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Testing for Prospect and Markowitz stochastic dominance efficiency

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Abstract

We develop non-parametric tests for prospect stochastic dominance Efficiency (PSDE) and Markowitz stochastic dominance efficiency (MSDE) using block bootstrap resampling. Under the appropriate conditions we show that they are asymptotically conservative and consistent. We employ Monte Carlo experiments to assess the finite sample size and power of the tests. We use the tests to empirically establish whether the value-weighted market portfolio is the best choice of every individual with preferences exhibiting certain patterns of local attitudes towards risk. Our results indicate that we cannot reject the hypothesis of prospect stochastic dominance efficiency for the market portfolio. This is supportive of the claim that the particular portfolio can be rationalized as the optimal choice for any S-shaped utility function. Instead, we reject the hypothesis for Markowitz stochastic dominance, which could imply that there exist reverse S-shaped utility functions that do not rationalize the market portfolio.

Keywords: Non parametric test, prospect stochastic dominance efficiency, Markowitz stochastic dominance efficiency, simplical complex, extremal point, Linear Programming, Mixed Integer Programming, Block Bootstrap, Consistency.

JEL Classification: C12, C13, C15, C44, D81, G11.

1 Introduction

Traditionally, in the context of portfolio theory, investors are assumed to act as non satiable and risk averse agents and thus their preferences are represented by increas-

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ing and globally concave utility functions. For this reason, most of the criteria used to statistically verify the efficiency of a given portfolio (see, among others, Gibbons, Ross, and Shenken [15]) are based on the second stochastic dominance rule, see, e.g., the papers by Kroll and Levy [20] and Levy [23], and the excellent monograph on the theory of stochastic dominance by Levy [24]. In the related literature, several nonparametric tests have been proposed for first, second, and higher order stochastic dominance, see, inter alia, McFadden [31], Barret and Donald [2], Scaillet and Topaloglou [40] and Linton, Maasoumi, and Whang [27], Linton, Post and Wang [28], Linton, Song, and Whang [29], and Donald and Hsu [8]. Recently, Gonzalo and Olmo [16] propose nonparametric consistent tests of conditional stochastic dominance of arbitrary order in a dynamic setting.

In constrast to the global risk aversion framework, Friedman and Savage [14] claim that the strictly concave functions may not be able to explain why investors buy insurance and/or lottery tickets. To address their concern, Markowitz [30] proposes a utility function that has convex and concave regions in both the positive and negative parts. In particular, he suggests that individuals are risk averse for losses and risk seeking for gains as long as the outcomes are not very extreme. A class of utility functions that partially[1] represents this kind of behavior is the one of reverse S-shaped (utility or value) functions.

Kahneman and Tversky [19] propose utilities that are concave for gains and convex for losses, yielding S-shaped functions w.r.t. a benchmark point. Their theory was further developed by Tversky and Kahneman [42] to cumulative prospect theory in order to be consistent with first-order stochastic dominance. Prospect theory recently gained much popularity among economists, and there is a stream of papers that build economic models based on it. There have been many empirical and experimental attempts to test prospect theory.

Accordingly, empirical evidence suggests that investors do not always act as risk averters. Instead, under certain circumstances they behave in a much more complex fashion exhibiting characteristics of both risk loving and risk-averting. Furthermore, they seem to evaluate wealth changes of assets w.r.t. to benchmark cases rather than final wealth positions. They behave differently on gains and losses, and one can say that they are more sensitive to losses than to gains (loss aversion). In addition, there are cases where the relevant utility (or value) function could be either concave for gains and convex for losses or convex for gains and concave for losses. Moreover, they seem to transform the objective probability measures to subjective ones using transformations that potentially increase the probabilities of negligible (and possibly averted) events, which in some cases share similar analytical characteristics to the aforementioned utility functions. Examples of risk orderings that (partially) reflect such findings are the dominance rules of behavioral finance (see Friedman and Savage [14], Baucells and Heukamp [3], Edwards [11], and the references therein).

Inspired by previous work, Levy and Levy [25] formulate the notions of prospect stochastic dominance (PSD) and Markowitz stochastic dominance (MSD). According to their definition, portfolio A prospect stochastically dominates portfolio B iff the expected utility of the return of A is greater than or equal to the expected utility of the return of B *for any* utility function in the set of increasing, convex on the neg-

[1] i.e. when the possibility of further changes of the risk attitude on extreme events is ignored.

ative part and concave on the positive part real functions (termed as S-shaped (at zero) utility functions). PSD efficiency is then the case when one considers *greatest* elements w.r.t. this ordering. Analogously, portfolio A Markowitz stochastically dominates portfolio B iff the expected utility of the return of A is greater than or equal to the expected utility of the return of B *for any* utility function in the set of increasing, concave on the negative part and convex on the positive part real functions (termed as reverse S-shaped (at zero) utility functions). Again, the notion of MSD efficiency follows naturally from the notion of greatest elements w.r.t. the particular ordering. Notice that PSD efficiency and MSD efficiency are not mutually exclusive (see for example the Monte Carlo section below).

The question that arises concerns the empirical analysis of the investors behavior towards risk. In practice, many institutional investors hold portfolios that mimic the behaviour of the market portfolio. They invest in Exchange-Traded Funds (ETFs) and mutual funds. These funds track stocks, commodities and bonds, or value-weighted equity indices which strongly resemble the market portfolio. Moreover, many actual funds, including total market index funds which are based on the Wilshire 5000 index, are very highly correlated with the market portfolio. Thus, it is interesting to investigate whether this behavior can be rationalized by preference relations inside the aforementioned classes of utility functions.

In view of the above, the main contribution of this paper is to develop consistent tests for prospect stochastic dominance (PSD) and Markowitz stochastic dominance (MSD) efficiency. We aim to establish statistically whether a given portfolio is the best choice of any individual within each of the aforementioned classes of preferences. We construct the test statistics using the preference free representations of those notions by Levy and Levy [25]. Under appropriate conditions we show that the tests are asymptotically conservative and consistent.

We use the tests to empirically establish whether a value-weighted market portfolio can be considered as efficient according to prospect and Markowitz stochastic dominance criteria when confronted to diversification principles made of risky assets. For this purpose, we use proxies of the individual assets in the investment universe. Thus, for the individual risky assets, we use benchmark portfolios formed on size, BE/ME, Momentum, and industry portfolios. These portfolios have been at the center of the asset pricing literature over the past two decades (see for example Post [36], Kuosmanen [22], Post and Levy [38], Post and Kopa [37], Gonzalo and Olmo [16] among others in the stochastic dominance framework). The size portfolios are of particular interest because empirical research, starting with Banz [2], suggest that small-cap stocks earn a return premium that defies rational explanation. Moreover, book-to-market based sorts are the basis for the factor model examined in Fama and French [12]. Additionally, academics and practitioners show strong interest in Momentum portfolios. Empirical evidence indicates that common stocks exhibit high returns on a period of 3-12 months will overperform on subsequent periods. This momentum phenomenon is an important challenge for the concept of market efficiency. Finally, industry sorted portfolios have posed a particularly challenging feature from the perspective of systematic risk measurement (see Fama and French [13]). To focus on the role of preferences and beliefs, we largely adhere to the assumptions of a single-period, portfolio-oriented model of a competitive capital market.

Given the above, the second contribution of the paper is the statistical finding that the value weighted market portfolio is Prospect stochastic dominance efficient compared to all possible sets of portfolios based on asset size, book to market value, Momentum and industry. This result is not true, though, for Markowitz stochastic dominance efficiency. Those results essentially indicate, decision errors apart, that the market portfolio is prefered to portfolio's formed inside the aforementioned classes of assets, by any investor with the s-shaped attitude towards riks, but also that there exist reverse s-shaped attitudes towards risk that do not rationalize such a choice.

As far as the relevant statistical literature is concerned, Prospect and Markowitz stochastic dominance efficiency criteria have not been extensivelly statistically tested, despite their appeal with experimental observations. This is the case, even though this research field seems particularly well suited for statistical analysis, given the availability of large datasets of historical returns.

Linton et al. [27] design a testing procedure for PSD efficiency assuming bounded supports and a finite number of prospects. In contrast, we construct procedures for PSD or MSD efficiency, while allowing for general supports and any (and thereby possibly uncountable) number of prospects.

Post and Levy [38] test for weaker versions of the aforementioned notions of stochastic dominance. More specifically, they allow for a portfolio A to be prospect (Markowitz) stochastically dominant to B iff *there exists* an S-shaped (reverse S-shaped) utility function that rationalizes the optimal choice of A over B. It is easy to see that changing the condition from any utility to the existence of one utility that rationalizes the relevant choice, they weaken the PSD and MSD notions of efficiency as defined in Levy and Levy [25] and discussed above. Then, they propose a non parametric test based on a test statistic constructed from first order conditions of utility maximization. They derive asymptotic critical values by an asymptotic normality argument in an iid framework.

In contrast, first, we use the stronger versions of PSD and MSD efficiency of Levy and Levy [25]. We do so motivated by the possibillity that an investor (e.g. a financial institution) being uncertain of the exact form of her utility function may find useful to have a test of whether a given portfolio can be considered as an optimal choice for any given S-shaped (reverse S-shaped) utility function. Second, we test for global optimality rather than using first-order conditions, something that introduces a considerable increase in the complexity of the numerical procedures required, compared to the linear programming ones used in the aforementioned paper. Third, we allow for dynamic time-series patterns (rather than serial iid-ness).

Our work is in the spirit of Scaillet and Topaloglou [40] who develop consistent tests for stochastic dominance efficiency at any order for time-dependent data (see also Linton, Post and Wang [28]). They are in turn inspired by the consistent procedures developed by Barrett and Donald [2] and extended by Horvath, Kokoszka, and Zitikis [18] to accommodate for non-compact support.

The paper is organized as follows. In section 2, we define prospect and Markowitz stochastic dominance relations and efficiency. We provide characterizations of efficiency by properties of suprema of appropriate functionals. We then use those characterizations to describe the relevant statistical hypotheses. In section 3, we obtain the test statistics as the appropriately scaled empirical analogs of the aforementioned suprema. Under a relevant assumption framework, in contrast to Scaillet and Topaloglou [40], we obtain the limit theories of the employed statistics under the null hypotheses. Furthermore, we derive useful properties of associated limit distributions. In section 4, we design the testing procedures based on approximations of the asymptotic p-values by a block bootstrap method, and we derive relevant asymptotic properties such as consistency. In section 5, we perform Monte Carlo experiments to evaluate the finite sample size and power of the tests allowing for optimization errors in the framework of conditional heteroskedasticity. In section 6, we provide the empirical illustration. We give some concluding remarks and provide some hints for further research in section 7. We discuss the computational aspects of mathematical programming formulations corresponding to the test statistics, and we give all the proofs in the Appendix.

2 Prospect and Markowitz Stochastic Dominance, Efficiency and Statistical Hypotheses

We work in the framework of a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that F is the cdf of a probability distribution on $\mathbb{R}^n$, which is the law of the random vector Y_0 defined on Ω. Given this, $G(z, \lambda, F)$ denotes $\int_{\mathbb{R}^n} \mathbb{I}\{\lambda' u \leq z\} dF(u)$, i.e. the cdf of the linear transformation $Y_0 \to \lambda' Y_0$ for $\lambda \in \Lambda = \left\{\lambda \in \mathbb{R}^n_+ : \mathbf{1}'\lambda = 1\right\}$. In our context, F represents the joint distribution of n basis asset returns and $G(z, \lambda, F)$ the distribution of the returns of the linear portfolio constructed from the basis assets with weights given by the vector λ. $\mathbb{L}$ denotes a non-empty closed subset of Λ, that represents a set of *feasible* portfolios, while τ is a *distinguished member* of $\mathbb{L}$ that is to be tested for the relevant notions of efficiency w.r.t. the members of $\mathbb{L}$.

Consider, for arbitrary $\lambda \in \Lambda$,

$$\mathcal{J}\left(z_1, z_2, \lambda, F\right) := \int_{z_1}^{z_2} G\left(u, \lambda, F\right) du,$$

where z_1, z_2 assume their values in the extended real line and $z_1 \leq z_2$. When $z_1 = -\infty$ and z_2 is finite, $\mathcal{J}\left(z_1, z_2, \lambda; F\right)$ is finite if $\mathbb{E}\left[(\lambda' Y_0)_-\right]$ exists as a real number. Analogously, when z_1 is finite and $z_2 = +\infty$, it is easy to see that $\mathcal{J}\left(z_1, +\infty, \tau, F\right) - \mathcal{J}\left(z_1, +\infty, \lambda, F\right)$ is finite if $\mathbb{E}\left[(\lambda' Y_0)_+\right]$ exists as a real number. Given this, and via the use of the preference free representations of prospect and Markowitz stochastic dominance efficiency of Levy and Levy [25], we have the following definitions, characterizing the two notions of stochastic dominance and the relevant notions of efficiency that we are occupied with.

Definition 1. τ *weakly Prospect Stochastically Dominates* λ, written as $\tau \succcurlyeq_P \lambda$, iff

$$p_1\left(z, \lambda, \tau, F\right) := \mathcal{J}\left(z, 0, \tau, F\right) - \mathcal{J}\left(z, 0, \lambda, F\right) \leq 0, \ \forall z \in \mathbb{R}_-,$$

and

$$p_2\left(z, \lambda, \tau, F\right) := \mathcal{J}\left(0, z, \tau, F\right) - \mathcal{J}\left(0, z, \lambda, F\right) \leq 0, \ \forall z \in \mathbb{R}_{++}.$$

Strict dominance, written as $\tau \succ_P \lambda$ occurs iff there exists some $z \in \mathbb{R}$ for which the relevant inequality is strict. Moreover, τ is *Prospect Stochastic Dominance Efficient* (PSD-efficient) w.r.t. $\mathbb{L}$, iff $\tau \succcurlyeq_P \lambda, \ \forall \lambda \in \mathbb{L}$.

Equivalence (3) of Levy and Levy [25] means that τ is PSD-efficient iff it is the optimal choice for any preference order in the class of *s-shaped* utility functions.

Definition 2. Suppose that $\mathbb{E}\left[\|Y_0\|\right] < +\infty$. τ *weakly Markowitz Stochastically Dominates* λ, written as $\tau \succcurlyeq_M \lambda$, iff

$$m_1\left(z, \lambda, \tau, F\right) := \mathcal{J}\left(-\infty, z, \tau, F\right) - \mathcal{J}\left(-\infty, z, \lambda, F\right) \leq 0,\ \forall z \in \mathbb{R}_-,$$

and

$$m_2\left(z, \lambda, \tau, F\right) := \mathcal{J}\left(z, +\infty, \tau, F\right) - \mathcal{J}\left(z, +\infty, \lambda, F\right) \leq 0,\ \forall z \in \mathbb{R}_{++}.$$

Strict dominance, written as $\tau \succ_M \lambda$ occurs iff there exists some $z \in \mathbb{R}$ for which the relevant inequality is strict. Moreover, τ is *Markowitz Stochastic Dominance Efficient* (MSD-efficient) w.r.t. $\mathbb{L}$, iff $\tau \succcurlyeq_M \lambda,\ \forall \lambda \in \mathbb{L}$.

