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Abstract

In this paper we are concerned with the issue of the existence of locally uniform Edgeworth expansions for the distributions of random vectors. Our motivation resides on the fact that this could enable subsequent uniform approximations of analogous moments and their derivatives. We derive sufficient conditions either in the case of stochastic processes exhibiting weak dependence, or in the case of smooth transformations of such expansions. The combination of the results can lead to the establishment of high order asymptotic properties for estimators of interest.

KEYWORDS: Locally uniform Edgeworth expansion, formal Edgeworth distribution, weak dependence, smooth transformations, moment approximations, GMM estimators, Indirect estimators, GARCH model.

JEL: C10, C13

1 Introduction

In this paper we are concerned with the issue of the approximation of the distributions of a sequence of random vectors by sequences of Edgeworth distributions *uniformly* with respect to a compact valued Euclidean parameter. Our motivation resides on the fact that this could enable subsequent uniform approximations of analogous moments and their derivatives with respect to the aforementioned parameter. This in turn can facilitate the extraction of higher order asymptotic properties of estimators that are defined by the use of such moments. A prominent example is the indirect estimator defined by Gourieroux et al. [9] (abbreviated as GMR2 estimator in Arvanitis and Demos [1], definition D.3.) as a minimizer of a criterion involving the expectation of an auxiliary estimator.

We will hereafter refer to the aforementioned approximation as a locally uniform Edgeworth expansion of the involved random vectors. We notice that analogous expansions have been studied by Bhattacharya and Ghosh [2] (see Theorem 3) in the i.i.d. case and Durbin [6] for the case where the random vectors are of the form of $\sqrt{n}$ times an arithmetic mean.

In what follows we will provide sufficient conditions for the *existence* of such an approximation in two cases. The first concerns the one where the random vectors are of the form of $\sqrt{n}$ times an arithmetic mean, the elements of which are members of a stochastic process exhibiting weak dependence, in the spirit of Gotze and Hipp [8]. There, the authros validate the pointwise (w.r.t. the parameter) *formal* Edgeworth expansions. We essentially follow their line of reasoning, whereby by strengthening their conditions we establish the result ensuring that the relevant remainders are independent of the parameter. In the second case we assume that a locally uniform Edgeworth expansion is valid, and given a sequence of smooth transformations for the random vector at hand, we provide sufficient conditions for an analogous expansion to exist for the *transformed* random vector. In this case our line of reasoning is close to the one in Skovgaard [13], but compared to this paper we utilize additional conditions concerning the *dependence* of the transformations on the parameter. Obviously the two cases can be combined for the establishment of valid locally uniform Edgeworth expansions in composite cases.

The structure of the paper is as follows. In the next two sections we are concerned with the aforementioned cases repsectively. In the fourth section we provide a simple example concerning a GARCH model involving estimators for the asymptotic analysis of which we utilize all the previous results. In the final section we conclude.

2 Valid Locally Uniform Formal Edgeworth Expansions Under Weak Dependence

In the following we denote with Θ a *compact* subset of $\mathbb{R}^p$ (w.r.t. the usual topology). The following assumption defines the form of the eligible stochastic processes for the results that follow.

Assumption A.1 *Let* (ε_j) *be a sequence of* iid *random variables,* $g : \mathbb{R}^{\mathbb{N}} \times \Theta \to \mathbb{R}$ *and* $h : \mathbb{R}^p \times \Theta \to \mathbb{R}^k$ *be Borel (jointly) measurable functions and suppose that* h *has uniformly equicontinuous (w.r.t.* Θ*) first order derivative w.r.t.* $(Z_j, \ldots, Z_{j+p-1})$*. Define for* $j \in \mathbb{Z}$*,* $\theta \in \Theta$

$$\begin{aligned} Z_j &:= g\left(\varepsilon_{j-i} : i \geq 0, \theta\right) \\ X_j &:= h\left(Z_j, \ldots, Z_{j+p-1}, \theta\right) \end{aligned} \tag{1}$$

where $\sup_{\theta \in \Theta} E \left\| D_Z h\left(Z_i, \ldots, Z_{i+p-1}, \theta\right)\right\| \leq K$ *for some constant* $K > 0$*.*

We supress the dependence of Z_j and X_j on θ for notational simplicity. Now, let $S_n(\theta) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(X_i(\theta) - EX_1(\theta)\right)$, for $r = 0, \ldots, s$ let $\chi_{r,n}(t)$ be the cumulants of $t^T S_n$ of order r, i.e.

$$\chi_{r,n}(t) = \left. \frac{d^r}{dx^r} \log E \exp\left(ixt^T S_n\right) \right|_{x=0}$$

Obviously $\chi_{r,n}$ depend on θ. Let $\Psi_{n,s}(t)$ be the *formal* Edgeworth measure of S_n of order $s-2$, $s \geq 3$, defined by its characteristic function $\widehat{\Psi}_{n,s}(t) = \exp\left(\chi_{2,s}\right) + \sum_{r=1}^{s-2} n^{-r/2} \widetilde{P}_{r,n}(t)$, where the functions $\widetilde{P}_{r,n}(t)$, $r = 1, 2, \ldots$ satisfy the formal identity

$$\exp\left(\chi_{2,n} + \sum_{r=3}^{\infty} \frac{1}{r!} \tau^{r-2} n^{(r-2)/2} \chi_{r,n}(t)\right) = \exp\left(\chi_{2,s}\right) + \sum_{r=1}^{\infty} \tau^r \widetilde{P}_{r,n}(t)$$

and $\mathcal{B}_c$ the collection of convex Borel set of $\mathbb{R}^k$.

Question Given A.1, under what conditions

$$\sup_{\theta \in \Theta} \sup_{A \in \mathcal{B}_c} \left| P\left(S_n(\theta) \in A\right) - \Psi_{n,s}(\theta)(A) \right| = o\left(n^{\frac{s-2}{2}}\right) \tag{2}$$

The next assumption provides with sufficient conditions so that the previous question is well-posed and has an affirmative answer. It essentialy corresponds to a uniform extension of the analogous conditions (2)-(4) in Gotze and Hipp [8]. The proof of sufficiency folows naturally the line of proof of Theorem 1.1 of Gotze and Hipp [8], by establishing that due to A.2 the terms appearing in the relevant bounds are independent of θ.

Assumption A.2 *Let the following conditions hold:*
-M ***(Existence of Moments)***

$$\sup_{\theta \in \Theta} E \left\| X_1 \right\|^{s+1} \leq \beta_{s+1}$$

-WD (***Weak Dependence***) *There exist constants* $K < \infty$ *and* $\alpha > 0$ *independent of* θ *such that for* $m \geq 1$*,*

$$E \left\| g \left(\varepsilon_j : j \geq 0, \theta \right) - g \left(\varepsilon_0, \ldots, \varepsilon_m, 0, \ldots, \theta \right) \right\| \leq K \exp \left(-\alpha m \right)$$

-SM (***Smoothness***) *There exist* $\alpha > 0$ *independent of* θ *and* $r > 0$ *such that for arbitrary large fixed* $\kappa > 1$ *and all* $n > m > \alpha^{-1}$ *and* $t \in \mathbb{R}^k$ *with* $n^\kappa > \|t\| > \alpha$*,*

$$E \left| E \left(\exp \left(\sqrt{-1} t' \left(X_0 + \ldots + X_{2m} \right) \right) / \varepsilon_j : |j - m| \geq r \right) \right| \leq \exp \left(-\alpha \right)$$

The next theorem provides the required result.

Theorem 2.1 *If assumptions A.1 and A.2 are valid then 2 holds.*

Auxiliary Lemmas For the proof of the previous theorem we will need the following auxiliary results. First, we denote with $\mathcal{D}_{j+p-1-m,j+p+1}$ the σ-algebra generated by $\left(\varepsilon_{j+p-1-m}, \ldots, \varepsilon_{j+p+1} \right)$ which is obviously independent of θ and satisfies conditions (2.4) and (2.6) of [7] for any m, n, p due to the definition of $\left(\varepsilon_j \right)$ in assumption A.1. The first auxiliary result is the uniform extension of Lemma 2.1 of [8].

Lemma 2.2 *Under assumptions A.1, A.2.*M *and A.2.*WD *there exist a constant* K_1 *independent of* θ *and a (for any* $\theta \in \Theta$*)* $\mathcal{D}_{j+p-1-m,j+p+1}$*-measurable random element* X_j^* *such that*

$$\sup_{\theta \in \Theta} E \left\| X_j - X_j^* \right\| \leq K_1 \exp \left[-\alpha \frac{s}{3 \left(s + 1 \right)} m \right] \tag{3}$$

Furthermore under A.1 and A.2 then

$$\inf_{\theta \in \Theta} \liminf_n \inf_{\|t\| = \alpha} \operatorname{var} \left(t' S_n \left(\theta \right) \right) > 0 \tag{4}$$

Remark R.1 *It is easy to see that 4 implies that* $\inf_{\theta \in \Theta} \liminf_n \lambda_n^{\min} \left(\theta \right) > 0$ *where* $\lambda_n^{\min} \left(\theta \right)$ *denotes the minimum eigenvalue of* $\operatorname{var} \left(S_n \left(\theta \right) \right)$ *implying that they are uniformly positive definite. Suppose the contrary, i.e. there exists* $x \neq 0_k$ *for which* $\inf_{\theta \in \Theta} \liminf_n x' \operatorname{var} \left(S_n \left(\theta \right) \right) x = 0$*. Then let* $z = \frac{a}{\|x\|} x$ *and* $0 = \frac{\|x\|^2}{a^2} \inf_{\theta \in \Theta} \liminf_n \operatorname{var} \left(z' S_n \left(\theta \right) \right)$ *which is impossible due to 4.*

Proof of Lemma 2.2. Let g_0 be the composition of $h \left(\cdot, \theta \right)$ and $g \left(\cdot, \theta \right)$ such that $X_j = g_0 \left(\varepsilon_{j+p-l} : l \geq 0, \theta \right)$. For $m > p - 1$ let

$$X_j' \left(\theta \right) := g_0 \left(\varepsilon_{j+p-1}, \ldots, \varepsilon_{j-m+p-1}, 0, \ldots, \theta \right)$$

Define the obviously $\mathcal{D}_{j+p-1-m,j+p+1}$-measurable random element $X_j^*(\theta)$

$$X_j^*(\theta) = X_j'(\theta)\, I\left(\left\|X_j'(\theta)\right\| \le a_m^{\frac{1}{s+1}}\right),$$

where $a_m = \exp\left(\frac{\alpha m}{3}\right)$. Further, let B_m be the set of sequences of ε_j such that

$$\sup_{\theta\in\Theta} \|D_Z h(Z_j,\ldots,Z_{j+p-1},\theta)\| \le K a_m$$

and

$$\sup_{\theta\in\Theta} \|Z_{j+v} - g(\varepsilon_{j+v},\ldots,\varepsilon_{j+v-m},0,\ldots,\theta)\| \le K\exp\left[\frac{\alpha m}{3} - \alpha(m-v)\right].$$

By the assumption of uniform equicontinuity, there exist m_0 such that for $m \ge m_0$ and ε_j, $j\in\mathbb{Z}$, in B_m, $\sup_{\theta\in\Theta}\|D_Z h\| \le K\exp\left(\frac{\alpha m}{3}\right)$ on the segment connecting the Z vectors of the infinite and truncated (at m) sequence of ε_j for any $\theta\in\Theta$. Now,

$$\begin{aligned}
& E\left\|X_j - X_j^*\right\| \\
\le\ & E\|X_j\|\,1_{\|X_j\|>a_m^{1/(s+1)}} + E\left\|X_j^*\right\|1_{\|X_j\|>a_m^{1/(s+1)}} + E\left\|X_j - X_j^*\right\|1_{\|X_j\|\le a_m^{1/(s+1)}} \\
\le\ & 2E\|X_j\|^{s+1} a_m^{-s/(s+1)} + E\left\|X_j - X_j^*\right\|1_{\|X_j\|\le a_m^{1/(s+1)}}
\end{aligned}$$

since $\|X_j\| > a_m^{1/(s+1)} \Leftrightarrow \|X_j\| < \|X_j\|^{s+1} a_m^{-s/(s+1)}$ and $P\left(\|X_j\| > a_m^{1/(s+1)}\right) \le E\|X_j\|^{s+1} a_m^{-(s+1)/(s+1)}$ due to Markov's inequality. Also

$$E\left\|X_j - X_j^*\right\|1_{\|X_j\|\le a_m^{1/(s+1)}} = E\left\|X_j - X_j^*\right\|1_{\|X_j\|\le a_m^{1/(s+1)}}1_{B_m} + E\left\|X_j - X_j^*\right\|1_{\|X_j\|\le a_m^{1/(s+1)}}1_{B_m^c}$$

and

$$\begin{aligned}
B_m^c \subseteq\ & \left\{(\varepsilon) : \sup_{\theta\in\Theta}\|Dh(Z_j,\ldots,Z_{j+p-1},\theta)\| > Ka_m\right\} \\
& \cup\left\{(\varepsilon) : \|Z_{t+p} - g(\varepsilon_{j+p},\ldots,\varepsilon_{j+p-m},0,\ldots,\theta)\| > Ka_m\exp(-a(m-p))\right\}
\end{aligned}$$

Due to 1 and the fact that $\alpha > 0$ we have that for any m

$$P\left(\left\{(\varepsilon) : \sup_{\theta\in\Theta}\|Dh(X_j,\ldots,X_{j+p})\| > Ka_m\right\}\right) = 0$$

and that due to A.2.WD and the inequality of Markov

$$\begin{aligned}
& P\left(\|Z_{t+p} - g(\varepsilon_{j+p},\ldots,\varepsilon_{j+p-m},0,\ldots,\theta)\| > Ka_m\exp(-\alpha(m-p))\right) \\
\le\ & \frac{E\|Z_{t+p} - g(\varepsilon_{j+p},\ldots,\varepsilon_{j+p-m},0,\ldots,\theta)\|}{Ka_m\exp(-a(m-p))} \le \frac{K\exp(-\alpha m)}{Ka_m\exp(-\alpha(m-p))} = \exp(-\alpha p)\, a_m^{-\frac{s+1}{s+1}}
\end{aligned}$$

Hence

$$P\left(B_m^c\right) \leq \exp\left(-\alpha p\right) a_m^{-\frac{s+1}{s+1}}$$

and therefore

$$E\left\|X_j - X_j^*\right\| 1_{\|X_j\| \leq a_m^{1/(s+1)}} 1_{B_m^c} \leq 2\exp\left(-\alpha p\right) a_m^{1/(s+1)} a_m^{-\frac{s+1}{s+1}} = 2\exp\left(-\alpha p\right) a_m^{-\frac{s}{s+1}}$$

Finally due to the Mean Value Theorem and the fact that $1_{\|X_j\| \leq a_m^{1/(s+1)}} 1_{B_m} = 1$ means that $1_{\left\|X_j'\right\| \leq a_m^{1/(s+1)}} = 1$

$$\begin{aligned} & E\left\|X_j - X_j^*\right\| 1_{\|X_j\| \leq a_m^{1/(s+1)}} 1_{B_m} \\ \leq\ & K a_m E \left\| \begin{array}{c} Z_j - g\left(\varepsilon_j, \ldots, \varepsilon_{j-m}, 0, \ldots, \theta\right) \\ \vdots \\ Z_{j+p-1} - g\left(\varepsilon_{j+p}, \ldots, \varepsilon_{j+p-m}, 0, \ldots, \theta\right) \end{array} \right\| 1_{B_m} \\ \leq\ & C K a_m p K \sum\nolimits_{i=1}^{p} \exp\left(i\alpha\right) \exp\left(-\alpha m\right) a_m P\left(B_m\right) = C^* a_m^{-1} \end{aligned}$$

for $C^* = pCK^2 \frac{\exp(p\alpha) - \exp(\alpha)}{\exp(\alpha) - 1} P\left(B_m\right) > 0$ and independent of θ. Hence

$$\begin{aligned} & E\left\|X_j - X_j^*\right\| \\ \leq\ & 2E\left\|X_j\right\|^{s+1} a_m^{-\frac{s}{s+1}} + 2\exp\left(-\alpha p\right) a_m^{-\frac{s}{s+1}} + C^* a_m^{-1} \\ =\ & K_1 a_m^{-\frac{s}{s+1}}, \end{aligned}$$

where $K_1 = \left(2E\left\|X_j\right\|^{s+1} + 2\exp\left(-\alpha p\right) + C^* a_m^{-\frac{1}{s+1}}\right)$ independent of θ.

Now let $\gamma = \frac{s-1}{2s}$ and $C_{s,\alpha} = \frac{1}{1-\exp(-2\gamma)} \sqrt{\frac{1-\exp(-2\alpha)}{2}}$ and $q = r+1+\left\lfloor \frac{1}{\gamma\alpha} \log\left(K_1 \alpha^{s+3} C_{s,\alpha} E^{\frac{1}{s}} \left\|X_1\right\|^{s+1}\right)\right\rfloor$, where $\lfloor x \rfloor$ denotes the integral part of x. Notice that all constants, γ, $C_{s,\alpha}$ and q, are independent of θ. For $n \geq 2q$ let $\mathcal{D}_j$ denote the σ-algebra generated by ε_l, $l \leq jq+p-1$, and define

$$\Delta_j = n^{1/2} t' \left[E\left(S_n | \mathcal{D}_j\right) - E\left(S_n | \mathcal{D}_{j-1}\right)\right].$$

Writing $n = Lq + N$ with $N < q$, the following variance decomposition holds:

$$\operatorname{var}\left(t' S_n\right) = n^{-1} \sum_{j=1}^{L} E\Delta_j^2 + E\left[t'\left(S_n - E\left(S_n | \varepsilon_l, l \leq n+p-1-N\right)\right)\right]^2.$$

Since $X_v, v \leq jq$, are $\mathcal{D}_j$-measurable for any $\theta \in \Theta$, we obtain

$$\begin{aligned} \Delta_j &= t'\left[\sum_{v=jq-q+1}^{jq-1} \left(X_v - E\left(X_v | \mathcal{F}_{j-1}\right)\right) + \sum_{v=jq}^{n} \left(E\left(X_v | \mathcal{F}_j\right) - E\left(X_v | \mathcal{F}_{j-1}\right)\right)\right] \\ &\doteqdot V_j + R_j \end{aligned}$$

Define $\varepsilon_{v,M} = (\varepsilon_{v+p-1}, \varepsilon_{v+p-2}, \ldots, \varepsilon_{Mq+p-1}, 0, \ldots)$ for $v \geq Mq$ and $\varepsilon_v = (\varepsilon_{v+p-l} : l \geq 1)$. We have that

$$E^{1/2}\Delta_j^2 \geq E^{1/2}V_j^2 - E^{1/2}R_j^2$$

and emloying Holder's inequality

$$E^{1/2}R_j^2 \leq C\alpha^2 K \left(1 - \exp\left(-\alpha\gamma\right)\right)^{-1} \exp\left(-\alpha\gamma q\right),$$

where $C = E^{1/s} \left|t' X_1\right|^{s+1}$ independent of θ. By definition of q we get

$$ER_j^2 \leq \frac{1}{2}\left(1 - \exp\left(-2\alpha\gamma\right)\right)$$

independent of θ. On the other hand the inequality $x^2/2 \geq 2\sin^2(x/2) = 1 - \cos(x)$ together with

$$\operatorname{var}(Z) = \frac{1}{2}E\left(Z - Z^*\right)^2 \geq 1 - \left|E\exp\left(iZ\right)\right|^2$$

for any r.v. Z and an independent copy, say Z^*, as well as condition A.2.SM with $\|t\| = \alpha$ we get

$$EV_j^2 \geq 1 - \exp\left(-2\alpha\right)$$

which concludes the proof of equation 4, as all bounds are independent of θ. ■

Hence, the following corollary.

