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Abstract

We develop non-parametric tests for prospect stochastic dominance Efficiency (PSDE) and Markowitz stochastic dominance efficiency (MSDE) with rejection regions determined by block bootstrap resampling techniques. Under the appropriate conditions we show that they are asymptotically conservative and consistent. We engage into Monte Carlo experiments to assess the finite sample size and power of the tests allowing for the presence of numerical errors. We use them to empirically analyze investor preferences and beliefs by testing whether the value-weighted market portfolio can be considered as efficient according to prospect and Markowitz stochastic dominance criteria when confronted to diversification principles made of risky assets. Our results indicate that we cannot reject the hypothesis of prospect stochastic dominance efficiency for the market portfolio. This is supportive of the claim that the particular portfolio can be rationalized as the optimal choice for any S-shaped utility function. Instead, we reject the hypothesis for Markowitz stochastic dominance, which could imply that there exist reverse S-shaped utility functions that do not rationalize the market portfolio.

Key words and phrases: Non parametric test, prospect stochastic dominance efficiency, Markowitz stochastic dominance efficiency, simplical complex, extremal point, Linear Programming, Mixed Integer Programming, Block Bootstrap, Consistency.

JEL Classification: C12, C13, C15, C44, D81, G11.

1 Introduction

In classical decision models, the utility of wealth is everywhere increasing and concave and hence investors are considered non-satiable and globally risk averse. However, there is empirical as well as experimental evidence that decision-makers are not always globally risk averse, but instead they seem to exhibit local risk-seeking behaviour (i.e., the utility function has convex segments). The aim of this paper is to develop statistical procedures for testing for efficiency under some cases of locally convex utility schemes (convex-concave, or concave-convex). Using these tests, we analyze investor preferences and beliefs by testing whether investors are risk seeking for losses and risk averse for gains, or the opposite, thus analyzing how investors behave in bull vs bear markets.

Traditionally, in the context of portfolio theory, investors are assumed to act as non satiable and risk averse agents and thus their preferences are represented by increasing and globally concave utility functions. For this reason, most of the criteria used to verify the efficiency of a given portfolio (see, among others, Gibbons, Ross, and Shenken (1989)) are based on the first and second stochastic dominance rules, see e.g. the papers by Kroll and Levy (1980) and Levy (1992), and the excellent monograph on the theory of stochastic dominance by Levy (2006). Recently, Gonzalo and Olmo (2014) propose nonparametric consistent tests of conditional stochastic dominance of arbitrary order in a dynamic setting.

A portfolio is first order stochastic dominance efficient if it is optimal w.r.t. any increasing utility function. Thus, this is a notion of efficiency w.r.t. non satiation. A portfolio is second order stochastic dominance efficient if it is optimal w.r.t. any increasing and globally concave utility function. This is a notion of efficiency w.r.t. non satiation and global risk aversion.

Given non satiation, empirical evidence suggests that investors do not always act as risk-averters. Instead, under certain circumstances they behave in a much more complex fashion exhibiting characteristics of both risk-loving and risk-hating. Furthermore, they seem to evaluate wealth changes of assets w.r.t. to benchmark cases, rather than final wealth positions. They behave differently on gains and losses, and one can say that they are more sensitive to losses than to gains (loss aversion). In addition, there are cases where the relevant utility (or value) function could be either concave for gains and convex for losses or convex for gains and concave for losses. Moreover, they seem to transform the objective probability measures to subjective ones using transformations that potentially increase the probabilities of negligible (and possibly averted) events, and which in some cases share similar analytical characteristics to the aforementioned utility functions. Examples of risk orderings that (partially) reflect such findings are the dominance rules of behavioral finance (see Friedman and Savage (1948), Baucells and Heukamp (2006), Edwards (1996), and the references therein).

A seminal instance of an analysis that incorporates the previous, developed in an experimental framework, is the prospect theory of Kahneman and Tversky (1979). In addition to the above mentioned theory, preferences can be characterized by S-shaped value functions w.r.t. a benchmark point. The theory was further developed by Tversky and Kahneman (1992) to cumulative prospect theory in order to be consistent with first-order stochastic dominance.

In a different context, related to the spirit of Friedman and Savage (1948), Markowitz (1952) suggests that individuals are risk averse for losses and risk seeking for gains as long as the outcomes are not very extreme. A class of utility functions that partially[1] represents this kind of behavior is the one of reverse S-shaped (utility or value) functions.

Inspired by previous work, Levy and Levy (2002) formulate the notions of prospect stochastic dominance (PSD) and Markowitz stochastic dominance (MSD). These are essentially obtained by the consideration of two partial orders on th e set of probability measures on $\mathbb{R}$. According to their definition, portfolio A prospect stochastically dominates portfolio B iff the expected utility of the return of A is greater than or equal to the expected utility of the return of B *for any* utility function in the set of increasing, convex on the negative part and concave on the positive part real functions (termed as S-shaped (at zero) utility functions). PSD efficiency is then derived when one considers maximal elements w.r.t. this ordering. Analogously, portfolio A Markowitz stochastically dominates portfolio B iff the expected utility of the return of A is greater than or equal to the expected utility of the return of B *for any* utility function in the set of increasing, concave on the negative part and convex on the positive part real functions (termed as reverse S-shaped (at zero) utility functions). Again, the notion of MSD efficiency follows naturally from the notion of maximality w.r.t. the particular ordering. PSD efficiency and MSD efficiency are not mutually exclusive. That is, one portfolio could be prospect as well as Markowitz stochastic dominance efficient (see the Monte Carlo experiment).

The main contribution of this paper is to develop consistent tests for prospect stochastic dominance (PSD) and Markowitz stochastic dominance (MSD) efficiency (where full diversification is allowed), to analyse *observed* investor behavior in a statistical context. We construct the test statistics using the principle of analogy along with the preference free representations of those notions by Levy and Levy (2002). We construct stochastic rejection regions based on approximations of asymptotic critical values using block bootstrap. Under the appropriate conditions we show that they are asymptotically conservative and consistent, a minimal result that is typical in the relevant literature. Both the statistics as well as the rejection regions are defined by complex optimization procedures. We approximate the optimal solutions by

[1] i.e. when the possibility of further changes of the risk attitude on extreme events is ignored.

reducing them to appropriate sets of linear or mixed integer optimization problems. We engage into Monte Carlo experiments to evaluate the finite sample size and power of the tests allowing for optimization errors in the framework of conditional heteroskedasticity.

We study whether observed portfolio choices can be characterized as efficient w.r.t. any of the two notions of stochastic dominance described in the previous paragraph. Prospect and Markowitz stochastic dominance efficiency criteria have not been well statistically tested, despite their appeal with experimental observations. This is the case, even though this research field seems particularly well suited for statistical analysis, given the availability of large datasets of historical returns.

Post and Levy (2005) test for weaker versions of the aforementioned notions of stochastic dominance. More specifically, they allow for a portfolio A to be prospect (Markowitz) stochastically dominant to B iff *there exists* an S-shaped (reverse S-shaped) utility function that rationalizes the optimal choice of A over B. It is easy to see that by substituting the universal with the existential quantifier they weaken the PSD and MSD notions as defined in Levy and Levy (2002) and discussed above. Then, they propose a non parametric test based on a test statistic constructed from first order conditions of utility maximization. They derive asymptotic critical values by an asymptotic normality argument in an iid framework.[2]

In contrast, first, as noted above, we use the stronger versions of PSD and MSD efficiency of Levy and Levy (2002). We do so motivated by the possibillity that an investor (e.g. a financial institution) being uncertain of the exact form of her utility function, may find useful to have a test of whether a given portfolio can be considered as an optimal choice for any given S-shaped (reverse S-shaped) utility function.

Second, we test for global optimality rather than using first-order conditions, something that among others complexifies our numerical procedures in comparison to the linear programming ones used in the aforementioned paper.

Third, we allow for dynamic time-series patterns (rather than serial IIDness). These research objectives are interesting and can have important effects on empirical research in portfolio analysis and asset pricing.

Our work is in the spirit of Scaillet and Topaloglou (2010) who develop consistent tests

[2] that is generally inconsistent with the empirical characteristics of financial data.

for stochastic dominance efficiency at any order for time-dependent data (see also Linton, Post and Wang (2013)), relying on weighted Kolmogorov-Smirnov type statistics in testing for stochastic dominance. They are in turn inspired by the consistent procedures developed by Barrett and Donald (2003) and extended by Horvath, Kokoszka, and Zitikis (2006) to accommodate for non-compact support.

The paper is organized as follows. In Section 2, we discuss the general hypotheses for testing prospect and Markowitz stochastic dominance efficiency. We describe the testing procedures, and derive their minimal asymptotic properties. The procedures are based on approximations of the asymptotic rejection regions[3] by a block bootstrap method. Notice here that other resampling methods such as subsampling are also available (see Linton, Maasoumi and Whang (2005) for the standard stochastic dominance tests). Linton, Post and Whang (2013) follow this route in the context of testing procedures for second order stochastic dominance efficiency. They use subsampling to estimate the p -values, and discuss power issues of the testing procedures. We use block bootstrap instead of subsampling, since our asymptotic considerations under the null, are based on random variables that are upper bounds for the test statistics at hand. Those bounds cannot be subsampled since they depend on a generally unknown distribution, instead they can be bootstraped when the latter is replaced by the empirical distribution. Their weak limits provide with possibly small asymptotic rejection regions hence the characterization of the testing procedures as asymptotically conservative.[4] The derivation of the exact limits under the null and the subsequent ability to use the subsampling method so as to obtain asymptotically exact procedures is delegated to further research. However the block bootstrap method enables the use of the full sample information. Hence it can be in some cases preferable especially in the presence of samples with a limited number of time-dependent data: we have 996 monthly observations in our empirical application. We note here that the testing procedures for both prospect and Markowitz stochastic dominance efficiency are algorithmically formulated in terms of linear and mixed integer programming respectively.

In Section 3, we design a Monte Carlo study to evaluate for the finite sample size and power of the proposed tests in a framework of conditional heteroskedasticity.

[3] actually of the limiting p-values.

[4] A minimal property which allows for the existence of local alternatives under which the tests are biased.

In Section 4, we provide an empirical illustration. We analyze investor preferences and beliefs by testing whether the value-weighted market portfolio can be considered as efficient according to prospect and Markowitz stochastic dominance criteria when confronted to diversification principles made of risky assets. For this pupropse, we use proxies of the individual assets in the investment universe. Thus, for the individual risky assets, we use four different sets of benchmark portfolios: The 6, the 25 and the 100 Fama and French benchmark portfolios constructed as the intersections of portfolios formed on size and book-to-market equity ratios, and the 49 Fama and French Industry portfolios. Although these datasets have been well studied in the literature, we use a novel analyses that leads to new scientific insights. The usual assumption in the literature is that the utility function of wealth is everywhere concave, where in this paper we test for utility schemes which are locally convex (convex-concave, or concave-convex). To focus on the role of preferences and beliefs, we largely adhere to the assumptions of a single-period, portfolio-oriented model of a frictionless and competitive capital market.

The problem is interesting, given the fact that many institutional investors hold portfolios that mimic the behaviour of the market portfolio. Many institutional investors invest in Exchange-Traded Funds (ETFs) and mutual funds. These funds track stocks, commodities and bonds, or value-weighted equity indices which strongly resemble the market portfolio. Moreover, many actual funds, including total market index funds which are based on the Wilshire 5000 index, are very highly correlated with the market portfolio. Thus, it is interesting to see whether this behaviour can be rationalized by preferences represented inside the aforementioned classes of utility functions.

We find that we cannot reject the PSD hypothesis for the market portfolio, in any of the different datasets. However, the MSD hypothesis for the market portfolio can be rejected in every case. These findings are in contrast to the normative implications inherent within classical expected utility frameworks, that assume that investors are universaly risk averse. Given the possibility of decision errors, our findings predict that individuals tend to be risk averse in a domain of gains, and risk seeking in a domain of losses.

We give some concluding remarks and provide some hints for further research in Section 5. We discuss in detail the computational aspects of mathematical programming formulations corresponding to the test statistics, and we give all the proofs in the Appendix.

2 Consistent Tests for Prospect and Markowitz Stochastic Dominance Efficiency

2.1 Notation, Assumption Framework and Hypotheses Structures

We describe and formulate consistent yet infeasible testing procedures for both criteria. We approximate them by feasible ones that retain consistency.

In this respect, consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, upon which a *strictly stationary* process $(\boldsymbol{Y}_t)_{t\in\mathbb{Z}}$ taking values in $\mathbb{R}^n$ is defined. $\boldsymbol{Y}_{t_i}$ denotes the i^{th} element of $\boldsymbol{Y}_t$. The sample is a realization of the random element $(\boldsymbol{Y}_t)_{t=1,\ldots,T}$. In our context it represents observed returns of n financial assets. F denotes the cdf of $\boldsymbol{Y}_0$ and $\hat{F}_T$ the *empirical* cdf associated with the random element $(\boldsymbol{Y}_t)_{t=1,\ldots,T}$. $(x)_+ = \max\{x, 0\}$ and $(x)_- = \min\{x, 0\}$ and when x is a vector, they are to be interpreted in the coordinatewise sense. In what follows, absolute continuity is considered w.r.t. the relevant Lebesgue measure.

Assumption A. *F is absolutely continuous with convex support and $\max_{i=1,\ldots,n} \mathbb{E}\left|(\boldsymbol{Y}_{0_i})_-\right| < +\infty$. Furthermore, $(\boldsymbol{Y}_t)_{t\in\mathbb{Z}}$ is a-mixing with mixing coefficients a_t such that $a_T = O(T^{-a})$ for some $a > 1$ as $T \to \infty$.*

The mixing part of the previous assumption is readily implied by concepts such as geometric ergodicity which holds for many stationary models used in the context of financial econometrics. Prominent examples are the strictly stationary versions of ARMA or GARCH and stochastic volatility type of models. Counter-examples are stationary models that exhibit long memory, etc. See Doukhan (1994) for the relevant rigorous definition and further examples.

Assumption B. *Assumption A holds and for some $\delta > 0$, $\max_{i=1,\ldots,n} \mathbb{E}\left[(\boldsymbol{Y}_{0_i})_+\right]^{2+\delta} < +\infty$. Furthermore $a > 1 + \frac{2}{\delta}$.*

The previous strengthens the requirements of Assumption A so that moment conditions that enable the validity of a mixing CLT hold. Those are readily established in models such as the ones mentioned above usually in the form of stricter restrictions on the properties of building blocks and the parameters of the processes involved.

Assumption C. $\mathbb{L}$ *is a simplicial complex comprised of a finite number of simplices of* $\Lambda = \{\boldsymbol{\lambda} \in \mathbb{R}^n_+ : \mathrm{e}'\boldsymbol{\lambda}= 1, \}$. *It contains all the extreme points of* Λ.

The parameter set $\mathbb{L}$ represents the portfolio collection at hand via the identification $\boldsymbol{\lambda} \to \boldsymbol{\lambda}'\boldsymbol{Y}_t$. $\boldsymbol{\tau}$ denotes some *distinguished* element of $\mathbb{L}$ that represents the portfolio to be tested for the relevant notion of efficiency w.r.t. the elements of $\mathbb{L}$. The structure of $\mathbb{L}$ allows for it to be non-convex and possibly disconnected while it is obviously compact. It enables the establishment of the limit theory of the procedures to be defined in relation to n i.e., the number of the extreme points of $\boldsymbol{\lambda}$, while its structure as a simplicial complex facilitates our numerical formulation. We obviously suppose that $n > 1$ since in the opposite case $\Lambda = \mathbb{L} = \{\boldsymbol{\tau}\}$ in which case this distinguished element is efficient w.r.t. any notion of stochastic dominance.

