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On financial bubbles, investment decisions and investor's utility

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Abstract

This paper is concerned with the consequences of the financial bubbles on investment decisions and on the expected utility of the typical investor. Under the bubble effect, the typical investor undertakes more-than-optimal risks, for which there is no proper compensation, given the actual capital-market line. This leads to significant expected utility losses. Since the risk-free return is a measure of the time value of money, the issuers of risk-free assets can tame the bubble beast by being disciplined enough to maintain the return on risk-free assets at its pre-bubble equilibrium level. The role of financial analysts and capital-market authorities is important nontheless.

Key words: Financial bubbles; capital-market line; risk premia; expected utility; financial analysts; capital-market authorities; risk-free asset issuers.

JEL classification: G01 (Financial Crises), G11 (Portfolio Choice—Investment Decisions), G12 (Asset Pricing), G18 (Government Policy and Regulation).

1 Introduction

Financial "bubbles" are the results of errorneous market pricing of financial assets (see, e.g., Bernanke (1983), Brunnermeier and Oehmke (2012), Yan et al. (2012), *inter alia*). Any event of temporary errorneous asset-pricing, however, cannot be identified as a bubble. Thus, financial bubbles are considered to have lasting errorneous-pricing results and obey a set of specific characteristics. In fact, under the bubble effect, asset pricing seems to follow an exponential form, which is to be specified shortly.

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There are four indicative stages in a financial-bubble insident: (i) the misspricing of financial assets over their fundamental values, (ii) the short run realisation of higher-than-normal returns, (iii) the massive investing by the typical economic agents, and (iv) the bursting of the bubble (see Sornette and Woodard (2010)).

Financial bubbles are impotant because they affect both, the allocation of resorces and the level of economic activity, which, in turn, determin vital economic variables, such as profitability, employment and growth. The creation of the bubbles can be attributed to psycological, social, monetary, credit, managerial and supervisionary causes (see Bhattacharya and Yu (2008), Kaizoji and Sornette (2009)).

This paper is an elaborate exercise with the scope to investigate the consequences of the financial bubbles on investment decisions and on the expected utility of the typical investor. Due to a misconception of financial-market condistions, the typical investor expects higher real returns on risky assets and, because of this, is willing to undertake greater risks. This, however, does not icrease the ability of the market to compansate risks accordingly, and the investor sufferes expected utility losses in the long run.

The role of financial analysts, capital-market authorities (see Allen and Gale (1999), White (2011)) and issuers of risk-free assets is discussed with a view to evaluate their responsibilities and capabilities to prevent the negative consequences of financial-bubble insidents.

This paper is organised as follows: Section 2 presents the notation and assumptions. The problem is formulated in Section 3, and its consequences are analysed in Section 4. The results are discussed in Section 5, while Section 6 provides some concludind remarks. All proofs are gothered in the Appendix.

2 Notation and Assumptions

In modern capitalist societies, economic agents can choose between the following three types of investment:

Risk-free assets: they consist of a very small minority of assets, which, for practical purposes, can be considered to bear no risk at all. As a consequence of their zero risk, the real return of these assets compensate the investors only for the time they have decided to give up present consumption in order to invest their money. Risk-free assets are either zero-risk money deposits, or bonds issued by *highest-ranked* institutions, such as a limited group of countries and private-sector companies, which seem to bear no default risk. As an example, countries with large surpluses, as well as banks performing only traditional financial intermediation, may be considered to belong to this category of zero-risk bond issuers.

risky bonds: they are issued by the majority of countries and private-sector companies, which, evaluated by the world financial markets, seem to possibly bear a certain amount of default risk. The magnitude of this risk corresponds to the corresponding risk premium charged, over and above the risk-free real return, so that their lenders be fully compensated.

risky stocks: they are issued by the vast majority of private-sector companies. These assets certainly bear a considerable amount of risk, due to the uncertainty of their real returns. This uncertainty is caused by their ever-changing evaluation, made by the global community of stock-exchange investors. The uncertainty of these assets is expressed by the variability of their market prices.

In the rest of this section, we analytically present the notation[1] and assumptions used in the paper.

Let r_f denote the real return of the risk-free assets, and r_1 and r_2 denote the real returns of the risky bonds and stocks, respectively.

Further, let P be a typical investor's portfolio on the efficient frontier (see Markowitz (1952, 1991)). Given the existence of risk-free assets, the efficient frontier is the well-known "capital-market line." Let w_0 be the percentage of investor's capital invested in risk-free assets, and w_1, w_2 be the percentages of his/her capital invested in risky bonds and stocks, respectively. Obviously,

$$w_0 + w_1 + w_2 = 1 \implies w_0 = 1 - w_1 - w_2. \tag{1}$$

The utility function of the typical investor is assumed to be a quadratic function of the real return, r_P, of his/her portfolio, i.e.,

$$U(r_P) = \alpha_1 r_P - \alpha_2 r_P^2, \tag{2}$$

where

$$\alpha_1 > 0, \quad \alpha_2 > 0 \quad \text{and} \quad 0 < r_P < \frac{\alpha_1}{2\alpha_2}. \tag{3}$$

Let $\bar{r}_P \equiv \mathrm{E}(r_P)$ and $\sigma_P^2 \equiv \mathrm{var}(r_P)$ be the expectation and variance, respectively, of the real return of portfolio P. Since $\mathrm{E}(r_P^2) = \bar{r}_P^2 + \sigma_P^2$, the expected utility of the typical investor, denoted as $V(\bar{r}_P, \sigma_P^2) \equiv \mathrm{E}[U(r_P)]$, can be written as

$$V(\bar{r}_P, \sigma_P^2) = \alpha_1 \bar{r}_P - \alpha_2 \bar{r}_P^2 - \alpha_2 \sigma_P^2. \tag{4}$$

Let $\mathcal{J}$ be a given set of indices and $j \in \mathcal{J}$ be an index with values $j = I, II$, etc. For any specified constant value, $\bar{\bar{V}}_I$ say, of $V(\bar{r}_P, \sigma_P^2)$, equation (4) gives the locus of equal-utility combinations of $\bar{r}_P$ and σ_P^2 on the particular indifference curve cited below:

$$\bar{\bar{V}}_I \equiv \bar{\bar{V}}_I(\bar{r}_P, \sigma_P^2) = (\alpha_1 \bar{r}_P - \alpha_2 \bar{r}_P^2 - \alpha_2 \sigma_P^2)_I, \tag{5}$$

which is specified by the particular value I of index j. Further, equation (5) can be easily solved for either σ_P^2 or $\bar{r}_P$ as follows:

(i) For any given value of expected real return, $\bar{r}_P$ say, the function

$$\sigma_{P(I)}(\bar{\bar{V}}_I, \bar{r}_P) = \left[\frac{\alpha_1 \bar{r}_P - \alpha_2 \bar{r}_P^2 - \bar{\bar{V}}_I}{\alpha_2} \right]^{1/2}, \tag{6}$$

with values $\sigma_{P(I)}$, gives an estimation of the risk undertaken by the investor, in order to assure a prespecified level, $\bar{\bar{V}}_I$ say, of his/her expected utility.

[1] Throughout this paper, we use the notational standard proposed by Abadir and Magnus (2002), for which clarifications are provided whenever needed.