Corollary 2.3 *Under assumption A.2 conditions (2.2)-(2.6) of Gotze and Hipp [7] hold with constants independent of θ.*

Proof of Corollary 2.3. By assumption A.2 conditions (2.2), (2.4), (2.5) and (2.6) of Gotze and Hipp [7] obviously hold with constants independent of θ. Condition of (2.3) of Gotze and Hipp [7] follows from lemma 2.2. ■

Now we need to show how the uniform versions of conditions (2.2)-(2.6) of Gotze and Hipp [7] imply intermediate results that lead to the proof of therem 2.1. Again these are uniform extensions of the analogous results in Gotze and Hipp [7]. We shall employ the following notation. Define

$$T(x) = \begin{cases} x & \text{if } \|x\| \leq n^{\beta} \\ \frac{x n^{\beta}}{\|x\|}\psi\left(\|x\| n^{-\beta}\right) & \text{otherwise} \end{cases}$$

where $\psi \in C^{\infty}(0, \infty)$ independent of θ, satisfying

$$\begin{aligned} \psi(r) &= r \quad \text{if } r \leq 1 \\ & \psi \quad \text{is increasing} \\ \psi(r) &= 2 \quad \text{if } r \geq 2. \end{aligned}$$

For $j=1,...,n$ let $Y_j = T(X_j)$ and $Z_j^* = Y_j - E(Z_j)$. Define $S_n^* = n^{-1/2}(Z_1^* + ... + Z_n^*)$ and $H_n(t) = E\exp\left(it^T S_n^*\right)$. Notice that $\widetilde{P}_{r,n}$, Y_j, Z_j^*, S_n^* and H_n depend on θ.

Let

$$E_t U = EU\exp\left(it^T S_n^*\right)/H_n(t)$$

and define the cumulant of order p

$$\kappa_t\left(a_1^T S_n^*, ..., a_p^T S_n^*\right) = \frac{\partial}{\partial\varepsilon_1}...\frac{\partial}{\partial\varepsilon_p}\bigg|_{\varepsilon_1=...=\varepsilon_p=0} \ln H_n\left(t+\varepsilon_1 a_1 + .. + \varepsilon_p a_p\right),$$

where $a_1, ..., a_p \in \mathbb{R}^k$. Write

$$\kappa_t\left(\underbrace{a^T S_n^*, ..., a^T S_n^*}_{j-times}, \underbrace{b^T S_n^*, ..., b^T S_n^*}_{l-times}\right) = \kappa_t\left(a^T S_n^{*j}, b^T S_n^{*l}\right),$$

the Taylor expansion of $\ln H_n(t)$ can be written

$$\begin{aligned} \ln H_n(t) &= \sum_{r=2}^{s}\frac{1}{r!}\kappa_0\left(it^T S_n^{*r}\right) + R_{s+1}(t) \quad \text{where} \qquad (5) \\ R_{s+1}(t) &= \frac{1}{s!}\int_0^1 (1-\eta)^s \kappa_{\eta t}\left(it^T S_n^{*(s+1)}\right) d\eta \end{aligned}$$

Lemma 2.4 *Under assumption A.2 lemma (3.33) of Gotze and Hipp [7] holds with constant c independent of* θ, *i.e. for every t with* $\|t\| < cn^{\bar{\varepsilon}}$, *we have that*

$$\begin{aligned} &\left|D^\alpha\left(H_n(t) - \widehat{\Psi}_{n,s}(t)\right)\right| \\ \leq\ & c(r,d,s,|\alpha|)\left(1+\beta_{s+1}\right)\left(1+\|t\|^{3(s-1)+|\alpha|}\right)\exp\left(-c(d)\|t\|^2\right) n^{-(s-2)/2-\bar{\varepsilon}} \end{aligned}$$

Proof of Lemma 2.4. From Lemma (3.28) of Gotze and Hipp [7] we have that for $2 \leq r \leq s$

$$\left|\kappa_0\left(a_1^T S_n^*, ..., a_r^T S_n^*\right)\right| \leq c(r,d,s)\, n^{-(r-2)/2}\beta_{s+1}^{r/(s+1)}\|a_1\| ... \|a_r\|$$

where c^* depents on r and d but not on θ. Now for $\|t\| \leq cn^{\varepsilon}$ we have that

$$\left|D^\alpha \sum_{r=3}^{s}\frac{\kappa_0\left(it^T S_n^{*r}\right)}{r!}\right| = \begin{cases} 1 & \text{for } s \leq |\alpha| \\ n^{-(|\alpha|-2)/2} & \text{for } 3 \leq |\alpha| < s \\ n^{-1/2}\|t\|^{3-|\alpha|} & \text{for } |\alpha| < 3 \end{cases}.$$

Now using equation 5 we get

$$\begin{aligned}\Delta &= D^{\alpha}\left(H_n(t)-\exp\left[\sum_{r=3}^{s}\frac{\kappa_0\left(it^TS_n^{*r}\right)}{r!}\right]\right)\\ &= \sum_{\alpha_1+\alpha_2=\alpha} c_{\alpha_1\alpha_2}D^{\alpha_1}\exp\left[\sum_{r=3}^{s}\frac{\kappa_0\left(it^TS_n^{*r}\right)}{r!}\right]D^{\alpha_2}\left(\exp\left[R_{s+1}(t)\right]-1\right).\end{aligned}$$

Now from Lemma (3.20) of Gotze and Hipp [7] we have that

$$\left|\left.\frac{\partial^l}{\partial\varepsilon^l}\right|_{\varepsilon=0}R_{s+1}(t+\varepsilon a)\right|\leq c(r,d)\left(1+\beta_{s+1}\right)n^{-\frac{s-2}{2}-\varepsilon^*}\left(1+\theta_n(t)^{s+1+l}\right)\left(1+\|t\|^{s+1}\right)$$

where $0<\varepsilon^*<1$, for every t satisfying

$$\theta_n(t)=\left(v_m^{(n)}(t)+\exp\left(-c(d)\,n^{\varepsilon^*/2}\right)\right)/H_n(t)<\infty,\qquad \|t\|\leq\varepsilon^*n^{-\varepsilon^*-\beta+1/2} \tag{6}$$

where $a\in\mathbb{R}^k$ with $\|a\|<1$ and

$$v_m^{(n)}(t)=\sup\left\{\left|E\exp\left(S_I^{(p)}\right)\right|:p\leq m,|I|\leq r\right\}$$

where l such that $s+1\leq r\leq s+1+l$, $S_I^{(p)}=in^{-1/2}t^T\sum^* Z_j$ and $\sum^*$ extends over all $1\leq j\leq n$ such that $|j-j_1|>mr$. Hence we get that

$$|\Delta|\leq c(r)\left(1+\|t\|^{|\alpha|}\right)\|t\|^{s+1}\left(1+\beta_{s+1}\right)n^{-(s-1)/2-\varepsilon^*}\exp\left(-c\|t\|^2\right)\left(1+\theta_n(t)^{|\alpha|}\right)$$

Now for complex $a_i\in\mathbb{C}^k$ with $\|\operatorname{Im}a_i\|<\eta$ from Lemma (3.30) of Gotze and Hipp [7] we have that

$$\begin{aligned}&\left|\kappa_0\left(a_1^TS_n^*,...,a_r^TS_n^*\right)-\kappa_0\left(a_1^TS_n,...,a_r^TS_n\right)\right| \qquad (7)\\ \leq\;& c(r)\,n^{-(s-2)/2-\varepsilon^{**}}\beta_{s+1}^{r/(s+1)}\|a_1\|...\|a_r\|,\quad \text{for } 1\leq r\leq s \text{ and } 0<\varepsilon^{**}<1\end{aligned}$$

Hence we have that

$$\begin{aligned}\exp\left(\sum_{r=2}^{s}\kappa_0\left(\frac{i}{r!}t^TS_n^{*r}\right)\right) &= \exp\left[o\left(n^{-(s-1)/2+\varepsilon^{**}}\|t\|\right)\right]\left(\widehat{\Psi}_{n,s}(t)+\overline{R}_s(t)\right)\\ &= \widehat{\Psi}_{n,s}(t)+\widetilde{R}_s(t)\end{aligned}$$

where

$$\overline{R}_s(t)\leq c(r,d,s)\,n^{-(s-1)/2}\left(1+\beta_{s+1}\right)\left(\|t\|^{s+1}+\|t\|^{3(s-1)}\right)\exp\left(-c_2\|\operatorname{Re}t\|^2+c_3\eta^2\right)$$

Since $\widehat{\Psi}_{n,s}(t)$ and $\widetilde{R}_s(t)$ are analytic in t, the Cauchy's inequalities can be employed to estimate the derivatives $D^\alpha \widetilde{R}_s(t)$ as

$$\max\left\{\left|\widetilde{R}_s(t)\right| : \|z_j - t_j\| = \eta,\ z \in \mathbb{C}^k,\ j = 1, ..., k\right\} \eta^{-|\alpha|}$$

Noting that due to A.2 and remark R.1 $\left|\widehat{\Psi}_{n,s}(t)\right| \le c \exp\left(-c \|\mathrm{Re}(t)\|^2\right)$, for some $c > 0$ independent of θ, $\|\mathrm{Im}(t)\| \le \eta$ and $\|\mathrm{Re}(t)\| \le cn^{\varepsilon^{**}}$ we get

$$\left|D^\alpha \widetilde{R}_s(t)\right| \le c(r,d,s)\, n^{-(s-2)/2-\varepsilon^{**}} \left(1 + \|t\|^{3(s-1)}\right) \exp\left(-c\|t\|^2\right)$$

for every $t \in \mathbb{R}^k$ with $\|t\| \le cn^{\varepsilon^{**}}$. Hence

$$\begin{aligned} &\left|D^\alpha \left(H_n(t) - \widehat{\Psi}_{n,s}(t)\right)\right| \\ \le\ & c(r,d,s)\left(1 + \beta_{s+1}\right)\left(1 + \|t\|^{3(s-1)+|\alpha|}\right)\left(1 + \theta_n(t)^{|\alpha|+1}\right) \exp\left(-c\|t\|^2\right) n^{-(s-2)/2-\bar{\varepsilon}} \end{aligned}$$

Now for $\theta_n(t)$ in equation 6, let $T_l = n^{-1/2}\sum_{p=1}^{l} Z_{j_p}$, for $1 \le l \le n$ for any sequence $j_1 \le j_2 \le ... \le j_l$ and define $H(T_l, t) = E\exp\left(it^T T_l\right)$. Now equation 5, Lemma (3.20) and Lemma (3.28) of Gotze and Hipp [7] can be employed together to prove that for $1 \le l \le n$ and $\alpha = 0$:

$$H(T_l, t) = \exp\left[-\frac{1}{2}\kappa_0\left(t^T T_l^2 + cn^{-3/2-\bar{\varepsilon}}\|t\|^3 l\theta_{l,3}(t)\right)\right]$$

where

$$|\theta_{l,3}(t)| \le \sup\left\{\left|\left(\exp(-cn^{\varepsilon}) + H\left(T_{I,l}^{(p)}, t\right)\right) / H(T_l, t)\right|^3 : |I| \le 3, 0 \le p \le m, T_l\right\}$$

where $T_{I,l}^{(p)} = in^{-1/2} t^T \sum^* Z_j$ where $\sum^*$ extends over all $1 \le j \le l$ such that $|j - j_1| > pm$ for every $j_1 \in I \subset \{1, 2, ..., l\}$. Notice that here ε, c and m are independent of l, $1 \le l \le n$. The claim is that

$$\sup\left\{|\theta_{l,3}(t)| : \|t\| \le n^{\bar{\varepsilon}}\right\} \le 2 \quad for\ l = 1, 2, ..., n \tag{8}$$

provided that n is sufficiently large (depending on s, k, d, β_{s+1}) but not on θ. For $l = 1$ the above inequality holds trivially. Suppose that it is true for $1 \le l \le r$. Assuming that it does not hold for $l = r + 1$, i.e. there exists a t_0, such that for $\|t_0\| \le n^{\bar{\varepsilon}}$ and $|\theta_{r+1,3}(t_0)| = 2$, we have that

$$2 \le \sup_{I,p}\left|\left(\exp\left(-cn^{2\bar{\varepsilon}}\right) + H\left(T_{I,r+1}^{(p)}, t_0\right)\right) / H(T_{r+1}, t_0)\right|^3$$

Substituting out $H\left(T^{(p)}_{I,r+1}, t_0\right)$ and $H\left(T_{r+1}, t_0\right)$ and employing $\left|\theta_{r+1,3}\left(t_0\right)\right| = 2$, we get that, for large n,

$$2 \leq \sup_{I,r} \left| \exp\left(-cn^{2\bar{\varepsilon}}\right) + \exp\left[\frac{1}{2}\kappa_0\left(t^T T^2_{r+1}\right) - \frac{1}{2}\kappa_0\left(t^T T^{(p)2}_{r+1}\right) - 4cn^{-3/2-\bar{\varepsilon}}\left(r+1\right)\|t\|^3\right]\right|^3 \tag{9}$$

Since

$$\begin{aligned} \left|\kappa_0\left(t^T T^2_{r+1}\right) - \kappa_0\left(t^T T^{(p)2}_{r+1}\right)\right| &\leq c_2 \|t\|^2 E\left\|T_{r+1} - T^{(p)}_{r+1}\right\| \left\|T_{r+1} - T^{(p)}_{r+1}\right\| \\ &\leq c\|t\|^2 mn^{-1/2} \leq n^{\bar{\varepsilon}-1/2}, \quad \bar{\varepsilon} < \frac{1}{2}. \end{aligned}$$

However, for large n, this contradicts equation 9, proving that equation 8 is true for $l = r+1$ and completing its proof. Hence,

$$\left|\theta_n\left(t\right)\right| \leq 2^{(s+|\alpha|)/3} \quad \text{for every } \|t\| \leq n^{\bar{\varepsilon}}$$

completing the proof of inequality of the Lemma. Furthermore, notice that all the employed constants depend on s, the order of the cumulants r, d, and the order of the derivative α and in any case are independent of θ. ■

Lemma 2.5 *Under the assumptions of the previous Lemma, Lemma (3.3) of Gotze and Hipp [7] holds with constant c and $o\left(n^{\frac{s-2+\delta}{2}}\right)$ independent of θ, i.e. for $f : \mathbb{R}^k \to \mathbb{R}$ such that $|f(x)| \leq M\left(1+\|x\|^{s_0}\right)$ for every $x \in \mathbb{R}^k$ and M independent of θ. Then for $\kappa > 0$*

$$\begin{aligned} & \sup_{\theta\in\Theta}\left|Ef\left(S_n\right) - \int f d\Psi_{n,s}\right| \\ \leq\ & c\left(k,s,\beta_{s+1}\right) M \sup_{|\alpha|\leq k+1+s_0} \sup_{\theta\in\Theta} \int \left|D^\alpha\left[\left(H_n\left(t\right) - \widehat{\Psi}_{n,s}\left(t\right)\right)\widehat{K}\left(n^{-\kappa}t\right)\exp\left(it^T e_n\right)\right]\right| dt \\ & + c\left(k,s,\beta_{s+1}\right)\sup_{\theta\in\Theta}\omega\left(g : n^{-\kappa}\right) + o\left(n^{-\frac{s-2+\delta}{2}}\right) \end{aligned}$$

where

$$\begin{aligned} \delta &> 0,\ g\left(x\right) = f\left(x\right)/\left(1+\|x\|^{s_0}\right), \\ \omega\left(g : n^{-\kappa}\right) &= \int \sup\left\{\left|g\left(x+y\right) - g\left(x\right)\right| : \|y\| \leq n^{-\kappa}\right\}\Phi_{\Sigma_n}\left(dx\right), \\ \Sigma_n &= \operatorname{var}\left(S_n\right) \end{aligned}$$

and $\widehat{K}$ is a continuous function with compact support independent of θ.

Proof of Lemma 2.5. Following the proof in Gotze and Hipp [7], define $S_n^{/} = n^{-1/2} \sum_1^n Y_j$ and letting $\overline{\varepsilon} > 0$, to be defined later, define

$$A = \left\{ \sup_{\theta \in \Theta} \|S_n\| \leq n^{\varepsilon} \right\}, \qquad B = \left\{ \sup_{\theta \in \Theta} \left\|S_n^{/}\right\| \leq n^{\varepsilon} \right\}.$$

Then employing that $(x+y)^n \leq 2^n (x^n + y^n)$, n, x, $y > 0$ we get

$$\begin{aligned} & \sup_{\theta \in \Theta} E \left\| S_n - S_n^{/} \right\|^{s_0} \\ \leq \; & c(s) \left(\sup_{\theta \in \Theta} E \|S_n\|^{s_0} 1_{A^c} + \sup_{\theta \in \Theta} E \left\|S_n^{/}\right\|^{s_0} 1_{B^c} + n^{\overline{\varepsilon} s_0} \sup_{\theta \in \Theta} P \left\{ S_n \neq S_n^{/} \right\} \right). \end{aligned}$$

Furthermore, due to equation 7 we have that there exists $0 < \delta$ independent of θ, such that

$$\begin{aligned} & \sup_{\theta \in \Theta} E \|S_n\|^{s_0} 1_{A^c} \\ \leq \; & c\left(s, \beta_{s+1}\right) o\left(n^{-(s-2+\delta)/2}\right) + 2 n^{\overline{\varepsilon} s_0} \sup_{\theta \in \Theta} P\left\{S_n \neq S_n^{/}\right\} + \sup_{\theta \in \Theta} E \|S_n\|^{s_0} 1_{B^c} \end{aligned}$$

and consequently,

$$\begin{aligned} & E \left\| S_n - S_n^{/} \right\|^{s_0} \\ \leq \; & c\left(s, \beta_{s+1}\right) \left(\sup_{\theta \in \Theta} E \left\|S_n^{/}\right\|^{s_0} 1_{B^c} + n^{\overline{\varepsilon} s_0} \sup_{\theta \in \Theta} P\left\{S_n \neq S_n^{/}\right\} \right) + o\left(n^{-(s-2+\delta)/2}\right). \end{aligned}$$

Now Lemma 2.4 imply that for arbitrary positive integer r we have that $\sup_{\theta \in \Theta} \sup_n E \left\| S_n^{/} \right\|^r < \infty$ and consequently for $\overline{\varepsilon}$ such that

$$0 < \overline{\varepsilon} < (s+1) \beta_{s+1} - (s-2)/2$$

-hence independent of θ-we get that

$$E \left\| S_n - S_n^{/} \right\|^{s_0} \leq c\left(s, \beta_{s+1}\right) o\left(n^{-(s-2)/2}\right).$$

This along with the definition of f imply that

$$\left| Ef(S_n) - Ef\left(S_n^{/}\right) \right| \leq c\left(s, \beta_{s+1}\right) o\left(n^{-(s-2)/2}\right).$$

Notice that for

$$e_n = O\left(n^{1/2} n^{-s\beta_{s+1}}\right) = o\left(n^{-(s-2+\delta)/2}\right)$$

which is independent of θ and

$$\sup_{\theta\in\Theta}\left|\int f\left(\bullet, e_n\right) d\Psi_{n,s} - \int f d\Psi_{n,s}\right| = o\left(n^{-(s-2+\delta)/2}\right).$$

Now from Lemma 11.6 in Bhattacharya and Rao [3], by applying the Sweeting Smoothing Inequality (Lemma 5 of [14]) and by noting that this inequality involves constants that depend solely on the properties of g and therefore are independent of θ along with a similar choice of $\widehat{K}$ we get the result. ■

We are now ready to prove theorem 2.1.

Proof of Theorem 2.1. Consider the functions f and g as defined in Lemma 2.4. Then, from Lemma 2.4 and Lemma 2.5 and for $\kappa = (s-2+\delta)/2$ we obtain that

$$\left|Ef\left(S_n\right) - \int f d\Psi_{n,s}\right| \leq c\left(k, s, \beta_{s+1}\right)\sup_{\theta\in\Theta}\omega\left(g : n^{-(s-2+\delta)/2}\right) + c\left(k, s, \beta_{s+1}\right) Mo\left(n^{-\frac{s-2+\delta}{2}}\right)$$

Now for $f(x) = 1_C(x)\left(1+\|x\|^{s_0}\right)$,[1] where $C \in \mathcal{B}_C$, and $1_C(x)$ is its indicator function we have that

$$\sup_{y\leq n^{-k}}\left|1_C(x+y) - 1_C(x)\right| \leq 1_{(\partial C)^{n^{-k}}}(x)$$

for large enough n (where ∂ is the boundary operator and the superscript $\cdot^{n^{-k}}$ denotes the analogous enlargement), hence

$$\begin{aligned}
&\sup_{\theta\in\Theta}\omega\left(g : n^{-(s-2+\delta)/2}\right)\\
= &\sup_{\theta\in\Theta}\int\sup_{y}\left\{\left|g(x+y) - g(x)\right| : \|y\| \leq n^{-(s-2+\delta)/2}\right\}\Phi_{\Sigma_n}(dx)\\
\leq &\sup_{\theta\in\Theta}\int 1_{(\partial C)^{n^{-(s-2+\delta)/2}}}(x)\,\Phi_{\Sigma_n}(dx) = \sup_{\theta\in\Theta}\int 1_{\Sigma_n^{-1/2}(\partial C)^{n^{-(s-2+\delta)/2}}}(x)\,\Phi(dx)
\end{aligned}$$

and due to remark R.1 for Σ^* the diagonal matrix with elements consisting of the inverse of $\inf_{\theta\in\Theta}\liminf_n\sqrt{\lambda_n^{\min}(\theta)}$, due to the fact that ∂ and enlargement commutes with Σ^*, the linear transformation of a convex set is convex the we have that the last term is less than or equal to

$$\int 1_{(\partial(\Sigma^* C))^{n^{-(s-2+\delta)/2}}}(x)\,\Phi(dx) = o\left(n^{-(s-2)/2}\right)$$

due to Corollary 3.2 of Bhattacharya and Rao [3]. ■

[1] Notice that in this case $M = 1$ and therefore independent of the choice of C.

