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TESTING FOR PROSPECT AND MARKOWITZ STOCHASTIC DOMINANCE EFFICIENCY

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Abstract

We consider non parametric tests for Prospect Stochastic Dominance Efficiency (PSDE) and Markowitz Stochastic Dominance Efficiency (MSDE) based on block bootstrap resampling schemes. The PSDE and the MSDE criteria determine stochastic dominance of benchmark portfolios over any other portfolio with respect to the set of prospect S-shaped utility functions and the set of reverse S-shaped utility functions analogously. We first derive consistency and then formulate algorithms for the computation of the test statistics and the approximation of the asymptotic critical values by the use of linear and mixed integer programs. We engage into Monte Carlo experiments. Empirical results indicate that the market portfolio is only prospect stochastic dominance efficient. This implies that there are S-shaped utility functions that rationalize the market portfolio, while the market portfolio is not efficient relative to reverse S-shaped utility functions.

Key words and phrases: Non parametric test, Stochastic Dominance, Prospect Stochastic Dominance Efficiency, Markowitz Stochastic Dominance Efficiency, Linear Programming, Mixed Integer Programming, Block Bootstrap, Consistency.

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1 Introduction

The portfolio problem is essentially an optimal choice problem between probability measures. The theory of the choice under uncertainty conditions has been mainly developed under the assumption that investors maximize their expected utility. In particular, many studies assume that investors act as non satiable and risk averse agents and thus they should have increasing and concave utility functions. For this reason most of the criteria to verify the efficiency of a given portfolio (see, among others, Gibbons, Ross, and Shenken (1989)) are based on the first and second stochastic dominance rules, see e.g. the papers by Kroll and Levy (1980) and Levy (1992), and the excellent monograph on the theory of stochastic dominance by Levy (2006).

In the literature several parametric methods have been proposed to test the mean risk efficiency of a given portfolio. However, the return distributions could depend on many parameters. In addition, the investor's behavior is not known, except in some obvious circumstances. As a matter of fact, while it is obvious that investors should prefer more to less, several behavioral finance analysis indicate that investors are neither risk preferring nor risk averting. Furthermore, it is not clear that the preferences of investors remain constant under periods of financial distress, in which the likelihood of adverse events is bound to increase. On the other hand, economic theory gives only minimal guidance for selecting the appropriate risk measure.

Examples of risk orderings are the dominance rules of behavioral finance (see Friedman and Savage (1948), Baucells and Heukamp (2006), Edwards(1996), and the references therein). Markowitz (1952) suggests that individuals are risk averse for losses and risk seeking for gains, as long as the outcomes are not very extreme. Kahneman and Tversky (1979) proposed prospect theory for decision making under ucncertainty. The theory was further developed by Tversky and Kahneman (1992) into cumulative prospect theory in order to be consistent with first-order stochastic dominance. Levy and Levy (2002) tested prospect theory using an experimental design with risky gambles that involved both gains and losses (mixed gambles). They found severe violations of cumulative prospect theory and they argue that the choices of their subjects suggest risk aversion for losses and risk seeking for gains. This is captured by their Markowitz stochastic dominance criterion. It should be noted that Levy and Levy (2002) assumed linear weighting function. Hence, their notion of SD cannot be properly called prospect stochastic dominance (see Wakker (2003)).

In the context of financial asset allocation, investors evaluate assets in comparison with certain benchmarks, such as the market portfolio, rather than on final wealth positions. Moreover, it is now true that investors behave differently on gains and losses, they are not uniformly risk averse or risk lovers, and one can say that they are more sensitive to losses than to gains. In addition, the value function (or utility function) could be either concave for gains and convex for losses (such as the S-shaped function) or convex for gains and concave for losses (such as the reverse S-shaped function).

All these observations motivate the use of prospect stochastic dominance (PSD) and Markowitz stochastic dominance (MSD) to analyze investor behavior. In this paper we develop non-parametric tests for Prospect Stochastic Dominance Efficiency (PSDE) and Markowitz Stochastic Dominance (MSDE). PSDE is a criterion that determines the dominance of the benchmark portfolio over any other portfolio that can be constructed from a set of assets for all prospect S-shaped utility functions. MSDE is a criterion that determines the dominance of the benchmark portfolio over any other portfolio that can be constructed for all reverse S-shaped utility functions.

The two testing procedures developed, allow us to infer whether a given portfolio lying on a set of portfolios can be considered as an optimal choice if investor preferences are characterized by the Prospect theory or the Markowitz paradigms respectively. In an empirical application, we test whether the market portfolio is efficient under PSD and MSD criteria, relative to benchmark portfolios formed on size and book-to-market equity ratio (BE/ME). Given that all individual investors hold efficient portfolios, it is under question whether the market, which is the weighted average of all individual portfolios, is efficient. The motivation to test for the efficiency of the market portfolio is that many institutional investors invest in Exchange-Traded Funds (ETFs) and mutual funds. These funds track stocks, commodities and bonds, or value-weighted equity indices which strongly resemble the market portfolio. Thus, it is interesting to ask what kind of utility functions could rationalize such behavior.

The goal of this paper is to develop consistent tests for prospect and Markowitz stochastic dominance efficiency for *time-dependent* data. Serial correlation is known to pollute financial data (see the empirical section), and to alter, often severely, the size and power of testing procedures when neglected. We rely on statistics that are maxima between elementwise Kolmogorov-Smirnov type statistics on vector valued processes. They are inspired by the consistent procedures developped by Barrett and Donald (2003) and extended by Horvath, Kokoszka, and Zitikis (2006) to accommodate non-compact support. Scaillet and Topaloglou (2012) develop consistent tests for stochastic dominance efficiency at any order for time-dependent data (see also Linton, Post and Wang (2005)), relying on weighted Kolmogorov-Smirnov type statistics in testing for stochastic dominance. Post and Levy (2005) test for SSD, PSD and MSD efficiency of the market portfolio relative to portfolios formed on size, BE/ME, and momentum. They find that the market portfolio is only MSD efficient.

The paper is organized as follows. In Section 2, having in mind the notion of stochastic dominance efficiency introduced by Kuosmanen (2004) and Post (2003), we discuss the general hypotheses for testing prospect and Markowitz stochastic dominance efficiency. We describe the test statistics, and analyse the asymptotic properties of the testing procedures. We also use simulation based procedures to compute p-values. We rely on a block bootstrap method, and explain this in Section 3. Note that other resampling methods such as subsampling are also available (see Linton, Maasoumi and Whang (2005) for the standard stochastic dominance tests). Linton, Post and Whang (2005) follow this route in the context of testing procedures for stochastic dominance efficiency. They use subsampling to estimate the p -values, and discuss power issues of the testing procedures. We prefer block bootstrap to subsampling since the former uses the full sample information. The block bootstrap is better suited to samples with a limited number of time-dependent data: we have 996 monthly observations in our empirical application.

The test statistics for both prospect and Markowitz stochastic dominance efficiency are formulated in terms of linear as well as mixed integer programming. Widely available algorithms can be used to compute both test statistics. We discuss in detail the computational aspects of mathematical programming formulations corresponding to the test statistics in Section 4.

In Section 5 we design a Monte Carlo study to evaluate actual size and power of the proposed tests in finite samples. In Section 6 we provide an empirical illustration. We analyze whether the Fama and French market portfolio can be considered as efficient according to prospect and Markowitz stochastic dominance criteria when confronted to diversification principles made of six Fama and French benchmark portfolios formed on size and book-to-market equity ratio. We find that the market portfolio is prospect stochastic dominance efficient but Markowitz stochastic dominance inefficient. We give some concluding remarks in Section 7. Proofs and detailed mathematical programming formulations are gathered in an appendix.

2 Consistent Tests for Prospect and Markowitz Stochastic Dominance Efficiency

2.1 Assumption Framework and Hypotheses Structures

Consider a *strictly stationary* process $(\boldsymbol{Y}_t)_{t\in\mathbb{Z}}$ taking values in $\mathbb{R}^n$. The observations consist of a realization of the random element $(\boldsymbol{Y}_t)_{t=1,\ldots,T}$. They correspond to observed returns of n financial assets. Let F denote the cdf of $\boldsymbol{Y}_0$ and $\hat{F}_T$ the *empirical* cdf associated to with the random element $(\boldsymbol{Y}_t)_{t=1,\ldots,T}$.

Assumption A.1. *F is everywhere continuous. Furthermore, $(\boldsymbol{Y}_t)_{t\in\mathbb{Z}}$ is a-mixing with mixing coefficients a_t such that $a_T = O(T^{-a})$ for some $a > 1$ as $T \to \infty$ (see Doukhan (1994) for relevant definition and examples).*

Assumption A.2. *Assumption A.1 holds and for some $\delta > 0$, $\mathbb{E}\,\|\boldsymbol{Y}_0\|^{2+\delta} < +\infty$. Furthermore $a > 1 + \frac{2}{\delta}$.*

Let $\mathbb{L}$ be an *infinite* and *compact* subset of $\{\lambda \in \mathbb{R}^n_+ : \mathbf{e}'\lambda = 1,\}$ with $\mathbf{e} = (1,\ldots,1)'$. For F^* the distribution function of some probability measure on $\mathbb{R}^n$ denote by $G(z,\lambda;F^*)$ the $\int_{\mathbb{R}^n} \mathbb{I}\{\lambda'\boldsymbol{u} \le z\}dF^*(\boldsymbol{u})$. The following are functionals that are useful for the definition and the derivation of the properties of the testing procedures that we later implement. Let

$$\mathcal{J}_2(z,\lambda;F^*) := \int_{-\infty}^{z} G(u,\lambda;F^*)du. \tag{1}$$

These are finite if $\mathbb{E}^*\left[(-\lambda'\boldsymbol{Y}_0)_+\right]$ exists, where $(x)_+ = \max(x,0)$ (Horvath, Kokoszka, and Zitikis (2006)) and $\mathbb{E}^*$ denotes the expectation operator w.r.t. F^*. From Davidson and Duclos (2000) Equation (2), we know that

$$\mathcal{J}_2(z,\lambda;F^*) = \int_{-\infty}^{z} (z-u)dG(u,\lambda,F^*),$$

which can be rewritten as

$$\mathcal{J}_2(z,\lambda;F^*) = \int_{\mathbb{R}^n} (z-\lambda'\boldsymbol{u})\mathbb{I}\{\lambda'\boldsymbol{u} \leq z\}dF^*(\boldsymbol{u}) \tag{2}$$

$\mathcal{J}_2$ is associated with the notion of second order stochastic dominance. Let also

$$\mathcal{J}\left(z,\lambda,\tau,F^*\right) \doteqdot \mathcal{J}_2(0,\tau;F^*) - \mathcal{J}_2(z,\tau;F^*) - (\mathcal{J}_2(0,\lambda;F^*) - \mathcal{J}_2(z,\lambda,F^*)) \tag{3}$$

For the following, let $\tau \in \mathbb{L}$. Let

$$\mathcal{J}_2^c(z,\lambda,\tau;F^*) := \int_z^{+\infty} \left(G\left(u,\lambda;F^*\right) - G\left(u,\tau;F^*\right)\right)du \tag{4}$$

It is easy to see (Lemma AL.1 in the Appendix) that $\mathcal{J}_2^c(z,\lambda;F^*)$ is finite if $\mathbb{E}^*\left[\lambda'\boldsymbol{Y}_0\right]$ and $\mathbb{E}^*\left[\tau'\boldsymbol{Y}_0\right]$ exist. When $\mathbb{E}\left[\boldsymbol{Y}_0\right]$ exists let $\mu_{\lambda'\boldsymbol{Y}} = \mathbb{E}\left[\lambda'\boldsymbol{Y}_0\right]$.