Suppose that F^* denotes the distribution function of some finite measure on $\mathbb{R}^n$. Let $G(z, \boldsymbol{\lambda}, F^*)$ be $\int_{\mathbb{R}^n} \mathbb{I}\{\boldsymbol{\lambda}'\boldsymbol{u} \leq z\}dF^*(\boldsymbol{u})$, i.e. the cdf of the linear transformation $x \to \boldsymbol{\lambda}'x$. The following is a list of linear functionals useful for the definition and the derivation of the properties of the testing procedures that we later implement. Let

$$\mathcal{J}_2(z, \boldsymbol{\lambda}, F^*) := \int_{-\infty}^{z} G(u, \boldsymbol{\lambda}, F^*)du. \tag{1}$$

This is finite if $\mathbb{E}^* \left[(-\boldsymbol{\lambda}'\boldsymbol{Y}_0)_+\right]$ exists (see Horvath, Kokoszka, and Zitikis (2006)) where $\mathbb{E}^*$ denotes the expectation operator w.r.t. F^*. Assumption A implies the existence of $\mathcal{J}_2(z, \boldsymbol{\lambda}, F)$ for any $z \in \mathbb{R}$, $\boldsymbol{\lambda} \in \boldsymbol{\lambda}$. From Davidson and Duclos (2000) Equation (2), we obtain that

$$\mathcal{J}_2(z, \boldsymbol{\lambda}, F^*) = \int_{-\infty}^{z} (z - u)dG(u, \boldsymbol{\lambda}, F^*),$$

which can be rewritten as

$$\mathcal{J}_2(z, \boldsymbol{\lambda}, F^*) = \int_{\mathbb{R}^n} (z - \boldsymbol{\lambda}'\boldsymbol{u})\mathbb{I}\{\boldsymbol{\lambda}'\boldsymbol{u} \leq z\}dF^*(\boldsymbol{u}). \tag{2}$$

$\mathcal{J}_2$ can be used to represent second order stochastic dominance. The following transformations and "complements" of $\mathcal{J}_2$ are associated with representations of the notions of stochastic dominance that are considered below and are useful for either the derivation of asymptotic properties and/or computational facilitation. The first, relevant to the prospect stochastic dominance, is

$$\mathcal{J}\left(z,\boldsymbol{\lambda},\boldsymbol{\tau},F^{*}\right)\doteqdot\int_{z}^{0}G(u,\boldsymbol{\tau},F^{*})du-\int_{z}^{0}G(u,\boldsymbol{\lambda},F^{*})du.$$

Under assumption A it can be rewritten as

$$\mathcal{J}_{2}(0,\boldsymbol{\tau},F^{*})-\mathcal{J}_{2}(z,\boldsymbol{\tau},F^{*})-\left(\mathcal{J}_{2}(0,\boldsymbol{\lambda},F^{*})-\mathcal{J}_{2}(z,\boldsymbol{\lambda},F^{*})\right). \tag{3}$$

The second, relevant to Markowitz stochastic dominance, is

$$\mathcal{J}_{2}^{c}(z,\boldsymbol{\lambda},\boldsymbol{\tau},F^{*}):=\int_{z}^{+\infty}\left(G\left(u,\boldsymbol{\lambda},F^{*}\right)-G\left(u,\boldsymbol{\tau},F^{*}\right)\right)du.$$

It not hard to see (Lemma 8 in the Appendix) that $\mathcal{J}_{2}^{c}(z,\boldsymbol{\lambda},\boldsymbol{\tau},F^{*})$ is finite if $\mathbb{E}^{*}[(\boldsymbol{\lambda}'\boldsymbol{Y}_{0})_{+}]$ and $\mathbb{E}^{*}[(\boldsymbol{\tau}'\boldsymbol{Y}_{0})_{+}]$ exist. Assumption B implies the existence of $\mathcal{J}_{2}^{c}(z,\boldsymbol{\lambda},\boldsymbol{\tau},F^{*})$ for any $\boldsymbol{\lambda}\in\mathbb{L}$.

Prospect stochastic dominance efficiency We use the preference free representation of prospect stochastic dominance efficiency of Levy and Levy (2002), and express it with respect to the aforementioned functionals so as to formulate the hypotheses structure for this form of efficiency. This also facilitates the numerical formulation of the relevant tests.

Remember that the dominant portfolio for this form of efficiency is interpreted as the optimal choice for any function in the class of s-shaped utility functions. The following definition draws from equivalence (3) of Levy and Levy (2002) and it is simply expressed in terms of some iterated optimization functional applied on $\mathcal{J}$.

Definition I. *$\boldsymbol{\tau}$ is PSD-efficient iff*

$$S\left(\boldsymbol{\tau}\right):=\max\left(S^{\alpha}\left(\boldsymbol{\tau}\right),S^{\beta}\left(\boldsymbol{\tau}\right)\right)=0,$$

where

$$S^{\alpha}\left(\boldsymbol{\tau}\right):=\sup_{z\geq0,\boldsymbol{\lambda}\in\mathbb{L}}\mathcal{J}\left(z,-\boldsymbol{\lambda},-\boldsymbol{\tau},F\right),$$

and

$$S^{\beta}\left(\boldsymbol{\tau}\right):=\sup_{z\geq0,\boldsymbol{\lambda}\in\mathbb{L}}-\mathcal{J}\left(z,\boldsymbol{\lambda},\boldsymbol{\tau},F\right).$$

$\boldsymbol{\tau}$ is PSD-inefficient iff $S\left(\boldsymbol{\tau}\right)>0$.

This directly identifies the hypothesis structure for any statistical test on the $\mathcal{P}$-efficiency of $\boldsymbol{\tau}$ as follows.

$$\mathbb{H}_0^{(\mathcal{P})}: \; S(\boldsymbol{\tau}) = 0,$$
$$\mathbb{H}_1^{(\mathcal{P})}: \; S(\boldsymbol{\tau}) > 0.$$

Markowitz stochastic dominance efficiency Similarly to the previous case we formulate the hypotheses structure for the Markowitz stochastic dominance efficiency based on the utility free representation of Levy and Levy (2002), equivalently expressed w.r.t. the functionals introduced in previous paragraphs due to reasons stated above.

Again, the dominant portfolio is by construction the optimal choice when the preferences are represented by any utility function in the class of reverse shaped ones. The following definition draws from equivalence (4) of Levy and Levy (2002) which is expressed in terms of some nested optimization procedure that involves $\mathcal{J}_2$ and $\mathcal{J}_2^c$.

Definition II. *If $\mathcal{J}_2$ and $\mathcal{J}_2^c$ exist then $\boldsymbol{\tau}$ is MSD-efficient (see Levy and Levy (2002)), equation (4) iff*

$$\Upsilon(\boldsymbol{\tau}) := \max\left(\hat{\Upsilon}^{\alpha}(\boldsymbol{\tau}), \hat{\Upsilon}^{\beta}(\boldsymbol{\tau})\right) = 0$$

where

$$\Upsilon^{\alpha}(\boldsymbol{\tau}) := \sup_{z \leq 0, \boldsymbol{\lambda} \in \mathbb{L}} \left(\mathcal{J}_2(z, \boldsymbol{\tau}, F) - \mathcal{J}_2(z, \boldsymbol{\lambda}, F)\right),$$

and

$$\Upsilon^{\beta}(\boldsymbol{\tau}) := \sup_{z \geq 0, \boldsymbol{\lambda} \in \mathbb{L}} \left[\int_{\mathbb{R}^n} \left((\boldsymbol{\lambda}'\boldsymbol{u})_+ - (\boldsymbol{\tau}'\boldsymbol{u})_+\right) dF(\boldsymbol{u}) + \mathcal{J}(z, \boldsymbol{\lambda}, \boldsymbol{\tau}, F)\right].$$

$\boldsymbol{\tau}$ is MSD-inefficient iff $\Upsilon(\boldsymbol{\tau}) > 0$.

This again establishes the current hypothesis structure for $\mathcal{M}$-efficiency of as follows.

$$\mathbb{H}_0^{(\mathcal{M})}: \; \Upsilon(\boldsymbol{\tau}) = 0,$$
$$\mathbb{H}_1^{(\mathcal{M})}: \; \Upsilon(\boldsymbol{\tau}) > 0.$$

2.2 Generally Infeasible Tests

Given the hypotheses structure in any of the aforementioned notions of efficiency, we construct test statistics via the use of the principle of analogy. In every occurrence inside the expressions of the null hypothesis we replace the cdf with its empirical counterpart. We then form appropriate differences of the empirical versions of the resulting linear functionals and maximise them w.r.t. z and $\boldsymbol{\lambda}$. Under the appropriate limit theory (enabled by our assumption framework) we obtain a well defined asymptotic distribution for a sequence of random variables that bound from above the previous maximum under the null. Thereby, we obtain valid asymptotic critical values for appropriate levels of significance, so that the limiting probability of rejection under the null is bounded above by the chosen significance level while under the alternative becomes unity.

The test statistics are obtained by two "internal" and a trivial "external" maximization procedure in each case. The former are usually analytically intractable and are thereby numerically approximated. The details of the relevant numerical procedures are explained in the Appendix. Finally, the asymptotic critical values are also usually analytically intractable since they are maxima of complicated Gaussian processes depending on usually unknown properties of F. Hence, they must also be approximated. This is accomplished by bootstrap resampling and numerical optimization and it is explained in the subsequent section. Thus, in this section we present the infeasible tests and their limit theory. The former are then approximated by feasible versions with the same (first order) asymptotic properties.

Prospect stochastic dominance efficiency In the case of the PSD efficiency the procedure defined above results in the following random variable which is the empirical analogue of the functional appearing in the definition of the null:

$$\hat{S}_T(\boldsymbol{\tau}) = \max\left(\hat{S}_T^{\alpha}(\boldsymbol{\tau}), \hat{S}_T^{\beta}(\boldsymbol{\tau})\right)$$

where

$$\hat{S}_T^{\alpha}(\boldsymbol{\tau}) = \sup_{z \geq 0, \boldsymbol{\lambda} \in \mathbb{L}} \sqrt{T}\mathcal{J}\left(z, -\boldsymbol{\lambda}, -\boldsymbol{\tau}, \hat{F}_T\right),$$

and

$$\hat{S}_T^{\beta}(\boldsymbol{\tau}) = \sup_{z \geq 0, \boldsymbol{\lambda} \in \mathbb{L}} -\sqrt{T}\mathcal{J}\left(z, \boldsymbol{\lambda}, \boldsymbol{\tau}, \hat{F}_T\right).$$

Furthermore, given the one sided form of the alternative, for some $c_{\mathcal{P}}$ consider the decision rule

$$\text{reject } \mathbb{H}_0^{(\mathcal{P})} \text{iff } \hat{S}_T(\boldsymbol{\tau}) > c_{\mathcal{P}}. \tag{4}$$

The following proposition demonstrates the minimal asymptotic properties for the test based on (4). The condition (5) appearing in addition to assumption A is identical to the one used in Horvath, Kokoszka, and Zitikis (2006) and Scaillet and Topaloglou (2010) and it is implied by the existence of $\mathbb{E}\left\|\boldsymbol{Y}_0\right\|^{1+\epsilon}$ for some $\epsilon > 0$.

Proposition 1. *Suppose that Assumptions A and C hold and that G satisfies*

$$\int_{\mathbb{R}} \sqrt{G\left(u, \boldsymbol{\lambda}, F\right)\left(1 - G\left(u, \boldsymbol{\lambda}, F\right)\right)} du < +\infty, \text{ for all } \boldsymbol{\lambda} \in \mathbb{L}. \tag{5}$$

1. *If $\mathbb{H}_0^{(\mathcal{P})}$ is true, then there exists a random variable $\bar{S}\left(\boldsymbol{\tau}\right)$ with the following properties. Its law has support $[0, +\infty)$. If $\boldsymbol{\tau}$ is an extreme point of $\boldsymbol{\lambda}$ then it may have an atom at zero of probability at most $\frac{1}{n}$, and it is absolutely continuous when restricted to $(0, +\infty)$. If $\boldsymbol{\tau}$ is not an extreme point of $\boldsymbol{\lambda}$ then its law is absolutely continuous. Given this, for the test based on the decision rule (4) and any $\alpha \in (0, 1)$, there exists a $c_{\mathcal{P}}$ such that:*

$$\limsup_{T \to \infty} \mathbb{P}\left(\text{reject } \mathbb{H}_0^{(\mathcal{P})}\right) \leq \mathbb{P}\left(\bar{S}\left(\boldsymbol{\tau}\right) > c_{\mathcal{P}}\right) \doteqdot \alpha\left(c_{\mathcal{P}}\right).$$

2. *If $\mathbb{H}_0^{(\mathcal{P})}$ is false, then*

$$\lim_{T \to \infty} \mathbb{P}\left(\text{reject } \mathbb{H}_0^{(\mathcal{P})}\right) = 1.$$

Thereby, the test is both asymptotically conservative and consistent. The fact that the asymptotic rejection regions are potentially smaller to the ones possibly related to the given level of significance is attributed to the fact that the derivation goes through via the construction of upper bounds for the test statistic. This is a result that also repeats itself in all the remaining cases and procedures. Hence we will refer to it generally as a minimal asymptotic property. Furthermore, the result specifies the absolute continuity of the limit law, at least when restricted to the interior of its support. Hence, via the connectedness of its support, it implies the continuity of its quantile function when $\alpha < \frac{n-1}{n}$. Notice that this condition implies that in the non trivial cases (i.e. when $n > 1$), this is satisfied when $\alpha \leq \frac{1}{2}$ which is obviously a slack restriction in the standard cases. Both those results are used in order to obtain consistency for the feasible analogues of those testing procedures based on bootstrap when the significance leven is chosen according to this information. Again, those properties also appear in the analogous results for the remaining test for the same reasons. Hence, given the previous comment they will not be further discussed.

Markowitz stochastic dominance efficiency Analogously to the previous paragraph, the following random variable has a similar form and thereby a suchlike structure to the one used in the prospect theory testing procedure. This is then used as the test statistic for the Markowitz type of efficiency. It is

$$\hat{\Upsilon}_T(\boldsymbol{\tau}) = \max\left(\hat{\Upsilon}_T^{\alpha}(\boldsymbol{\tau}), \hat{\Upsilon}_T^{\beta}(\boldsymbol{\tau})\right)$$

where

$$\hat{\Upsilon}_T^{\alpha}(\boldsymbol{\tau}) = \sup_{z\leq 0, \boldsymbol{\lambda}\in\mathbb{L}} \sqrt{T}\left(\mathcal{J}_2(z,\boldsymbol{\tau},\hat{F}_T) - \mathcal{J}_2(z,\boldsymbol{\lambda},\hat{F}_T)\right),$$

and

$$\hat{\Upsilon}_T^{\beta}(\boldsymbol{\tau}) = \sup_{z\geq 0, \boldsymbol{\lambda}\in\mathbb{L}} \left[\frac{1}{\sqrt{T}}\sum_{i=1}^{T}\left((\boldsymbol{\lambda}'Y_i)_+ - (\boldsymbol{\tau}'Y_i)_+\right) + \sqrt{T}\mathcal{J}\left(z,\boldsymbol{\lambda},\boldsymbol{\tau},\hat{F}_T\right)\right].$$

Furthermore, given the one sided form of the relevant alternative, for some $c_{\mathcal{M}}$ consider the decision rule

$$\text{reject } \mathbb{H}_0^{(\mathcal{M})} \text{iff } \hat{\Upsilon}_T(\boldsymbol{\tau}) > c_{\mathcal{M}}. \tag{6}$$

The proposition that follows similarly derives the minimal asymptotic and the bootstrap related properties for the test based on (6). Similarly to the previous, the condition (7) appearing in addition to assumption B implies along with the assumption (5). It is implied by the existence of $\max_{i=1,\ldots,n}\mathbb{E}\left|(\boldsymbol{Y}_{0_i})_-\right|^{1+\epsilon}$ for some $\epsilon > 0$.

