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Non-Emptyness of Stochastic Dominance Efficient Sets via Stochastic Spanning

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Abstract

We derive sufficient conditions for the non-emptyness of the efficient set for a class of Stochastic Dominance Relations, commonly applied in Economics and Finance, over sets of distributions on the real line. We do so via the use of the concept of stochastic spanning and its characterization via a saddle type property. Under the appropriate framework sufficiency takes the form of semi-continuity of some related functional. In some cases this boils down to mild uniform moment existence conditions.

JEL subject codes: C02, D81, D31, G11.

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1 Introduction

Stochastic dominance relations (hereafter SDRs) are (pre-) orders on sets of Borel probability distributions on the real line. Their study has gained importance in the fields of economics, finance and statistics/econometrics (see inter alia Kroll and Levy [4], McFadden [8], Levy [5, 6], Mosler and Scarsini [10]), since among others it enables inference on issues regarding optimal choice under uncertainty, without parametric specification for preferences.

Usually, SDRs are defined by complicated functional inequalities, that have also characterizations in terms of classes of utility functions (see Levy [5, 6], Levy and Levy [7]-hereafter L&L). This implies that when a distribution dominates another w.r.t. such a relation, it is simultaneously prefered by any utility in the relevant class and vice versa. In this respect, order characteristics of the relation can be connected to properties of optimal choices.

One such property is efficiency. A distribution is efficient w.r.t. an SDR when it is a maximal element of the order. When a utility class characterization holds this is equivalent to that it is prefered by some utility in the class. Thereby, the efficient set of a suchlike SDR is essentially the set of optimal choices of associated utilities, and its mathematical structure has obvious importance for Decision Theory. Despite its importance, to the best of our knowledge, there exists no general result in the literature that provides with sufficient conditions even for the non-emptyness of the efficient set at least for the SDRs that are commonly used in the related applications. This is in stark contrast to the exhaustive study of the efficient sets in the Mean-Variance setting in finance (see Merton [9]), and demostrates the inherent complexities of such orderings.

Thus, the purpose of the current note is to develop easily applicable sufficient conditions for the non-emptyness of the efficient set for as many ”commonly applied” SDRs as possible. We derive those by utilizing another order characteristic of an SDR, that of stochastic spanning. This is a brilliant idea of Thierry Post, influenced by the notion of Mean-Variance spanning in Huberman and Kandell [3], that was formulated in the context of second order stochastic dominance in Arvanitis, Hallam, Post and Topaloglou [1] (hereafter AHPT), but is easily extendable to arbitrary SDRs. Vaguely, a subset of the original set of distributions is spanning, when optimal choices are not lost, when choices are restricted from the latter to the former. It is not difficult to see that modulo equivalences, the efficient set is a minimal spanning set. Hence, our strategy is to develop analyticaly tractable conditions that ensure that the collection of spanning sets has minimal elements. We do so via, among others, the analytical characterization of spanning via a ”saddle type” property of an appropriate functional.

We present our framework and derive the results in the following section. In the final one, we discuss issues for further research.

2 Framework and Result

Consider a set of Borel probability measures on $\mathbb{R}$, say $\mathcal{P} := \{\mathbb{P}_\lambda, \lambda \in \Lambda\}$, parameterized by $\lambda \in \Lambda$ a non empty compact subset of $\mathbb{R}^m$. We assume that the mapping $\lambda \to \mathbb{P}_\lambda$ is a continuous bijection w.r.t. the weak topology on $\mathcal{P}$. Hence $\mathcal{P}$ is weakly compact. This is a quite general formulation as it encompasses several frameworks encountered in applications.