Similarly to the previous case, equivalence (4) of Levy and Levy [25] means that τ is MSD-efficient iff it is the optimal choice for any preference order in the class of *reverse s-shaped* utility functions.

In both the previous definitions efficiency does not hold iff there exists some element of $\mathbb{L}$ different that τ, that either strictly dominates, or is incomparable to τ w.r.t. the analogous relation. This then implies that the τ can be equivalent to some (possibly all) the portfolios in the reference set. Consider the following extreme examples. Suppose that $\mathbb{L}$ is only comprised by two equivalent (w.r.t. some of the considered preorders) portfolios. Then both are accordingly efficient. Suppose analogously that $\mathbb{L}$ is only comprised by two portfolios which now are incomparable. Then neither is accordingly efficient.
The following proposition in each case characterizes efficiency via the use of suprema of appropriate functionals.

Proposition 1. *τ is PSD-Efficient w.r.t. $\mathbb{L}$ iff,*

$$p\left(\tau, F\right) := \sup_{\lambda \in \mathbb{L}} \max_{i=1,2} \sup_{z \in A_i} p_i\left(z, \lambda, \tau, F\right) = 0,$$

where $A_1 = \mathbb{R}_-$, $A_2 = \mathbb{R}_{++}$. τ is not PSD-efficient w.r.t. $\mathbb{L}$ iff $p\left(\tau, F\right) > 0$. Furthermore,$\tau$ is MSD-Efficient w.r.t. $\mathbb{L}$ iff,

$$m\left(\tau, F\right) := \sup_{\lambda \in \mathbb{L}} \max_{i=1,2} \sup_{z \in A_i} m_i\left(z, \lambda, \tau, F\right) = 0.$$

τ is not MSD-efficient w.r.t. $\mathbb{L}$ iff $m\left(\tau, F\right) > 0$.

The results above cannot usually be directly employed for the characterization of τ since F is in most cases unknown. However, given the availability of statistical information on F, it is possible to be utilized for the construction of analogous testing procedures. Hence, in the context of the aforementioned framework and in the light of the previous lemma, the relevant hypotheses structures can be expressed as:

$$\mathbb{H}_0^{(P)}:\ p\left(\tau, F\right) = 0,$$
$$\mathbb{H}_1^{(P)}:\ p\left(\tau, F\right) > 0,$$

for PSD-efficiency, and as:

$$\mathbb{H}_0^{(M)} : m(\tau, F) = 0,$$
$$\mathbb{H}_1^{(M)} : m(\tau, F) > 0,$$

for MSD-efficiency.
Notice here, that if any other portfolio in $\mathbb{L}$ is "suspect" of efficiency, the following testing procedures can be also performed by considering the latter as a benchmark portfolio in place of τ. For example if $\tau, \tau' \in \mathbb{L}$ are tested for, say PSD efficiency, and in both cases the null is not rejected, then, and given the previous comments on the notion of efficiency, one can conclude in the relevant significance level that τ, τ' are also equivalent w.r.t. the PSD ordering.

3 Assumption Framework, Test Statistics, and Null Limit Theories

We employ Proposition 1 in order to construct statistical tests for the hypotheses structures above. In order to proceed, we extend our framework as follows. We assume the existence of a *strictly stationary* process $(Y_t)_{t\in\mathbb{Z}}$ taking values in $\mathbb{R}^n$. The sample is the random element $(Y_t)_{t=1,\ldots,T}$. In our context a sample value represents a time series of observed returns of the n financial basis assets. F is the cdf of Y_0 and $\hat{F}_T$ the *empirical* cdf associated with the random element $(Y_t)_{t=1,\ldots,T}$. $\mathbb{I}\{\cdot\}$ denotes the relevant indicator function.

We now present the test statistics for PSD and MSD efficiency. Those are obtained as the appropriately scaled[2] empirical analogues of the functionals appearing in Proposition 1. They are

$$p_T := p\left(\tau, \sqrt{T}\hat{F}_T\right),$$

for PSD, and

$$m_T := m\left(\tau, \sqrt{T}\hat{F}_T\right).$$

for MSD efficiency.
The commutativity of the $\sup_{\lambda\in\mathbb{L}}$ and the $\max_{i=1,2}$ operators, and the integration by parts formula imply that p_T and m_T can be equivalently expressed as:

$$p_T = \max_{i=1,2} \sup_{\lambda\in\mathbb{L}} \sup_{z\in A_i} \frac{(-1)^i}{\sqrt{T}} \sum_{t=1}^{T} v(z, \lambda, \tau, Y_t), \tag{1}$$

and

$$m_T = \max_{i=1,2} \sup_{\lambda\in\mathbb{L}} \sup_{z\in A_i} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} q_i(z, \lambda, \tau, Y_t), \tag{2}$$

where

$$q_i(z, \lambda, \tau, Y_t) := \begin{cases} K(z, \lambda, \tau, Y_t), & i = 1 \\ \left[(\lambda' Y_t)_+ - (\tau' Y_t)_+ - v(z, \lambda, \tau, Y)\right], & i = 2 \end{cases},$$

[2] The $\sqrt{T}$ scaling is justified by the assumption framework that follows.

and $v(z, \lambda, \tau, Y) := K(z, \lambda, \tau, Y) - K(0, \lambda, \tau, Y)$, and $K(z, \lambda, \tau, Y) := (z - \tau' Y)_+ - (z - \lambda' Y)_+$.

The underlying optimizations are usually analytically intractable and thereby we resort to numerical techniques for the evaluation of the statistics. We provide details about numerical implementations of the optimization procedures in the Appendix. We proceed with the description of an assumption framework that will enable the derivation of the asymptotic distributions of p_T and m_T under the respective null hypotheses, which will obviously facilitate the design of the analogous testing procedures. The first assumption concerns probabilistic properties of the random elements involved.

Assumption 1. *F is absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}^n$ with convex support, and for some $0 < \delta$, $\mathbb{E}\left[\|Y_0\|^{2+\delta}\right] < +\infty$. $(Y_t)_{t\in\mathbb{Z}}$ is a-mixing with mixing coefficients $a_T = O(T^{-a})$ for some $a > 1 + \frac{2}{\eta}$, $0 < \eta < 2$, as $T \to \infty$. Furthermore,*

$$\mathbb{V} = \mathbb{E}\left[(Y_0 - \mathbb{E}(Y_0))(Y_0 - \mathbb{E}(Y_0))'\right] + 2\sum_{t=1}^{\infty} \mathbb{E}\left[(Y_0 - \mathbb{E}(Y_0))(Y_t - \mathbb{E}(Y_t))'\right]$$

is positive definite.

The mixing part of the previous assumption is readily implied by concepts such as geometric ergodicity which holds for many stationary models used in the context of financial econometrics. Prominent examples are the strictly stationary versions of ARMA or GARCH and stochastic volatility type of models. Counter-examples are stationary models that exhibit long memory, etc (see inter alia, Doukhan [10] for the relevant rigorous definition and further examples). Along with the moment existence condition it facilitates the validity of limiting arguments about partial sums of mixing processes as well as ”continuity” arguments of particular transformations of the latter (see the proof of Lemma 2 in the Appendix). The moment existence condition is usually established by analogous restrictions on moments of the innovation processes appearing as building blocks in the aforementioned examples along with parameter restrictions. The positive definiteness of the ”long run” covariance matrix above facilitates the extraction of properties of the cdf of appropriate limiting random variables at zero (see the third part of Proposition 2 below), and it is thereby connected to properties of asymptotic rejection regions for the testing procedures to be established below. For instance, if $(Y_t)_{t\in\mathbb{Z}}$ is a vector martingale difference process, this part of the assumption can be verified if the elements of Y_0 are linearly independent random variables.

The second assumption concerns topological properties of the ”portfolio parameter space” $\mathbb{L}$.

Assumption 2. *$\mathbb{L}$ is a simplicial complex comprised of a finite number of simplices of $\Lambda = \left\{\lambda \in \mathbb{R}^n_+ : \mathbf{1}'\lambda = 1\right\}$.*

The assumed structure of $\mathbb{L}$ allows for it to be non-convex and possibly disconnected while it is obviously compact. It enables the establishment of the limit theory of the procedures to be defined in relation to $n^\star$, i.e., the number of the extreme points of $\mathbb{L}$,[3] while its structure as a simplicial complex facilitates our numerical formulation, and it implies that the inclusion of τ in $\mathbb{L}$ is application-wise non-restrictive, since it allows for it to be an isolated point. Notice also that our definition of $\mathbb{L}$ is compatible with portfolio spaces further restricted by several kinds of market frictions, liquidity constraints and/or any other form of economic, legal, etc restrictions as long as Assumption 2 is satisfied. Typically we have that $\mathbb{L} = \Lambda$ but this set up allows for example for the realistic cases where, the portfolio space is comprised by a finite, yet possibly quite large number of portfolios due to divisibility issues. Notice finally that the basis assets are not restricted to be individual securities. Abstractly those can be defined as the most extreme feasible combinations of the individual securities. This for example essentially allows for short selling, since some of the basis assets could in turn be portfolios constructed via short selling.

In the following proposition, we obtain the relevant limit theories, as well as some properties of the limit distributions that will be useful for the design of the statistical procedures below and the establishment of their asymptotic properties. We denote with $\rightsquigarrow$ convergence in distribution as $T \to \infty$.

Proposition 2. *Suppose that Assumption 1 holds.*

1. *Then,*

$$p\left(\tau, \sqrt{T}\left(\hat{F}_T - F\right)\right) \rightsquigarrow p^{\star}_{\infty} := \sup_{\lambda \in \mathbb{L}} \max_{i=1,2} \sup_{z \in A_i} p_i\left(z, \lambda, \tau, \mathcal{G}_F\right), \tag{3}$$

where $\mathcal{G}_F$ is a centered Gaussian process with covariance kernel given by $\text{Cov}(\mathcal{G}_F(x), \mathcal{G}_F(y)) = \sum_{t \in \mathbb{Z}} \text{Cov}\left(\mathbb{I}\left\{Y_0 \leq x\right\}, \mathbb{I}\left\{Y_t \leq y\right\}\right)$ and almost surely uniformly continuous sample paths defined on $\mathbb{R}^n$ (see Theorem 7.3 of Rio [39]), *where $\leq$ denotes the pointwise order on $\mathbb{R}^n$. If furthermore $\mathbb{H}_0^{(P)}$ is true, then*

$$p_T \rightsquigarrow p_{\infty} := \max_{i=1,2} \sup_{(\lambda, z) \in \Gamma_i^{(P)}} p_i\left(z, \lambda, \tau, \mathcal{G}_F\right), \tag{4}$$

where

$$\Gamma_i^{(P)} := \left\{(\lambda, z), \lambda \in \mathbb{L}, z \in A_i : p_i\left(z, \lambda, \tau, F\right) = 0\right\}, \ i = 1, 2.$$

2. *Also,*

$$m\left(\tau, \sqrt{T}\left(\hat{F}_T - F\right)\right) \rightsquigarrow m^{\star}_{\infty} := \sup_{\lambda \in \mathbb{L}} \max_{i=1,2} \sup_{z \in A_i} m_i\left(z, \lambda, \tau, \mathcal{G}_F\right), \tag{5}$$

and, if furthermore $\mathbb{H}_0^{(M)}$ is true, then

$$m_T \rightsquigarrow m_{\infty} := \max_{i=1,2} \sup_{(\lambda, z) \in \Gamma_i^{(M)}} m_i\left(z, \lambda, \kappa, \mathcal{G}_F\right), \tag{6}$$

where

$$\Gamma_i^{(M)} := \left\{(\lambda, z), \lambda \in \mathbb{L}, z \in A_i : m_i\left(z, \lambda, \tau, F\right) = 0\right\}, \ i = 1, 2.$$

[3] Notice that $n^{\star} > 1$ since in the opposite case $\mathbb{L} = \{\tau\}$ in which case this distinguished element is efficient w.r.t. any notion of stochastic dominance.

3. *Suppose furthermore that* Assumption 2 *holds. If τ is not an extreme point of $\mathbb{L}$ then the laws of $p^{\star}_{\infty}$ and $m^{\star}_{\infty}$ are absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}$.* If *τ is an extreme point of $\mathbb{L}$ then the laws of $p^{\star}_{\infty}$ and $m^{\star}_{\infty}$ are absolutely continuous when restricted to $(0, +\infty)$, and each one may have an atom at zero of probability less than or equal to $\frac{1}{n^{\star}}$, where $n^{\star}$ is the number of extreme points of $\mathbb{L}$.*

Remark 1. The limiting random variables have the form of suprema of Gaussian processes w.r.t. subsets of the relevant "parameter spaces". First notice that $p_i\left(z, \lambda, \tau, \mathcal{G}_F\right)$ and $m_i\left(z, \lambda, \tau, \mathcal{G}_F\right)$ can for any $i = 1, 2$ be equivalently expressed as appropriate Riemann integrals of $\mathcal{G}_{\lambda F}(u) - \mathcal{G}_{\tau F}(u)$, where for any $\lambda \in \mathbb{L}$, $\mathcal{G}_{\lambda F}$ is a zero mean Gaussian process with covariance kernel $\sum_{t\in\mathbb{Z}} \text{Cov}\left(\mathbb{I}\left\{\lambda' Y_0 \leq x\right\}, \mathbb{I}\left\{\lambda' Y_t \leq y\right\}\right)$. Those integrals are well defined zero mean Gaussian processes due to the fact that

$$\mathbb{V}\text{ar} \int_0^{+\infty} \mathcal{G}_{\lambda F}(u)\, du = \int_0^{+\infty} \sum_{t\in\mathbb{Z}} \text{Cov}\left(\mathbb{I}\left\{\lambda' Y_0 \leq u\right\}, \mathbb{I}\left\{\lambda' Y_t \leq u\right\}\right) du$$

$$\leq 2 \sum_{t=0}^{\infty} \sqrt{a_T} \int_0^{+\infty} \sqrt{1 - G(u, \lambda, F)}\, du < +\infty,$$

while

$$\mathbb{V}\text{ar} \int_{-\infty}^{0} \mathcal{G}_{\lambda F}(u)\, du = \int_{-\infty}^{0} \sum_{t\in\mathbb{Z}} \text{Cov}\left(\mathbb{I}\left\{\lambda' Y_0 \leq u\right\}, \mathbb{I}\left\{\lambda' Y_t \leq u\right\}\right) du$$

$$\leq 2 \sum_{t=0}^{\infty} \sqrt{a_T} \int_{-\infty}^{0} \sqrt{G(u, \lambda, F)}\, du < +\infty,$$

where the first inequalities in each of the previous displays follow from inequality 1.12b in Rio [39], and the second ones follow from Assumption 1 (see also p. 196 of Horvath et al. [18]).