Proposition 2. *Suppose that Assumptions B and C hold and that G satisfies*

$$\int_{-\infty}^{0}\sqrt{G(u,\boldsymbol{\lambda},F)(1-G(u,\boldsymbol{\lambda},F))}du < +\infty, \text{ for all } \boldsymbol{\lambda}\in\mathbb{L}. \tag{7}$$

1. *If $\mathbb{H}_0^{(\mathcal{M})}$ is true, then there exists a random variable $\bar{\Upsilon}(\boldsymbol{\tau})$ with the following properties. Its law has support $[0,+\infty)$. If $\boldsymbol{\tau}$ is an extreme point of $\boldsymbol{\lambda}$ then it may have an atom at zero of probability at most $\frac{1}{n}$, and it is absolutely continuous when restricted to $(0,+\infty)$. If $\boldsymbol{\tau}$ is not an extreme point of $\boldsymbol{\lambda}$ then its law is absolutely continuous. Given this, for the test based on the decision rule (6) and $\alpha\in(0,1)$, there exists a $c_{\mathcal{M}}$ such that:*

$$\limsup_{T\to\infty} \mathbb{P}\left(\textit{reject} \ : \mathbb{H}_0^{(\mathcal{M})}\right) \leq \mathbb{P}\left(\bar{\Upsilon}\left(\boldsymbol{\tau}\right) > c_{\mathcal{M}}\right) \doteqdot \alpha\left(c_{\mathcal{M}}\right).$$

2. If $\mathbb{H}_0^{(\mathcal{M})}$ is false, then

$$\lim_{T\to\infty} \mathbb{P}\left(\textit{reject}\ \mathbb{H}_0^{(\mathcal{M})}\right) = 1.$$

2.3 K-S type Tests Based on Block Bootstrap

The previous testing procedures are generally non implementable due to the fact that in most cases the critical values are unknown. This is first due to the fact that the covariance kernels of the limiting Gaussian processes that enable the previous results are generally unknown and cannot be estimated without further parametric assumptions. Furthermore, even if such estimates were available, the optimization of the resulting processes is generally analytically intractable, and thereby must be as well performed by numerical procedures.

In this section, in order to avoid such assumptions, and given that in any case the numerical burden employed in what follows would be comparable to the one implied in the previous paragraph (see the relevant comment after Proposition 2 of Scaillet and Topaloglou (2010)) we consider approximations based on bootstrap resampling techniques that manage to incorporate the assumed dependence.[5]

Block bootstrap methods are based on "blocking" arguments, in which data are divided into blocks and those, rather than individual data, are resampled in order to mimic the time dependent structure of the original data.[6] Let b_T, l_T denote integers such that $T = b_T l_T$. b_T denotes the number of blocks and l_T the block size. The following assumption rests on Theorem 2.2 of Peligrad (1998).

Assumption D. *For some $0 < \rho < \frac{1}{3}$ and some $0 < h < \frac{1}{3} - \rho$, $T^h \ll l_T \ll T^{\frac{1}{3}-\rho}$ and $l_T = l_{2^k}$ for $2^k \leq T < 2^{k+1}$.*

[5] As also mentioned before, the subsampling procedure employed in Linton et al. [26] is not implementable in our case since we do not have an exact limit theory of our test statistics under the relevant null hypotheses. Since only asymptotic upper bounds are available, bootstrap resampling schemes are implementable on the empirical analogues of those limits conditional on the sample. This is not true for subsampling schemes since F is unknown.

[6] i.e. the relevant empirical measure on the powerset of the sample is essentially used.

We consider only the case of non-overlapping blocks. This is due to the fact that the bias reducting centering of the relevant statistics would imply further serious numerical burden.[7] In any case, and due to the fact that we are only concerned with first order asymptotic properties it would be easy to see that the overlapping case would also have those properties. Let $(\boldsymbol{Y}_t^*)_{t=1,\ldots,T}$ denote a bootstrap sample arising by either methodology and let $\hat{F}_T^*$ denote its empirical distribution. Denote with $\mathbb{E}_T^*[\cdot]$ the expectation operator with respect to the probability measure induced by the sampling scheme. Under the current methodology we have that $\mathbb{E}_T^*[\mathcal{J}_2\left(z, \boldsymbol{\lambda}, \hat{F}_T^*\right)] = \mathcal{J}_2\left(z, \boldsymbol{\lambda}, \hat{F}_T\right)$ and $\mathbb{E}_T^*[\frac{1}{T}\sum_{i=1}^T \boldsymbol{Y}_i^*] = \frac{1}{T}\sum_{i=1}^T \boldsymbol{Y}_i$ (see Scaillet and Topaloglou(2010)). Hence we are able to define and study the consistency of the following approximations to the testing procedures defined previously.

Prospect stochastic dominance efficiency Consider the bootstrapped analogues of the random variables appearing in the infeasible testing procedure for the $\mathcal{P}$ efficiency. Those are obtained when the linear functionals appearing in the objective functions of each part of the test statistic are evaluated at the difference between the empirical measure of the bootstrapped sample and the empirical measure of the original one. Given the analogous optimizations the decision rule is then formulated by an approximation of the relevant p-value. In this respect we obtain:

$$\hat{S}_T^*\left(\boldsymbol{\tau}\right) = \max\left(\hat{S}_T^{\alpha*}\left(\boldsymbol{\tau}\right), \hat{S}_T^{\beta*}\left(\boldsymbol{\tau}\right)\right),$$

where

$$\hat{S}_T^{\alpha*}\left(\boldsymbol{\tau}\right) = \sup_{z\geq 0, \boldsymbol{\lambda}\in\mathbb{L}} \sqrt{T}\left(\mathcal{J}\left(z, -\boldsymbol{\lambda}, -\boldsymbol{\tau}, \hat{F}_T^*\right) - \mathcal{J}\left(z, -\boldsymbol{\lambda}, -\boldsymbol{\tau}, \hat{F}_T\right)\right),$$

and

$$\hat{S}_T^{\beta*}\left(\boldsymbol{\tau}\right) = \sup_{z\geq 0, \boldsymbol{\lambda}\in\mathbb{L}} \sqrt{T}\left(\mathcal{J}\left(z, \boldsymbol{\lambda}, \boldsymbol{\tau}, \hat{F}_T\right) - \mathcal{J}\left(z, \boldsymbol{\lambda}, \boldsymbol{\tau}, \hat{F}_T^*\right)\right).$$

Define $p_{\mathcal{P}}^* := \mathbb{P}[\hat{S}_T^*\left(\boldsymbol{\tau}\right) > \hat{S}_T\left(\boldsymbol{\tau}\right)]$ and consider the decision rule

$$\text{reject } \mathbb{H}_0^{(\mathcal{P})} \text{iff } p_{\mathcal{PSD}}^* < \alpha. \tag{8}$$

We then obtain the following result.

[7] At least for the second test, the recentering makes the test statistics very difficult to compute, since the optimization for Markowitz stochastic dominance involves a large number of binary variables (see the section on the numerical implementation).

Proposition 3. *Suppose that Assumptions A, C and D hold, G satisfies the condition (5) in Proposition 1 for all $\boldsymbol{\lambda} \in \boldsymbol{\lambda}$. Suppose that $0 < \alpha < \frac{n-1}{n}$ when $\boldsymbol{\tau}$ is an extreme point of $\boldsymbol{\lambda}$ and that $\alpha \in (0,1)$ when it is not. Then, for such a choice of the significance level the test based on decision rule (8) is asymptotically conservative and consistent.*

The restriction on the choice of the significance level is of negligible practical importance since when $n > 1$, the usual choices of α nessesarily satisfy it. Again, the approximation of the p-value is analytically intractable. It is approximated by an empirical frequency argument based on several bootstrap samples. More specifically, given $R \geq 1$ bootstrap samples, an approximation of the rejection probability of the bootstrap test of PSD and for significance level α is $R\hat{P}_{\mathsf{PSD}}(\alpha) = \dfrac{1}{R}\displaystyle\sum_{r=1}^{R} \mathbb{I}\{\hat{S}_{T,r}(\boldsymbol{\tau}) < \hat{Q}^*_{\mathsf{PSD}}(\alpha)\}$ where the test statistics $\hat{S}_{T,r}(\boldsymbol{\tau})$ are obtained for $r = 1, \cdots, R$, and $\hat{Q}^*_{\mathsf{PSD}}(\alpha)$ is the empirical α-quantile of the sample of bootstrap statistics $\hat{S}^*_{T,r}(\boldsymbol{\tau}), r = 1, \cdots, R$ (see also Davidson and MacKinnon (2006a,b)). The asymptotic theory used for the proof of those propositions along with an application of the CMT imply also the consistency of those procedures for any R, given $T \to \infty$. Obviously, the value of R is expected to affect higher order properties of the resulting procedures.

Markowitz stochastic dominance efficiency Again, we consider the bootstrapped analogues of the random variables appearing in the infeasible testing procedure for the $\mathcal{M}$ efficiency.

$$\hat{\Upsilon}^*_T(\boldsymbol{\tau}) = \max\left(\hat{\Upsilon}^{\alpha_*}_T(\boldsymbol{\tau}), \hat{\Upsilon}^{\beta_*}_T(\boldsymbol{\tau})\right),$$

where

$$\hat{\Upsilon}^{\alpha_*}_T(\boldsymbol{\tau}) = \sup_{z\leq 0, \boldsymbol{\lambda}\in\mathbb{L}} \sqrt{T}\left(\mathcal{J}_2\left(z, \boldsymbol{\tau}, \hat{F}^*_T\right) - \mathcal{J}_2\left(z, \boldsymbol{\tau}, \hat{F}_T\right) - \mathcal{J}_2\left(z, \boldsymbol{\lambda}, \hat{F}^*_T\right) + \mathcal{J}_2\left(z, \boldsymbol{\lambda}, \hat{F}_T\right)\right),$$

and

$$\hat{\Upsilon}^{\beta_*}_T(\boldsymbol{\tau}) = \sup_{z\geq 0, \boldsymbol{\lambda}\in\mathbb{L}} \left[\kappa^*_T(\boldsymbol{\lambda}, \boldsymbol{\tau}) - \sqrt{T}\left(\mathcal{J}\left(z, \boldsymbol{\lambda}, \boldsymbol{\tau}, \hat{F}_T\right) - \mathcal{J}\left(z, \boldsymbol{\lambda}, \boldsymbol{\tau}, \hat{F}^*_T\right)\right)\right],$$

where

$$\kappa^*_T(\boldsymbol{\lambda}, \boldsymbol{\tau}) = \frac{1}{\sqrt{T}} \sum\nolimits_{i=1}^{T} \left[\left((\boldsymbol{\lambda}'\boldsymbol{Y}^*_i)_+ - (\boldsymbol{\lambda}'\boldsymbol{Y}_i)_+\right) - \left((\boldsymbol{\tau}'\boldsymbol{Y}^*_i)_+ - (\boldsymbol{\tau}'\boldsymbol{Y}_i)_+\right)\right].$$

Define $p^*_{\mathcal{M}} := \mathbb{P}[\hat{\Upsilon}^*_T(\boldsymbol{\tau}) > \hat{\Upsilon}_T(\boldsymbol{\tau})]$ and consider the decision rule

$$\text{reject } \mathbb{H}_0^{(\mathcal{M})} \text{iff } p^*_{\mathcal{MSD}} < \alpha. \tag{9}$$

The following proposition establishes the analogous limit theory.

Proposition 4. *Suppose that Assumptions B, C, D hold, and that G satisfies condition (7) for all $\boldsymbol{\lambda} \in \boldsymbol{\lambda}$. Furthermore, let $0 < \alpha < \frac{n-1}{n}$ when $\boldsymbol{\tau}$ is an extreme point of $\boldsymbol{\lambda}$ and that $\alpha \in (0,1)$ when it is not. Then, for such a choice of the significance level the test based on decision rule (8) is asymptotically conservative and consistent.*

Again, as in the previous case, the computation of the p-values appearing in each proposition is also analytically intractable in most cases. Analogously to the PSD, we use the rejection probability approximation $\hat{RP}_{\mathsf{MSD}}(\alpha) = \frac{1}{R}\sum_{r=1}^{R} \mathbb{I}\{\hat{\Upsilon}_{T,r}(\boldsymbol{\tau}) < \hat{Q}^*_{MSD}(\alpha)\}$ for $r = 1, \cdots, R$ where the test statistics $\hat{\Upsilon}_{T,r}(\boldsymbol{\tau})$ are obtained from the R subsamples, and $\hat{Q}^*_{\mathsf{MSD}}(\alpha)$ is the empirical α-quantile of the sample of bootstrap statistics $\hat{\Upsilon}^*_{T,r}(\boldsymbol{\tau}), r = 1, \cdots, R$.

3 Monte Carlo study

Remember that the "feasible" versions of both tests depend on every step of their formulation on complex optimization procedures. The numerical formulation of those programs, operating by reduction to appropriate sets of simpler optimization problems, are explained in detail in the Appendix.

In this section we design a set of Monte Carlo experiments to evaluate the size and power of the proposed tests in finite samples, in the context of the aforementioned numerical approximation of the test statistics and the critical (or p-values), as well as w.r.t. the choice of the block size for which the assumption framework provides only asymptotic guidance.

We do so in a framework of conditional heteroskedasticity that is consistent with empirical findings on returns of financial data that are similar to the empirical application that follows. The $(\boldsymbol{Y}_t)_{t\in\mathbb{Z}}$ process is constructed as a vector GARCH(1,1) process that also contains an appropriately transformed element. Under the appropriate restrictions, this allows for both temporal as well as cross sectional dependence between the random variables that constitute the vector process. In the following paragraph we describe the process, and using the analogous propositions, we establish efficient and non-efficient portfolios w.r.t to any of the criteria. We engage to the experiment and present the results in the final paragraph of this section.

3.1 GARCH Type Processes and Efficiency Considerations

Suppose that

$$z_t \overset{\text{iid}}{\sim} N(0,1), \; t \in \mathbb{Z}.$$

Furthermore for all $t \in \mathbb{Z}$, for $i = 1, 2, 3$, $\omega_i, \alpha_i, \beta_i \in \mathbb{R}_{++}$, $\mu_i \in \mathbb{R}_{+}$ define

$$\begin{aligned} y_{i_t} &= \mu_i + z_t h_{i_t}^{1/2}, \\ h_{i_t} &= \omega_i + \left(\alpha_i z_{t-1}^2 + \beta_i\right) h_{i_{t-1}}, \; \mathbb{E}\left(\alpha_i z_0^2 + \beta_i\right)^{1+\epsilon} < 1, \end{aligned}$$

for some $\epsilon > 0$ while for $i = 4$ and $v_1, v_2 \in \mathbb{R}$ define

$$y_{4_t} = v_1 \left(z_t h_{3_t}^{1/2}\right)_+ + v_2 \left(z_t h_{3_t}^{1/2}\right)_- .$$

Suppose that $\mathbf{Y}_t = (y_{1_t}, y_{2_t}, y_{3_t}, y_{4t})'$. Then, Corollary 1 and Theorem 8 of Lindner (2003), the definition of strong mixing along with the measurability of $(\cdot)_+$, $(\cdot)_-$ and their independence from t, imply that assumptions A and B hold for $(\mathbf{Y}_t)$. They also imply covariance stationarity. Notice that the vector process of interest also exhibits contemporaneous dependence between its elements. Furthermore, trivial calculations show that $\text{Cov}\left(y_{i_t}, y_{i_{t-k}}\right) = 0$ for all non zero k and $i \neq 4$, while this is not true for $i = 4$. Let $\boldsymbol{\tau} = (0,0,1,0)$, $\boldsymbol{\tau}^* = (0,0,0,1)$ and $\mathbb{L} = \{(\boldsymbol{\lambda}, 1-\boldsymbol{\lambda}, 0, 0), \boldsymbol{\lambda} \in [0,1], \boldsymbol{\tau}, \boldsymbol{\tau}^*\}$. Obviously, Assumption C also holds.

The first proposition establishes that $\boldsymbol{\tau}^*$ *is a portfolio that is both Markowitz and prospect efficient* w.r.t. $\mathbb{L}$ when the structuring coefficients are appropriately chosen so that the negative part of $\boldsymbol{\tau}^*$ has smaller variance and the positive part of $\boldsymbol{\tau}^*$ has larger variance when compared to the other portfolios in $\mathbb{L}$.

