets of parameters, $\beta = (\beta_1, \ldots \beta_\ell)'$ and $\gamma = (\gamma_1, \ldots, \gamma_m)$ appearing in the m constraint functions. All vectors are column vectors with the exception of $h(x, \beta) = (h^1(x, \beta), \ldots, h^m(x, \beta))$ and γ.

We will assume that (I) is well behaved with interior solutions, that f and h are twice continuously differentiable and that the constraints are independent[2]. Finally, for simplicity we assume a unique solution of (I).

The first and second-order conditions characterizing a maximum of (I) are given by

$$\left\{\begin{array}{l} f_x(x,\alpha)=\sum_{j=1}^{m}\lambda_j h_x^j(x,\beta)\equiv H_x(x,\beta)\lambda \\ h(x,\beta)=\gamma \end{array}\right\} \tag{5}$$

and

$$\left\{\begin{array}{l} \eta' A\eta<0 \text{ for all } \eta\in R^n, \eta\neq 0_n, \\ \text{satisfying } h_x^j(x,\beta)'\eta=0,\ j=1,\dots,m \end{array}\right\}, \tag{6}$$

where $H_x(x,\beta)=(h_{x_i}^j(x,\beta))$, $i=1,\dots n$, $j=1,\dots,m$, λ is the vector of lagrangean multipliers and $A=F_{xx}-H_{xx}$ is the hessian matrix of (I) in x, with $F_{xx}=(f_{xx}(x,\alpha))$ and $H_{xx}=\sum_{j=1}^{m}\lambda_j h_{xx}^j(x,\beta)$[3].

We can apply the implicit function theorem and obtain the optimal path $x(\alpha,\beta,\gamma)$, $\lambda(\alpha,\beta,\gamma)$ and the maximum value function $\varphi(\alpha,\beta,\gamma)\equiv f(x(\alpha,\beta,\gamma),\alpha)$, as twice continuously differentiable functions of (α,β,γ).

If we denote by $X_\alpha=[x_{a_j}^i(\alpha,\beta,\gamma)], X_\beta, X_\gamma, \Lambda_\alpha, \Lambda_\beta$, and Λ_γ the matrices of the rates of change of $x(\alpha,\beta,\gamma)$ and $\lambda(\alpha,\beta,\gamma)$ with respect to (α,β,γ), then we have:

<u>Theorem 1</u>: (first-order derivative properties of φ)

(i) $\varphi_\alpha(\alpha,\beta,\gamma)=f_\alpha(x(\alpha,\beta,\gamma),\alpha)$, (ii) $\varphi_\beta(\alpha,\beta,\gamma)=-H_\beta(x(\alpha,\beta,\gamma),\beta)\lambda(\alpha,\beta,\gamma)$ and

(iii) $\varphi_\gamma(\alpha,\beta,\gamma)=\lambda(\alpha,\beta,\gamma)$

---

[2] Namely, their gradients in x, $h_x^i(x,\beta)$, $i=1\dots m$, are linearly independent.

[3] Capital letters are used as symbols for matrices. Note the difference between matrices H_x or $H_\beta=(h_{\beta_i}^j(x,\beta))$, on the one hand, and H_{xx}, $H_{x\beta}=\sum_{j=1}^{m}\lambda_j h_{x\beta}^j(x,\beta)$ or $H_{\beta\beta}=\sum_{j=1}^{m}\lambda_j h_{\beta\beta}^j$, on the other.

and

Theorem 2: (second-order derivative properties of *φ)*

$$\Phi = \begin{bmatrix} \Phi_{\alpha\alpha}, \Phi_{\alpha\beta}, \Phi_{\alpha\gamma} \\ \Phi_{\beta\alpha}, \Phi_{\beta\beta}, \Phi_{\beta\gamma} \\ \Phi_{\gamma\alpha}, \Phi_{\gamma\beta}, \Phi_{\gamma\gamma} \end{bmatrix} =$$

$$= \begin{bmatrix} F_{\alpha x} X_\alpha + F_{\alpha\alpha} & , & F_{\alpha x} X_\beta & , & F_{\alpha x} X_\gamma \\ -H_{\beta x} X_\alpha - H_\beta \Lambda_\alpha & , & -H_{\beta x} X_\beta - H_\beta \Lambda_\beta - H_{\beta\beta} & , & -H_{\beta x} X_\gamma - H_\beta \Lambda_\gamma \\ \Lambda_\alpha & , & \Lambda_\beta & , & \Lambda_\gamma \end{bmatrix}$$

is a symmetric matrix.

