ON THE OPTIMAL LIFETIME OF ASSETS

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Abstract

We show that the “abandonment” model emphasized by researchers in capital budgeting and the “steady state” replacement model emphasized by economic theorists constitute sub-cases of a more general class of models in which the horizon of reinvestments is determined endogenously along with the other decision variables. Moreover, comparisons between our model and that of steady state replacement revealed that there are considerable differences. In particular, we found that: i) the two models lead to different estimates concerning the profit horizon, the duration of replacements, the timing of abandonment or scrapping, and the impact of productive capacity and market structure on service lives, as these are determined by various parameters, ii) even though the steady state replacement policy may result in higher total profit, it does so at great expense in flexibility for the planner, because the replacements are built into the model from the beginning, and iii) the transitory replacement policy seems more realistic in that the replacements are undertaken only if forced on the planner by decreasing profits.

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1. Introduction

The economic life of assets is a key variable in many fields of decision sciences. In capital budgeting, for example, if the economic life of assets under consideration is unknown, their relevant cash flows and scrap values cannot be ascertained. So computing the standard criteria of net present value and internal rate of return becomes untenable and this in turn inhibits the accurate planning of investments. Similarly, in accounting and finance, if the economic life of assets is unknown, their depreciation cannot be accounted for with any precision and all approximations, say, to the user cost of capital are rendered uncertain. Lastly, in economics, if the useful life of assets is unknown, as Haavelmo (1962) has established, we cannot derive consistent aggregates of the stock of capital of a firm, a sector or an economy. No wonder therefore that the issue of establishing the optimal life of assets occupies a central place in economic theory and policy.

Preinreich (1940) was the first to show how the optimal life of assets can be determined. More specifically, according to his theorem, in order for the economic life of a single machine to be optimal, it should be computed jointly with the economic life of each machine in the chain of future replacements extending as far into the future as the owner's profit horizon. But he formulated it under two crucial assumptions. The first of them abstracted from technological progress and postulated that older machines are replaced by newer machine of identical type (like-for-like). As such it contradicted casual observation and was ultimately relaxed by Smith (1962) who generalized the above result to the case where older machines are replace d by more productive machines embodying the most recent advances in science and technology. The second assumption concerned the horizon of the reinvestment process and left it upon the owner of the machine to decide its duration on the basis of his perception on how long the investment opportunity might remain profitable. Thus, depending on the specification of the owner's profit horizon, there emerged a continuum of models for the determination of the optimal lifetime of assets.

In particular, by limiting the owner's profit horizon to a single investment cycle, researchers in the field of capital budgeting obtained the so-called "abandonment" class of models and used it to derive sharp rules regarding optimal asset life. Initially Robichek and Van Horne (1967) suggested that an asset should be abandoned in any period in which the present value of future cash flows does not exceed its abandonment value. Then, drawing on the possibility that the function of cash flows may not have a single peak, Dyl and Long (1969) argued that abandonment should not occur at the earliest possible date that the above abandonment condition is satisfied, but rather at the date that yields the highest net present value over all future abandonment opportunities. Lastly, Howe and McCabe (1983): i) highlighted the patterns of cash flows and scrap values under which the "abandonment" model leads to a unique global optimum of the abandonment time, ii) characterized the complete range of models that can be obtained by varying the owner's profit horizon, and iii) clarified the circumstances which should guide in practice the choice between "abandonment" and "replacement" models.

Research economists, on the other hand, continued to work in the tradition of Terborgh (1949) and Smith (1962) by assuming invariably that the owner's profit horizon is infinite. This in turn led them to concentrate exclusively on a single class of "replacement" models, all of which presume that reinvestments take place at equal time intervals. Just to indicate how pervasive this conceptualization has been, it suffices to mention that it has been adopted in all significant contributions in this area from Brems (1968) to Nickel (1975), Rust (1987) and Van Hilten (1991), and to Mauer and Ott (1995), more recently. The question then arises as to the importance of the implications that may result if, instead of treating the owner's profit horizon as given while solving for the optimal lifetime assets, we make it an endogenous variable which is decided along with all other variables in the optimization process. Our objective in this paper is to investigate the implications of the proposed generalization and characterize their significance.

The paper is organized as follows. In Section 2 we set up the model and analyze the properties of the replacement policies that emanate from its solution. The structure of the model allows for a number of reinvestment cycles, which terminate with abandonment or terminal scrapping of the asset under consideration. Thus, with all other variables and parameters given, the owner of the asset is presumed to determine the profit horizon and the dates of reinvestments, if any, so as to maximize the net present value of overall profits. In Section 3 we compare the policy from our model and that from the steady state replacement model by means of a numerical example. Then, in Section 4, we draw the implications of the analysis for the two types of replacement policies, and, finally, in Section 5 we conclude with a synopsis of the main results and a suggestion for further research.

2. The model

At the end of the service life of equipment, there are always two options, to replace it and continue doing so up to some profit horizon or to abandon or scrap it and terminate operations. To examine them we formulate the service life problem for a multiple series of operating periods and we compare the following two alternative approaches:

1. *Transitory replacements with terminal scraping,* where the equipment is replaced a finite number of times, ending with terminal scrapping.[1]
2. *Steady state replacements,* where the equipment is replaced at equal time intervals, indefinitely.