Prospect Stochastic Dominance Efficiency

Definition D.1. *τ is $\mathcal{PSD}$-efficient (see Levy and Levy (2002), equation (3)) iff*

$$\begin{aligned} \mathcal{J}\left(z,\lambda,\tau,F\right) &\leq 0, \textit{ for all } (z,\lambda) \in \mathbb{R}^- \times \mathbb{L}, \\ \textit{and } \mathcal{J}\left(z,\lambda,\tau,F\right) &\geq 0, \textit{ for all } (z,\lambda) \in \mathbb{R}^{++} \times \mathbb{L}. \end{aligned} \tag{5}$$

Any statistical test on the $\mathcal{PSD}$-efficiency of τ must comply to the hypothesis structure consisted of 5 as the null (say $H_0^{(\mathcal{PSD})}$), with alternative $H_1^{(\mathcal{PSD})}$:

$$\begin{aligned} \mathcal{J}\left(z,\lambda,\tau,F\right) &> 0, \text{for some } (z,\lambda) \in \mathbb{R}^- \times \mathbb{L}, \\ \text{or } \mathcal{J}\left(z,\lambda,\tau,F\right) &< 0, \text{for some } (z,\lambda) \in \mathbb{R}^{++} \times \mathbb{L}. \end{aligned} \tag{6}$$

Markowitz Stochastic Dominance Efficiency

Definition D.2. *τ is $\mathcal{MSD}$-efficient (see Levy and Levy (2002), equation (4) iff*

$$\begin{aligned} \mathcal{J}_2(z,\boldsymbol{\lambda};F) &\geq \mathcal{J}_2(z,\tau;F), \textit{ for all } (z,\lambda) \in \mathbb{R}^- \times \mathbb{L} \\ \textit{and } \mathcal{J}_2^c\left(z,\boldsymbol{\lambda},\tau;F\right) &\geq 0, \textit{ for all } (z,\lambda) \in \mathbb{R}^{++} \times \mathbb{L}. \end{aligned} \tag{7}$$

Similarly any statistical test on the $\mathcal{MSD}$-efficiency of τ must comply to the hypothesis structure consisted of 7 as the null (say $H_0^{(\mathcal{MSD})}$), with alternative $H_1^{(\mathcal{MSD})}$:

$$\begin{aligned} \mathcal{J}_2(z,\boldsymbol{\lambda};F) &< \mathcal{J}_2(z,\tau;F), \text{for some } (z,\lambda) \in \mathbb{R}^- \times \mathbb{L}, \\ \text{or } \mathcal{J}_2^c\left(z,\boldsymbol{\lambda},\tau;F\right) &< 0, \text{for some } (z,\lambda) \in \mathbb{R}^{++} \times \mathbb{L}. \end{aligned} \tag{8}$$

2.2 Infeasible K-S type Tests

Prospect Stochastic Dominance Efficiency Consider the following random variable, in the form of the maximum between a pair of Kolmogorov-Smirnov type statistics. It constitutes the statistic for an infeasible consistent test for the first form of efficiency studied here.

$$\hat{S}_T(\tau) = \max\left(\hat{S}_T^{\alpha}(\tau), \hat{S}_T^{\beta}(\tau)\right). \tag{9}$$

where

$$\hat{S}_T^{\alpha}(\tau) = \sup_{z\geq 0, \lambda\in\mathbb{L}} \sqrt{T}\mathcal{J}\left(z, -\lambda, -\tau, \hat{F}_T\right). \tag{10}$$

and

$$\hat{S}_T^{\beta}(\tau) = \sup_{z\geq 0, \lambda\in\mathbb{L}} -\sqrt{T}\mathcal{J}\left(z, \lambda, \tau, \hat{F}_T\right), \tag{11}$$

Furthermore for some $c_{\mathcal{PSD}} > 0$ consider the decision rule

$$\text{reject } H_0^{(\mathcal{PSD})} \text{iff } \hat{S}_T(\tau) > c_{\mathcal{PSD}}. \tag{12}$$

The following proposition demonstrates the consistency of the test based on 12.

Proposition 1. *Suppose that Assumption A.1 holds and that G satisfies*

$$\int_{\mathbb{R}} \sqrt{G(u, \boldsymbol{\lambda}, F)(1 - G(u, \boldsymbol{\lambda}, F))}du < +\infty, \text{for all } \lambda \in \mathbb{L}. \tag{13}$$

Then for the test based on decision rule 12, there exists a random variable $\bar{S}(\tau)$ such that:

1. *if $H_0^{(\mathcal{PSD})}$ is true, then*

$$\lim_{T\to\infty} P\left(\text{reject } H_0^{(\mathcal{PSD})}\right) \leq \lim_{T\to\infty} P\left(\bar{S}(\tau) > c_{\mathcal{PSD}}\right) \doteqdot \alpha(c_{\mathcal{PSD}}),$$

2. *if $H_0^{(\mathcal{PSD})}$ is false, then*

$$\lim_{T\to\infty} P\left(\text{reject } H_0^{(\mathcal{PSD})}\right) = 1.$$

Markowitz Stochastic Dominance Efficiency Analogously to the previous paragraph, consider the following random variable that has a similar form to the one used in the prospect theory testing procedure.

$$\hat{\Upsilon}_T(\tau) = \max\left(\hat{\Upsilon}_T^{\alpha}(\tau), \hat{\Upsilon}_T^{\beta}(\tau)\right). \tag{14}$$

where

$$\hat{\Upsilon}_T^{\alpha}(\tau) = \sup_{z\leq 0,\lambda\in\mathbb{L}} \sqrt{T}\left(\mathcal{J}_2(z,\tau;\hat{F}_T) - \mathcal{J}_2(z,\boldsymbol{\lambda},\hat{F}_T)\right), \tag{15}$$

and

$$\hat{\Upsilon}_T^{\beta}(\tau) = \sup_{z\geq 0,\lambda\in\mathbb{L}} \left[\frac{1}{\sqrt{T}}\sum\nolimits_{i=1}^{T}\left(\lambda' Y_i - \tau' Y_i\right) - \sqrt{T}\mathcal{J}\left(z,\lambda,\tau,\hat{F}_T\right)\right], \tag{16}$$

Furthermore for some $c_{\mathcal{MSD}} > 0$ consider the decision rule

$$\text{reject } H_0^{(\mathcal{MSD})} \text{iff } \hat{\Upsilon}_T(\tau) > c_{\mathcal{MSD}}. \tag{17}$$

The following proposition demonstrates the consistency of the test based on 17.

Proposition 2. *Suppose that Assumption A.2 holds. Then for the test based on decision rule 17, there exists a random variable* $\bar{\Upsilon}(\tau)$ *such that:*

1. *if* $H_0^{(\mathcal{MSD})}$ *is true, then*

$$\lim_{T\to\infty} P\left(\textit{reject } H_0^{(\mathcal{MSD})}\right) \leq \lim_{T\to\infty} P\left(\bar{\Upsilon}(\tau) > c_{\mathcal{MSD}}\right) \doteqdot \alpha\left(c_{\mathcal{MSD}}\right),$$

2. *if* $H_0^{(\mathcal{MSD})}$ *is false, then*

$$\lim_{T\to\infty} P\left(\textit{reject } H_0^{(M\mathcal{PSD})}\right) = 1.$$

2.3 K-S type Tests Based on Block Bootstrap

The previous testing procedures are generally non implementable due to the fact that in most cases the critical values are unknown. In this section we consider approximations based on bootstrap resampling techniques that incorporate the dependence that the data may exhibit. This is done in an analogous manner to the procedures described in section 3 of Scaillet and Topaloglou (2012).

Block bootstrap methods are based on "blocking" arguments, in which data are divided into blocks and those, rather than individual data, are resampled in order to mimick the time dependent structure of the original data. Let b_T, l_T denote integers such that $T = b_T l_T$. b_T denotes the number of blocks and l_T the block size. The following assumption rests on Theorem 2.2 of Peligrad (1998).

Assumption A.3. *For some* $0 < \rho < \frac{1}{3}$ *and some* $0 < h < \frac{1}{3} - \rho$, $T^h \ll l_T \ll T^{\frac{1}{3}-\rho}$ *and* $l_T = l_{2^k}$ *for* $2^k \leq T < 2^{k+1}$.

Presently we only allow for the case of non-overlapping the blocks (see the section on the numerical implementation). Let $(\boldsymbol{Y}_t^*)_{t=1,\ldots,T}$ denote a bootstrap sample arising by either methodology and let $\hat{F}_T^*$ denote its empirical distribution. Denote with $\mathbb{E}_T^*$ the expectation operator with respect to the probability measure induced by the sampling scheme. Under the current metodology we have that $\mathbb{E}_T^* \mathcal{J}_2\left(z, \lambda; \hat{F}_T^*\right) = \mathcal{J}_2\left(z, \lambda; \hat{F}_T\right)$ and $\mathbb{E}_T^* \frac{1}{T}\sum_{i=1}^T Y_i^* = \frac{1}{T}\sum_{i=1}^T Y_i$ (see Scaillet and Topaloglou(2012)). Hence we are able to define and study the consistency of the following approximations to the testing procedures defined previously.

Prospect Stochastic Dominance Efficiency Consider the bootstrapped analogues of the random variables appearing in the infeasible testing procedure for the $\mathcal{PSD}$ efficiency.

$$\hat{S}_T^*(\tau) = \max\left(\hat{S}_T^{\alpha*}(\tau), \hat{S}_T^{\beta*}(\tau)\right), \tag{18}$$

where

$$\hat{S}_T^{\alpha*}(\tau) = \sup_{z\geq 0, \lambda\in\mathbb{L}} \sqrt{T}\left(\mathcal{J}\left(z, -\lambda, -\tau, \hat{F}_T\right) - \mathcal{J}\left(z, -\lambda, -\tau, \hat{F}_T^*\right)\right), \tag{19}$$

and

$$\hat{S}_T^{\beta*}(\tau) = \sup_{z\geq 0, \lambda\in\mathbb{L}} -\sqrt{T}\left(\mathcal{J}\left(z, \lambda, \tau, \hat{F}_T^*\right) - \mathcal{J}\left(z, \lambda, \tau, \hat{F}_T\right)\right), \tag{20}$$

Define $p_{\mathcal{PSD}}^* := P[\hat{S}_T^*(\tau) > \hat{S}_T(\tau)]$ and consider the decision rule

$$\text{reject } H_0^{(\mathcal{PSD})} \text{iff } p_{\mathcal{PSD}}^* < \alpha. \tag{21}$$

The following proposition demonstrates the consistency of the test based on 21.

Proposition 3. *Suppose that Assumptions A.1 and A.3 hold,* G *satisfies the condition 13 in Proposition 1 and* $\alpha < \frac{1}{2}$. *Then the test based on decision rule 21 is consistent.*

Markowitz Stochastic Dominance Efficiency Again we consider the bootstrapped analogues of the random variables appearing in the infeasible testing procedure for the $\mathcal{MSD}$ efficiency.

$$\hat{\Upsilon}_T^*(\tau) = \max\left(\hat{\Upsilon}_T^{\alpha*}(\tau), \hat{\Upsilon}_T^{\beta*}(\tau)\right), \tag{22}$$

where

$$\hat{\Upsilon}_T^{\alpha_*}(\tau) = \sup_{z\leq 0,\lambda\in\mathbb{L}} \sqrt{T}\left(\mathcal{J}_2\left(z,\tau;\hat{F}_T^*\right) - \mathcal{J}_2\left(z,\tau;\hat{F}_T\right) - \mathcal{J}_2\left(z,\boldsymbol{\lambda},\hat{F}_T^*\right) + \mathcal{J}_2\left(z,\boldsymbol{\lambda},\hat{F}_T\right)\right), \tag{23}$$

and

$$\hat{\Upsilon}_T^{\beta_*}(\tau) = \sup_{z\geq 0,\lambda\in\mathbb{L}} \left[\frac{(\lambda' - \tau')}{\sqrt{T}}\sum\nolimits_{i=1}^{T}(Y_i^* - Y_i) - \sqrt{T}\left(\mathcal{J}\left(z,\lambda,\tau,\hat{F}_T^*\right) - \mathcal{J}\left(z,\lambda,\tau,\hat{F}_T\right)\right)\right], \tag{24}$$

Define $p^*_{\mathcal{MSD}} := P[\hat{\Upsilon}_T^*(\tau) > \hat{\Upsilon}_T(\tau)]$ and consider the decision rule

$$\text{reject } H_0^{(\mathcal{MSD})} \text{iff } p^*_{\mathcal{MSD}} < \alpha. \tag{25}$$

The following proposition demonstrates the consistency of the test based on 25.

Proposition 4. *Suppose that Assumptions A.2 and A.3 hold, and $\alpha < \frac{1}{2}$. Then the test based on decision rule 25 is consistent.*

3 Implementation with mathematical programming

In this section we present the final mathematical programming formulations corresponding to the test statistics for prospect and Markowitz stochastic dominance efficiency. In the appendix we provide the detailed derivation of the formulations.

3.1 Formulation for prospect stochastic dominance

We have a total number of T monthly return observations. The return on the benchmark portfolio $\tau'\mathbf{Y}_t$ is positive in T_p and negative in $T_n = T - T_p$ observations.

For the derivation of the test statistic $\hat{S}_T^{\alpha}(\tau)$ for prospect stochastic dominance efficiency, we only consider positive values of $\tau'\mathbf{Y}$. It is proven in the appendix that the optimal value of z is one of the ranked values of $\tau'\mathbf{Y}_t, t = 1, ..., T_p$. We can see that there is a set of at most T_p values, say $\mathcal{R}_+ = \{r_1, r_2, ..., r_{T_p}\}$, containing the optimal value of the variable z. A direct consequence is that we can solve prospect stochastic dominance efficiency by solving the smaller problems $P(r)$, $r \in \mathcal{R}_+$, in which z is fixed to rThen we can take the value for z that yields the best total result. The advantage is that the optimization model is linear.