2.1 Sufficient Conditions For Smoothness

In this paragraph we provide uniform versions of conditions 2.3.(i)-(iii) of Gotze and Hipp [8] and prove that they imply assumption A.2.SM in the case that X_j in assumption A.1 is represented as

$$X_j = g\left(\varepsilon_{j-i} : i \in \mathbb{Z}, \theta\right), \qquad j \in \mathbb{Z} \tag{10}$$

(obviously g in the current representation could be the composition of $h(\cdot, \theta)$ with $g(\cdot, \theta)$ in the language of assumption A.1). Again the proof of sufficiency traces the arguments in the proof of Lemma 2.3 of Gotze and Hipp [8], establishing that the analogous bounds can be chosen independent of θ. Again we employ the auxiliary notation of the aforementioned paper as well as extend to uniform versions the auxiliary results. Hence, for $j \in \mathbb{Z}$ and $y \in \mathbb{R}^{\mathbb{Z}}$, $x \in \mathbb{R}$ let $(y, x)^j$ be the sequence with coordinates

$$\begin{cases} y_i, & i < j \\ x, & i = j \\ y_{i-1} & i > j. \end{cases}$$

Consider the following assumption.

Assumption A.3 *Let the following conditions hold:*
-EL *(**Exponential small Locally Lipschitz**) There exist $K < \infty$, $\eta > 0$ and $\alpha > 0$, not depending on θ, such that for $j \in \mathbb{Z}$ and $x_1 \in \mathbb{R}$, $x_2 \in \overline{\mathcal{O}}(x_1, \eta)$,*[2]

$$\sup_{\theta \in \Theta} E \left\| g\left((\varepsilon, x_1)^j, \theta \right) - g\left((\varepsilon, x_2)^j, \theta \right) \right\| \leq K \left| x_1 - x_2 \right| \exp\left(-\alpha \left| j \right|\right)$$

-CPD *(**Almost sure continuity of partial derivatives**) For $j \in \mathbb{Z}$ there exists $G_j \subset \mathbb{R}$, $P(G_j) = 1$ independent of θ, such that for all $x_0 \in G_j$, η, $\delta > 0$ there exists $\tau > 0$ independent of θ satisfying*

$$P\left\{ \begin{array}{c} y \in \mathbb{R}^{\mathbb{Z}} : \forall x \in \mathbb{R}, |x - x_0| < \tau, \frac{\partial}{\partial \varepsilon_0} X_j \text{ exists at the point } (y, x)^j \text{ and} \\ \sup_{\theta \in \Theta} \left| \frac{\partial}{\partial \varepsilon_0} X_j \left((y, x)^j, \theta \right) - \frac{\partial}{\partial \varepsilon_0} X_j \left((y, x_0)^j, \theta \right) \right| \leq \delta \end{array} \right\} \geq 1 - \eta$$

-NDD *(**Nondegenerate derivatives on a set of positive probability**) For some distinct $l_1, \ldots l_k \geq 0$ independent of θ,*

$$\inf_{\theta \in \Theta} \det \left(\sum_{j=0}^{\infty} \frac{\partial}{\partial \varepsilon_{l_\nu}} X_j : \nu = 1, \ldots, k \right) \neq 0$$

on a set of positive P-probability independent of θ.

[2] Notice that this is weaker than the condition 2.3.(i) in page 2073 of Gotze and Hipp [8].

The required result is the following.

Lemma 2.6 *Under 10, assumption A.3 and if ε_j admits a positive continuous density, then assumption A.2.*SM *holds for the sequence (X_j), $j \in \mathbb{Z}$.*

For its proof, we shall need the following lemma which is the uniform extension of lemma 2.2 of Gotze and Hipp [8].

Lemma 2.7 *Let $\mathcal{O}_r \subset \mathbb{R}^k$ denote an open ball with radius r and let $F : \mathbb{R}^k \times \Theta \to \mathbb{R}^k$ denote a measurable function that is injective and continuously differentiable function on $\mathcal{O}_r$ w.r.t. the first argument for any $\theta \in \Theta$, such that for constants $\eta > 0$ and $M < \infty$ that do not depend on θ and for all $x \in O$,*

$$\eta \leq \left|\det F^{/}(x,\theta)\right| \leq M \text{ and } \left\|F^{/}(x,\theta)\right\| \leq M.$$

where $F^{/}$ denotes the aforementioned derivative. Let h denote a density on $\mathbb{R}^k$ satisfying $h(x) \geq \eta$, $x \in \mathcal{O}_r$, and fix $\delta > 0$. Then there exists $\rho < 1$ depending only on η, δ, M and r such that for $t \in \mathbb{R}^k$ with $\|t\| \geq \delta$,

$$\sup_{\theta\in\Theta}\left|\int \exp\left[it^T F(x,\theta)\right] h(x)\,dx\right| \leq \rho(k,\delta,M,r).$$

Proof of Lemma 2.7. By a change of variables we have that

$$\int_{\mathcal{O}_r} \exp\left[it^T F(x,\theta)\right] h(x)\,dx = \int_{F_\theta(\mathcal{O}_r)} \exp\left[it^T u\right] \frac{h\left(F_\theta^{-1}(u)\right)}{\left|\det F^{/}\left(F_\theta^{-1}(u),\theta\right)\right|} du.$$

with $F_\theta(\mathcal{O}_r) = F(\mathcal{O}_r,\theta)$ and $F_\theta^{-1}(u) = F^{-1}(u,\theta)$. Now for $x \in O$ we have that $\inf_{\theta\in\Theta} \frac{h(x)}{\left|\det F^{/}(x,\theta)\right|} \geq \frac{\eta}{M}$ and therefore

$$\left|\int_{F_\theta(\mathcal{O}_r)} \exp\left[it^T u\right]\left(\frac{h\left(F_\theta^{-1}(u)\right)}{\left|\det F^{/}\left(F_\theta^{-1}(u),\theta\right)\right|} - \frac{\eta}{M}\right) du\right| \leq \int_{\mathcal{O}_r} h(x)\,dx - \int_{F_\theta(\mathcal{O}_r)} \frac{\eta}{M} du.$$

Fix $1 \leq j \leq k$ and $u_1, ..., u_{j-1}, u_{j+1}, ..., u_k$. Then $\{u_j : (u_1, ..., u_k) \in F_\theta(\mathcal{O}_r)\}$ is an interval with endpoints, say, $a(\theta) < b(\theta)$ and

$$\left|\int_{a(\theta)}^{b(\theta)} \exp\left[it^T u\right] du_j\right| = \left|\exp\left[i\sum_{m\neq j}^{k} t_m u_m\right] \frac{1}{it_j}\left(\exp\left[it_j b(\theta)\right] - \exp\left[it_j(\theta)\right]\right)\right| \leq \frac{2}{|t_j|}.$$

Let $A(\theta) = \{(u_1, ..., u_{j-1}, u_{j+1}, ..., u_k) : \exists u_j \in \mathbb{R} : (u_1, ..., u_k) \in F_\theta(\mathcal{O}_r)\}$. We have

$$\left|\int_{F_\theta(\mathcal{O}_r)} \exp\left[it^T u\right] du\right| \leq \frac{2}{|t_j|}\int_{A(\theta)} du_1 ... du_{j-1} du_{j+1} ... du_k.$$

Since now $A(\theta)$ is the projection of $F_\theta(\mathcal{O}_r)$ onto $\mathbb{R}^{k-1}$ and $F_\theta(\mathcal{O}_r)$ is contained in a ball of radius Mr, we get that

$$\left|\int_{F_\theta(\mathcal{O}_r)} \exp\left[it^T u\right] du\right| \leq \frac{2}{|t_j|}(2Mr)^{k-1}.$$

Since $1 \leq j \leq k$ was arbitrary, we can find $\xi > 0$ depending on k, η, M and r only such that for $t \in \mathbb{R}^k$ and $\|t\| \geq \xi$,

$$\left|\int_{F_\theta(\mathcal{O}_r)} \exp\left[it^T u\right] du\right| \leq \frac{1}{2}\int_{F_\theta(\mathcal{O}_r)} dx.$$

Hence for these t,

$$\begin{aligned} & \left|\int \exp\left[it^T F(x,\theta)\right] h(x)\, dx\right| \\ \leq & \int_{\mathcal{O}_r^c} h(x)\, dx + \int_{\mathcal{O}_r} h(x)\, dx - \frac{\eta}{M}\int_{F_\theta(\mathcal{O}_r)} dx + \frac{\eta}{2M}\int_{F_\theta(\mathcal{O}_r)} dx \\ = & \; 1 - \frac{\eta}{2M}\int_{F_\theta(\mathcal{O}_r)} dx \leq 1 - \frac{\eta^2}{2M}\int_{\mathcal{O}_r} dx = 1 - \frac{\eta^2}{2M}\frac{\pi^{k/2} r^k}{\Gamma\left(\frac{k}{2}+1\right)} = \rho(k,\eta,M,r). \end{aligned}$$

Now for $\|t\| < \xi$, the multivariate version of theorem 1 in Petrov [12], page 10, yields that for any characteristic function $f(t)$, $t \in \mathbb{R}^k$, with $|f(t)| \leq c < 1$ for $\|t\| \geq \xi$,

$$|f(t)| \leq 1 - \frac{1-c^2}{8\xi^2}\|t\|^2 = \rho(c,\xi) \quad for \;\; \|t\| < \xi$$

proving the assertion. ■

We are now ready to prove lemma 13.

Proof of Lemma 2.6. By assumption A.3.NDD we can find a number $\eta > 0$ and a set A of sequences $y \in \mathbb{R}^{\mathbb{Z}}$, independent of θ, with $P(A) > 0$ such that for $y \in A$ the partial derivatives $\frac{\partial}{\partial \varepsilon_0} X_j$, $j \in \mathbb{Z}$, are defined at y, and

$$\inf_{\theta\in\Theta} \det\left|\left(\sum_{j=0}^{\infty} \frac{\partial}{\partial \varepsilon_{l_\nu}} X_j : \nu = 1, ..., k\right)\right| \geq \eta.$$

Let $\|B\| = \sup\{\|Bx\| : \|x\| \leq 1\}$ for any $k \times k$ matrix B and let $c(k)$ denote a universal constant satisfying

$$|\det B_1 - \det B_2| \leq c(k)\left(\|B_1\|^{k-1} + \|B_2\|^{k-1}\right)\|B_1 - B_2\|$$

for arbitrary $k \times k$ matrices B_1 and B_2. From assumption A.3.EL we get that

$$\sup_{\theta\in\Theta}\sum_{j=0}^{\infty}E\left(\left\|\frac{\partial}{\partial\varepsilon_0}X_j(y)\right\|1_A(y)\right) \tag{11}$$

$$\leq \sup_{\theta\in\Theta}\sum_{j=0}^{\infty}E\left(\sup_{y_0^*\in\overline{\mathcal{O}}(y_0,\delta),y_0^*\neq y_0}\frac{\|X_j(\ldots,y_0^*,\ldots)-X_j(\ldots,y_0,\ldots)\|}{|y_0^*-y_0|}1_A(y)\right)$$

$$\leq \sup_{\theta\in\Theta}E(1_A(y))K\sum_{j=0}^{\infty}\exp(-\alpha|j|)$$

$$< \frac{2K}{1-\exp(-\alpha)} \tag{12}$$

It follows that there exists a measurable subset $A^{/}\subset A$, independent of θ, with $P\left(A^{/}\right)>0$ and $m_0>0$ such that for $y\in A^{/}$,

$$\sup_{\theta\in\Theta}\left\|\sum_{j=m_0+1}^{\infty}\frac{\partial}{\partial\varepsilon_{l_\nu}}X_j(y):\nu=1,...,k\right\|\leq\frac{\eta}{4C(k,K,\alpha)}$$

where $C(k,K,\alpha)=2C(k)(2K/(1-\exp[-\alpha]))^k$. Further estimating the difference of determinants for all $y\in A^{/}$ we get

$$\sup_{\theta\in\Theta}\left|\det\left(\sum_{j=0}^{m_0}\frac{\partial}{\partial\varepsilon_{l_\nu}}X_j(y):\nu=1,...,k\right)\right|\leq\frac{3\eta}{4}. \tag{13}$$

For $y\in\mathbb{R}^{\mathbb{Z}}$ and $x=(x_1,...,x_k)$ let (y,x) be the sequence, such that for $\nu=1,...,k$, x_ν is inserted at place l_ν and all other places are filled with the components of y :

$$(y,x)_j=\begin{cases}x_j, & \text{if } j\in\{l_1,...,l_k\},\\ y_{j-i}, & \text{if } i=1,...,k \text{ and } l_{i-1}<j<l_{i+1}.\end{cases}$$

Here $l_0=-\infty$ and $l_{k+1}=+\infty$. For $x\in\mathbb{R}^k$ let

$$A(x)=\left\{y\in\mathbb{R}^{\mathbb{Z}}:(y,x)\in A\right\}.$$

Since $P(A)>0$, we can find $x^{(0)}\in G_{l_1}\times...\times G_{l_k}$ such that $P\left(A\left(x^{(0)}\right)\right)>0$. Assumption A.3.CPD implies that for $\delta>0$ there exists a small ball $B\subset\mathbb{R}^k$ containing $x^{(0)}$ ans a set $A^{//}\subset A^{/}$ with $P\left(A^{//}\right)>0$ such that for $y\in A^{//}$

and $x \in B$ and for $\nu = 1, ..., k$ and $j = 0, ..., m_0$, $\frac{\partial}{\partial \epsilon_{l_\nu}} X_j$ exists at (y, x) for all $\theta \in \Theta$ and

$$\sup_{\theta \in \Theta} \left\| \frac{\partial}{\partial \varepsilon_{l_\nu}} X_j \left(\left(y, x^{(0)} \right), \theta \right) - \frac{\partial}{\partial \varepsilon_{l_\nu}} X_j \left(\left(y, x \right), \theta \right) \right\| \leq \delta.$$

Choosing appropriately small δ, we obtain (estimating the change of determinants) from this and equation 13, $x \in B$ and $y \in A^{//}$,

$$\inf_{\theta \in \Theta} \left| \det \left(\sum_{j=0}^{m_0} \frac{\partial}{\partial \varepsilon_{l_\nu}} X_j \left(\left(y, x \right), \theta \right) : \nu = 1, ..., k \right) \right| \geq \frac{\eta}{2}.$$

This implies that for $m \geq m_0$ and $x \in B$, $y \in A^{//}$ we have by equation **??** again estimating the change of determinants

$$\inf_{\theta \in \Theta} \left| \det \left(\sum_{j=0}^{m} \frac{\partial}{\partial \varepsilon_{l_\nu}} X_j \left(\left(y, x \right), \theta \right) : \nu = 1, ..., k \right) \right| \geq \frac{\eta}{4}.$$

By lemma 2.7 we get that for $\delta > 0$ there exists $\rho < 1$ depending only on δ, η and B (not on m and θ) such that for $m > m_0$ and all $y \in A^{//}$ and $\|t\| \geq \delta$,

$$\sup_{\theta \in \Theta} \left| \int_B \exp \left[i t^T \left(X_0 + ... + X_m \right) \right] \prod_{\nu=1}^{k} h \left(\varepsilon_{l_\nu} \right) d\varepsilon_{l_1} ... d\varepsilon_{l_k} \right| \leq \rho \left(\delta, \eta, B \right)$$

where b is the radius of the ball B. Now the left-hand side is an upper bound for

$$\left| E \left(\exp \left[i t^T \left(X_0 + ... + X_m \right) \right] | \varepsilon_j : |j - m| \geq r \right) \right|$$

where $r = \max \left(l_1, ..., l_k \right)$. This implies

$$E \left| E \left(\exp \left[i t^T \left(X_0 + ... + X_m \right) \right] | \varepsilon_j : |j - m| \geq r \right) \right| \leq 1 - P \left(A^{//} \right) + \rho P \left(A^{//} \right),$$

which proves the lemma. ■

3 Sequences of Smooth Transformations of Valid Locally Uniform Edgeworth Expansions

We are now considering the question of whether appropriately smooth transformations of sequences of random elements-with distributions that admit locally uniform Edgeworth expansions as the one described in 2-admit analogous expansions. In this paragraph we follow the line of reasoning of Skovgaard [13]. We suppose that $(S_n(\theta))$ is a sequence of random elements *not* *nesessarily* of the form described immediately after assumption A.1. Furthermore the distribution of $S_n(\theta)$ admits *a locally uniform Edgeworth expansion of order* $s-2$, i.e. there exists some θ_0 and $\varepsilon > 0$ for which equation 2 holds with $\Psi_{n,s}(\theta)$ an Edgeworth distribution (*not necessarily the formal one*).[3]

Question Let $f_n : \mathbb{R}^q \to \mathbb{R}^p$. Find sufficient conditions for the validity of

$$\sup_{\theta\in\Theta}\sup_{A\in B_c^*}\left|P\left(f_n\left(S_n\left(\theta\right)\right)\in A\right)-\Psi^*_{n,s}\left(\theta\right)\left(A\right)\right| = o\left(n^{-\frac{s-2}{2}}\right) \tag{14}$$

where $\Psi^*_{n,s}(\theta)$ is an Edgeworth distribution of order $s-1$ ($s \geq 3$) on $\mathbb{R}^p$ and B_c^* is the collections of the convex Borel subsets of $\mathbb{R}^p$.

In the example of the following section we will utilize the forthcoming answer to this in order to establish locally uniform Edgeworth expansions of statistical functions of interest. We first make the following assumption.

Assumption A.4 *Let the following conditions hold:*
-POL $f_n(x,\theta) = \sum_{i=0}^{s-2}\frac{A_{i_n}(\theta)\left(x^{i+1}\right)}{n^{i/2}}$ *where* $A_{i_n} : \Theta \times \mathbb{R}^{q^{i+1}} \to \mathbb{R}^p$ *is* $(i+1)$-*linear* $\forall\theta\in\Theta$, $x^i = \left(\underbrace{x,...,x}_{i-times}\right)$, $A_{0_n}(\theta) = A_0(\theta)$, $\operatorname{rank} A_0(\theta) = p$ $\forall\theta\in\Theta$, A_{i_n} *equicontinuous on* Θ, $\forall x^{i+1}$.
-EEQ *The* i^{th} *polynomial, say,* $\pi_i(z,\theta)$ *of* $\Psi_{n,s}(\theta)$ *is equicontinuous on* Θ $\forall z\in\mathbb{R}^q$, *for* $i = 1,\ldots,s-2$, *and if* $\Sigma(\theta)$ *denotes the variance matrix in the density of* $\Psi_{n,s}(\theta)$ *then it is continuous on* Θ *and positive definite.*

Remark R.2 *Obviously assumption A.4.*POL *implies that* $p \leq q$ *while if* $\Psi_{n,s}(\theta)$ *is the formal Edgeworth distribution the required equicontinuity in A.4.*EEQ *would follow from the continuity of* $E\left(K_i\left(S_n\left(\theta\right)\right)\right)$ *on* Θ *for* $i = 1,\ldots,s+1$ *and* K_i *any i-linear real function on* $\mathbb{R}^{q^i}$, *while lemma 2.2.4 provides with sufficient conditions for the validity of the eigenvalue condition in A.4.*EEQ. *Furthermore continuity and compactness imply that* $\inf_{\theta\in\Theta}\lambda_{\min}\left(A_0\left(\theta\right)\right), \inf_{\theta\in\Theta}\lambda_{\min}\left(\Sigma\left(\theta\right)\right) > 0$ *where* $\lambda_{\min}(\cdot)$ *denotes the smallest absolute eigenvalue.*

The following theorem provides the first result of this section.

