**APPENDIX ONE**: ***Proofs of Theorems***

Any smooth feasible policy satisfies the following two Lemmas.

**Lemma 1**: For $y(\alpha, \beta, \gamma) \in C^1$ we have

(a) $Y_\alpha' H_x = 0_{\kappa m}$, (β) $Y_\beta' H_x + H_\beta = 0_{lm}$ and

(γ) $Y_\gamma' H_x = I_{mm}$.

proof: obvious, by differentiating $h^j(y(\alpha,\beta,\gamma), \beta) = \gamma_j$, $j = 1, \ldots, m$, with respect to $\alpha$, $\beta$, or $\gamma$ and forming the matrices.

**Lemma 2**: For $y(\alpha, \beta, \gamma) \in C^2$ we have:

(α,α) $Y_a' H_{xx} Y_a + [y'_{\alpha_i \alpha_j} H_x \lambda] = 0_{\kappa\kappa}$,

(α,β) $Y_\alpha' H_{xx} Y_\beta + Y_\alpha' H_{x\beta} + [y'_{\alpha_j \beta_j} H_x \lambda] = 0_{\kappa l}$ ,

(α,γ) $Y_\alpha' H_{xx} Y_\gamma + [y'_{\alpha_i \gamma_j} H_x \lambda] = 0_{\kappa m}$,

(β,α) $Y_\beta' H_{xx} Y_a + [y'_{\beta_i \alpha_j} H_x \lambda] + H_{\beta x} Y_\alpha = 0_{l\kappa}$,

(β,β) $Y_\beta' H_{xx} Y_\beta + Y_\beta' H_{x\beta} + [y'_{\beta_i \beta_j} H_x \lambda] + H_{\beta x} Y_\beta + H_{\beta\beta} = 0_{ll}$ ,

(β,γ) $Y_\beta' H_{xx} Y_\gamma + [y'_{\beta_i \gamma_j} H_x \lambda] + H_{\beta x} Y_\gamma = 0_{lm}$ ,

(γ,α) $Y_\gamma' H_{xx} Y_\alpha + [y'_{\gamma_i \alpha_j} H_x \lambda] = 0_{m\kappa}$,

(γ,β) $Y_\gamma' H_{xx} Y_\beta + Y_\gamma' H_{x\beta} + [y'_{\gamma_i \beta_j} H_x \lambda] = 0_{ml}$,

(γ,γ) $Y_\gamma' H_{xx} Y_\gamma + [y'_{\gamma_i \gamma_j} H_x \lambda] = 0_{mm}$.

Each matrix within the brackets has elements consisting of inner products of vectors like e.g. $y'_{\alpha_i \alpha_j} = (y^1_{\alpha_i \alpha_j}, \ldots, y^n_{\alpha_i \alpha_j})$ and $H_x \lambda$

proof: We differentiate $Y_\alpha' h_x^j = 0_\kappa, Y_\beta' h_x^j = 0_l$ and $Y_\gamma' h_x^j = 0_m, j = 1,..,m,$ with respect to $\alpha$, $\beta$, or $\gamma$. Thus e.g. we get

$Y_\alpha' h_{xx}^j Y_\alpha + [y'_{\alpha_i\alpha_j}\ h_x^j] = 0_{\kappa\kappa}, j = 1,...,m,$ which we multiply by $\lambda_j$ and add over $j$.

Thus we obtain

$$Y_\alpha' H_{xx}\ Y_\alpha + [y'_{\alpha_i\alpha_j}\ H_x\lambda] = 0_{\kappa\kappa}.$$

The multiplication by $\lambda$ has no other purpose but to present the second-order derivative properties of feasible paths in a way immediately usuable in further proofs.

**Theorem 1** is well known. We give here its direct proof which requires only three lines

(α) $\varphi_\alpha = X_\alpha' f_x + f_a = X_\alpha' H_x\lambda + f_\alpha = f_\alpha,$

(β) $\varphi_\beta = X_\beta' f_x = X_\beta' H_x\lambda = -H_\beta\lambda,$

(γ) $\varphi_\gamma = X_\gamma' f_x = X_\gamma' H_x\lambda = \lambda.$

In all of these, the second equality uses the f-o-c of problem (I) while the third results from *Lemma 1*.

**Theorem 2** is easily proved. Using the last expression for (α), (β), or (γ), we differentiate $\varphi_\alpha$, $\varphi_\beta$ and $\varphi_\gamma$ with respect to $\alpha$, $\beta$, or $\gamma$.

If we used, instead, the expressions for $\varphi_\alpha$, $\varphi_\beta$, or $\varphi_\gamma$ after the second equality, then we would have e.g.