Theorems 1 and 2 establish an important feature of the maximum value function. The first (second) partial derivatives of $\varphi(\alpha, \beta, \gamma)$ are expressed in terms of $x(\alpha, \beta, \gamma)$ and $\lambda(\alpha, \beta, \gamma)$ (in terms of the first partials of $x(\alpha, \beta, \gamma)$ and $\lambda(\alpha, \beta, \gamma)$). This is due to the following properties of X_α, X_β and X_γ, which hold for all (α, β, γ):

$$X_\alpha(\alpha,\beta,\gamma)' f_x(x(\alpha,\beta,\gamma),\alpha) = 0_\kappa,$$
$$X_\beta(\alpha,\beta,\gamma)' f_x(x(\alpha,\beta,\gamma),\alpha) = -H_\beta(x(\alpha,\beta,\gamma),\beta)\lambda(\alpha,\beta,\gamma) \qquad (7)$$
and $X_\gamma(\alpha,\beta,\gamma)' f_x(x(\alpha,\beta,\gamma),\alpha) = \lambda(\alpha,\beta,\gamma)$.

Theorems 1 and 2 can also be used to derive the properties of variations in *β*, *with γ responding appropriately*, so that all constraints continue to pass through the optimal solution at *(α, β, γ)*. Thus it is immediate that the following corollaries hold:

Corollary 1:

From Theorem 1 (ii) and (iii) and from (7) we get

$$\varphi_\beta(\alpha, \beta, \gamma) + H_\beta(x(\alpha, \beta, \gamma), \beta)\, \varphi_\gamma(\alpha, \beta, \gamma) =$$

$$\{X'_\beta + H_\beta X'_\gamma\} f_x(x(\alpha, \beta, \gamma), \alpha) = 0_l$$

Corollary 2:

The matrix

$$\lfloor I_{\ell\ell}, H_\beta \rfloor \begin{bmatrix} \Phi_{\beta\beta}, \Phi_{\beta\gamma} \\ \Phi_{\gamma\beta}, \Phi_{\gamma\gamma} \end{bmatrix} \begin{bmatrix} I_{\ell\ell} \\ , \\ H'_\beta \end{bmatrix} \quad \text{is symmetric}^4$$

and equal to

$$-H_{\beta x}\{X_\beta + X_\gamma H'_\beta\} - H_{\beta\beta}$$

We will see in Appendix 2, where we consider the "compensated" version of (I), that the *nxl* matrix

$$S_\beta \equiv X_\beta + X_\gamma H'_\beta \tag{8}$$

is the Slutsky substitution matrix. Thus from Corollary 1 we have

$$S'_\beta\ f_x(x(\alpha,\beta,\gamma),a) = 0_\iota \quad , \tag{9}$$

while from Corollary 2 and the symmetry of $H_{\beta\beta}$, we get the symmetry of

$$H_{\beta x} S_\beta \ . \tag{10}$$

We conclude that Theorem 2 and Corollary 2 are the source of all *reciprocity relations* obtained in comperative static analysis: all three matrices

$$F_{\alpha x} X_\alpha \quad , \quad H_{\beta x} X_\beta + H_\beta \Lambda_\beta \quad \text{and} \quad \Lambda_\gamma \ , \quad \text{as well as} \quad H_{\beta x} S_\beta \ .$$

are symmetric[5]. We also see that

$$\Lambda_\alpha = X'_\gamma F_{x\alpha} \quad , \quad \Lambda_\beta + \Lambda_\gamma H'_\beta = -X'_\gamma H_{x\beta} \ ,$$

$$-H_{\beta x} X_\alpha = \{X'_\beta + H_\beta X'_\gamma\} F_{x\alpha}$$

---

[4] If B is an $n \times n$ symmetric matrix and P is $n \times s$ then $P'BP$ is symmetric, since $(P'BP)' = P'B'(P')' = P'BP$

[5] X_α or S_β are not even square matrices in general. Their symmetry can be established only when we examine problems with simple linear versions of f and h^j, $j = 1, \ldots, m$, respectively.

