4. The Envelope Theorem in the Literature

As we all know, the Envelope Theorem is given in the literature by the equalities

$$\varphi_\alpha(\alpha, \beta, \gamma) = f_a(x^o,\alpha) = L_\alpha (x^o, \lambda^o, \alpha, \beta, \gamma) \tag{18}$$

$$\varphi_\beta(\alpha, \beta, \gamma) = -H_\beta(x^o, \beta)\lambda^o = L_\beta (x^o, \lambda^o, \alpha, \beta, \gamma), \tag{19}$$

and

$$\varphi_\gamma(\alpha, \beta, \gamma) = \lambda^o = L_\gamma (x^o, \lambda^o, \alpha, \beta, \gamma), \tag{20}$$

where $L (x, \lambda, \alpha, \beta, \gamma) = f(x, \alpha) + \sum_{j=1}^{m} \lambda_j ,(\gamma_j - h^j (x, \beta))$ is the Lagrangian of (I) and $x^o = x(\alpha, \beta, \gamma)$, $\lambda^o = \lambda(\alpha, \beta, \gamma)$. In other words, the Theorem establishes the *equality of slopes of* φ, where variations in α or β or γ induce appropriate changes in $x(\alpha, \beta, \gamma)$ and $\lambda(\alpha, \beta, \gamma)$, *to those of* L, where x^o and λ^o are treated as constant and only α or β or γ vary.

When only parameters α vary, the maximum value function $\varphi(\alpha, \beta^o, \gamma^o)$ is the upper envelope of $f(x^o, \alpha)$ at $\alpha = \alpha^o$ and $x^o = x(\alpha^o, \beta^o, \gamma^o)$ [7]. As illustrated in *Figure 2*, when a single parameter α_i varies around a_i^o, with $\alpha_j = a_j^o$, $j \neq i$, the maximum value function is compared with the value function of the degenerate feasible policy, x^o; the latter reaches the former at $\alpha_i = a_i^o$ where they become tangent to one another. Similarly if α_i varies around a_i^1, then $\varphi(\alpha^o_{(i)}, \alpha_i, \beta^o, \gamma^o)$ is tangential to $f(x^1, a^o_{(i)}, a_i)$, at a_i^1, where $x^1 = x(a^o_{(i)}, a_i^1, \beta^o, \gamma^o)$.

This is the Envelope Theorem as presented by *Samuelson* (1947). Figure 2 is its usual diagrammatic interpretation.

[7] This is the reason for the second order curvature property $\Phi_{\alpha\alpha}(\alpha^o, \beta^o, \gamma^o) > F_{\alpha\alpha}(x(\alpha^o, \beta^o, \gamma^o), \alpha^o)$.

Figure 2

Difficulties appear, however, when we consider variations in parameters that enter the constraint functions: the latter will be violated if β or γ vary, while x remain unchanged.

Thus, when parameters β vary, the literature simply points to the equality of slopes $\varphi_\beta(\alpha^o, \beta, \gamma^o)$ and $L_\beta(x^o, \lambda^o, \alpha^o, \beta, \gamma^o)$ at $\beta = \beta^o$. But, is $\varphi(\alpha^o, \beta, \gamma^o)$ an upper envelope tangential at $\beta = \beta^o$ to some other surface? Can we have a diagram like Figure 2 when a single β_i varies around β_i^o ? The answer is surely not given by interpreting (19) as an Envelope tangency by analogy to (18); *for no such analogy exists*![8]

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[8] We must, of course, be careful and not think of L as being the performance function of the other policy, which is compared to $\varphi(\alpha^o, \beta, \gamma^o)$. It is apparent that for any $\beta \neq \beta^o$ $L(x^o, \lambda^o, \alpha^o, \beta, \gamma^0) \neq f(x^o, \alpha^o)$, since feasibility is violated. On the contrary, no such problem appears in (18); here $h(x^o, \beta^o) = \gamma^o$ continue to hold despite any variation in α and, thus, $L(x^o, \lambda^o, \alpha, \beta^o, \gamma^0) = f(x^o, \alpha)$.