[3] For the definition of the general form of an Edgeworth distribution see equations (3.7) and (3.8) of Magdalinos [10].

Theorem 3.1 *Under assumption A.4 there exist an Edgeworth distribution $\Psi^*_{n,s}(\theta)$ for which equation 14 is valid, with polynomials that satisfy A.4.*EEQ. *Furthermore, if K is a m-linear real function on $\mathbb{R}^p$ then*

$$\sup_{\theta\in\Theta}\left|\int_{\mathbb{R}^p} K(x^m)\, d\Psi^*_{n,s}(\theta) - \int_{\mathbb{R}^q} K\left((f_n(x))^m\right) d\Psi_{n,s}(\theta)\right| = o\left(n^{\frac{s-2}{2}}\right)$$

In order to prove the theorem we will utilize the following auxiliary results.

Lemma AL.1 *Suppose that S_n admits an Edgeworth expansion of order $s-2$. Then for any $i<j:1,\dots q$, $\mathrm{pr}_{i,j}(S_n) \doteqdot \left(S_{n_i}, S_{n_{i+1}},\dots,S_{n_j}\right)'$ admits an analogous expansion of the same order.*

Proof. The density of $\Psi_{n,s}(\theta)$ is of the form $\left(1+\sum_{i=1}^{s-2}\frac{1}{n^{\frac{i}{2}}}\pi_i(x,\theta)\right)\varphi_{\Sigma(\theta)}(x)$. For A a convex Borel set in $\mathbb{R}^{j-i+1}$we have

$$\begin{aligned} P\left(\mathrm{pr}_{i,j}(S_n(\theta))\in A\right) &= P\left(S_n(\theta)\in \mathrm{pr}^{-1}_{i,j}(A)\right) \\ &= \int_{\mathbb{R}\times\ldots\times A\times\ldots\mathbb{R}}\left(1+\sum_{i=1}^{s-2}\frac{1}{n^{\frac{i}{2}}}\pi_i(x,\theta)\right)\varphi_{\Sigma(\theta)}(x)\,dx + o\left(n^{-\frac{s-2}{2}}\right) \\ &= \int_A\left(1+\sum_{i=1}^{s-2}\frac{1}{n^{\frac{i}{2}}}\pi^*_i(v,\theta)\right)\varphi_{I\Sigma(\theta)I'}dv + o\left(n^{-\frac{s-2}{2}}\right) \end{aligned}$$

where $v=\mathrm{pr}_{i,j}(x)$, $x=(v,v^*)$, $I=(0,\mathrm{Id}_{j-i+1\times j-i+1},0)$, and $\pi^*_i(v) = \int_{\mathbb{R}^{q-p}}\pi_i(v,v^*,\theta)\,\varphi_{\Sigma(\theta)}(v,v^*)\,dv^*$ and $o\left(n^{-\frac{s-2}{2}}\right)$ is independent of A and θ. ■

Lemma AL.2 *Suppose that $S_n(\theta)$ admits a locally uniform Edgeworth expansion of order $s-2$ for which assumption A.4.*EEQ *holds. Then there exists a constant C independent of θ such that*

$$\sup_{\theta\in\Theta} P\left(\|S_n(\theta)\| > C\ln^{1/2} n\right) = o\left(n^{\frac{s-2}{2}}\right)$$

Proof. Let $C = \frac{\sqrt{s-1}}{\sup_{\theta\in\Theta}\left\|\Sigma^{-1/2}(\theta)\right\|}$, $\mathcal{H}_n(C) = \left\{x\in\mathbb{R}^q : \|x\| > C\ln^{1/2} n\right\}$. Then

$$P\left(\|S_n(\theta)\| > C\ln^{1/2} n\right) = \int_{\mathcal{H}_n(C)}\left(1+\sum_{i=1}^{s-2}\frac{1}{n^{\frac{i}{2}}}\pi_i(x,\theta)\right)\varphi_{\Sigma(\theta)}(x)\,dx + o\left(n^{-\frac{s-2}{2}}\right)$$

where the last term on the right is independent of θ. Now

$$\begin{aligned}
&\left|\int_{\mathcal{H}_n(C)}\left(1+\sum_{i=1}^{s-2}\frac{1}{n^{\frac{i}{2}}}\pi_i\left(x,\theta\right)\right)\varphi_{\Sigma(\theta)}\left(x\right)dx\right| \\
\leq\ &\int_{\Sigma^{-1/2}(\theta)\mathcal{H}_n(C)}\left(1+\sum_{i=1}^{s-2}\frac{1}{n^{\frac{i}{2}}}\sup_{\theta\in\Theta}\left|\pi_i\left(\Sigma^{1/2}\left(\theta\right)z,\theta\right)\right|\right)\exp\left(-\frac{1}{2}\left\|z\right\|^2\right)dz \\
\leq\ &\int_{\mathcal{H}_n(1)}\left(1+\sum_{i=1}^{s-2}\frac{1}{n^{\frac{i}{2}}}\sup_{\theta\in\Theta}\left|\pi_i\left(\Sigma^{1/2}\left(\theta\right)z,\theta\right)\right|\right)\exp\left(-\frac{1}{2}\left\|z\right\|^2\right)dz
\end{aligned}$$

Assumption A.4.EEQ (see also tha last part of R.2) implies that $\sup_{\theta\in\Theta}\left|\pi_i\left(\Sigma^{1/2}\left(\theta\right)z,\theta\right)\right|\leq\sum_{i=1}^{m}c_i\left\|z\right\|^i$ for some finite m and c_i $i=1,\ldots,m$ independent of θ. The result follows from equation (**A.8**) in the proof of Lemma 2 in Magdalinos [10]. ■

Lemma AL.3 *Let f_n be as in assumption A.4.*POL *and $p=q$. Then there exists a function $h_n:\Theta\times\mathbb{R}^p\to\mathbb{R}^p$ of the form $h_n\left(x,\theta\right)=\sum_{i=0}^{s-2}\frac{B_{i_n}(\theta)\left(x^{i+1}\right)}{n^{i/2}}$ where the $B_{i_n}\left(\theta\right)$ have the same properties with $A_{i_n}\left(\theta\right)$, $h_n\left(f_n\left(\cdot,\theta\right),\theta\right)=id_{\mathbb{R}^p}\left(\cdot\right)+M_n\left(\cdot,\theta\right)$ where $M_n\left(x,\theta\right)$ is polynomial in x of finite degree $\forall\theta\in\Theta$, and* $\sup_{\theta\in\Theta}\left\|M_n\left(x,\theta\right)\right\|=o\left(n^{\frac{s-2}{2}}\right)$ $\forall y\in\mathbb{R}^p$.

Proof. Proceed inductively. Let $s=3$, and therefore

$$f_n\left(x,\theta\right)=A_0\left(\theta\right)x+\frac{A_{1_n}\left(\theta\right)\left(x^2\right)}{n^{1/2}}$$

and choose $B_0\left(\theta\right)=A_0^{-1}\left(\theta\right)$ and $B_{1_n}\left(\theta\right)\left(x,x\right)=-A_0^{-1}\left(\theta\right)A_{1_n}\left(\theta\right)\left(A_0^{-1}\left(\theta\right)x,A_0^{-1}\left(\theta\right)x\right)$. Obviously h_n is unique. Then due to the compactness of Θ the equicontinuity of $A_{i_n}\left(\theta\right)$ and the boundness away from zero of $A_0\left(\theta\right)$ unifromly w.r.t. θ the properties described in A.4.POL hold h_n. Furthemore

$$\begin{aligned}
M_n\left(x,\theta\right) &= -\frac{1}{n}A_0^{-1}\left(\theta\right)A_{1_n}\left(\theta\right)\left(x,A_{1_n}\left(\theta\right)\left(x^2\right)\right) \\
&\quad-\frac{1}{n}A_0^{-1}\left(\theta\right)A_{1_n}\left(\theta\right)\left(A_{1_n}\left(\theta\right)\left(x^2\right),x\right) \\
&\quad-\frac{1}{n^{3/2}}A_0^{-1}\left(\theta\right)A_{1_n}\left(\theta\right)\left(A_{1_n}\left(\theta\right)\left(x^2\right),A_{1_n}\left(\theta\right)\left(x^2\right)\right)
\end{aligned}$$

which due to the previous clearly is polynomial in x of fourth degree $\forall\theta\in\Theta$, and $\sup_{\theta\in\Theta}\left\|M_n\left(x,\theta\right)\right\|=o\left(n^{\frac{1}{2}}\right)$ $\forall x\in\mathbb{R}^p$. Then suppose that the result holds for $s=k$. For $s=k+1$ choose $B_0\left(\theta\right)=A_0^{-1}\left(\theta\right)$, $B_{i_n}\left(\theta\right)=B_{i_n}^{*}\left(\theta\right)$ for

$i = 1, \ldots, k-2$, where $B^*_{i_n}(\theta)$ is the i^{th} coefficient of the h_n in the previous step and identify $B_{k-1_n}(\theta)$ by

$$\begin{aligned}\frac{B_{k-1_n}(\theta)\left((A_0(\theta)\,x)^k\right)}{n^{(k-1)/2}} &= -A_0^{-1}(\theta)\,\frac{A_{k-1_n}(\theta)\left(x^k\right)}{n^{(k-1)/2}} \\ &\quad -\sum_{i=1}^{k-2}\frac{B^*_{i_n}(\theta)\left(\sum_{i=0}^{k-1}\frac{A_{i_n}(\theta)\left(x^{i+1}\right)}{n^{i/2}}\right)^{i+1}}{n^{i/2}} \\ &\quad -A_0^{-1}(\theta)\,M^*_n(x,\theta) \mod \left(\frac{1}{n^{k-1/2}}\right)\end{aligned}$$

where $\mod\left(\frac{1}{n^{k-1/2}}\right)$ signifies that only terms of order $O\left(\frac{1}{n^{k-1/2}}\right)$ are considered, while terms of lower order have been considered in the previous steps and terms of higher order are placed in the M_n, while M^*_n is the relevant remainder of the previous step. Notice that due to the properties of $A_0(\theta)$ the solution exists and is unique. The attributed properties follow by the same reasoning as the ones for $s = 3$. ■

Remark R.3 *In the notation of the proof of lemma AL.2 the previous result implies that for* $x \in \mathcal{H}^c_n(C)$

$$h_n(f_n(x,\theta),\theta) = x + o\left(n^{\frac{s-2}{2}}\right)$$

where the $o\left(n^{\frac{s-2}{2}}\right)$ *is independent of* x *and* θ*.*

We are now ready to prove the aforementioned theorem.

Proof of Theorem 3.1. Assume first without loss of generality that $p = q$, for if $p < q$, consider

$$f^*_n(\theta, x) = \begin{pmatrix} A_0(\theta) & 0 \\ 0 & I_{q-p}\end{pmatrix} + \sum_{i=1}^{s-1}\frac{1}{n^{i/2}}A^*_{i_n}(\theta)\left(x^{i+1}\right)$$

where $A^*_{i_n}(\theta)(x^{i+1}) = \begin{pmatrix} A_{i_n}(\theta)(x^{i+1}) \\ 0_{q-p}\end{pmatrix}$. If $f^*_n(S_n)$ admits a valid Edgeworth expansion with the prescribed property, then so does $f_n(S_n)$ by lemma AL.1. It suffices to prove that

$$\sup_{\theta\in\Theta}\left|P(f_n(S_n(\theta)) \in A) - \Psi^*_{n,s}(\theta)(A)\right| = o\left(n^{-\frac{s-2}{2}}\right)$$

for an arbitrary Borel set A, due to the fact that any Normal distribution attributes to the boundary of such a set zero measure, hence then 14 would follow from Theorem 2.11 (and the subsequent Remark) of Bhattacharya and Rao [3]. Now

$$\begin{aligned} P\left(f_n\left(S_n,\theta\right)\in A\right) &= P\left(S_n\in f_n^{-1}\left(A,\theta\right)\right) \\ &= \int_{f_n^{-1}(A,\theta)}\left(1+\sum_{i=1}^{s-2}\frac{\pi_i\left(z,\theta\right)}{n^{i/2}}\right)\varphi_{\Sigma(\theta)}\left(z\right)dz+o\left(n^{-\frac{s-2}{2}}\right) \end{aligned}$$

uniformly over $A\in B_{\mathbb{R}^p}$ and θ. Then notice that

$$\begin{aligned} &\int_{f_n^{-1}(A,\theta)}\left(1+\sum_{i=1}^{s-2}\frac{\pi_i\left(z,\theta\right)}{n^{i/2}}\right)\varphi_{\Sigma(\theta)}\left(z\right)dz \\ =\ &\int_{f_n^{-1}(A,\theta)\cap\mathcal{H}_n^c(C)}\left(1+\sum_{i=1}^{s-2}\frac{\pi_i\left(z,\theta\right)}{n^{i/2}}\right)\varphi_{\Sigma(\theta)}\left(z\right)dz+o\left(n^{-\frac{s-2}{2}}\right) \end{aligned}$$

where the last term is independent of θ due to the fact that

$$\begin{aligned} &\left|\int_{f_n^{-1}(A,\theta)\cap\mathcal{H}_n(C)}\left(1+\sum_{i=1}^{s-2}\frac{\pi_i\left(z,\theta\right)}{n^{i/2}}\right)\varphi_{\Sigma(\theta)}\left(z\right)dz\right| \\ \leq\ &\int_{\mathcal{H}_n(C)}\left(1+\sum_{i=1}^{s-2}\frac{\left|\pi_i\left(z,\theta\right)\right|}{n^{i/2}}\right)\varphi_{\Sigma(\theta)}\left(z\right)dz=o\left(n^{-\frac{s-2}{2}}\right) \end{aligned}$$

as in the proof of lemma AL.2. Now notice that due to A.4.POL and the compactness of Θ, if $z\in f_n^{-1}\left(A,\theta\right)\cap\mathcal{H}_n^c\left(C\right)$ then $f_n\left(z,\theta\right)\in A$ and $\left\|f_n\left(z\right)\right\|<C^*\ln^{1/2}n$ for some C^* independent of θ. Hence, substituting for $u=f_n\left(z,\theta\right)$ we have that due to remark R.3 $z=h_n\left(u,\theta\right)+o\left(n^{-\frac{s-2}{2}}\right)$ where the last term does not depend on θ or z, when $z\in\mathcal{H}_n^c\left(C^*\right)$. Hence

$$\begin{aligned} &\int_{f_n^{-1}(A,\theta)\cap\mathcal{H}_n^c(C)}\left(1+\sum_{i=1}^{s-2}\frac{\pi_i\left(z,\theta\right)}{n^{i/2}}\right)\varphi_{\Sigma(\theta)}\left(z\right)dz \\ =\ &\int_{A\cap\mathcal{H}_n^c(C^*)}\left(1+\sum_{i=1}^{s-2}\frac{\pi_i\left(h_n\left(u,\theta\right)+o\left(n^{-\frac{s-2}{2}}\right),\theta\right)}{n^{i/2}}\right)\kappa\left(u,\theta\right)du \\ =\ &\int_{A\cap\mathcal{H}_n^c(C^*)}\left(1+\sum_{i=1}^{s-2}\frac{\pi_i^*\left(u,\theta\right)}{n^{i/2}}\right)\varphi_{K(\theta)}\left(u\right)du+o\left(n^{-\frac{s-2}{2}}\right) \end{aligned}$$

where $K(\theta) = A_0(\theta) V(\theta) A_0(\theta)$, $\kappa(u,\theta) = \frac{\varphi_{\Sigma(\theta)}\left(h_n(u,\theta)+o\left(n^{-\frac{s-2}{2}}\right)\right)}{\det\left(A_0(\theta)+O\left(n^{-\frac{1}{2}}\right)\right)}$ and the $\pi_i^* s$ are obtained by expanding and holding terms of the relevant order. Due to assumptions A.4, A.4 and the definition of $B_{i_n}(\theta)(x^{i+1})\, s$, the $\pi_i^* s$ are equicontinuous in θ $\forall u \in \mathbb{R}^p$ and the terms in $o\left(n^{-\frac{s-2}{2}}\right)$ are independent of θ. Finally notice that since by an argument analogous to that of the proof of lemma AL.2

$$\left|\int_{A\cap\mathcal{H}_n(C^*)}\left(1+\sum_{i=1}^{s-2}\frac{\pi_i^*(u,\theta)}{n^{i/2}}\right)\varphi_{K(\theta)}(u)\,du\right|$$
$$\leq \left|\int_{\mathcal{H}_n(C^*)}\left(1+\sum_{i=1}^{s-2}\frac{\pi_i^*(u,\theta)}{n^{i/2}}\right)\varphi_{K(\theta)}(u)\,du\right| = o\left(n^{-\frac{s-2}{2}}\right)$$

where the last term is independent of θ, we obtain that

$$P(f_n(S_n,\theta)\in A) = \int_A\left(1+\sum_{i=1}^{s-2}\frac{\pi_i^*(u,\theta)}{n^{i/2}}\right)\varphi_{K(\theta)}(u)\,du + o\left(n^{-\frac{s-2}{2}}\right)$$

uniformly over Θ. For the second part of the theorem notice that

$$\int_{\mathbb{R}^p} K\left((f_n(z,\theta))^m\right)\left(1+\sum_{i=1}^{s-2}\frac{\pi_i(z,\theta)}{n^{i/2}}\right)\varphi_{\Sigma(\theta)}(z)\,dz$$
$$= \int_{\mathcal{H}_n^c(C)} K\left((f_n(z,\theta))^m\right)\left(1+\sum_{i=1}^{s-2}\frac{\pi_i(z,\theta)}{n^{i/2}}\right)\varphi_{\Sigma(\theta)}(z)\,dz$$
$$+\int_{\mathcal{H}_n(C)} K\left((f_n(z,\theta))^m\right)\left(1+\sum_{i=1}^{s-2}\frac{\pi_i(z,\theta)}{n^{i/2}}\right)\varphi_{\Sigma(\theta)}(z)\,dz$$

and again due to an analogous argument as in the proof of lemma AL.2, the fact that K is multilinear and the form of f_n in assumption A.4.POL, the last integral is $o\left(n^{-\frac{s-2}{2}}\right)$ which is independent of θ. Now exactly as in the previous part for $u = f_n(z,\theta)$

$$\int_{\mathcal{H}_n^c(C)} K\left((f_n(z,\theta))^m\right)\left(1+\sum_{i=1}^{s-2}\frac{\pi_i(z,\theta)}{n^{i/2}}\right)\varphi_{\Sigma(\theta)}(z)\,dz$$
$$= \int_{\mathcal{H}_n^c(C^*)} K(u^m)\left(1+\sum_{i=1}^{s-2}\frac{\pi_i\left(h_n(u,\theta)+o\left(n^{-\frac{s-2}{2}}\right),\theta\right)}{n^{i/2}}\right)\kappa(u,\theta)\,du$$

and the result follows exactly as the previous one, by simply noticing that (some of) the remainders will also depend on $K(u^m)$ which is nevertheless polynomial. ■

The final result of this section, is partially a consequence of the previous theorem, and can be of convenience for the establishment of valid locally uniform Edgeworth expansions for estimators that asymptotically satisfy sufficiently smooth first order conditions with sufficiently high probability.

