APPENDIX TWO: The Compensated Version of Problem (I)

We have seen in the paper the need for considering problem

\[ \psi(\alpha, \beta; z) = \max_x \{ f(x, \alpha) \mid h(x, \beta) = h(z, \beta) \} \tag{I^c} \]

This compensated maximization problem requires all constraints to pass through some \( z \in \mathbb{R}^n \) for all values of \((\alpha, \beta)\).

We can be very brief here, since (I^c) has been the focus of attention of Hatta (1980). We simply want to (i) introduce compensated feasible policies, (ii) present the general Envelope Theorem for (I^c) and (iii) show that we do not have to examine (I^c) separately, since all relevant results follow directly from Theorems 1 – 4.

The solution of (I^c) is given by

\[ s(\alpha, \beta; z) \equiv x(\alpha, \beta, h(z, \beta)) \quad \text{and} \quad \mu(\alpha, \beta; z) = \lambda(\alpha, \beta, h(z, \beta)). \]

It is immediate that \( S_\alpha(\alpha, \beta; z) = X_\alpha(\alpha, \beta, h(z, \beta)), M_\alpha(\alpha, \beta; z) = A_\alpha(\alpha, \beta, h(z, \beta)) \) and \( S_\beta(\alpha, \beta; z) = X_\beta(\alpha, \beta, h(z, \beta)) + X_\gamma(\alpha, \beta, h(z, \beta)) H_\beta(z, \beta)' \).

Any smooth compensated feasible policy, \( t(\alpha, \beta; z) \equiv y(\alpha, \beta, h(z, \beta)) \), satisfies all constraints \( h_j(t(\alpha, \beta; z), \beta) = h_j(z, \beta), j = 1, \ldots, m \), and thus \( T_\alpha(\alpha, \beta; z) = Y_\alpha(\alpha, \beta, h(z, \beta)) \) and \( T_\beta(\alpha, \beta; z) = Y_\beta(\alpha, \beta, h(z, \beta)) H_\beta(z, \beta)' \) need not be zero, unless \( t(\alpha, \beta; z) \) is a degenerate policy staying at \( z \) for all \((\alpha, \beta)\).

**Lemma 1^c:** For \( t(\alpha, \beta; z) \in C^1 \) and any \( z \) we have

\[ T_\alpha' H_x = 0_{km} \quad \text{and} \quad T_\beta' H_x + H_\beta = H_\beta(z, \beta). \]

**Lemma 2^c:** For \( t(\alpha, \beta; z) \in C^2 \) and any \( z \) we have

\[
\begin{align*}
(\alpha, \alpha) & \quad T_\alpha' H_{xx} T_\alpha + [t'_{\alpha, \alpha_j} H_x \mu] = 0_{kk}, \\
(\alpha, \beta) & \quad T_\alpha' H_{xx} T_\beta + T_\alpha' H_{x\beta} + [t'_{\alpha, \beta_j} H_x \mu] = 0_{kl}, \\
(\beta, \alpha) & \quad T_\beta' H_{xx} T_\alpha + [t'_{\beta, \alpha_j} H_x \mu] + H_{\beta\beta} T_\alpha = 0_{1k}, \\
(\beta, \beta) & \quad T_\beta' H_{xx} T_\beta + T_\beta' H_{x\beta} + [t'_{\beta, \beta_j} H_x \mu] + H_{\beta\beta} T_\beta + H_{\beta\beta} = \sum_{j=1}^m \mu_{j, \beta} h_{\beta j}(z, \beta). 
\end{align*}
\]
The proof of both Lemmas follows that given for Lemmas 1 and 2. \( H_\beta = H_\beta(z, \beta) \) and
\[
H_{\beta\beta} = \sum_{j=1}^{m} \mu_j h_{\beta\beta}^j(z, \beta) \quad \text{only when} \quad z = t(\alpha, \beta; z).
\]

From \( \psi(\alpha, \beta; z) = \varphi(\alpha, \beta, h(z, \beta)) = f(s(\alpha, \beta; z), \alpha) \), we see that
\[
\psi_\alpha(\alpha, \beta; z) = \varphi_\alpha(\alpha, \beta, h(z, \beta)) = S_\alpha f_\alpha + f_\alpha = f_\alpha(s(\alpha, \beta; z), \alpha'), \quad \psi_\beta(\alpha, \beta; z) = \varphi_\beta(\alpha, \beta, h(z, \beta)) + H_\beta(z, \beta) \varphi_\gamma(\alpha, \beta, h(z, \beta)) =
\]
\[
= \{-H_\beta(s(\alpha, \beta; z), \beta) - H_\beta(z, \beta)\} \mu(\alpha, \beta; z) = S_\beta f_\gamma(s(\alpha, \beta; z), \alpha).
\]

Thus \( S_\alpha' f_\gamma(s(\alpha, \beta; z), \alpha) = 0 \) but
\( S_\beta' f_\gamma(s(\alpha, \beta; z), \alpha) \neq 0 \)

unless \( z = s(\alpha, \beta; z) \). It is clear that we could proceed and consider any given \( z \in R^n \). Our interest, however, is in the special case where \( z = x(\alpha, \beta, \gamma) = s(\alpha, \beta; z) \) and all constraints continue to pass through the previously optimal solution as \( (\alpha, \beta) \) vary around their former values (and \( \gamma \) respond accordingly). Thus we have:

**Theorem 1**:  
(i) \( \psi_\alpha(\alpha, \beta; x(\alpha, \beta, \gamma)) = f_\alpha(x(\alpha, \beta, \gamma), \alpha) \)

(ii) \( \psi_\beta(\alpha, \beta; x(\alpha, \beta, \gamma)) = 0 \)

and

**Theorem 2**:  
\[
\psi = \begin{bmatrix}
F_{\alpha\alpha} s_\alpha + F_{\alpha\alpha} \cdot & F_{\alpha\beta} s_\beta \\
- H_{\beta\alpha} s_\alpha & - H_{\beta\beta} s_\beta
\end{bmatrix}
\]
is a symmetric matrix.