3. The Envelope Theorem in its Full Generality

The concept of *feasible paths* was introduced in § 1. Whenever the number of constraints of (I) is smaller than the number of choice variables, $m < n$, there are infinitely many $y(\alpha,\beta,\gamma)$ sufficiently smooth and satisfying

$$h^j\ (y\ (\alpha, \beta, \gamma)\, , \beta\,) = \gamma_i\ ,\ j = 1, \ldots, m \tag{11}$$

The value of the objective function $f(\gamma(\alpha,\beta,\gamma),\alpha)$ under $y(\alpha,\beta,\gamma),)$, is taken as the measure of performance of the feasible path to be contrasted with the maximum value function. Thus the *gain function*, defined by

$$g(\alpha,\beta,\gamma) = \varphi(\alpha,\beta,\gamma) - f(y(\alpha,\beta,\gamma),\alpha), \tag{12}$$

is only a function of the parameters and measures the excess of φ over the value attained by the particular feasible path that is examined[6].

In general $g(\alpha,\beta,\gamma) > 0$ for all (α,β,γ), unless the feasible path coincides with the optimal one at some $(\alpha^o,\beta^o,\gamma^o)$,

$$y(a^o,\beta^o,\gamma^o) = x(\alpha^o,\beta^o,\gamma^o). \tag{13}$$

At such a point, *if it exists,* the gain function has a minimum of *zero*. But g is sufficiently smooth and thus its minimum is characterized by first and second – order necessary conditions.

Differentiating $g(\alpha,\beta,\gamma)$ we get

$$\begin{aligned} g_\alpha &= \varphi_\alpha - Y_\alpha{}' f_x(y(\alpha,\beta,\gamma),\alpha) - f_a(y(\alpha,\beta,\gamma),\alpha), \\ g_\beta &= \varphi_\beta - Y_\beta{}' f_x(y(\alpha,\beta,\gamma)) \qquad \text{and} \\ g_\gamma &= \varphi_\gamma - Y_\gamma{}' f_x(y(\alpha,\beta,\gamma),\alpha) \end{aligned} \tag{14}$$

---

6 If $m = 0$ and (I) is an unconstrained problem, any smooth $y(\alpha) \in R^n$ is admissible with $g(a) = \varphi(\alpha) - f(y(\alpha),\alpha)$.

where Y_α , Y_β and Y_γ are the matrices of the rates of change of $y(\alpha,\beta,\gamma)$.

Nothing definite can of course be said about the signs of derivatives in (14) at any (α,β,γ), since $y(\alpha,\beta,\gamma)$ may be quite different from $x(\alpha,\beta,\gamma)$. But at $(\alpha^o,\beta^o,\gamma^o)$, the situation is different! Severe restrictions on Y^o_α, Y^o_β and Y^o_γ are imposed, since (13) necessarily implies that $\varphi(\alpha,\beta,\gamma)$ is the upper envelope of $f(y(\alpha,\beta,\gamma),\alpha)$ at $(\alpha^o,\beta^o,\gamma^o)$. Indeed, letting G denote the matrix of second - order partial derivatives of g, we get the main results of the paper **:**

Theorem 3. *Envelope Theorem* (first-order tangencies)

At $(\alpha^o,\beta^o,\gamma^o)$ with $f_x^o = f_x(y(\alpha^o,\beta^o,\gamma^o),\alpha^o) = f_x(x^o,\alpha^o)$ we have

(i) $g^o_\alpha = -Y^{o\prime}_\alpha f^o_x = 0_\kappa$, (ii) $g^o_\beta = -H^o_\beta \lambda^o - Y^{o\prime}_\beta f^o_x = 0_\ell$ and

(iii) $g^o_\gamma = \lambda^o - Y^{o\prime}_\gamma f^o_x = 0_m$.

Theorem 4. *Envelope Theorem* (second-order curvature properties)

At $(\alpha^o,\beta^o,\gamma^o)$, G^0 can be expressed only in terms of the rates of change of x and y *and* is given in Table 1. G^o is symmetric and positive semi-definite.