(α,α) $\Phi_{\alpha\alpha} = X_\alpha' F_{xx} X_\alpha + X_\alpha' F_{x\alpha} + [x'_{\alpha_i\alpha_j} f_x] + F_{\alpha x} X_\alpha + F_{\alpha\alpha}$

and subtracting the corresponding term that appears in *Lemma 2,* we get

(α,α) $\Phi_{\alpha\alpha} = X_\alpha' A X_\alpha + X_\alpha' F_{x\alpha} + [x'_{\alpha_i\alpha_j}(f_x - H_x\lambda)] + F_{\alpha x} X_a + F_{\alpha\alpha}$

Thus we see, using again the f-o-c of (I), that the matrix in brackets is a zero matrix. Finally, comparing $\Phi_{\alpha\alpha}$ as given above and in *Theorem 2* we see that the optimal policy must satisfy the equalities

(α,α) $X_a{}'AX_\alpha + X_\alpha' F_{x\alpha} = 0_{\kappa\kappa}$ .

Following exactly the same steps, we prove the following:

**Lemma 3**: The solution of problem (I) satisfies the equalities

(α,α) $X_a{}'AX_\alpha + X_\alpha' F_{x\alpha} = 0_{\kappa\kappa}$ ,

(α,β) $X_a{}'AX_\beta - X_\alpha' H'_{x\beta} = 0_{\kappa l}$ ,

(α,γ) $X_a{}'AX_\gamma = 0_{\kappa m}$ ,

(β,α) $H_\beta \Lambda_\alpha + X_\beta' AX_\alpha + X_\beta{}' F_{x\alpha} = 0_{l\kappa}$ ,

(β,β) $H_\beta \Lambda_\beta + X_\beta' AX_\beta - X'_\beta H_{x\beta} = 0_{ll}$ ,

(β,γ) $H_\beta \Lambda_\gamma + X_\beta' AX_\gamma = 0_{lm}$ ,

(γ,α) $\Lambda_\alpha = X_\gamma' AX_\alpha + X_\gamma' F_{x\alpha}$ ,

(γ,β) $\Lambda_\beta = X_\gamma' AX_\beta - X_\gamma' H_{x\beta}$ ,

(γ,γ) $\Lambda_\gamma = X_\gamma' AX_\gamma$ ,

*Theorems 1, 2* and *Lemma 3* show how optimality and feasibility properties of *x(α, β, γ)* are combined in $(\varphi_\alpha, \varphi_\beta, \varphi_\gamma)'$ and $\Phi$. This matrix can be expressed wholly in terms of $X_\alpha$, $X_\beta$ and $X_\gamma$, if we so wish.

We come finally to *g(α, β, γ)* and its first-order derivatives in (14). The proof of **Theorem 4** follows from the differentiation of (14) with respect to *α, β,* or *γ* and the use of *Lemma 2*. Thus e.g.

$$G_{\alpha\alpha} = \Phi_{\alpha\alpha} - Y_\alpha' F_{xx} Y_\alpha - Y_\alpha' F_{x\alpha} - [\, y'_{\alpha_i \alpha_j} \; f_x \,] - F_{\alpha x} Y_\alpha - F_{\alpha\alpha}$$

If we add the corresponding term in *Lemma 2*, we get

$$G_{\alpha\alpha} = \Phi_{\alpha\alpha} - Y_{\alpha}^{'} A Y_{\alpha} - Y_{\alpha}^{'} F_{x\alpha} - [\, y'_{\alpha_i \alpha_j} \,(f_x - H_x \lambda)] - F_{\alpha x} Y_{\alpha} - F_{\alpha\alpha}$$

and thus at $(\alpha^o, \beta^o, \gamma^o)$, where $y(\alpha^o, \beta^o, \gamma^o) = x(\alpha^o, \beta^o, \gamma^o)$, the matrix in brackets drops out and we have

$$G_{\alpha\alpha}^{o} = F_{\alpha x}^{o} \{ X_{a}^{o} - Y_{a}^{o} \} - Y_{\alpha}^{o'} A^{o} Y_{a}^{o} - Y_{a}^{o'} F_{x\alpha}^{o} \,.$$

The same process is used in deriving all the terms of $G^o$.

Our proof of *Theorem 4* shows clearly why the second-order partial derivatives of $f(y(\alpha, \beta, \gamma), \alpha)$ are expressed solely in terms of $Y_\alpha$, $Y_\beta$, or $Y_\gamma$, at $(\alpha^o, \beta^o, \gamma^o)$ and only there.