Proposition 5. *If* $\mu_i = 0$ *for* $i = 1, 2, 3$, $|v_1| > \sqrt{\frac{\max\{\omega_i, \alpha_i, \beta_i, \; i=1,2,3\}}{\min\{\omega_i, \alpha_i, \beta_i, \; i=1,2,3\}}}$ *and* $|v_2| < \sqrt{\frac{\min\{\omega_i, \alpha_i, \beta_i, \; i=1,2,3\}}{\max\{\omega_i, \alpha_i, \beta_i, \; i=1,2,3\}}}$ *then* $\boldsymbol{\tau}^*$ *is both* $\mathcal{M}$ *and* $\mathcal{P}$*-efficient w.r.t.* $\mathbb{L}$*.*

Proof. Define $\mathcal{F}_t = \sigma\{z_{t-1}, z_{t-2}, \ldots\}$ and notice that due to the definition of $\boldsymbol{\lambda}$, the almost sure positivity of h_{i_t} for all i and Jensen's inequality

$$\min\{h_{1_t}, h_{2_t}, h_{3t}\} \leq v_{\boldsymbol{\lambda}_t} \leq \max\{h_{1_t}, h_{2_t}, h_{3t}\} \;\; \mathbb{P} \text{ a.s.}$$

where $v_{\boldsymbol{\lambda}_t} \triangleq \text{Var}\left(\boldsymbol{\lambda} y_{1_t} + (1-\boldsymbol{\lambda})\, y_{2_t}/\mathcal{F}_t\right)$ or $v_{\boldsymbol{\lambda}_t} \triangleq h_{3t}$. Define the auxiliary processes by

$$\begin{aligned} h_{*_t} &= \alpha_*\left(1+\left(z_{t-1}^2+1\right)h_{*_{t-1}}\right) \\ h_t^* &= \alpha^*\left(1+\left(z_{t-1}^2+1\right)h_{t-1}^*\right), \end{aligned}$$

for $\alpha_* = \min\{\omega_i, \alpha_i, \beta_i,\ i=1,2,3\}$, $\alpha^* = \max\{\omega_i, \alpha_i, \beta_i,\ i=1,2,3\}$ and notice that

$$h_{*_t} \le \min\{h_{1_t}, h_{2_t}, h_{3t}\} \le \max\{h_{1_t}, h_{2_t}, h_{3t}\} \le h_t^*,\ \mathbb{P}\,\text{a.s.}$$

Hence, when $v_2^2\alpha^* < \alpha_*$ then $|v_2|\sqrt{h_{3_t}} < \sqrt{v_{\boldsymbol{\lambda}_t}}$ $\mathbb{P}$ a.s. and when $v_1^2\alpha_* > \alpha^*$ then $|v_1|\sqrt{h_{3_t}} > \sqrt{v_{\boldsymbol{\lambda}_t}}$, $\mathbb{P}$ a.s. Furthermore, the distribution function of $\boldsymbol{\lambda}$ equals $\mathbb{E}\Phi\left(\frac{x}{\sqrt{v_{\boldsymbol{\lambda}_t}}}\right)$ due to the law of iterated expectations. In an analogous manner it is easy to see that the distribution function of $\boldsymbol{\tau}^*$ equals $\begin{cases} \mathbb{E}\Phi\left(\frac{x}{|v_2|\sqrt{h_{3_t}}}\right), & x \le 0 \\ \mathbb{E}\Phi\left(\frac{x}{|v_1|\sqrt{h_{3_t}}}\right), & x > 0 \end{cases}$. The monotonicity of the integral along with the relevant property of Φ imply that both distribution functions are strictly increasing. Hence for $z \le 0$ we have that

$$\begin{aligned} &\mathcal{J}_2(z,\boldsymbol{\lambda},F) - \mathcal{J}_2(z,\boldsymbol{\tau},F) \\ = &\int_{-\infty}^{z}\left(\mathbb{E}\Phi\left(\frac{x}{\sqrt{v_{\boldsymbol{\lambda}_t}}}\right) - \mathbb{E}\Phi\left(\frac{x}{|v_2|\sqrt{h_{3_t}}}\right)\right)dx > 0 \end{aligned}$$

due to the previous and the fact that x assumes non positive values except for sets of Lebesgue measure zero. Analogously, for any $z > 0$

$$\mathcal{J}_2^c(z,\boldsymbol{\lambda},\boldsymbol{\tau},F) = \int_z^{+\infty}\left(\mathbb{E}\Phi\left(\frac{x}{\sqrt{v_{\boldsymbol{\lambda}_t}}}\right) - \mathbb{E}\Phi\left(\frac{x}{|v_1|\sqrt{h_{3_t}}}\right)\right)dx > 0$$

which holds due to the fact that the integral exists from lemma 8, the previous and the fact that x assumes positive values. The result for $\mathcal{M}$ efficiency follows from the definition above. The same arguments show that when $z \le 0$

$$\int_z^{0}\left(\mathbb{E}\Phi\left(\frac{x}{\sqrt{v_{\boldsymbol{\lambda}_t}}}\right) - \mathbb{E}\Phi\left(\frac{x}{|v_2|\sqrt{h_{3_t}}}\right)\right)dx > 0$$

and when $z > 0$

$$\int_0^{z}\left(\mathbb{E}\Phi\left(\frac{x}{\sqrt{v_{\boldsymbol{\lambda}_t}}}\right) - \mathbb{E}\Phi\left(\frac{x}{|v_1|\sqrt{h_{3_t}}}\right)\right)dx > 0$$

establishing $\mathcal{P}$-efficiency via the relevant definition above. ■

The following proposition provides with a counterexample by describing conditions under which $\boldsymbol{\tau}$ is neither $\mathcal{M}$ or $\mathcal{P}$-efficient w.r.t. $\mathbb{L}$. Notice that an analogous result is directly obtained by the previous proposition in a more restricted setting.

Proposition 6. *If $\mu_i = 0$ for $i = 1, 3$, $\omega_1 < \omega_3$, $a_1 < a_3$ and $\beta_1 < \beta_3$ then $\boldsymbol{\tau}$ is both $\mathcal{P}$ and $\mathcal{M}$-inefficient w.r.t. $\mathbb{L}$.*

Proof. Let $\boldsymbol{\lambda} = 1$ whence $h_{3_t} > v_{\mathbf{1}} = h_{1_t}$ $\mathbb{P}$ a.s. Using analogous arguments as before we have that for $z \leq 0$

$$\begin{aligned}&\mathcal{J}_2(z, 1, F) - \mathcal{J}_2(z, \boldsymbol{\tau}, F)\\ =\ &\int_{-\infty}^{z} \left(\mathbb{E}\Phi\left(\frac{x}{\sqrt{h_{1_t}}}\right) - \mathbb{E}\Phi\left(\frac{x}{\sqrt{h_{3_t}}}\right)\right) dx < 0,\end{aligned}$$

which implies that the first part of definition II is not valid. Analogously

$$\begin{aligned}&\mathcal{J}\left(z, 1, \boldsymbol{\tau}, F\right)\\ =\ &\int_{z}^{0} \left(\mathbb{E}\Phi\left(\frac{x}{\sqrt{h_{3_t}}}\right) - \mathbb{E}\Phi\left(\frac{x}{\sqrt{h_{1_t}}}\right)\right) dx > 0\end{aligned}$$

invalidating the first part of definition I. ■

Finally, the existence of moments of order $2 + \delta$ implies the validity of (5) and (7) for any of the considered portfolios.

Choice of Monte Carlo and Bootstrap Parameters

In each case the relevant data generating process DGP is used to draw realizations of the asset returns using the GARCH process described above (with different parameters for each case to evaluate size and power). We generate $R = 300$ original samples with size $T = 500$. For each one of these original samples we generate a block bootstrap (nonoverlapping case) data generating process $\widehat{\text{DGP}}$. Once $\widehat{\text{DGP}}$ is obtained for each replication r, a new set of random numbers, independent of those used to obtain $\widehat{\text{DGP}}$, is drawn. Then, using these numbers we draw R original samples and R block bootstrap samples to compute $\hat{S}_{T,r}(\boldsymbol{\tau})$, $\hat{S}^*_{T,r}(\boldsymbol{\tau})$, $\hat{\Upsilon}_{T,r}(\boldsymbol{\tau})$ and $\hat{\Upsilon}^*_{T,r}(\boldsymbol{\tau})$ to get the estimates $\hat{RP}_{\mathsf{PSD}}(\alpha)$ and $\hat{RP}_{\mathsf{MSD}}(\alpha)$ respectively.

Size evaluation. To approximate the fixed T size, we test for PSE and MSD efficiency of portfolio $\boldsymbol{\tau}^*$ containing the fourth asset, with respect to all other possible portfolios $\mathbf{w} \in \mathbb{L} - \{\boldsymbol{\tau}^*\}$. We set $\mu_i = 0$ for $i = 1, 2, 3$, $\omega_1 = 0.5$, $\omega_2 = 0.5$, and $\omega_3 = 0.5$, $a_1 = 0.4$, $a_2 = 0.45$, and $a_3 = 0.5$ and $\beta_1 = 0.5$, $\beta_2 = 0.45$, and $\beta_3 = 0.4$. Then, we set $v_1 = 1.5$ and $v_2 = 0.5$. In this case, we have that $|v_1| > \sqrt{\frac{\max\{\omega_i, \alpha_i, \beta_i,\ i=1,2,3\}}{\min\{\omega_i, \alpha_i, \beta_i,\ i=1,2,3\}}}$ and $|v_2| < \sqrt{\frac{\min\{\omega_i, \alpha_i, \beta_i,\ i=1,2,3\}}{\max\{\omega_i, \alpha_i, \beta_i,\ i=1,2,3\}}}$.

We set the significance level α equal to 5%, and the block size to $l = 10$. We get $\hat{RP}_{\text{PSD}}(5\%) = 2.8\%$ for the prospect stochastic dominance efficiency test, while we get $\hat{RP}_{\text{MSD}}(5\%) = 4.1\%$ for the Markowitz stochastic dominance efficiency test. Hence, we may conclude that both bootstrap tests perform well in terms of size properties.

Power evaluation. To approximate the fixed T power, we test for PSE and MSD efficiency of portfolio $\boldsymbol{\tau}$ containing the third asset, with respect to all other possible portfolios $\mathbf{w} \in \mathbb{L} - \{\boldsymbol{\tau}\}$. We set $\mu_i = 0$ for $i = 1, 2, 3$, $\omega_1 = 0.5$, $\omega_2 = 0.5$, and $\omega_3 = 0.8$, $a_1 = 0.3$, $a_2 = 0.4$, and $a_3 = 0.45$ and $\beta_1 = 0.3$, $\beta_2 = 0.4$, and $\beta_3 = 0.45$. Then, we set $v_1 = 2$ and $v_2 = 0.2$. In this case, we have that $\omega_1 < \omega_3$, $a_1 < a_3$ and $\beta_1 < \beta_3$.

We find positive evidence for the power of both tests. Indeed, we have that $\hat{RP}_{\text{PSD}}(5\%) = 97.1\%$ for the prospect stochastic dominance efficiency test when we take wrongly as efficient the portfolio $(\boldsymbol{\tau} = (0, 0, 1))$. Similarly we find $\hat{RP}_{\text{MSD}}(5\%) = 95.2\%$ for the Markowitz stochastic dominance efficiency.

We present our Monte Carlo results in Table 1 on the sensitivity to the choice of block length. We investigate block sizes ranging from $l = 4$ to $l = 12$ by step of 4. This covers the suggestions of Hall, Horowitz, and Jing (1995), who show that optimal block sizes are multiple of $T^{1/3}$, $T^{1/4}$, $T^{1/5}$, depending on the context. According to our experiments the choice of the block size does not seem to dramatically alter the performance of our methodology. Finally, we investigate the sensitivity of the tests to the choice of the number of original samples (R) and size (T). Three different cases are presented in Table 1. The tests seem to perform well in every case.

Computational Resources and Time We solve all the optimization problems using the General Algebraic Modeling System (GAMS), which is a high-level modeling system for mathematical programming and optimization. This language calls special solvers (GUROBI in our case) that are specialized in linear and mixed integer programs. GUROBI uses the branch and bound technique to solve the MIP program. The Matlab code (where the simulations run) calls the specific GAMS program, which calls the GUROBI solver to solve each optimization.

The problems are optimized on an iMac (i7 processor, 2.9 GHz Power, 16Gb of RAM). We note the almost exponential increase in solution time with the increasing number of observations.

The computational time for each PSD case which involves only linear optimization problems varies from 10 minutes in case 1, to 30 minutes in case 3. On the other hand, the computational time for each MSD case which involves linear as well as MIP problems varies from 2 hours in case 1, to 30 hours in case 3.

4 Empirical application

In this section we present the results of an empirical application. To illustrate the potential of the proposed test statistics, we analyze investor preferences and beliefs by testing whether the value-weighted market portfolio can be considered as efficient according to prospect and Markowitz stochastic dominance criteria when confronted to diversification principles made of risky assets. For this purpose, we use proxies of the individual assets in the investment universe. Thus, for the individual risky assets, we use four different sets of benchmark portfolios: The 6, the 25 and the 100 Fama and French benchmark portfolios constructed as the intersections of portfolios formed on size and book-to-market equity ratios, and the 49 Fama and French Industry portfolios.

4.1 Description of the data

We use three different datasets of the Fama and French (FF) benchmark portfolios as well as industry portfolios as our sets of risky assets.

- **The 6 FF Benchmark portfolios**: They are constructed at the end of each June, and correspond to the intersections of two portfolios formed on size (market equity, ME) and three portfolios formed on the ratio of book equity to market equity (BE/ME). The size breakpoint for year t is the median NYSE market equity at the end of June of year t. BE/ME for June of year t is the book equity for the last fiscal year end in $t-1$ divided by ME for December of $t-1$. Firms with negative BE are not included in any portfolio. The annual returns are from January to December.
- **The 25 FF Benchmark portfolios**: They are constructed at the end of each June, are the intersections of 5 portfolios formed on size (market equity, ME) and 5 portfolios

formed on the ratio of book equity to market equity (BE/ME).

- **The 100 FF Benchmark portfolios**: They are constructed at the end of each June, are the intersections of 10 portfolios formed on size (market equity, ME) and 10 portfolios formed on the ratio of book equity to market equity (BE/ME).

- **The 49 Industry portfolios**: They are constructed by assigning each NYSE, AMEX, and NASDAQ stock to an industry portfolio at the end of June of year t based on its four-digit SIC code at that time. The Compustat SIC codes are used for the fiscal year ending in calendar year t-1. Whenever Compustat SIC codes are not available, the CRSP SIC codes for June of year t are used.

For each dataset we use data on monthly excess returns (month-end to month-end) from January 1930 to December 2012 (996 monthly observations) obtained from the data library on the homepage of Kenneth French (http://mba.turc.dartmouth.edu/pages/faculty/ken.french). The test portfolio is the Fama and French market portfolio, which is the value-weighted average of all non-financial common stocks listed on NYSE, AMEX, and Nasdaq, and covered by CRSP and COMPUSTAT.

Table 2 presents some sample characteristics of the Market portfolio and the 6 FF portfolios[8] covering the period from January 1930 to December 2012 (996 monthly observations) that are used in the test statistics.

As we can see from Table 2 the sample skewness and kurtosis provide evidence against marginal normality. If this is true and the investor utility function is not quadratic, then preference relation of any such investor cannot be represented by the variance-covariance matrix of these portfolios. At this point it is perhaps interesting to note that Scaillet and Topaloglou (2010) show that the Fama and French market portfolio is not mean-variance efficient, compared to the 6 benchmark portfolios. This motivates us to test whether the market portfolio is efficient when different preferences are taken into account.

[8] Analogous statistical characteristics are also available for the other datasets

4.2 Results of the stochastic dominance efficiency tests

We find a significant autocorrelation of order one at a 5% significance level in some benchmark portfolios, while ARCH effects are also present at a 5% significance level. This indicates that a block bootstrap approach should be favored over a standard i.i.d. bootstrap approach. Furthermore, estimation of GARCH type models provide evidence in favor of the mixing and moment conditions appearing in our assumption framework. Indeed, both for the market portfolio as well as for each benchmark portfolio i, the estimates of the sum of the GARCH and the ARCH coefficients are less than 1. We choose a block size of 10 observations following the suggestions of Hall, Horowitz, and Jing (1995), who show that optimal block sizes are multiple of $T^{1/3}$, where in our case, $T = 996$. The p-values are approximated as shown before.

