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Abstract

We establish the existence of a unique stationary and ergodic solution for systems of stochastic recurrence equations defined by stochastic self-maps on Polish metric spaces based on the fixed point theorem of Matkowski. The results can be useful in cases where the stochastic Lipschitz coefficients implied by the currently used method either do not exist, or lead to the imposition of unecessarily strong conditions for the derivation of the solution.

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1 Introduction

The present note concerns the establishment of the existence of a unique stationary and ergodic solution over the set of integral numbers for systems of stochastic recurrence equations defined by stochastic self-maps on Polish metric spaces and its representation as a limit of relevant Picard iterates. It is essentially based on the fixed point theorem of Matkowski (see Matkowski [4]) and thereby extends current results depending on the analogous use of the classical fixed point theorem of Banach (see Theorem 20 of Bougerol [1]) that are currently used heavily for the study of time series models defined by non-linear recursions (for a survey article see Diaconis and Freedman [3]). The results presented in the following section could be useful in cases where the stochastic Lipschitz coefficients implied by the currently used method either do not exist, or lead to the imposition of unecessarily strong conditions for the derivation of the solution.

2 Existence and Uniqueness of Stationary and Ergodic Solution to SRE's

In what follows $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, (E, d) is a Polish metric space, $\mathcal{B}_E$ its Borel σ-algebra, Θ is an arbitrary non empty set, $\Phi_{t,\theta} : \Omega \times E \to E$, $t \in \mathbb{Z}$, $\theta \in \Theta$ are $\mathcal{B}_E / \mathcal{F} \otimes \mathcal{B}_E$-measurable self maps on E, and $g_{t,\theta} : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ are $\mathcal{B}_{\mathbb{R}_+} / \mathcal{F} \otimes \mathcal{B}_{\mathbb{R}_+}$-measurable self maps on $\mathbb{R}_+$. The relevant supremum metric is denoted by d_Θ. $\overset{\text{eas}}{\to}$ denotes exponentially almost sure convergence (see paragraph 2.5 in Straumann [5]), $\to_{\mathbb{P}\text{a.s.}}$ denotes $\mathbb{P}$ a.s. convergence, and $\overset{\text{as}}{=}$ almost sure equality w.r.t. $\mathbb{P}$. $\mathbb{E}$ denotes integration w.r.t. $\mathbb{P}$. If $\{X_1, X_2, ...\}$ is a collection of random E-valued random elements defined on Ω, then $\sigma_{X_t} \equiv \sigma(X_1, X_2, ...)$ denotes the σ-algebra generated by that collection. Finally for $m \in \mathbb{N}$,

$$\Phi_{t,\theta}^{(m)} \equiv \begin{cases} \text{id}_E, m = 0 \\ \Phi_{t,\theta} \circ \Phi_{t-1,\theta} \circ ... \circ \Phi_{t-m+1,\theta}, m > 0 \end{cases}.$$

The following theorem establishes the existence of a unique, up to indistinguishability, stationary and ergodic solution to the stochastic recurrence system defined by $x_{t+1} = \Phi_{t,\theta}(x_t)$, its continuity properties w.r.t. θ, the form by which it approximates any other solution as well as the issue of its invertibility. In part, it is essentially based on the fixed point theorem of Matkowski (see Matkowski [4]) in the particular probabilistic setting of a stochastic flow defined by stochastic recurrences. As such it generalizes the analogous result used in the time series literature that is based on the Banach fixed point theorem (see Theorem 20 of Bougerol [1] or equivalently Theorem 2.6.1 of Straumann [5]).

Theorem 1. *Suppose that $(\Phi_{t,\theta})_{t\in\mathbb{Z}}$ is stationary and ergodic for any $\theta \in \Theta$. Furthermore:*

a. there exists a $y \in E$ such that,

$$\mathbb{E}\left[\log^+ d_\Theta\left(\Phi_{0,\theta}(y), y\right)\right] < +\infty, \ \mathbb{P} \textit{ almost surely,} \tag{1}$$

b. for any t and θ, $\mathbb{P}$ almost surely, for any $x, y \in E$,

$$d\left(\Phi_{t,\theta}(x), \Phi_{t,\theta}(y)\right) \leq g_{t,\theta}\left(d(x, y)\right), \tag{2}$$

and,

c. for any $t \in \mathbb{Z}$ and $\theta \in \Theta$, $g_{t,\theta}$ is $\mathbb{P}$ a.s. increasing, and for any $z \in \mathbb{R}_+$,

$$\left|g_{t,\theta}^{(m)}(z)\right|_\Theta \overset{eas}{\to} 0 \textit{ as } m \to \infty, \tag{3}$$

while for at least one $t \in \mathbb{Z}$ the convergence is locally uniform in $\mathbb{R}_+$.