(ii) For any given value, σ_P, of the risk undertaken by the investor, the required expected return, $\bar{r}_{P(I)}$ say, which assures a prespecified level, $\bar{\bar{V}}_I$ say, of his/her expected utility, can be easily calculated by solving the following trinomial:

$$-\alpha_2 \bar{r}_{P(I)}^2 + \alpha_1 \bar{r}_{P(I)} - \left(\alpha_2 \sigma_P^2 + \bar{\bar{V}}_I\right) = 0. \tag{7}$$

The real return of portfolio P can be written as

$$r_P = w_0 r_f + w_1 r_1 + w_2 r_2. \tag{8}$$

Since r_f is non-stochastic, whilst r_1 and r_2 are random variables, the expected real return of P can be expressed as

$$\bar{r}_P = r_f + w_1(\bar{r}_1 - r_f) + w_2(\bar{r}_2 - r_f), \tag{9}$$

where $\bar{r}_i \equiv \mathrm{E}(r_i)$ $(i = 1, 2)$, and $\bar{r}_1 - r_f$, $\bar{r}_2 - r_f$ are the real risk premia of the risky bonds and stocks, respectively. Moreover, since r_f is non-stochastic, it corresponds to zero risk, i.e., $\sigma_f^2 \equiv \mathrm{var}(r_f) \equiv 0$, which implies that the covariances of the risk-free assets with the risky bonds and stocks, denoted as σ_{f1} and σ_{f2}, respectively, are both, by definition, equal to zero, i.e., $\sigma_{f1} \equiv \sigma_{f2} \equiv 0$.[2] Thus, the variance of the real return of portfolio P can be expressed as

$$\sigma_P^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}, \tag{10}$$

where $\sigma_i^2 \equiv \mathrm{var}(r_i)$ $(i = 1, 2)$, and $\sigma_{12} \equiv \mathrm{cov}(r_1, r_2)$ is the covariance between the real returns r_1 and r_2.

Define the vectors $\boldsymbol{w}$, $\bar{\boldsymbol{r}}$, $\tilde{\boldsymbol{r}}$, $\boldsymbol{\imath}$ and the symmetric, non-singular matrix $\boldsymbol{\Sigma}$ as follows:[3]

$$\boldsymbol{w} := \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \ \bar{\boldsymbol{r}} := \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \end{bmatrix}, \ \tilde{\boldsymbol{r}} := \begin{bmatrix} \bar{r}_1 - r_f \\ \bar{r}_2 - r_f \end{bmatrix}, \ \boldsymbol{\imath} := \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \boldsymbol{\Sigma} := \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}. \tag{11}$$

The capital-market line, which in our case is the efficient frontier, results in as the solution of the following conditional minimisation problem:

$$\min_{w_1, w_2} \sigma_P^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12} \tag{12}$$

subject to the restriction (9). The corresponding Lagrangean function is

$$\begin{aligned} \mathscr{L} &= \frac{1}{2}\left[w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}\right] \\ &\quad + \lambda\left[\bar{r}_P - r_f - w_1(\bar{r}_1 - r_f) - w_2(\bar{r}_2 - r_f)\right], \end{aligned} \tag{13}$$

where scalar λ is the Lagrange multiplier. First-order conditions imply that

$$\boldsymbol{w} = \lambda \boldsymbol{\Sigma}^{-1} \tilde{\boldsymbol{r}} \quad \text{and} \quad \lambda = \frac{(\bar{r}_P - r_f)}{b}, \tag{14}$$

where $b = \tilde{\boldsymbol{r}}' \boldsymbol{\Sigma}^{-1} \tilde{\boldsymbol{r}}$.[4] Therefore, the resulting capital-market line is

$$\bar{r}_P = r_f + \sigma_P \sqrt{b} \tag{15}$$

[2] Obviously, we use the standard notation: $\sigma_{f1} \equiv \mathrm{cov}(r_f, r_1)$ and $\sigma_{f2} \equiv \mathrm{cov}(r_f, r_2)$.

[3] The expression $a := b$ denotes the definition of a in terms of b.

[4] The symbol $\tilde{\boldsymbol{r}}'$ denotes the transpose of vector $\tilde{\boldsymbol{r}}$.

with slope equal to

$$\sqrt{b} = \frac{(\bar{r}_P - r_f)}{\sigma_P}. \tag{16}$$

This means that, for any possible portfolio P on the capital-market line, the typical investor is willing to undertake a one-percent-increased risk as long as its compensation, in terms of the expected real risk premium for his/her portfolio, denoted as $(\bar{r}_P - r_f)$, is increased by $\sqrt{b}$. In other words, a one-percent increase in the expected real risk premium of the portfolio can lead the typical investor to undertake a $(\frac{1}{\sqrt{b}})$-percent-increased risk, by choosing the corresponding alternative portfolio on the capital-market line.

Moreover, by using the definitions of scalar b and vector $\tilde{\boldsymbol{r}}$, we can write

$$b = b_0 - 2b_1 r_f + b_2 r_f^2, \tag{17}$$

where

$$b_0 = \bar{\boldsymbol{r}}'\boldsymbol{\Sigma}^{-1}\bar{\boldsymbol{r}}, \quad b_1 = \bar{\boldsymbol{r}}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\imath}, \quad b_2 = \boldsymbol{\imath}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\imath}. \tag{18}$$

On the capital-marlet line the following two portfolios can be defined:

(i) the portfolio made of risk-free assets only, denoted as F, whose real return is r_f, and

(ii) the so-called "market" (or "tangent") portfolio made exclusively of riky assets, denoted as M, whose real return is r_M. Since portfolio M contains no risk-free assets, its vector of weights, defined as $\boldsymbol{w}_M := (w_{1_M}, w_{2_M})'$, satisfies the following restriction:

$$\boldsymbol{w}_M'\boldsymbol{\imath} = 1 \implies w_{1_M} + w_{2_M} = 1, \tag{19}$$

which combined with (14) and (18) implies that

$$\begin{bmatrix} w_{1_M} \\ w_{2_M} \end{bmatrix} = \frac{1}{b_1 - b_2 r_f}\boldsymbol{\Sigma}^{-1}\begin{bmatrix} \tilde{r}_1 \\ \tilde{r}_2 \end{bmatrix}, \tag{20}$$

where $\tilde{r}_i = \bar{r}_i - r_f$ $(i = 1, 2)$. The expected return and risk associated with the market portfolio can be written as

$$\bar{r}_M = \frac{b_0 - b_1 r_f}{b_1 - b_2 r_f} \quad \text{and} \quad \sigma_M = \frac{\sqrt{b}}{b_1 - b_2 r_f}, \tag{21}$$

respectively, where $\bar{r}_M \equiv \mathrm{E}(r_M)$ and $\sigma_M^2 \equiv \mathrm{var}(r_M)$.

Portfolios F and M are the two most important of all portfolios located on the capital-market line, because it is known that any portfolio P, located on the capital-market line, can be constructed as a linear combination of the form

$$P = \varphi F + (1 - \varphi)M. \tag{22}$$

Note that, if $0 < \varphi < 1$, portfolio P contains a combination of both, risk-free and risky assets. On the other hand, if $\varphi < 0$, the amount invested on risky assets exceeds the capital available to the investor, the difference being borrowed from the banking system.

It is obvious that the real return of portfolio P can be calculated in terms of the real returns of portfolios F and M as follows:

$$r_P = \varphi r_f + (1-\varphi) r_M. \tag{23}$$

Given that r_f is non-stochastic, it is straightforward that $\sigma_f^2 \equiv 0$ and $\sigma_{fM} \equiv \operatorname{cov}(r_f, r_M) \equiv 0$, which, in turn, imply that the expected real return and risk associated with portfolio P can be calculated as

$$\bar{r}_P = \varphi r_f + (1-\varphi)\bar{r}_M \quad \text{and} \quad \sigma_P = (1-\varphi)\sigma_M, \tag{24}$$

respectively.