The 6 FF Benchmark portfolios. For the prospect stochastic dominance efficiency, we cannot reject the hypothesis that the market portfolio is efficient. The p-value, $\tilde{p} = 0.647$ is way above the significance level of 5%. We divide the full period into two sub-periods, the first one from January 1930 to June 1971, a total of 498 monthly observations, and the second one from July 1971 to December 2012, 498 monthly observations. We test for prospect stochastic dominance of the market portfolio to each sub-period. We find that the p-value for the first sub-period is $\tilde{p}_1 = 0.597$ and the p-value for the second sub-period is $\tilde{p}_2 = 0.713$.

On the other hand, we find that the MSD criterion cannot be accepted at the aforementioned significance level. The p-value, $\tilde{p} = 0.039$ is below the significance level of 5%. Additionally, the p-value $\tilde{p}_1 = 0.027$ for the first sub-period and the p-value $\tilde{p}_2 = 0.043$ for the second sub-period indicate that the market portfolio is not Markowitz stochastic dominance efficient in each sub-period as well as in the full period.

The 25 FF Benchmark portfolios. As before, for the prospect stochastic dominance efficiency, we cannot reject the hypothesis that the market portfolio is efficient. The p-value, $\tilde{p} = 0.437$ is way above the significance level of 5%. We find that the p-value for the first sub-period is $\tilde{p}_1 = 0.523$ and the p-value for the second sub-period is $\tilde{p}_2 = 0.614$.

We additionally find that the MSD criterion cannot be accepted. The p-value, $\tilde{p} = 0.047$ is below the significance level of 5%. Additionally, the p-value $\tilde{p}_1 = 0.036$ for the first sub-period and the p-value $\tilde{p}_2 = 0.055$ for the second sub-period indicate that the market portfolio is not Markowitz stochastic dominance efficient in each sub-period as well as in the full period.

The 100 FF Benchmark portfolios. Again, for the prospect stochastic dominance efficiency, we cannot reject the hypothesis that the market portfolio is efficient. The p-value, $\tilde{p} = 0.514$ is above the significance level of 5%. We find that the p-value for the first sub-period is $\tilde{p}_1 = 0.631$ and the p-value for the second sub-period is $\tilde{p}_2 = 0.597$.

As before, we find that the MSD criterion cannot be accepted. The p-value, $\tilde{p} = 0.043$ is below the significance level of 5%. Additionally, the p-value $\tilde{p}_1 = 0.029$ for the first sub-period and the p-value $\tilde{p}_2 = 0.046$ for the second.

The 49 Industry portfolios. We cannot reject the hypothesis that the market portfolio is prospect stochastic dominance efficient. The p-value is $\tilde{p} = 0.623$. We find that the p-value for the first sub-period is $\tilde{p}_1 = 0.517$ and the p-value for the second sub-period is $\tilde{p}_2 = 0.679$.

Finally, we find that the MSD criterion cannot be accepted. The p-value is $\tilde{p} = 0.034$, which is below the significance level of 5%. Additionally, the p-value $\tilde{p}_1 = 0.031$ for the first sub-period and p-value $\tilde{p}_2 = 0.064$ for the second sub-period.

The results provide evidence in favor of the claim that the market portfolio is prospect stochastic dominance efficient in each sub-period as well as in the whole period. If this holds, it implies that any S-shaped utility function rationalizes the market portfolio as an optimal choice. On the other hand, in all cases the market portfolio is not Markowitz stochastic dominance efficient in each sub-period as well as in the full period. If this holds, *it does not imply* that no reverse S-shaped utility function can rationalize the market portfolio, but only the existence of at least one such function that fails to do so.

Experimental evidence suggests that decision makers subjectively transform the true return distribution and use subjective decision weights that overweight or underweight the true probabilities. The most common pattern of probability transformation overweights small probabilities of large gains and losses, and underweights large and intermediate probabilities of small and intermediate gains and losses (Tversky and Kahneman, (1992)). The prospect stochastic dominance efficiency of the market portfolio we found here, is not affected by transformations that are increasing and convex over losses and increasing and concave over gains, that is, S-shaped transformations. Moreover, if the market portfolio is non dominated w.r.t. PSD, then it is also non dominated w.r.t. the weaker condition given by Baucells and Heukamp (2006).

4.3 Rolling window analysis

We carry out an additional analysis to validate the prospect stochastic dominance efficiency of the market portfolio and the stability of the model results. It is possible that the efficiency of the market portfolio as a weighted average varies over time due to changes in the weights constructing it from the universe of assets. [9] Furthermore the temporal extend of our sample could imply the non validity of the stationarity assumption due to possible changes in the DGP. To account for the above, we perform a rolling window analysis, using a window width of 20 years. The test statistic is calculated separately for 63 overlapping 20-year periods, (January 1930-December 1949), (January 1931-December 1951),...,(January 1993-December 2012). The time series in this case is smaller (240 monthly observations) so that a maintained assumption of stationarity is more credible.

Figure 1 shows the corresponding p-values for the prospect stochastic dominance efficiency test (upper graph) and for the Markowitz stochastic dominance efficiency test (lower graph) using the 6 FF benchmark portfolios. We observe that the market portfolio is prospect stochastic dominance efficient in the total sample period. The prospect stochastic dominance efficiency is not rejected on any subsample. The p-values are always greater than 25%, and in some cases they reach the 70%. This result is in consonance to that prospect stochastic dominance efficiency that was not rejected in the previous subsection, for the full period. On the other hand, we observe that the Markowitz stochastic dominance efficiency is rejected on 46 out of 63 subsamples. The p-values are most of the cases lower than 5%. This result is in accordance with the rejection of the Markowitz stochastic dominance efficiency that was found in the previous subsection. If this is true, it implies that for those subsamples there exist portfolios constructed from the set of the six benchmark portfolios that dominates the market portfolio w.r.t. at least one reverse S-shaped utility function.

Figure 2 shows the corresponding p-values for the prospect stochastic dominance efficiency test (upper graph) and for the Markowitz stochastic dominance efficiency test (lower graph) using the 49 Industry portfolios. Interestingly, we observe that the market portfolio is prospect stochastic dominance efficient in the total sample period. Again, the prospect stochastic dominance efficiency is not rejected on any subsample. The p-values are always greater than

[9] It is also possible that the degree of efficiency may change over time, as pointed by Post (2003).

17%, and in some cases they reach the 55%. This result is in consonance to that prospect stochastic dominance efficiency that was not rejected in the previous subsection. On the other hand, we observe that the Markowitz stochastic dominance efficiency is rejected on 50 out of 63 subsamples. The p-values are most of the cases lower than 5%. This result is in accordance with the rejection of the Markowitz stochastic dominance efficiency that was found in the previous subsection. It implies that for those subsamples there exist portfolios constructed from the set of the 49 Industry portfolios that dominates the market portfolio w.r.t. at least one reverse S-shaped utility function.

5 Concluding remarks

In this paper we develop *consistent* and feasible statistical tests for prospect and Markowitz stochastic dominance efficiency for *time-dependent* data. We use a block bootstrap formulation to achieve valid asymptotic inference in a setting of temporal dependence. Linear as well as mixed integer programs are used to compute the test statistics. The PSD and MSD criteria allow us to test the prospect theory S-shaped value function hypothesis and the Markowitz reverse S-shaped hypothesis in a framework where full diversification is allowed.

To illustrate the potential of the proposed test statistics, we test whether the two stochastic dominance efficiency criteria rationalize the Fama and French market portfolio over three different data sets of Fama and French benchmark portfolios constructed as the intersections of ME portfolios and BE/ME portfolios, as well as over 49 Industry portfolios. Empirical results support the claim that the market portfolio is prospect stochastic dominance efficient. Analogously they are not in favor of the claim that the market portfolio is Markowitz stochastic dominance efficient, indicating that utility functions with global risk aversion for losses and risk seeking over gains cannot rationalize the market portfolio.

Both the asymptotic analysis and the numerical implementation revealed an interesting asymmetry between the two testing procedures. The MSD test uses stricter assumptions and is carried out with more computational intensity. A possibly interesting future research could focus on whether those asymmetries could be weakened.

These tests could possibly be used as initial steps for the statistical decoupling of the form of the utility or value function to the transformation of the probability measures that characterize many theories of choice under uncertainty. For example non rejection of the MSD efficiency using the previous methodology could support the validity of cumulative prospect theory when the curvature of the S-shaped utility is dominated by the reverse S-shaped probability transformation (see Post and Levy (2005)) as this theory suggests. The construction of inferential procedures that statistically disentangle the two could be of importance and is delegated to future research.

Finally, the methodology used could be also relevant for the construction of tests of efficiency w.r.t. notions of stochastic dominance that are representable by utility functions with more complex behavior (e.g. attitudes towards risks may exhibit additional changes on extreme events). Such considerations are also delegated to future research.

APPENDIX

Numerical Implementation

In this section we present the numerical implementation of the feasible testing procedures for PSD and MSD efficiency. Remember that, as noted above, both the computation of the statistics appearing in any of the tests, as well as the approximation of the asymptotic critical or p-values are analytically infeasible and thereby performed numerically.

In what follows, we describe a procedure applicable for the computation of every statistic in every testing procedure. This is essentially a reduction that involves a sequence of equivalent (w.r.t. optimization) problems one for each statistic. The transition from the previous to the next problem in each case involves numerical simplification and is essentially based on results similar to proposition 7 that appears below. A completely analogous procedure is used for the approximation of the critical values but it is by construction more tediously describable and thereby omitted to economize on space.

We also assume that $\mathbb{L}$ is actually convex in order to facilitate the presentation. The formulation is easily generalized to the cases covered by Assumption C since in that case the parameter space is a finite union of convex sets.

Let T denote the sample size. Denote with T_p both an indexing set and its cardinality, representing the part of the sample for which $\boldsymbol{\tau}'\mathbf{Y}_t$ is positive. Given this convention $T_n = T-$

T_p denotes the analogous notion for the part of the sample of strictly negative observations.[10]

Formulation for prospect stochastic dominance

Given the previous notation, the statistic[11] $\hat{S}_T^{\alpha}(\boldsymbol{\tau})$ is for a fixed T, equivalent to

$$\sup_{z\geq 0, \boldsymbol{\lambda}\in\mathbb{L}} \sum\nolimits_{t=1}^{T_p} \left((z - \boldsymbol{\tau}'\boldsymbol{Y}_t)_+ - (z - \boldsymbol{\lambda}'\boldsymbol{Y}_t)_+ \mathbb{I}_{0\leq \boldsymbol{\lambda}'\boldsymbol{Y}_t} \right).$$

An equivalent (see below) programming formulation is the following:

$$\max_{\mathbf{z}\geq 0, \boldsymbol{\lambda}\in\mathbb{L}} \quad \frac{1}{\sqrt{T}} \sum_{t=1}^{T_p} (L_t - W_t) \tag{10a}$$

$$\text{s.t.} \quad M(F_t - 1) \leq z - \boldsymbol{\tau}'\mathbf{Y}_t \leq MF_t, \qquad \forall t \in T_p, \tag{10b}$$

$$-M(1 - F_t) \leq L_t - (z - \boldsymbol{\tau}'\mathbf{Y}_t) \leq M(1 - F_t), \qquad \forall t \in T_p, \tag{10c}$$

$$-MF_t \leq L_t \leq MF_t, \qquad \forall t \in T_p, \tag{10d}$$

$$W_t \geq z - \boldsymbol{\lambda}'\mathbf{Y}_t, \qquad \forall t \in T_p, \tag{10e}$$

$$\boldsymbol{\tau}'\mathbf{Y}_t \geq 0, \qquad \forall t \in T_p, \tag{10f}$$

$$\boldsymbol{\lambda}'\mathbf{Y}_t \geq 0, \qquad \forall t \in T_p, \tag{10g}$$

$$W_t \geq 0,\ F_t \in \{0,1\}, \qquad \forall t \in T_p. \tag{10h}$$

with M being a large constant that is allowed to depend on z.

This is a mixed integer programming formulation maximizing the distance between the sum over all scenarios of two variables, $\sum_{t=1}^{T_p} L_t$ and $\sum_{t=1}^{T_p} W_t$ which represent the difference between $(z - \boldsymbol{\tau}'\boldsymbol{Y}_t)_+$ and $(z - \boldsymbol{\lambda}'\boldsymbol{Y}_t)_+$ respectively. This is difficult to solve since it is the maximization of the difference of two convex functions. We use a binary variable F_t, which, according to inequalities (10b), equals 1 for each scenario $t \in T_p$ for which $z \geq \boldsymbol{\tau}'\mathbf{Y}_t$, and 0 otherwise. Then, inequalities (10c) and (10d) ensure that the variable L_t equals $z - \boldsymbol{\tau}'\mathbf{Y}_t$ for the scenarios for which this difference is positive, and 0 for all the other scenarios. Inequalities (10e) and (10h) ensure that *the optimal* W_t equals exactly the difference $z - \boldsymbol{\lambda}'\mathbf{Y}_t$ for the

[10] Notice that T_p would remain unaltered when defined on a bootstrapped sample due to the non-overlapping scheme considered here.

[11] or the analogous one that appears in the resampling procedure for the approximation of the asymptotic critical values.

scenarios for which this difference is positive, and 0 otherwise. Inequalities (10f) and (10g) ensure that both $\boldsymbol{\tau}'\mathbf{Y}_t$ and $\boldsymbol{\lambda}'\mathbf{Y}_t$ are greater than zero.

The model is easily transformed to a linear one, which is also very easy to solve. The key lies in the following proposition.

Proposition 7. *The optimal value of z belongs to the finite set $\mathcal{R} = \{0, r_1, r_2, ..., r_{T_p^*}\}$ that is containing zero and the mutually different strictly positive realizations of the benchmark portfolio ordered increasingly.*

A direct consequence is that we can reduce the solution of the previous problem to the solution of T_{p^*} smaller problems $P(r)$, $r \in \mathcal{R}$, in which z is fixed to r. Then we take take the value for z that yields the best total result. The advantage is that the optimal values of the L_t variables are known in $P(r)$. Precisely, $\sum_{t=1,...,T_p} L_t$ is equal to the number of t such that $\boldsymbol{\tau}'\mathbf{Y}_t \leq r$. Hence problem $P(r)$ boils down to the following simpler problem.

$$
\begin{aligned}
\max_{\mathbf{z}\geq 0, \boldsymbol{\lambda}\in\mathbb{L}} \quad & \frac{1}{\sqrt{T}}\sum_{t=1}^{T_p}(L_t - W_t) \\
\texttt{s.t.} \quad & W_t \geq r - \boldsymbol{\lambda}'\mathbf{Y}_t, \qquad \forall t \in T_p \\
& L_t = (r - \boldsymbol{\tau}'\mathbf{Y}_t)_+, \qquad \forall t \in T_p \\
& \boldsymbol{\lambda}'\mathbf{Y}_t \geq 0, \qquad \forall\, t \in T_p, \\
& W_t \geq 0, \qquad \forall\, t \in T_p.
\end{aligned}
$$

The constant M and the binary variable F_t do not appear in any of those problems. Hence the previous is reduced to the solution of a finite number of linear programs. The optimal portfolio $\boldsymbol{\lambda}$ and the optimal value r of variable z are those that give the maximum objective value. Something analogous (given the relevant reformulations) is also true for $\hat{S}_T^{\beta}$ and thereby the analogous formulation is not presented for reasons of economy of space.