Then the SRE defined by

$$x_{t+1} = \Phi_{t,\theta}(x_t), \tag{4}$$

admits a stationary and ergodic solution $(Y_{t,\theta})_{t\in\mathbb{Z}}$ *for any* $\theta \in \Theta$ *that has the representation*

$$Y_{t+1,\theta} \overset{as}{=} \lim_{m\to\infty} \Phi^{(m)}_{t,\theta}(y), \tag{5}$$

and the convergence is uniform w.r.t. θ*. If* $(Y^*_{t,\theta})_{t\in\mathbb{Z}}$ *denotes any other stationary solution then*

$$\mathbb{P}\left(d\left(Y_{t,\theta}, Y^*_{t,\theta}\right) = 0\right) = 1 \text{ for any } t \text{ and } \theta. \tag{6}$$

The random element $Y_{t+1,\theta}$ *is measurable w.r.t.* $\sigma(\Phi_{t,\theta}, \Phi_{t-1,\theta}, \ldots), \theta \in \Theta$*. If* $\Phi_{t,\theta} = \Phi_\theta(X_t)$ *for some stationary and ergodic* $(X_t)_{t\in\mathbb{Z}}$ *where* X_t *assumes values in* E*, and* $\Phi_\theta : E \to E$ *is* $\mathcal{B}_E/\mathcal{B}_E$*-measurable, then the random element* $Y_{t+1,\theta}$ *is measurable w.r.t.* σ_{X_t}*. If* Θ *is a compact topological space,* $\Phi_{t,\theta}(y)$ *is* $\mathbb{P}$ *a.s. continuous w.r.t.* θ*, then the random element* $Y_{t+1,\theta}$ *is* $\mathbb{P}$ *a.s. continuous w.r.t.* θ*. Finally, if* Θ *is Polish, and* $(Y^*_{t,\theta})_{t\in\mathbb{Z}}$ *denotes any solution, for which* $\mathbb{E}\left[\log^+ d_\Theta\left(y^*, Y^*_{t,\theta}\right)\right] < +\infty$ *and* $\mathbb{E}\left[\log^+ d_\Theta\left(y^*, Y_{t,\theta}\right)\right] < +\infty$ *for some* $t \in \mathbb{Z}$ *and* $y^* \in E$*, then*

$$d_\Theta\left(Y_{t,\theta}, Y^*_{t,\theta}\right) \overset{eas}{\to} 0 \text{ as } t \to \infty. \tag{7}$$

Proof. Fix $y \in E$. Suppose first that the $\mathbb{P}$ a.s. limit in (5) exists. Then from the continuity of the metric for any θ

$$d\left(\lim_{m\to\infty} \Phi^{(m)}_{t,\theta}(y), \Phi_{t,\theta}(Y_t)\right) = \lim_{m\to\infty} d\left(\Phi^{(m)}_{t,\theta}(y), \Phi_{t,\theta}(Y_t)\right), \ \mathbb{P} \text{ a.s.,}$$

and that due to (2), (4) and (5)

$$\begin{aligned} d\left(\Phi^{(m)}_{t,\theta}(y), \Phi_{t,\theta}(Y_t)\right) &\leq g_{t,\theta}\left(d\left(\Phi^{(m-1)}_{t-1,\theta}(y), Y_t\right)\right), \ \mathbb{P} \text{ a.s.,} \\ &\leq g_{t,\theta}\left(d\left(\Phi^{(m-1)}_{t-1,\theta}(y), \lim_{n\to\infty} \Phi^{(n)}_{t-1,\theta}(y)\right)\right), \ \mathbb{P} \text{ a.s.,} \\ &= \lim_{n\to\infty} g_{t,\theta}\left(d\left(\Phi^{(m-1)}_{t-1,\theta}(y), \Phi^{(n)}_{t-1,\theta}(y)\right)\right), \ \mathbb{P} \text{ a.s.,} \end{aligned}$$