Theorem 3.2 *Suppose that:*
-POLFOC $M_n(\theta)$ *satisfies* $0_{p\times 1} = \sum_{i=0}^{s-2} \frac{1}{n^{i/2}} \sum_{j=0}^{i+1} C_{ij_n}(\theta) \left(M_n(\theta)^j, S_n(\theta)^{i+1-j} \right) + R_n(\theta)$ *with probability* $1 - o\left(n^{-\frac{s-2}{2}}\right)$ *independent of* θ *where* $C_{ij_n} : \Theta \times \mathbb{R}^{q^{i+1}} \to \mathbb{R}^p$ *is* $(i+1)$*-linear* $\forall \theta \in \Theta$, $C_{00_n}(\theta), C_{01_n}(\theta)$ *are independent of* n *and have rank* p $\forall \theta \in \Theta$, C_{ij_n} *are equicontinuous on* Θ, $\forall x^{i+1}$,
-LUE $S_n(\theta)$ *admits a locally uniform Edgeworth expansion that satisfies assumption A.4.*EEQ,
-UAT $\sup_{\theta\in\Theta} P\left(\|M_n(\theta)\| > C \ln^{1/2} n\right) = o\left(n^{-\frac{s-2}{2}}\right)$ *for some* $C > 0$ *independent of* θ,
-USR $\sup_{\theta\in\Theta} P(\|R_n(\theta)\| > \gamma_n) = o\left(n^{-\frac{s-2}{2}}\right)$ *for some real sequence* $\gamma_n = o\left(n^{-\frac{s-2}{2}}\right)$ *independent of* θ.
Then $M_n(\theta)$ *admits a locally uniform Edgeworth expansion that satisfies assumption A.4.*EEQ.

In order to prove this we need the following result.

Lemma 3.3 *Suppose that* $S_n(\theta)$ *admits a locally uniform Edgeworth expansion that satisfies A.4.*EEQ *and that* $U_n(\theta)$ *is such that*

$$\sup_{\theta\in\Theta} P(U_n(\theta) = S_n(\theta) + R_n(\theta)) = 1 - o\left(n^{-\frac{s-2}{2}}\right)$$

where R_n *satisfies condition 3.2.*USR. *Then* $U_n(\theta)$ *admits the same locally uniform Edgeworth expansion.*

Proof of Lemma 3.3. Let $a = \frac{s-2}{2}$. Notice first that

$$\begin{aligned}
& P(U_n(\theta) \in A) \\
\leq\ & P(U_n(\theta) \in A, U_n(\theta) = S_n(\theta) + R_n(\theta)) + P(U_n(\theta) \neq S_n(\theta) + R_n(\theta)) \\
\leq\ & P(S_n(\theta) + R_n(\theta) \in A, \|R_n(\theta)\| \leq \gamma_n) + P(\|R_n(\theta)\| > \gamma_n) + o\left(n^{-\frac{s-2}{2}}\right) \\
=\ & P(S_n(\theta) \in A - \gamma_n) + o\left(n^{-\frac{s-2}{2}}\right)
\end{aligned}$$

where all the remainders in the previous display are independent of θ. Now, as $S_n(\theta)$ admits a locally uniform Edgeworth expansion, we have that for an arbitrary Borel set A

$$\begin{aligned} P\left(S_n(\theta) \in A - \gamma_n\right) &= \int_{A-\gamma_n i_q} \left(1 + \sum_{i=1}^{s-2} \frac{\pi_i(z,\theta)}{n^{i/2}}\right) \varphi_{\Sigma(\theta)}(z)\, dz + o\left(n^{-\frac{s-2}{2}}\right) \\ &= \int_{(A-\gamma_n i_q)\cap \mathcal{H}_n^c(C)} \left(1 + \sum_{i=1}^{s-2} \frac{\pi_i(z,\theta)}{n^{i/2}}\right) \varphi_{\Sigma(\theta)}(z)\, dz + o\left(n^{-\frac{s-2}{2}}\right) \end{aligned}$$

where i_q is a $q \times 1$ vector of 1's, $A - \gamma_n i_q$ denotes translation by $-\gamma_n i_q$, and the last term is independent of θ and A (see also the begining of the proof of theorem 3.1) and $\mathcal{H}_n^c(C)$ was defined in the proof of lemma AL.2. Now by a change of variables we have that for large enough n

$$\begin{aligned} &\int_{A\cap(\mathcal{H}_n^c(C)+\gamma_n i_q)} \left(1 + \sum_{i=1}^{s-2} \frac{\pi_i(z,\theta)}{n^{i/2}}\right) \varphi_{\Sigma(\theta)}(z)\, dz \\ = &\int_{A\cap(\mathcal{H}_n^c(C)+\gamma_n i_q)} \left(1 + \sum_{i=1}^{s-2} \frac{\pi_i(z-\gamma_n i_q,\theta)}{n^{i/2}}\right) \varphi_{\Sigma(\theta)}(z-\gamma_n i_q)\, dz \end{aligned}$$

. Expanding terms using the mean value theorem, holding terms of the relevant order due to the fact that A.4.EEQ we obtain that

$$\begin{aligned} \varphi_{\Sigma(\theta)}(z-\gamma_n i_q) &= \varphi_{\Sigma(\theta)}(z) + \gamma_n \varphi_{\Sigma(\theta)}(z-\gamma_n^* i_q)\left(\gamma_n^* i_q' \Sigma^{-1}(\theta)\, i_q - i_q' \Sigma^{-1}(\theta)\, z\right) \\ \pi_i(z-\gamma_n i_q,\theta) &= \pi_i(z,\theta) - \gamma_n i_q' \frac{\partial \pi_i(z-\gamma_n^{**} i_q,\theta)}{\partial z} \end{aligned}$$

where $\gamma_n^*, \gamma_n^{**}$ lie between $0_{q\times 1}$, $\frac{\partial \pi_i(z,\theta)}{\partial z}$ is also polynomial in z, and there exists some $C^* > C$ such that consequently

$$\begin{aligned} &\int_{A\cap(\mathcal{H}_n^c(C)+\gamma_n)} \left(1 + \sum_{i=1}^{s-2} \frac{\pi_i(z-\gamma_n i_q,\theta)}{n^{i/2}}\right) \varphi_{\Sigma(\theta)}(z-\gamma_n i_q)\, dz \\ = &\int_{A\cap(\mathcal{H}_n^c(C)+\gamma_n)} \left(1 + \sum_{i=1}^{s-2} \frac{\pi_i(z,\theta)}{n^{i/2}}\right) \varphi_{\Sigma(\theta)}(z)\, dz + R_n^*(\theta) \end{aligned}$$

where

$$
\begin{aligned}
& R_n^*(\theta) \\
\leq\ & \sum_{i=1}^{s-2} \int_{A\cap(\mathcal{H}_n^c(C)+\gamma_n)} \frac{|\gamma_n|}{n^{i/2}} \sqrt{q} \left| \frac{\partial \pi_i (z - \gamma_n^{**} i_q, \theta)}{\partial z} \right| \varphi_{\Sigma(\theta)}(z)\, dz \\
& + \sum_{i=1}^{s-2} \int_{A\cap(\mathcal{H}_n^c(C)+\gamma_n)} \frac{|\gamma_n|}{n^{i/2}} |\pi_i(z,\theta)| \left\| \gamma_n^* i_q' \Sigma^{-1}(\theta) i_q - i_q' \Sigma^{-1}(\theta) z \right\| \\
& \times \varphi_{\Sigma(\theta)}(z - \gamma_n^* i_q)\, dz \\
& + \sum_{i=1}^{s-2} \int_{A\cap(\mathcal{H}_n^c(C)+\gamma_n)} \frac{|\gamma_n|^2}{n^{i/2}} \sqrt{q} \left\| \frac{\partial \pi_i (z - \gamma_n^{**} i_q, \theta)}{\partial z} \right\| \\
& \times \left\| \gamma_n^* i_q' \Sigma^{-1}(\theta) i_q - i_q' \Sigma^{-1}(\theta) z \right\| \varphi_{\Sigma(\theta)}(z - \gamma_n^* i_q)\, dz \\
& + \int_{A\cap(\mathcal{H}_n^c(C)+\gamma_n)} |\gamma_n| \sqrt{q} \left\| \gamma_n^* i_q' \Sigma^{-1}(\theta) i_q - i_q' \Sigma^{-1}(\theta) z \right\| \\
& \times \varphi_{\Sigma(\theta)}(z - \gamma_n^* i_q)\, dz
\end{aligned}
$$

which is less than or equal to

$$
\begin{aligned}
& \sum_{i=1}^{s-2} \int_{\Sigma^* A\cap(\mathcal{H}_n^c(\|\Sigma^*\|C)+\Sigma^*\gamma_n)} \frac{|\gamma_n|}{n^{i/2}} \sqrt{q} \\
& \times \left| \frac{\partial \pi_i (\Sigma(\theta) z - \Sigma(\theta) \gamma_n^{**} i_q, \theta)}{\partial z} \right| \varphi(z)\, dz \\
& + \sum_{i=1}^{s-2} \int_{\Sigma^* A\cap(\mathcal{H}_n^c(\|\Sigma^*\|C)+\Sigma^*\gamma_n)} \frac{|\gamma_n|}{n^{i/2}} |\pi_i(\Sigma(\theta) z, \theta)| \\
& \times \left\| \gamma_n^* i_q' \Sigma^{-1}(\theta) i_q - i_q' \Sigma^{-1}(\theta) z \right\| \varphi\left(z - \Sigma^{-1/2}(\theta) \gamma_n^* i_q\right) dz \\
& + \sum_{i=1}^{s-2} \int_{\Sigma^* A\cap(\mathcal{H}_n^c(\|\Sigma^*\|C)+\Sigma^*\gamma_n)} \frac{|\gamma_n|^2}{n^{i/2}} \sqrt{q} \left\| \frac{\partial \pi_i (\Sigma(\theta) z - \Sigma(\theta) \gamma_n^{**} i_q, \theta)}{\partial z} \right\| \\
& \times \left\| \gamma_n^* i_q' \Sigma^{-1}(\theta) i_q - i_q' \Sigma^{-1/2}(\theta) z \right\| \varphi\left(z - \Sigma^{-1/2}(\theta) \gamma_n^* i_q\right) dz \\
& + \int_{\Sigma^* A\cap(\mathcal{H}_n^c(\|\Sigma^*\|C)+\Sigma^*\gamma_n)} |\gamma_n| \sqrt{q} \left\| \gamma_n^* i_q' \Sigma^{-1}(\theta) i_q - i_q' \Sigma^{-1}(\theta) z \right\| \\
& \times \varphi\left(z - \Sigma^{-1/2}(\theta) \gamma_n^* i_q\right) dz
\end{aligned}
$$

Σ^{*} the diagonal matrix with elements consisting of the inverse of $\lambda_{*} = \inf_{\theta\in\Theta}\sqrt{\lambda_{\min}\Sigma(\theta)}$ and notice that on $\mathcal{H}_{n}^{c}(\|\Sigma^{*}\| C) + \Sigma^{*}\gamma_{n}$

$$\begin{aligned}
& \varphi\left(z - \Sigma^{-1/2}(\theta)\gamma_{n}^{*} i_{q}\right) \\
= & \varphi(z)\exp\left(z'\Sigma^{-1/2}(\theta)\gamma_{n}^{*} i_{q}\right)\exp\left((\gamma_{n}^{*})^{2} i_{q}'\Sigma^{-1}(\theta) i_{q}\right) \\
\leq & \varphi(z)\exp\left(\frac{q}{\lambda_{*}}\left(C\left(\ln^{1/2} n\right)\gamma_{n}^{*} + (\gamma_{n}^{*})^{2}\right)\right)
\end{aligned}$$

furthermore due to the properties of $\Sigma(\theta)$ and the fact that the π_{i} and their derivatives are polynomials there exist positive constants independent of z and θ such that $|\pi_{i}(\Sigma(\theta)z, \theta)| \leq \sum_{j=1}^{q_{i}} c_{ij}\|z\|^{j}$, $\left\|\frac{\partial\pi_{i}(\Sigma(\theta)z - \Sigma(\theta)\gamma_{n}^{*}{}^{*} i_{q}, \theta)}{\partial z}\right\| \leq \sum_{j=1}^{q_{i}^{*}} c_{ij}^{*}\left(\|z\|^{j} + |\gamma_{n}^{*}|^{j}\right)$, hence we obtain that

$$\begin{aligned}
& |R_{n}^{*}(\theta)| \\
\leq & \sum_{i=1}^{s-2}\sum\nolimits_{j=1}^{q_{i}^{*}}\int_{\mathcal{H}_{n}^{c}(C^{*})}\frac{|\gamma_{n}|}{n^{i/2}}\sqrt{q}\left(c_{ij}^{*}\left(\|z\|^{j} + |\gamma_{n}^{*}|^{j}\right)\right)\varphi(z)\,dz \\
& + \sum_{i=1}^{s-2}\sum\nolimits_{j=1}^{q_{i}}\int_{\mathcal{H}_{n}^{c}(C^{*})}\frac{|\gamma_{n}|}{n^{i/2}}c_{ij}\|z\|^{j}\frac{q}{\lambda_{*}^{2}}\left(|\gamma_{n}^{*}| + \|z\|\right) \\
& \times\exp\left(\frac{q}{\lambda_{*}}\left(C\left(\ln^{1/2} n\right)\gamma_{n}^{*} + (\gamma_{n}^{*})^{2}\right)\right)\varphi(z)\,dz \\
& + \sum_{i=1}^{s-2}\sum\nolimits_{j=1}^{q_{i}^{*}}\int_{\mathcal{H}_{n}^{c}(C^{*})}\frac{|\gamma_{n}|^{2}}{n^{i/2}}\sqrt{q}\left(c_{ij}^{*}\left(\|z\|^{j} + |\gamma_{n}^{*}|^{j}\right)\right) \\
& \times\frac{q}{\lambda_{*}^{2}}\left(|\gamma_{n}^{*}| + \|z\|\right)\exp\left(\frac{q}{\lambda_{*}}\left(C\left(\ln^{1/2} n\right)\gamma_{n}^{*} + (\gamma_{n}^{*})^{2}\right)\right)\varphi(z)\,dz \\
& + \int_{\mathcal{H}_{n}^{c}(C^{*})}|\gamma_{n}|\sqrt{q}\frac{q}{\lambda_{*}^{2}}\left(|\gamma_{n}^{*}| + \|z\|\right)\exp\left(\frac{q}{\lambda_{*}}\left(C\left(\ln^{1/2} n\right)\gamma_{n}^{*} + (\gamma_{n}^{*})^{2}\right)\right)\varphi(z)\,dz
\end{aligned}$$

for some large enough C^{*} and each of these terms is $o\left(n^{-\frac{s-2}{2}}\right)$ and independent of θ due to the properties of γ_{n} and equation (**A.8**) in the proof of Lemma 2 in Magdalinos [10] that implies

$$\int_{\mathcal{H}_{n}^{c}(C^{*})}\|z\|^{j}\varphi(z)\,dz = K_{j}\left(1 - o\left(n^{-\frac{s-2}{2}}\right)\right)$$

for C^* large enough and any j. Finally

$$\int_{A\cap(\mathcal{H}_n^c(C)+\gamma_n)} \left(1+\sum_{i=1}^{s-2}\frac{\pi_i(z,\theta)}{n^{i/2}}\right)\varphi_{\Sigma(\theta)}(z)\,dz$$

$$= \int_A \left(1+\sum_{i=1}^{s-2}\frac{\pi_i(z,\theta)}{n^{i/2}}\right)\varphi_{\Sigma(\theta)}(z)\,dz + o\left(n^{-\frac{s-2}{2}}\right)$$

where the last term is independent of θ due to the fact that

$$\int_{A\cap(\mathcal{H}_n^c(C)+\gamma_n)^c} \left(1+\sum_{i=1}^{s-2}\frac{\pi_i(z,\theta)}{n^{i/2}}\right)\varphi_{\Sigma(\theta)}(z)\,dz$$

$$\leq \int_{\mathcal{H}_n(C)} \left(1+\sum_{i=1}^{s-2}\frac{\pi_i(z,\theta)}{n^{i/2}}\right)\varphi_{\Sigma(\theta)}(z)\,dz = o\left(n^{-\frac{s-2}{2}}\right)$$

and the last term is independent of θ, due to the properties of γ_n and equation (**A.8**) in the proof of Lemma 2 in Magdalinos [10]. ■

Proof of Theorem 3.2. We first show that

$$M_n(\theta) = \sum_{i=0}^{s-2}\frac{1}{n^{i/2}}A_{i_n}(\theta)\left(S_n(\theta)^{i+1}\right) + Q_n(\theta)$$

with probability $1-o\left(n^{-\frac{s-2}{2}}\right)$ independent of θ with unique $A_{i_n}(\theta)$ satisfying assumption A.4.POL and $Q_n(\theta)$ satisfying condition 3.2.USR. We proceed inductively. When $s=3$, we have that with probability $1-o\left(n^{-\frac{s-2}{2}}\right)$ independent of θ

$$\begin{aligned} M_n(\theta) &= -C_{00_n}^{-1}(\theta)\,C_{01_n}(\theta)\,S_n(\theta) \\ &\quad -\frac{1}{n^{1/2}}C_{00_n}^{-1}(\theta)\,C_{10_n}(\theta)\left(S_n(\theta)^2\right) \\ &\quad -\frac{1}{n^{1/2}}C_{00_n}^{-1}(\theta)\,C_{11_n}(\theta)\left(-C_{00_n}^{-1}(\theta)\,C_{01_n}(\theta)\,S_n(\theta),S_n(\theta)\right) \\ &\quad +Q_n(\theta)\end{aligned}$$

where

$$
\begin{aligned}
Q_n(\theta) &= \frac{1}{n} C_{00_n}^{-1}(\theta) C_{11_n}(\theta) \left(C_{00_n}^{-1}(\theta) C_{10_n}(\theta) \left(S_n(\theta)^2\right), S_n(\theta)\right) \\
&+ \frac{1}{n} C_{00_n}^{-1}(\theta) C_{11_n}(\theta) \left(C_{00_n}^{-1}(\theta) C_{11_n}(\theta) (M_n(\theta), S_n(\theta)), S_n(\theta)\right) \\
&+ \frac{1}{n} C_{00_n}^{-1}(\theta) C_{11_n}(\theta) \left(C_{00_n}^{-1}(\theta) C_{12_n}(\theta) \left(M_n(\theta)^2\right), S_n(\theta)\right) \\
&- \frac{1}{n} C_{00_n}^{-1}(\theta) C_{12_n}(\theta) \left(C_{00_n}^{-1}(\theta) C_{01_n}(\theta) S_n(\theta), C_{00_n}^{-1}(\theta) C_{10_n}(\theta) \left(S_n(\theta)^2\right)\right) \\
&- \frac{1}{n} C_{00_n}^{-1}(\theta) C_{12_n}(\theta) \left(C_{00_n}^{-1}(\theta) C_{10_n}(\theta) \left(S_n(\theta)^2\right), C_{00_n}^{-1}(\theta) C_{01_n}(\theta) S_n(\theta)\right) \\
&- \frac{1}{n} C_{00_n}^{-1}(\theta) C_{12_n} \left(C_{00_n}^{-1}(\theta) C_{01_n}(\theta) S_n(\theta), C_{00_n}^{-1}(\theta) C_{11_n}(\theta) (M_n(\theta), S_n(\theta))\right) \\
&- \frac{1}{n} C_{00_n}^{-1}(\theta) C_{12_n} \left(C_{00_n}^{-1}(\theta) C_{11_n}(\theta) (M_n(\theta), S_n(\theta)), C_{00_n}^{-1}(\theta) C_{01_n}(\theta) S_n(\theta)\right) \\
&- \frac{1}{n^{3/2}} C_{00_n}^{-1}(\theta) C_{12_n}(\theta) \left(C_{00_n}^{-1}(\theta) C_{01_n}(\theta) S_n(\theta), C_{00_n}^{-1}(\theta) C_{12_n}(\theta) \left(M_n(\theta)^2\right)\right) \\
&- \frac{1}{n^{3/2}} C_{00_n}^{-1}(\theta) C_{12_n}(\theta) \left(C_{00_n}^{-1}(\theta) C_{12_n}(\theta) \left(M_n(\theta)^2\right), C_{00_n}^{-1}(\theta) C_{01_n}(\theta) S_n(\theta)\right) \\
&- \frac{1}{n^{3/2}} C_{00_n}^{-1}(\theta) C_{12_n}(\theta) \left(C_{00_n}^{-1}(\theta) C_{10_n}(\theta) \left(S_n(\theta)^2\right), C_{00_n}^{-1}(\theta) C_{11_n}(\theta) (M_n(\theta), S_n(\theta))\right) \\
&- \frac{1}{n^{3/2}} C_{00_n}^{-1}(\theta) C_{12_n}(\theta) \left(C_{00_n}^{-1}(\theta) C_{11_n}(\theta) (M_n(\theta), S_n(\theta)), C_{00_n}^{-1}(\theta) C_{10_n}(\theta) \left(S_n(\theta)^2\right)\right) \\
&- \frac{1}{n^{3/2}} C_{00_n}^{-1}(\theta) C_{12_n}(\theta) \left(C_{00_n}^{-1}(\theta) C_{10_n}(\theta) \left(S_n(\theta)^2\right), C_{00_n}^{-1}(\theta) C_{12_n}(\theta) \left(M_n(\theta)^2\right)\right) \\
&- \frac{1}{n^{3/2}} C_{00_n}^{-1}(\theta) C_{12_n}(\theta) \left(C_{00_n}^{-1}(\theta) C_{12_n}(\theta) \left(M_n(\theta)^2\right), C_{00_n}^{-1}(\theta) C_{10_n}(\theta) \left(S_n(\theta)^2\right)\right) \\
&- \frac{1}{n^{3/2}} C_{00_n}^{-1}(\theta) C_{12_n}(\theta) \left(C_{00_n}^{-1}(\theta) C_{11_n}(\theta) (M_n(\theta), S_n(\theta)), C_{00_n}^{-1}(\theta) C_{12_n}(\theta) \left(M_n(\theta)^2\right)\right) \\
&- \frac{1}{n^{3/2}} C_{00_n}^{-1}(\theta) C_{12_n}(\theta) \left(C_{00_n}^{-1}(\theta) C_{12_n}(\theta) \left(M_n(\theta)^2\right), C_{00_n}^{-1}(\theta) C_{11_n}(\theta) (M_n(\theta), S_n(\theta))\right) \\
&- C_{00_n}^{-1}(\theta) R_n(\theta)
\end{aligned}
$$