Finally, let E be the equilibrium (or optimal) portfolio, and define its vector of equilibrium weights as $\boldsymbol{w}_E := (w_{1_E}, w_{2_E})'$. Portfolio E is located at the point of tangency of the capital-market line with the highest compatible indifference curve of the typical investor. Let (6) be this particular indifference curve, expressed as a function of the undertaken risk, σ_E, in terms of the associated expected real return, $\bar{r}_E$, and rewrite (16) accordingly as follows:

$$\sigma_E = \frac{(\bar{r}_E - r_f)}{\sqrt{b}}. \tag{25}$$

Then, equating the slope of (6), that is

$$\frac{\mathrm{d}\sigma_{E(I)}}{\mathrm{d}\bar{r}_E} = \frac{1}{2}\left(\sigma^2_{E(I)}\right)^{-1/2} \frac{(\alpha_1 - 2\alpha_2 \bar{r}_E)}{\alpha_2}, \tag{26}$$

with the slope of (25), that is

$$\frac{\mathrm{d}\sigma_{E(I)}}{\mathrm{d}\bar{r}_E} = \frac{1}{2}\left(\sigma^2_{E(I)}\right)^{-1/2} 2\frac{(\bar{r}_E - r_f)}{b}, \tag{27}$$

the expected real return and associated risk of portfolio E can be expressed as

$$\bar{r}_E = \frac{1}{1+b}\left[r_f + \frac{b\alpha_1}{2\alpha_2}\right] \quad \text{and} \quad \sigma_E = \frac{\sqrt{b}}{1+b}\left[\frac{\alpha_1}{2\alpha_2} - r_f\right], \tag{28}$$

respectively. Moreover, the vector of weights of portfolio E can be written as

$$\begin{bmatrix} w_{1_E} \\ w_{2_E} \end{bmatrix} = \frac{1}{1+b}\left[\frac{\alpha_1}{2\alpha_2} - r_f\right] \boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{r}}, \tag{29}$$

where $\bar{r}_E \equiv \mathrm{E}(r_E)$ and $\sigma_E^2 \equiv \operatorname{var}(r_E)$.

3 The problem: the occurence of a bubble

Let us suppose that, due to a misconception of the actual market conditions, the typical investor believes that the real returns on the risky assets are multiplied by a positive factor greater than unity, while the associated risks remain unchanged. This means that the bubbled real returns on the risky assets can be expressed as follows:

$$r_{i*} = (1+l) r_i \quad (i = 1, 2), \tag{30}$$

where $l > 0$ is a common[5] multiplicative factor, while the variances of r_{1*} and r_{2*}, and their covariance remain at their actual levels, i.e.,

$$\operatorname{var}(r_{i*}) = \sigma_i^2 \quad (i = 1, 2) \quad \text{and} \quad \operatorname{cov}(r_{1*}, r_{2*}) = \sigma_{12}, \tag{31}$$

respectively. Thus, the expected bubbled real returns are

$$\bar{r}_{i*} \equiv \mathrm{E}(r_{i*}) = (1 + l)\,\mathrm{E}(r_i) = (1 + l)\bar{r}_i \quad (i = 1, 2). \tag{32}$$

By using (11) and (32), we can write vector $\bar{\boldsymbol{r}}_*$, of bubbled real returns, and vector $\tilde{\boldsymbol{r}}_*$, of bubbled risk premia, as

$$\bar{\boldsymbol{r}}_* := \begin{bmatrix} \bar{r}_{1*} \\ \bar{r}_{2*} \end{bmatrix} = (1 + l)\bar{\boldsymbol{r}} \quad \text{and} \quad \tilde{\boldsymbol{r}}_* := \begin{bmatrix} \bar{r}_{1*} - r_f \\ \bar{r}_{2*} - r_f \end{bmatrix} = \bar{\boldsymbol{r}}_* - \boldsymbol{\imath} r_f = \tilde{\boldsymbol{r}} + l\bar{\boldsymbol{r}}, \tag{33}$$

respectively. Since $l > 0$, it is obvious that

$$\tilde{\boldsymbol{r}}_{i*} = (1 + l)\bar{r}_i - r_f > \bar{r}_i - r_f = \tilde{\boldsymbol{r}}_i \quad (i = 1, 2), \tag{34}$$

given that $\bar{r}_1$ and $\bar{r}_2$ are positive. Equation (34) implies that, the norms (lengths or measures) of vectors $\tilde{\boldsymbol{r}}_*$ and $\tilde{\boldsymbol{r}}$ satisfy the following relationship:

$$\|\tilde{\boldsymbol{r}}_*\| = (\tilde{\boldsymbol{r}}_*' \tilde{\boldsymbol{r}}_*)^{1/2} > (\tilde{\boldsymbol{r}}' \tilde{\boldsymbol{r}})^{1/2} = \|\tilde{\boldsymbol{r}}\|. \tag{35}$$

Moreover, equation (30) implies that the bubbled real return of portfolio P can be written as

$$\begin{aligned} r_P^* := w_0 r_f + w_1 r_{1*} + w_2 r_{2*} &= w_0 r_f + w_1(1 + l) r_1 + w_2(1 + l) r_2 \\ &= r_P + l(w_1 r_1 + w_2 r_2) \\ &> r_P, \end{aligned} \tag{36}$$

given that $l > 0$, and r_i, w_i $(i = 1, 2)$ are both positive. This means that, under the bubble effect, the typical investor conceives the return of portfolio P to be larger than its actual value. Taking expectations in (36) we can easily find that, when a bubble occures, the expected return of portfolio P is, accordingly, misconceived to be larger than its actual value, i.e.,

$$\begin{aligned} \bar{r}_P^* \equiv \mathrm{E}(r_P^*) &= \mathrm{E}(r_P) + l[w_1\,\mathrm{E}(r_1) + w_2\,\mathrm{E}(r_2)] \\ &= \bar{r}_P + l(w_1 \bar{r}_1 + w_2 \bar{r}_2) \\ &> \bar{r}_P. \end{aligned} \tag{37}$$

This, in turn, means that, since the risk measures remain unchanged, under the bubble effect the risky assets seem more attractive to the typical investor, who, therefore, becomes more willing to undertake greater risks.

[5]It might seem preferable to use two distinct positive multiplicative factors, namely $l_1 > 0$ and $l_2 > 0$, in which case $r_{i*} = (1 + l_i) r_i \quad (i = 1, 2)$. This would only perplex things, however, without the gain of any more insight. To see this, suppose that $l_1 > l_2$. In such a case, there would be a market distorsion leading the typical investor to overinvest in the first asset and underinvest in the second. This would increase the market price of the first asset relative to the market price of the second, which, in turn, would lower r_1 relative to r_2, thus leading both l_1 and l_2 towards their weighted mean, denoted as l. Nevertheless, the use of a single multiplicative factor is not without theoretical implications, since it suggests that the occurence of a bubble does not disturb the relative assessment of the risky assets by the typical investor.