Remark 2. $\Gamma_i^{(P)} \neq \emptyset$ ($\Gamma_i^{(M)} \neq \emptyset$), since $\{\tau\} \times A_i \subseteq \Gamma_i^{(P)}$ ($\{\tau\} \times A_i \subseteq \Gamma_i^{(M)}$), and that due to Assumption 1 and the Dominated Convergence Theorem they are closed, for all $i = 1, 2$. In the prospect case, and if the support of F is bounded, then for any $\lambda \in \mathbb{L} : \mathbb{E}\left(\lambda' Y_0\right)_{s(i)} = \mathbb{E}\left(\tau' Y_0\right)_{s(i)}$, $\exists z \in A_i : (\lambda, z) \in \Gamma_i^{(P)}$,[4] for all $i = 1, 2$, where $s(i) = \begin{cases} -, & i = 1 \\ +, & i = 2 \end{cases}$. In the Markowitz case, if the support of F is bounded, we obtain the stronger result that for any $\lambda \in \mathbb{L}$, $\exists z \in A_i : (\lambda, z) \in \Gamma_i^{(M)}$, for all $i = 1, 2$.

Remark 3. It is easy to see that $p_{\infty} \leq p^{\star}_{\infty}$ and $m_{\infty} \leq m^{\star}_{\infty}$. When $\Gamma_i^{(P)} = \{\tau\} \times A_i$ for all $i = 1, 2$, then the distribution of p_{∞}, is degenerate at zero, while when $\Gamma_i^{(M)} = \{\tau\} \times A_i$, for all $i = 1, 2$, then the distribution of m_{∞}, is degenerate at zero.

[4] for example, since the support is bounded, it can be covered by some hypercube of the form $[z_l, z_u]^n$ where z_l can be chosen as negative. Obviously $(\lambda, z_l) \in \Gamma_1^{(P)}$, for any λ that satisfies the restriction above for $i = 1$.

Remark 4. The laws of $p^\star_\infty$ and $m^\star_\infty$ are absolutely continuous, at least when restricted to the interior of the respective supports. This, via the connectedness of the supports, implies the continuity of the relevant quantile functions at any $\alpha < \frac{n^\star-1}{n^\star}$. Notice that in the non trivial cases (i.e. when $n^\star > 1$) this condition is satisfied when $\alpha \leq \frac{1}{2}$ which is obviously a slack restriction.

In what follows, and in order to avoid the possibility of asymptotic degeneracy, we employ $p^\star_\infty$ and $m^\star_\infty$ respectively, for the facilitation of the decision process. Obviously, the laws of $p^\star_\infty$ and $m^\star_\infty$ are usually analytically intractable since they are maxima of complicated Gaussian processes depending on the usually unknown F and the the dependence structure of $(Y_t)_{t\in\mathbb{Z}}$. Hence, for the design of feasible testing procedures they must be approximated. This is accomplished by bootstrap resampling and numerical optimization and it is explained in the following section.

4 Consistent Testing Procedures Based on Block Bootstrap

As mentioned above, the laws of $p^\star_\infty$ and $m^\star_\infty$ generally depend on the usually unknown covariance kernels of the limiting Gaussian processes. Hence, we cannot use numerical techiniques in order to provide with the relevant rejection regions once and for all, without further strong parametric assumptions.

In this section, in order to avoid such assumptions, we consider approximations based on block bootstrap resampling techniques that manage to incorporate the assumed dependence. Those are based on arguments by which data are divided into blocks and those, rather than individual data, are resampled in order to mimic the time dependent structure of the original data.[5] Let b_T, l_T denote integers such that $T = b_T l_T$. b_T denotes the number of blocks and l_T the block size. The following assumption concerns the choice of l_T and it is consistent to the relevant choice appearing in Theorem 2.2 of Peligrad [34].

Assumption 3. *For some* $0 < q < \frac{1}{3}$ *and some* $0 < h < \frac{1}{3} - q$*,* l_T *satisfies* $T^h \ll l_T \ll T^{\left(\frac{1}{3}-q\right)}$*, as* $T \to \infty$*.*

We consider only the case of non-overlapping blocks. This is due to the fact that the bias reducting centering of the relevant statistics would imply further serious numerical burden.[6] In any case, and due to the fact that we are only concerned with first order asymptotic properties, it would be easy to see that the overlapping case would also have those properties. In what follows, let $(Y^\star_t)_{t=1,\ldots,T}$ denote a bootstrap sample in the context of the non-overlapping blocks methodology, and let $\hat{F}^\star_T$ denote its empirical distribution. Denote by $\mathbb{P}^\star_T$ the relevant probability distribution that represents the law of $(Y^\star_t)_{t=1,\ldots,T}$ conditional on $(Y_t)_{t=1,\ldots,T}$. Let

$$p^\star_T := p\left(\tau, \sqrt{T}\left(\hat{F}^\star_T - \hat{F}_T\right)\right)$$

[5] i.e. the relevant empirical measure on the powerset of the sample is essentially used.

[6] At least for the second test, the recentering makes the test statistics very difficult to compute, since the optimization for Markowitz stochastic dominance involves a large number of binary variables (see the section on the numerical implementation).

and

$$m_T^\star := m\left(\tau, \sqrt{T}\left(\hat{F}_T^\star - \hat{F}_T\right)\right).$$

For $\alpha \in (0,1)$, consider the following decision rules:

1. Let $\rho_{T,P}^\star := \mathbb{P}_T^\star\left(p_T^\star > p_T\right)$ and

$$\text{reject } \mathbb{H}_0^{(P)} \text{iff } \rho_{T,P}^\star < \alpha. \tag{7}$$

2. Let $\rho_{T,M}^\star := \mathbb{P}_T^\star\left(m_T^\star > m_T\right)$ and

$$\text{reject } \mathbb{H}_0^{(M)} \text{iff } \rho_{T,M}^\star < \alpha. \tag{8}$$

The following result establishes asymptotic properties of the decision rules above.

Proposition 3. *Suppose that Assumptions 1, 2 and 3 hold. Suppose that* $0 < \alpha < \frac{n^\star - 1}{n^\star}$ *when* τ *is an extreme point of* $\mathbb{L}$ *and that* $\alpha \in (0,1)$ *when it is not. Then the tests based on decision rules (7) and (8), respectively, are asymptotically conservative and consistent.*

The restriction on the choice of the significance level is of negligible practical importance since the usual choices of α nessesarily satisfy it as mentioned before. Furthermore, the tests are in any case consistent.

The p-values appearing in (7)-(8) are usually analytically intractable. They are in both cases approximated by an empirical frequency argument based on several bootstrap samples. More specifically, given $R \geq 1$ bootstrap samples $\left\{\left(Y_{t,r}^\star\right)_{t=1,\dots,T}\right\}_{r=1,\dots,R}$, approximations of the aforementioned p-values are provided by

$$\hat{\rho}_{T,j}^\star(\alpha) = \frac{1}{R}\sum_{r=1}^{R} \mathbb{I}\{k_{T,r}^\star(j) > k_T(j)\},$$

where $j = P, M$, $k_{T,r}^\star(j) = \begin{cases} p\left(\tau, \sqrt{T}\left(\hat{F}_{T,r}^\star - \hat{F}_T\right)\right), & j = P \\ m\left(\tau, \sqrt{T}\left(\hat{F}_{T,r}^\star - \hat{F}_T\right)\right), & j = M \end{cases}$, for $r = 1, \cdots, R$, and $k_T(j) = \begin{cases} p_T, & j = P \\ m_T, & j = M \end{cases}$ (see also Davidson and MacKinnon, [6, 7]). The asymptotic theory used for the proof of the proposition above, along with an application of the relevant to Assumption 1 LLN imply also the stated above asymptotic properties of those procedures as $R \to \infty$, and then $T \to \infty$. Obviously, the value of R is expected to affect higher order (and/or fixed sample) properties of the resulting procedures.

In the case of asymptotic non-degeneracy (utilizing, among others, Theorem 3.5.1.i of Politis et al. [35] and the results in Proposition 2), it is easy to construct analogous testing procedures based on subsampling that would be *asymptotically exact* and consistent, in the spirit of Linton, et al. [27]. Furthermore, in this context, it would also be possible to form testing procedures based on a block bootstrap design without the recentering that appears in the definition of $p_T^\star$ and $m_T^\star$. Using the proof methodology of Proposition 2, it could be possible that such procedures would also be asymptotically exact if the null hypothesis is strengthened so as to hold also in some weak neighborhood of F. Consistency would also hold if the bootstrap sample size would diverge to infinity at a slower rate than T.[7]

We do not currently engage into such considerations due to the following reasons. First, except for cases such as the ones described in Remark 2, and since F is unknown, non-degeneracy cannot be easily established. Second, partial Monte Carlo evidence for the subsampling procedures shows finite sample properties that crucially depend on the subsample choice and seem inferior to the analogous properties of the tests defined above (for the latter see the following section). Third, even if the block bootstrap procedures without recentering have the aforementioned properties, those seem to result from much stronger forms of the null hypotheses, while it could be possible that the restriction of the boostrap sample size for consistency implies analogously poor finite sample properties.

5 Monte Carlo study

In this section we design a set of Monte Carlo experiments to evaluate the size and power of the proposed tests in finite samples, in the context of the aforementioned numerical approximation of the test statistics and the p-values, as well as w.r.t. the choice of the block size for which the assumption framework provides only asymptotic guidance.

We do so in a framework of conditional heteroskedasticity that is partially consistent with empirical findings on returns of financial data that are similar to the empirical application that follows. The $(Y_t)_{t\in\mathbb{Z}}$ process is constructed as a vector GARCH(1,1) process that also contains an appropriately transformed element. Under the relevant restrictions, this allows for both temporal as well as cross sectional dependence between the random variables that constitute the vector process. In the following paragraph we describe the process, formulate $\mathbb{L}$, and by deriving relevant results, we establish efficient and non-efficient portfolios w.r.t to both criteria. We engage to the experiment and then present the results in the final paragraph of this section.

5.1 GARCH Type Processes and Efficiency Considerations

Suppose that

$$z_t \overset{\text{iid}}{\sim} N(0,1), t \in \mathbb{Z}.$$

[7] In the case of asymptotic degeneracy, it can be proven that, similarly to the relevant results of Linton, Maasoumi and Whang [27], such procedures are also asymptotically conservative.

Furthermore for all $t \in \mathbb{Z}$, for $i = 1,2,3$, $\omega_j, a_j, \beta_j \in \mathbb{R}_{++}$, such that $\mathbb{E}\left[\left(a_j z_0^2 + \beta_i\right)^{1+\epsilon}\right] < 1$ for some $\epsilon > 0$, $\mu_i \in \mathbb{R}_+$ define

$$\begin{aligned} y_{j_t} &= \mu_j + z_t h_{j_t}^{1/2}, \\ h_{i_t} &= \omega_j + \left(a_j z_{t-1}^2 + \beta_j\right) h_{i_{t-1}}, \end{aligned}$$

while for $j = 4$ and $v_1, v_2 \in \mathbb{R}$ define

$$y_{4_t} = v_1 h_{3_t}^{1/2} \left(z_t\right)_+ + v_2 h_{3_t}^{1/2} \left(z_t\right)_- .$$

Let $Y_t = (y_{1_t}, y_{2_t}, y_{3_t}, y_{4t})'$. The construction of y_{4_t} in comparison to the other elements, and the consideration of $\mathbb{L}$ below, facilitates the verification of the relevant dominance conditions. Corollary 1 and Theorem 8 of Lindner [26], the definition of strong mixing along with the measurability of $(\cdot)_+$, $(\cdot)_-$ and their independence from t, the existence of moments of order $2+\delta$ for all the univariate processes involved, imply that Assumption 1, except for the part of it about the $\mathbb{V}$ matrix, holds for (Y_t). Furthermore, notice that by a linear independence argument, $\mathbb{V}$ can be shown to be positive definite, if for instance $a_j \neq a_{j^\star}$ and $\beta_j \neq \beta_{j^\star}$ when $j \neq j^\star$ and $v_1 \neq v_1$. Notice that the fact that all the involved processes are constructed by the same innovations, allows the modeling of contemporaneous dependence between the elements of the vector process, without further complicating the form of the conditional variances recursions. Furthermore, trivial calculations show that Cov $\left(y_{i_t}, y_{i_{t-k}}\right) = 0$ for all non zero k and $i \neq 4$, while this is not true for $i = 4$. Let $\tau = (0,0,1,0)$, $\tau^\star = (0,0,0,1)$ and $\mathbb{L} = \{(\lambda, 1-\lambda, 0, 0), \lambda \in [0,1], \tau, \tau^\star\}$. For this choice of $\mathbb{L}$,[8] Assumption 2 also holds, while we can easily specify portfolios for which the relevant null hypotheses are valid.

In this respect, the first proposition establishes that $\tau^\star$ *is a portfolio that is both Markowitz and prospect efficient* w.r.t. $\mathbb{L}$ when the structuring coefficients are appropriately chosen so that the negative part of $\tau^\star$ has smaller variance and the positive part of $\tau^\star$ has larger variance when compared to the other portfolios in $\mathbb{L}$.

Proposition 4. *If $\mu_i = 0$ for $i = 1,2,3$, $|v_1| > \sqrt{\frac{\max\{\omega_i, a_i, \beta_i,\, i=1,2,3\}}{\min\{\omega_i, a_i, \beta_i,\, i=1,2,3\}}}$ and $|v_2| < \sqrt{\frac{\min\{\omega_i, a_i, \beta_i,\, i=1,2,3\}}{\max\{\omega_i, a_i, \beta_i,\, i=1,2,3\}}}$ then $\tau^\star$ is both PSD and MSD-efficient w.r.t. $\mathbb{L}$.*

The following proposition establishes the ineficiency of τ w.r.t. $\mathbb{L}$ for both relations. Notice that an analogous result is directly obtained by the previous proposition in a more restricted setting.

