APPENDIX THREE : ***The Le Chatelier Principle***

Our purpose here is to show how much simpler the curvature properties of the *Envelope Theorem* become, when the feasible policy that is examined is in fact the optimal policy of a problem like (I) or (I^c) which, however, contains additional constraints[14].

Let us then consider the following problem

$$\varphi(\alpha,\beta,\tilde{\gamma}) \equiv \max_x \{ f(x,\alpha) \mid \tilde{h}(x,\beta)=\tilde{\gamma} \} \qquad (\tilde{I})$$

where $\tilde{h}(x,\beta)=(h(x,\beta),h^+(x,\beta)=\tilde{\gamma}$ and $\tilde{\gamma}=(\gamma,\gamma^+)$.

The number of new constraints is m^+, with $0 \le m < m + m^+ = \tilde{m} < n$ [15].

The first-order conditions characterizing ($\tilde{I}$)

$$\begin{Bmatrix} f_x(x,\alpha)=\tilde{H}_x(x,\beta)\nu \\ \tilde{h}(x,\beta)=\tilde{\gamma} \end{Bmatrix} \qquad (\tilde{5})$$

can be solved for the optimal path $x(\alpha,\beta,\tilde{\gamma})\equiv\tilde{x}$ and $\nu(\alpha,\beta,\tilde{\gamma})=(\lambda(\alpha,\beta,\tilde{\gamma}),\lambda^+(\alpha,\beta,\tilde{\gamma}))'\equiv(\tilde{\lambda},\tilde{\lambda}^+)'\equiv\tilde{\nu}$ of choice variables and lagrangear multipliers[16].

Of our various matrices, $\tilde{H}_x=(H_x,H_x^+)$ and $\tilde{H}_\beta=(H_\beta\ H_\beta^+)$ contain additional columns, while the elements of $\tilde{H}_{xx}=\sum_{j=1}^{\tilde{m}}\tilde{\nu}_j h_{xx}^j(\tilde{x},\beta)$, $\tilde{H}_{x\beta}$ and $\tilde{H}_{\beta\beta}$ contain additional terms. Finally, $\tilde{F}_{xx}=F_{xx}(\tilde{x},\alpha)$ and similarly for $\tilde{F}_{\alpha x}$ and $\tilde{F}_{\alpha\alpha}$.

The maximum value function of $(\tilde{I})$, $\tilde{\varphi}\equiv\varphi(\alpha,\beta,\tilde{\gamma})\equiv f(x(\alpha,\beta,\tilde{\gamma}),\alpha)$, has exactly the same derivative properties as $\varphi(\alpha,\beta,\gamma)$ with the appropriate reinterpretations :

[14] The most recent analysis of the *Le Chatelier Principle* is in *Hatta* (1980) and (1987). Our formulation of the gain function removes its last shortcoming : it permits the consideration of variations in the levels of all constraints, old and new.

[15] The gradients in x of *all* constraints $h_x^j(x,\beta)$, $j = 1,\dots,m+m^+$, are linearly independent.

[16] $\tilde{\nu}$ is an $\tilde{m}$ dimensional vector including not only the original m multipliers, $\tilde{\lambda}$, but also the m^+ new ones, $\tilde{\lambda}^+$.

$\widetilde{X}_\alpha, \widetilde{X}_\beta, \widetilde{X}_\gamma = (X_\gamma, X_\gamma^+)$ and $\widetilde{N}_\alpha = \begin{pmatrix} \widetilde{\Lambda}_\alpha \\ \widetilde{\Lambda}_\alpha^+ \end{pmatrix}, \widetilde{N}_\beta = \begin{pmatrix} \widetilde{\Lambda}_\beta \\ \widetilde{\Lambda}_\beta^+ \end{pmatrix}, \widetilde{N}_\gamma = \begin{pmatrix} \widetilde{\Lambda}_\gamma & \widetilde{\Lambda}_{\gamma^+} \\ \widetilde{\Lambda}_\gamma^+ & \widetilde{\Lambda}_{\gamma^+}^+ \end{pmatrix}$ are the matrices of the rates of change of $\widetilde{x}$ and $\widetilde{v}$.

Indeed, Lemmas 1-3 and Theorems 1-2 hold for $(\widetilde{I})$ as well.

The *gain function* is given here by

$$\widetilde{g} \equiv g(\alpha, \beta, \widetilde{\gamma}) \equiv \varphi(\alpha, \beta, \gamma) - \varphi(\alpha, \beta, \widetilde{\gamma}) \qquad (1\widetilde{2})$$

and is positive in general, since the addition of new constraints leads to inferior results.