The reduced form of the problem is as follows (see the appendix for the derivation of this formulation and details on practical implementation):

$$
\begin{aligned}
\lambda \in \mathbb{L} \quad & \frac{1}{\sqrt{T}} \sum_{t=1}^{T_p} (L_t - W_t) \\
\texttt{s.t.} \quad & W_t \geq r - \lambda' \mathbf{Y}_t, \qquad \forall t \in T_p \\
& L_t = (r - \tau' \mathbf{Y}_t)_+, \qquad \forall t \in T_p \\
& \mathbf{e}' \lambda = 1, \\
& \lambda_i \geq 0, \quad \forall i, \\
& \lambda' \mathbf{Y}_t \geq 0, \qquad \forall\, t \in T_p, \\
& W_t \geq 0,\, F_t \in \{0, 1\}, \qquad \forall\, t \in T_p.
\end{aligned}
\tag{26a}
$$

This is a linear program. The optimal portfolio λ and the optimal value r of variable z are those that give the maximum objective value. It takes less than a minute to solve a number of T_p optimization problems and get the optimal solution.

Analogously, for the derivation of the test statistic $\hat{S}_T^{\beta}(\tau)$ for prospect stochastic dominance efficiency, we only consider negative values of $\tau' \mathbf{Y}$. Similarly, the optimal value of z is one of the ranked values of $\tau' \mathbf{Y}_t, t = 1, ..., T_n$. There is a set of at most T_n values, say $\mathcal{R}_- = \{r_1, r_2, ..., r_{T_n}\}$, containing the optimal value of the variable z. Thus, solve prospect stochastic dominance efficiency by solving the smaller problems $P(r)$, $r \in \mathcal{R}_-$, in which z is fixed to r. The reduced form of the problem is as follows:

$$
\begin{aligned}
\lambda \in \mathbb{L} \quad & \frac{1}{\sqrt{T}} \sum_{t=1}^{T_n} (L_t - W_t) \\
\texttt{s.t.} \quad & W_t \geq \lambda' \mathbf{Y}_t - r, \qquad \forall t \in T_n \\
& L_t = (r - \tau' \mathbf{Y})_-, \qquad \forall t \in T_n \\
& \mathbf{e}' \lambda = 1, \\
& \lambda_i \geq 0, \quad \forall i, \\
& \lambda' \mathbf{Y}_t \leq 0, \qquad \forall\, t \in T_n, \\
& W_t \geq 0, \qquad \forall t \in T_n.
\end{aligned}
\tag{27a}
$$

This is again a linear program. The optimal portfolio λ and the optimal value r of variable z are those that give the maximum objective value. It takes less than a minute to solve a number of T_n optimization problems and get the optimal solution.

3.2 Formulation for Markowitz stochastic dominance

As before, to derive the test statistic $\hat{\Upsilon}_T^{\alpha}(\tau)$ for Markowitz stochastic dominance efficiency we solve a number of T_n smaller problems $P(r)$, $r \in \mathcal{R}_-$, in which z is fixed to r. Then we can take the value for z that yields the best total result. The reduced form of the problems is as follows:

$$
\begin{aligned}
\max_{\lambda \in \mathbb{L}} \quad & \sqrt{T} \sum_{t=1}^{T_n} (L_t - W_t) \\
\texttt{s.t.} \quad & W_t \geq r - \lambda' \mathbf{Y}_t, \qquad \forall t \in T_n \\
& L_t = (r - \tau' \mathbf{Y})_+, \qquad \forall t \in T_n \\
& \mathbf{e}' \lambda = 1, \\
& \lambda_i \geq 0, \quad \forall i, \\
& W_t \geq 0, \qquad \forall t \in T_n.
\end{aligned}
\tag{28a}
$$

This is a linear program. The optimal portfolio λ and the optimal value r of variable z are those that give the maximum objective value. It takes less than a minute to solve a number of T_n optimization problems and get the optimal solution.

Finally, for the test statistic $\hat{\Upsilon}_T^{\beta}(\tau)$ for Markowitz stochastic dominance efficiency we solve T_p smaller problems $P(r)$, $r \in \mathcal{R}_+$, in which z is fixed to rThe model to derive the is the following:

$$
\begin{aligned}
& \max_{\lambda \in \mathbb{L}} \frac{1}{\sqrt{T}}[\sum_{t=1}^{T}(\lambda'\mathbf{Y}_t - \tau'\mathbf{Y}_t) + \sum_{t=1}^{T_p}(W_t - L_t)] \\
\text{s.t.} \quad & M(F_t - 1) \leq r - \lambda'\mathbf{Y}_t \leq MF_t, \qquad \forall\, t \in T_p, \\
& -M(1 - F_t) \leq W_t - (r - \lambda'\mathbf{Y}_t) \leq M(1 - F_t), \qquad \forall\, t \in T_p, \\
& -MF_t \leq W_t \leq MF_t, \qquad \forall\, t \in T_p, \\
& L_t = (r - \tau'\mathbf{Y})_+, \qquad \forall t \in T_p \\
& W_t = z - \lambda'\mathbf{Y}_t, \qquad \forall\, t \in T_p, \\
& \mathbf{e}'\lambda = 1, \\
& \lambda'\mathbf{Y}_t \geq 0, \qquad \forall\, t \in T_p, \\
& \lambda \geq 0, \\
& F_t \in \{0, 1\}, \qquad \forall\, t \in T_p.
\end{aligned}
\tag{29a}
$$

with M being a large constant. This is a Mixed Integer program. The optimal portfolio λ and the optimal value r of variable z are those that give the maximum objective value. It takes about two hours to solve a number of T_p optimization problems and get the optimal solution.

4 Monte Carlo study

In this section we design two sets of Monte Carlo experiments to evaluate actual size and power of the proposed tests in finite samples. In the first one the $(\boldsymbol{Y}_t)_{t \in \mathbb{Z}}$ process is constructed as a vector MA(1) process and in the second as a vector GARCH(1,1) one. Because of the computational burden of evaluating bootstrap procedures in a highly complex optimization environment, we implement the suggestion of Davidson and McKinnon (2006a,b) to get approximate rejection probabilities.

4.1 MA Processes

Suppose that

$$(z_{1_t}, z_{2_t}, z_{3_t})' \overset{\text{iid}}{\sim} N(\mathbf{0}, Id_3)$$

and let $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$, $\mu_1, \mu_2, \mu_3 \in \mathbb{R}^+$, $n = 3$, $\mathbf{Y}_t = (y_{1_t}, y_{2_t}, y_{3_t})'$ where for all $t \in \mathbb{Z}$ and $i = 1, 2, 3$

$$y_{i_t} = \mu_i + \theta_i z_{i_{t-1}} + z_{i_t}.$$

The normality assumption along with the 1-dependence specified by the $MA(1)$ structures immediately imply that $(\mathbf{Y}_t)_{t \in \mathbb{Z}}$ conforms to assumptions A.1 and A.2. Suppose moreover that $\tau = (0, 0, 1)$ and $\mathbb{L} = \{(\lambda, 1 - \lambda, 0), \lambda \in [0, 1]\}$. The first proposition establishes a portfolio that is efficient w.r.t. the Markowitz stochastic dominance criterion.

Proposition 5. *If $\mu_i = 0$ for $i = 1, 2, 3$ and $\min(\theta_1^2, \theta_2^2) > (1 + 2\theta_3^2)$ then τ is $\mathcal{MSD}$-efficient (w.r.t. $\mathbb{L}$).*

Proof. τ is distributed with $N(0, v_\tau)$ and any λ portfolio in $\mathbb{L}$ is distributed with $N(0, v_\lambda)$ with $v_\tau = 1 + \theta_3^2 < \lambda^2 (1 + \theta_1^2) + (1 - \lambda)^2 (1 + \theta_2^2) \doteqdot v_\lambda$. Hence for $z \leq 0$ we have that

$$\begin{aligned} & \mathcal{J}_2(z, \boldsymbol{\lambda}, F) - \mathcal{J}_2(z, \tau, F) \\ = & \int_{-\infty}^{z} \int_{-\infty}^{u} \left(\frac{1}{\sqrt{2\pi v_\lambda}} \exp\left(-\frac{1}{2} \frac{s^2}{v_\lambda} \right) - \frac{1}{\sqrt{2\pi v_\tau}} \exp\left(-\frac{1}{2} \frac{s^2}{v_\tau} \right) \right) ds du \\ = & \int_{-\infty}^{z} \left(\Phi\left(\frac{u}{\sqrt{v_\lambda}} \right) - \Phi\left(\frac{u}{\sqrt{v_\tau}} \right) \right) du. \end{aligned}$$

Due to the relation between the variances, the fact that u assumes non positive values and the monotonicity of $\Phi(\cdot)$ we have that the integrand in the last integral is non negative establishing the first part of definition D.2. For the second part notice that for any $z > 0$ due to lemma AL.1

$$\mathcal{J}_2^c(z, \boldsymbol{\lambda}, \tau; F) = \int_0^z \left(\Phi\left(\frac{u}{\sqrt{v_\tau}} \right) - \Phi\left(\frac{u}{\sqrt{v_\lambda}} \right) \right) du.$$

Using the previous along with the fact that u assumes positive values we have that $\Phi\left(\frac{u}{\sqrt{v_\tau}}\right) > \Phi\left(\frac{u}{\sqrt{v_\lambda}}\right)$ implying that $\mathcal{J}_2^c(z, \boldsymbol{\lambda}, \tau; F) > 0$ for all $z > 0$ and $\lambda \in \mathbb{L}$. ■

The following proposition establishes a portfolio that is efficient according to the Prospect Stochastic Dominance Efficiency.

Proposition 6. *If* $\mu_1, \mu_2 > 0$, $\mu_3 > \mu_1 + \mu_2$ *and* $\min\left(\theta_1^2, \theta_2^2\right) > \left(1 + 2\theta_3^2\right)$ *then* τ *is* $\mathcal{PSD}$*-efficient (w.r.t.* $\mathbb{L}$*).*

Proof. Consider first the definition D.2. Obviously the τ is distributed with $N\left(\mu_\tau, v_\tau\right)$ and any λ portfolio in $\mathbb{L}$ is distributed with $N\left(\mu_\lambda, v_\lambda\right)$ with $\mu_\lambda \doteqdot \lambda\mu_1 + (1-\lambda)\,\mu_2 < \mu_\tau \doteqdot \mu_3$ and $v_\tau = 1 + \theta_3^2 < \lambda^2\left(1+\theta_1^2\right) + (1-\lambda)^2\left(1+\theta_2^2\right) \doteqdot v_\lambda$. Hence for $z \leq 0$ we have that

$$
\begin{aligned}
&\mathcal{J}\left(z, \lambda, \tau, F\right)\\
=\;& \int_z^0 \int_{-\infty}^u \left(\frac{1}{\sqrt{2\pi v_\tau}} \exp\left(-\frac{1}{2}\frac{\left(s-\mu_\tau\right)^2}{v_\tau}\right) - \frac{1}{\sqrt{2\pi v_\lambda}} \exp\left(-\frac{1}{2}\frac{\left(s-\mu_\lambda\right)^2}{v_\lambda}\right)\right) dsdu\\
=\;& \int_z^0 \left(\Phi\left(\frac{u-\mu_\tau}{\sqrt{v_\tau}}\right) - \Phi\left(\frac{u-\mu_\lambda}{\sqrt{v_\lambda}}\right)\right) du.
\end{aligned}
$$

The fact that $\mu_\lambda\sqrt{v_\tau} < \mu_\tau\sqrt{v_\lambda}$, along with the monotonicity of $\Phi\left(\cdot\right)$ implies the first part of definition D.1. For the second part notice that for any $z > 0$

$$
\begin{aligned}
&\mathcal{J}\left(z, \lambda, \tau, F\right)\\
=\;& \int_0^z \int_{-\infty}^u \left(\frac{1}{\sqrt{2\pi v_\lambda}} \exp\left(-\frac{1}{2}\frac{\left(s-\mu_\lambda\right)^2}{v_\lambda}\right) - \frac{1}{\sqrt{2\pi v_\tau}} \exp\left(-\frac{1}{2}\frac{\left(s-\mu_\tau\right)^2}{v_\tau}\right) - \right) dsdu\\
=\;& \int_0^z \left(\Phi\left(\frac{u-\mu_\lambda}{\sqrt{v_\lambda}}\right) - \Phi\left(\frac{u-\mu_\tau}{\sqrt{v_\tau}}\right)\right) du.
\end{aligned}
$$

and the result follows analogously. ■

The final proposition establishes a portfolio that is inefficient w.r.t. both criteria.

