APPENDIX TWO: ***The Compensated Version of Problem (I)***

We have seen in the paper the need for considering problem

$\psi(\alpha, \beta; z) = \max_x \{f(x, \alpha) \mid h(x, \beta) = h(z, \beta)\}$ (I^c).

This compensated maximization problem requires all constraints to pass through some $z \in R^n$ for all values of (α, β).

We can be very brief here, since (I^c) has been the focus of attention of *Hatta* (1980). We simply want to (i) introduce compensated feasible policies, (ii) present the general Envelope Theorem for (I^c) and (iii) show that we do not have to examine (I^c) separately, since all relevant results follow directly from Theorems 1 – 4.

The solution of (I^c) is given by

$s(\alpha, \beta; z) \equiv x(\alpha, \beta, h(z, \beta))$ and $\mu(\alpha, \beta; z) = \lambda(\alpha, \beta, h(z, \beta))$.

It is immediate that $S_\alpha(\alpha, \beta; z) = X_\alpha(\alpha, \beta, h(z,\beta))$, $M_\alpha(\alpha, \beta;z) = \Lambda_\alpha(\alpha, \beta, h(z, \beta))$ and $S_\beta(\alpha, \beta; z) = X_\beta(\alpha, \beta, h(z,\beta))+X_\gamma(\alpha, \beta, h(z, \beta))H_\beta(z, \beta)'$, $M_\beta(\alpha, \beta; z)=\Lambda_\beta(\alpha, \beta, h(z, \beta))+ \Lambda_\gamma(\alpha, \beta, h(z, \beta))H_\beta(z, \beta)'$.

Any smooth compensated feasible policy, $t(\alpha, \beta;z) \equiv y(\alpha, \beta, h(z, \beta))$, satisfies all constraints $h^j(t(\alpha, \beta; z), \beta) = h^j(z, \beta)$, $j = 1, \ldots, m$, and thus $T_\alpha(\alpha, \beta; z) = Y_\alpha(\alpha, \beta, h(z, \beta))$ and $T_\beta(\alpha, \beta; z) = Y_\beta(\alpha, \beta, h(z, \beta)) + Y_\gamma(\alpha, \beta, h(z, \beta))\, H_\beta(z, \beta)'$ need not be zero, unless $t(\alpha, \beta; z)$ is a degenerate policy staying at z for all (α, β).

<u>Lemma 1^c</u>: For $t(\alpha, \beta; z) \in C^1$ and any z we have

$T_\alpha' H_x = 0_{\kappa m}$ and

$T'_\beta H_x + H_\beta = H_\beta(z, \beta)$.

Lemma 2^c: For $t(\alpha, \beta; z) \in C^2$ and any z we have

(α,α) $T_\alpha' H_{xx} T_\alpha + [t'_{\alpha_i\alpha_j} H_x \mu] = 0_{\kappa\kappa}$,

(α,β) $T_\alpha' H_{xx} T_\beta + T_\alpha' H_{x\beta} + [t'_{\alpha_i\beta_j} H_x \mu] = 0_{\kappa l}$,

(β,α) $T_\beta' H_{xx} T_\alpha + [t'_{\beta_i\alpha_j} H_x \mu] + H_{\beta x} T_\alpha = 0_{l\kappa}$,

(β,β) $T_\beta' H_{xx} T_\beta + T_\beta' H_{x\beta} + [t'_{\beta_i\beta_j} H_x \mu] + H_{\beta x} T_\beta + H_{\beta\beta} = \sum_{j=1}^{m} \mu_j h^j_{\beta\beta}(z, \beta)$.

The proof of both Lemmas follows that given for *Lemmas 1 and 2*. $H_\beta = H_\beta(z, \beta)$ and

$H_{\beta\beta} = \sum_{j=1}^{m} \mu_j h^j_{\beta\beta}(z, \beta)$ only when $z = t(\alpha, \beta; z)$.

From $\psi(\alpha, \beta; z) \equiv \varphi(\alpha, \beta, h(z, \beta)) \equiv f(s(\alpha, \beta; z), \alpha)$, we see that

$$\psi_\alpha(\alpha, \beta; z) = \varphi_\alpha(\alpha, \beta, h(z, \beta)) = S_\alpha' f_x + f_\alpha = f_a(s(\alpha, \beta; z), a'),$$

$$\psi_\beta(\alpha, \beta; z) = \varphi_\beta(\alpha, \beta, h(z, b)) + H_\beta(z, \beta)\, \varphi_\gamma(\alpha, \beta, h(z, \beta)) =$$

$$=-\{H_\beta(s(\alpha, \beta; z), \beta) - H_\beta(z, \beta)\}\, \mu(\alpha, \beta; z)$$

$$= S_\beta' f_x(s(\alpha, \beta; z), \alpha).$$

Thus $S_\alpha' f_x(s(\alpha, \beta; z), \alpha) = 0_\kappa$ but

$S_\beta' f_x(s(\alpha, \beta; z), \alpha) \neq 0_l,$

unless $z = s(\alpha, \beta; z)$. It is clear that we could proceed and consider any given $z \in R^n$. Our interest, however, is in the special case where $z = x(\alpha, \beta, \gamma) = s(\alpha, \beta; z)$ and all constraints continue to pass through the previously optimal solution as (α, β) vary around their former values (and γ respond accordingly). Thus we have:

Theorem 1^c:

(i) $\psi_\alpha(\alpha, \beta; x(\alpha, \beta, \gamma)) = f_a(x(\alpha, \beta, \gamma), \alpha)$

(ii) $\psi_\beta(\alpha, \beta; x(\alpha, \beta, \gamma) = 0_l.$

and

Theorem 2^c:

$$\psi = \begin{bmatrix} F_{\alpha x} S_\alpha + F_{\alpha\alpha}, & F_{\alpha x} S_\beta \\ -H_{\beta x} S_\alpha & , & -H_{\beta x} S_\beta \end{bmatrix}$$

is a symmetric matrix.