and analogously

$$\begin{aligned} g_{t,\theta}\left(d\left(\Phi^{(m-1)}_{t-1,\theta}(y), \lim_{n\to\infty} \Phi^{(m)}_{t-1,\theta}(y)\right)\right) &\leq g^{(2)}_{t,\theta}\left(d\left(\Phi^{(m-2)}_{t-2,\theta}(y), \Phi^{(m-1)}_{t-2,\theta}(y)\right)\right), \ \mathbb{P} \text{ a.s.,} \\ &\leq g^{(m)}_{t,\theta}\left(d\left(\Phi^{(0)}_{t-m,\theta}(y), \Phi_{t-m,\theta}(y)\right)\right), \ \mathbb{P} \text{ a.s.,} \\ &= g^{(m)}_{t,\theta}\left(d\left(y, \Phi_{t-m,\theta}(y)\right)\right), \ \mathbb{P} \text{ a.s.} \end{aligned}$$

This along with (3) implies that $d\left(Y_{t+1,\theta}, \Phi_{t,\theta}(Y_t)\right) = 0$, $\mathbb{P}$ a.s., which implies that the process $(Y_{t,\theta})_{t\in\mathbb{Z}}$ is a solution to (4). Furthermore if the limit exists then the stationarity, ergodicity and the measurability w.r.t. $\sigma(\Phi_{t,\theta}, \Phi_{t-1,\theta}, \ldots)$ of $Y_{t,\theta}$ follows from Corollary

2.1.3. of Straumann [5] while measurability w.r.t. σ_{X_t} follows trivially. Due to completeness the proof of the existence of the limit, reduces to the proof that $\left(\Phi_{t,\theta}^{(m)}(y)\right)_{m\in\mathbb{N}}$ is a Cauchy sequence for any t,θ. Using the same reasoning as before we have that

$$\begin{aligned}\lim_{m\to\infty} d\left(\Phi_{t,\theta}^{(m+1)}(y),\Phi_{t,\theta}^{(m)}(y)\right) &\leq \lim_{m\to\infty} g_{t,\theta}\left(d\left(\Phi_{t-1,\theta}^{(m)}(y),\Phi_{t,\theta}^{(m-1)}(y)\right)\right),\ \mathbb{P}\text{ a.s.,}\\ &\leq \lim_{m\to\infty} g_{t,\theta}^{(m)}\left(d\left(\Phi_{t-m,\theta}(y),y\right)\right),\ \mathbb{P}\text{ a.s.,}\end{aligned}$$

and the latter is due to monotonicity $\mathbb{P}$ a.s. less than or equal

$$\lim_{m\to\infty} g_{t,\theta}^{(m)}\left(d_\Theta\left(\Phi_{t-m,\theta}(y),y\right)\right).$$

Stationarity and (1) imply that $d_\Theta\left(\Phi_{t-m,\theta}(y),y\right) < +\infty$ $\mathbb{P}$ a.s. and then (3) implies that the last limit is zero $\mathbb{P}$ a.s. Hence the limit exists. For the uniqueness up to indistinguishability result in (6) suppose again without loss of generality that the locally uniform version of (3) holds for $t=0$. Then

$$d\left(Y_{1,\theta},Y_{1,\theta}^*\right) \leq \lim_{m\to\infty} g_{0,\theta}^{(m)}\left(d\left(\Phi_{-m,\theta}(y),Y_1^*\right)\right),\ \mathbb{P}\text{ a.s.}$$