Proposition 5. *If $\mu_i = 0$ for $i = 1,3$, $\omega_1 < \omega_3$, $a_1 < a_3$ and $\beta_1 < \beta_3$ then τ is neither PSD, nor MSD-inefficient w.r.t. $\mathbb{L}$.*

5.2 Scenarios, Computational Issues and Results

Scenarios We use as DGPs instances of the GARCH processes described above, by choosing the parameters according to Propositions 4 and 5, to approximate the fixed

[8] $\mathbb{L}$ is obviously disconnected.

T size and power. For $B = 300$ we generate independent across $b = 1, \ldots, B$ samples $\left(Y_t^{(b)}\right)_{t=1,\ldots,T}$ for several values of T. For each b, we use the non-overlapping block bootstrap methodology described above to evaluate $\hat{\rho}_{T,j}^{\star(b)}(\alpha)$, $j = P, M$ and decide according to decision rules 7 and 8 respectively, by choosing $\alpha = 0.05$, and several values of R and l_T. We approximate the fixed T size by $\alpha_{B,T,j} := \frac{1}{B}\sum_{b=1}^{B} \mathbb{I}\{\hat{\rho}_{T,j}^{\star(b)}(0.05) < 0.05\}$ when the DGP is such that $\mathbb{H}_0^{(j)}$ holds and the fixed T power by $1 - \beta_{B,T,j} := \frac{1}{B}\sum_{b=1}^{B} \mathbb{I}\{\hat{\rho}_{T,j}^{\star(b)}(0.05) < 0.05\}$ when the DGP is such that $\mathbb{H}_1^{(j)}$ holds for $j = P, M$.

Size Evaluation Scenario-Parameters Selection: To approximate the fixed T size, we test for PSE and MSD efficiency of portfolio τ^* by setting $\mu_i = 0$ for $i = 1, 2, 3$, $\omega_1 = 0.5$, $\omega_2 = 0.5$, and $\omega_3 = 0.5$, $a_1 = 0.4$, $a_2 = 0.45$, and $a_3 = 0.5$ and $\beta_1 = 0.5$, $\beta_2 = 0.45$, $\beta_3 = 0.4$, $v_1 = 1.5$ and $v_2 = 0.5$. In this case, we have that $|v_1| > \sqrt{\frac{\max\{\omega_i, a_i, \beta_i, i=1,2,3\}}{\min\{\omega_i, a_i, \beta_i, i=1,2,3\}}}$ and $|v_2| < \sqrt{\frac{\min\{\omega_i, a_i, \beta_i, i=1,2,3\}}{\max\{\omega_i, a_i, \beta_i, i=1,2,3\}}}$.

Power evaluation Scenario-Parameters Selection: To approximate the fixed T power, we test for PSE and MSD efficiency of portfolio τ by setting $\mu_i = 0$ for $i = 1, 2, 3$, $\omega_1 = 0.5$, $\omega_2 = 0.5$, and $\omega_3 = 0.8$, $a_1 = 0.3$, $a_2 = 0.4$, and $a_3 = 0.45$ and $\beta_1 = 0.3$, $\beta_2 = 0.4$, $\beta_3 = 0.45$, $v_1 = 2$ and $v_2 = 0.2$. In this case, we have that $\omega_1 < \omega_3$, $a_1 < a_3$ and $\beta_1 < \beta_3$.

Computational Issues We numerically solve all the optimization problems, according to the formulations presented in the Appendix, using the General Algebraic Modeling System (GAMS), which is a high-level modeling system for mathematical programming and optimization. This language calls special solvers (GUROBI in our case) that are specialized in linear and mixed integer programs. GUROBI uses the branch and bound technique to solve the MIP program. The Matlab code (where the simulations run) calls the specific GAMS program, which calls the GUROBI solver to solve each optimization. The optimizations are performed on a number of computers (with i7 processors, 3.2 GHz Power, 16Gb of RAM). We note the almost exponential increase in solution time with the increasing number of observations.

The computational time, which involves linear as well as MIP problems, varies for both PSD and MSD models from 10 minutes in case 1 (linear models) to 30 hours in case 3 (MIP models).

Results We present our Monte Carlo results in Table 1. Given the non-informative nature of Assumption 3 on the choice of l_T for fixed T, for the case where $T = 500$, we investigate cases where l_T ranges from 4 to 12 by a step size of 4, choices motivated by the suggestions of Hall, Horowitz, and Jing [17], who suggest as optimal block sizes multiples of $T^{1/3}$, $T^{1/4}$, and $T^{1/5}$. Our experiments show that the choice of the block size according to the previous, does not seem to dramatically alter the performance of our methodology even for moderately smaller and larger values of T. We also investigate the sensitivity of the tests to the choice of the number of bootstrap samples and sample size by allowing for $(R, T) = (100, 200), (300, 500), (500, 1000)$. The tests seem to perform well in every case. For example, for $l_T = 10,\ R = 300,\ T = 500$, we get $\alpha_{B,T,P} = 0.048$ and $\alpha_{B,T,M} = 0.039$ for the Markowitz stochastic dominance efficiency test in the first scenario of parameters selection and $1 - \beta_{B,T,P} = 0.942$, and $1 - \beta_{B,T,M} = 0.938$ in the second scenario.

Tab. 1: Monte Carlo Results

Block size l_T :	4	8	10	12
Case 1:	R=100	T=200		
Size Scenario:				
$\alpha_{B,T,P}$	3.1%	3.8%	4.7%	3.6%
$\alpha_{B,T,M}$	4.2%	6.3%	3.5%	3.7%
Power Scenario:				
$1 - \beta_{B,T,P}$	93.2%	92.7%	92.6%	94.1%
$1 - \beta_{B,T,M}$	96.3%	93.5%	92.8%	92.9%
Case 2:	R=300	T=500		
Size Scenario:				
$\alpha_{B,T,P}$	3.9%	4.2%	4.8%	5.5%
$\alpha_{B,T,M}$	3.5%	5.9%	3.9%	4.7%
Power Scenario:				
$1 - \beta_{B,T,P}$	93.4%	92.9%	94.2%	97.6%
$1 - \beta_{B,T,M}$	92.9%	94.0%	93.8%	92.8%
Case 3:	R=500	T=1000		
Size Scenario:				
$\alpha_{B,T,P}$	4.6%	4.1%	5.5%	5.1%
$\alpha_{B,T,M}$	4.7%	4.3%	3.9%	5.2%
Power Scenario:				
$1 - \beta_{B,T,P}$	96.1%	95.7%	96.4%	95.9%
$1 - \beta_{B,T,M}$	96.6%	95.1%	94.3%	94.2%

6 Empirical application

In the empirical application, we test for the aforementioned notions of efficiency of the market portfolio relative to the space of all possible portfolios that can be constructed upon a set of basis assets excluding the market portfolio. More specifically, we use as basis assets either several instances of the Fama and French (FF) benchmark portfolios, a set of Momentum portfolios, or a set of industry portfolios as described below, along with the market portfolio. If the number of basis assets equals n, then $\mathbb{L}$ is essentially the union of the relevant $n-2$ subsimplex of the standard $n-1$ simplex with $\{(0, \cdots, 1)\}$, where the latter signifies the market portfolio. The FF benchmark portfolios formed on market capitalization (size) and book-to-market equity ratio (BE/ME) (Fama and French [12]). To check whether our results are specific to the BE/ME shorted portfolios, we use three different datasets of the Fama and French (FF) benchmark portfolios. We also use the 10 Momentum portfolios, which contain the returns for 10 prior-return portfolios. Finally, we use the 49 industry portfolios from the US market (Fama and French 1997). The assignment of the NYSE, AMEX, and NASDAQ stocks into industry portfolios are based on their four-digit SIC code. The industry portfolio returns are value weighted, i.e. based on the market capitalisation. All these portfolios have been at the center of the asset pricing literature over the past two decades.

- **The FF Benchmark portfolios**: They are constructed at the end of each June, and correspond to the intersections of portfolios formed on size (market equity, ME) and portfolios formed on the ratio of book equity to market equity (BE/ME). ME is the stock price times the number of shares, while BE is the book value of shareholders equity, plus balance sheet deferred taxes and investment tax credit (if available), minus the book value of preferred stocks. The size breakpoint for year t is the median NYSE market equity at the end of June of year t. BE/ME for June of year t is the book equity for the last fiscal year end in $t-1$ divided by ME for December of $t-1$. Firms with negative BE are not included in any portfolio. Also, only firms with ordinary common equity (as classified by CRSP) are included in the tests. We use three different data sets:
  - The 6 FF Benchmark portfolios: They are constructed as the intersections of 2 portfolios formed on size (market equity, ME) and 3 portfolios formed on the ratio of book equity to market equity (BE/ME).
  - The 25 FF Benchmark portfolios: They are constructed as the intersections of 5 portfolios formed on size (market equity, ME) and 5 portfolios formed on the ratio of book equity to market equity (BE/ME).
  - The 100 FF Benchmark portfolios: They are constructed as the intersections of 10 portfolios formed on size (market equity, ME) and 10 portfolios formed on the ratio of book equity to market equity (BE/ME).
- **The 10 Momentum portfolios**: They are constructed monthly using NYSE prior (2-12) return decile breakpoints. The portfolios include NYSE, AMEX, and NASDAQ stocks with prior return data. To be included in a portfolio for month t (formed at the end of month t-1), a stock must have a price for the end of month t-13 and a good return for t-2.
- **The 49 Industry portfolios**: They are constructed by assigning each NYSE, AMEX, and NASDAQ stock to an industry portfolio at the end of June of year t based on its four-digit SIC code at that time. The Compustat SIC codes are used for the fiscal year ending in calendar year t-1. Whenever Compustat SIC codes are not available, the CRSP SIC codes for June of year t are used. The industries are defined with the goal of having a manageable number of distinct industries that cover all NYSE, AMEX, and NASDAQ stocks.

For each dataset we use data on monthly excess returns (month-end to month-end) from January 1930 to December 2012 (996 monthly observations) obtained from the data library on the homepage of Kenneth French (http://mba.turc.dartmouth.edu/pages/faculty/ken.french). The test portfolio is the Fama and French market portfolio, which is the value-weighted average of all non-financial common stocks listed on NYSE, AMEX, and Nasdaq, and covered by CRSP and COMPUSTAT.

Table 2 presents some sample characteristics of the Market portfolio and the 6 FF portfolios[9] covering the period from January 1930 to December 2012 (996 monthly observations) that are used in the test statistics.

Descriptive Statistics (January 1930 to December 2012)						
No.	Mean	Std. Dev.	Skewness	Kurtosis	Minimum	Maximum
Market Portfolio	0.604	5.413	0.237	7.593	-29.98	37.77
1	1.016	7.825	1.026	7.270	-32.32	65.63
2	1.288	7.139	1.310	11.660	-31.10	64.12
3	1.493	8.367	2.175	18.810	-33.06	85.24
4	0.847	5.308	-0.023	2.231	-28.08	32.55
5	0.936	5.823	1.303	14.227	-28.01	51.52
6	1.161	7.327	1.547	14.926	-35.45	68.25

Tab. 2: *Descriptive statistics of monthly returns in % from January 1930 to December 2012 (996 monthly observations) for the Fama and French market portfolio and the six Fama and French benchmark portfolios formed on size and book-to-market equity ratio. Portfolio 1 has low BE/ME and small size, portfolio 2 has medium BE/ME and small Size, portfolio 3 has high BE/ME and small size, ..., portfolio 6 has high BE/ME and large size.*

As we can see from Table 2, the sample skewness and kurtosis provide evidence against marginal normality. If this is true and the investor utility function is not quadratic, then preference relation of any such investor cannot be represented by the variance-covariance matrix of these portfolios. At this point, it is perhaps interesting to note that Scaillet and Topaloglou [40] show that the Fama and French market portfolio is not mean-variance efficient, compared to the 6 benchmark portfolios. This motivates us to test whether the market portfolio is efficient when different preferences are taken into account.

6.1 Results of the stochastic dominance efficiency tests

We find a significant autocorrelation of order one at a 5% significance level in some benchmark portfolios, while ARCH effects are also present at a 5% significance level. This indicates that a block bootstrap approach should be favored over a standard i.i.d. bootstrap approach. Furthermore, estimation of GARCH type models provide evidence in favor of the mixing and moment conditions appearing in our assumption framework. Indeed, both for the market portfolio as well as for each benchmark portfolio i, the estimates of the sum of the GARCH and the ARCH coefficients are less than 1. We choose a block size of 10 observations following the suggestions of Hall, Horowitz, and Jing [17], who show that optimal block sizes are multiple of $T^{1/3}$, where in our case, $T = 996$. The p-values are approximated as shown before.

[9] Analogous statistical characteristics are also available for the other datasets.

- **The FF Benchmark portfolios:**
  - The 6 FF Benchmark portfolios: For the prospect stochastic dominance efficiency, we cannot reject the hypothesis that the market portfolio is efficient. The p-value, $\tilde{p} = 0.743$, is way above the significance level of 5%. We divide the full period into two sub-periods, the first one from January 1930 to June 1971, a total of 498 monthly observations, and the second one from July 1971 to December 2012, 498 monthly observations. We test for prospect stochastic dominance of the market portfolio to each sub-period. We find that the p-value for the first sub-period is $\tilde{p}_1 = 0.654$ and the p-value for the second sub-period is $\tilde{p}_2 = 0.687$.

    On the other hand, we find that the MSD criterion cannot be accepted at the aforementioned significance level. The p-value, $\tilde{p} = 0.043$ is below the significance level of 5%. Additionally, the p-value, $\tilde{p}_1 = 0.061$, for the first sub-period and the p-value, $\tilde{p}_2 = 0.029$, for the second sub-period indicate that the market portfolio is not Markowitz stochastic dominance efficient in each sub-period as well as in the full period.
  - The 25 FF Benchmark portfolios: As before, for the prospect stochastic dominance efficiency, we cannot reject the hypothesis that the market portfolio is efficient. The p-value, $\tilde{p} = 0.564$, is way above the significance level of 5%. We find that the p-value for the first sub-period is $\tilde{p}_1 = 0.729$ and the p-value for the second sub-period is $\tilde{p}_2 = 0.483$.

    We additionally find that the MSD criterion cannot be accepted. The p-value, $\tilde{p} = 0.034$, is below the significance level of 5%. Additionally, the p-value, $\tilde{p}_1 = 0.047$, for the first sub-period and the p-value, $\tilde{p}_2 = 0.051$, for the second sub-period indicate that the market portfolio is not Markowitz stochastic dominance efficient in each sub-period as well as in the full period.
  - The 100 FF Benchmark portfolios: Again, for the prospect stochastic dominance efficiency, we cannot reject the hypothesis that the market portfolio is efficient. The p-value, $\tilde{p} = 0.479$, is above the significance level of 5%. We find that the p-value for the first sub-period is $\tilde{p}_1 = 0.384$ and the p-value for the second sub-period is $\tilde{p}_2 = 0.516$.