If however, we observe

$$x(\alpha^0, \beta^0, \widetilde{\gamma}^0) = x(\alpha^0, \beta^0, \gamma^0) \qquad (1\widetilde{3})$$

at some $(\alpha^0, \beta^0, \gamma^0, \gamma^{+0})$, with γ^{+0} appropriately set as to satisfy

$$h^+(x(\alpha^0, \beta^0, \gamma^0), \beta^0) = \gamma^{+0},$$

then comparing (5) and $(\widetilde{5})$ we are led to the conclusion that

$$\lambda(\alpha^0, \beta^0, \widetilde{\gamma}^0), = \lambda(\alpha^0, \beta^0, \gamma^0)$$

and $\lambda^+(\alpha^0, \beta^0, \widetilde{\gamma}^0) = 0_{m^+}$

In other words, the additional constraints are "just binding" at $(\alpha^0, \beta^0, \widetilde{\gamma}^0)$ and, thus,

$$g(\alpha^0, \beta^0, \widetilde{\gamma}^0) = 0.$$

We can then establish the following

<u>Theorem $\widetilde{4}$</u>: At $(\alpha^0, \beta^0, \widetilde{\gamma}^0)$, $\widetilde{G}^0$ is symmetric and positive semi-definite. Letting δ stand for either α, β, or γ, its various submatrices are :

(α, δ) $\quad \widetilde{G}^0_{\alpha\delta} = F^0_{\alpha x}\{X^0_\delta - \widetilde{X}^0_\delta\}$,

(α, γ^+) $\quad \widetilde{G}^0_{\alpha\gamma^+} = -F^0_{\alpha x}\ \widetilde{X}^0_{\gamma^+}$,

(β, δ) $\quad \widetilde{G}^0_{\beta\delta} = -H^0_{\beta x}\{X^0_\delta - \widetilde{X}^0_\delta\} - H^0_\beta\{\Lambda^0_\delta - \widetilde{\Lambda}^0_\delta\} + H^{+0}_\beta \widetilde{\Lambda}^{+0}_\delta$,

(β, γ^+) $\quad \widetilde{G}^0_{\beta\gamma^+} = H^0_{\beta x}\widetilde{X}^0_{\gamma^+} + H^0_\beta \widetilde{\Lambda}^0_{\gamma^+} + H^{+0}_\beta \widetilde{\Lambda}^{+0}_{\gamma^+}$,

(γ, δ) $\quad \widetilde{G}^0_{\gamma\delta} = \Lambda^0_\delta - \widetilde{\Lambda}^0_\delta$,

(γ, γ^+) $\qquad \tilde{G}^0_{\gamma\gamma^+} = -\tilde{\Lambda}^0_{\gamma^+}$,

(γ^+, δ) $\qquad \tilde{G}^{+0}_{\gamma^+\delta} = -\tilde{\Lambda}^{+0}_{\delta}$,

(γ^+, γ^+) $\qquad \tilde{G}^0_{\gamma^+\gamma^+} = -\tilde{\Lambda}^{+0}_{\gamma^+}$.

The proof follows that of Theorem 4. We note that at $(\alpha^0, \beta^0, \tilde{\gamma}^0)$ we have $\tilde{H}^0_\beta \tilde{\nu}^0 = H^0_\beta \tilde{\lambda}^0 + H^{+0}_\beta \tilde{\lambda}^+ = H^0_\beta \lambda^0$, $\tilde{H}^0_{\beta x} = H^0_{\beta x}$ and, similarly, for all other matrices.

All submatrices in the main diagonal of $\tilde{G}^0$ are symmetric and positive semidefinite : of them

(α, α) $\qquad F^o_{\alpha x}\{X^0_\alpha - \tilde{X}^0_\alpha\}$,

(γ, γ) $\qquad \Lambda^0_\gamma - \tilde{\Lambda}^0_\gamma$, and

(γ^+, γ^+) $\qquad -\tilde{\Lambda}^{+0}_{\gamma^+}$

provide useful comparative statio results. Using Lemma 1 we see that $\tilde{X}^{0\prime}_\alpha H^0_x = O_{\kappa m}$ and $\tilde{X}^{0\prime}_\alpha H^{+0}_x = O_{\kappa m^+}$. Thus the rank of $\tilde{X}^0_\alpha$ is smaller than that of X^0_α

We can also premultiply $\tilde{G}^0$ by

$$\begin{bmatrix} I_{\kappa\kappa} & 0_{\kappa l} & 0_{\kappa m} & 0_{\kappa m^+} \\ 0_{\ell\kappa} & I_{\ell\ell} & H^0_\beta & H^{+0}_\beta \end{bmatrix}$$

and postmultiply it by its transpose to get

$$\tilde{G}^{c0} = \begin{bmatrix} F^0_{\alpha x}\{S^0_\alpha - \tilde{S}^0_\alpha\} \ , & F^0_{\alpha x}\{S^0_\beta - \tilde{S}^0_\beta\} \\ -H^0_{\beta x}\{S^0_\alpha - \tilde{S}^0_\alpha\} \ , & -H^0_{\beta x}\{S^0_\beta - \tilde{S}^0_\beta\} \end{bmatrix}$$

the matrix of the second-order partials of

$\tilde{g}^c(\alpha, \beta; x(\alpha, \beta, \gamma))$ at $(\alpha^0, \beta^0, \tilde{\gamma}^0)$.

Here $\tilde{S}_\beta = \tilde{X}_\beta + \tilde{X}_\gamma H^{\prime}_\beta + \tilde{X}_\gamma + H^{+\prime}_\beta$ and it can be shown, using Lemmas 1 and 1[c], that $\tilde{S}^{0\prime}_\beta H^0_x = 0_{lm}$ and $\tilde{S}^{0\prime}_\beta H^{+0}_x = 0_{lm^+}$. Thus $\tilde{S}^0_\beta$ has a smaller rank than S^0_β .