3.1 Capital-market line under the bubble effect

When a bubble occures, the misconceived capital-market line results in as the solution of the following conditional minimisation problem:

$$\min_{w_{1*},w_{2*}} \sigma_{P_*}^2 = w_{1*}^2\sigma_1^2 + w_{2*}^2\sigma_2^2 + 2w_{1*}w_{2*}\sigma_{12} \tag{38}$$

subject to the restriction

$$\bar{r}_{P_*} = r_f + w_{1*}(\bar{r}_{1*} - r_f) + w_{2*}(\bar{r}_{2*} - r_f), \tag{39}$$

where the percentage of capital invested in risk-free assets is

$$w_{0*} = 1 - w_{1*} - w_{2*}. \tag{40}$$

Under the bubble effect, the Lagrangean function becomes

$$\begin{aligned} \mathscr{L}_* &= \frac{1}{2}\left[w_{1*}^2\sigma_1^2 + w_{2*}^2\sigma_2^2 + 2w_{1*}w_{2*}\sigma_{12}\right] \\ &\quad +\lambda_*\left[\bar{r}_{P_*} - r_f - w_{1*}(\bar{r}_{1*} - r_f) - w_{2*}(\bar{r}_{2*} - r_f)\right], \end{aligned} \tag{41}$$

where scalar λ_* is the corresponding Lagrange multiplier. First-order conditions imply that

$$\boldsymbol{w}_* = \lambda_* \boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{r}}_* \quad \text{and} \quad \lambda_* = \frac{(\bar{r}_{P_*} - r_f)}{b_*}, \tag{42}$$

where $b_* = \tilde{\boldsymbol{r}}_*'\boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{r}}_*$. Thus, the bubbled capital-market line is

$$\bar{r}_{P_*} = r_f + \sigma_{P_*}\sqrt{b_*} \tag{43}$$

and its slope is

$$\sqrt{b_*} = \frac{(\bar{r}_{P_*} - r_f)}{\sigma_{P_*}}. \tag{44}$$

By using equations (17) and (18), and the definitions of $\bar{\boldsymbol{r}}$, $\tilde{\boldsymbol{r}}$ and $\tilde{\boldsymbol{r}}_*$, scalar b_* can be written as follows:

$$\begin{aligned} b_* &= b + 2l(b_0 - b_1 r_f) + l^2 b_0 \\ &= b_0(1+l)^2 - 2b_1(1+l)r_f + b_2 r_f^2 \\ &= b_{*0} - 2b_{*1}r_f + b_2 r_f^2\,, \end{aligned} \tag{45}$$

where

$$b_{*0} = (1+l)^2 b_0 \quad \text{and} \quad b_{*1} = (1+l)b_1. \tag{46}$$

The relationship between b and b_* is given by the following

Theorem 1. *The occurence of a bubble increases the conceived slope of the capital-market line, i.e., b_* is always larger than b.*

Denote as M_* the bubbled market portfolio, whose vector of weights, defined as $\boldsymbol{w}_{M*} := (w_{1_{M_*}}, w_{2_{M_*}})'$, satisfies the following restriction:

$$\boldsymbol{w}_{M_*}'\boldsymbol{\imath} = 1 \implies w_{1_{M_*}} + w_{2_{M_*}} = 1. \tag{47}$$

Along the same lines as in (20), the weights of portfolio M_* are given by

$$\begin{bmatrix} w_{1_{M_*}} \\ w_{2_{M_*}} \end{bmatrix} = \frac{1}{(1+l)b_1 - b_2 r_f} \boldsymbol{\Sigma}^{-1} \begin{bmatrix} \tilde{r}_{1*} \\ \tilde{r}_{2*} \end{bmatrix}. \tag{48}$$

In accordance to (21), the expected return and risk associated with portfolio M_* can be written as

$$\bar{r}_{M_*} = \frac{(1+l)^2 b_0 - (1+l) b_1 r_f}{(1+l)b_1 - b_2 r_f} \quad \text{and} \quad \sigma_{M_*} = \frac{\sqrt{b_*}}{(1+l)b_1 - b_2 r_f}, \tag{49}$$

respectively, where $\bar{r}_{M_*} \equiv \mathrm{E}(r_{M_*})$ and $\sigma^2_{M_*} \equiv \mathrm{var}(r_{M_*})$.

Any portfolio P_*, located on the bubbled capital-market line, can be constructed as a linear combination of the form

$$P_* = \varphi_* F + (1 - \varphi_*) M_*. \tag{50}$$

Since r_f in non-stochastic, the expected real return and risk associated with portfolio P_* can be calculated as

$$\bar{r}_{P_*} = \varphi_* r_f + (1 - \varphi_*) \bar{r}_{M_*} \quad \text{and} \quad \sigma_{P_*} = (1 - \varphi_*) \sigma_{M_*}, \tag{51}$$

respectively, given that $\sigma_f^2 \equiv \mathrm{var}(r_f) \equiv 0$ and $\sigma_{fM_*} \equiv \mathrm{cov}(r_f, r_{M_*}) \equiv 0$.

3.2 Equilibrium portfolio under the bubble effect

In accordance to (2), the utility function of the typical investor becomes

$$U(r_{P_*}) = \alpha_1 r_{P_*} - \alpha_2 r^2_{P_*}, \tag{52}$$

where

$$\alpha_1 > 0, \quad \alpha_2 > 0 \quad \text{and} \quad 0 < r_{P_*} < \frac{\alpha_1}{2\alpha_2}.^{6} \tag{53}$$

Equation (4) implies that the expected utility of the typical investor under the bubble effect can be written as follows:

$$V(\bar{r}_{P_*}, \sigma^2_{P_*}) = \alpha_1 \bar{r}_{P_*} - \alpha_2 \bar{r}^2_{P_*} - \alpha_2 \sigma^2_P, \tag{54}$$

where $\bar{r}_{P_*} \equiv \mathrm{E}(r_{P_*})$ and $\sigma^2_{P_*} \equiv \mathrm{var}(r_{P_*})$.

Let E_* and $\boldsymbol{w}_{E_*} := (w_{1_{E_*}}, w_{2_{E_*}})'$ be the equilibrium portfolio and its vector of weights, respectively, under the bubble effect. By equating the slopes of the bubbled capital-market line with the corresponding highest compatible indifference curve of the typical investor, we can express the expected real return and associated risk of portfolio E_* as

$$\bar{r}_{E_*} = \frac{1}{1+b_*} \left[r_f + \frac{b_* \alpha_1}{2\alpha_2} \right] \quad \text{and} \quad \sigma_{E_*} = \frac{\sqrt{b_*}}{1+b_*} \left[\frac{\alpha_1}{2\alpha_2} - r_f \right], \tag{55}$$

respectively, and its vector of weights as

$$\begin{bmatrix} w_{1_{E_*}} \\ w_{2_{E_*}} \end{bmatrix} = \frac{1}{1+b_*} \left[\frac{\alpha_1}{2\alpha_2} - r_f \right] \boldsymbol{\Sigma}^{-1} \tilde{\boldsymbol{r}}_*, \tag{56}$$

where $\bar{r}_{E_*} \equiv \mathrm{E}(r_{E_*})$ and $\sigma^2_{E_*} \equiv \mathrm{var}(r_{E_*})$.

[6] The third inequality in (53) implies that, for the values of α_1 and α_2 in (52), the ratio $\alpha_1/2\alpha_2$ is large enough so that the utility function can accommodate for the increased amounts of capital invested in risky assets. Therefore, α_1 and α_2 are assumed to be exogenous, i.e., independent from r_P or r_{P_*}. If this is not the case, the utility function has to be transformed accordingly, to render the typical investor capable of decision making under the bubble effect.

4 Consequences of the occurence of a bubble

To evaluate the consequences of the occurence of a financial-market bubble, we are going to compare the levels of expected utility enjoied by the typical investor in the following four distinct situations:

(i) portfolio E, with initial weights w_{0_E}, w_{1_E} and w_{2_E}, and real returns r_f, $\bar{r}_1$ and $\bar{r}_2$, respectively.

(ii) portfolio $\grave{E}$, with initial weights w_{0_E}, w_{1_E} and w_{2_E}, and bubbled real returns on risky assets, i.e., r_f, $\bar{r}_{1*}$ and $\bar{r}_{2*}$, respectively.

(iii) portfolio E_*, with adjusted weights $w_{0_{E_*}}$, $w_{1_{E_*}}$ and $w_{2_{E_*}}$, and bubbled real returns on risky assets, i.e., r_f, $\bar{r}_{1*}$ and $\bar{r}_{2*}$, respectively.