For an example, suppose that $\mathbb{P}$ is a probability measure on $\mathbb{R}^n$, and $f : \Lambda \times \mathbb{R}^n \to \mathbb{R}$ continuous. $\mathbb{P}_\lambda$ is represented by $\int_{\mathbb{R}^n} \mathbb{I}\{f(\lambda, u)\} \leq \cdot\} d\mathbb{P}$, i.e. the cdf of the measurable transformation $\mathbb{R}^n \ni x \to f(\lambda, x)$. If for any $\lambda,\ z$, $\int_{\mathbb{R}^n} \mathbb{I}\{f(\lambda, u)\} = z\} d\mathbb{P} = 0$, then the aforementioned continuity holds by dominated convergence. The latter framework is usual in the context of financial economics and econometrics. Specifically, there $m = n$, Λ is a subset of the standard simplex of $\mathbb{R}^n$, and $f(\lambda, x) = \lambda' x$ which represents a linear portfolio constructed on n base assets with random returns that assume values inside $\mathbb{R}^n$ (see Post and Levy [12], Scaillet and Topaloglou [13]-hereafter ST). Another example involves income distributions, whence Λ is simply a finite index set and the rest follow easily (see McFadden [8], Horvath, Kokoszka, and Zitikis [2]).

Hereafter λ, κ, potentially super-(sub-)scripted, denote generic elements of Λ. The previous allow us to identify $\mathbb{P}_\lambda$ with its index λ. K denotes a non empty closed subset of Λ. $\mathcal{J}$ is a finite partition of $\mathbb{R}$ by intervals such that every $I \in \mathcal{J}$ is equipped with an orientation $o_I \in \{-1,1\}$. When $z \in I$ then $I_z := \begin{cases} \{x \in I, x \leq z\} & \text{if } o_I = 1 \\ \{x \in I, x \geq z\} & \text{if } o_I = -1 \end{cases}$. For $s \in \mathbb{N}^\star$, $I \in \mathcal{J}$ and $z \in I$, define recursively

$$\mathcal{J}_I(z,\lambda,s) \equiv \begin{cases} \int_{I_z} \mathcal{J}(I,u,\lambda,s-1)\,du, & s>1 \\ \mathbb{P}_\lambda((-\infty,z]), & s=1 \end{cases}. \quad (1)$$

Using integration by parts it is easy to establish that when $s=1$, or $s>1$ and $o_I=1$, $\mathcal{J}_I$ is well defined if $\mathbb{E}_\lambda\left(|X|^{s-1}\mathbb{I}\{X \in I_z\}\right) < \infty$ for some $z \in I$, where $\mathbb{E}_\lambda$ denotes expectation w.r.t. $\mathbb{P}_\lambda$. When $s>1$ and $o_I=-1$ this holds when I is bounded.

SDRs on $\mathcal{P}$ are usually defined via functional inequalities. A quite general formulation is the following.

Definition 1. For $\kappa, \lambda \in \Lambda$, κ *weakly SD*$(\mathcal{J},s)$*-dominates* λ, denoted by $\kappa \succcurlyeq_{SD} \lambda$, iff $\forall I \in \mathcal{J}, \forall z \in I$

$$\Delta_I(z,\kappa,\lambda,s) \equiv \mathcal{J}_I(z,\kappa,s) - \mathcal{J}_I(z,\lambda,s) \leq 0. \quad (2)$$

Strict *SD*$(\mathcal{J},s)$-dominance, $\kappa \succ_{SD} \lambda$, occurs when in addition to (2), $\exists I \in \mathcal{J}$, $\exists z \in I$ such that $\Delta_I(z,\kappa,\lambda,s) < 0$. *SD*$(\mathcal{J},s)$-equivalence, $\kappa \sim_{SD} \lambda$, occurs iff $\kappa \succcurlyeq_{SD} \lambda$ and $\lambda \succcurlyeq_{SD} \kappa$.

This encompasses several SDRs that appear in the literature. For example when $\mathcal{J} = \{\mathbb{R}\}$, $o_{\mathbb{R}} = 1$, then *SD*$(\mathcal{J},s)$ is the s^{th}- order SDR (see ST [13]). When $s=1$, $\mathcal{J} = \{\mathbb{R}^-, \mathbb{R}^{++}\}$ and $o_{\mathbb{R}^-} - 1$, $o_{\mathbb{R}^{++}} = 1$, then *SD*$(\mathcal{J},s)$ is the Prospect SD, while when $o_{\mathbb{R}^-} = 1$, $o_{\mathbb{R}^{++}} - 1$, we obtain the Markowitz SDR (see L&L [5]). We note here that the aforementioned SDRs can be characterized via optimal choices w.r.t. classes of utilities. E.g. $\kappa \succcurlyeq_{SD} \lambda$ w.r.t. the 2^{nd}- order SDR is equivalent to that κ is weakly prefered to λ by every non-satiable and risk averse utility (see L&L [5]).