    As before, we find that the MSD criterion cannot be accepted. The p-value, $\tilde{p} = 0.030$, is below the significance level of 5%. Additionally, the p-value $\tilde{p}_1 = 0.049$ for the first sub-period and the p-value $\tilde{p}_2 = 0.028$ for the second.
- **The 10 Momentum portfolios:** We cannot reject the hypothesis that the market portfolio is prospect stochastic dominance efficient. The p-value is $\tilde{p} = 0.387$. We find that the p-value for the first sub-period is $\tilde{p}_1 = 0.416$ and the p-value for the second sub-period is $\tilde{p}_2 = 0.465$.

  The hypothesis that the market portfolio is MSD efficient is rejected. The p-value is $\tilde{p} = 0.049$, which is below the significance level of 5%. Additionally, the p-value $\tilde{p}_1 = 0.057$ for the first sub-period and p-value $\tilde{p}_2 = 0.028$ for the second sub-period.

- **The 49 Industry portfolios:** The hypothesis that the market portfolio is prospect stochastic dominance efficient is not rejected. The p-value is $\tilde{p} = 0.519$. We find that the p-value for the first sub-period is $\tilde{p}_1 = 0.623$ and the p-value for the second sub-period is $\tilde{p}_2 = 0.414$.

  Finally, we find that the MSD criterion is rejected. The p-value is $\tilde{p} = 0.054$, which is below the significance level of 5%. Additionally, the p-value $\tilde{p}_1 = 0.039$ for the first sub-period and p-value $\tilde{p}_2 = 0.040$ for the second sub-period.

The results provide evidence in favor of the claim that the market portfolio is prospect stochastic dominance efficient. If this holds, it implies that any S-shaped utility function rationalizes the market portfolio as an optimal choice. If investors are risk seeking for losses and risk averse for gains, then they will pay a premium for stocks that have low downside risk in bear markets and high upside potential in bull markets. Prospect type investors will have an ”abnormal” demand for assets that offer systematic downside protection (due to loss aversion and the overweighting of small probabilities of large losses) or systematic upside potential (due to loss aversion and the overweighting of small probabilities of large gains).

Prospect theory also involves the concept of probability transformations that overweight small probabilities of large gains and losses, and underweight large and intermediate probabilities of small and intermediate gains and losses (Tversky and Kahneman [42]). The prospect stochastic dominance efficiency of the market portfolio we found here is not affected by transformations that are increasing and convex over losses and increasing and concave over gains, i.e. S-shaped transformations. Moreover, if the market portfolio is non dominated w.r.t. PSD, then it is also non dominated w.r.t. the weaker condition given by Baucells and Heukamp [3].

On the other hand, in all cases our implementation does not provide support for the Markowitz stochastic dominance efficiency of the market portfolio. If this holds, *it does not necessarily imply* that no reverse S-shaped utility function can rationalize the market portfolio, but only the existence of at least one such function that fails to do so.

6.2 Rolling window analysis

We carry out an additional analysis to validate the prospect stochastic dominance efficiency of the market portfolio and the stability of the model results. It is possible that the efficiency of the market portfolio as a weighted average varies over time due to changes in the weights constructing it from the universe of assets. [10] Furthermore the temporal extend of our sample could imply the non validity of the stationarity assumption due to possible structural changes in the DGP. To account for the above, we perform a rolling window analysis, using a window width of 20 years. The test statistic is calculated separately for 63 overlapping 20-year periods (January 1930-December 1949), (January 1931-December 1951),...,(January 1993-December 2012). The time series in this case is smaller (240 monthly observations) so that a maintained assumption of stationarity is more credible.

[10] It is also possible that the degree of efficiency may change over time, as pointed by Post [36].

Figure 1 shows the corresponding p-values for the prospect stochastic dominance efficiency test (upper graph) and for the Markowitz stochastic dominance efficiency test (lower graph) using the 6 FF benchmark portfolios. We observe that the market portfolio is prospect stochastic dominance efficient in the total sample period. The prospect stochastic dominance efficiency is not rejected on any subsample. The p-values are always greater than 22% and in some cases they reach the 74%. This result is in accordance to that prospect stochastic dominance efficiency that was not rejected in the previous subsection, for the full period. On the other hand, we observe that the Markowitz stochastic dominance efficiency is rejected on 51 out of 63 subsamples. The p-values are most of the cases lower than 5%. This result is in accordance with the rejection of the Markowitz stochastic dominance efficiency that was found in the previous subsection. If this is true, it implies that for those subsamples there exist portfolios constructed from the set of the six benchmark portfolios that dominates the market portfolio w.r.t. at least one reverse S-shaped utility function.

Figure 2 exhibits the p-values for the prospect (upper graph) and for the Markowitz stochastic dominance efficiency test (lower graph) using the 10 Momentum portfolios. We observe again that the market portfolio is prospect stochastic dominance efficient in the total sample period. The prospect stochastic dominance efficiency is not rejected on any subsample. The p-values are always greater than 30%, and in some cases they reach the 70%. This result is in consonance to that prospect stochastic dominance efficiency that was not rejected in the previous subsection. On the other hand, we observe that the Markowitz stochastic dominance efficiency is rejected on 48 out of 63 subsamples. The p-values are most of the cases lower than 5%. This result is in accordance with the rejection of the Markowitz stochastic dominance efficiency and implies that for those subsamples there exist portfolios constructed from the set of the 10 Momentum portfolios that dominates the market portfolio w.r.t. at least one reverse S-shaped utility function.

Finally, Figure 3 shows the corresponding p-values for the prospect (upper graph) and for the Markowitz stochastic dominance efficiency test (lower graph) using the 49 Industry portfolios. We observe once again that the market portfolio is prospect stochastic dominance efficient in the total sample period. The prospect stochastic dominance efficiency is not rejected on any subsample. The p-values are always greater than 25%, and in some cases they reach the 70%. On the other hand, we observe that the Markowitz stochastic dominance efficiency is rejected on 49 out of 63 subsamples. The p-values are most of the cases lower than 5%. This result implies that for those subsamples there exist portfolios constructed from the set of the 49 Industry portfolios that dominates the market portfolio w.r.t. at least one reverse S-shaped utility function.

7 Concluding Remarks

In this paper we develop *consistent* statistical tests for prospect and Markowitz stochastic dominance efficiency for *time-dependent* data. We use a block bootstrap formulation to achieve valid asymptotic inference in a setting of temporal dependence. Mixed integer and linear programming are used to facilitate the computational aspects of the procedures.

To illustrate the potential of the proposed test statistics, we test whether the two stochastic dominance efficiency criteria rationalize the Fama and French market portfolio over three different data sets of Fama and French benchmark portfolios constructed as the intersections of ME portfolios and BE/ME portfolios, as well as over 49 Industry portfolios. The results support the claim that the market portfolio is prospect stochastic dominance efficient. In contrast, they are not in favor of the claim that the market portfolio is Markowitz stochastic dominance efficient, indicating that there might exist utility functions with global risk aversion for losses and risk seeking over gains that cannot rationalize the market portfolio as optimal.

The theoretical interpretation of the aforementioned empirical results is a quite interesting question. For example, they seem consistent with financial equilibria involving a generic representative investor with risk aversion for gains and risk seeking for losses. However, such interpretations should also take into account theoretical results such as the ones concerning the possibility of non-existence of equilibria in financial markets with prospect theory preferences-see for example De Giorgi, Hens, and Rieger [9].

The tests could possibly be used as initial steps for the statistical decoupling of the form of the utility or value function to the transformation of the probability measures that characterize many theories of choice under uncertainty. For example, non rejection of the MSD efficiency could support the validity of cumulative prospect theory when the curvature of the S-shaped utility is dominated by the reverse S-shaped probability transformation (see Post and Levy [38]) as this theory suggests. The construction of inferential procedures that statistically disentangle the two could be of importance.

The methodology used could be also relevant for the construction of tests of efficiency w.r.t. notions of stochastic dominance that are representable by utility functions with more complex behavior (e.g., attitudes towards risks may exhibit additional changes on extreme events).

In any case, we delegate the above considerations to future research. We hope that our results provide a stimulus for further theoretical and empirical examination of decision under prospect and Markowitz type preferences, as well as that this study contributes to the further proliferation of the SD methodology.

APPENDIX

Numerical Implementation

We describe a procedure applicable for the computation of p_T and m_T. This essentially works by reductions to equivalent (w.r.t. optimization) problems one for each statistic. Those are essentially based on Lemma 1 that appears below. A completely analogous procedure is used for the approximation of the statistics evaluated at the bootstrap samples, but it is by construction more tediously describable and thereby omitted to economize on space.

We also assume that $\mathbb{L}$ is convex in order to facilitate the presentation. The formulation is easily generalized to the cases covered by Assumption 2 since in its most general form the parameter space is a *finite* union of simplices.

In what follows we denote with Y_p the set $\{\tau' Y_t : \tau' Y_t > 0\}$ and with Y_n the complement $\{Y_t, t = 1, \ldots, T\} - Y_p$ along with zero.

Prospect Positive Part

$\sup_{z \in \mathbb{R}_{++}} \sup_{\lambda \in \mathbb{L}} p_2\left(z, \lambda, \tau, \sqrt{T} F_T\right)$ equals

$$\sup_{z \in \mathbb{R}_{++}} \sup_{\lambda \in \mathbb{L}} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left\{ \left[(z - \tau' Y_t)_+ - (-\tau' Y_t)_+ \right] - \left[(z - \lambda' Y_t)_+ - (-\lambda' Y_t)_+ \right] \right\},$$

and due to Lemma 1, for $r \in Y_p$ we have that

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left[(r - \tau' Y_t)_+ - (-\tau' Y_t)_+ \right] + \sup_{\lambda \in \mathbb{L}} -\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left[(r - \lambda' Y_t)_+ - (-\lambda' Y_t)_+ \right]$$

$$= \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left[(r - \tau' Y_t)_+ - (-\tau' Y_t)_+ \right] + \sup_{\lambda \in \mathbb{L}} -\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left[(r - \lambda' Y_t)_+ \mathbb{I}\{\lambda' Y_t > 0\} + r \mathbb{I}\{\lambda' Y_t \leq 0\} \right].$$

The last expression in the previous display equals

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left[(r - \tau' Y_t) \mathbb{I}\{0 < \tau' Y_t \leq r\} + r \mathbb{I}\{\tau' Y_t \leq 0\} \right]$$
$$- \inf_{\lambda \in \mathbb{L}} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left[(r - \lambda' Y_t) \mathbb{I}\{0 < \lambda' Y_t \leq r\} + r \mathbb{I}\{\lambda' Y_t \leq 0\} \right].$$

Since

$$\left[(r - \lambda' Y_t) \mathbb{I}\{0 < \lambda' Y_t \leq r\} + r \mathbb{I}\{\lambda' Y_t \leq 0\} \right]$$
$$= \min(r - \lambda' Y_t, r) \mathbb{I}\{\lambda' Y_t \leq r\},$$

$p_2\left(z_2, \lambda, \tau, \sqrt{T} F_T\right)$ becomes

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \min(r - \tau' Y_t, r) 1_{\{\tau' Y_t \leq r\}} - \inf_{\lambda \in \mathbb{L}} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \min(r - \lambda' Y_t, r) 1_{\{\lambda' Y_t \leq r\}}.$$

Hence, we need to solve the optimization problem

$$\inf_{\lambda \in \mathbb{L}} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \min\left(r - \lambda' Y_t, r\right) 1_{\{\lambda' Y_t \le r\}}.$$

We represent the previous by the following MIP program:

$$\begin{aligned}
\min_{\lambda \in \mathbb{L}} \quad & \frac{1}{\sqrt{T}} \sum_{t=1}^{T} (X_t + c b_t) && (9)\\
\text{s.t.} \quad & M(F_t - 1) \le r - \lambda' Y_t \le M F_t \quad \forall t \in T, && (10)\\
& -M(1 - F_t) \le L_t - (r - \lambda' Y_t) \le M(F_t - 1) \quad \forall t \in T, && (11)\\
& -M F_t \le L_t \le M F_t \quad \forall t \in T, && (12)\\
& X_t = L_t b_t + r(1 - b_t) \quad \forall t \in T, && (13)\\
& L_t - r + M b_t > 0 \quad \forall t \in T, && (14)\\
& \lambda' \mathbf{1} = 1, && (15)\\
& \lambda \ge 0, && (16)\\
& F_t, b_t \in \{0, 1\} \quad \forall t \in T, . && (17)
\end{aligned}$$

where $\mathbf{1}$ is a vector of ones, $c \in \mathbb{R}$ and M is a large real number.

X_t which is a linearization of the $\min(r - \lambda' Y_t, r)$ function. We use a binary variable F_t, which, according to the inequalities (10), equals 1 for each $t \in T$ for which $r \ge \lambda' Y_t$, and 0 otherwise. Then, the following two sets of inequalities, (11) and (12), ensure that the variable L_t equals $r - \lambda' Y_t$ for each $t \in T$ for which this difference is positive, and 0 otherwise. Constraints (13) and (14) ensure that X_t takes the minimum value between L_t and r. To get that to happen, we use a binary variable b_t which is equal to 1 if L_t is lower than r, or 0 otherwise.