this defines uniquely $A_{0_n}(\theta)$, and $A_{1_n}(\theta)$ which have the required properties due to analogous properties of $C_{ij_n}(\theta)$. Furthermore due to the conditions structuring the theorem, lemma AL.2 and the compactness of Θ it is easy to see that there exists a sequence $\gamma_n' = o\left(n^{-\frac{s-2}{2}}\right)$ such that condition 3.2.USR holds for $Q_n(\theta)$. Suppose now that this holds for $s = k$, then for $s = k+1$ we obtain that $A_{i_n}(\theta) = A_{i_n}^*(\theta)$ for $i = 0, \ldots, k-2$, where $A_{i_n}^*(\theta)$ denotes the analogous term of the previous induction step and $A_{(k-1)_n}(\theta)$ is uniquely determined by the $O\left(\frac{1}{n^{(k-2)/2}}\right)$ terms of the following expression

$$
\begin{aligned}
&-C_{00_n}^{-1}(\theta)\sum_{i=1}^{k-3}\frac{1}{n^{i/2}}\sum\nolimits_{j=0}^{i+1}C_{ij_n}(\theta)\left(\left(\sum_{i=0}^{k-3}\frac{A_{i_n}^{*}(\theta)\left(S_n(\theta)^{i+1}\right)}{n^{i/2}}\right)^{j},S_n(\theta)^{i+1-j}\right)\\
&-\frac{1}{n^{(k-2)/2}}C_{00_n}^{-1}(\theta)\sum\nolimits_{j=0}^{i+1}C_{ij_n}(\theta)\left(\left(-C_{00_n}^{-1}(\theta)C_{01_n}(\theta)S_n(\theta)\right)^{j},S_n(\theta)^{i+1-j}\right)\\
&-C_{00_n}^{-1}(\theta)Q_n^{*}(\theta)
\end{aligned}
$$

where $Q_n^{*}(\theta)$ denotes the remainder of the previous induction step. Again $A_{(k-1)_n}$ has the required properties due to analogous properties of $C_{ij_n}(\theta)$. Furthermore Q_n is determined by the $o\left(\frac{1}{n^{(k-2)/2}}\right)$ terms of

$$
\begin{aligned}
&-C_{00_n}^{-1}(\theta)\sum_{i=1}^{k-3}\frac{1}{n^{i/2}}\sum\nolimits_{j=0}^{i+1}C_{ij_n}(\theta)\left(\left(\sum_{i=0}^{k-3}\frac{A_{i_n}^{*}(\theta)\left(S_n(\theta)^{i+1}\right)}{n^{i/2}}\right)^{j},S_n(\theta)^{i+1-j}\right)\\
&-\frac{1}{n^{(k-2)/2}}C_{00_n}^{-1}(\theta)\sum\nolimits_{j=0}^{i+1}C_{ij_n}(\theta)\left(\left(\sum_{i=0}^{k-2}\frac{A_{i_n}^{*}(\theta)\left(S_n(\theta)^{i+1}\right)}{n^{i/2}}\right)^{j},S_n(\theta)^{i+1-j}\right)\\
&-C_{00_n}^{-1}(\theta)Q_n^{*}(\theta)
\end{aligned}
$$

(the largest order of which is $O\left(\frac{1}{n^{(k-1)/2}}\right)$) and again due to the conditions structuring the theorem, lemma AL.2 and the compactness of Θ it is easy to see that there exists a sequence $\gamma_n' = o\left(n^{-\frac{s-2}{2}}\right)$ such that condition 3.2.USR holds for $Q_n(\theta)$. In the light of lemma 3.3 the result would follow if $\sum_{i=0}^{s-2}\frac{1}{n^{i/2}}A_{i_n}(\theta)\left(S_n(\theta)^{i+1}\right)$ admits an analogous Edgeworth expansion. This in turn follows by Theorem 3.1, which applies due to the properties of the $A_{i_n}(\theta)$ established previously and due to condition 3.2.LUE. ■

We will make repeated use of this result in the following section.

4 Example

In this section we present a simple example that utilizes the previous results in order to establish the validity of several M-type (mainly GMM and indirect) estimators in the context of a GARCH$(1,1)$ model.

Assumption A.5 *Consider the set of stationary ergodic and covariance stationary processes defined by the recursion*

$$\begin{aligned} y_j^2 &= \varepsilon_j^2 h_j \\ h_j &= \theta_1 (1-\theta_2-\theta_3) + \left(\theta_2 z_{j-1}^2 + \theta_3\right) h_{j-1} \end{aligned}$$

where the (ε_j) *are* iid, *with* $E\varepsilon_0 = 0$, $E\varepsilon_0^2 = 1$, $E|\varepsilon_0|^{2s+2} < +\infty$ *the distribution of* ε_0 *admits a positive continuous density and* $\theta = (\theta_1, \theta_2, \theta_3)' \in \Theta = \left[\underline{\eta}_\omega, \overline{\eta}_\omega\right] \times \left[\underline{\eta}_\alpha, \overline{\eta}_\alpha\right] \times \left[\underline{\eta}_\beta, \overline{\eta}_\beta\right]$ *where* $\underline{\eta}_\omega, \underline{\eta}_\alpha, \underline{\eta}_\beta > 0$ *and for any* $\theta \in \Theta$, $E\left(\theta_2 |\varepsilon_0|^2 + \theta_3\right)^{s+1} < 1$.

This assumption implies that $E\left(h_j^{s+1}(\theta)\right)$ exists and is independent of j and thereby due to Theorem 3.1 of Bougerol [4] that the recursion defines almost surely unique stationary and ergodic processes represented by

$$y_j^2 = \varepsilon_j^2 \theta_1 (1-\theta_2-\theta_3)\left(1 + \sum_{r=0}^{\infty} \prod\nolimits_{p=0}^{r} \left(\theta_2 \varepsilon_{j-p-1}^2 + \theta_3\right)\right)$$

For any $\theta \in \times$ let $X_j(\theta) = \left(\begin{array}{cccc} y_j^2 & y_j^4 & y_j^2 y_{j-1}^2 & y_j^2 y_{j-2}^2 \end{array}\right)'$.

Proposition 1 *Under assumption A.5* $S_n(\theta)$ *admits a locally uniform Edgeworth expansion of order* $s-2$ *over* Θ, *with* $\Psi_{n,s}(\theta)$ *the formal Edgeworth distribution. Moreover the polynomials of the density of* $\Psi_{n,s}(\theta)$ *satisfy assumption A.4.*EEQ.

Proof. First notice that a dominated convergence argument along with the condition $E\left(\theta_2 |\varepsilon_0|^2 + \theta_3\right)^{s+1} < 1$ and the monotonicity of h w.r.t. θ in A.5, imply that $E\left(y_j^m(\theta)\right)$ exists and is continuous on Θ for any $m = 1, \ldots, 2s+2$. Therefore $\sup_{\theta \in \times} E\|X_0(\theta)\|^{s+1} < +\infty$ establishing A.2.M. This also implies that if the formal Edgeworth expansion is valid, the polynomials of its density, which are equicontinuous functions of these moments and the covariance matrix would satisfy assumption A.4.EEQ. For the establishment of validity we have that for $A_j = \left(\theta_2 \varepsilon_j^2 + \theta_3\right)$, $\Omega_m = \left(1 + \sum_{r=0}^{m-1} \prod_{p=0}^{r} A_{j-p-1}\right)$, $\omega = \theta_1 (1-\theta_2-\theta_3)$

$$\begin{aligned} E\left|y_j^2 - \omega \varepsilon_j^2 \Omega_m\right| &= \theta_1 (\theta_2+\theta_3)(\theta_2+\theta_3)^m \\ &\leq \overline{\eta}_\omega \left(\overline{\eta}_\alpha + \overline{\eta}_\beta\right)\left(\overline{\eta}_\alpha + \overline{\eta}_\beta\right)^m \end{aligned}$$

Analogously, for any $0 \leq k^* < m$

$$h_j h_{j-k^*} - \omega^2 \Omega_m \Omega_{m-k^*} = \omega^2 B_{j,m,k^*}$$

where

$$B_{j,m,k^*} = \frac{h_j}{\omega} \sum_{r=m-k^*}^{\infty} \prod_{p=0}^{r} A_{j-k^*-p-1} + \frac{h_{j-k^*}}{\omega} \sum_{r=m}^{\infty} \prod_{p=0}^{r} A_{j-p-1}$$

and therefore due to the inequality of Cauchy-Schwarch, which is applicable due to the moment existence conditions described before, we have that

$$\begin{aligned}
& E\left|y_j^2 y_{j-k^*}^2 - \omega^2 \varepsilon_j^2 \varepsilon_{j-k^*}^2 \Omega_m \Omega_{m-k^*}\right| \\
= \;& \omega E\left(\varepsilon_{j-k^*}^2 h_j \sum_{r=m-k^*}^{\infty} \prod_{p=0}^{r} A_{j-k^*-p-1}\right) + \omega E\left(\varepsilon_{j-k^*}^2 h_{j-k^*} \sum_{r=m}^{\infty} \prod_{p=0}^{r} A_{j-p-1}\right) \\
\leq \;& \omega E^{1/4}\left(\varepsilon_0^8\right) E^{1/4}\left(h_0^4\right) \left(\sum_{r=m-k^*}^{\infty} \sum_{r^*=m-k^*}^{\infty} \prod_{p=0}^{r} E\left(A_{j-k^*-p-1} \prod_{p^*=0}^{r^*} A_{j-k^*-p^*-1}\right)\right)^{1/2} \\
& + \omega E^{1/4}\left(\varepsilon_0^8\right) E^{1/4}\left(h_0^4\right) \left(\sum_{r=m}^{\infty} \sum_{r^*=m}^{\infty} E \prod_{p=0}^{r} A_{j-p-1} \prod_{p^*=0}^{r^*} A_{j-p^*-1}\right)^{1/2}
\end{aligned}$$

Now

$$E\left(\prod_{p=0}^{r} A_{j-p-1} \prod_{p^*=0}^{r^*} A_{j-p^*-1}\right) = \left(\theta_2^2 E\left(\varepsilon_0^4\right) + \theta_3^2 + 2\theta_2\theta_3\right)^{\min(r,r^*)} \left(\theta_2 + \theta_3\right)^{\max(r,r^*) - \min(r,r^*) + 1}$$

hence the previous expected value is less than or equal to

$$\begin{aligned}
& \omega E^{1/4}\left(\varepsilon_0^8\right) E^{1/4}\left(h_0^4\right) \sqrt{(\theta_2 + \theta_3)} \\
& \times \left(\sum_{r=m-k^*}^{\infty} \left(\left(\frac{\theta_2^2 E\left(\varepsilon_0^4\right) + \theta_3^2 + 2\theta_2\theta_3}{\theta_1 + \theta_2}\right)^r \sum_{r^*=m-k^*}^{\infty} (\theta_2 + \theta_3)^{r^*}\right)\right)^{1/2} \\
& + \omega E^{1/4}\left(\varepsilon_0^8\right) E^{1/4}\left(h_0^4\right) \sqrt{(\theta_2 + \theta_3)} \\
& \times \left(\sum_{r=m}^{\infty} \left(\left(\frac{\theta_2^2 E\left(\varepsilon_0^4\right) + \theta_3^2 + 2\theta_2\theta_3}{\theta_2 + \theta_3}\right)^r \sum_{r^*=m}^{\infty} (\theta_2 + \theta_3)^{r^*}\right)\right)^{1/2}
\end{aligned}$$

which in turn is less than or equal to

$$\begin{aligned}
\leq \;& \overline{\eta}_\omega E^{1/4}\left(\varepsilon_0^8\right) \sup_{\theta \in \Theta} E^{1/4}\left(h_0^4\right) \sqrt{\frac{\left(\overline{\eta}_\alpha + \overline{\eta}_\beta\right)}{1 - \eta_{\max}}} \\
& \times \left(\begin{array}{c} \left(\sqrt{\overline{\eta}_\alpha^2 E\left(\varepsilon_0^4\right) + \overline{\eta}_\beta^2 + 2\overline{\eta}_\alpha \overline{\eta}_\beta}\right)^{m-k^*} \\ + \left(\sqrt{\overline{\eta}_\alpha^2 E\left(\varepsilon_0^4\right) + \overline{\eta}_\beta^2 + 2\overline{\eta}_\alpha \overline{\eta}_\beta}\right)^{m} \end{array}\right) \\
\leq \;& \frac{2\overline{\eta}_\omega E^{1/4}\left(\varepsilon_0^8\right) \sup_{\theta \in \Theta} E^{1/4}\left(h_0^4\right) \sqrt{\left(\overline{\eta}_\alpha + \overline{\eta}_\beta\right)}}{\left(\sqrt{\overline{\eta}_\alpha^2 E\left(\varepsilon_0^4\right) + \overline{\eta}_\beta^2 + 2\overline{\eta}_\alpha \eta_\beta}\right)^{k^*} \sqrt{1 - \eta_{\max}}} \left(\sqrt{\overline{\eta}_\alpha^2 E\left(\varepsilon_0^4\right) + \overline{\eta}_\beta^2 + 2\overline{\eta}_\alpha \overline{\eta}_\beta}\right)^{m}
\end{aligned}$$

where $\eta_{\max}$ is the maximum of $\frac{\left(\theta_2^2 E\left(\varepsilon_0^4\right)+\theta_3^2+2\theta_2\theta_3\right)}{(\theta_2+\theta_3)}$ on Θ which exists and is less than 1 due to the compactness of Θ the continuity of this function and the fact that it is less than 1 for any $\theta \in \Theta$. Finally, for $0 < m < k^*$ and any $k^* > 0$ we have that

$$\begin{aligned}
& h_j h_{j-k^*} - \omega^2 \Omega_m \\
= \;& \omega h_{j-k^*} \left(\sum_{r=m}^{\infty} \prod_{p=0}^{r} A_{j-p+1} \right) + \omega^2 \Omega_m \left(\sum_{r=0}^{\infty} \prod_{p=0}^{r} A_{j-k^*-p+1} \right)
\end{aligned}$$

and therefore

$$\begin{aligned}
& E\left| y_j^2 y_{j-k^*}^2 - \omega^2 \varepsilon_j^2 \varepsilon_{j-k^*}^2 \Omega_m \right| \\
\leq \;& \omega E\left(\varepsilon_{j-k^*}^2 h_{j-k^*} \left(\sum_{r=m}^{\infty} \prod_{p=0}^{r} A_{j-p+1} \right) \right) \\
& + \omega^2 E\left(\varepsilon_{j-k^*}^2 \Omega_m \left(\sum_{r=0}^{\infty} \prod_{p=0}^{r} A_{j-k^*-p+1} \right) \right) \\
\leq \;& \omega E^{1/4}\left(\varepsilon_0^8\right) E^{1/4}\left(h_0^4\right) \left(\sum_{r=m}^{\infty} \sum_{r^*=m}^{\infty} E \prod_{p=0}^{r} A_{j-p-1} \prod_{p^*=0}^{r^*} A_{j-p^*-1} \right)^{1/2} \\
& + \omega E^{1/4}\left(\varepsilon_0^8\right) E^{1/4}\left(h_0^4\right) E^{1/2}\left(h_0^2\right) \\
\leq \;& \overline{\eta}_\omega E^{1/4}\left(\varepsilon_0^8\right) \sup_{\theta\in\Theta} E^{1/4}\left(h_0^4\right) \sqrt{\frac{\left(\overline{\eta}_\alpha + \overline{\eta}_\beta\right)}{1-\eta_{\max}}} \left(\sqrt{\overline{\eta}_\alpha^2 E\left(\varepsilon_0^4\right) + \overline{\eta}_\beta^2 + 2\overline{\eta}_\alpha \overline{\eta}_\beta} \right)^m \\
& \times \left(1 + \frac{\sup_{\theta\in\Theta} E^{1/2}\left(h_0^2\right)}{\sqrt{\frac{1-\eta_{\max}}{\left(\overline{\eta}_\alpha + \overline{\eta}_\beta\right)}} \left(\sqrt{\overline{\eta}_\alpha^2 E\left(\varepsilon_0^4\right) + \overline{\eta}_\beta^2 + 2\overline{\eta}_\alpha \overline{\eta}_\beta} \right)^m} \right) \\
\leq \;& \overline{\eta}_\omega E^{1/4}\left(\varepsilon_0^8\right) \sup_{\theta\in\Theta} E^{1/4}\left(h_0^4\right) \sqrt{\frac{\left(\overline{\eta}_\alpha + \overline{\eta}_\beta\right)}{1-\eta_{\max}}} \left(\sqrt{\overline{\eta}_\alpha^2 E\left(\varepsilon_0^4\right) + \overline{\eta}_\beta^2 + 2\overline{\eta}_\alpha \overline{\eta}_\beta} \right)^m \\
& \times \left(1 + \frac{\sup_{\theta\in\Theta} E^{1/2}\left(h_0^2\right)}{\sqrt{\frac{1-\eta_{\max}}{\left(\overline{\eta}_\alpha + \overline{\eta}_\beta\right)}} \left(\sqrt{\overline{\eta}_\alpha^2 E\left(\varepsilon_0^4\right) + \overline{\eta}_\beta^2 + 2\overline{\eta}_\alpha \overline{\eta}_\beta} \right)^{k^*}} \right)
\end{aligned}$$

These imply that assumption A.2.WD holds. Now $\frac{\partial y_j^2}{\partial \varepsilon_{j-m}^2}$ equals

$$\left\{ \begin{array}{c} h_j \text{ when } m = 0 \\ \varepsilon_j^2 \omega \theta_2 \sum_{r=m-1}^{\infty} \prod_{p=0}^{r-\{m-1\}} \left(\theta_2 \varepsilon_{j-p-1}^2 + \theta_3\right) \text{ when } m > 0 \end{array} \right. \tag{15}$$