In particular, we see that the matrices on the main diagonal of G^0 are given by

(i) $G^o_{\alpha\alpha} = F^o_{\alpha x}\{X^o_\alpha - Y^o_\alpha\} - Y^{o\prime}_\alpha F^o_{x\alpha} - Y^{o\prime}_\alpha A^o Y^o_\alpha$, (15)

(ii) $G^o_{\beta\beta} = -H^o_{\beta x}\{X^o_\beta - Y^o_\beta\} - H^o_\beta \Lambda^o_\beta + Y^{o\prime}_\beta H^o_{x\beta} - Y^{o\prime}_\beta A^o Y^o_\beta$, (16)

and (iii) $G^o_{\gamma\gamma} = \Lambda^o_\gamma - Y^{o\prime}_\gamma A^o Y^o_\gamma$, (17)

which are all symmetric and positive semi-definite.

The importance of the above theorems cannot be exaggerated!

Table 1

The matrix $G^o = \begin{bmatrix} G^{\mathrm{o}}_{\alpha\alpha}, & G^{\mathrm{o}}_{\alpha\beta}, & G^{\mathrm{o}}_{\alpha\gamma} \\ G^{\mathrm{o}}_{\beta\alpha}, & G^{\mathrm{o}}_{\beta\beta}, & G^{\mathrm{o}}_{\beta\gamma} \\ G^{\mathrm{o}}_{\gamma\alpha}, & G^{\mathrm{o}}_{\gamma\beta}, & G^{\mathrm{o}}_{\gamma\gamma} \end{bmatrix} =$

$$= \begin{bmatrix} F^{\mathrm{o}}_{\alpha x}\{X^o_a - Y^o_a\} - Y^{\mathrm{o}\prime}_a F^{\mathrm{o}}_{x\alpha} - \mathrm{Y}^{\mathrm{o}\prime}_{\alpha} \mathrm{A}^{\mathrm{o}} \mathrm{Y}^{\mathrm{o}}_{\alpha}, & F^{\mathrm{o}}_{\alpha x}\{\mathrm{X}^{\mathrm{o}}_{\beta} - \mathrm{Y}^{\mathrm{o}}_{\beta}\} + Y^{\mathrm{o}\prime}_a H^{\mathrm{o}}_{x\beta} - \mathrm{Y}^{o\prime}_{\alpha} \mathrm{A}^{\mathrm{o}} \mathrm{Y}^{\mathrm{o}}_{\beta}, & F^{\mathrm{o}}_{\alpha x}\{\mathrm{X}^{\mathrm{o}}_{\gamma} - \mathrm{Y}^{\mathrm{o}}_{\gamma}\} - \mathrm{Y}^{\mathrm{o}\prime}_{\alpha} \mathrm{A}^{\mathrm{o}} \mathrm{Y}^{\mathrm{o}}_{\gamma} \\ -\mathrm{H}^{\mathrm{o}}_{\beta x}\{\mathrm{X}^{\mathrm{o}}_{\alpha} - \mathrm{Y}^{\mathrm{o}}_{\alpha}\} - H^o_{\beta} \Lambda^{\mathrm{o}}_{\alpha} - \mathrm{Y}^{o\prime}_{\beta} F^{\mathrm{o}}_{x\alpha} - \mathrm{Y}^{\mathrm{o}\prime}_{\beta} A^o Y^o_a, & -\mathrm{H}^{\mathrm{o}}_{\beta x}\{\mathrm{X}^{\mathrm{o}}_{\beta} - \mathrm{Y}^{\mathrm{o}}_{\beta}\} + \mathrm{Y}^{\mathrm{o}\prime}_{\beta} \mathrm{H}^{\mathrm{o}}_{x\beta} - \mathrm{Y}^{o\prime}_{\beta} \mathrm{A}^{\mathrm{o}} \mathrm{Y}^{\mathrm{o}}_{\beta} - \mathrm{H}^{\mathrm{o}}_{\beta} \Lambda^{\mathrm{o}}_{\beta}, & -H^{\mathrm{o}}_{\beta x}\{\mathrm{X}^{\mathrm{o}}_{\gamma} - \mathrm{Y}^{\mathrm{o}}_{\gamma}\} - \mathrm{H}^{\mathrm{o}}_{\beta} \Lambda^{\mathrm{o}}_{\gamma} - \mathrm{Y}^{o\prime}_{\beta} \mathrm{A}^{\mathrm{o}} \mathrm{Y}^{\mathrm{o}}_{\gamma} \\ \Lambda^{\mathrm{o}}_{\alpha} - Y^{o\prime}_{\gamma} F^{\mathrm{o}}_{x\alpha} - \mathrm{Y}^{o\prime}_{\gamma} \mathrm{A}^{\mathrm{o}} \mathrm{Y}^{\mathrm{o}}_{\alpha}, & \Lambda^{\mathrm{o}}_{\beta} + \mathrm{Y}^{\mathrm{o}\prime}_{\gamma} \mathrm{H}x^{\mathrm{o}}_{x\beta} - \mathrm{Y}^{\mathrm{o}\prime}_{\gamma} \mathrm{A}^{\mathrm{o}} \mathrm{Y}^{\mathrm{o}}_{\beta}, & \Lambda^{\mathrm{o}}_{\gamma} - \mathrm{Y}^{\mathrm{o}\prime}_{\gamma} \mathrm{A}^{\mathrm{o}} \mathrm{Y}^{\mathrm{o}}_{\gamma} \end{bmatrix}$$