Prospect Negative Part

$\sup_{z \in \mathbb{R}_-} \sup_{\lambda \in \mathbb{L}} p_1 \left(z_1, \lambda, \tau, \sqrt{T} F_T\right)$ equals

$$\sup_{z \in \mathbb{R}_-} \sup_{\lambda \in \mathbb{L}} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left\{ \left[(-\tau' Y_t)_+ - (z - \tau' Y_t)_+ \right] - \left[(-\lambda' Y_t)_+ - (z - \lambda' Y_t)_+ \right] \right\},$$

and due to Lemma 1, for $r \in Y_n$ we have that

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left[(-\tau' Y_t)_+ - (z - \tau' Y_t)_+ \right] + \sup_{\lambda \in \mathbb{L}} -\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left[(-\lambda' Y_t)_+ - (z - \lambda' Y_t)_+ \right]$$

$$= \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left[(-\tau' Y_t)_+ - (z - \tau' Y_t)_+ \right] + \sup_{\lambda \in \mathbb{L}} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left[\left(\lambda' Y_t + (r - \lambda' Y_t)_+ \right) \mathbb{I}\left\{ \lambda' Y_t \le 0 \right\} \right].$$

The last expression in the previous display equals

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left[\tau' Y_t \mathbb{I}\left\{\tau' Y_t < r\right\} + r\mathbb{I}\left\{r \le \tau' Y_t \le 0\right\}\right]$$
$$+\sup_{\lambda\in\mathbb{L}}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left[\lambda' Y_t \mathbb{I}\left\{\lambda' Y_t < r\right\} + r\mathbb{I}\left\{r \le \lambda' Y_t \le 0\right\}\right].$$

Since

$$\left[\lambda' Y_t \mathbb{I}\left\{\lambda' Y_t < r\right\} + r\mathbb{I}\left\{r \le \lambda' Y_t \le 0\right\}\right]$$
$$= \min\left(\lambda' Y_t, r\right)\mathbb{I}\left\{\lambda' Y_t \le 0\right\},$$

$p_1\left(z_1, \lambda, \tau, \sqrt{T}F_T\right)$ becomes

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\min\left(\tau' Y_t, r\right)1_{\{\tau' Y_t \le 0\}} + \sup_{\lambda\in\mathbb{L}}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\min\left(\lambda' Y_t, r\right)1_{\{\lambda' Y_t \le 0\}}.$$

Hence, we need to solve the optimization problem

$$\sup_{\lambda\in\mathbb{L}}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\min\left(\lambda' Y_t, r\right)1_{\{\lambda' Y_t \le 0\}}.$$

We represent the previous by the MIP program:

$$\begin{aligned}
\max_{\lambda\in\mathbb{L}} \quad & \frac{1}{\sqrt{T}}\sum_{t=1}^{T}(X_t - cb_t) && (18)\\
\text{s.t.} \quad & M(F_t - 1) \le -\lambda' Y_t \le MF_t \quad \forall t \in T, && (19)\\
& -M(1 - F_t) \le L_t - \lambda' Y_t \le M(F_t - 1) \quad \forall t \in T, && (20)\\
& -MF_t \le L_t \le MF_t \quad \forall t \in T, && (21)\\
& X_t = L_t b_t + r(1 - b_t) \quad \forall t \in T, && (22)\\
& L_t - r + Mb_t > 0 \quad \forall t \in T, && (23)\\
& \lambda' \mathbf{1} = 1, && (24)\\
& \lambda \ge 0, && (25)\\
& F_t, b_t \in \{0, 1\} \quad \forall t \in T, && (26)\\
& && (27)
\end{aligned}$$

where $\mathbf{1}, c, M$ as before.

Analogously, X_t is a linearization of the min$(\lambda' Y_t, r)$ function. We use a binary variable F_t, which, according to Inequalities (19) equals 1 for each $t \in T$ for which $\lambda' Y_t \le 0$, and 0 otherwise. Then, the following two sets of inequalities, (20) and (21), ensure that the variable L_t equals $\lambda' Y_t$ for each $t \in T$ for which this is negative, and 0 otherwise. Constraints (22) ensure that X_t takes the minimum value between L_t and r. To get that to happen, we use a binary variable b_t which is equal to 1 if L_t is lower than r, or 0 otherwise.

Markowitz Positive Part

$\sup_{z\in\mathbb{R}_{++}} \sup_{\lambda\in\mathbb{L}} m_2\left(z,\lambda,\tau,\sqrt{T}F_T\right)$ equals

$$\sup_{\lambda\in\mathbb{L}} \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left\{(\lambda' Y_t)_+ - (-\lambda' Y_t)_+ + (z-\lambda' Y_t)_+\right\} - \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left\{(\tau' Y_t)_+ - (-\tau' Y_t)_+ + (z-\tau' Y_t)_+\right\},$$

and due to Proposition 1, for $r \in Y_p$ the latter equals

$$\sup_{\lambda\in\mathbb{L}} \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left\{(\lambda' Y_t)_+ - (-\lambda' Y_t)_+ + (r-\lambda' Y_t)_+\right\} - \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left\{(\tau' Y_t)_+ - (-\tau' Y_t)_+ + (r-\tau' Y_t)_+\right\}.$$

The previous becomes

$$\sup_{\lambda\in\mathbb{L}} \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left\{(\lambda' Y_t)\,\mathbb{I}\left\{\lambda' Y_t > r\right\} + r\mathbb{I}\left\{\lambda' Y_t \le r\right\}\right\} - \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left\{(\tau' Y_t)\,\mathbb{I}\left\{\tau' Y_t > r\right\} + r\mathbb{I}\left\{\tau' Y_t \le r\right\}\right\}.$$

Since

$$\begin{aligned}&\left\{(\lambda' Y_t)\,\mathbb{I}\left\{\lambda' Y_t > r\right\} + r\mathbb{I}\left\{\lambda' Y_t \le r\right\}\right\}\\ &\quad = \max\left(\lambda' Y_t, r\right) = r - (r-\lambda' Y_t)_-,\end{aligned}$$

$m_2\left(z_2,\lambda,\tau,\sqrt{T}F_T\right)$ becomes

$$\sup_{\lambda\in\mathbb{L}} \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\max\left(\lambda' Y_t, r\right) - \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\max\left(\tau' Y_t, r\right)$$

Hence, we need to solve the optimization problem

$$\sup_{\lambda\in\mathbb{L}} \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\max\left(\lambda' Y_t, r\right).$$

We represent it by the following MIP program:

$$\begin{aligned}
\max_{\lambda\in\mathbb{L}} \quad & \frac{1}{\sqrt{T}}\sum_{t=1}^{T}(X_t - c b_t) & \\
\text{s.t.} \quad & X_t = \lambda' Y_t b_t + r(1-b_t) \quad \forall\, t\in T, & (28)\\
& r - \lambda' Y_t + M b_t > 0 \quad \forall\, t \in T, & (29)\\
& \lambda' \mathbf{1} = 1, & (30)\\
& \lambda \ge 0, & (31)\\
& b_t \in \{0,1\} \quad \forall\, t\in T. & (32)
\end{aligned}$$

where $\mathbf{1}, c, M$ as before.

In the above formulation X_t is a linearization of the $\max(\lambda' Y_t, r)$ function. The first two constraints, (28) and (29)) ensure that X_t takes the maximum value between $\lambda' Y_t$ and r. To get that to happen, we use a binary variable b_t which is equal to 1 if $\lambda' Y_t$ is higher than r, or 1 otherwise.

Due to the dependence of L_t on λ the smaller problems depend also on the binary variable. Hence each one of them is a Mixed Integer program. It usually takes significantly multiple time for the solution of each such a problem compared to the linear ones.

Finally, notice that in every case the practical implementation of any test using R bootstrap samples involves $2(R+1)$ internal numerical optimizations and $R+1$ trivial ones. Hence, the usual trade off between possibly desirable higher order properties and numerical burden is obviously present in our considerations.

Markowitz Negative Part

As previously mentioned, the numerical approximation of $\sup_{z\in\mathbb{R}_-}\sup_{\lambda\in\mathbb{L}} m_1\left(z_1,\lambda,\tau,\sqrt{T}F_T\right)$ can be also reduced to the solution of a finite number of linear programming problems via the use of an analogous formulation as above or via the results for the mathematical implementation of the SSD test in Scaillet and Topaloglou [40]. Given Lemma 1 the formulation used is the following:

$$
\begin{aligned}
\max_{z\geq 0,\lambda\in\mathbb{L}} \quad & \frac{1}{\sqrt{T}}\sum_{t=1}^{T}(L_t - W_t)\\
\text{s.t.} \quad & W_t \geq r - \lambda' Y_t, \qquad \forall t\in T_n\\
& L_t = (r-\tau' Y)_+, \qquad \forall t\in T_n\\
& \lambda'\mathbf{1} = 1,\\
& \lambda_i \geq 0, \quad \forall i,\\
& W_t \geq 0, \qquad \forall t\in T_n.
\end{aligned}
\tag{33}
$$

Auxiliary Lemmata

In what follows $\overset{p}{\rightsquigarrow}$ denotes (conditional) weak convergence in probability (see among others Paragraph 3.6.1 of van der Vaart and Wellner [44]).

Lemma 1. $\sup_{z\in\mathbb{R}_-} p_1\left(z,\lambda,\tau,\sqrt{T}F_T\right) = p_1\left(r,\lambda,\tau,\sqrt{T}F_T\right)$ *for some* $r\in Y_n$ *(resp.* $\sup_{z\in\mathbb{R}_-} m_1\left(z,\lambda,\tau,\sqrt{T}F_T\right) = m_1\left(r,\lambda,\tau,\sqrt{T}F_T\right)$ *for some* $r\in Y_n$*). Analogously,* $\sup_{z\in\mathbb{R}_{++}} p_2\left(z,\lambda,\tau,\sqrt{T}F_T\right) = p_2\left(r,\lambda,\tau,\sqrt{T}F_T\right)$ *for some* $r\in Y_p$
(resp. $\sup_{z\in\mathbb{R}_{++}} m_2\left(z,\lambda,\tau,\sqrt{T}F_T\right) = m_2\left(r,\lambda,\tau,\sqrt{T}F_T\right)$ *for some* $r\in Y_p$*).*

Proof. Consider $\sup_{z\in\mathbb{R}_{++}} p_2\left(z,\lambda,\tau,\sqrt{T}F_T\right)$ and assume that Y_p is increasingly ordered. If z such that $r_i\leq z\leq r_{i+1},\ i=1,\cdots T_p-1,\ T_p := |Y_p|,\ \sum_{t=1}^{T}(z-\tau' Y_t)_+$ is constant. Furthermore, the maximum value of $-\sum_{t=1}^{T}(z-\lambda' Y_t)_+$ is reached for $z=r_i$. Analogous considerations are easily obtained when $z<r_1$ or $z>r_{T_p}$. Hence, we can restrict z to belong to the set Y_p. All other cases are analogously obtained. □

Lemma 2. *Under Assumption 1 ,*

$$\begin{pmatrix} p_1\left(z_1,\lambda,\tau,\sqrt{T}\left(F_T-F\right)\right) \\ p_2\left(z_2,\lambda,\tau,\sqrt{T}\left(F_T-F\right)\right) \end{pmatrix} \rightsquigarrow \begin{pmatrix} p_1\left(z_1,\lambda,\tau,\mathcal{G}_F\right) \\ p_2\left(z_2,\lambda,\tau,\mathcal{G}_F\right) \end{pmatrix}$$

and

$$\begin{pmatrix} m_1\left(z_1,\lambda,\tau,\sqrt{T}\left(F_T-F\right)\right) \\ m_2\left(z_2,\lambda,\tau,\sqrt{T}\left(F_T-F\right)\right) \end{pmatrix} \rightsquigarrow \begin{pmatrix} m_1\left(z_1,\lambda,\tau,\mathcal{G}_F\right) \\ m_2\left(z_2,\lambda,\tau,\mathcal{G}_F\right) \end{pmatrix}$$

as random elements with values on the space of $\mathbb{R}^2$-valued bounded functions on $\mathbb{L}\times\mathbb{R}_-\times\mathbb{R}_{++}$ equiped with the sup-norm. The limiting processes have continuous sample paths.

Proof. Let $\theta := (\lambda, z_1, z_2) \in \Theta := \mathbb{L}\times\mathbb{R}_-\times\mathbb{R}_{++}$, ρ any non zero element of $\mathbb{R}^2$, and consider $P(\theta,\cdot) := \rho_1 p_1(z_1,\lambda,\tau,\cdot) + \rho_2 p_2(z_1,\lambda,\tau,\cdot)$. Notice that Theorem 7.3 of Rio [39], due to Assumption 1, implies that $\sqrt{T}(F_T-F) \rightsquigarrow \mathcal{G}_F$. This implies that $\sqrt{T}(F_T-F)$ also weakly hypo-converges to $\mathcal{G}_F$ (see for example Knight [21]). Both are upper semi-continuous (usc) $\mathbb{P}$ a.s. and the space of usc functions with the topology of epiconvergence can be metrized as complete and separable (see again Knight [21]). Due to separability and the Skorokhod Representation Theorem (see for example Theorem 1 in Cortissoz [5]) there exists a suitable probability space and random elements with values in the aforementioned function space such that $f_T^* \overset{d}{=} \sqrt{T}(F_T-F)$, $f^* \overset{d}{=} \mathcal{G}_F$, and $f_T^* \to f^*$ a.s.. Let $J := \overline{\text{span}}\{f_T^*, f^*, T=1,2,\cdots\}$ equipped with the metrizable topology of weak convergence.[11] Consider $P(\cdot,\cdot)$ restricted to J with values in the linear space of stochastic processes, equipped with the topology of convergence in distribution, with values in the space of bounded real functions defined on Θ equipped with the sup-norm. From Assumption 1, Corollary 4.1, and Theorem 7.3 of Rio [39] we also have that

$$\sup_{\theta\in\Theta}\sup_{T}\mathbb{E}\left[\left(P\left(\theta,\sqrt{T}\left(F_T-F\right)\right)\right)^2\right] + \sup_{\theta\in\Theta}\mathbb{E}\left[\left(P\left(\theta,\mathcal{G}_F\right)\right)^2\right] < +\infty.$$

The latter inequality along with Theorem 6.5.2 in Narici and Beckenstein [33], the metrization of convergence in distribution by the bounded Lipschitz metric (see for example p. 73, van der Vaart [43]) which is bounded from above by $\sup_\theta \mathbb{E}\left[(x-y)^2\right]$, for x, y members of the aforementioned space of processes, imply that $P(\cdot,\cdot)$ as restricted above is continuous. Hence the CMT implies that $P(\theta, f_T^*) \rightsquigarrow P(\theta, f^*)$ which means that $P\left(\theta,\sqrt{T}(F_T-F)\right) \rightsquigarrow P(\theta,\mathcal{G}_F)$. This and the Cramer-Wold Theorem imply the needed result. The final assertion follows from that $\sup_{\theta\in\Theta}\mathbb{E}\left[(P(\theta,\mathcal{G}_F))^2\right] < +\infty$, the discussion in Example 1.5.10 of van der Vaart and Wellner [44], and the continuity of $\mathbb{E}\left[(P(\theta,\mathcal{G}_F))^2\right]$ w.r.t. θ. The second result is completely analogous. □

[11] Here $\overline{\text{span}}$ denotes the closure w.r.t. the particular topology of the linear span.