(iv) portfolio E_*, with adjusted weights $w_{0_{E_*}}$, $w_{1_{E_*}}$ and $w_{2_{E_*}}$, and initial real returns on risky assets, i.e., r_f, $\bar{r}_1$ and $\bar{r}_2$, respectively.

4.1 The initial equilibrium portfolio

By using the indifference curve (5), we find that the expected utility level enjoyed by the typical investor at the initial equilibrium portfolio E is

$$\bar{\bar{V}}_E = \alpha_1 \bar{r}_E - \alpha_2 \bar{r}_E^2 - \alpha_2 \sigma_E^2, \tag{57}$$

where $\bar{r}_E \equiv \mathrm{E}(r_E)$ and $\sigma_E^2 \equiv \mathrm{var}(r_E)$.

4.2 The initial equilibrium portfolio with bubbled returns on risky assets

Suppose that, due to the occurence of a bubble, the typical investor expects to collect returns r_f, $\bar{r}_{1*}$ and $\bar{r}_{2*}$ for his/her initial equilibrium portfolio E with weights w_{0_E}, w_{1_E} and w_{2_E}, respectively. In such a case, the typical investor misconceives his/her actual situation. As a matter of fact, the typical investor confuses the initial equilibrium portfolio E (on the actual capital-market line), with portfolio $\grave{E}$, which has the initial weights w_{0_E}, w_{1_E} and w_{2_E}, and unchanged associated risk, i.e.,

$$\sigma_{\grave{E}} \equiv \sigma_E, \tag{58}$$

but lies on the bubbled capital-market line (43), on which the bubbled equilibrium portfolio E_* is located. This means that the typical investor expects to collect increased real returns on the risky assets of his/her portfolio.

Therefore, since the slope of the bubbled capital-market line is $\sqrt{b_*}$, the expected real return of portfolio $\grave{E}$ is (see (43))

$$\begin{aligned} \bar{r}_{\grave{E}} &= r_f + \sigma_E \sqrt{b_*} = r_f + \sigma_E \sqrt{b} + \sigma_E (\sqrt{b_*} - \sqrt{b}) \\ &= \bar{r}_E + k_0 \sigma_E \\ &> \bar{r}_E, \end{aligned} \tag{59}$$

where

$$k_0 = \sqrt{b_*} - \sqrt{b} > 0, \tag{60}$$

i.e., k_0 is a positive scalar because $b_* > b$ (see Theorem 1). This means that, since $\grave{E}$ is located on the bubbled capital-market line (43), its expected real return, $\bar{r}_{\grave{E}} \equiv \mathrm{E}(r_{\grave{E}})$, is greater than the expected real return, $\bar{r}_E \equiv \mathrm{E}(r_E)$, of the initial equilibrium portfolio, E. As a consequence of this result, the following theorem holds.

Theorem 2. *The occurence of a bubble increases the expected utility enjoied by the typical investor for the initial equilibrium portfolio E, i.e., $\bar{\bar{V}}_{\grave{E}} > \bar{\bar{V}}_E$.*

Note that, under the bubble effect, the typical investor misconceives the actual market situation, and believes that his/her initial portfolio is located at point $\grave{E}$, which is the intersection of the bubbled capital-market line (43) and a higher indifference curve, denoted as $\bar{\bar{V}}_{\grave{E}}$. But since this higher indifference curve, is not tangent to the bubbled capital-market line, the typical investor has the motive to select the bubbled equilibrium portfolio E_*, in order to maximise his/her expected utility.

4.3 The bubbled equilibrium portfolio

According to (57), we find that the expected utility level enjoyed by the typical investor at the bubbled equilibrium portfolio E_* is

$$\bar{\bar{V}}_{E_*} = \alpha_1 \bar{r}_{E_*} - \alpha_2 \bar{r}^2_{E_*} - \alpha_2 \sigma^2_{E_*}, \tag{61}$$

where $\bar{r}_{E_*} \equiv \mathrm{E}(r_{E_*})$ and $\sigma^2_{E_*} \equiv \mathrm{var}(r_{E_*})$.

Portfolio E_* maximises the expected utility of the typical investor under the assumption that the bubbled capital-market line (43) is valid. And according to Theorem 1, the slope, $\sqrt{b_*}$, of (43) is greater than the slope, $\sqrt{b}$, of the initial capital-market line (15). This result implies the following

Theorem 3. *The expected utility enjoyed by the typical investor for the bubbled equilibrium portfolio E_* exceeds the expected utilities of both portfolios $\grave{E}$ and E, i.e., $\bar{\bar{V}}_{E_*} > \bar{\bar{V}}_{\grave{E}}$ and $\bar{\bar{V}}_{E_*} > \bar{\bar{V}}_E$.*

Thus, due to the misconception of capital-market conditions, the resulting equilibrium portfolio, E_*, under the bubble effect, is located on a higher indifference curve than the initial equilibrium portfolio E. This makes the typical investor willing to accept greater risk and invest a greater percentage of his/her capital in risky assets, decreasing, at the same time, his/her investments in risk-free assets. What is not at all clear, however, at the time of investing in portfolio E_*, is the fact that the only compensations possibly collected by the typical investor are, in the long run, those indicated by the initial (not bubbled) capital-market line with slope $\sqrt{b}$. The implications of this will be pursued in the following subsection.

4.4 The bubbled equilibrium portfolio with initial returns

Suppose that, due to the occurence of a bubble, and the corresponding expected real returns r_f, $\bar{r}_{1*}$ and $\bar{r}_{2*}$, the typical investor undertakes greater risk, equal to σ_{E_*}, and invests his/her capital in portfolio E_*. Whatever the expectations, however, the actual capabilities of financial markets to compensate risks are, in the long run, only those indicated by the real returns r_f, $\bar{r}_1$ and $\bar{r}_2$, paid by the actual (not bubbled) capital-market line, whose slope is $\sqrt{b} < \sqrt{b_*}$.

Thus, as a result of his/her misconception of actual market conditions, the typical investor is trapped in an awkward situation. As a matter of fact, the typical investor comes to understand that he/she has actually invested, not in portfolio E_*, but in portfolio $\grave{E}_*$, with weights $w_{0_{E_*}}$, $w_{1_{E_*}}$ and $w_{2_{E_*}}$, and associated risk, i.e.,

$$\sigma_{\grave{E}_*} \equiv \sigma_{E_*}, \tag{62}$$

which lies on the actual (not bubbled) capital-market line. This means that, although the typical investor expects to collect returns on the bubbled capita-market line (43), he/she is actually confined to collect returns on the initial capital-market line (15), on which the initial equilibrium portfolio, E, is located.

Therefore, since the slope of the actual capital-market line is $\sqrt{b}$, the expected real return of portfolio $\grave{E}_*$ is (see (15))

$$\begin{aligned} \bar{r}_{\grave{E}_*} &= r_f + \sigma_{\grave{E}_*}\sqrt{b} = r_f + \sigma_{\grave{E}_*}\sqrt{b_*} + \sigma_{\grave{E}_*}(\sqrt{b} - \sqrt{b_*}) \\ &= \bar{r}_{E_*} - k_0\sigma_E \\ &< \bar{r}_{E_*}, \end{aligned} \tag{63}$$

where $k_0 = \sqrt{b_*} - \sqrt{b}$ is the positive scalar defined in (60). This means that, since $\grave{E}_*$ is located on the initial capital-market line (15), its expected real return, $\bar{r}_{\grave{E}_*} \equiv \mathrm{E}(r_{\grave{E}_*})$, is smaller than the expected real return, $\bar{r}_{E_*} \equiv \mathrm{E}(r_{E_*})$, of equilibrium portfolio E_*, under the bubble effect.