Given an SDR several order characteristics are of importance. An example is the efficient set comprised by elements that are not dominated by any other member of Λ, i.e. they are *SD*$(\mathcal{J},s)$*-maximal*.

Definition 2. $\kappa \in \Lambda$ is an *SD*$(\mathcal{J},s)$*-efficient element of* Λ *iff* $\forall \lambda \in \Lambda$,

$$\exists I \in \mathcal{J}\ \exists z \in I: \ \Delta_I(z,\kappa,\lambda,s) < 0, \text{ or } \kappa \sim_{SD} \lambda.$$

The set of the *SD*$(\mathcal{J},s)$*-efficient elements of* Λ *is denoted by* $\mathcal{E}_{SD}(\Lambda)$.

Under utility class interpretation, efficiency is global preference by *some* utility function in the class when Λ is convex (see Post [11]). Hence $\mathcal{E}_{SD}(\Lambda)$ is the set of optimal choices. The issue of its non-emptyness may be non trivial when supports are not compact, even when the class contains appropriately continuous utilities. In order to obtain sufficient conditions for it, we exploit the order theoretic notion of spanning.

Definition 3. K *SD*$(\mathcal{J},s)$*-spans* Λ (say $K \succcurlyeq_{SD} \Lambda$) iff $\forall \lambda \in \Lambda$, $\exists \kappa \in K: \ \kappa \succcurlyeq_{SD} \lambda$.

Under a utility class interpretation, K is spanning iff the excision of $\Lambda - K$ from the choice set Λ does not affect optimality for all utilities in the class. If moreover a spanning set is singleton then it represents an identical optimal choice for *every* utility in the class, which is thereby an *SD*$(\mathcal{J}, s)$-*greatest* element. Statistical inference for greatest elements is the subject of a strand of the analogous literature that generalizes the traditional pairwise comparisons (see ST [13]). As is evident from the previous, greatest elements are *generically* difficult to exist.

Obviously $\Lambda \succcurlyeq_{SD} \Lambda$. Hence (closed) spanning sets always exist. Consider $R_{SD}(\Lambda) = \{K \subseteq \Lambda, K \succcurlyeq_{SD} \Lambda, K \text{ closed}\}$ and equip it with the $\subseteq$-partial order. We thus obtain a useful but not easily verifiable condition for non-emptyness.

Lemma 1. If $(R_{SD}(\Lambda), \subseteq)$ has a minimal element then $\mathcal{E}_{SD}(\Lambda) \neq \emptyset$.

Proof. Let $K^\star$ be minimal, and suppose that $\mathcal{E}_{SD}(\Lambda) = \emptyset$. The latter implies that for any $\kappa \in K^\star$, $\exists \lambda_\kappa \in \Lambda$ such that $\lambda_\kappa \succ_{SD} \kappa$. Since $K^\star$ is spanning, then $\lambda_\kappa \in K$. Then $K^\star - \{\kappa\}$ is also spanning which is impossible due to minimality. □

We combine the result above with a saddle type characterization for spanning to obtain tractable analytical conditions. Suchlike results can also be useful for the design of inferential procedures for spanning via empirical analogues (see AHPT [1]). Its proof is obvious and thereby omitted.

Lemma 2. $K \succcurlyeq_{SD} \Lambda$ iff $\sup_{\lambda\in\Lambda} \inf_{\kappa\in K} \rho(\kappa, \lambda) \leq 0$, where

$$\rho(\kappa, \lambda) := \max_{I\in\mathcal{J}} \sup_{z\in I} \Delta_I(z, \kappa, \lambda, s).$$

Our main result then links non-emptyness to semi-continuity for ρ via Zorn's Lemma and the finite intersection property (fip).

Theorem 1. *If for any* $\lambda \in \Lambda$, $\rho(\lambda, \cdot) : \Lambda \to \mathbb{R}$ *is lower semi-continuous (lsc), then* $\mathcal{E}_{SD}(\Lambda) \neq \emptyset$.