We will see in Appendix 2, where we examine the "compensated" version of problem (I), that an Envelope tangency between the maximum value function of that version and $f(x^o, \alpha^o)$ appears – quite similarly to that in Figure 2 – whenever variations in β induce accommodating responses in γ, so that all constraints continue to pass through the previously optimal solution. This is the approach pioneered in *Mc Kenzie's* (1957) seminal paper and completed by *Hatta* (1980). *Hatta* has been able to overcome *Silberberg's* (1974) difficulties in applying his primal – dual method to variations in β. *Figure 3* illustrates the impact of such variations in β_i , with $\beta_j = \beta^o_j, j \neq i,$ and $x^o = x(\alpha^o, \beta^o, \gamma^o)$. It shows both $\varphi(\alpha^o, \beta^o_{(i)}, \beta_i , \gamma^o)$, which is not tangentail to the value of any degenerate path, and the maximum value function $\psi(\alpha^o, \beta^o_{(i)}, \beta_i; x^o)$, which has a horizontal tangency with $f(x^o, \alpha^o)$ at its minimum at β^o_i. Similarly when β_i varies around β^1_i, $\psi(\alpha^o, \beta^o_{(i)}, \beta_i; x^1)$ is tangent to $f(x^1, \alpha^o)$ at β^1_i, with $x^1 = x(\alpha^o, \beta^o_{(i)}, \beta^1_i, \gamma^o)$[9].

Figure 3

We come finally to examine variation in γ. (20) is the familiar "intepretation of lagrangean multipliers", known in the economic literature even before the advent of the Envelope Theorem.

[9] $f(x^o, \alpha^o)$ and $f(x^1, \alpha^o)$ are horizontal lines since neither x nor α vary.

And, yet, it is here that we meet an unsurmountable difficulty: the question about whether $\varphi(\alpha^o, \beta^o, \gamma)$ is an upper envelope of other value functions cannot be dealt with the method of the previous paragraph and, indeed, has not yet been given an answer[10] In fact, a conceptual chasm can be discerned between (20), on the one hand, and (18) & (19) on the other: quite frequently when the Envelope Theorem is examined γ is treated as a constant, while $\varphi_\gamma (\alpha, \beta, \gamma) = \lambda(\alpha, \beta, \gamma)$ is discussed separately, usually with a note that this is really a special case of the Envelope Theorem.

The contrast between the general form of the Theorem given in this paper and the traditional one is quite evident. Theorem & Corollary 1, on the derivative properties of maximum value functions, are presented as the Envelope Theorem for no other reason but the fact that Theorem 3(i) and its Corollary reduce to Theorem 1(i) and its Corollary in a very particular case; namely, that of degenerate feasible paths. When feasible paths of any kind are considered and their performance is compared to that of the optimal path, then we obtain the general form of the Theorem. The results of this new perspective are clearly shown in the differences between Figures 1 versus Figures 2 & 3. In the latter we have tangencies between two value functions: the maximum value and the value functions of degenerate feasible paths. In the former we have an infinite number of mutual tangencies between the upper envelope and all other value functions of feasible policies coinciding with the optimal policy at the specified values of $\alpha, \beta,$ or γ.

5. <u>On the scope of the Envelope Theorem</u>

The Envelope Theorem is quite powerful when it works, but it does not spring into action unless $y(\alpha, \beta, \gamma)$ coincides with $x(\alpha, \beta, \gamma)$ at some configuration of parameter values. When however $f(y(\alpha, \beta, \gamma), \alpha)$ is below $\varphi(\alpha, \beta, \gamma)$, within whole regions of parameter values, then the Theorem remains silent. Can we then say anything about possible relations between $Y_\alpha(\alpha, \beta, \gamma)$, $Y_\beta(\alpha, \beta, \gamma)$, or $Y_\gamma(\alpha, \beta, \gamma)$ and the corresponding rates of change along the optimal path? Is it possible to extend the theorem´s scope beyond its usual setting? Indeed in many cases the value functions of comparison paths are well below the upper envelope for observed values of parameters.

[10] Variations in γ cannot be examined along the lines of *Hatta* (1980).

This is a big problem the solution of which cannot be attempted here. It will suffice therefore to search the literature and see whether some work has already been done in this direction. As a matter of fact we can point to two such examples, which will be presented quite briefly and in a way reflecting our point of view.

The first refers to *behavior under quantity rationing* as examined in *Neary & Roberts* (1980) and *Deaton* (1981). Let us consider cost minimization by a competitive firm in the long and the short – run, or problem

$$c(w, r, u) \equiv \min_{x,z} \{w'x + r'z \mid f(x, z) = u\} \qquad \text{(II)}$$

in comparison with

$$c(w, r, u, \bar{z}) \equiv \min_{x,z} \{w'x + r'z \mid f(x, z) = u,\ \bar{z} = z\} \qquad \text{(II}^s\text{)}$$

where w, r are the existing prices of variable and fixed inputs, u = output level and $\bar{z}$ = quantities of fixed inputs. The solutions $x(w, r, u)$ & $z(w, r, u)$ of (II) compared with $x(w, u, \bar{z})$ of (II^s) may be such that short-run costs exceed long-run ones, or