Moreover, the weights $w_{0_{E_*}}$, $w_{1_{E_*}}$ and $w_{2_{E_*}}$ of portfolio $\grave{E}_*$ are chosen so as to maximise the expected real return of portfolio E_* on the bubbled capital-market line (43). This implies that the risk undertaken by the typical investor is $\sigma_{\grave{E}_*} \equiv \sigma_{E_*} > \sigma_E$ (see (62)). That is, portfolio $\grave{E}_*$ is located at the right-hand side of equilibrium portfolio E, on the initial capital-market line (15). As a matter of fact, portfolio $\grave{E}_*$ is located at the intersection of the initial capital-market line (15) with an indifference curve, denoted as $\bar{\bar{V}}_{\grave{E}_*}$, which is lower than the tangent indifference curve $\bar{\bar{V}}_E$. The following theorem describes the situation.

Theorem 4. *The expected utility enjoyed by the typical investor for portfolio $\grave{E}_*$ is inferior to the expected utility of the initial equilibrium portfolio E, i.e., $\bar{\bar{V}}_{\grave{E}_*} < \bar{\bar{V}}_E$.*

This theorem proves that the occurence of a bubble in the financial markets worsens the position of the typical investor. In fact, the typical investor misunderstands the actual market conditions and, as a consequence, he/she expects to collect greater returns on the risky assets. This, in turn leads to overinvestment in risky assets. However, the long run capabilities of the financial markets to compansate risks *cannot* be changed, due to misconceptions of market conditions. As a result, the typical investor can only collect returns on the initial, i.e., the actual capital-market line, and these returns *do not* fully compensate the investor for the excessive risks undertaken, due to the occurence of the bubble. In other words, the typical investor ends in worse off, under the occurence of a financial bubble.

5 Discussion

The analysis up til now signifies that there are four portfolios of outmost importance for the typical investor:

(i) portfolio E, which is the optimal portfolio on the actual capital-market line before the occurence of the bubble,

(ii) portfolio $\grave{E}$, on the bubbled capital-market line, which depicts the misconception of market conditions by the typical investor under the bubble effect,

(iii) portfolio E_*, which is the optimal portfolio on the bubbled capital-market line, and

(iv) portfolio $\grave{E}_*$, on the actual capital-market line, which depicts the actual (long run) situation of the typical investor under the bubble effect.

These four portfolios, taken together, describe the full cycle of events during any bubble-occurence insident as follows:

Before the occurence of the bubble, the typical investor holds the optimal portfolio, E, and expects to collect real returns $\bar{r}_E$ on the actual capital-market line. The occurence of the bubble, however, leads the typical investor to believe that he/she holds portfolio $\grave{E}$, on the bubbled capital-market line, with expected real returns $\bar{r}_{\grave{E}} > \bar{r}_E$.

Since the bubbled risk premia, $\bar{r}_{i*} - r_f$ $(i = 1, 2)$, are greater than the corresponding actual risk premia, $\bar{r}_i - r_f$, the typical investor is significantly motivated to expand his/her exposure to risk, in order to maximise the expected real returns, given the risks undertaken. As a consequence, under the bubble effect, the typical investor moves along the new, misconceived (bubbled) capital-market line towards the corresponding optimal portfolio, E_*.

Although, in the short run, the bubble effect seems to be self-confirming, giving rise to higher-than-normal, actually collected real returns, the situation becomes very different in the long run. Due to the misconception of market conditions, the higher-than-normal real returns, realised in the short run, cannot be maintained for long periods of time. In fact, any possible misconception of market conditions vanishes in the long run, due to the accumulation of all available information.

Therefore, the only real returns, possibly collected in the long run, are those determined by the actual (not bubbled) capital-market line. This means that the actually collected real returns on portfolio E_* are those corresponding to portfolio $\grave{E}_*$, which is located on the initial capital-market line. But since portfolio $\grave{E}_*$ corresponds to higher exposure to risk compared to the initial optimal portfolio E, the typical investor suffers expected utility losses, in the long run, due to the occurence of the bubble.

These thoughts bring into perspective an inquiry into the precautionary actions needed to possibly avoid the negative consequences of the occurence of financial bubbles. Such precautionary actions involve the role of capital-market analysts, the responsibilities of the capital-market authorities, and the actions of the issuers of the risk-free assets, discussed in the rest of this section.

5.1 The role of analysts

In modern capitalist economies, there is intensive competition between investment projects to attract funding, and this competition becomes even more severe, especially for new investment projects of high-technology industries. This competition triggers insentives, which, if not tamed properly, might make it almost imperative for the financial analysts to over-state the expected real returns on these investments. But since the actual market evaluation of these investments does not necessarily confirm the analysts' predictions, there will be significant possibilities of over-optimistic expectations. These, in turn, lead to greater-than-normal expected real returns and increase the probability of occurence of a financial bubble.

5.2 The responsibilities of authorities

Since the motivation of the individual analyst might not be self-contained, this necessity must be the responsibility of the capital-market authorities. However, in modern capitalis economies, the capital-market authorities, more often than not, tend to act in favour of deregulation—or medium regulation—of the financial markets, in order to facilitate the highest possible advantages of (almost) perfect competition. Therefore, given the insentives of the financial analysts and the behavioural tendencies of the capital-market authorities, the most important factor, capable to prevent the occurence of financial bubbles, seems to be the behaviour of the issuers of the risk-free assets, discussed in the sequel.

5.3 The actions of the issuers of the risk-free assets

The discipline of the issuers of risk-free assets, and their decision to maintain the risk-free return at level r_f, seem to make it possible for the financial bubble to end, without putting in jeopardy the prospects of the economy. Let us elaborate on this matter.

Due to the bubble effect, the increased risk premia, $\tilde{\boldsymbol{r}}_{i*} = \bar{r}_{i*} - r_f$ $(i = 1, 2)$, make risky investments seem more promising and preferable relative to their risk-free counterparts. So, the issuers of risk-free assets have to increase the risk-free return from its initial level, r_f, to a higher level, r_{f*} say, in order to maintain the share of risk-free investments. In fact, given the new, increased risk-free return, r_{f*}, the bubbled risk premia, $\tilde{\boldsymbol{r}}_{i*}$ $(i = 1, 2)$, will decrease to a lower level, $\tilde{\boldsymbol{r}}_{i*} = \bar{r}_{i*} - r_{f*}$ $(i = 1, 2)$. And the risk-free return will continue to increase until the risk premia decrease to their initial—before the bubble—level, i.e., $\tilde{\boldsymbol{r}}_{i*} = \bar{r}_{i*} - r_{f*} = \bar{r}_i - r_f = \tilde{\boldsymbol{r}}_i\,(i = 1, 2)$. This means that the risk-free return will continue to increase until the slope of the capital-market line decreases from the misconceived, bubbled value, $\sqrt{b_*}$, to its actual value, $\sqrt{b}$.

Although the increase of the risk-free return seems to solve the bubble problem, the solution comes at a very high cost, since it undermines any possibility to restore the market equilibrium in the pre-bubble setting of real risk premia. This happens for two reasons:

Fisrt, in the short run, the increase of the risk-free return decreases the risk premia, which, in turn, blurs the actual picture of the financial markets by creating a veil that covers the bubble effect. Thus, it is becoming more difficult for the typical investor to measure the actual magnitude of the risk undertaken, and, as a result, it leads to the realisation of all-the-more risky investments, over and above the optimal level for the economy.

Second, since the new, bubbled risk-free return, r_{f*}, is errorneously considered to represent the time value of monetary capital, it may produce catastrofic effets on the economy, in the long run. Because, this increased time value of money provokes the increase of the risky real returns in even higher levels, in order to maintain the equilibrium level of the corresponding risk premia. And theses higher real returns on risky assets render unprofitable a large number of investment projects at the same time they are mostly needed, since the increased amount of capital invested would diminish the risky returns to their long run (equilibrium) level.