Proof. Due to Lemma (1) the result would follow if $(R_{SD}(\Lambda), \subseteq)$ has a minimal element. By Zorn's Lemma such a minimal element would exist if any chain inside $(R_{SD}(\Lambda), \subseteq)$ has a lower bound that is also a spanning set. Consider an arbitrary chain $\{K_s, s \in S\}$ where S is an appropriate ordered index set, as well as $\cap_{s\in S} K_s$, which is obviously a lower bound for the chain. We will prove that $\cap_{s\in S} K_s \succcurlyeq_{SD} \Lambda$. Consider an arbitrary $\lambda \in \Lambda$ as well as an arbitrary element of the chain, say $K_{s^\star}$ and the set $K^\star_{s^\star} = \{\kappa \in K_{s^\star} : \kappa \succcurlyeq_{SD} \lambda\}$. This is non empty since $K_{s^\star}$ is a spanning set. If $(\kappa_m)_{m\in\mathbb{N}}$ is any convergent sequence in $K^\star_{s^\star}$ then its limit lies also inside $K^\star_{s^\star}$ since $\rho(\lambda, \cdot)$ is lsc, by Lemma (2). Hence $K^\star_{s^\star}$ is non-empty compact. Furthermore $K^\star_{s'} \subseteq K^\star_{s^\star}$ when $s^\star \leq s'$. Hence the monotone collection of compact subsets of Λ, $\{K^\star_s, s \in S\}$ has the fip, i.e. $\cap_{s\in S} K^\star_s \neq \emptyset$, which implies that $\cap_{s\in S} K_s$ has an element that weakly *SD*$(\mathcal{J}, s)$-dominates λ, hence it is spanning. □

Notice that for any $\lambda, \kappa \in \Lambda$ and any sequence $\Lambda \ni \kappa_m \to \kappa$, we have that

$$\lim\inf_{m\to\infty} \rho(\lambda, \kappa_m) \geq \max_{I\in\mathcal{J}} \sup_{z\in I} \lim\inf_{m\to\infty} \Delta_I(z, \kappa_m, \lambda, s). \quad (3)$$

Thereby, a sufficient condition for (lsc) is that $\Delta_I(z, \cdot, \lambda, s)$ is lsc for *some* $z \in I$ and some $I \in \mathcal{J}$. When $s = 1$, or $s > 1$ and $o_I = 1$ for all $I \in \mathcal{J}$, by continuity of $\lambda \to \mathbb{P}_\lambda$, (3) holds if for some $\delta > 0$, $\sup_{\kappa\in\Lambda} \mathbb{E}_\kappa \left(|X|^{s-1+\delta} \mathbb{I}\{X \in I_z\}\right) < \infty$ for some $z \in I, I \in \mathcal{J}$, due to uniform integrability. When $s > 1$ and additionally to the previous $o_I = -1$ for some I it holds if moreover this is bounded.

Such conditions are easily met when $\cup_{\lambda\in\Lambda}$supp $(\mathbb{P}_\lambda)$ is bounded. This is the case for every application concerning the usual 2^{nd} order, Prospect, or Markowitz SDR for income distributions, or in financial applications involving base assets with compact supports. When Λ is convex, this agrees to that there exist proper upper semi-continuous utilities in the relevant characterizing class which hence admit optimal choices. For those SDRs and non-bounded supports it is sufficient that $\sup_{\kappa\in\Lambda} \mathbb{E}_\kappa \left(|X|^{1+\delta}\right) < \infty$ which is mild for many applications.

3 Conclusions

We have essentially shown that under the appropriate framework, mild uniform moment existence conditions are sufficient for the non-emptyness of the efficient set for some commonly applied SDRs. We have done so via the use of the order theoretic notion of spanning. The study of the efficient sets is obviously far from complete. In constrast to the Mean-Variance framework, efficient sets for suchlike SDRs are possibly more complex entities.

Given their importance for decision theory, and since a spanning set can be perceived as an ”outer approximation” via Lemma (1), it could be possible that properties of efficient sets can be approximated by properties of sequences of spanning sets that *appropriately converge to them*. If this is true then the concept of spanning can also play a crucial role in the further study of optimal choices in such complex environments. We obviously leave such considerations for future research.

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