Proposition 7. *If* $\mu_i = 0$ *for* $i = 1, 2, 3$ *and* $\theta_1^2 < \theta_3^2$ *then* τ *is* $\mathcal{PSD}$ *and* $\mathcal{MSD}$*-inefficient (w.r.t.* $\mathbb{L}$*).*

Proof. Let $\lambda = 1$ whence $v_\tau = 1 + \theta_3^2 > 1 + \theta_1^2 = v_{\mathbf{1}}$. Hence for $z \leq 0$ we have that

$$
\begin{aligned}
&\mathcal{J}_2(z, \mathbf{1}, F) - \mathcal{J}_2(z, \tau, F)\\
=\;& \int_{-\infty}^z \left(\Phi\left(\frac{u}{\sqrt{v_{\mathbf{1}}}}\right) - \Phi\left(\frac{u}{\sqrt{v_\tau}}\right)\right) du < 0,
\end{aligned}
$$

due to the monotonicity of $\Phi\left(\cdot\right)$ which implies that the first part of definition D.2 is not valid. Analogously

$$
\begin{aligned}
&\mathcal{J}\left(z, \mathbf{1}, \tau, F\right)\\
=\;& \int_z^0 \left(\Phi\left(\frac{u}{\sqrt{v_\tau}}\right) - \Phi\left(\frac{u}{\sqrt{v_{\mathbf{1}}}}\right)\right) du > 0
\end{aligned}
$$

invalidating the first part of definition D.1. ■

Furthermore the existence of exponential moments for the normal distribution implies the validity of 13 for any of the considered portfolios. Given the above we perform experiments for size and power considerations.

4.2 GARCH Processes

We construct another set of experiments that is based on GARCH type processes. Suppose that

$$z_{i_t} \overset{\text{iid}}{\sim} \mathcal{D} \text{ for all } t \in \mathbb{Z} \text{ and } i = 1, 2, 3$$

where $\mathcal{D}$ has strictly increasing distribution function, $\mathbb{E}z_{i_0} = 0$, $\mathbb{E}z_{i_0}^2 = 1$, and z_{i_t} is independent of $z_{j_{t'}}$ for $i \neq j$ and all (t, t'). Suppose furthermore that $\omega_i, a_i, \beta_i \in \mathbb{R}^{++}$, $\mu_1, \mu_2, \mu_3 \in \mathbb{R}^+$, $n = 3$, $\mathbf{Y}_t = (y_{1_t}, y_{2_t}, y_{3_t})'$ where for all $t \in \mathbb{Z}$ and $i = 1, 2, 3$

$$\begin{aligned} y_{i_t} &= \mu_i + z_{i_{t-1}} h_{i_t}^{1/2}, \\ h_{i_t} &= \omega_i + \left(a_i z_{i_{t-1}}^2 + \beta_i\right) h_{i_{t-1}}, \ \mathbb{E}\left(a_i z_{i_{t-1}}^2 + \beta_i\right)^{\delta} < 1. \end{aligned}$$

Then, Corollary 1 and Theorem 8 of Lindner (2003) imply that assumptions A.1 and A.2 hold for $(\mathbf{Y}_t)$ for any a. As before suppose that $\tau = (0, 0, 1)$ and $\mathbb{L} = \{(\lambda, 1-\lambda, 0), \lambda \in [0, 1]\}$. The following propositions are completely analogous to propositions 5, 6 and 7 respectively in the previous section.

Proposition 8. *If $\mu_i = 0$ for $i = 1, 2, 3$, $\min(\omega_1, \omega_2) > \omega_3$, $\min(a_1, a_2) > a_3$ and $\min(\beta_1, \beta_2) > \beta_3$, then τ is $\mathcal{MSD}$-efficient (w.r.t. $\mathbb{L}$).*

Proposition 9. *If $\mu_3 > \mu_1 + \mu_2$ $\min(\omega_1, \omega_2) > \omega_3$, $\min(a_1, a_2) > a_3$ and $\min(\beta_1, \beta_2) > \beta_3$ then τ is $\mathcal{PSD}$-efficient (w.r.t. $\mathbb{L}$).*

Proposition 10. *If $\mu_i = 0$ for $i = 1, 2, 3$ $\omega_1 < \omega_3$, $a_1 < a_3$ and $\beta_1 < \beta_3$ then τ is $\mathcal{PSD}$ and $\mathcal{MSD}$-inefficient (w.r.t. $\mathbb{L}$).*

Proof. The proofs follow from the ones of propositions 5, 6 and 7 respectively, by simply noticing that in the present cases we have that $v_\tau = h_{3_t}$ and $v_\lambda = \lambda^2 h_{1_t} + (1-\lambda)^2 h_{2_t}$, by replacing $\Phi\left(\frac{u-\mu_\tau}{\sqrt{v_\tau}}\right)$ and $\Phi\left(\frac{u-\mu_\lambda}{\sqrt{v_\lambda}}\right)$ by $\mathbb{E}\left(\mathcal{D}\left(\frac{u-\mu_\tau}{\sqrt{v_\tau}}\right)\right)$ and $\mathbb{E}\left(\mathcal{D}\left(\frac{u-\mu_\lambda}{\sqrt{v_\lambda}}\right)\right)$ respectively and by noticing that the monotonicity of $\mathcal{D}$ and of the integral imply the required monotonicity of $\mathbb{E}(\mathcal{D}(\cdot))$. ■

Finally, the existence of moments of order $2+\delta$ implies the validity of 13 for any of the considered portfolios.

Approximate rejection probabilities

According to Davidson and MacKinnon (2006a,b), a simulation estimate of the rejection probability of the bootstrap test of PSD and for significance level α is $\hat{RP}_{PSD}(\alpha) = \frac{1}{R}\sum_{r=1}^{R}\mathbb{I}\{\hat{S}_{T,r}(\tau) < \hat{Q}^*_{PSD}(\alpha)\}$ where the test statistics $\hat{S}_{T,r}(\tau)$ are obtained under the true data generating process on R subsamples, and $\hat{Q}^*_{PSD}(\alpha)$ is the α-quantile of the bootstrap statistics $\hat{S}^*_{T,r}(\tau)$.

Analogously, for MSD we have $\hat{RP}_{MSD}(\alpha) = \frac{1}{R}\sum_{r=1}^{R}\mathbb{I}\{\hat{\Upsilon}_{T,r}(\tau) < \hat{Q}^*_{MSD}(\alpha)\}$ where the test statistics $\hat{\Upsilon}_{T,r}(\tau)$ are obtained under the true data generating process on R subsamples, and $\hat{Q}^*_{MSD}(\alpha)$ is the α-quantile of the bootstrap statistics $\hat{\Upsilon}^*_{T,r}(\tau)$.

So, for each one of the two cases, the data generating process DGP_0 is used to draw realizations of the three asset returns, using either the autoregressive process or the GARCH process described above (with different parameters for each case to evaluate size and power). We generate $R = 300$ original samples with size $T = 500$. For each one of these original samples we generate a block bootstrap (nonoverlapping case) data generating process $\widehat{DGP}$. Once $\widehat{DGP}$ is obtained for each replication r, a new set of random numbers, independent of those used to obtain $\widehat{DGP}$, is drawn. Then, using these numbers we draw R original samples and R block bootstrap samples to compute $\hat{S}_{T,r}(\tau)$, $\hat{S}^*_{T,r}(\tau)$, $\hat{\Upsilon}_{T,r}(\tau)$ and $\hat{\Upsilon}^*_{T,r}(\tau)$ to get the estimates $\hat{RP}_{PSD}(\alpha)$ and $\hat{RP}_{MSD}(\alpha)$ respectively.

Results

MA Experiment. To evaluate for actual size, we test for PSE and MSD efficiency of portfolio τ containing the third asset ($\tau = (0,0,1)$) with respect to all other possible portfolios λ containing the first two assets, when $\theta_1 = 0.5$, $\theta_2 = 0.4,$ and $\theta_3 = 0.1$. In this case, we have that $\min\left(\theta_1^2, \theta_2^2\right) > \left(1+2\theta_3^2\right)$. For the PSD case we set $\mu_i = 0$ for $i = 1,2,3$, while for the Markowitz case we set $\mu_1 = 0.2$, $\mu_2 = 0.3$ and $\mu_3 = 1$, so that $\mu_3 > \mu_1 + \mu_2$.

We set the significance level α equal to 5%, and the block size to $l = 10$. We get

$\hat{RP}_{PSD}(5\%) = 3.8\%$ for the prospect stochastic dominance efficiency test, while we get $\hat{RP}_{MSD}(5\%) = 4.6\%$ for the Markowitz stochastic dominance efficiency test. Hence we may conclude that both bootstrap tests perform well in terms of size properties.

To evaluate for actual power, we take an inefficient portfolio as the benchmark portfolio τ, and we compare it to all other possible portfolios λ with positive weights summing to one. We compare portfolio $(\tau = (0,0,1))$ with respect to all other possible portfolios λ containing the first two assets, when $\theta_1 = 0.5$, $\theta_2 = 0.4,$ and $\theta_3 = 1$. In this case, we have that $\theta_1^2 < \theta_3^2$.

We find that the power of both tests is large. Indeed, we find $\hat{RP}_{PSD}(5\%) = 97.2\%$ for the prospect stochastic dominance efficiency test when we take wrongly as efficient the portfolio $(\tau = (0,0,1))$. Similarly we find $\hat{RP}_{MSD}(5\%) = 95.4\%$ for the Markowitz stochastic dominance efficiency.

GARCH Experiment. To evaluate for actual size, we test for PSE and MSD efficiency of portfolio τ containing the third asset $(\tau = (0,0,1))$ with respect to all other possible portfolios λ containing the first two assets. We set $\omega_1 = 0.3$, $\omega_2 = 0.2,$ and $\omega_3 = 0.1$, $a_1 = 0.3$, $a_2 = 0.2,$ and $a_3 = 0.1$ and $\beta_1 = 0.3$, $\beta_2 = 0.2,$ and $\beta_3 = 0.1$. In this case, we have that $\min(\omega_1, \omega_2) > \omega_3$, $\min(a_1, a_2) > a_3$ and $\min(\beta_1, \beta_2) > \beta_3$. For the PSD case we set $\mu_i = 0$ for $i = 1, 2, 3$, while for the Markowitz case we set $\mu_1 = 0.2, \mu_2 = 0.3$ and $\mu_3 = 1$, so that $\mu_3 > \mu_1 + \mu_2$.

As before, we set the significance level α equal to 5%, and the block size to $l = 10$. We get $\hat{RP}_{PSD}(5\%) = 3.6\%$ for the prospect stochastic dominance efficiency test, while we get $\hat{RP}_{MSD}(5\%) = 3.0\%$ for the Markowitz stochastic dominance efficiency test. Hence we may conclude that both bootstrap tests perform well in terms of size properties.

To evaluate for actual power, we compare the inefficient portfolio $(\tau = (0,0,1))$ with respect to all other possible portfolios λ containing the first two assets, when We set $\mu_i = 0$ for $i = 1, 2, 3$ and $\omega_1 = 0.1$, $\omega_2 = 0.2,$ and $\omega_3 = 0.3$, $a_1 = 0.1$, $a_2 = 0.2,$ and $a_3 = 0.3$ and $\beta_1 = 0.1$, $\beta_2 = 0.2,$ and $\beta_3 = 0.3$. In this case, we have that $\omega_1 < \omega_3$, $a_1 < a_3$ and $\beta_1 < \beta_3$.

We find that the power of both tests is large. Indeed, we find $\hat{RP}_{PSD}(5\%) = 96.6\%$ for the prospect stochastic dominance efficiency test when we take wrongly as efficient the portfolio $(\tau = (0,0,1))$. Similarly we find $\hat{RP}_{MSD}(5\%) = 96.8\%$ for the Markowitz stochastic dominance efficiency.

Finally we present Monte Carlo results in Table 1 on the sensitivity to the choice of block length. We investigate block sizes ranging from $l = 4$ to $l = 12$ by step of 4. This covers the suggestions of Hall, Horowitz, and Jing (1995), who show that optimal block sizes are multiple of $T^{1/3}$, $T^{1/4}$, $T^{1/5}$, depending on the context. According to our experiments the choice of the block size does not seem to dramatically alter the performance of our methodology.

		MA	Process	
Block size l :	4	8	10	12
Size:				
$\hat{RP}_{PSD}$	4.0%	3.6%	3.8%	4.4%
$\hat{RP}_{MSD}$	4.2%	2.8%	4.6%	3.8%
Power:				
$\hat{RP}_{PSD}$	96.0%	97.0%	97.2%	96.6%
$\hat{RP}_{MSD}$	95.8%	97.8%	95.4%	98.6%

		GARCH	Process	
Block size l :	4	8	10	12
Size:				
$\hat{RP}_{PSD}$	4.2%	4.0%	3.6%	3.8%
$\hat{RP}_{MSD}$	3.6%	3.4%	3.0%	4.6%
Power:				
$\hat{RP}_{PSD}$	97.6%	98.0%	96.6%	98.4%
$\hat{RP}_{MSD}$	97.2%	98.6%	96.8%	98.0%

Table 1: Sensitivity analysis of size and power to the choice of block length using the MA and Garch processes for the asset returns. We compute the actual size and power of the prospect and Markowitz stochastic dominance efficiency tests for block sizes ranging from $l = 4$ to $l = 12$.

5 Empirical application

In this section we present the results of an empirical application. To illustrate the potential of the proposed test statistics, we test whether different stochastic dominance efficiency criteria (Prospect and Markowitz) rationalize the market portfolio. Thus, we test for the stochastic dominance efficiency of the market portfolio with respect to all possible portfolios constructed from a set of assets, namely six risky assets ($n = 6$).