is symmetric and positive semidefinite

Theorem 3 shows that *such feasible paths*, no matter how different from $x(\alpha,\beta,\gamma)$ around $(\alpha^0,\beta^0,\gamma^0)$, *must share at* $(\alpha^0,\beta^0,\gamma^0)$ *a property that the optimal path has at any* (α,β,γ). Namely, Envelope tangency at $(\alpha^0,\beta^0,\gamma^0)$ means that

$$Y_\alpha' f_x = 0_\kappa\,,\qquad Y_\beta' f_x = -H_\beta\,\lambda \qquad \text{and} \qquad Y_\gamma' f_x = \lambda$$

must hold there, exactly as

$$X_\alpha' f_x = 0_\kappa\,,\quad X_\beta' f_x = -H_\beta\,\lambda \qquad \text{and} \qquad X_\gamma' f_x = \lambda$$

hold at any (α,β,γ).

Theorem 4 shows that *all second-order partial derivatives of g can be expressed at* $(\alpha^0,\beta^0,\gamma^0)$ *only in terms of the first partials* Y_α, Y_β *and* Y_γ.

This is in stark contrast with what happens away from $(\alpha^o, \beta^o, \gamma^o)$. When the feasible path does not coincide with the optimal one, the first partials of *g* depend on those of $y(\alpha, \beta, \gamma)$ while the second partial derivatives of *g* cannot be evaluated without knowledge of $Y^i_{aa}(\alpha, \beta, \gamma)$, or $Y^i_{\beta\beta}(\alpha, \beta, \gamma)$, or $Y^i_{\gamma\gamma}(\alpha, \beta, \gamma)$.

When we look at (15) – (17) or any submatrix of G^o, we see that the partials of $x(\alpha, \beta, \gamma)$ and $\lambda(\alpha, \beta, \gamma)$ appear in a smaller number of terms than the partials of $y(\alpha, \beta, \gamma)$. This is nice and is due to the structure imposed on the optimal policy: as *Lemma 3* of Appendix 1 shows along the optimal path we have

$$X_\alpha' F_{x\alpha} + X'_\alpha A X_\alpha = O_{\kappa\kappa}$$

$$-H_\beta \Lambda_\beta = X_\beta' A X_\beta - X_\beta' H_{x\beta}$$

and $\quad \Lambda_\gamma = X_\gamma' A X_\gamma,$

at all (α, β, γ).

There is therefore complete symmetry in G^o between the partials of $x(\alpha, \beta, \gamma)$ and those of $y(\alpha, \beta, \gamma)$ at $(\alpha^o, \beta^o, \gamma^o)$.