$\frac{\partial y_j^4}{\partial \varepsilon_{j-m}^2}$ equals

$$\left\{ \begin{array}{c} 2\varepsilon_j^2 h_j^2 \text{ when } m=0 \\ 2\theta_2 \omega \varepsilon_j^4 h_j \left(\sum_{r=m-1}^{\infty} \prod_{p=0}^{r-\{m-1\}} \left(\theta_2 \varepsilon_{j-p-1}^2 + \theta_3 \right) \right) \text{ when } m>0 \end{array} \right. \tag{16}$$

and for any $k^* > 0$, $\frac{\partial y_j^2 y_{j-k^*}^2}{\partial \varepsilon_{j-m}^2}$ equals

$$\left\{ \begin{array}{c} \varepsilon_{j-k^*}^2 h_j h_{j-k^*} \text{ when } m=0 \\ \omega \varepsilon_j^2 \varepsilon_{j-k^*}^2 \theta_2 \left(\sum_{r=m-1}^{\infty} \prod_{p=0}^{r-\{m-1\}} \left(\theta_2 \varepsilon_{j-p-1}^2 + \theta_3 \right) \right) h_{j-k^*} \text{ when } 0<m<k^* \\ \varepsilon_j^2 h_j h_{j-k^*} + \omega \varepsilon_j^2 \varepsilon_{j-k^*}^2 \theta_2 \left(\sum_{r=k^*-1}^{\infty} \prod_{p=0}^{r-\{k^*-1\}} \left(\theta_2 \varepsilon_{j-p-1}^2 + \theta_3 \right) \right) h_{j-k^*} \text{ when } m=k^* \\ \omega^2 \varepsilon_j^2 \varepsilon_{j-k^*}^2 \theta_2 \left(\begin{array}{c} h_{j-k^*} \sum_{r=m-1}^{\infty} \prod_{p=0}^{r-\{m-1\}} \left(\theta_2 \varepsilon_{j-p-1}^2 + \theta_3 \right) \\ +h_j \sum_{r=m+k^*-1}^{\infty} \prod_{p=0}^{r-\{m-1\}} \left(\theta_2 \varepsilon_{j-k^*-p-1}^2 + \theta_3 \right) \end{array} \right) \text{ when } m>k^* \end{array} \right. \tag{17}$$

hence

$$E\left|\frac{\partial y_j^2}{\partial \varepsilon_0^2}\right| \leq \left\{ \begin{array}{c} \overline{\eta}_\omega \text{ when } j=0 \\ \frac{\overline{\eta}_\omega \overline{\eta}_\alpha}{\overline{\eta}_\alpha + \overline{\eta}_\beta} \left(\overline{\eta}_\alpha + \overline{\eta}_\beta \right)^j \text{ when } j>0 \end{array} \right.$$

$$E\left|\frac{\partial y_j^4}{\partial \varepsilon_0^2}\right|$$
$$\leq \left\{ \begin{array}{c} 2\overline{\eta}_\omega \left(1 + \overline{\eta}_\alpha + \overline{\eta}_\beta \right) \text{ when } j=0 \\ 2\eta_\alpha \overline{\eta}_\omega E\left(\varepsilon_0^4 \right) \sup_{\theta \in \Theta} E^{1/2}\left(h_j^2 \right) \sqrt{\frac{\overline{\eta}_\alpha + \overline{\eta}_\beta}{(1-\eta_{\max})M}} M^j \text{ when } j>0 \end{array} \right.$$

where $M = \left(\eta_\alpha^2 E\left(\varepsilon_0^4 \right) + \eta_\beta^2 + 2\eta_\alpha \eta_\beta \right)$,

$$E\left|\frac{\partial y_j^2 y_{j-k^*}^2}{\partial \varepsilon_0^2}\right|$$
$$\leq \left\{ \begin{array}{c} \sup_{\theta \in \Theta} E\left(h_0 \varepsilon_{-k^*}^2 h_{-k^*} \right) \text{ when } j=0 \\ \omega \theta_2 E^{1/4}\left(\varepsilon_0^8 \right) \sup_{\theta \in \Theta} E^{1/4}\left(h_0^4 \right) \frac{\sqrt{M}^j}{\sqrt{Q}} \text{ when } 0<j<k^* \\ \sup_{\theta \in \Theta} E\left(h_j h_{j-k^*} \right) + \overline{\eta}_\omega^2 \overline{\eta}_\alpha E^{1/4}\left(\varepsilon_0^8 \right) \sup_{\theta \in \Theta} E^{1/4}\left(h_0^4 \right) \frac{\sqrt{M}^j}{\sqrt{Q}} \text{ when } j=k^* \\ \overline{\eta}_\omega^2 \overline{\eta}_\alpha E^{1/4}\left(\varepsilon_0^8 \right) \sup_{\theta \in \Theta} E^{1/4}\left(h_0^4 \right) \left(\frac{\sqrt{M}^j + \sqrt{M}^{j+k^*}}{\sqrt{Q}} \right) \text{ when } j>k^* \end{array} \right.$$

where $Q = M\left(1 - \left(\overline{\eta}_\alpha + \overline{\eta}_\beta\right)\right)\left(1 - \eta_{\max}\right)$. Notice that these do not necessarily imply assumption A.2.EL but are sufficient for the verification of equation 11. The form of 15-17 imply condition A.2.CPD. Finally notice that 15-17 imply

$$\sum_{j=0}^{\infty} \frac{\partial y_j^2}{\partial \varepsilon_0^2} = h_0 + \sum_{j=1}^{\infty} \varepsilon_j^2 \frac{\partial h_j}{\partial \varepsilon_0^2} \tag{18}$$

$$\sum\nolimits_{j=0}^{\infty} \frac{\partial y_j^4}{\partial \varepsilon_0^2} = 2\varepsilon_0^2 h_0 + 2\sum\nolimits_{j=1}^{\infty} \varepsilon_j^4 h_j \frac{\partial h_j}{\partial \varepsilon_0^2} \tag{19}$$

$$\begin{aligned} \sum\nolimits_{j=0}^{\infty} \frac{\partial y_j^2 y_{j-1}^2}{\partial \varepsilon_0^2} &= \varepsilon_{-1}^2 h_0 h_{-1} + \varepsilon_1^2 h_1 h_0 + \varepsilon_1^2 \varepsilon_0^2 \frac{\partial h_1}{\partial \varepsilon_0^2} h_0 \\ &\quad + \sum\nolimits_{j=1}^{\infty} \varepsilon_j^2 \varepsilon_{j-1}^2 \left(\frac{\partial h_j}{\partial \varepsilon_0^2} h_{j-1} + \frac{\partial h_{j-1}}{\partial \varepsilon_0^2} h_j \right) \end{aligned} \tag{20}$$

and

$$\begin{aligned} \sum\nolimits_{j=0}^{\infty} \frac{\partial y_j^2 y_{j-2}^2}{\partial \varepsilon_0^2} &= \varepsilon_{-2}^2 h_0 h_{-2} + \varepsilon_2^2 h_2 h_0 + \varepsilon_1^2 \varepsilon_{-1}^2 \frac{\partial h_1}{\partial \varepsilon_0^2} h_{-1} + \varepsilon_2^2 \varepsilon_0^2 \frac{\partial h_2}{\partial \varepsilon_0^2} h_0 \\ &\quad + \sum\nolimits_{j=1}^{\infty} \varepsilon_j^2 \varepsilon_{j-2}^2 \left(\frac{\partial h_j}{\partial \varepsilon_0^2} h_{j-2} + \frac{\partial h_{j-2}}{\partial \varepsilon_0^2} h_j \right) \end{aligned} \tag{21}$$

and that the right hand sides of 18-20 are series of polynomial functions of (ε^2), hence are linearly dependent iff there exist λ_i, $i = 1, \ldots, 4$ that are independent of j such that the relevant linear combinations of the coefficients of the terms of the same monomials are zero for every j. A simple inspection reveals that this is impossible for P almost all (ε^2) and for all $\theta \in \Theta$. This in turn implies that the determinant of the matrix

$$\sum\nolimits_{j=0}^{\infty} \left(\begin{array}{cccc} \frac{\partial y_j^2}{\partial \varepsilon_{l_i}^2} & \frac{\partial y_j^4}{\partial \varepsilon_{l_i}^2} & \frac{\partial y_j^2 y_{j-1}^2}{\partial \varepsilon_{l_i}^2} & \frac{\partial y_j^2 y_{j-2}^2}{\partial \varepsilon_{l_i}^2} \end{array}, i = 1, \ldots, 4 \right)$$

with distinct l_i where $0 = l_i$ for some i is different from zero for all $\theta \in \Theta$. This along with the continuity of these four terms on Θ the continuity of the determinant and the compactness of Θ imply that condition A.2.NDD holds. Finally lemma 2.6 and theorem 2.1 yield the result. ■

Let

$$b(\theta) = \left(\theta_1, \frac{\theta_2 \left(1 - (\theta_2 + \theta_3)\theta_3\right)}{1 - 2\theta_2\theta_3 - \theta_3^2}, \theta_2 + \theta_3 \right)'$$

and for some compact $B \supseteq b(\Theta)$ define

$$\varphi_n \in \arg\min_{\varphi \in B} \frac{1}{2} \left\| \left(\overline{y^2}, \widehat{\rho}_1, \frac{\widehat{\rho}_2}{\widehat{\rho}_1} \right)' - \varphi \right\|^2$$

where $\overline{y^2} = \frac{1}{n}\sum_{j=1}^{n} y_j^2$, $\widehat{\rho}_i = \frac{\frac{1}{n}\sum_{j=1}^{n}\left(y_t^2 y_{t-i}^2\right) - \left(\overline{y^2}\right)^2}{\frac{1}{n}\sum_{j=1}^{n}\left(y_t^4\right) - \left(\overline{y^2}\right)^2}$. Furthermore define

$$\theta_n \in \arg\min_{\theta \in \Theta} \frac{1}{2} \left\| \varphi_n - b(\theta) \right\|^2$$

It is easy to see that due to the joint measurability, continuity and boundness from below of the criteria, φ_n and θ_n exist (see for example Theorem 2.13 of Molchanov [11]). θ_n essentially corresponds to the definition of the first of the indirect estimators in Gourieroux et. al. [9]. For any compact $\Theta' \subset \operatorname{Int}(\Theta)$ we obtain the following proposition.

Proposition 2 *Under assumption A.5 $\sqrt{n}\,(\varphi_n - b(\theta))$ and $\sqrt{n}\,(\theta_n - \theta)$ admit locally uniform Edgeworth expansions of order $s-2$ over Θ', the polynomials of the density of which satisfy assumption A.4.*EEQ.

Proof. Notice first that with probability $1 - o\left(n^{-\frac{s-2}{2}}\right)$ that does not locally depend on θ, $\varphi_n = \left(\overline{y^2}, \widehat{\rho}_1, \frac{\widehat{\rho_2}}{\widehat{\rho_1}}\right)'$ since due to proposition 1 and the fact that $f(x) = \left(x_1, \frac{x_3 - x_1^2}{x_2 - x_1^2}, \frac{x_4 - x_1^2}{x_2 - x_1^2}\right)$ is continuous hence $\left(\overline{y^2}, \widehat{\rho}_1, \frac{\widehat{\rho_2}}{\widehat{\rho_1}}\right)' \in b(\Theta)$ with probability $1 - o\left(n^{-\frac{s-2}{2}}\right)$ that does not locally depend on θ. Hence due to theorem 3.3 the first result would follow if $\sqrt{n}\left(\left(\overline{y^2}, \widehat{\rho}_1, \frac{\widehat{\rho_2}}{\widehat{\rho_1}}\right)' - b(\theta)\right)$ admits a *locally uniform Edgeworth expansion of order* $s-2$ over Θ' (in the notation of the theorem $S_n = \sqrt{n}\left(\left(\overline{y^2}, \widehat{\rho}_1, \frac{\widehat{\rho_2}}{\widehat{\rho_1}}\right)' - b(\theta)\right)$ and R_n is zero). A Taylor expansion of f-which is independent of θ- around $E(X_0(\theta))$ of order $s-1$ implies that (S_n is as in proposition 1)

$$\sqrt{n}\left(\left(\overline{y^2}, \widehat{\rho}_1, \frac{\widehat{\rho_2}}{\widehat{\rho_1}}\right)' - b(\theta)\right) = \sum_{i=0}^{s-2} \frac{1}{n^{i/2}} D^{(i+1)} f\left(E\left(X_0(\theta)\right)\right)\left(S_n(\theta)\right)^{i+1} + R_n(\theta)$$

where

$$R_n(\theta) = \frac{1}{n^{(s-2)/2}}\left(D^{(s-1)} f\left(R_n^+(\theta)\right)\left(S_n(\theta)\right)^{s-1} - D^{(s-1)} f\left(E\left(X_0(\theta)\right)\right)\left(S_n(\theta)\right)^{s-1}\right)$$

$R_n^+(\theta)$ lies between $\frac{1}{n}\sum_{j=1}^{n} X_j(\theta)$ and $E(X_0(\theta))$ with probability $1 - o\left(n^{-\frac{s-2}{2}}\right)$ that does not depend on θ. Due to the continuity of $D^{(s-1)} f$ on some compact neighborhood of $E(X_0(\theta))$ we have that

$$\|R_n(\theta)\| \leq \frac{\|R_n^+(\theta)\| \, \|S_n(\theta)\|^{s-1}}{n^{(s-2)/2}}$$

Hence the definition of $R_n^+(\theta)$, along with proposition 1, lemma AL.2, and theorem 3.3 imply that the result will hold if $\sum_{i=0}^{s-2} \frac{1}{n^{i/2}} D^{(i+1)} f\left(E\left(X_0(\theta)\right)\right)\left(S_n(\theta)\right)^{i+1}$ admits the relevant Edgeworth expansion. But this holds due to the fact that

$Df(E(X_0(\theta)))$ has rank 3 for any θ hence assumption A.3.POL is satisfied, while assumption A.3.EEQ is satisfied due to proposition 1, hence theorem 3.1 is applicable. For the second case initially observe that due to the first part, for some $\Theta^* = \left[\underline{\eta}^*_\omega, \overline{\eta}^*_\omega\right] \times \left[\underline{\eta}^*_\alpha, \overline{\eta}^*_\alpha\right] \times \left[\underline{\eta}^*_\beta, \overline{\eta}^*_\beta\right]$ where $0 < \underline{\eta}^*_m < \underline{\eta}_m, \overline{\eta}^*_m > \overline{\eta}_m$ for $m = \omega, \alpha, \beta$, such that $\mathrm{Int}(\Theta) \supset \Theta^* \supset \Theta'$

$$\sup_{\theta \in \overline{\mathcal{O}}(\theta_0, \delta)} P\left(\varphi_n(\theta) \in \overline{\mathcal{O}}(\theta_0, \delta^*)\right) = 1 - o\left(n^{-\frac{s-2}{2}}\right)$$

and it is easy to see that $\frac{\partial b}{\partial \theta'}$ has full rank for any θ in $\overline{\mathcal{O}}(\theta_0, \delta^*)$, hence with probability $1 - o\left(n^{-\frac{s-2}{2}}\right)$ that does not locally depend on θ, θ_n satisfies $\varphi_n = b(\theta_n)$. The mean value theorem along with the constant full rank and continuity of $\frac{\partial b}{\partial \theta'}$ on Θ' imply that for some $c > 0$ independent of θ

$$\sup_{\theta \in \overline{\mathcal{O}}(\theta_0, \delta)} P\left(\sqrt{n}\,\|\theta_n - \theta\| \leq c\sqrt{n}\,\|\varphi_n - b(\theta)\|\right) = 1 - o\left(n^{-\frac{s-2}{2}}\right)$$

which along with the result of the first part and lemma AL.2 imply that for some $C^* > 0$ independent of θ

$$\sup_{\theta \in \overline{\mathcal{O}}(\theta_0, \delta)} P\left(\sqrt{n}\,\|\theta_n - \theta\| > C^* \ln^{1/2} n\right) = o\left(n^{-\frac{s-2}{2}}\right) \tag{22}$$

A Taylor expansion of $b(\theta_n)$ around $b(\theta)$ of order $s-1$ implies that (S_n is as in proposition 1)

$$0_{3\times 1} = \sqrt{n}\left(\varphi_n - b(\theta)\right) + \sqrt{n}\sum_{i=0}^{s-2} \frac{1}{n^{i/2}} D^{(i+1)} b(\theta)\left(\sqrt{n}(\theta_n - \theta)\right)^{i+1} + R_n(\theta)$$

where

$$R_n(\theta) = \frac{1}{n^{(s-2)/2}}\left(D^{(s-1)} b\left(\theta_n^+\right)\left(\sqrt{n}(\theta_n - \theta)\right)^{s-1} - D^{(s-1)} b(\theta)\left(\sqrt{n}(\theta_n - \theta)\right)^{s-1}\right)$$

θ_n^+ lies between θ_n and θ with probability $1 - o\left(n^{-\frac{s-2}{2}}\right)$ that does not depend on θ. Due to the continuity of $D^{(s-1)} b$ on some compact neighborhood of θ we have that

$$\|R_n(\theta)\| \leq \frac{\left\|\theta_n^+ - \theta\right\| \left\|\sqrt{n}(\theta_n - \theta)\right\|^{s-1}}{n^{(s-2)/2}}$$

Hence the definition of θ_n^+, along with proposition 2, equation 22, and theorem 3.3 imply that the result will hold if condition POLFOC holds since in this case theorem 3.2 is applicable. But this holds due to the constant full rank of the Jacobian of b. ■

For the final part of this section we utilize the following assumption.

Assumption A.6 *$s \geq 5$ and $\varepsilon_0 \sim N(0,1)$.*

Remember that φ_n and θ_n depend on X_i which in turn depends on θ. When needed this dependence will be expicitely denoted. Define

$$\varphi_n^* \in \arg\min_{\theta \in \Theta} \frac{1}{2} \left\| \varphi_n - E\left(\varphi_n(\theta)\right) \right\|^2$$

and

$$\theta_n^* \in \arg\min_{\theta \in \Theta} \frac{1}{2} \left\| \theta_n - E\left(\theta_n(\theta)\right) \right\|^2$$

These correspond to the definition of the second of the indirect estimators in Gourieroux et. al. [9]. The boundness of B and Θ imply that the relevant expectations exist for any θ. The P almost everywhere continuity of X_i w.r.t. θ along with the definitions of φ_n and θ_n and the boundness of B and Θ ensure via the dominated convergence theorem that the expectations are also continuous and this along with joint measurability and the boundness from below of the criteria imply again the existence of φ_n^* and θ_n^*. Similarly to the previous case we obtain the following proposition.

Proposition 3 *Suppose that $\sqrt{n}\left(\varphi_n - b(\theta)\right)$ and $\sqrt{n}\left(\theta_n - \theta\right)$ admit locally uniform Edgeworth expansions of order $s-2$ over Θ' the polynomials of the densities of which satisfy assumption A.4.*EEQ *and A.6 holds. Then$\sqrt{n}\left(\varphi_n^* - \theta\right)$ and $\sqrt{n}\left(\theta_n^* - \theta\right)$ admit locally uniform Edgeworth expansions of order $s-3$ over Θ'' for any compact $\Theta'' \subset \Theta'$.*

For the proof of the previous proposition we use the following auxiliary results. In the following lemma $m_n(\theta)$ denotes a generic random element admitting values in a bounded subset of some Euclidean space.

Lemma 4.1 *Suppose that $\sqrt{n}m_n(\theta)$ admits a locally uniform Edgeworth expansion of order $s-2$ over Θ', the polynomials of the density of which satisfy assumption A.4.*EEQ. *Then $\sqrt{n}\left(m_n(\theta) - Em_n(\theta)\right)$ admits a locally uniform Edgeworth expansion of order $s-3$ over $\overline{\mathcal{O}}(\theta_0, \delta)$, the polynomials of the density of which satisfy assumption A.4.*EEQ.