The existence of the limit in (5) along with the locally uniform nature of (3) imply that the left handside is $\mathbb{P}$ almost surely zero which then implies that $\mathbb{P}\left(d\left(Y_{1,\theta},Y_{1,\theta}^*\right)=0\right)=1$. Stationarity and measurability of d imply (6). The uniformity over Θ in (5) and the compactness of Θ imply the continuity result. Now, let Θ be Polish and $\left(Y_{t,\theta}^*\right)_{t\in\mathbb{Z}}$ denote any other solution of (4). Suppose without loss of generality that the log-moment conditions described in the additional prerequisites of (7) are valid for $t=0$. In a completely analogous manner to the previous we obtain that

$$\lim_{t\to\infty} d_\Theta\left(Y_{t+1,\theta},Y_{t+1,\theta}^*\right) \leq \lim_{t\to\infty}\left|g_{t,\theta}^{(t)}\right|_\Theta\left(d_\Theta\left(Y_{0,\theta},Y_{0,\theta}^*\right)\right),\ \mathbb{P}\text{ a.s.}$$

Due to the monotonicity of $g_{t,\theta}$ this implies that

$$\lim_{t\to\infty} d_\Theta\left(Y_{t+1,\theta},Y_{t+1,\theta}^*\right) \leq \lim_{t\to\infty}\left|g_{t,\theta}^{(t)}\right|_\Theta\left(d_\Theta\left(y^*,Y_{0,\theta}^*\right)+d_\Theta\left(Y_{0,\theta},y^*\right)\right),\ \mathbb{P}\text{ a.s.,}$$

and the log-moment conditions along with (3) imply (7). □

The $g_{t,\theta}$ is essentially a random $\mathbb{P}$ a.s. comparison (or Matkowski) function and $\Phi_{t,\theta}$ is analogously a random $\mathbb{P}$ a.s. Matkowski contraction (see for example Cădariu and Radu [2]), and the unique, in the sense of idistinguishibility, stationary and ergodic solution is characterized as a $\mathbb{P}$ a.s. limit of Picard iterations. This sollution is adapted to the $\left(\sigma\left(\Phi_{t,\theta},\Phi_{t-1,\theta},\ldots\right)\right)_{t\in\mathbb{Z}}$ filtration, and subsequently adapted to the "richer" $\left(\sigma\left(\Phi_{t,\theta},\Phi_{t-1,\theta},\ldots,\theta\in\Theta\right)\right)_{t\in\mathbb{Z}}$ filtration. When the recursion is constructed by the $(X_t)_{t\in\mathbb{Z}}$ process then the aforementioned solution is *invertible*, i.e. adapted to the $\left(\sigma_{X_t}\right)_{t\in\mathbb{Z}}$ filtration. The continuity property w.r.t. θ of the solution characterization result in (5) can be equivalently described as follows.

Corollary 1. *Suppose that* Θ *is a compact topological space. Then for any* $\theta, \theta_n \in \Theta$*, such that* $\theta_n \to \theta$ *as* $n \to \infty$*, the unique stationary and ergodic solution characterized in (5) of the recursion defined by* $x_{t+1} = \Phi_{t,\theta_n}(x_t)$*,* $\mathbb{P}$ *a.s. converges to the analogous solution of the recursion defined by* $x_{t+1} = \Phi_{t,\theta}(x_t)$*.*

Finally, when Θ is a Polish space, then we obtain an even stronger version of the uniqueness property, in the sense that any *other* solution of the recursion, converges exponentially fast to the aforementioned one.

As mentioned above, the previous theorem admits as a particular case the standard Banach type argument in which $g_{t,\theta}(z) = \Lambda_{t,\theta} z$ and $\mathbb{E}\left[\sup_{\Theta} \ln \Lambda_{t,\theta}\right] < 0$. As such it can be used to obtain weaker sufficient conditions in cases where the aforementioned result yields Lipschitz coefficients that are inadequate or with properties that imply strong restrictions.

For an easy example suppose that $E = \mathbb{R}_+$, $\Theta = \mathbb{R}_{++}$, $d(x,y) = |x-y|$, and $\Phi_{t,\theta}(x) = \frac{\theta X_{t-1} x}{1+\theta X_{t-1} x}$, where $(X_t)_{t \in \mathbb{Z}}$ is a stationary and ergodic sequence of $\mathbb{R}_+$-valued random variables. Then all the assertions of Theorem (1) hold with $g_{t,\theta}(r) = \frac{\theta X_{t-1} r}{1+\theta X_{t-1} r}$, $r \in \mathbb{R}_+$.

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