$$c(w, r, u, \bar{z}) > c(w, r, u).$$

In such a case it is shown that $r^v \neq r$ exist under which

$$c(w,r^v,u)=c(w,r^v,u,\bar{z})$$

and thus

$$x(w,r^v,u)=x(w,u,\bar{z}) \qquad \text{and} \qquad z(w,r^v,u)=\bar{z},$$

making possible the use of the Envelope Theorem in the form of Le Chatelier Principle. Thus "virtual" prices, r^v, restore the application of Le Chatelier Principle in a situation in which it would not otherwise apply. *Figure 4* illustrates all cost curves, when the price of some variable input varies around w_i^o, with $w_j=w_j^o, j \neq i$, and $r=r^o$ or $r=r^v$. As shown, in the Figure, the imposition of rationing increases the use of variable inputs like x_i and results in $r^v>r^o$ and

$$c(w^o,r^v,u)=c(w^o,r^v,u,\bar{z})>c(w^o,r^o,u,\bar{z})>c(w^o,r^o,u)\text{ [11].}$$

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[11] If $\bar{z} > z(w^o,r^o,u)$ and the slope of $c(w^o,r^o,u,\bar{z})$ with respect to w_i is smaller than that of $c(w^o,r^o,u)$ at w_i^o then the two dotted curves which are tangent at w_i^o will not be above but below $c(w^o,r^o,u)$, since $r^v < r^o$.

Figure 4

The second example examines the *Illyrian firm* which maximizes non-labor returns per worker; see *Kahana* (1989) or *Neary* (1988). In a model, with competitively priced variable and fixed inputs, such behavior is contrasted with that of a competitive firm having the same technology and maximizing profits. If the maximum profit of the latter firm is taken as our criterion, the profits achieved by the illyrian firm are lower at competitive prices. But if maximum non-labor returns per worker is considered as the "virtual" price of labor and we compute the profits of both firms, then they both become equal to zero and tangent to one another. *Figure 5* illustrates this situation when the price of output, p , varies around p^o and (w,ω,r) are the prices of non-labor variable inputs, labor and fixed inputs, respectively. At the competitive wage, ω^o, the illyrian firm's performance is inferior to that of the profit - maximizing firm at all p; but as ω

increases towards ω^v, both profit curves shift downward, the distance between them gets smaller and, finally, they become tangent to one another as profits are extinguished[12].

Figure 5

[12] As should be expected in problems with different objective functions, we could take as our criterion the maximum value of the objective of the illyrian firm. Then that firm achieves higher returns than the competitive firm at ω^o ! As ω increases to ω^v, the competitive firm´s returns increase until, at ω^v, we have a tangency between them.

The studies referred to in the last two paragraphs, examine simple models which are linear in parameters and choice variables. However the possibility of finding "virtual" prices is not due to linearities but, rather, *to the presence of an adequate number of parameters and their strategic positioning vis-a-vis the choice variables.*

This observation seems crucial for insuring similar success for the envelope theorem in more complex problems. The same is true for generalized duality, as examined by *Epstein* (1981). His basic assumption, that for any bundle of choice variables there exist appropriate values of parameters for which the bundle becomes a solution of the constrained optimization problem, clearly sounds more reasonable and less heroic as the number of parameters increases and their positioning *vis-α-vis* each choice variable improves.

It is clear, on the other hand, that such extensions of the scope of the envelope theorem can only be attempted in the context of specific models. In our abstract problem (I) we said nothing about the number of parameters α or β; nor could we say anything about their relationship with choice variables.

Concluding Remarks

We have seen in this paper that the Envelope Theorem can be generalized and shown to work whenever the maximum (or minimum) value function of an optimization problem can be reached from below (or above) by the value function of any policy which satisfies the constraints of the problem.

Our concept is clearly more general than *Samuelson's* Envelope Theorem, in which $\varphi(\alpha, \beta^o, \gamma^o)$ is compared to $f(x^o, \alpha)$, $x^o = x(\alpha^o, \beta^o, \gamma^o)$, i.e., to the value function of a degenerate feasible policy $y(\alpha, \beta^o, \gamma^o) = x^o$.