Here the gain function is given by
\[
g^e(\alpha, \beta; z) = \psi(\alpha, \beta; z) - f(t(\alpha, \beta; z), \alpha)
\]

and is in general positive, unless at some \( (\alpha, \beta) \) it happens that
\[ t(\alpha, \beta; z) = s(\alpha, \beta; z)^{13} \]

Then \( g^c \) has a minimum of zero and we can exploit the first and second-order necessary conditions that characterize it.

In the special case where, at \((\alpha^o, \beta^o; x^o)\), \( z = x^o = x(\alpha^o, \beta^o, \gamma^o) = s(\alpha^o, \beta^o; x^o) \), we deduce

**Theorem 3c:**

(i) \[ g^c_\alpha(\alpha^o, \beta^o; x^o) = -T^o_\alpha f^o_x = 0_k, \]

(ii) \[ g^c_\beta(\alpha^o, \beta^o; x^o) = -T^o_\beta f^o_x = 0_l \]

and

**Theorem 4c:**

The matrix \( G^o \), with submatrices

\[
G^{co}_{aa} = F^{o}_{oa\{ S^{o}_{a} - T^{o}_{a}\} - T^{o}_{a} A^{o} T^{o}_{a} - T^{o}_{a} F^{o}_{xa} }, \quad G^{co}_{ab} = F^{o}_{oa\{ S^{o}_{b} - T^{o}_{b}\} - T^{o}_{a} A^{o} T^{o}_{b} + T^{o}_{b} H^{o}_{xb} }, \\
G^{co}_{ba} = -H^{o}_{kb\{ S^{o}_{a} - T^{o}_{a}\} - T^{o}_{b} A^{o} T^{o}_{a} - T^{o}_{b} F^{o}_{xa} }, \quad G^{co}_{bb} = -H^{o}_{kb\{ S^{o}_{b} - T^{o}_{b}\} - T^{o}_{b} A^{o} T^{o}_{b} + T^{o}_{b} H^{o}_{xb} },
\]

is symmetric and positive semidefinite.

Envelope tangency requires that

\[
S^o_\alpha f^o_x = T^o_\alpha f^o_x = 0_k, \]

\[
S^o_\beta f^o_x = T^o_\beta f^o_x = 0_l,
\]

although \( S^o_\alpha f^o_x = 0_k \) holds in general while \( S^o_\beta (\alpha, \beta; z)^\prime f_x(s(\alpha, \beta; z), \alpha) = 0_l \) holds whenever \( z = x(\alpha, \beta, \gamma) \).

As for the curvature properties of an Envelope, not only second-order derivative terms drop out at \((\alpha^o, \beta^o; x^o)\), but also terms involving \( M^{o}_\alpha \) and \( M^{o}_\beta \); the latter terms drop out whenever \( z = x(\alpha^o, \beta^o, \gamma^o) \).

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13 The main difference between Hatta’s (1980) approach and ours can be indicated in terms of the difference between our gain function

\[ \phi(\alpha, \beta, h(z, \beta)) - f(t(\alpha, \beta; z), \alpha) \]

and his own

\[ \phi(\alpha, \beta, h(z, \beta)) - f(z, \alpha), \]

expressed in terms of our model and notation. Thus Hatta considers only degenerate compensated feasible paths.
Finally, let us note that Theorems 1 – 4 can be used to prove the Theorems of this Appendix. Indeed consider the matrix

\[
B = \Phi + \begin{bmatrix}
0_{KK}, & 0_{kl}, & 0_{km} \\
0_{lk}, & H_{\beta\beta}, & 0_{lm} \\
0_{mk}, & 0_{ml}, & 0_{mm}
\end{bmatrix}
\]

in which the second term is due to the changes in the constraints from (I) to (I\(^c\)), with

\[
H_{\beta\beta} = \sum_{j=1}^{m} \mu_j h_j^{(\beta)}(x(\alpha, \beta, \gamma), \beta). \quad \text{If } B \text{ is pre-multiplied by } \begin{bmatrix} I_{KK}, & 0_{kl}, & 0_{km} \\
0_{lk}, & I_{ll}, & H_{\beta}\end{bmatrix} \text{ with }
\]

\[
H_{\beta} = H_{\beta}(x(\alpha, \beta, \gamma), \beta), \quad \text{and is also post-multiplied by its transpose, then we get the symmetric (\(\kappa + l\), (\(\kappa + l\) matrix } \Psi. \quad \text{Similarly, if } G^o \text{ is pre-multiplied by the above matrix and is post-multiplied by its transpose, we get the symmetric and positive semidefinite } G^{o}.\]