Proofs of Main Results

Proof of Proposition 1. For the part concerning PSD-efficiency we have that: i. If $\tau \succcurlyeq_P \mathbb{L}$ then, from Definition 1 we have that for any λ, $\sup_{z\leq 0} p_1(z,\lambda,\tau,F) \leq 0$ and $\sup_{z>0} p_2(z,\lambda,\tau,F) \leq 0$. This implies that

$$\max_{i=1,2} \sup_{z\in A_i} p_i(z,\lambda,\tau,F) \leq 0,$$

which in turn implies that $p(\tau,F) \leq 0$. The required equality follows from that,

$$p(\tau,F) \geq p_2(z,\tau,\tau,F) = 0.$$

If $\tau \not\succcurlyeq_P \mathbb{L}$ then there exists some $\lambda^\star$, some i and subsequently some $z^\star \in A_i$, such that $p_i(z^\star,\lambda^\star,\tau,F) > 0$. This directly implies that $p(\tau,F) > 0$. ii. If $p(\tau,F) = 0$ then for any $\lambda \in \mathbb{L}$ we get that $\max_{i=1,2} \sup_{z\in A_i} p_i(z,\lambda,\tau,F) \leq 0$. Hence, $p_i(z,\lambda,\tau,F) \leq 0$, for every $z \in A_i$, $i = 1,2$. If $p(\tau,F) > 0$ then there exists some $\lambda^\star$, some i and subsequently some $z^\star \in A_i$, such that $p_i(z^\star,\lambda^\star,\tau,F) > 0$ which then implies that $\tau \not\succcurlyeq_P \mathbb{L}$. The part concerning MSD-efficiency follows analogously. □

Proof of Proposition 2. The results in 3 and 5 follow from Lemma 2 and the CMT. The results in Lemma 2 imply that $\begin{pmatrix} p_1\left(z_1,\lambda,\tau,\sqrt{T}(F_T - F)\right) \\ p_2\left(z_2,\lambda,\tau,\sqrt{T}(F_T - F)\right) \end{pmatrix}$ weakly converges to $\begin{pmatrix} p_1(z_1,\lambda,\tau,\mathcal{G}_F) \\ p_2(z_2,\lambda,\tau,\mathcal{G}_F) \end{pmatrix}$ w.r.t. to the product topology of hypo-convergence on the product of the relevant spaces of usc real valued functions (see e.g. Knight [21] for the dual notion of epi-convergence). This product space is metrizable as complete and separable (see again Knight [21]). Hence, Skorokhod representations are applicable (as above, see for example Theorem 1 in Cortissoz [5]) and thereby there exists an enhanced probability space and processes $\begin{pmatrix} P_{1,T}(\theta_1) \\ P_{2,T}(\theta_2) \end{pmatrix} \overset{d}{=} \begin{pmatrix} p_1\left(z_1,\lambda,\tau,\sqrt{T}(F_T - F)\right) \\ p_2\left(z_2,\lambda,\tau,\sqrt{T}(F_T - F)\right) \end{pmatrix}$, $\begin{pmatrix} P_1(\theta_1) \\ P_2(\theta_2) \end{pmatrix} \overset{d}{=} \begin{pmatrix} p_1(z_1,\lambda,\tau,\mathcal{G}_F) \\ p_2(z_2,\lambda,\tau,\mathcal{G}_F) \end{pmatrix}$, defined on it such that $\begin{pmatrix} P_{1,T} \\ P_{2,T} \end{pmatrix} \to \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$ almost surely, w.r.t. to the product topology of hypo-convergence, where $\overset{d}{=}$ denotes equality in distribution, and $\theta_i = (\lambda, z_i) \in \mathbb{L} \times A_i$, $i = 1,2$. Notice that

$$\begin{pmatrix} p_1\left(z_1,\lambda,\tau,\sqrt{T}F_T\right) \\ p_2\left(z_2,\lambda,\tau,\sqrt{T}F_T\right) \end{pmatrix} \overset{d}{=} K_T(\theta_1,\theta_2) := \begin{pmatrix} P_{1,T}(\theta_1) \\ P_{2,T}(\theta_2) \end{pmatrix} + \sqrt{T}\begin{pmatrix} p_1(z_1,\lambda,\tau,F) \\ p_2(z_2,\lambda,\tau,F) \end{pmatrix},$$

and that under $\mathbb{H}_0^{(P)}$, almost surely, for any $\theta_i = (\lambda, z_i) \in \mathbb{L} \times A_i$, $i = 1,2$ and any $\theta_{T,i} \to \theta_i$, due to that $\Gamma_i^{(P)}$ is closed (see Remark 2), for any $i = 1,2$,

$$\limsup_{T\to\infty} \left(P_{i,T}(\theta_{T,i}) + \sqrt{T} p_i(z_{T,i},\lambda_T,\tau,F)\right) \leq \begin{cases} P_i(\theta_i), & \theta_i \in \Gamma_i^{(P)}, \theta_{T,i} \in_{T^\star} \Gamma_i^{(P)} \\ P_i(\theta_i), & \theta_i \in \Gamma_i^{(P)}, \theta_{T,i} \notin \Gamma_i^{(P)} \\ -\infty, & \theta_i \notin \Gamma_i^{(P)}, \theta_{T,i} \notin \Gamma_i^{(P)} \end{cases},$$

where $\in_{T^\star}$ denotes "eventually belongs", and,

$$\liminf_{T\to\infty}\left(P_{i,T}\left(\theta_i\right)+\sqrt{T}p_i\left(z_i,\lambda,\tau,F\right)\right)=\begin{cases}P_i\left(\theta_i\right), & \theta_i\in\Gamma_i^{(P)},\\ -\infty, & \theta_i\notin\Gamma_i^{(P)},\end{cases},$$

hence, $\begin{pmatrix}P_{1,T}\left(\theta_1\right)\\ P_{2,T}\left(\theta_2\right)\end{pmatrix}+\sqrt{T}\begin{pmatrix}p_1\left(z_1,\lambda,\tau,F\right)\\ p_2\left(z_2,\lambda,\tau,F\right)\end{pmatrix}$ almost surely converges w.r.t. to the product topology of hypo-convergence to the limit $K\left(\theta_1,\theta_2\right)=\begin{pmatrix}K_1\left(\theta_1\right)\\ K_2\left(\theta_2\right)\end{pmatrix}$, with $K_i\left(\theta_i\right)=\begin{cases}P_i\left(\theta_i\right), & \theta_i\in\Gamma_i^{(P)}\\ -\infty, & \theta_i\notin\Gamma_i^{(P)}\end{cases}$, due to the (dual version of) Proposition 3.2 (ch. 5, p. 337) of Molchanov [32]. Furthermore, since almost surely $\sup_{T\in\mathbb{N},\theta_i\in\mathbb{L}\times A_i}P_{i,T}\left(\theta_{T,i}\right)<+\infty$, and due to the form of $\mathbb{H}_0^{(P)}$, we have that

$$\limsup_{T\to\infty}\sup_{\theta_i\in\mathbb{L}\times A_i}\left(P_{i,T}\left(\theta_i\right)+\sqrt{T}p_i\left(z_i,\lambda,\tau,F\right)\right)\leq\sup_{\theta_i\in\mathbb{L}\times A_i}P_i\left(\theta_i\right)$$

$$=\sup_{\theta_i\in\mathbb{L}\times A_i}K_i\left(\theta_i\right)=\sup_{\theta_i\in\Gamma_i^{(P)}}P_i\left(\theta_i\right),$$

and thereby due to (the dual version of) Theorem 3.4 (ch. 5, p. 338) of Molchanov [32], and the CMT

$$\max_i\sup_{\theta_i\in\mathbb{L}\times A_i}\left(P_{i,T}\left(\theta_i\right)+\sqrt{T}p_i\left(z_i,\lambda,\tau,F\right)\right)\to\max_i\sup_{\theta_i\in\Gamma_i^{(P)}}P_i\left(\theta_i\right),\ \text{almost surely},$$

and 4 follows. 6 follows analogously. For the third part, we have that for any $\lambda\in\mathbb{L}$ and any T,

$$\max_{i=1,2}\sup_{z\in A_i}m_i\left(z,\lambda,\tau,\sqrt{T}\left(\hat{F}_T-F\right)\right)\geq$$

$$\frac{1}{2}\int_{-\infty}^{0}\left[G\left(u,\tau,\sqrt{T}\left(\hat{F}_T-F\right)\right)-G\left(u,\tau,\sqrt{T}\left(\hat{F}_T-F\right)\right)\right]du$$

$$+\frac{1}{2}\int_{0}^{+\infty}\left[G\left(u,\tau,\sqrt{T}\left(\hat{F}_T-F\right)\right)-G\left(u,\tau,\sqrt{T}\left(\hat{F}_T-F\right)\right)\right]du$$

$$\geq\frac{1}{2}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left[\left(\left(\lambda'Y_t\right)_- -\mathbb{E}\left(\lambda'Y_0\right)_-\right)-\left(\left(\tau'Y_t\right)_- -\mathbb{E}\left(\tau'Y_0\right)_-\right)\right]\right)$$

$$+\frac{1}{2}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left[\left(\left(\lambda'Y_t\right)_+ -\mathbb{E}\left(\lambda'Y_0\right)_+\right)-\left(\left(\tau'Y_t\right)_- -\mathbb{E}\left(\tau'Y_0\right)_-\right)\right]\right)$$

$$=\frac{1}{2}\left(\lambda-\tau\right)'\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left[Y_t-\mathbb{E}\left(Y_0\right)\right]\rightsquigarrow\frac{1}{2}\left(\lambda-\tau\right)'Z,$$

where $Z\sim N\left(0_{n\times 1},\mathbb{V}\right)$, and the latter limiting argument follows from Assumption 1 and the CLT for strongly mixing stationary sequences (e.g. see Theorem 4.2 in Rio

[39]). Hence from the monotonicity of the supremum and the Portmanteau Theorem, we have that

$$\mathbb{P}\left(m_{\infty}^{\star} \geq 0\right) \geq \limsup_{T \to \infty} \mathbb{P}\left(m\left(\tau, \sqrt{T}\left(\hat{F}_T - F\right)\right) \geq 0\right)$$

$$\geq \limsup_{T \to \infty} \mathbb{P}\left(\frac{1}{2}\sup_{\lambda \in \mathbb{L}}\left((\lambda - \tau)' \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left[Y_t - \mathbb{E}\left(Y_0\right)\right]\right) \geq 0\right) = 1,$$

since $\tau \in \mathbb{L}$ and thereby the support of the law of $m_{\infty}^{\star}$ is $[0, +\infty)$, and then Corollary 4.4.2.(i)-(ii) of Bogachev [4] implies that the law restricted to $(0, +\infty)$ is absolutely continuous with a possible atom at zero. Furthermore, using 5 and the previous convergence we have that

$$\mathbb{P}\left(m_{\infty}^{\star} = 0\right) \leq \mathbb{P}\left(\frac{1}{2}\sup_{\lambda \in \mathbb{L}}\left((\lambda - \tau)' Z\right) = 0\right).$$

Due to the non-degeneracy of $\mathbb{V}$ the latter probability equals exactly the probability that the minimum of the random vector Z occurs at a coordinate that corresponds to the intersection of the set of the extreme points of $\mathbb{L}$ with $\{\tau\}$. If this is non empty, and by using Theorem 2 in chapter 3 (p. 37) of Sidak et al. [41] by (in their notation) letting p be the density of the n-variate standard normal distribution it is easy to see that $\mathbb{P}\left(\frac{1}{2}\sup_{\lambda \in \mathbb{L}}\left((\lambda - \tau)' Z\right) = 0\right) = \frac{1}{n^{\star}}$. Moreover, for any $\lambda \in \mathbb{L}$, any T, and any $z_p > 0$,

$$\max_{i=1,2}\sup_{z \in A_i} p_i\left(z, \lambda, \tau, \sqrt{T}\left(\hat{F}_T - F\right)\right) \geq$$

$$\frac{1}{2}\int_{-z_p}^{0}\left[G\left(u, \tau, \sqrt{T}\left(\hat{F}_T - F\right)\right) - G\left(u, \tau, \sqrt{T}\left(\hat{F}_T - F\right)\right)\right] du$$

$$+\frac{1}{2}\int_{0}^{z_p}\left[G\left(u, \tau, \sqrt{T}\left(\hat{F}_T - F\right)\right) - G\left(u, \tau, \sqrt{T}\left(\hat{F}_T - F\right)\right)\right] du,$$

and thereby letting $z_p \to \infty$, we obtain

$$\max_{i=1,2}\sup_{z \in A_i} p_i\left(z, \lambda, \tau, \sqrt{T}\left(\hat{F}_T - F\right)\right) \geq$$

$$\frac{1}{2}\int_{-\infty}^{0}\left[G\left(u, \tau, \sqrt{T}\left(\hat{F}_T - F\right)\right) - G\left(u, \tau, \sqrt{T}\left(\hat{F}_T - F\right)\right)\right] du$$

$$+\frac{1}{2}\int_{0}^{+\infty}\left[G\left(u, \tau, \sqrt{T}\left(\hat{F}_T - F\right)\right) - G\left(u, \tau, \sqrt{T}\left(\hat{F}_T - F\right)\right)\right] du,$$

and the result about $p_{\infty}^{\star}$ follows in exactly the same manner as the one about $m_{\infty}^{\star}$. □

Proof of Proposition 3. The third part of Proposition 2 implies that $c_P(\alpha)$ is well defined by $\mathbb{P}\left(p_{\infty}^{\star} > c_P(\alpha)\right) = \alpha$ and strictly positive if α satisfies the stated restrictions. It also implies the continuity of the quantile function of the law of $p_{\infty}^{\star}$ at $1 - \alpha$ when α satisfies the stated restrictions. Now, Assumptions 1, 3 and Theorem 2.3 of Peligrad [34] imply that conditionally on the sample,

$$\sqrt{T}\left(\hat{F}_T^{\star} - \hat{F}_T\right) \overset{p}{\rightsquigarrow} \mathcal{G}_F^{*}$$

where $\mathcal{G}_F^*$ is an independent version of the Gaussian process in Proposition 2. Analogously to the proofs of Lemmata 2 and 2,

$$\sup_{\lambda\in\mathbb{L}}\max_{i=1,2}\sup_{z\in A_i} p_i\left(z,\lambda,\tau,\sqrt{T}\left(\hat{F}_T^\star-\hat{F}_T\right)\right)\overset{p}{\rightsquigarrow}\sup_{\lambda\in\mathbb{L}}\max_{i=1,2}\sup_{z\in A_i} p_i\left(z,\lambda,\tau,\mathcal{G}_F^*\right).$$

Hence the $1-\alpha$ quantile of the law of $p_T^\star$ converges in probability to the $1-\alpha$ quantile of the law of $p_\infty^\star$. This along with that under $\mathbb{H}_0^{(P)}$, $p_T \rightsquigarrow p_\infty$, along with that $p_\infty \leq p_\infty^\star$ and the fact that $\mathbb{P}\left(\rho_{T,P}^\star<\alpha\right)$ equals the probability that p_T is greater than the $1-\alpha$ quantile of the law of $p_T^\star$, establish the asymptotic conservatism for the test based on decision rule (7).