Here the gain function is given by

$$g^c(\alpha, \beta; z) = \psi(\alpha, \beta; z) - f(t(\alpha, \beta; z), \alpha)$$

and is in general positive, unless at some (α, β) it happens that

$t(\alpha, \beta; z) = s(\alpha, \beta; z)$[13]

Then g^c has a minimum of zero and we can exploit the first and second-order necessary conditions that characterize it.

In the special case where, at $(\alpha^o, \beta^o; x^o)$, $z = x^o = x(\alpha^o, \beta^o, \gamma^o) = s(\alpha^o, \beta^o; x^o)$, we deduce

Theorem 3^c:

(i) $g^c_\alpha(\alpha^o, \beta^o; x^o) = -T^{o'}_\alpha f^o_x = 0_\kappa$,

(ii) $g^c_\beta(\alpha^o, \beta^o; x^o) = -T^{o'}_\beta f^o_x = 0_l$

and

Theorem 4^c:

The matrix G^o, with submatrices

$$G^{co}_{\alpha\alpha} = F^o_{ax}\{S^o_\alpha - T^o_\alpha\} - T^{o'}_a A^o T^o_a - T^{o'}_a F^o_{xa} \quad , \quad G^{co}_{\alpha\beta} = F^o_{ax}\{S^o_\beta - T^o_\beta\} - T^{o'}_a A^o T^o_\beta + T^{o'}_a H^o_{x\beta},$$
$$G^{oo}_{\beta\alpha} = -H^o_{\beta x}\{S^o_\alpha - T^o_\alpha\} - T^{o'}_\beta A^o T^o_\alpha - T^{o'}_\beta F^o_{xx} \ , \quad G^{co}_{\beta\beta} = -H^o_{\beta x}\{S^o_\beta - T^o_\beta\} - T^{o'}_\beta A^o T^o_\beta + T^{o'}_\beta H^o_{x\beta},$$

is symmetric and positive semidefinite.

Envelope tangency requires that

$$S^{o'}_\alpha f^o_x = T^{o'}_\alpha f^o_x = 0_\kappa,$$

$$S^{o'}_\beta f^o_x = T^{o'}_\beta f^o_x = 0_l,$$

although $S^{o'}_\alpha f^o_x = 0_\kappa$ holds in general while $S_\beta(\alpha, \beta; z)' f_x(s(\alpha, \beta; z), \alpha) = 0_l$ holds whenever $z = x(\alpha, \beta, \gamma)$.

As for the curvature properties of an Envelope, not only second-order derivative terms drop out at $(\alpha^o, \beta^o; x^o)$, but also terms involving M^o_α and M^o_β ; the latter terms drop out whenever $z = x(\alpha^o, \beta^o, \gamma^o)$.

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[13] The main difference between *Hatta's* (1980) approach and ours can be indicated in terms of the difference between our gain function

$$\varphi(\alpha, \beta, h(z, \beta)) - f(t(\alpha, \beta; z), \alpha)$$

and his own

$$\varphi(\alpha, \beta, h(z, \beta)) - f(z, \alpha),$$

expressed in terms of our model and notation. Thus *Hatta* considers only degenerate compensated feasible paths.

Finally, let us note that *Theorems 1 – 4* can be used to prove the Theorems of this Appendix. Indeed consider the matrix

$$B \equiv \Phi + \begin{bmatrix} 0_{\kappa\kappa}, & 0_{\kappa l}, & 0_{\kappa m} \\ 0_{l\kappa}, & H_{\beta\beta}, & 0_{lm} \\ 0_{m\kappa}, & 0_{ml}, & 0_{mm} \end{bmatrix}$$

in which the second term is due to the changes in the constraints from (I) to (I^c), with

$H_{\beta\beta} = \sum_{j=1}^{m} \mu_j h^j_{\beta\beta}(x(\alpha,\beta,\gamma),\beta)$. If B is pre-multiplied by $\begin{bmatrix} I_{\kappa\kappa}, & 0_{\kappa l}, & 0_{\kappa m} \\ 0_{l\kappa}, & I_{ll}, & H_{\beta} \end{bmatrix}$, with

$H_\beta = H_\beta(x(\alpha, \beta, \gamma), \beta)$, and is also post-multiplied by its transpose, then we get the symmetric $(\kappa + l), (\kappa + l)$ matrix Ψ. Similarly, if G^o is pre-multiplied by the above matrix and is post-multiplied by its transpose, we get the symmetric and positive semidefinite G^{co}.