5.1 Description of the data

We use six Fama and French benchmark portfolios as our set of risky assets. They are constructed at the end of each June, and correspond to the intersections of two portfolios formed on size (market equity, ME) and three portfolios formed on the ratio of book equity to market equity (BE/ME). The size breakpoint for year t is the median NYSE market equity at the end of June of year t. BE/ME for June of year t is the book equity for the last fiscal year end in $t - 1$ divided by ME for December of $t - 1$. Firms with negative BE are not included in any portfolio. The annual returns are from January to December. We use data on monthly excess returns (month-end to month-end) from January 1930 to December 2012 (996 monthly observations) obtained from the data library on the homepage of Kenneth French (http://mba.turc.dartmouth.edu/pages/faculty/ken.french). The test portfolio is the Fama and French market portfolio, which is the value-weighted average of all non-financial common stocks listed on NYSE, AMEX, and Nasdaq, and covered by CRSP and COMPUSTAT.

First we analyze the statistical characteristics of the data covering the period from January 1930 to December 2012 (996 monthly observations) that are used in the test statistics. As we can see from Table 2, portfolio returns exhibit considerable variance. Moreover, the skewness and kurtosis indicate that normality cannot be accepted for the majority of them. These observations suggest adopting the stochastic dominance efficiency tests which account for the full return distribution and not only the mean and the variance.

One interesting feature is the comparison of the behavior of the market portfolio with that of the individual portfolios. Scaillet and Topaloglou(2012) show that the Fama and French market portfolio is not mean-variance efficient, compared to the 6 benchmark portfolios.Thus,

Descriptive Statistics (January 1930 to December 2012)						
No.	Mean	Std. Dev.	Skewness	Kurtosis	Minimum	Maximum
Market Portfolio	0.604	2.413	0.237	7.593	-29.98	37.77
1	1.016	7.825	1.026	7.270	-32.32	65.63
2	1.288	7.139	1.310	11.660	-31.10	64.12
3	1.493	8.367	2.175	18.810	-33.06	85.24
4	0.847	5.308	-0.023	2.231	-28.08	32.55
5	0.936	5.823	1.303	14.227	-28.01	51.52
6	1.161	7.327	1.547	14.926	-35.45	68.25

Table 2: Descriptive statistics of monthly returns in $\%$ from January 1930 to December 2012 (996 monthly observations) for the Fama and French market portfolio and the six Fama and French benchmark portfolios formed on size and book-to-market equity ratio. Portfolio 1 has low BE/ME and small size, portfolio 2 has medium BE/ME and small Size, portfolio 3 has high BE/ME and small size, ..., portfolio 6 has high BE/ME and large size.

If the investor utility function is not quadratic, then the risk profile of the benchmark portfolios cannot be totally captured by the variance of these portfolios. Generally, the variance is not a satisfactory measure. It is a symmetric measure that penalizes gains and losses in the same way. Moreover, the variance is inappropriate to describe the risk of low probability events. This motivates us to test whether the market portfolio is efficient when different preferences are taken into account.

5.2 Results of the stochastic dominance efficiency tests

We find a significant autocorrelation of order one at a 5% significance level in benchmark portfolios 1 to 3, while ARCH effects are present in benchmark portfolio 4 at a 5% significance level. This indicates that a block bootstrap approach should be favoured over a standard i.i.d.

bootstrap approach. Since the autocorrelations die out quickly, we may take a block of small size to compute the p-values of the test statistics. We choose a size of 10 observations following the suggestions of Hall, Horowitz, and Jing (1995), who show that optimal block sizes are multiple of $T^{1/3}$, where in our case, $T = 996$. We use the nonoverlapping rule because we need to recenter the test statistics in the overlapping rule. The recentering makes the test statistics very difficult to compute, since the optimization for Markowitz stochastic dominance involves a large number of binary variables. The p-values are approximated with an averaging made on $R = 300$ replications. This number guarantees that the approximations are accurate enough, given time and computer constraints.

For the prospect stochastic dominance efficiency, we cannot reject that the market portfolio is efficient. The p-value $\tilde{p} = 0.443$ is way above the significance level of 5%. We divide the full period into two subperiods, the first one from January 1930 to June 1971, a total of 498 monthly observations, and the second one from July 1971 to December 2012, 498 monthly observations. We test for prospect stochastic dominance of the market portfolio to each subperiod. We find that the p-value for the first subperiod is $\tilde{p}_1 = 0.403$ and the p-value for the second subperiod is $\tilde{p}_2 = 0.526$. The results indicate that the market porfolio is prospect stochastic dominance efficient in each subperiod as well as in the whole period. This implies that there are S-shaped utility functions that rationalize the market portfolio. Risk seeking for losses and risk aversion for gains helps to explain the pattern of stock returns.

Experimental evidence suggests that decision makers subjectively transform the true return distribution and use subjective decision weights that overweight or underweight the true probabilities.The most common pattern of probability transformation overweights small probabilities of large gains and losses, and underweights large and intermediate probabilities of small and intermediate gains and losses (Tverskyand Kahneman, (1992)). The prospect stochastic dominance efficiency of the market portfolio we found here, is not affected by transformations that are increasing and convex over losses and increasing and concave over gains,thatis, S-shaped transformations. Moreover, if the market portfolio is undominated by PSDE, then it is also undominated by the weaker condition given by Baucells and Heukamp (2006).

On the other hand, we find that the MSD criterion is rejected. The p-value $\tilde{p} = 0.013$ is below the significance level of 5%. Additionaly, the p-value $\tilde{p}_1 = 0.003$ for the first subperiod and p-value $\tilde{p}_2 = 0.006$ for the second subperiod indicate that the market porfolio is not Markowitz stochastic dominance efficient in each subperiod as well as in the full period. No reverse S-shaped utility function can rationalize the market portfolio. This implies that we can form portfolios of the six benchmark portfolios that dominate the market.

Our efficiency finding cannot be attributed to a potential lack of power of the testing procedures. Indeed, we use a long enough time series of 996 return observations, and a relatively narrow cross-section of six benchmark portfolios. Further even if our test concerns a necessary and not a sufficient condition for optimality of the market portfolio (Post (2005)), this does not influence the output of our results.

5.3 Rolling window analysis

We carry out an additional test to validate the prospect and Markowitz stochastic dominance efficiency of the market portfolio and the stability of the model results. It is possible that the efficiency of the market portfolio changes over time, as the risk and preferences of investors change. Therefore, the market portfolio may be efficient in the total sample, but inefficient in some subsamples. Moreover, the degree of efficiency may change over time, as pointed by Post (2003). To control for that, we perform a rolling window analysis, using a window width of 20 years. The test statistic is calculated separately for 62 overlapping 20-year periods, (January 1930-December 1949), (January 1931-December 1951),...,(January 1993-December 2012).

Figure 1 shows the corresponding p-values for the prospect stochastic dominance efficiency. Interestingly, we observe that the market portfolio is prospect stochastic dominance efficient in the total sample period. The prospect stochastic dominance efficiency is not rejected on any subsamples. The p-values are always greater than 18%, and in some cases they reach the 60%. This result confirms the prospect stochastic dominance efficiency that was found in the previous subsection, for the full period. This means that we cannot form an optimal portfolio from the set of the six benchmark portfolios that dominates the market portfolio by prospect stochastic dominance. The line exhibits large fluctuations; thus the degree of efficiency is changing over time, but remains always above the critical level of 5%.

The time series in this case is smaller (240 onthly observations) so that a maintained assumption of stationarity is credible.

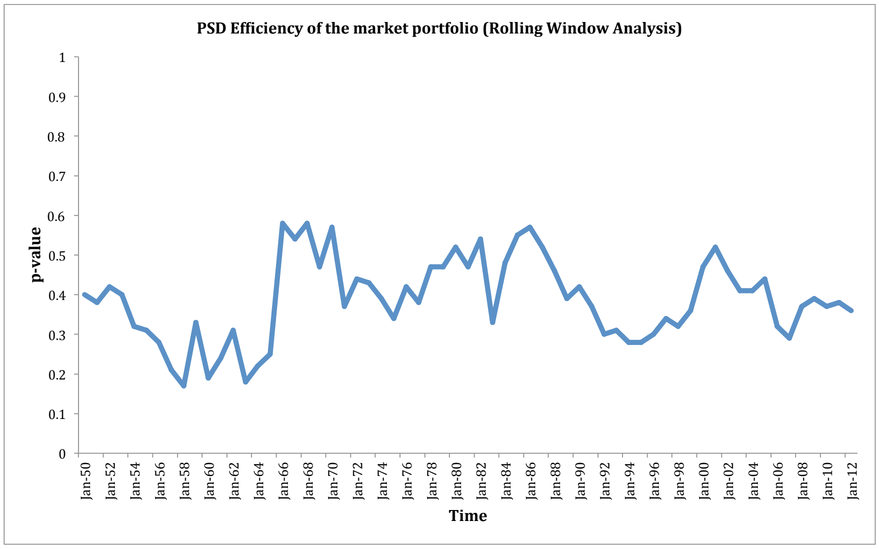

Figure 1: p-values for the prospect stochastic dominance efficiency test using a rolling window of 20 years. The test statistic is calculated separately for 62 overlapping 20-year periods, (January 1930-December 1949), (January 1931-December 1951),...,(January 1993-December 2012). The prospect stochastic dominance efficiency is not rejected.

Figure 2 shows the corresponding p-values for the Markowitz stochastic dominance efficiency. We observe that the market portfolio is not Markowitz stochastic dominance efficient. The Markowitz stochastic dominance efficiency is rejected on 58 out of 63 subsamples. The p-values are almost always lower than 5%. This result confirms the rejection of the Markowitz stochastic dominance efficiency that was found in the previous subsection. This means that every year, a new optimal portfolio λ is obtained from the set of the six benchmark portfolios that dominates the market portfolio by Markowitz stochastic dominance.

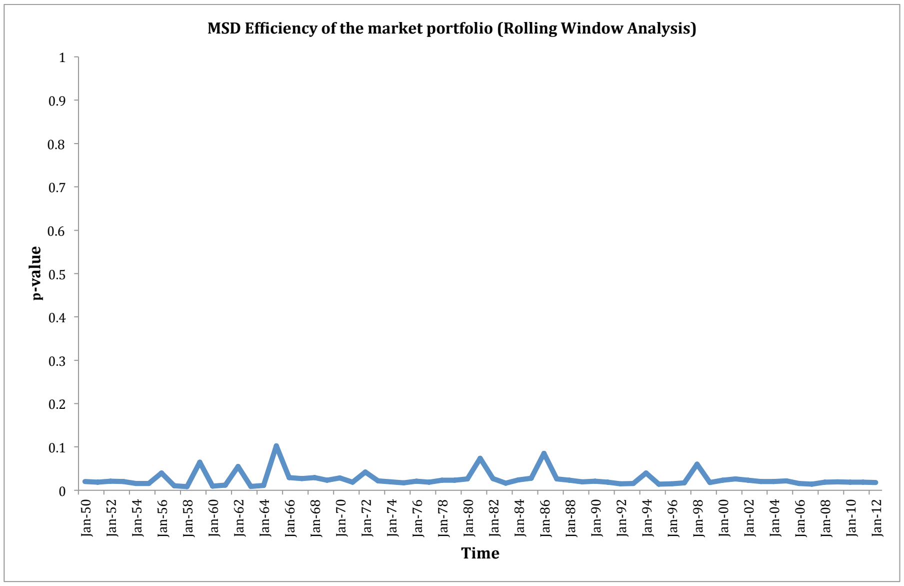

Figure 2: p-values for the Markowitz stochastic dominance efficiency test using a rolling window of 20 years. The test statistic is calculated separately for 62 overlapping 20-year periods, (January 1930-December 1949), (January 1931-December 1951),...,(January 1993-December 2012). The Markowitz stochastic dominance efficiency is rejected.

6 Concluding remarks

In this paper we develop *consistent* tests for prospect and Markowitz stochastic dominance efficiency for *time-dependent* data. We study tests for stochastic dominance efficiency of a given portfolio with respect to all possible portfolios constructed from a set of risky assets. We justify block bootstrap approaches to achieve valid inference in a time series setting. Linear as well as mixed integer programs are formulated to compute the test statistics.

To illustrate the potential of the proposed test statistics, we test whether the two stochastic dominance efficiency criteria rationalize the Fama and French market portfolio over six Fama and French benchmark portfolios constructed as the intersections of two ME portfolios and three

BE/ME portfolios. Empirical results indicate that the market portfolio is prospect stochastic dominance efficient. Morover, the market portfolio is not Markowitz stochastic dominance efficient. The results are also confirmed in a rolling window analysis. This implies that there are S-shaped utility functions that rationalize the market portfolio, while the market portfolio is not efficient relative to reverse S-shaped utility functions.

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APPENDIX

Helpful Lemmata and Proofs

In what follows $\rightsquigarrow$ denotes weak convergence and $\overset{p}{\rightsquigarrow}$ (conditional) weak convergence in probability (see among others Paragraph 3.6.1 of van der Vaart and Wellner (1996)). Analogously $\overset{p}{\rightarrow}$ denotes convergence in probability. CMT abbreviates the continuous mapping theorem in the relevant context.