For the economy to be speared of all these negative events, it is imperative that the issuers of risk-free assets, i.e., countries and private-sector companies with no default risk, be disciplined enough to maintain the risk-free return to its actual, pre-bubbled equilibrium level. Given that the risk-free return is, in fact, the time value of money, it measures the return which compensates for holding the risk-free assets up to maturity. And any increase of the risk-free return over the time-value-of-money level, can only be interpreted as an indication of the corresponding assets being risky. This means that only the long run share of risk-free assets is to be maintained by their issuers.

Therefore, it is the ability of the issuers of the risk-free assets to understand the situation and accept short run decreases under their optimal risk-free assets share, that can prevent the vicius circle of prolonged bubbled risk premia, which could lead to an ever lasting series of bubble occurences.

6 Concluding remarks

The misconception of economic condistions lead the typical investor to adjust his/her expected real returns on risky assets given the risks undertaken. This triggers a series of events, known as a financial bubble. Under the bubble effect, the increased risk premia motivate the expansion of the exposure to risk and the realisation of more-than-optimal risky investments. However, since the only risky returns, possibly collected in the long run, are those produced by the initial (pre-bubbled) capital-market line, the typical investor sufferes expected utility losses due to financial-bubble insidents.

Financial analysts, capital-market authorities and issuers of risk-free assets have a role to play in order to prevent the lengthening of the duration of the bubble. Since only the behaviour of the issuers of risk-free assets involves actual money investments, their role in managing financial bubbles seems to be of outmost importance. For a financial bubble to end in a short time and without any severe consequences, it is imperative that the issuers of risk-free assets be disciplined enough to accept short run decreases of their market share under the equilibrium level of risk-free investments.

Appendix

Proof of Theorem 1. Since the weights of the market portfolio and its associated risk premia are positive quantities, equation (20) implies that

$$b_1 - b_2 r_f > 0, \tag{A.1}$$

which, in turn, implies that

$$b_0 - b_1 r_f > 0, \tag{A.2}$$

since the expected real return on market portfolio, given in equation (21), is also positive. Then, the first equality in (45) completes the proof, since b_0, which is defined in (18), and l are both positive. □

Preliminary results

In the next theorems, we provide some useful preliminary results, in order to facilitate the proof of the main results given ... .

Lemma A.1. *Define the scalars*

$$k_1 = \frac{b_*(1+b)}{b(1+b_*)} \quad \text{and} \quad k_2 = \frac{\sqrt{b_*}(1+b)}{\sqrt{b}(1+b_*)}. \tag{A.3}$$

The following results hold:

$$k_1^2 - 1 = \frac{b_*^2(1+b) - b^2(1+b_*) + bb_*(b_* - b)}{b^2(1+b_*)^2}, \tag{A.4}$$

$$k_1(k_1 - 1) = \frac{b_*^2 - bb_* + bb_*^2 - b^2 b_*}{b^2(1+b_*)^2}, \tag{A.5}$$

$$k_2^2 - 1 = \frac{(b_* - b)(1 - bb_*)}{b(1+b_*)^2}. \tag{A.6}$$

Proof of Lemma A.1. Since both b and b_* are positive scalars and $b_* > b$ (see Theorem 1), the first definition in (A.3) implies that

$$k_1 = \frac{b_* + bb_*}{b + bb_*} > 1 \implies k_1 - 1 = \frac{b_* - b}{b(1+b_*)} > 0. \tag{A.7}$$

Similarly, the second definition in (A.3) implies that

$$k_2^2 = \frac{b_*(1+b)^2}{b(1+b_*)^2} = \frac{b}{b_*}\frac{b_*^2(1+b)^2}{b^2(1+b_*)^2} = \frac{b}{b_*}k_1^2 \implies k_2^2 < k_1^2. \tag{A.8}$$

Given that

$$k_1^2 - 1 = \frac{b_*^2(1+b)^2 - b^2(1+b_*)^2}{b^2(1+b_*)^2}, \tag{A.9}$$

simple algebra completes the proof of (A.4). Since, by using (A.8), we can write

$$k_1(k_1 - 1) = \frac{b_*(1+b)}{b(1+b_*)}\frac{b_* - b}{b(1+b_*)}, \tag{A.10}$$

the proof of (A.5) is straightforward. The result (A.6) can be easily proven, given that

$$k_2^2 - 1 = \frac{b_*(1+b)^2 - b(1+b_*)^2}{b(1+b_*)^2}. \tag{A.11}$$

□

Lemma A.2. *For scalars k_1 and k_2, defined in* (A.3)*, the following results hold:*

$$b(k_1^2-1)(k_2^2-1) = \frac{(b_*-b)(1+b)}{b(1+b_*)} > 0 \tag{A.12}$$

and

$$\frac{b(k_1^2-1)(k_2^2-1)}{b} = (k_1-1)\frac{1+b}{b} > 0. \tag{A.13}$$

Proof of Lemma A.2. By using (A.4), (A.6) and simple algebra, we can easily prove the equality in (A.12), which, combined with (A.7), implies the equality in (A.13). Moreover, given that both b and b_* are positive, $b_* > b$ (see Theorem 1) and $k_1 - 1 > 0$ (see (A.7)), the inequalities in (A.12) and (A.13) are straightforward. □

Main results

In what follows we provide proofs of the main results of the paper.

Proof of Theorem 2. Equation (59) implies that

$$\bar{r}_{\grave{E}}^2 = \bar{r}_E^2 + 2k_0\sigma_E\bar{r}_E + k_0^2\sigma_E^2. \tag{A.14}$$

By using the third inequality in (53) we can show that

$$\alpha_1 - 2\alpha_2\bar{r}_{\grave{E}} > 0, \tag{A.15}$$

which, combined with (59), implies that

$$\begin{aligned} & \alpha_1 - 2\alpha_2(\bar{r}_E + k_0\sigma_E) > 0 \\ \Longrightarrow\ & (\alpha_1 - 2\alpha_2\bar{r}_E) - 2\alpha_2k_0\sigma_E \\ \Longrightarrow\ & (\alpha_1 - 2\alpha_2\bar{r}_E) > 2\alpha_2k_0\sigma_E > \alpha_2k_0\sigma_E > 0, \end{aligned} \tag{A.16}$$

given that a_2, k_0 and σ_E are all positive. By using this result we can show that

$$k_0\sigma_E(\alpha_1 - 2\alpha_2\bar{r}_E - \alpha_2k_0\sigma_E) > 0. \tag{A.17}$$

Since $\sigma_{\grave{E}} \equiv \sigma_E$, equations (57), (60), (A.14) and (A.17) imply that

$$\begin{aligned} \bar{\bar{V}}_{\grave{E}} &= \alpha_1\bar{r}_{\grave{E}} - \alpha_2\bar{r}_{\grave{E}}^2 - \alpha_2\sigma_{\grave{E}}^2 \\ &= \alpha_1(\bar{r}_E + k_0\sigma_E) - \alpha_2\bar{r}_E^2 + 2k_0\sigma_E\bar{r}_E + k_0^2\sigma_E^2) - \alpha_2\sigma_E^2 \\ &= [\alpha_1\bar{r}_E - \alpha_2\bar{r}_E^2 - \alpha_2\sigma_E^2] + k_0\sigma_E(\alpha_1 - 2\alpha_2\bar{r}_E - \alpha_2k_0\sigma_E) \\ &= \bar{\bar{V}}_E + k_0\sigma_E(\alpha_1 - 2\alpha_2\bar{r}_E - \alpha_2k_0\sigma_E) \\ &> \bar{\bar{V}}_E. \end{aligned} \tag{A.18}$$