On the other hand, current literature presents the Envelope Theorem as involving a comparison between $\varphi(\alpha, \beta, \gamma)$ and the Langrangean $L(x^o, \lambda^o, \alpha, \beta, \gamma)$, $x^o = x(\alpha^o, \beta^o, \gamma^o)$, $\lambda^o = \lambda(\alpha^o, \beta^o, \gamma^o)$. If L is taken seriously as the value function of α comparison policy when parameters entering the constraints vary then, although $\varphi(\alpha^o, \beta^o, \gamma^o) = L(x^o, \lambda^o, \alpha^o, \beta^o, \gamma^o)$ and $\varphi_\beta = L_\beta$ or $\varphi_\gamma = L_\gamma$ hold at $(\alpha^o, \beta^o, \gamma^o)$, one cannot escape the predicament that L does not preserve feasibility away from the point of tangency. Thus L may very well be *above* $\varphi(\alpha, \beta, \gamma)$ at points $(\alpha^o, \beta, \gamma^o)$ or $(\alpha^o, \beta^o, \gamma)$

around $(\alpha^o, \beta^o, \gamma^o)$! Consequently, g does not necessarily reach a minimum at that point and Theorem 4 does not necessarily hold there. In sum, L is not the value function of an economically relevant policy to be compared to $\varphi(\alpha, \beta, \gamma)$.

Of course, we may follow a method that has long been proved fruitful; i.e., we may consider variations in β that preserve feasibility. This line of proof of the properties of the substitution matrix stems from the seminal paper of *Mc Kenzie* (1957). As noted in *Takayama* (1994), the method has been generally adopted in Economics, with most authors considering compensations `a la *Hicks* while *Hatta* (1980) follows *Mc Kenzie* more closely and considers compensations a la *Slutsky.*

The fact remains, however, that L violates feasibility when parameters γ vary. There is no way we can present $\varphi_\gamma(\alpha, \beta, \gamma) = \lambda(\alpha, \beta, \gamma)$ as an Envelope Tangency!

Paul Samuelson (1947) concludes his discussion of the Envelope Theorem with the following observation on the derivative properties of $\varphi(\alpha) \equiv \max_x f(x, \alpha)$:

> *"The changes in [φ(α), brought about by a change in a parameter] are of higher order than the [corresponding] changes in [the decision variables]. In fact, the nth derivative of [φ(α)] depends at most upon the (n-1)th derivative of [x(α)]".*

This is a keen insight characterizing the derivatives of any maximum (or minimum) value functions. Remarkably enough, we have seen above that this feature of $\varphi(\alpha, \beta, \gamma)$ extends to $f(y(\alpha, \beta, \gamma), \alpha)$ as well, whenever the two are tangent and no matter which feasible policy we care to consider. Theorems 3 and 4 and their Corollaries utilize exactly this fact.

Theorem 4 allows us to explore another property of problem (I). If we consider the diagonal elements of G^o, like e.g.

$$g^o_{\gamma_i\gamma_i} = \lambda^o_{i\gamma_i} - y^o_{\gamma_i} A^o y^o_{\gamma_i} =$$

$$= x^o_{\gamma_i} A^o x^o_{\gamma_i} - y^o_{\gamma_i} A^o y^o_{\gamma_i} \geq 0 \ ,$$

we see that the first term in the r-h-s is greater than or equal to the second, for any such feasible path. This is indeed true as can be shown by solving the following "second-order" problem:

$$\max_{v} \left\{ v' A^{o} v \,\middle|\, H_{x}^{o'} v = e_{i} \right\},$$

where $v = y^{o}_{\gamma i}$, e_i is a unit vector and the constraints result from differentiating the constraints of (I) with respect to γ_i. It can be shown that the solution of this problem is indeed $x^{o}_{\gamma i}$ and that

$x^{o'}_{\gamma_i} A^{o} x^{o}_{\gamma_i} = \lambda^{o}_{i\gamma_i}$.

Thus it appears that, at any x^o, λ^o, each rate of change vector $x^{o}_{\gamma_i}$ is so selected as to maximize $\lambda^{o}_{i\gamma_i}$ over all feasible rates of change, $y^{o}_{\gamma_i}$. Quite similarly, each $x^{o}_{\alpha_i}$ or $x^{o}_{\beta_i}$ is so selected as to maximize $\varphi^{0}_{\alpha_i \alpha_i}$ or $\varphi^{0}_{\beta_i \beta_t}$.

Finally, it is certainly quite interesting that research has already been undertaken with the purpose of extending the scope of the Envelope Theorem. It is hoped that the present paper will lead to further research in this direction. It is important to realize that models involving quantity or point rationing, or some form of regulation, or any other types of additional constraints, can be related to a model with the smallest number of constraints. Then the Envelope Theorem will be activated and a lot of information about the former problems can be provided by the solution of the latter problem. The same is true for models related as those of the competitive and the illyrian firm. In simple cases, like those referred to in §5, the solution of one of the extremum problems can be expressed completely in terms of the solution of the other. But even when this is not exactly possible, the ensuring benefits are obvious.