Lemma AL.1. *If $\mathbb{E}\left[\lambda' \boldsymbol{Y}_0\right]$ and $\mathbb{E}\left[\tau' \boldsymbol{Y}_0\right]$ exist then $\mathcal{J}_2^c\left(z, \lambda, F\right)$ is finite and equals*

$$\mu_{\tau' Y} - \mu_{\lambda' Y} + \mathcal{J}_2(z, \tau, F) - \mathcal{J}_2(0, \tau, F) - \mathcal{J}_2(z, \boldsymbol{\lambda}, F) + \mathcal{J}_2(0, \boldsymbol{\lambda}, F).$$

Proof. Remember that iff $E\left[\left|\lambda' \boldsymbol{Y}\right|\right] < +\infty$ then we have that $\mu_{\lambda' Y} \triangleq E\left[\lambda' \boldsymbol{Y}\right] = \int_0^{+\infty}\left(1 - G\left(u, \lambda, F\right)\right) du$ and therefore

$$\begin{aligned} & \int_z^{+\infty}\left(G\left(u, \boldsymbol{\lambda}, F\right) - G\left(u, \tau, F\right)\right) du \\ = & \int_z^{+\infty}\left(1 - G\left(u, \tau, F\right)\right) - \left(1 - G\left(u, \boldsymbol{\lambda}, F\right)\right) du \\ = & \ \mu_{\tau' Y} - \mu_{\lambda' Y} + \int_0^z G\left(u, \tau, F\right) du - \int_0^z G\left(u, \lambda, F\right) du. \end{aligned}$$

■

In the following let

$$x_T = \begin{pmatrix} \sqrt{T}\sup_{z\geq 0, \lambda\in\mathbb{L}}\left(\mathcal{J}\left(z, -\lambda, -\tau, \hat{F}_T\right) - \mathcal{J}\left(z, -\lambda, -\tau, F\right)\right) \\ \sqrt{T}\sup_{z\geq 0, \lambda\in\mathbb{L}}\left(\mathcal{J}\left(z, \lambda, \tau, F\right) - \mathcal{J}\left(z, \lambda, \tau, \hat{F}_T\right)\right) \end{pmatrix},$$

$$y_T = \begin{pmatrix} \sqrt{T}\sup_{z\leq 0, \lambda\in\mathbb{L}}\left(\mathcal{D}_2\left(z, \tau; \lambda, \hat{F}_T\right) - \mathcal{D}_2\left(z, \tau; \lambda, F\right)\right) \\ \sup_{z\geq 0, \lambda\in\mathbb{L}}\left[\frac{(\lambda-\tau)'}{\sqrt{T}}\sum_{i=1}^T\left(Y_i - \mathbb{E}Y_0\right) - \sqrt{T}\left(\mathcal{J}\left(z, \lambda, \tau, \hat{F}_T\right) - \mathcal{J}\left(z, \lambda, \tau, F\right)\right)\right] \end{pmatrix},$$

where

$$\mathcal{D}_2\left(z, \tau; \lambda, F^*\right) = \mathcal{J}_2(z, \tau; F^*) - \mathcal{J}_2(z, \boldsymbol{\lambda}, F^*).$$

Lemma AL.2. 1. *Suppose that for any $\lambda \in \mathbb{L}$, $\mathbb{E}\left[\left(-\lambda' \boldsymbol{Y}_0\right)_+\right] < +\infty$, G satisfies the condition 13 in Proposition 1 and assumption A.1 holds. Then as $T \to \infty$*

$$x_T \rightsquigarrow \begin{pmatrix} \sup_{z\geq 0, \lambda\in\mathbb{L}}\mathcal{J}\left(z, -\lambda, -\tau, \mathcal{B}\circ F\right) \\ \sup_{z\geq 0, \lambda\in\mathbb{L}} -\mathcal{J}\left(z, \lambda, \tau, \mathcal{B}\circ F\right) \end{pmatrix} \quad (30)$$

where $\mathcal{B} \circ F$ denotes a zero mean Gaussian process with well defined covariance kernel, in the space of continuous functions on $\mathbb{R}^n$. **2.** *Suppose that G satisfies the condition 13 in Proposition 1 and assumption A.2 holds. Then as $T \to \infty$*

$$y_T \rightsquigarrow \begin{pmatrix} \sup_{z \leq 0, \lambda \in \mathbb{L}} \mathcal{D}_2 \left(z, \tau; \lambda, \mathcal{B} \circ F\right) \\ \sup_{z \geq 0, \lambda \in \mathbb{L}} \left[(\lambda - \tau)' \mathcal{Z} - \mathcal{J} \left(z, \lambda, \tau, \mathcal{B} \circ F\right)\right] \end{pmatrix} \tag{31}$$

where $\mathcal{B} \circ F$ is as before, and $\mathcal{Z}$ denotes a zero mean Gaussian random vector.

Proof. **1.** Notice first that due to assumption A.1 we have that $\sqrt{T}\left(\hat{F}_T - F\right) \rightsquigarrow \mathcal{B} \circ F$ (see e.g. the multivariate functional central limit theorem for stationary strongly mixing sequences stated in Rio (2000)). Then we have that due to the previous, the relations 1, 2, 3 and the CMT, for any $(z, \lambda) \in \mathbb{R}^+ \times \mathbb{L}$,

$$\sqrt{T}\left(\mathcal{J}\left(z, -\lambda, -\tau, \hat{F}_T\right) - \mathcal{J}\left(z, -\lambda, -\tau, F\right)\right) \rightsquigarrow \mathcal{J}\left(z, -\lambda, -\tau, \mathcal{B} \circ F\right)$$

and,

$$\sqrt{T}\left(\mathcal{J}\left(z, \lambda, \tau, F\right) - \mathcal{J}\left(z, \lambda, \tau, \hat{F}_T\right)\right) \rightsquigarrow -\mathcal{J}\left(z, \lambda, \tau, \mathcal{B} \circ F\right).$$

Due to linearity both limits are well defined Gaussian processes. Then an analysis similar to the one in the proof of the first part of Proposition 2.2 of Scaillet and Topaloglou (2012) (which evolves along the lines of the proof of Theorem 1 of Horvath, Kokoszka, and Zitikis (2006)) implies that $x_{1,T} \rightsquigarrow \sup_{z \geq 0, \lambda \in \mathbb{L}} \mathcal{J}\left(z, -\lambda, -\tau, \mathcal{B} \circ F\right)$ and $x_{2,T} \rightsquigarrow \sup_{z \geq 0, \lambda \in \mathbb{L}} -\mathcal{J}\left(z, \lambda, \tau, \mathcal{B} \circ F\right)$. Then the limiting property in 30 follows from Theorem 1.4.8 of van der Vaart and Wellner (1996)vaart. **2.** Again assumption A.2 implies that $\sqrt{T}\left(\hat{F}_T - F\right) \rightsquigarrow \mathcal{B} \circ F$ and that $\frac{1}{\sqrt{T}} \sum_{i=1}^{T} \left(Y_i - \mathbb{E} Y_0\right) \rightsquigarrow \mathcal{Z}$ (see e.g. the multivariate functional central limit theorem for stationary strongly mixing sequences stated in Rio (2000), or combine the Martingale approximation in Gordin (1969) with tha CLT for stationary squared integrable m.d. processes to obtain the second result). Then the previous along with linearity implied by 1, 2, 3, and the CMT guarantee that for any $(z, \lambda) \in \mathbb{R}^+ \times \mathbb{L}$, $\sqrt{T}\left(\mathcal{D}_2\left(z, \tau; \lambda, \hat{F}_T\right) - \mathcal{D}_2\left(z, \tau; \lambda, F\right)\right) \rightsquigarrow \mathcal{D}_2\left(z, \tau; \lambda, \mathcal{B} \circ F\right)$ and for any $(z, \lambda) \in \mathbb{R}^+ \times \mathbb{L}$, $\sqrt{T}\left(\mathcal{J}\left(z, \lambda, \tau, \hat{F}_T\right) - \mathcal{J}\left(z, \lambda, \tau, F\right)\right) \rightsquigarrow \mathcal{J}\left(z, \lambda, \tau, \mathcal{B} \circ F\right)$. Theorem 1.4.8 of van der Vaart and Wellner (1996) implies that this convergence holds jointly with well defined covariances due to the previous and the moment condition in A.2. The fact that $y_{1,T} \rightsquigarrow$

$\sup_{z\leq 0,\lambda\in\mathbb{L}}\mathcal{D}_2\left(z,\tau;\lambda,\mathcal{B}\circ F\right)$ follows from Proposition 2.2 of Scaillet and Topaloglou (2012) and the CMT. Again an analysis similar to the one in the proof of the first part of Proposition 2.2 of Scaillet and Topaloglou (2012) implies that for any $\lambda \in \mathbb{L}$ $\sup_{z\geq 0}\left[-\sqrt{T}\left(\mathcal{J}\left(z,\lambda,\tau,\hat{F}_T\right)-\mathcal{J}\left(z,\lambda,\tau,F\right)\right)\right] \rightsquigarrow \sup_{z\geq 0}\left[-\mathcal{J}\left(z,\lambda,\tau,\mathcal{B}\circ F\right)\right]$. Again the CMT along with Theorem 1.4.8 of van der Vaart and Wellner (1996) imply that this convergence holds jointly with $\frac{(\lambda-\tau)'}{\sqrt{T}}\sum_{i=1}^{T}\left(Y_i-\mathbb{E}Y_0\right) \rightsquigarrow \left(\lambda-\tau\right)'\mathcal{Z}$. The compactness of $\mathbb{L}$ implies then that $y_{3,T} \rightsquigarrow \sup_{z\geq 0,\lambda\in\mathbb{L}}\left[\left(\lambda-\tau\right)'\mathcal{Z}-\mathcal{J}\left(z,\lambda,\tau,\mathcal{B}\circ F\right)\right]$. The limiting property in 31 follows from another application of the aforementioned Theorem. ■

Proof of Proposition 1. First notice that

$$\begin{aligned}\hat{S}^{a}_{T}\left(\tau\right) &= \sup_{z\geq 0,\lambda\in\mathbb{L}}\sqrt{T}\mathcal{J}\left(z,-\lambda,-\tau,\hat{F}_T\right)\\ &= \frac{1}{\sqrt{T}}\sup_{z\leq 0,\lambda\in\mathbb{L}}\sum_{i=1}^{T}\left(\left(z-\tau'Y_i\right)\mathbb{I}_{z\leq\tau'Y_i\leq 0}-\left(z-\lambda'Y_i\right)\mathbb{I}_{z\leq\lambda'Y_i\leq 0}\right)\\ &= \sup_{z\leq 0,\lambda\in\mathbb{L}}\sqrt{T}\mathcal{J}\left(z,\lambda,\tau,\hat{F}_T\right).\end{aligned}$$

Then notice that

$$\begin{aligned}\hat{S}^{a}_{T}\left(\tau\right) &= \sqrt{T}\sup_{z\geq 0,\lambda\in\mathbb{L}}\left(\mathcal{J}\left(z,-\lambda,-\tau,\hat{F}_T\right)-\mathcal{J}\left(z,-\lambda,-\tau,F\right)+\mathcal{J}\left(z,-\lambda,-\tau,F\right)\right)\\ &\leq x_{1,T}+\sup_{z\geq 0,\lambda\in\mathbb{L}}\sqrt{T}\mathcal{J}\left(z,-\lambda,-\tau,F\right).\end{aligned}$$

and

$$\begin{aligned}\hat{S}^{\beta}_{T}\left(\tau\right) &= \sqrt{T}\sup_{z\geq 0,\lambda\in\mathbb{L}}\left(\mathcal{J}\left(z,\lambda,\tau,F\right)-\mathcal{J}\left(z,\lambda,\tau,\hat{F}_T\right)-\mathcal{J}\left(z,\lambda,\tau,F\right)\right)\\ &\leq x_{2,T}+\sup_{z\geq 0,\lambda\in\mathbb{L}}-\sqrt{T}\mathcal{J}\left(z,\lambda,\tau,F\right)\end{aligned}$$