□

Proof of Theorem 3. The formulae for $\bar{r}_E$ and $\bar{r}_{E_*}$ in (28) and (55), respectively, imply that

$$\bar{r}_E - r_f = \frac{b}{1+b}\left[\frac{\alpha_1}{2\alpha_2} - r_f\right] \quad \text{and} \quad \bar{r}_{E_*} - r_f = \frac{b_*}{1+b_*}\left[\frac{\alpha_1}{2\alpha_2} - r_f\right]. \tag{A.19}$$

This result, combined with the definition of scalar k_1 given in (A.3), implies that

$$\begin{aligned} \bar{r}_{E_*} - r_f &= \frac{b_*(1+b)}{b(1+b_*)}(\bar{r}_E - r_f) \\ &= k_1(\bar{r}_E - r_f). \end{aligned} \tag{A.20}$$

By using (16), (44) and (A.20), and the definition of scalar k_2 given in (A.3), we can write

$$\begin{aligned}\sigma_{E_*} &= \frac{\bar{r}_{E_*}-r_f}{\sqrt{b_*}} = \frac{1}{\sqrt{b_*}}\frac{b_*(1+b)}{b(1+b_*)}(\bar{r}_E - r_f) = \frac{1}{\sqrt{b_*}}\frac{b_*(1+b)}{b(1+b_*)}\sqrt{b}\sigma_E \\ &= \frac{\sqrt{b_*}(1+b)}{\sqrt{b}(1+b_*)}\sigma_E \\ &= k_2\sigma_E. \end{aligned} \tag{A.21}$$

Thus, since $\sigma_{\grave{E}} \equiv \sigma_E$ (see (58)), we can write

$$\sigma^2_{E_*} = k_2^2\sigma^2_{\grave{E}} = \sigma^2_{\grave{E}} + (k_2^2-1)\sigma^2_{\grave{E}}, \tag{A.22}$$

which, taken together with equations (15) and (43), and the first equality in (59), implies that

$$\begin{aligned}\bar{r}_{E_*} &= r_f + \sigma_{E_*}\sqrt{b_*} = r_f + k_2\sigma_{\grave{E}}\sqrt{b_*} = r_f + [\sigma_{\grave{E}} + (k_2-1)\sigma_{\grave{E}}]\sqrt{b_*} \\ &= (r_f + \sigma_{\grave{E}}\sqrt{b_*}) + (k_2-1)\sigma_{\grave{E}}\sqrt{b_*} \\ &= \bar{r}_{\grave{E}} + (k_2-1)\sigma_{\grave{E}}\sqrt{b_*} \end{aligned} \tag{A.23}$$

and

$$\bar{r}^2_{E_*} = \bar{r}^2_{\grave{E}} + 2(k_2-1)\sqrt{b_*}\,\bar{r}_{\grave{E}}\sigma_{\grave{E}} + (k_2-1)^2 b_*\sigma^2_{\grave{E}}. \tag{A.24}$$

Further, by using the definition of scalar k_2 given in (A.3), we take

$$k_2(1+b_*) = \frac{\sqrt{b_*}}{\sqrt{b}}(1+b). \tag{A.25}$$

Then, by combining equations (61), (A.18), (A.22), (A.23), (A.24) and (A.25), we can write

$$\begin{aligned}\bar{\bar{V}}_{E_*} &= \alpha_1\left[\bar{r}_{\grave{E}} + (k_2-1)\sigma_{\grave{E}}\sqrt{b_*}\right] \\ &\quad -\alpha_2\left[\bar{r}^2_{\grave{E}} + 2(k_2-1)\sqrt{b_*}(r_f+\sigma_{\grave{E}}\sqrt{b_*})\sigma_{\grave{E}} + (k_2-1)^2 b_*\sigma^2_{\grave{E}}\right] \\ &\quad -\alpha_2\left[\sigma^2_{\grave{E}} + (k_2^2-1)\sigma^2_{\grave{E}}\right] \\ &= \bar{\bar{V}}_{\grave{E}} + 2\alpha_2(k_2-1)\frac{\sqrt{b_*}}{\sqrt{b}}(1+b)\sigma^2_{\grave{E}} - \alpha_2(k_2-1)(k_2+1)(1+b_*)\sigma^2_{\grave{E}} \\ &= \bar{\bar{V}}_{\grave{E}} + \alpha_2(k_2-1)[2k_2(1+b_*) - (k_2+1)(1+b_*)]\sigma^2_{\grave{E}} \\ &= \bar{\bar{V}}_{\grave{E}} + \alpha_2(k_2-1)^2(1+b_*)\sigma^2_{\grave{E}} \\ &> \bar{\bar{V}}_{\grave{E}} \end{aligned} \tag{A.26}$$

because $\alpha_2 > 0$ and $b_* > 0$. Moreover, since $\bar{\bar{V}}_{\grave{E}} > \bar{\bar{V}}_E$ (see Theorem 2), equation (A.26) implies that $\bar{\bar{V}}_{E_*} > \bar{\bar{V}}_E$, which completes the proof. □

Proof of Theorem 4. Since $\sigma_{\grave{E}_*} \equiv \sigma_{E_*}$ (see (62)), equation (A.21) implies that

$$\sigma_{\grave{E}_*} = k_2\sigma_E \implies \sigma^2_{\grave{E}_*} = k_2^2\sigma^2_E = \sigma^2_E + (k_2^2-1)\sigma^2_E. \tag{A.27}$$

By using equations (15) we can write that

$$\begin{aligned}\bar{r}_{\grave{E}_*} &= r_f + \sigma_{\grave{E}_*}\sqrt{b} = r_f + k_2\sigma_E\sqrt{b} = r_f + [\sigma_E + (k_2-1)\sigma_E]\sqrt{b} \\ &= (r_f + \sigma_E\sqrt{b}) + (k_2-1)\sigma_E\sqrt{b} \\ &= \bar{r}_E + (k_2-1)\sigma_E\sqrt{b}, \end{aligned} \tag{A.28}$$

which, in turn, implies that

$$\bar{r}^2_{\hat{E}_*} = \bar{r}^2_E + 2(k_2 - 1)\sqrt{b}\,\bar{r}_E\sigma_E + (k_2 - 1)^2 b\sigma^2_E. \tag{A.29}$$

Then, by combining equations (57), (A.27), (A.28) and (A.29), we can write

$$\begin{aligned}
\bar{\bar{V}}_{\hat{E}_*} &= \alpha_1\bar{r}_{\hat{E}_*} - \alpha_2\bar{r}^2_{\hat{E}_*} - \alpha_2\sigma^2_{\hat{E}_*} \\
&= \alpha_1\left[\bar{r}_E + (k_2 - 1)\sigma_E\sqrt{b}\right] \\
&\quad -\alpha_2\left[\bar{r}^2_E + 2(k_2 - 1)\sqrt{b}(r_f + \sigma_E\sqrt{b})\sigma_E + (k_2 - 1)^2 b\sigma^2_E\right] \\
&\quad -\alpha_2\left[\sigma^2_E + (k^2_2 - 1)\sigma^2_E\right] \\
&= \bar{\bar{V}}_E + 2\alpha_2(k_2 - 1)(1 + b)\sigma^2_E - \alpha_2(k_2 - 1)(k_2 + 1)(1 + b)\sigma^2_E \\
&= \bar{\bar{V}}_E + \alpha_2(k_2 - 1)[2 - (k_2 + 1)](1 + b)\sigma^2_E \\
&= \bar{\bar{V}}_E - \alpha_2(k_2 - 1)^2(1 + b)\sigma^2_E \\
&< \bar{\bar{V}}_E
\end{aligned} \tag{A.30}$$

because $\alpha_2 > 0$ and $b > 0$. □

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