If $\mathbb{H}_1^{(P)}$ holds, then for arbitrary $z_i^\star\in A_i,\ i=1,2$, and $\lambda^\star\in\mathbb{L}$, and due to the commutativity of the $\sup_{\lambda\in\mathbb{L}}$ and $\max_{i=1,2}$ operators,

$$p\left(\tau,\sqrt{T}\hat{F}_T\right)\geq\max_{i=1,2} p_i\left(z_i^\star,\lambda^\star,\tau,\sqrt{T}\left(\hat{F}_T-F\right)\right)+\sqrt{T}p\left(\tau,F\right). \tag{34}$$

For the rhs of 34 we have that due to Lemma 2 and the CMT

$$\max_{i=1,2} p_i\left(z_i^\star,\lambda^\star,\tau,\sqrt{T}\left(\hat{F}_T-F\right)\right)\rightsquigarrow\max_{i=1,2} p_i\left(z_i^\star,\lambda^\star,\tau,\mathcal{G}_F\right),$$

and thereby $p\left(\tau,\sqrt{T}\hat{F}_T\right)$ is asymptotically non-tight. Given the previous considerations on the asymptotic behavior of the $1-\alpha$ quantile of the law of $p_T^\star$ we obtain the result on consistency. The results about the test based on decision rule (8) follow analogously. □

Proof of Proposition 4. Define $\mathcal{F}_t=\sigma\left\{z_{t-1},z_{t-2},\ldots\right\}$ and notice that due to the definition of λ, the almost sure positivity of h_{i_t} for all i and Jensen's inequality,

$$\min\left\{h_{1_t},h_{2_t},h_{3t}\right\}\leq v_{\lambda_t}\leq\max\left\{h_{1_t},h_{2_t},h_{3t}\right\}\ \mathbb{P}\text{ a.s.},$$

where $v_{\lambda_t}:=\text{Var}\left(\lambda y_{1_t}+(1-\lambda)\,y_{2_t}/\mathcal{F}_t\right)$ or $v_{\lambda_t}:=h_{3t}$. Define the auxiliary processes by

$$\begin{aligned}h_{*_t}&=a_*\left(1+\left(z_{t-1}^2+1\right)h_{*_{t-1}}\right),\\ h_t^*&=a^*\left(1+\left(z_{t-1}^2+1\right)h_{t-1}^*\right),\end{aligned}$$

for $a_*=\min\left\{\omega_i,a_i,\beta_i,i=1,2,3\right\}$, $a^*=\max\left\{\omega_i,a_i,\beta_i,i=1,2,3\right\}$ and notice that

$$h_{*_t}\leq\min\left\{h_{1_t},h_{2_t},h_{3t}\right\}\leq\max\left\{h_{1_t},h_{2_t},h_{3t}\right\}\leq h_t^*,\ \mathbb{P}\text{ a.s.}$$

Hence, when $v_2^2a^*<a_*$ then $|v_2|\sqrt{h_{3_t}}<\sqrt{v_{\lambda_t}}$ $\mathbb{P}$ a.s. and when $v_1^2a_*>a^*$ then $|v_1|\sqrt{h_{3_t}}>\sqrt{v_{\lambda_t}}$, $\mathbb{P}$ a.s. Furthermore, the distribution function of $\lambda\neq\tau^\star$ equals $\mathbb{E}\Phi\left(\frac{x}{\sqrt{v_{\lambda_t}}}\right)$ due to the law of iterated expectations. In an analogous manner it is easy to see that the distribution function of $\tau^\star$ equals $\begin{cases}\mathbb{E}\left[\Phi\left(\frac{x}{|v_2|\sqrt{h_{3_t}}}\right)\right], x\leq 0\\ \mathbb{E}\left[\Phi\left(\frac{x}{|v_1|\sqrt{h_{3_t}}}\right)\right], x>0\end{cases}$. The monotonicity of the integral along with the relevant property of Φ imply that both distribution functions are strictly increasing. Hence for $z \leq 0$ we have that,

$$m_1(z, \lambda, \tau^\star, F) = \int_{-\infty}^{z} \left(\mathbb{E}\left[\Phi\left(\frac{x}{\sqrt{v_{\lambda_t}}} \right) \right] - \mathbb{E}\left[\Phi\left(\frac{x}{|v_2| \sqrt{h_{3_t}}} \right) \right] \right) dx > 0,$$

due to the previous and the fact that x assumes non positive values except for sets of Lebesgue measure zero. Analogously, for any $z > 0$,

$$m_2(z, \lambda, \tau^\star, F) = \int_{z}^{+\infty} \left(\mathbb{E}\left[\Phi\left(\frac{x}{\sqrt{v_{\lambda_t}}} \right) \right] - \mathbb{E}\left[\Phi\left(\frac{x}{|v_1| \sqrt{h_{3_t}}} \right) \right] \right) dx > 0,$$

which holds due to the fact that x assumes positive values. The result on MSD-efficiency follows. The same arguments show that when $z \leq 0$,

$$\int_{z}^{0} \left(\mathbb{E}\left[\Phi\left(\frac{x}{\sqrt{v_{\lambda_t}}} \right) \right] - \mathbb{E}\left[\Phi\left(\frac{x}{|v_2| \sqrt{h_{3_t}}} \right) \right] \right) dx > 0,$$

and when $z > 0$

$$\int_{0}^{z} \left(\mathbb{E}\left[\Phi\left(\frac{x}{\sqrt{v_{\lambda_t}}} \right) \right] - \mathbb{E}\left[\Phi\left(\frac{x}{|v_1| \sqrt{h_{3_t}}} \right) \right] \right) dx > 0,$$

establishing PSD-efficiency. □

Proof of Proposition 5. Let $\lambda = 1$ whence $h_{3_t} > v_{\mathbf{1}} = h_{1_t}$ $\mathbb{P}$ a.s. Using analogous arguments as before we have that for $z \leq 0$,

$$m_1(z, \lambda, \tau, F) = \int_{-\infty}^{z} \left(\mathbb{E}\left[\Phi\left(\frac{x}{\sqrt{h_{3_t}}} \right) \right] - \mathbb{E}\left[\Phi\left(\frac{x}{\sqrt{h_{1_t}}} \right) \right] \right) dx > 0,$$

which implies that the first part of Definition 2 is not valid. Analogously,

$$p_2(z, \lambda, \tau, F) = \int_{z}^{0} \left(\mathbb{E}\left[\Phi\left(\frac{x}{\sqrt{h_{3_t}}} \right) \right] - \mathbb{E}\left[\Phi\left(\frac{x}{\sqrt{h_{1_t}}} \right) \right] \right) dx > 0,$$

invalidating the first part of Definition 1. □

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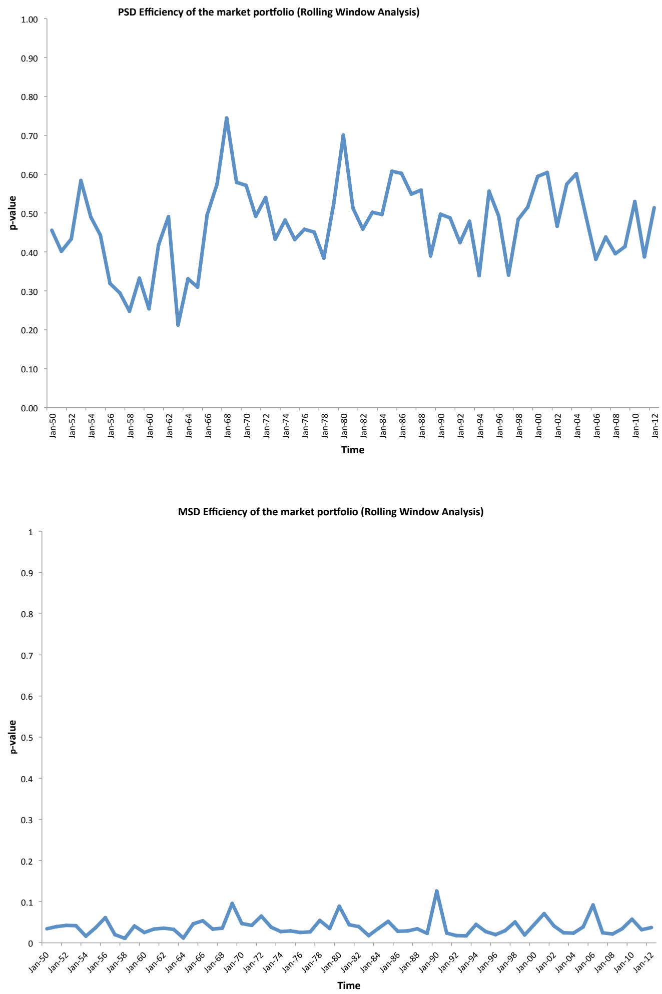

Fig. 1: **6 FF portfolios:** p-values for the prospect stochastic dominance efficiency test (upper graph) and for the Markowitz stochastic dominance efficiency test (lower graph) using a rolling window of 20 years. The test statistic is calculated separately for 63 overlapping 20-year periods, (January 1930-December 1949), (January 1931-December 1951),...,(January 1993-December 2012). The prospect stochastic dominance efficiency is not rejected in any subperiod, while the Markowitz stochastic dominance efficiency is rejected in 51 out of 63 subperiods.

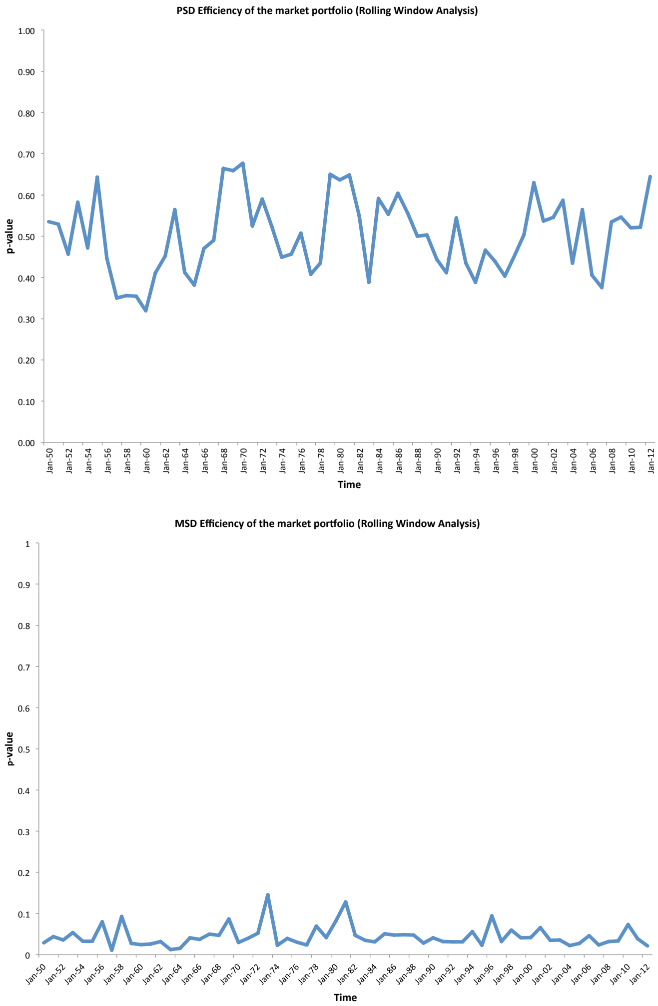

Fig. 2: **10 Momentum portfolios:** p-values for the prospect stochastic dominance efficiency test (upper graph) and for the Markowitz stochastic dominance efficiency test (lower graph) using a rolling window of 20 years. The test statistic is calculated separately for 63 overlapping 20-year periods, (January 1930-December 1949), (January 1931-December 1951),...,(January 1993-December 2012). The prospect stochastic dominance efficiency is not rejected in any subperiod, while the Markowitz stochastic dominance efficiency is rejected in 48 out of 63 subperiods.

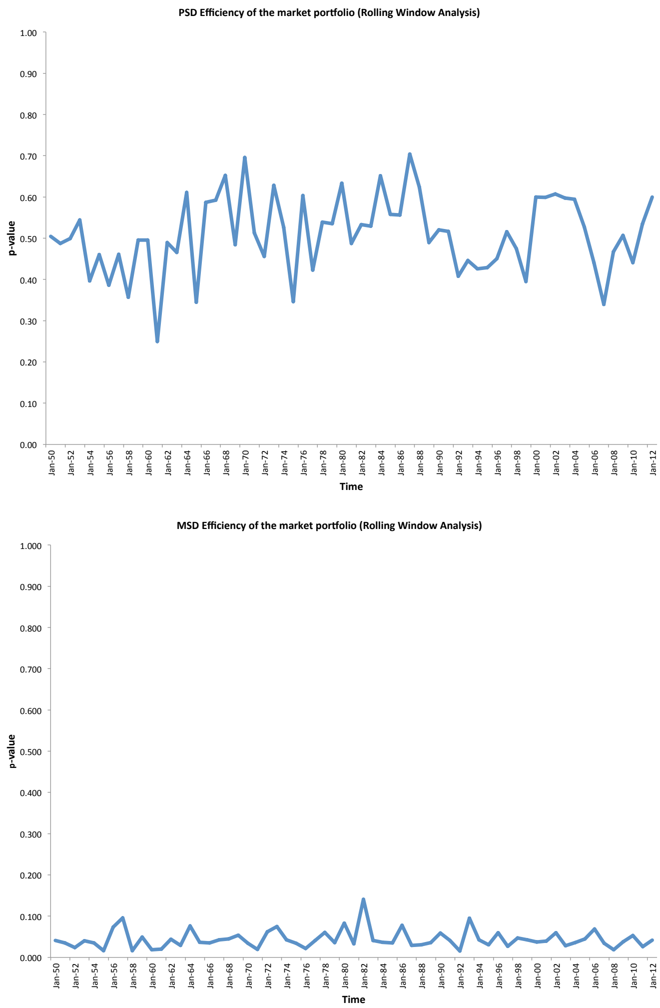

Fig. 3: **48 Industry portfolios:** p-values for the prospect stochastic dominance efficiency test (upper graph) and for the Markowitz stochastic dominance efficiency test (lower graph) using a rolling window of 20 years. The test statistic is calculated separately for 63 overlapping 20-year periods, (January 1930-December 1949), (January 1931-December 1951),...,(January 1993-December 2012). The prospect stochastic dominance efficiency is not rejected in any subperiod, while the Markowitz stochastic dominance efficiency is rejected in 49 out of 63 subperiods.