Formulation for Markowitz stochastic dominance

As previously mentioned, the numerical formulation for the statistic $\hat{\Upsilon}_T^{\alpha}(\boldsymbol{\tau})$ can be also reduced to the solution of a finite number of linear programming problems via the use of an analogous formulation as above or via the results for the mathematical implementation of the SSD test in Scaillet and Topaloglou (2010) and thereby the relevant details are also omitted. This seizes to be true for the final statistic in the MSD test. $\hat{\Upsilon}_T^{\beta}(\boldsymbol{\tau})$ can be equivalently written as

$$\sup_{z\geq 0,\boldsymbol{\lambda}\in\mathbb{L}}\left[\sum_{t=1}^{T_p}((\boldsymbol{\lambda}'Y_t)_+ - (\boldsymbol{\tau}'\boldsymbol{Y}_t)) + \sum_{t=1}^{T_p}\left((z-\boldsymbol{\lambda}'\boldsymbol{Y}_t)_+\mathbb{I}_{\boldsymbol{\lambda}'\boldsymbol{Y}_t\geq 0}\right) - (z-\boldsymbol{\tau}'\boldsymbol{Y}_i)_+\right].$$

An equivalent programming formulation is the following:

$$\max_{\mathbf{z}\geq 0,\boldsymbol{\lambda}\in\mathbb{L}} \quad \frac{1}{\sqrt{T}}[\sum_{t=1}^{T}(\boldsymbol{\lambda}'\boldsymbol{Y}_t - \boldsymbol{\tau}'\boldsymbol{Y}_t) + \sum_{t=1}^{T_p}(L_t - W_t)] \tag{12a}$$

$$\text{s.t.} \quad M(F_t-1)\leq z-\boldsymbol{\lambda}'\boldsymbol{Y}_t\leq MF_t, \qquad \forall t\in T_p, \tag{12b}$$

$$-M(1-F_t)\leq L_t-(z-\boldsymbol{\lambda}'\boldsymbol{Y}_t)\leq M(1-F_t), \qquad \forall t\in T_p, \tag{12c}$$

$$-MF_t\leq L_t\leq MF_t, \qquad \forall t\in T_p, \tag{12d}$$

$$W_t\geq z-\boldsymbol{\tau}'\boldsymbol{Y}_t, \qquad \forall t\in T_p, \tag{12e}$$

$$\boldsymbol{\tau}'\boldsymbol{Y}_t\geq 0, \qquad \forall t\in T_p, \tag{12f}$$

$$\boldsymbol{\lambda}'\boldsymbol{Y}_t\geq 0, \qquad \forall t\in T_p, \tag{12g}$$

$$\boldsymbol{\lambda}\geq 0, \tag{12h}$$

$$W_t\geq 0, \qquad \forall t\in T_p, \tag{12i}$$

$$F_t\in\{0,1\}, \qquad \forall t\in T_p. \tag{12j}$$

M is again a large constant that may also depend on $\boldsymbol{\lambda}$. Given that this assumes its values in a compact set, M can be chosen large enough to avoid this dependence. This is again a mixed integer programming formulation. Now we use a binary variable F_t, which, according to inequalities (12b), equals 1 for each scenario $t\in T_p$ for which $z\geq\boldsymbol{\lambda}'\boldsymbol{Y}_t$, and 0 otherwise. Then, inequalities (12c) and (12d) ensure that the variable L_t equals $z-\boldsymbol{\lambda}'\boldsymbol{Y}_t$ for the scenarios for which this difference is positive, and 0 for all the other scenarios. Inequalities (12e) and (12i) ensure that the optimal W_t equals exactly the difference $z-\boldsymbol{\tau}'\boldsymbol{Y}_t$ for the scenarios for which this difference is positive, and 0 otherwise. Inequalities (12f) and (12g) ensure that both $\boldsymbol{\tau}'\boldsymbol{Y}_t$ and $\boldsymbol{\lambda}'\boldsymbol{Y}_t$ are greater than zero.

As before this optimization program can be reduced to the solution of a finite number of smaller problems $P(r)$, $r\in\mathcal{R}$, in which z is fixed to r, and the above problem boils down to the following problem.

$$
\begin{aligned}
\max_{\mathbf{z}\geq 0,\boldsymbol{\lambda}\in\mathbb{L}} \quad & \frac{1}{\sqrt{T}}[\sum_{t=1}^{T}(\boldsymbol{\lambda}'\mathbf{Y}_t-\boldsymbol{\tau}'\mathbf{Y}_t)+\sum_{t=1}^{T_p}(L_t-W_t)] \\
\text{s.t.} \quad & M(F_t-1)\leq r-\boldsymbol{\lambda}'\mathbf{Y}_t\leq MF_t, \qquad \forall\, t\in T_p, \\
& -M(1-F_t)\leq L_t-(r-\boldsymbol{\lambda}'\mathbf{Y}_t)\leq M(1-F_t), \qquad \forall\, t\in T_p, \\
& -MF_t\leq L_t\leq MF_t, \qquad \forall\, t\in T_p, \\
& W_t=(r-\boldsymbol{\tau}'\mathbf{Y})_+, \qquad \forall t\in T_p \\
& \boldsymbol{\lambda}'\mathbf{Y}_t\geq 0, \qquad \forall\, t\in T_p, \\
& F_t\in\{0,1\}, \qquad \forall\, t\in T_p.
\end{aligned} \tag{13a}
$$

Due to the dependence of L_t on $\boldsymbol{\lambda}$ the smaller problems depend also on the binary variable. Hence each one of them is a Mixed Integer program. It usually takes significantly multiple time for the solution of each such a problem compared to the ones before. Even though the analytical formulations of PSD and MSD can be derived by particular combinations of the SSD and its dual, both the strength of the assumptions utilized as well as the complexity of the numerical implementations imply an asymmetry between the two notions.

Finally, notice that in every case the practical implementation of any test using R bootstrap samples involves $2(R+1)$ internal numerical optimizations and $R+1$ trivial ones. Hence, the usual trade off between possibly desirable higher order properties and numerical burden is obviously present in our considerations.

Helpful Lemmata and Proofs

In what follows $\rightsquigarrow$ denotes weak convergence and $\overset{p}{\rightsquigarrow}$ (conditional) weak convergence in probability (see among others Paragraph 3.6.1 of van der Vaart and Wellner (1996)). Analogously $\overset{p}{\to}$ denotes convergence in probability. CMT abbreviates the continuous mapping theorem in the relevant context.

Lemma 8. *If $\mathbb{E}^*\left(\boldsymbol{\lambda}'\boldsymbol{Y}_0\right)_+$ and $\mathbb{E}^*\left(\boldsymbol{\tau}'\boldsymbol{Y}_0\right)_+$ exist then $\mathcal{J}_2^c\left(z,\boldsymbol{\lambda},\boldsymbol{\tau},F\right)$ exists and equals*

$$\mathbb{E}^*\left(\boldsymbol{\tau}'\boldsymbol{Y}_0\right)_+-\mathbb{E}^*\left(\boldsymbol{\lambda}'\boldsymbol{Y}_0\right)_+-\mathcal{J}\left(z,\boldsymbol{\lambda},\boldsymbol{\tau},F^*\right).$$

Proof. Remember that the distribution function of $(\boldsymbol{\lambda}'\boldsymbol{Y}_0)_+$ is $\begin{cases} 0,\ u<0 \\ G(u,\boldsymbol{\lambda},F^*),\ u\geq 0 \end{cases}$ and thereby iff $\mathbb{E}^*(\boldsymbol{\lambda}'\boldsymbol{Y}_0)_+ < +\infty$ then we have that $\mathbb{E}^*(\boldsymbol{\lambda}'\boldsymbol{Y}_0)_+ = \int_0^{+\infty}(1-G(u,\boldsymbol{\lambda},F^*))\,du$ and therefore

$$\begin{aligned}
&\int_z^{+\infty}(G(u,\boldsymbol{\lambda},F^*)-G(u,\boldsymbol{\tau},F^*))\,du\\
=\ &\int_z^{+\infty}(1-G(u,\boldsymbol{\tau},F^*))-(1-G(u,\boldsymbol{\lambda},F^*))\,du\\
=\ &\mathbb{E}^*(\boldsymbol{\tau}'\boldsymbol{Y}_0)_+-\mathbb{E}^*(\boldsymbol{\lambda}'\boldsymbol{Y}_0)_+ + \int_0^z G(u,\boldsymbol{\tau},F^*)\,du-\int_0^z G(u,\boldsymbol{\lambda},F^*)\,du.
\end{aligned}$$

■

In the following let

$$x_T=\begin{pmatrix}\sqrt{T}\sup_{z\geq 0,\boldsymbol{\lambda}\in\mathbb{L}}\left(\mathcal{J}\left(z,-\boldsymbol{\lambda},-\boldsymbol{\tau},\hat{F}_T\right)-\mathcal{J}(z,-\boldsymbol{\lambda},-\boldsymbol{\tau},F)\right)\\ \sqrt{T}\sup_{z\geq 0,\boldsymbol{\lambda}\in\mathbb{L}}\left(\mathcal{J}(z,\boldsymbol{\lambda},\boldsymbol{\tau},F)-\mathcal{J}\left(z,\boldsymbol{\lambda},\boldsymbol{\tau},\hat{F}_T\right)\right)\end{pmatrix},$$

and

$$y_T=\begin{pmatrix}\sqrt{T}\sup_{z\leq 0,\boldsymbol{\lambda}\in\mathbb{L}}\left(\mathcal{D}_2\left(z,\boldsymbol{\tau},\boldsymbol{\lambda},\hat{F}_T\right)-\mathcal{D}_2(z,\boldsymbol{\tau},\boldsymbol{\lambda},F)\right)\\ \sup_{z\geq 0,\boldsymbol{\lambda}\in\mathbb{L}}\left[\kappa_T(\boldsymbol{\lambda},\boldsymbol{\tau})-\sqrt{T}\left(\mathcal{J}(z,\boldsymbol{\lambda},\boldsymbol{\tau},F)-\mathcal{J}\left(z,\boldsymbol{\lambda},\boldsymbol{\tau},\hat{F}_T\right)\right)\right]\end{pmatrix},$$

where

$$\mathcal{D}_2(z,\boldsymbol{\tau},\boldsymbol{\lambda},F^*)=\mathcal{J}_2(z,\boldsymbol{\tau},F^*)-\mathcal{J}_2(z,\boldsymbol{\lambda},F^*),$$

and

$$\kappa_T(\boldsymbol{\lambda},\boldsymbol{\tau})=\frac{1}{\sqrt{T}}\sum_{i=1}^T\left(\left((\boldsymbol{\lambda}'\boldsymbol{Y}_i)_+-(\boldsymbol{\tau}'\boldsymbol{Y}_i)_+\right)-\left(\mathbb{E}(\boldsymbol{\lambda}'\boldsymbol{Y}_0)_+-\mathbb{E}(\boldsymbol{\tau}'\boldsymbol{Y}_0)_+\right)\right).$$

Lemma 9. *1. Suppose that for any $\boldsymbol{\lambda}\in\mathbb{L}$, G satisfies the condition (5) and that assumption A holds. Then as $T\to\infty$*

$$x_T\rightsquigarrow\begin{pmatrix}\sup_{z\geq 0,\boldsymbol{\lambda}\in\mathbb{L}}\mathcal{J}(z,-\boldsymbol{\lambda},-\boldsymbol{\tau},\mathcal{B}_F)\\ \sup_{z\geq 0,\boldsymbol{\lambda}\in\mathbb{L}}-\mathcal{J}(z,\boldsymbol{\lambda},\boldsymbol{\tau},\mathcal{B}_F)\end{pmatrix}$$

where $\mathcal{B}_F$ denotes a zero mean Gaussian process in the Skorokhod space of real valued functions on $\mathbb{R}^n$ with uniformly continuous paths and covariance kernel given by Cov$(\mathcal{B}_F(x),\mathcal{B}_F(y))=\sum_{t\in\mathbb{Z}}$ Cov$(1(X_0\leq x,1(X_t\leq y)$.

2. *Suppose that for any $\boldsymbol{\lambda} \in \mathbb{L}$, G satisfies the condition (7) and that assumption B holds. Then as $T \to \infty$*

$$y_T \rightsquigarrow \begin{pmatrix} \sup_{z\leq 0, \boldsymbol{\lambda}\in\mathbb{L}} \mathcal{D}_2\left(z, \boldsymbol{\tau}, \boldsymbol{\lambda}, \mathcal{B}_F\right) \\ \sup_{z\geq 0, \boldsymbol{\lambda}\in\mathbb{L}} -\mathcal{J}_2^c\left(z, \boldsymbol{\lambda}, \mathcal{B}_F\right) \end{pmatrix}$$

where $\mathcal{B}_F$ is as before.

Proof. Notice first that due to assumption A (which holds in both frameworks) we have that

$$\sqrt{T}\left(\hat{F}_T - F\right) \rightsquigarrow \mathcal{B}_F$$

in the Skorokhod space of real valued functions on $\mathbb{R}^n$ (see e.g. Theorem 7.3 of Rio (2013)). Assumption A along with a trivial extension of Proposition 2.2 of Scaillet and Topaloglou (2010) (which evolves along the lines of the proof of Theorem 1 of Horvath, Kokoszka, and Zitikis (2006)) and relations (1), (2) imply that $\mathcal{J}_2\left(z, \boldsymbol{\lambda}, \cdot\right)$ is linear and continuous for any $z \in \mathbb{R}$, $\boldsymbol{\lambda} \in \mathbb{L} \cup -\mathbb{L}$ tangentially at $\mathbb{D}_0 \triangleq \left\{\sqrt{T}\left(\hat{F}_T - F\right), \mathcal{B}\circ F, T = 1, 2, \ldots\right\}$ due to the results of Theorem 1 of Horvath, Kokoszka, and Zitikis (2006). Thereby Theorem 20.8 of van der Vaart (functional Delta method) implies that for any $z \in \mathbb{R}$, $\boldsymbol{\lambda} \in \mathbb{L} \cup -\mathbb{L}$

$$\sqrt{T}\left(\mathcal{J}_2\left(z, \boldsymbol{\lambda}, \hat{F}_T\right) - \mathcal{J}_2\left(z, \boldsymbol{\lambda}, F\right)\right) \rightsquigarrow \mathcal{J}_2\left(z, \boldsymbol{\lambda}, \mathcal{B}_F\right),$$

which is a Gaussian process due to linearity.

1. The previous along Assumption A, the relation (3) and the CMT imply that for any $(z, \boldsymbol{\lambda}) \in \mathbb{R}^+ \times \mathbb{L}$,

$$\sqrt{T}\begin{pmatrix} \mathcal{J}\left(z, -\boldsymbol{\lambda}, -\boldsymbol{\tau}, \hat{F}_T\right) - \mathcal{J}\left(z, -\boldsymbol{\lambda}, -\boldsymbol{\tau}, F\right) \\ \mathcal{J}\left(z, \boldsymbol{\lambda}, \boldsymbol{\tau}, F\right) - \mathcal{J}\left(z, \boldsymbol{\lambda}, \boldsymbol{\tau}, \hat{F}_T\right) \end{pmatrix} \rightsquigarrow \begin{pmatrix} \mathcal{J}\left(z, -\boldsymbol{\lambda}, -\boldsymbol{\tau}, \mathcal{B}\circ F\right) \\ -\mathcal{J}\left(z, \boldsymbol{\lambda}, \boldsymbol{\tau}, \mathcal{B}\circ F\right) \end{pmatrix},$$

which is a well defined Gaussian process due to linearity. The result would follow from the CMT along with Theorem 1.4.8 of van der Vaart and Wellner (1996) if $\sup_{z\in\mathbb{R}, \boldsymbol{\lambda}\in\mathbb{L}\cup-\mathbb{L}} \mathcal{J}\left(z, \boldsymbol{\lambda}, -\boldsymbol{\tau}, F_n^*\right)$ is (asymptotically) tight for any sequence (F_n^*) of members of $\mathbb{D}_0$. The fact that (asymptotic) tightness is preserved by continuous transformations and it is equivalent to tightness of the normed sequence, the triangle inequality along with the results of Proposition 2.2 of Scaillet and Topaloglou (2010), and the representation of $\mathcal{J}$ in terms of $\mathcal{J}_2$ in (3) imply the result.