The general form of the *Envelope Theorem* is illustrated in *Figure 1*. No matter which parameter varies, α_i or β_i or γ_i, $\varphi(\alpha, \beta, \gamma)$ is the upper envelope tangential to $f(y^{\xi}(\alpha, \beta, \gamma), \alpha)$ at $(\alpha^o, \beta^o, \gamma^o)$ for any and all feasible paths $y^{\xi}(\alpha, \beta, \gamma)$ for which

$$y^{\xi}(\alpha^o, \beta^o, \gamma^o) = x(\alpha^o, \beta^o, \gamma^o), \text{ for all } \quad \xi = 1, 2, \ldots .$$

The upper envelope is more convex or less concave than any $f(y^{\xi}(\alpha, \beta, \gamma), \alpha)$ at $(\alpha^o, \beta^o, \gamma^o)$. These feasible paths may be such that their $f(y^{\xi}(\alpha, \beta, \gamma), \alpha)$ – except for their mutual tangency at $(\alpha^o, \beta^o, \gamma^o)$ – may cross one another at one or several points or exhibit further tangencies among them.

Figure 1

As in §2 with Corollaries 1 and 2, we derive the following:

Corollary 3:

$$g^{o}_{\beta} + \mathrm{H}^{o}_{\beta} g^{o}_{\gamma} = -\{\Upsilon^{o\prime}_{\beta} + \mathrm{H}^{o}_{\beta} \Upsilon^{o\prime}_{\gamma}\} f^{o}_{x} = 0_l$$

and

Corollary 4:

$$\text{The matric } \begin{bmatrix} I_{\ell\ell}, H^{o}_{\beta} \end{bmatrix} \begin{bmatrix} G^{o}_{\beta\beta}, G^{o}_{\beta\gamma} \\ G^{o}_{\gamma\beta}, G^{o}_{\gamma\gamma} \end{bmatrix} \begin{bmatrix} I_{\ell\ell} \\ , \\ H^{o}_{\beta} \end{bmatrix}$$

is symmetric, positive semidefinite and equal to

$$-H^{o}_{\beta x}\{S^{o}_{\beta} - T^{o}_{\beta}\} - T^{o\prime}_{\beta} A^{o} T^{o}_{\beta} + T^{o\prime}_{\beta} H^{o}_{x\beta} .$$

Here $T_{\beta} = Y_{\beta} + Y_{\gamma} H'_{\beta}$ is the matrix of the rates of change of the "compensated" feasible path, as explained in Appendix 2.

Theorems 3 – 4 and their corollaries provide a complete characterization of the *Envelope Theorem in its general form* of comparisons between maximum value functions and the value functions of feasible paths that coincide with the optimal path at $(\alpha^o, \beta^o, \gamma^o)$.

The expressions for the second partails of *g* in Theorem and Corollary 4 are certainly quite complex. However, this is to be expected under the minimum requirement – of feasibility – imposed on $y^{\xi}(\alpha, \beta, \gamma)$. As more structure is imposed on feasible paths, the above expressions get progressively simpler.

***Special Cases*:**

(a) if $y(\alpha, \beta, \gamma)$ is the optimal path of a problem similar to (I), but with additional constraints, then we will see in Appendix 3 on the *Le Chatelier Principle* that G^o becomes much simpler. In particular the matrix in (15) reduces to

$$F^{o}_{\alpha x}\{X^{o}_{\alpha} - Y^{o}_{\alpha}\}$$

while that in Corollary 4 reduces to

$$-H^{o}_{\beta x}\{S^{o}_{\beta} - T^{o}_{\beta}\},$$

with both matrices being symmetric and positive semidefinite.

(b) more drastically, if $y(\alpha, \beta, \gamma)$ is a degenerate path staying at $x^o = x(\alpha^o, \beta^o, \gamma^o)$ as α vary, or as β vary with accommodating responses in γ, then $Y_{\alpha}(\alpha, \beta, \gamma) = O_{n\kappa}$ or $T_{\beta}(\alpha, \beta; x^o) = O_{nl}$ and thus the matrices immediately above reduce to

$$F^{o}_{xx} X^{o}_{\alpha} \qquad \text{or} \qquad -H^{o}_{\beta x} S^{o}_{\beta}$$

being symmetric and positive semidefinite. It is in this case of degenerate feasible paths, that Theorem & Corollary 4 reduce to Theorem and Corollary 2 and, thus, complete all comparative static results that characterize the optimum solution of (I) and its "compensated" version.