1. If $H_0^{(\mathcal{PSD})}$ holds then the previous imply that

$$\hat{S}^{a}_{T}\leq x_{1,T} \text{ and } \hat{S}^{\beta}_{T}\left(\tau\right)\leq x_{2,T}$$

and thereby

$$\hat{S}_T\left(\tau\right)\leq\max\left(x_{1,T},x_{2,T}\right)$$

and due to the CMT and lemma AL.2.**1** $\max\left(x_{1,T},x_{2,T}\right)$ converges in distribution to

$$\bar{S}\left(\tau\right)\doteqdot\max\left(\begin{pmatrix}\sup_{z\geq 0,\lambda\in\mathbb{L}}\mathcal{J}\left(z,-\lambda,-\tau,\mathcal{B}\circ F\right)\\ \sup_{z\geq 0,\lambda\in\mathbb{L}}-\mathcal{J}\left(z,\lambda,\tau,\mathcal{B}\circ F\right)\end{pmatrix}'\right),$$

and the result follows. **2.** If $H_0^{(\mathcal{PSD})}$ is not true then $\sup_{z\geq 0,\lambda\in\mathbb{L}} -\sqrt{T}\mathcal{J}\left(z,\lambda,\tau,F\right)$ and/or $\sup_{z\geq 0,\lambda\in\mathbb{L}} \sqrt{T}\mathcal{J}\left(z,-\lambda,-\tau,F\right)$ converge to $+\infty$. If any of them does not then it does not also contribute to the relevant supremum. Due to Gaussianity and the non finite cardinality of $\mathbb{L}$ we therefore obtain that

$$\hat{S}_T^a \geq \sup_{z\geq 0,\lambda\in\mathbb{L}} \sqrt{T}\mathcal{J}\left(z,-\lambda,-\tau,F\right) \text{ and/or } \hat{S}_T^{\beta}\left(\tau\right) \geq \sup_{z\geq 0,\lambda\in\mathbb{L}} -\sqrt{T}\mathcal{J}\left(z,\lambda,\tau,F\right)$$

and thereby $\hat{S}_T\left(\tau\right)$ is greater than or equal to the maximum of the right hand sides of the previous display and the result follows. ■

Proof of Proposition 2. First notice that the moment condition in assumption A.2 implies that G satisfies the condition 13 in Proposition 1. Then from 15 and lemma AL.1 we obtain that

$$\begin{aligned}
\hat{\Upsilon}_T^{\beta}\left(\tau\right) &= \sup_{z\geq 0,\lambda\in\mathbb{L}} -\sqrt{T}\mathcal{J}_2^c\left(z,\lambda,\tau,\hat{F}_T\right) \\
&= \sup_{z\geq 0,\lambda\in\mathbb{L}} -\sqrt{T}\left(\mathcal{J}_2^c\left(z,\lambda,\tau,\hat{F}_T\right) - \mathcal{J}_2^c\left(z,\lambda,\tau,F\right)\right) - \sqrt{T}\mathcal{J}_2^c\left(z,\lambda,\tau,F\right) \\
&\leq \sup_{z\geq 0,\lambda\in\mathbb{L}} -\sqrt{T}\left(\mathcal{J}_2^c\left(z,\lambda,\tau,\hat{F}_T\right) - \mathcal{J}_2^c\left(z,\lambda,\tau,F\right)\right) + \sup_{z\geq 0,\lambda\in\mathbb{L}} -\sqrt{T}\mathcal{J}_2^c\left(z,\lambda,\tau,F\right) \\
&= y_{1,T} + \sup_{z\geq 0,\lambda\in\mathbb{L}} -\sqrt{T}\mathcal{J}_2^c\left(z,\lambda,\tau,F\right).
\end{aligned}$$

Analogously

$$\begin{aligned}
\hat{\Upsilon}_T^{a}\left(\tau\right) &= \sqrt{T}\sup_{z\leq 0,\lambda\in\mathbb{L}}\left(\mathcal{D}_2\left(z,\tau;\lambda,\hat{F}_T\right) \pm \mathcal{D}_2\left(z,\tau;\lambda,F\right)\right) \\
&\leq y_{1,T} + \sup_{z\leq 0,\lambda\in\mathbb{L}} \sqrt{T}\left(\mathcal{J}_2(z,\tau;F) - \mathcal{J}_2(z,\boldsymbol{\lambda};F)\right).
\end{aligned}$$

1. If $H_0^{(\mathcal{MSD})}$ holds then the previous imply that

$$\hat{\Upsilon}_T^{a} \leq y_{1,T} \text{ and } \hat{\Upsilon}_T^{\beta}\left(\tau\right) \leq y_{2,T}$$

and thereby

$$\hat{S}_T\left(\tau\right) \leq \max\left(y_{1,T}, y_{2,T}\right)$$

and due to the CMT and lemma AL.2.**2** $\max\left(y_{1,T}, y_{2,T}\right)$ converges in distribution to

$$\bar{\Upsilon}\left(\tau\right) \doteqdot \max\left(\begin{pmatrix} \sup_{z\leq 0,\lambda\in\mathbb{L}} \mathcal{D}_2\left(z,\tau;\lambda,\mathcal{B}\circ F\right) \\ \sup_{z\geq 0,\lambda\in\mathbb{L}}\left[\left(\lambda-\tau\right)'\mathcal{Z} - \mathcal{J}\left(z,\lambda,\tau,\mathcal{B}\circ F\right)\right] \end{pmatrix}'\right),$$

and the result follows. **2.** If $H_0^{(\mathcal{MSD})}$ is not true then $\sup_{z\leq 0,\lambda\in\mathbb{L}}\sqrt{T}\left(\mathcal{J}_2(z,\tau;F)-\mathcal{J}_2(z,\lambda;F)\right)$ and/or $\sup_{z\geq 0,\lambda\in\mathbb{L}}-\sqrt{T}\mathcal{J}_2^c\left(z,\lambda,\tau,F\right)$ converge to $+\infty$. If any of them does not then it does not also contribute to the relevant supremum. Due to Gaussianity and the non finite cardinality of $\mathbb{L}$ we therefore obtain that

$$\hat{\Upsilon}_T^a \geq \sup_{z\leq 0,\lambda\in\mathbb{L}}\sqrt{T}\left(\mathcal{J}_2(z,\tau;F)-\mathcal{J}_2(z,\boldsymbol{\lambda};F)\right) and/or \hat{\Upsilon}_T^{\beta}\left(\tau\right) \geq \sup_{z\geq 0,\lambda\in\mathbb{L}}-\sqrt{T}\mathcal{J}_2^c\left(z,\lambda,\tau,F\right)$$

and thereby $\hat{\Upsilon}_T\left(\tau\right)$ is greater than or equal to the maximum of the right hand sides of the previous display and the result follows. ■

Proof of Proposition 3. From assumptions A.1 and A.3 and Theorem 2.3 of Peligrad (1998) we have that conditionally on the sample

$$\sqrt{T}\left(\hat{F}_T^* - \hat{F}_T\right) \overset{p}{\rightsquigarrow} \mathcal{B}^*\circ F$$

where $\mathcal{B}^*\circ F$ is an independent version of the Gaussian process in lemma AL.2. From relations 3, 2, 3 the Delta method the CMT and the results of lemma AL.2 we obtain that

$$\begin{pmatrix} \hat{S}_T^{\alpha_*}\left(\tau\right) \\ \hat{S}_T^{\beta_*}\left(\tau\right) \end{pmatrix} \overset{p}{\rightsquigarrow} \begin{pmatrix} \sup_{z\geq 0,\lambda\in\mathbb{L}}\mathcal{J}\left(z,-\lambda,-\tau,\mathcal{B}^*\circ F\right) \\ \sup_{z\geq 0,\lambda\in\mathbb{L}}-\mathcal{J}\left(z,\lambda,\tau,\mathcal{B}^*\circ F\right) \end{pmatrix}$$

and due to the CMT we finally obtain

$$\hat{S}_T^*\left(\tau\right) \overset{p}{\rightsquigarrow} \max\left(\begin{pmatrix} \sup_{z\geq 0,\lambda\in\mathbb{L}}\mathcal{J}\left(z,-\lambda,-\tau,\mathcal{B}^*\circ F\right) \\ \sup_{z\geq 0,\lambda\in\mathbb{L}}-\mathcal{J}\left(z,\lambda,\tau,\mathcal{B}^*\circ F\right) \end{pmatrix}'\right).$$

1. Due to Gaussianity we have that **med** $\left(\sup_{z\geq 0,\lambda\in\mathbb{L}}\mathcal{J}\left(z,-\lambda,-\tau,\mathcal{B}^*\circ F\right)\right)$ and **med** $\left(\sup_{z\geq 0,\lambda\in\mathbb{L}}-\mathcal{J}\left(z,\lambda,\tau,\mathcal{B}^*\circ F\right)\right)$ are finite and positive. By the relevant property of quantile functions the same is true for the median of the weak limit (in probability) of $\hat{S}_T^*\left(\tau\right)$. Furthermore since the function $(a,b)\rightarrow\max\left(\sup\left(a\right),\sup\left(b\right)\right)$ is a norm on the space of bounded functions defined on $\mathbb{R}^+\times\mathbb{L}$ with values in $\mathbb{R}^+\times\mathbb{R}^+$, Corollary 4.4.2.(i)-(ii) of Bogachev (1991) implies that the cdf of the $\max\begin{pmatrix} \sup_{z\geq 0,\lambda\in\mathbb{L}}\mathcal{J}\left(z,-\lambda,-\tau,\mathcal{B}^*\circ F\right) \\ \sup_{z\geq 0,\lambda\in\mathbb{L}}-\mathcal{J}\left(z,\lambda,\tau,\mathcal{B}^*\circ F\right) \end{pmatrix}$ restricted to $(0,+\infty)$ is absolutely continuous. Furthermore using (among others) the proposition A.2.7 of van deer Vaart and Wellner (1996) and the relevant property of quantile functions we have that $c_{\mathcal{PSD}}$ as defined in proposition 1 is finite and strictly positive for $a<\frac{1}{2}$. Then the result follows exactly as in the proof of Proposition 3.1 of Scaillet and Topaloglou (2012) (or Propositions 2 or 3 of Barrett and Donald (2003)). **2.** follows directly from the second part of proposition 1 and the fact that $c_{\mathcal{PSD}}$ is finite. ■

Proof of Proposition 4. From assumptions A.1 and A.3, Theorems 2.3 of Peligrad (1998), 2.4 of Shao and Yu (1993) and 1.4.8 of van der Vaart and Wellner (1996) we have that conditionally on the sample

$$\sqrt{T}\begin{pmatrix} \hat{F}_T^* - \hat{F}_T \\ \frac{1}{T}\sum_{i=1}^{T}(Y_i^* - Y_i) \end{pmatrix} \overset{p}{\rightsquigarrow} \begin{pmatrix} \mathcal{B}^* \circ F \\ \mathcal{Z}^* \end{pmatrix}$$

where $\mathcal{B}^* \circ F$ and $\mathcal{Z}^*$ are independent versions of the Gaussian process and random vector in lemma AL.2. The rest follow as in the proof of proposition 3. ■

Mathematical programming formulations

Formulation for prospect stochastic dominance

The test statistic $\hat{S}_T^{\alpha}(\tau)$ for prospect stochastic dominance efficiency is given by

$$\hat{S}_T^{\alpha}(\tau) = \sup_{z \geq 0, \lambda \in \mathbb{L}} \sqrt{T}\mathcal{J}\left(z, -\lambda, -\tau, \hat{F}_T\right), \tag{32}$$

which is equivalent to

$$\hat{S}_T^{\alpha}(\tau) = \sup_{z \geq 0, \lambda \in \mathbb{L}} \frac{1}{\sqrt{T}} \sum_{t=1}^{T_p} \left(\left(z - \tau' Y_t\right)_+ \mathbb{I}_{0 \leq \tau' Y_t} - \left(z - \lambda' Y_t\right)_+ \mathbb{I}_{0 \leq \lambda' Y_t} \right) \tag{33}$$

The mathematical formulation is the following:

$$\max_{\mathbf{z}\geq 0, \lambda\in\mathbb{L}} \quad \hat{S}_T^{\alpha}(\tau) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T_p} (L_t - W_t) \tag{34}$$

$$\texttt{s.t.} \quad M(F_t - 1) \leq z - \tau'\mathbf{Y}_t \leq MF_t, \qquad \forall t \in T_p, \tag{35}$$

$$-M(1 - F_t) \leq L_t - (z - \tau'\mathbf{Y}_t) \leq M(1 - F_t), \qquad \forall t \in T_p, \tag{36}$$

$$-MF_t \leq L_t \leq MF_t, \qquad \forall t \in T_p, \tag{37}$$

$$W_t \geq z - \lambda'\mathbf{Y}_t, \qquad \forall t \in T_p, \tag{38}$$

$$\tau'\mathbf{Y}_t \geq 0, \qquad \forall t \in T_p, \tag{39}$$

$$\lambda'\mathbf{Y}_t \geq 0, \qquad \forall t \in T_p, \tag{40}$$

$$\mathbf{e}'\lambda = 1, \tag{41}$$

$$\lambda \geq 0, \tag{42}$$

$$W_t \geq 0, \; F_t \in \{0, 1\}, \qquad \forall t \in T_p. \tag{43}$$

with M being a large constant.