Proof. Due to Lemma 3.1 of Arvanitis and Demos [1] we have that

$$\begin{aligned}
& \sup_{\theta \in \Theta'} \left| \sqrt{n} E_\theta m_n - \int_{\mathbb{R}} z \left(1 + \sum_{i=1}^{s-3} \frac{\pi_i(z,\theta)}{n^{(i+1)/2}}\right) \varphi_{V(\theta)}(z)\, dz \right| \\
= & \sup_{\theta \in \Theta'} \left| \sqrt{n} E_\theta m_n - \sum\nolimits_{i=1}^{s-3} \frac{\mathcal{I}_V\left(k_i(z,\theta)\right)}{n^{i/2}} \right| = o\left(n^{-\frac{s-3}{2}}\right)
\end{aligned}$$

where $\left(1+\sum_{i=1}^{s-3}\frac{\pi_i(z,\theta)}{n^{i/2}}\right)\varphi_{V(\theta)}(z)$ denotes the density of the Edgeworth distribution of proposition 2 truncated up to the $O\left(n^{-\frac{s-3}{2}}\right)$ order, i.e. of the (obviously) valid locally uniform Edgeworth expansion of order $s-3$, given the one in 2, $k_i(z,\theta)=z\pi_i(z,\theta)$ and $\mathcal{I}_V(k_i(z,\theta))=\int_{\mathbb{R}}k_i(z,\theta)\varphi_{V(\theta)}(z)\,dz$. Using the fact that the π_i's satisfy assumption A.4.EEQ it is easy to see that so do the $\mathcal{I}_V(k_i(z,\theta))$. Now for an arbitrary Borel set A

$$
\begin{aligned}
& P\left(\sqrt{n}\left(m_n(\theta)-Em_n(\theta)\right)\in A\right)\\
= & P\left(\sqrt{n}m_n(\theta)\in A+\sum_{i=1}^{s-3}\frac{\mathcal{I}_V(k_i(z,\theta))}{n^{i/2}}+o\left(n^{-\frac{s-3}{2}}\right)\right)\\
= & \int_{A\cap\mathcal{H}_n^c(C)}\left(1+\sum_{i=1}^{s-3}\frac{\pi_i\left(z+\sum_{i=1}^{s-3}\frac{\mathcal{I}_V(k_i(z,\theta))}{n^{i/2}}+o\left(n^{-\frac{s-3}{2}}\right),\theta\right)}{n^{i/2}}\right)\\
& \times\varphi_{V(\theta)}\left(z+\sum_{i=1}^{s-3}\frac{\mathcal{I}_V(k_i(z,\theta))}{n^{i/2}}+o\left(n^{-\frac{s-3}{2}}\right)\right)dz+o\left(n^{-\frac{s-3}{2}}\right)
\end{aligned}
$$

where $\mathcal{H}_n^c(C^*)$ analogously to the relevant term in the proof of theorem 3.1. Expanding and holding terms of relevant order, by noticing that the π_i are polynomial in z, and that the $o\left(n^{-\frac{s-2}{2}}\right)$ are independent of θ we obtain the needed result. ■

The second auxiliary result is the only one making use of assumption A.6.

Lemma 4.2 *Suppose that $\sqrt{n}(\varphi_n-b(\theta))$ and $\sqrt{n}(\theta_n-\theta)$ admit locally uniform Edgeworth expansions of order $s-2$ over Θ' the polynomials of the densities of which satisfy assumption A.4.EEQ and A.6 holds. Then $E(\varphi_n(\theta))$ and $E(\theta_n(\theta))$ are two times differentiable on Θ' and for any $\theta\in\Theta'$ and any sequence $\theta_n\neq\theta$ with values in Θ' such that $\|\theta_n-\theta\|\leq C\frac{\ln^{1/2}n}{n^{1/2}}$ for $C>0$, $i=1,2$ $\left\|\frac{\partial M_{i_n}(\theta_n)}{\partial\theta'}-K_i(\theta)\right\|=o(1)$ where $M_{1_n}(\theta)=E(\varphi_n(\theta))$, $M_{2_n}(\theta)=E(\theta_n(\theta))$, $K_1=\frac{\partial b}{\partial\theta'}$, $K_2=\mathrm{id}_{\mathbb{R}^3}$.*

Proof. Consider first the case of $E(\varphi_n(\theta))$. Let $\sigma(\varepsilon_0)$ the smallest sub σ-algebra of $\mathcal{F}$ w.r.t. the $\varepsilon_0,\varepsilon_{-1},\ldots$ are measurable. We have that

$$E(\varphi_n(\theta))=E\left(E\left(\varphi_n(\theta)/\sigma(\varepsilon_0)\right)\right)$$

Now notice that

$$E\left(\varphi_n(\theta)/\sigma(\varepsilon_0)\right)=\int_{\mathbb{R}^n}\varphi_n\frac{1}{\sqrt{(2\pi)^n\prod_{j=1}^n h_i(\theta)}}\exp\left(-\frac{1}{2}\sum_{i=1}^n\frac{y_j^2(\theta)}{h_j(\theta)}\right)dz$$

and the differentiability result would follow via the dominated convergence theorem if

$$E\left(\sup_{\theta\in\Theta'}\left\|s_n(\theta)\right\|\right) \text{ and } E\left(\sup_{\theta\in\Theta'}\left\|H_n(\theta)\right\|\right)$$

are finite where $s_n(\theta) \doteqdot \sum_{j=1}^{n}\left(\varepsilon_j^2-1\right)\frac{1}{h_j(\theta)}\frac{\partial h_j(\theta)}{\partial\theta}$, $H_n(\theta) \doteqdot \sum_{j=1}^{n}\left(\varepsilon_j^2-1\right)\frac{1}{h_j(\theta)}\frac{\partial^2 h_j(\theta)}{\partial\theta\partial\theta'} - \sum_{j=1}^{n}\left(2\varepsilon_j^2-1\right)\frac{1}{h_j^2(\theta)}\frac{\partial h_j(\theta)}{\partial\theta}\frac{\partial h_j(\theta)}{\partial\theta'}$, $\overline{s}_n(\theta)=\frac{1}{n}s_n(\theta)$, $\overline{H}_n(\theta)=\frac{1}{n}H_n(\theta)$. First notice that $h_j(\theta)\geq\underline{\eta}_\omega\left(1-\overline{\eta}_\alpha-\overline{\eta}_\beta\right)\doteqdot c_*$ and due to the fact that

$$\begin{aligned}
\frac{\partial h_j(\theta)}{\partial\theta_1} &= (1-\theta_2-\theta_3)+\left(\theta_2\varepsilon_{j-1}^2+\theta_3\right)\frac{\partial h_{j-1}(\theta)}{\partial\theta_1}\\
\frac{\partial h_j(\theta)}{\partial\theta_2} &= -\theta_1+\varepsilon_{j-1}^2h_{j-1}(\theta)+\left(\theta_2\varepsilon_{j-1}^2+\theta_3\right)\frac{\partial h_{j-1}(\theta)}{\partial\theta_2}\\
\frac{\partial h_j(\theta)}{\partial\theta_3} &= -\theta_1+h_{j-1}(\theta)+\left(\theta_2\varepsilon_{j-1}^2+\theta_3\right)\frac{\partial h_{j-1}(\theta)}{\partial\theta_3}
\end{aligned}$$

hence

$$\begin{aligned}
& E\left(\sup_{\theta\in\Theta'}\left\|\sum_{j=1}^{n}\left(\varepsilon_j^2-1\right)\frac{1}{h_j(\theta)}\frac{\partial h_j(\theta)}{\partial\theta}\right\|\right)\\
\leq\ & \frac{1}{c_*}\sum_{j=1}^{n}E^{1/2}\left|\varepsilon_j^2-1\right|^2E^{1/2}\sup_{\theta\in\Theta'}\left\|\frac{\partial h_j(\theta)}{\partial\theta}\right\|^2
\end{aligned}$$

and for $\theta^*=\left(\overline{\eta}_\omega^*,\underline{\eta}_\alpha^*,\underline{\eta}_\beta^*\right)'$ it is easy to see that

$$E\sup_{\theta\in\Theta'}\left\|\frac{\partial h_j(\theta)}{\partial\theta}\right\|^2\leq E\left\|\frac{\partial h_j(\theta^*)}{\partial\theta}\right\|^2<+\infty$$

Furthermore, since

$$\begin{aligned}
\frac{\partial^2 h_j(\theta)}{\partial\theta_1^2} &= 0\\
\frac{\partial^2 h_j(\theta)}{\partial\theta_2^2} &= -\theta_1+\varepsilon_{j-1}^2\frac{\partial h_j(\theta)}{\partial\theta_2}+\varepsilon_{j-1}^2\frac{\partial h_{j-1}(\theta)}{\partial\theta_2}+\left(\theta_2\varepsilon_{j-1}^2+\theta_3\right)\frac{\partial h_{j-1}(\theta)}{\partial\theta_2^2}\\
\frac{\partial^2 h_j(\theta)}{\partial\theta_3^2} &= -\theta_1+2\frac{\partial h_{j-1}(\theta)}{\partial\theta_3}+\left(\theta_2\varepsilon_{j-1}^2+\theta_3\right)\frac{\partial^2 h_{j-1}(\theta)}{\partial\theta_3^2}\\
\frac{\partial^2 h_j(\theta)}{\partial\theta_1\partial\theta_2} &= -1+\varepsilon_{j-1}^2\frac{\partial h_{j-1}(\theta)}{\partial\theta_1}+\left(\theta_2\varepsilon_{j-1}^2+\theta_3\right)\frac{\partial h_{j-1}(\theta)}{\partial\theta_1\partial\theta_2}\\
\frac{\partial^2 h_j(\theta)}{\partial\theta_1\partial\theta_3} &= -1+\frac{\partial h_{j-1}(\theta)}{\partial\theta_1}+\left(\theta_2\varepsilon_{j-1}^2+\theta_3\right)\frac{\partial^2 h_j(\theta)}{\partial\theta_1\partial\theta_3}
\end{aligned}$$

we have that

$$
\begin{aligned}
& E\left(\sup_{\theta\in\Theta'}\left\|\sum\nolimits_{j=1}^{n}\left(\varepsilon_j^2-1\right)\frac{1}{h_j\left(\theta\right)}\frac{\partial^2 h_j\left(\theta\right)}{\partial\theta\partial\theta'}\right\|\right)\\
\leq\ & \frac{1}{c_*}\sum\nolimits_{j=1}^{n}E^{1/2}\left|\varepsilon_j^2-1\right|^2 E^{1/2}\left\|\frac{\partial^2 h_j\left(\theta^*\right)}{\partial\theta\partial\theta'}\right\|^2<+\infty
\end{aligned}
$$

and

$$
\begin{aligned}
& E\left(\sup_{\theta\in\Theta'}\left\|\sum\nolimits_{j=1}^{n}\left(2\varepsilon_j^2-1\right)\frac{1}{h_j^2\left(\theta\right)}\frac{\partial h_j\left(\theta\right)}{\partial\theta}\frac{\partial h_j\left(\theta\right)}{\partial\theta'}\right\|\right)\\
\leq\ & \frac{1}{c_*^2}\sum\nolimits_{j=1}^{n}E^{1/2}\left|2\varepsilon_j^2-1\right|^2 E^{1/2}\left\|\frac{\partial h_j\left(\theta^*\right)}{\partial\theta}\right\|^4<+\infty
\end{aligned}
$$

Next notice that for any θ in Θ' any $i=1,\ldots,3$, and any sequence θ_n as described above we have that

$$
\begin{aligned}
& \left\|\frac{\partial E\left(\varphi_n\left(\theta_n\right)\right)}{\partial\theta_i}-\frac{\partial b\left(\theta\right)}{\partial\theta_i}\right\|\\
\leq\ & 2\sup_{\theta^*\in\Theta'}\left\|\frac{\partial^2 E\left(\varphi_n\left(\theta^*\right)\right)}{\partial\theta_i\partial\theta'}\right\|\left\|\theta_n-\theta\right\|+\left\|\frac{E\left(\varphi_n\left(\theta_n\right)\right)-E\left(\varphi_n\left(\theta\right)\right)}{\theta_{i_n}-\theta_i}-\frac{\partial b\left(\theta\right)}{\partial\theta_i}\right\|
\end{aligned}
$$

Then lemma 2.3 of Arvanitis and Demos [1] implies that due to the behavior of θ_n the last term on the right hand side of the last display is $o\left(1\right)$. Hence the result would follow if $\sup_{\theta^*\in\Theta''}\left\|\frac{\partial^2 E\left(\varphi_n\left(\theta^*\right)\right)}{\partial\theta_i\partial\theta'}\right\|=o\left(\frac{\sqrt{n}}{\ln^{1/2}n}\right)$. The previous along with an application of the Cauchy-Schwarz and the triangle inequalities imply that for any i

$$
\begin{aligned}
& \sup_{\theta\in\Theta'}\left\|\frac{\partial^2 E\left(\varphi_n\left(\theta\right)\right)}{\partial\theta_i\partial\theta'}\right\|\\
\leq\ & \sup_{\theta\in\Theta'}E^{1/2}\left\|\varphi_n\left(\theta\right)-\theta\right\|^2\\
& \times\left(\sup_{\theta\in\Theta'}E^{1/2}\left\|s_n\left(\theta\right)s_n'\left(\theta\right)-EH_n\left(\theta\right)\right\|^2+\sup_{\theta\in\Theta'}E^{1/2}\left\|H_n\left(\theta\right)-EH_n\left(\theta\right)\right\|^2\right)
\end{aligned}
$$

Furthermore, due to assumed Edgeworth approximation for $\sqrt{n}\left(\varphi_n\left(\theta\right)-\theta\right)$, and the fact that $s\geq 5$ lemma 3.1 of Arvanitis and Demos [1] along with theorem 3.1 imply that $\sup_{\theta\in\Theta'}E^{1/2}\left\|\left(\varphi_n-b\left(\theta\right)\right)\right\|^2=O\left(\frac{1}{\sqrt{n}}\right)$. Hence the result would follow if

$$
\begin{aligned}
\sup_{\theta\in\Theta'}E\left\|n\overline{s}_n\left(\theta\right)\overline{s}_n'\left(\theta\right)+E\overline{H}_n\left(\theta\right)\right\|^2 &= o\left(\frac{n}{\ln n}\right)\\
\sup_{\theta\in\Theta'}E\left\|\overline{H}_n\left(\theta\right)-E\overline{H}_n\left(\theta\right)\right\|^2 &= o\left(\frac{n}{\ln n}\right)
\end{aligned}
$$

From the proof of Lemma A.1 of Corradi and Inglesias [5], we can prove that $\sqrt{n}\left(S_n^*(\theta) - E\left(S_n^*(\theta)\right)\right)$, where S_n^* contains stacked the elements of $\overline{s}_n$ and $\overline{H}_n$ admits *a locally uniform Edgeworth expansion of order* $s-4$ over Θ' by establishing the conditions A.2.M-WD and A.3.EL-CPD through the provision of bounds independent of θ using the compactness of Θ' and condition A.3.NDD using the result of the referenced proof, the P almost everywhere continuity of the elements of $S_n^*(\theta)$ on Θ', the continuity of det and the compactness of Θ'. Then the remark immediately after the proof of lemma 3.1 of Arvanitis and Demos [1] implies that

$$\begin{aligned}
\sup_{\theta\in\Theta'} E\left\|n\overline{s}_n(\theta)\overline{s}_n'(\theta) + E\overline{H}_n(\theta)\right\|^2 &= O(1) \\
\sup_{\theta\in\Theta'} E\left\|\overline{H}_n(\theta) - E\overline{H}_n(\theta)\right\|^2 &= O\left(\frac{1}{n}\right)
\end{aligned}$$

which establish the needed bounds. The result about $E\left(\theta_n(\theta)\right)$ is derived analogously. ■

We are now ready to prove the main proposition.

Proof. Notice first that uniform consistency of φ_n to $b(\theta)$ along with the boundeness of Θ imply by uniform integrability that

$$\sup_{\theta\in\Theta}\left|E_\theta\varphi_n - b(\theta)\right| = o(1) \tag{23}$$

hence for any $\varepsilon > 0$

$$\begin{aligned}
&\sup_{\theta^*\in\Theta} P\left(\sup_{\theta\in\Theta}\left|\left|\varphi_n - E\varphi_n(\theta)\right| - \left|b(\theta^*) - b(\theta)\right|\right| > \varepsilon\right) \\
\leq\ &\sup_{\theta^*\in\Theta} P\left(\left|\varphi_n - b(\theta^*)\right| + o(1) > \varepsilon\right) = o\left(n^{-\frac{s-2}{2}}\right)
\end{aligned}$$

due to the analogous consistency of φ_n. Hence

$$\sup_{\theta^*\in\Theta} P\left(\varphi_n^* \in \mathcal{O}(\theta^*,\varepsilon)\cap\Theta\right) = 1 - o\left(n^{-\frac{s-2}{2}}\right)$$

for any $\varepsilon > 0$. Then from lemma 4.2 and the proofs of lemma 2 and lemma 2.4 of Arvanitis and Demos [1] we obtain that

$$\sup_{\theta^*\in\Theta''} P\left(\sqrt{n}\left|\varphi_n^* - \theta\right| > C\ln^{1/2} n\right) = o\left(n^{-\frac{s-2}{2}}\right) \tag{24}$$

for some appropriate $C > 0$. Now by recursive examination it is easy to see that $Eh_0^m(\theta)$ is s times continuously differentiable for any θ in Θ'' for all $m = 1, \ldots, s+1$. This along the analogous differentiability of f in the proof of proposition 2 imply that the π_i there are also s times continuously differentiable for any θ in Θ'' for any $z \in \mathbb{R}$. Then dominated convergence implies the same for $\mathcal{I}_V(k_i(z,\theta))$ for all $i = 1, \ldots, s+2$. Then lemma 2.3 of Arvanitis and Demos [1] implies that for any stochastic sequence $\widetilde{\theta}_n$ for which

$$\sup_{\theta\in\Theta''} P\left(\sqrt{n}\left|\widetilde{\theta}_n - \theta\right| > C\ln^{1/2} n\right)$$

then

$$\sup_{\theta\in\Theta''} P\left(\left\|\sqrt{n}\left(E_{\widetilde{\theta}_n}\varphi_n - E_\theta\varphi_n\right) - A_n(\theta)\right\| > \gamma_n\right) = o\left(n^{-\frac{s-2}{2}}\right)$$

where

$$A_n(\theta) = \sum_{j=0}^{s-2} \frac{1}{n^{\frac{j}{2}}(j+1)!} D^{j+1}\left(b(\theta) + \sum_{k=1}^{s-2-j} \frac{\mathcal{I}_V(k_k(z,\theta))}{n^{\frac{k}{2}}}\right)\left(\sqrt{n}\left(\widetilde{\theta}_n - \theta\right)^{j+1}\right)$$

and $\gamma_n = o(n^{-a})$ independent of θ. This along with lemma 4.2 imply that $\frac{\partial E\varphi_n(\varphi_n^*)}{\partial\theta}$ converges to $\frac{\partial b(\theta)}{\partial\theta'}$ for any θ in Θ'' with probability $1 - o\left(n^{-\frac{s-2}{2}}\right)$ independent of θ, hence with the same probability φ_n^* satisfies $\varphi_n = E_{\varphi_n^*}\varphi_n$. Hence with probability $1 - o\left(n^{-\frac{s-2}{2}}\right)$ independent of θ, φ_n^* satisfies

$$0 = \sqrt{n}\left(\varphi_n - E_\theta\varphi_n^*\right) + A_n(\theta) + R_n(\theta)$$

and the result follows from 24, proposition 4.1, lemma AL.2 and theorem 3.2. The case of θ_n^* follows in complete analogy to the previous by simply replacing in the previous proof any invocation to f with $b^{-1}(\varphi) = \left(\varphi_1, \frac{1-\varphi_3^2-\sqrt{\left(1-(2\varphi_2-\varphi_3)^2\right)\left(1-\varphi_3^2\right)}}{2(\varphi_2-\varphi_3)}, \frac{-\left(1-2\varphi_2\varphi_3+\varphi_3^2\right)+\sqrt{\left(1-(2\varphi_2-\varphi_3)^2\right)\left(1-\varphi_3^2\right)}}{2(\varphi_2-\varphi_3)}\right)$ and of b with the identity. ■

Remark R.4 *Notice that θ_n^* can be shown to be locally uniformly second order unbiased (i.e. $\sup_{\theta\in\Theta''}\|E_\theta\theta_n^* - \theta\| = o(n^{-1})$) something that is not the case for θ_n or φ_n^*, even though the three estimators posess the same second order* MSE *uniformly over Θ'' (see Arvanitis and Demos [1] Corollary 2 and Lemma 3.6).*

5 Conclusions

We have established sufficient conditions for the existence of localy uniform Edgeworth expansions under weak dependence and/or smooth transformations. These extend analogous *pointwise* results in the relevant literature and can be applied for the establishment of high order asymptotic properties of estimators arising in the context of eligible stochastic processes. Special cases are M-estimators defined by the expectation of auxiliary ones. In these cases the results enable the polynomial approximation of the equations that are asymptoticaly satisfied by the estimators without the use of higher order derivatives, something that avoids the establishment of issues such as their rates of convergence. The interest on these estimators lies on the fact that under appropriate conditions they can possess desirable higher order properties. A question for future research concerns the issue of establishing Edgeworth type expansions (see Magdalinos [10]) when θ lies in the boundary of the parameter space.

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