2. Assumption B, and a trivial application of the multivariate functional central limit theorem for stationary strongly mixing sequences stated in Rio (2000) and the fact that the transformation $(\boldsymbol{\lambda}'\cdot)_+$ is measurable and independent of T imply that for any $\boldsymbol{\lambda} \in \mathbb{L}$, $\frac{1}{\sqrt{T}}\sum_{i=1}^{T}\left((\boldsymbol{\lambda}'\boldsymbol{Y}_i)_+ - \mathbb{E}\left(\boldsymbol{\lambda}'\boldsymbol{Y}_i\right)_+\right)$ converges weakly to a Gaussian random variable. This along with the representation of $\mathcal{J}_2^c$ in lemma 8 and Theorem 1 of Horvath, Kokoszka, and Zitikis (2006) imply that $\begin{pmatrix} \mathcal{D}_2\left(z, \boldsymbol{\tau}, \boldsymbol{\lambda}, \cdot\right) \\ -\mathcal{J}_2^c\left(z, \boldsymbol{\lambda}, \cdot\right) \end{pmatrix}$ is linear and continuous for all $z, \boldsymbol{\lambda}, \boldsymbol{\tau}$ tangentially at $\mathbb{D}_0$. Hence Theorem 20.8 of van der Vaart (functional Delta method) implies that for any $z \in \mathbb{R}$, $\boldsymbol{\lambda} \in \mathbb{L}$

$$\sqrt{T}\begin{pmatrix} \mathcal{D}_2\left(z, \boldsymbol{\tau}, \boldsymbol{\lambda}, \hat{F}_T\right) - \mathcal{D}_2\left(z, \boldsymbol{\tau}, \boldsymbol{\lambda}, F\right) \\ \mathcal{J}_2^c\left(z, \boldsymbol{\lambda}, \boldsymbol{\tau}, F\right) - \mathcal{J}_2^c\left(z, \boldsymbol{\lambda}, \boldsymbol{\tau}, \hat{F}_T\right) \end{pmatrix} \rightsquigarrow \begin{pmatrix} \mathcal{D}_2\left(z, \boldsymbol{\tau}, \boldsymbol{\lambda}, \mathcal{B}\circ F\right) \\ -\mathcal{J}_2^c\left(z, \boldsymbol{\lambda}, \mathcal{B}_F\right) \end{pmatrix},$$

which is a well defined Gaussian process due to linearity. Likewise to the previous case the result would follow from the CMT along with Theorem 1.4.8 of van der Vaart and Wellner (1996) if $\sup_{z\in\mathbb{R},\boldsymbol{\lambda}\in\mathbb{L}} \mathcal{D}_2\left(z, \boldsymbol{\lambda}, -\boldsymbol{\tau}, F_n^*\right)$ and $\sup_{z\in\mathbb{R},\boldsymbol{\lambda}\in\mathbb{L}} \mathcal{J}_2^c\left(z, \boldsymbol{\lambda}, -\boldsymbol{\tau}, F_n^*\right)$ are $\sup_{z\in\mathbb{R},\boldsymbol{\lambda}\in\mathbb{L}\cup-\mathbb{L}} \mathcal{J}\left(z, \boldsymbol{\lambda}, -\boldsymbol{\tau}, F_n^*\right)$ is (asymptotically) tight for any sequence (F_n^*) of members of $\mathbb{D}_0$. This follows directly from the results of Proposition 2.2 of Scaillet and Topaloglou (2010), the analogous results of the previous part and the compactness of $\mathbb{L}$, the Lipschitz continuity of $\boldsymbol{\lambda} \to (\boldsymbol{\lambda}'\boldsymbol{Y}_0)_+$, the analogous continuity of $\boldsymbol{\lambda} \to \mathbb{E}\left(\boldsymbol{\lambda}'\boldsymbol{Y}_0\right)_+$ due to assumption B and dominated convergence, a trivial application of Markov's inequality and Yokoyama's (1980) strong mixing inequality for $p = 2$, imply the asymptotic equicontinuity in probability for $\frac{1}{\sqrt{T}}\sum_{i=1}^{T}\left((\boldsymbol{\lambda}'\boldsymbol{Y}_i)_+ - \mathbb{E}\left(\boldsymbol{\lambda}'\boldsymbol{Y}_i\right)_+\right)$ w.r.t. $\boldsymbol{\lambda}$, since $\mathbb{L} \cup -\mathbb{L}$ is compact and remains so w.r.t. any weaker topology. This also implies that $\frac{1}{\sqrt{T}}\sum_{i=1}^{T}\left((\boldsymbol{\lambda}'\boldsymbol{Y}_i)_+ - \mathbb{E}\left(\boldsymbol{\lambda}'\boldsymbol{Y}_i\right)_+\right)$ converges to a Gaussian process in the space of continuous real functions on $\mathbb{L} \cup -\mathbb{L}$ which means that the supremum w.r.t. $\boldsymbol{\lambda}$ of the limit is also tight. This implies tightness for $\sup_{z\in\mathbb{R},\boldsymbol{\lambda}\in\mathbb{L}} -\mathcal{J}_2^c\left(z, \boldsymbol{\lambda}, \mathcal{B}_F\right)$. ■

Proof of Proposition 1. First notice that

$$\begin{aligned}
\sup_{z\geq 0,\boldsymbol{\lambda}\in\mathbb{L}} \mathcal{J}\left(z, -\boldsymbol{\lambda}, -\boldsymbol{\tau}, F^*\right) &= \sup_{z\geq 0,\boldsymbol{\lambda}\in\mathbb{L}} \left(\int_z^0 G(u, -\boldsymbol{\tau}, F^*)du - \int_z^0 G(u, -\boldsymbol{\lambda}, F^*)du\right) \\
&= \sup_{z\geq 0,\boldsymbol{\lambda}\in\mathbb{L}} \left(\int_0^z G(u, -\boldsymbol{\lambda}, F^*)du - \int_0^z G(u, -\boldsymbol{\tau}, F^*)du\right) \\
&= \sup_{z\leq 0,\boldsymbol{\lambda}\in\mathbb{L}} \left(\int_{\mathbb{R}^n} \left((z - \boldsymbol{\tau}'\boldsymbol{Y}_i)\,\mathbb{I}_{z\leq \boldsymbol{\tau}'\boldsymbol{Y}_i\leq 0} - (z - \boldsymbol{\lambda}'\boldsymbol{Y}_i)\,\mathbb{I}_{z\leq \boldsymbol{\lambda}'\boldsymbol{Y}_i\leq 0}\right) dF^*\left(\boldsymbol{u}\right)\right) \\
&= \sup_{z\leq 0,\boldsymbol{\lambda}\in\mathbb{L}} \mathcal{J}\left(z, \boldsymbol{\lambda}, \boldsymbol{\tau}, F^*\right).
\end{aligned}$$

Due to the previous

$$\hat{S}_T^a(\boldsymbol{\tau}) = \sqrt{T} \sup_{z\geq 0,\boldsymbol{\lambda}\in\mathbb{L}} \left(\mathcal{J}\left(z,-\boldsymbol{\lambda},-\boldsymbol{\tau},\hat{F}_T\right) - \mathcal{J}\left(z,-\boldsymbol{\lambda},-\boldsymbol{\tau},F\right) + \mathcal{J}\left(z,-\boldsymbol{\lambda},-\boldsymbol{\tau},F\right)\right) \tag{14}$$

which is less than or equal to

$$x_{1,T} + \sup_{z\leq 0,\boldsymbol{\lambda}\in\mathbb{L}} \mathcal{J}\left(z,\boldsymbol{\lambda},\boldsymbol{\tau},F\right).$$

and similarly

$$\hat{S}_T^\beta(\boldsymbol{\tau}) = \sqrt{T} \sup_{z\geq 0,\boldsymbol{\lambda}\in\mathbb{L}} \left(\mathcal{J}\left(z,\boldsymbol{\lambda},\boldsymbol{\tau},F\right) - \mathcal{J}\left(z,\boldsymbol{\lambda},\boldsymbol{\tau},\hat{F}_T\right) - \mathcal{J}\left(z,\boldsymbol{\lambda},\boldsymbol{\tau}F\right)\right) \tag{15}$$

which is analogously less than or equal to

$$x_{2,T} + \sup_{z\geq 0,\boldsymbol{\lambda}\in\mathbb{L}} -\sqrt{T}\mathcal{J}\left(z,\boldsymbol{\lambda},\boldsymbol{\tau},F\right).$$

1. If $\mathbb{H}_0^{(\mathcal{P})}$ holds then the previous imply that

$$\hat{S}_T^a \leq x_{1,T} \text{ and } \hat{S}_T^\beta(\boldsymbol{\tau}) \leq x_{2,T}$$

and thereby

$$\hat{S}_T(\boldsymbol{\tau}) \leq \max\left(x_{1,T}, x_{2,T}\right)$$

and due to the CMT and lemma 9.**1** $\max\left(x_{1,T}, x_{2,T}\right)$ converges in distribution to

$$\bar{S}(\boldsymbol{\tau}) \doteqdot \max\begin{pmatrix} \sup_{z\leq 0,\boldsymbol{\lambda}\in\mathbb{L}} \mathcal{J}\left(z,\boldsymbol{\lambda},\boldsymbol{\tau},\mathcal{B}_F\right) \\ \sup_{z\geq 0,\boldsymbol{\lambda}\in\mathbb{L}} -\mathcal{J}\left(z,\boldsymbol{\lambda},\boldsymbol{\tau},\mathcal{B}_F\right)\end{pmatrix}',$$

and the result follows. Furthermore since the $(a,b) \to \max(\sup(a),\sup(b))$ function when well defined on the product of some space of real functions is convex, Corollary 4.4.2.(i)-(ii) of Bogachev (1991) implies that the cdf of the law of $\max\begin{pmatrix} \sup_{z\geq 0,\boldsymbol{\lambda}\in\mathbb{L}} \mathcal{J}\left(z,-\boldsymbol{\lambda},-\boldsymbol{\tau},\mathcal{B}\circ F\right) \\ \sup_{z\geq 0,\boldsymbol{\lambda}\in\mathbb{L}} -\mathcal{J}\left(z,\boldsymbol{\lambda},\boldsymbol{\tau},\mathcal{B}\circ F\right)\end{pmatrix}$ restricted to $(c,+\infty)$ is absolutely continuous with a possible atom on some $c\in\mathbb{R}$ defined in the aforementioned corollary as the argument that minimizes of the set of positive values of the relevant cdf. We will identify this with zero and provide an upper bound for the probability attributed to the possible atom. Assumption A implies that for every $\varepsilon > 0$ there exists a $z_\varepsilon > 0$ such that $\inf_{\boldsymbol{\lambda}} G(z_\varepsilon,\boldsymbol{\lambda},F) \geq 1-\varepsilon$ for all $\boldsymbol{\lambda}$. Let F_ε denote the cdf of the probability measure obtained by restriction of F to the set $\cup_{\boldsymbol{\lambda}\in\mathbb{L}^\star}\{\boldsymbol{u}\in\mathbb{R}^n : \boldsymbol{\lambda}'\boldsymbol{y}\leq z_\varepsilon\}$ where $\mathbb{L}^\star$ is a countable dense subset of $\mathbb{L}$. Notice first that 0 lies in the support of $\bar{S}$. Furthermore it is easy to see that $\bar{S}$ is greater than or equal to $T = \sup_{\boldsymbol{\lambda}\in\mathbb{L}} -\mathcal{J}(z_\varepsilon, \boldsymbol{\lambda}, \boldsymbol{\tau}, \mathcal{B}_F)$ and the support of the latter is $[0, +\infty)$. This along with the previous imply that $c = 0$. Consider the random variables $\bar{S}_\varepsilon$ and T_ε defined as $\bar{S}$ and T respectively by replacing F with F_ϵ in the relevant definitions. The previous arguments hold also for $\bar{S}_\varepsilon$ and T_ε and their relation. Furthermore, as $\varepsilon \to \infty$ we have that $\mathbb{P}\left(\bar{S}_\varepsilon = 0\right) \to \mathbb{P}\left(\bar{S} = 0\right)$. Notice that using an analogous to the previous argument for the support of $\sup_{\mathbb{L}} \int_{\mathbb{R}^n} ((\boldsymbol{\lambda}'\boldsymbol{u})_- - (\boldsymbol{\tau}'\boldsymbol{u})_-)\, d\mathcal{B}_{F_\varepsilon}(\boldsymbol{u})$,

$$T_\varepsilon \geq \sup_{\mathbb{L}}(\boldsymbol{\lambda} - \boldsymbol{\tau})'\boldsymbol{Y}_\varepsilon + \sup_{\mathbb{L}} \int_{\mathbb{R}^n} ((\boldsymbol{\lambda}'\boldsymbol{u})_- - (\boldsymbol{\tau}'\boldsymbol{u})_-)\, d\mathcal{B}_{F_\varepsilon}(u) \geq \sup_{\mathbb{L}}(\boldsymbol{\lambda} - \boldsymbol{\tau})'\boldsymbol{Y}_\varepsilon,$$

where $\boldsymbol{Y}_\varepsilon$ follows an n-variate zero mean normal distribution with positive definite variance that depends on ε. Thereby $\mathbb{P}\left(\bar{S}_\varepsilon = 0\right) \leq \mathbb{P}\left(\sup_{\mathbb{L}}(\boldsymbol{\lambda} - \boldsymbol{\tau})'\boldsymbol{Y}_\varepsilon = 0\right)$ and the latter probability equals exactly the probability that the maximum of the random vector $\boldsymbol{Y}_\varepsilon$ occurs at a coordinate that represents an element in the intersection of the set of extreme points of the simplices of $\boldsymbol{\lambda}$ and $\boldsymbol{\tau}$ due to Assumption C. If the intersection is empty then this probability is zero. If it not then using Theorem 2 in chapter 3 (p. 37) of Sidak et al. [32] by (in their notation) letting p be the density of the n-variate standard normal distribution it is easy to see that this is less than or equal to $\frac{1}{n}$. In both case the result follows by passing to the limit as $\varepsilon \to \infty$.