The model is a mixed integer program maximizing the distance between the sum over all scenarios of two variables, $\sum_{t=1}^{T_p} L_t$ and $\sum_{t=1}^{T_p} W_t$ which represent the difference between $(z - \tau' Y_t)_+$ and $(z - \lambda' Y_t)_+$ respectively. This is difficult to solve since it is the maximization of the difference of two convex functions. We use a binary variable F_t, which, according to Inequalities (58b), equals 1 for each scenario $t \in T_p$ for which $z \geq \tau'\mathbf{Y}_t$, and 0 otherwise. Then, Inequalities (36) and (58d) ensure that the variable L_t equals $z - \tau'\mathbf{Y}_t$ for the scenarios for which this difference is positive, and 0 for all the other scenarios. Inequalities (58e) and (58h) ensure that W_t equals exactly the difference $z - \lambda'\mathbf{Y}_t$ for the scenarios for which this difference is positive, and 0 otherwise. Inequalities (39) and (40) ensure that both $\tau'\mathbf{Y}_t$ and $\lambda'\mathbf{Y}_t$ are greater than zero. Equation (58f) defines the sum of all portfolio weights to be unity, while Inequality (58g) disallows for short positions in the available assets.

The model is easily transformed to a linear one, which is very easy to solve. The steps are the following:

Proposition 11. *There is a set of at most T_p positive values, say $\mathcal{R} = \{r_1, r_2, ..., r_{T_p}\}$, containing the optimal value of the variable z.*

Proof. Vectors τ and $\mathbf{Y}_t$, $t = 1, ..., T_p$ being given, we can rank the values of $\tau'\mathbf{Y}_t, t = 1, ..., T_p$, by increasing order. Let us call $r_1, ..., r_{T_p}$ the possible different values of $\tau'\mathbf{Y}_t$, with $r_1 < r_2 < ... < r_{T_p}$ (actually there may be less than T_p different values). Now, for any z such that $r_i \leq z \leq r_i + 1$, $\sum_{t=1,...,T_p} L_t$ is constant (it is equal to the number of t such that $\tau'\mathbf{Y}_t \leq r_i$). Further, when $r_i \leq z \leq r_i + 1$, the maximum value of $-\sum_{t=1,...,T_p} W_t$ is reached for $z = r_i$. Hence, we can restrict z to belong to the set $\mathcal{R}$. ■

A direct consequence is that we can solve prospect stochastic dominance efficiency by solving the smaller problems $P(r)$, $r \in \mathcal{R}$, in which z is fixed to r. Then we take take the value for z that yields the best total result. The advantage is that the optimal values of the L_t variables are known in $P(r)$. Precisely, $\sum_{t=1,...,T_p} L_t$ is equal to the number of t such that $\tau'\mathbf{Y}_t \leq r$. Hence problem $P(r)$ boils down to the linear problem (26).

The test statistic $\hat{S}_T^{\beta}(\tau)$ for prospect stochastic dominance efficiency is given by

$$\hat{S}_T^{\beta}(\tau) = \sup_{z \geq 0, \lambda \in \mathbb{L}} -\sqrt{T}\mathcal{J}\left(z, \lambda, \tau, \hat{F}_T\right), \tag{44}$$

which is equivalent to

$$\hat{S}_T^{\beta}(\tau) = \sup_{z \leq 0, \lambda \in \mathbb{L}} \frac{1}{\sqrt{T}} \sum_{t=1}^{T_p} \left(\left(z - \tau' Y_t\right)_{-} \mathbb{I}_{0 \geq \tau' Y_t} - \left(z - \lambda' Y_t\right)_{-} \mathbb{I}_{0 \geq \lambda' Y_t} \right) \tag{45}$$

The mathematical formulation is the following:

$$\max_{z \leq 0, \lambda \in \mathbb{L}} \quad \hat{S}_T^{\beta}(\tau) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T_n} (L_t - W_t) \tag{46}$$

$$\text{s.t.} \quad M(F_t - 1) \leq \tau' \mathbf{Y}_t - z \leq M F_t, \qquad \forall t \in T_n, \tag{47}$$

$$-M(1 - F_t) \leq L_t - (\tau' \mathbf{Y}_t - z) \leq M(1 - F_t), \qquad \forall t \in T_n, \tag{48}$$

$$-M F_t \leq L_t \leq M F_t, \qquad \forall t \in T_n, \tag{49}$$

$$W_t \geq \lambda' \mathbf{Y}_t - z, \qquad \forall t \in T_n, \tag{50}$$

$$\tau' \mathbf{Y}_t \leq 0, \qquad \forall t \in T_n, \tag{51}$$

$$\lambda' \mathbf{Y}_t \leq 0, \qquad \forall t \in T_n, \tag{52}$$

$$\mathbf{e}' \lambda = 1, \tag{53}$$

$$\lambda \geq 0, \tag{54}$$

$$W_t \geq 0, \; F_t \in \{0, 1\}, \qquad \forall t \in T_n. \tag{55}$$

with M being a large constant.

The model is a mixed integer program maximizing the distance between the sum over all scenarios of two variables, $\sum_{t=1}^{T_n} L_t$ and $\sum_{t=1}^{T_n} W_t$ which represent the difference between $(z - \tau' Y_t)_-$ and $(z - \lambda' Y_t)_-$ respectively. We use a binary variable F_t, which, according to Inequalities (47), equals 1 for each scenario $t \in T_n$ for which $z \leq \tau' \mathbf{Y}_t$, and 0 otherwise. Then, Inequalities (48) and (49) ensure that the variable L_t equals $z - \tau' \mathbf{Y}_t$ for the scenarios for which this difference is negative, and 0 for all the other scenarios. Inequalities (50) and (55) ensure that W_t equals exactly the difference $z - \lambda' \mathbf{Y}_t$ for the scenarios for which this difference is negative, and 0 otherwise. Inequalities (51) and (52) ensure that both $\tau' \mathbf{Y}_t$ and $\lambda' \mathbf{Y}_t$ are lower than zero. Equation (53) defines the sum of all portfolio weights to be unity, while Inequality (54) disallows for short positions in the available assets.

Analogusly, we can solve a number of smaller problems $P(r)$, $r \in \mathcal{R}$, in which z is fixed to r, and the above problem boils down to the linear problem (27).

Formulation for Markowitz stochastic dominance

The test statistic $\hat{\Upsilon}_T^{\alpha}(\tau)$ for Markowitz stochastic dominance efficiency is given by

$$\hat{\Upsilon}_T^{\alpha}(\tau) = \sup_{z \le 0, \lambda \in \mathbb{L}} \sqrt{T} \left(\mathcal{J}_2(z, \tau; \hat{F}_T) - \mathcal{J}_2(z, \boldsymbol{\lambda}, \hat{F}_T) \right), \tag{56}$$

which is equivalent to

$$\hat{\Upsilon}_T^{\alpha}(\tau) = \sup_{z \le 0, \lambda \in \mathbb{L}} \frac{1}{\sqrt{T}} \sum\nolimits_{t=1}^{T_n} \left((z - \tau' Y_t)_+ - (z - \lambda' Y_t)_+ \right) \tag{57}$$

The mathematical formulation is the following:

$$\max_{\mathbf{z} \le 0, \boldsymbol{\lambda}} \quad \hat{\Upsilon}_T^{\alpha}(\tau) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T_n} (L_t - W_t) \tag{58a}$$

$$\text{s.t.} \quad M(F_t - 1) \le z - \boldsymbol{\tau}' \boldsymbol{Y}_t \le M F_t, \qquad \forall t \in T_n, \tag{58b}$$

$$-M(1 - F_t) \le L_t - (z - \boldsymbol{\tau}' \boldsymbol{Y}_t) \le M(1 - F_t), \qquad \forall t \in T_n, \tag{58c}$$

$$-M F_t \le L_t \le M F_t, \qquad \forall t \in T_n, \tag{58d}$$

$$W_t \ge z - \boldsymbol{\lambda}' \boldsymbol{Y}_t, \qquad \forall t \in T_n, \tag{58e}$$

$$\boldsymbol{e}' \boldsymbol{\lambda} = 1, \tag{58f}$$

$$\boldsymbol{\lambda} \ge 0, \tag{58g}$$

$$W_t \ge 0, \; F_t \in \{0, 1\}, \qquad \forall t \in T_n. \tag{58h}$$

with M being a large constant.

The model is a mixed integer program. We use a binary variable F_t, which, according to Inequalities (58b), equals 1 for each scenario $t \in T_n$ for which $z \ge \boldsymbol{\tau}' \boldsymbol{Y}_t$, and 0 otherwise. Then, Inequalities (58c) and (58d) ensure that the variable L_t equals $z - \boldsymbol{\tau}' \boldsymbol{Y}_t$ for the scenarios for which this difference is positive, and 0 for all the other scenarios. Inequalities (58e) and (58h) ensure that W_t equals exactly the difference $z - \boldsymbol{\lambda}' \boldsymbol{Y}_t$ for the scenarios for which this difference is positive, and 0 otherwise. Equation (58f) defines the sum of all portfolio weights to be unity, while Inequality (58g) disallows for short positions in the available assets.

We can solve a number of smaller problems $P(r)$, $r \in \mathcal{R}$, in which z is fixed to r, and the above problem boils down to the linear problem (28).

The test statistic $\hat{\Upsilon}_T^{\beta}(\tau)$ for Markowitz stochastic dominance efficiency is given by

$$\hat{\Upsilon}_T^{\beta}(\tau) = \sup_{z \ge 0, \lambda \in \mathbb{L}} \left[\frac{1}{\sqrt{T}} \sum\nolimits_{i=1}^{T} (\lambda' Y_i - \tau' Y_i) - \sqrt{T} \mathcal{J} \left(z, \lambda, \tau, \hat{F}_T \right) \right], \tag{59}$$

which is equivalent to

$$\hat{\Upsilon}_T^{\beta}(\tau) = \frac{1}{\sqrt{T}} \sup_{z\geq 0, \lambda \in \mathbb{L}} \left[\sum\nolimits_{t=1}^{T} \left(\lambda' Y_t - \tau' Y_t\right) + \sum\nolimits_{t=1}^{T_p} \left(\left(z - \lambda' Y_t\right)_+ \mathbb{I}_{\lambda' Y_t \geq 0} - \left(z - \tau' Y_i\right)_+ \mathbb{I}_{\tau' Y_t \geq 0} \right) \right] \tag{60}$$

The mathematical formulation is the following:

$$\begin{aligned}
\max_{\mathbf{z}\geq 0, \boldsymbol{\lambda}} \quad & \hat{\Upsilon}_T^{\beta}(\tau) = \frac{1}{\sqrt{T}} [\sum_{t=1}^{T} (\boldsymbol{\lambda}' \boldsymbol{Y}_t - \boldsymbol{\tau}' \boldsymbol{Y}_t) + \sum_{t=1}^{T_p} (W_t - L_t)] && \text{(61a)}\\
\text{s.t.} \quad & M(F_t - 1) \leq z - \boldsymbol{\lambda}' \boldsymbol{Y}_t \leq M F_t, \qquad \forall t \in T_p, && \text{(61b)}\\
& -M(1 - F_t) \leq W_t - (z - \boldsymbol{\lambda}' \boldsymbol{Y}_t) \leq M(1 - F_t), \qquad \forall t \in T_p, && \text{(61c)}\\
& -M F_t \leq W_t \leq M F_t, \qquad \forall t \in T_p, && \text{(61d)}\\
& L_t \geq z - \boldsymbol{\tau}' \boldsymbol{Y}_t, \qquad \forall t \in T_p, && \text{(61e)}\\
& \boldsymbol{e}' \boldsymbol{\lambda} = 1, && \text{(61f)}\\
& \boldsymbol{\tau}' \boldsymbol{Y}_t \geq 0, \qquad \forall t \in T_p, && \text{(61g)}\\
& \boldsymbol{\lambda}' \boldsymbol{Y}_t \geq 0, \qquad \forall t \in T_p, && \text{(61h)}\\
& \boldsymbol{\lambda} \geq 0, && \text{(61i)}\\
& W_t \geq 0,\ F_t \in \{0, 1\}, \qquad \forall t \in T_p. && \text{(61j)}
\end{aligned}$$

with M being a large constant.

The model is again a mixed integer program. We use a binary variable F_t, which, according to Inequalities (61b), equals 1 for each scenario $t \in T_p$ for which $z \geq \boldsymbol{\tau}' \boldsymbol{Y}_t$, and 0 otherwise. Then, Inequalities (61c) and (61d) ensure that the variable L_t equals $z - \boldsymbol{\tau}' \boldsymbol{Y}_t$ for the scenarios for which this difference is positive, and 0 for all the other scenarios. Inequalities (61e) and (61j) ensure that W_t equals exactly the difference $z - \boldsymbol{\lambda}' \boldsymbol{Y}_t$ for the scenarios for which this difference is positive, and 0 otherwise. Inequalities (61g) and (61h) ensure that both $\boldsymbol{\tau}' \boldsymbol{Y}_t$ and $\boldsymbol{\lambda}' \boldsymbol{Y}_t$ are greater than zero. Equation (61f) defines the sum of all portfolio weights to be unity, while Inequality (61i) disallows for short positions in the available assets.

We can solve a number of smaller problems $P(r)$, $r \in \mathcal{R}$, in which z is fixed to r, and the above problem boils down to the linear problem (29).

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