2. If $\mathbb{H}_0^{(\mathcal{P})}$ is not true then $\sup_{z\geq 0, \boldsymbol{\lambda}\in\mathbb{L}} -\sqrt{T}\mathcal{J}(z, \boldsymbol{\lambda}, \boldsymbol{\tau}, F)$ and/or $\sup_{z\geq 0, \boldsymbol{\lambda}\in\mathbb{L}} \sqrt{T}\mathcal{J}(z, -\boldsymbol{\lambda}, -\boldsymbol{\tau}, F)$ diverge to $+\infty$. If any of them does not then it does not also contribute to the relevant maximum. By choosing admissible yet arbitrary $z, \boldsymbol{\lambda}, \boldsymbol{\tau}$ in each case, and by using relations (14)and (15) and due to the results of lemma 9 there exist asymptotically Gaussian random variables, $s_{1,T}$, $s_{2,T}$ such that

$$\hat{S}_T^a \geq s_{1,T} + \sup_{z\geq 0, \boldsymbol{\lambda}\in\mathbb{L}} \sqrt{T}\mathcal{J}(z, -\boldsymbol{\lambda}, -\boldsymbol{\tau}, F) \text{ and/or } \hat{S}_T^\beta(\boldsymbol{\tau}) \geq s_{2,T} + \sup_{z\geq 0, \boldsymbol{\lambda}\in\mathbb{L}} -\sqrt{T}\mathcal{J}(z, \boldsymbol{\lambda}, \boldsymbol{\tau}, F)$$

and thereby $\hat{S}_T(\boldsymbol{\tau})$ is greater than or equal to the maximum of the right hand sides of the previous display and the result follows. ■

Proof of Proposition 2. First notice that the moment condition in assumption B implies that G satisfies the condition (5) in Proposition 1. Then from lemma 8 we obtain that

$$\hat{\Upsilon}_T^\beta(\boldsymbol{\tau}) = \sup_{z\geq 0, \boldsymbol{\lambda}\in\mathbb{L}} -\sqrt{T}\left(\mathcal{J}_2^c\left(z, \boldsymbol{\lambda}, \boldsymbol{\tau}, \hat{F}_T\right) - \mathcal{J}_2^c(z, \boldsymbol{\lambda}, \boldsymbol{\tau}, F)\right) - \sqrt{T}\mathcal{J}_2^c(z, \boldsymbol{\lambda}, \boldsymbol{\tau}, F) \quad (16)$$

which is less than or equal to

$$y_{2,T} + \sup_{z \geq 0, \boldsymbol{\lambda} \in \mathbb{L}} -\sqrt{T} \mathcal{J}_2^c \left(z, \boldsymbol{\lambda}, \boldsymbol{\tau}, F\right).$$

Similarly

$$\hat{\Upsilon}_T^a \left(\boldsymbol{\tau}\right) = \sqrt{T} \sup_{z \leq 0, \boldsymbol{\lambda} \in \mathbb{L}} \left(\mathcal{D}_2 \left(z, \boldsymbol{\tau}, \boldsymbol{\lambda}, \hat{F}_T\right) \pm \mathcal{D}_2 \left(z, \boldsymbol{\tau}, \boldsymbol{\lambda}, F\right)\right) \tag{17}$$

which is again less than or equal to

$$y_{1,T} + \sup_{z \leq 0, \boldsymbol{\lambda} \in \mathbb{L}} \sqrt{T} \left(\mathcal{J}_2(z, \boldsymbol{\tau}, F) - \mathcal{J}_2(z, \boldsymbol{\lambda}, F)\right).$$

1. If $\mathbb{H}_0^{(\mathcal{M})}$ holds then the previous imply that

$$\hat{\Upsilon}_T^a \leq y_{1,T} \text{ and } \hat{\Upsilon}_T^{\beta} \left(\boldsymbol{\tau}\right) \leq y_{2,T}$$

and thereby

$$\hat{S}_T \left(\boldsymbol{\tau}\right) \leq \max \left(y_{1,T}, y_{2,T}\right)$$

and due to the CMT and lemma 9.**2** $\max \left(y_{1,T}, y_{2,T}\right)$ converges in distribution to

$$\bar{\Upsilon} \left(\boldsymbol{\tau}\right) \doteqdot \max \begin{pmatrix} \sup_{z \leq 0, \boldsymbol{\lambda} \in \mathbb{L}} \mathcal{D}_2 \left(z, \boldsymbol{\tau}, \boldsymbol{\lambda}, \mathcal{B}_F\right) \\ \sup_{z \geq 0, \boldsymbol{\lambda} \in \mathbb{L}} \left[-\mathcal{J}_2^c \left(z, \boldsymbol{\lambda}, \boldsymbol{\tau}, \mathcal{B}_F\right)\right] \end{pmatrix}',$$

and the result follows. The results on the properties of the distribution of the previous random variable follow exactly as in the previous proposition using analogous comparisons between the limit random variable and the first element of the random vector in its aforementioned representation.

2. If $\mathbb{H}_0^{(\mathcal{M})}$ is not true then $\sup_{z \leq 0, \boldsymbol{\lambda} \in \mathbb{L}} \sqrt{T} \left(\mathcal{J}_2(z, \boldsymbol{\tau}, F) - \mathcal{J}_2(z, \boldsymbol{\lambda}, F)\right)$ and/or $\sup_{z \geq 0, \boldsymbol{\lambda} \in \mathbb{L}} -\sqrt{T} \mathcal{J}_2^c \left(z, \boldsymbol{\lambda}, \boldsymbol{\tau}, F\right)$ diverge to $+\infty$. If any of them does not then it does not also contribute to the relevant supremum. By choosing admissible yet arbitrary $z, \boldsymbol{\lambda}, \boldsymbol{\tau}$ in each case, and by using relations (16)and (17) and due to the results of lemma 9 there exist asymptotically Gaussian random variables, $m_{1,T}$, $m_{2,T}$ such that

$$\hat{\Upsilon}_T^a \geq m_{1,T} + \sup_{z \leq 0, \boldsymbol{\lambda} \in \mathbb{L}} \sqrt{T} \left(\mathcal{J}_2(z, \boldsymbol{\tau}, F) - \mathcal{J}_2(z, \boldsymbol{\lambda}, F)\right)$$

and/or

$$\hat{\Upsilon}_T^{\beta} \left(\boldsymbol{\tau}\right) \geq m_{2,T} + \sup_{z \geq 0, \boldsymbol{\lambda} \in \mathbb{L}} -\sqrt{T} \mathcal{J}_2^c \left(z, \boldsymbol{\lambda}, \boldsymbol{\tau}, F\right)$$

and thereby $\hat{\Upsilon}_T \left(\boldsymbol{\tau}\right)$ is greater than or equal to the maximum of the right hand sides of the previous display and the result follows. ∎

Proof of Proposition 3. From assumptions A and D and Theorem 2.3 of Peligrad (1998) we have that conditionally on the sample

$$\sqrt{T}\left(\hat{F}_T^* - \hat{F}_T\right) \overset{p}{\rightsquigarrow} \mathcal{B}_F^*$$

where $\mathcal{B}_F^*$ is an independent version of the Gaussian process in lemma 9. Analogously to the results of lemma 9 we obtain that

$$\begin{pmatrix} \hat{S}_T^{\alpha_*}(\boldsymbol{\tau}) \\ \hat{S}_T^{\beta_*}(\boldsymbol{\tau}) \end{pmatrix} \overset{p}{\rightsquigarrow} \begin{pmatrix} \sup_{z\geq 0,\boldsymbol{\lambda}\in\mathbb{L}} \mathcal{J}(z,-\boldsymbol{\lambda},-\boldsymbol{\tau},\mathcal{B}_F^*) \\ \sup_{z\geq 0,\boldsymbol{\lambda}\in\mathbb{L}} -\mathcal{J}(z,\boldsymbol{\lambda},\boldsymbol{\tau},\mathcal{B}_F^*) \end{pmatrix}$$

and due to the CMT we finally obtain

$$\hat{S}_T^*(\boldsymbol{\tau}) \overset{p}{\rightsquigarrow} \max\left(\begin{pmatrix} \sup_{z\geq 0,\boldsymbol{\lambda}\in\mathbb{L}} \mathcal{J}(z,-\boldsymbol{\lambda},-\boldsymbol{\tau},\mathcal{B}_F^*) \\ \sup_{z\geq 0,\boldsymbol{\lambda}\in\mathbb{L}} -\mathcal{J}(z,\boldsymbol{\lambda},\boldsymbol{\tau},\mathcal{B}_F^*) \end{pmatrix}'\right).$$

1. The absolute continuity of the CDF when appropriately restricted due to the results of the first part of proposition 1 implies the continuity of the quantile function when $\alpha < 1 - \frac{1}{n}$, and the result follows exactly as in the proof of Proposition 3.1 of Scaillet and Topaloglou (2010) (or Propositions 2 or 3 of Barrett and Donald (2003)). This is due to the equivalence of weak convergence (in probability) to locally uniform convergence at continuity points of the limit CDF (see Bhattacharya and Rao (2010)) and the subsequent pointwise convergence (in probability) of quantile functions at continuity points.

2. follows directly from the previous part of the current proof, the second part of proposition 1 and the fact that $c_{\mathcal{P}}$ is finite. ■

Proof of Proposition 4. The results follow as in the proof of proposition 3 in view of the second part of lemma 9. ■

Proof of Proposition 7. If z such that $r_i \leq z \leq r_{i+1}$, $i = 1, \cdots T_{p^*}$, $\sum_{t=1,\ldots,T_p} L_t$ is constant (it is equal to the maximum value of t such that $\boldsymbol{\tau}'\mathbf{Y}_t \leq r_i$). Further, when $r_i \leq z \leq r_i + 1$, the maximum value of $-\sum_{t=1,\ldots,T_p} W_t$ is reached for $z = r_i$. Analogous considerations are easily obtained when $z < r_1$ or $z > r_{T_{p^*}+1}$. Hence, we can restrict z to belong to the set $\mathcal{R}$. ■

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Tab. 1: *Sensitivity analysis of size and power to the choice of block length using the* GARCH *process for the asset returns. We compute the actual size and power of the prospect and Markowitz stochastic dominance efficiency tests for block sizes ranging from* $l = 4$ *to* $l = 12$.

Block size l :	4	8	10	12
Case 1:	R=100	T=200		
Size:				
$\hat{RP}_{\text{PSD}}$	4.5%	4.8%	3.7%	3.9%
$\hat{RP}_{\text{MSD}}$	5.1%	5.3%	4.7%	4.8%
Power:				
$\hat{RP}_{\text{PSD}}$	94.8%	95.1%	95.7%	96.2%
$\hat{RP}_{\text{MSD}}$	95.4%	96%	94.6%	95.3%
Case 2:	R=300	T=500		
Size:				
$\hat{RP}_{\text{PSD}}$	3.8%	4.7%	2.8%	3.5%
$\hat{RP}_{\text{MSD}}$	4.5%	4.2%	4.1%	3.9%
Power:				
$\hat{RP}_{\text{PSD}}$	95.4%	96.2%	97.1%	98.0%
$\hat{RP}_{\text{MSD}}$	97.1%	96.8%	95.2%	96.1%
Case 3:	R=500	T=1000		
Size:				
$\hat{RP}_{\text{PSD}}$	2.6%	3.1%	2.5%	3.1%
$\hat{RP}_{\text{MSD}}$	2.7%	3.3%	2.9%	2.1%
Power:				
$\hat{RP}_{\text{PSD}}$	97.1%	98.7%	98.4%	98.9%
$\hat{RP}_{\text{MSD}}$	98.6%	99.1%	96.3%	98.2%

Descriptive Statistics (January 1930 to December 2012)						
No.	Mean	Std. Dev.	Skewness	Kurtosis	Minimum	Maximum
Market Portfolio	0.604	5.413	0.237	7.593	-29.98	37.77
1	1.016	7.825	1.026	7.270	-32.32	65.63
2	1.288	7.139	1.310	11.660	-31.10	64.12
3	1.493	8.367	2.175	18.810	-33.06	85.24
4	0.847	5.308	-0.023	2.231	-28.08	32.55
5	0.936	5.823	1.303	14.227	-28.01	51.52
6	1.161	7.327	1.547	14.926	-35.45	68.25

Tab. 2: *Descriptive statistics of monthly returns in % from January 1930 to December 2012 (996 monthly observations) for the Fama and French market portfolio and the six Fama and French benchmark portfolios formed on size and book-to-market equity ratio. Portfolio 1 has low BE/ME and small size, portfolio 2 has medium BE/ME and small Size, portfolio 3 has high BE/ME and small size, ..., portfolio 6 has high BE/ME and large size.*

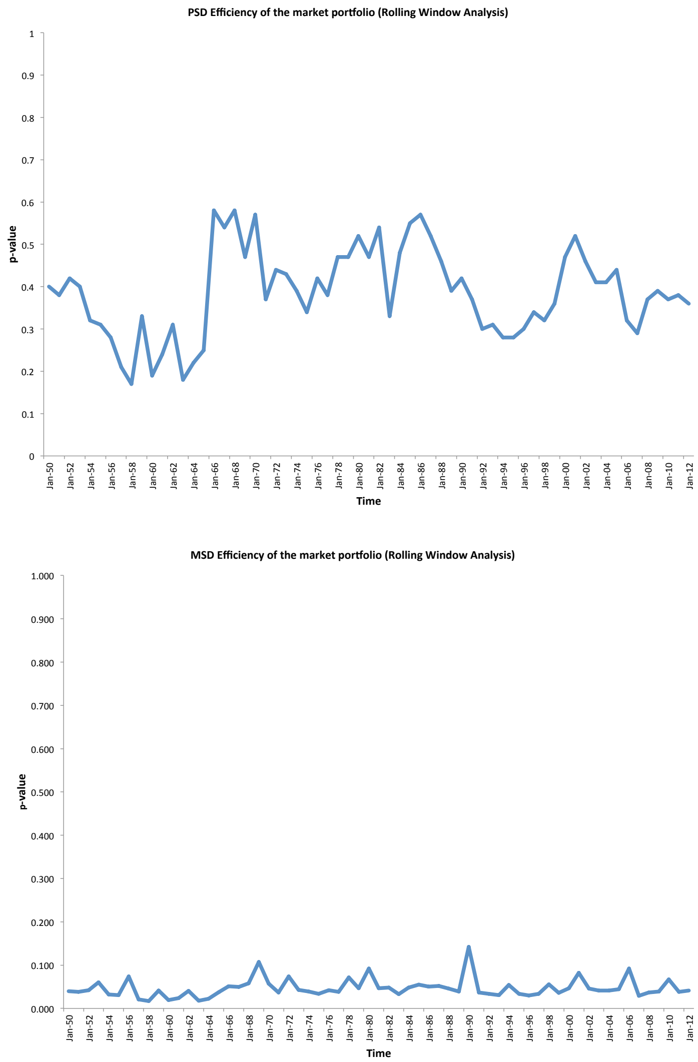
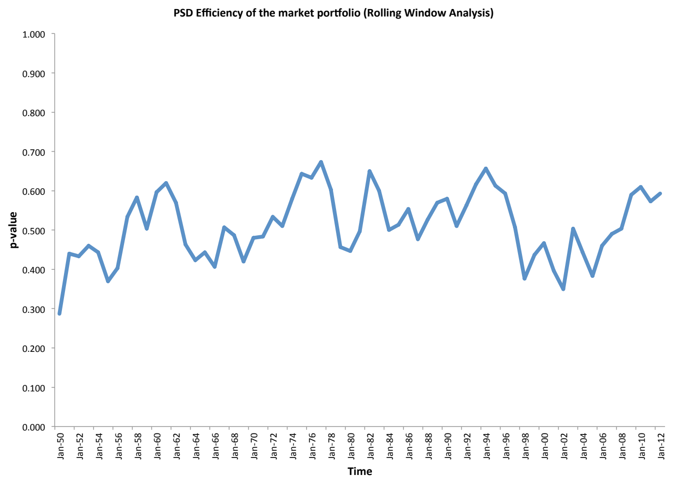


Fig. 1: **6 FF portfolios:** p-values for the prospect stochastic dominance efficiency test (upper graph) and for the Markowitz stochastic dominance efficiency test (lower graph) using a rolling window of 20 years. The test statistic is calculated separately for 63 overlapping 20-year periods, (January 1930-December 1949), (January 1931-December 1951),...,(January 1993-December 2012). The prospect stochastic dominance efficiency is not rejected in any subperiod, while the Markowitz stochastic dominance efficiency is rejected in 46 out of 63 subperiods.



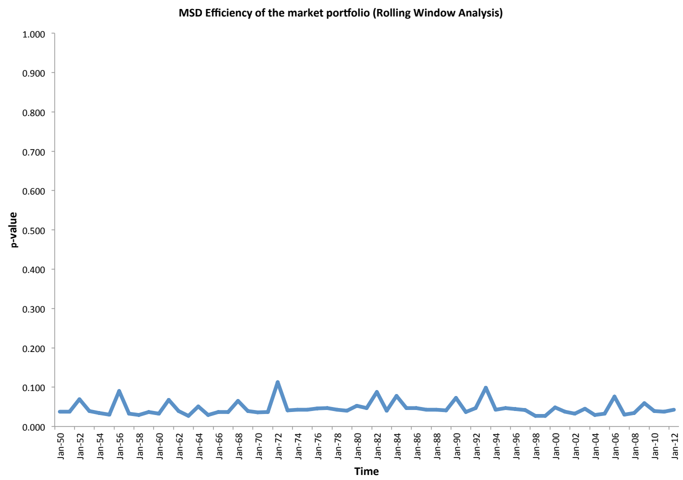


Fig. 2: **48 Industry portfolios:** p-values for the prospect stochastic dominance efficiency test (upper graph) and for the Markowitz stochastic dominance efficiency test (lower graph) using a rolling window of 20 years. The test statistic is calculated separately for 63 overlapping 20-year periods, (January 1930-December 1949), (January 1931-December 1951),...,(January 1993-December 2012). The prospect stochastic dominance efficiency is not rejected in any subperiod, while the Markowitz stochastic dominance efficiency is rejected in 50 out of 63 subperiods.