To keep the analysis as simple as possible we adopt the following simplifying assumptions: 1) Time invariance, in the sense that only relative time matters and in particular all operating periods are similar and hence we can examine each one starting at zero time; 2) The effects of *minor* technological advances are counterbalanced by maintenance of the *upgrading* type, whereas those of *major* technological breakthroughs are incorporated in the discount factor,[2] and 3) Impatience on the part of the owner in the sense that he does not accept even a temporary drop in total profits, terminating operations when this happens. Among other reasons this could be attributed to the uncertainty caused by the possibility of an obsolescence effect eliminating all revenue and scrap value thereafter.

We use the term equipment in a general sense, by not adopting any particular model. Thus, considering first a single operating period of duration T, we assume that the maximum total profit in present values, is given by some *profit function*:

$$A(T) = Q(T) + e^{-\sigma T} S(T) - P\,, \text{ for } 0 \le T \le \infty\,, \tag{1}$$

where $Q(T)$ is the net operating revenue, $S(T)$ is the scrap value of the equipment, P is the price of new equipment, and σ is the discount rate. In general, $A(T)$ is the result of some optimization procedure that may also include uncertainties in which case it refers to expected values. Also, $S(T)$ may be zero if we have abandonment, negative if we have disposal costs, and in special cases it may even be higher than P if we have upgrading. The term:

$$A(0) = S(0) - P \le 0\,, \tag{2}$$

denotes the cost of new as opposed to unused equipment. It will be called *transactions cost.* We can separate it from the variable part by setting:

$$VA(T) = A(T) - A(0) = Q(T) + e^{-\sigma T}S(T) - S(0) \;\Rightarrow A(T) = VA(T) + A(0) \qquad \textbf{(3)}$$

We will be examining first the variable part by ignoring the transactions cost. Then we will include the transactions cost as a correction term. Given the above, we consider also the terminal profit rate in current values given by:

$$\alpha(T) = A'(T)e^{\sigma T} = VA'(T)e^{\sigma T} \qquad \textbf{(4)}$$

2.1 Transitory replacements

Applying the procedure of dynamic programming we reorder the operating periods, starting with the last period leading to scrapping, which we refer to as $0-$period, and going backwards. Thus, $1-$period is the period before the last replacement, etc. In view of the impatience assumption, we will say that the equipment is *profitable*[3] if it starts with strictly positive profit rate: $\alpha(0) > 0$.[4] Then the scrapping period duration T_0, determined by the first maximum of $VA(T)$, will be nonzero:

$$\alpha(0) > 0 \Rightarrow V\Pi_0 = VA(T_0) > 0, \text{ where } 0 < T_0 \leq \infty . \qquad \textbf{(5)}$$

Assuming profitability, we consider the last replacement period before the scrapping period, and the corresponding $1-$replacement total variable profit function:[5]

$$V\Pi_1(T) = VA(T) + e^{-\sigma T}V\Pi_0 \qquad \textbf{(6)}$$

As previously, we will say that the equipment is *scrapping* $1-$*replaceable*, if this profit function starts with a strictly positive profit rate: $V\Pi_1'(0) = \alpha(0) - \sigma V\Pi_0 > 0$. Then the first maximum will be strictly larger than $V\Pi_0$, and the corresponding replacement time T_1 will be nonzero. Using the notation $V\Pi_1 = V\Pi_1(T_1)$, we have:

$$\alpha(0) > \sigma V\Pi_0 \Rightarrow V\Pi_1 = VA(T_1) + e^{-\sigma T_1}V\Pi_0 > V\Pi_0 , \qquad \textbf{(7)}$$

In the same way we define inductively the notion of $v-$*replaceable* equipment with replacement time T_v, period variable profit $VA(T_v)$ and total variable profit $V\Pi_v$. We collect the main properties in:

Lemma 1

1. If $\alpha(t) \le \bar{\alpha}$*, then* $VA(T) \le \frac{\bar{\alpha}}{\sigma}(1-e^{-\sigma T})$.

2. If the equipment is $v-$*replaceable, then it satisfies:*

$$V\Pi_v = VA(T_v) + e^{-\sigma T_v} V\Pi_{v-1}, \text{ and } V\Pi_{v-1} = \frac{\alpha(T_v)}{\sigma} < \frac{\alpha(0)}{\sigma}.$$

3. Profitable replacements have successively strictly decreasing durations,[6] *strictly decreasing period variable profits and strictly increasing total variable profits:*

$$T_v < T_{v-1},\ VA(T_v) < VA(T_{v-1}),\ V\Pi_v > V\Pi_{v-1}.$$

4. If the equipment has profitable replacements of every order, then because of the monotonicities involved, we will have the following limits:

$$T_v \downarrow T_\infty,\ VA(T_v) \downarrow VA(T_\infty),\ V\Pi_v \uparrow V\Pi_\infty = VA(T_\infty) + e^{-\sigma T} V\Pi_\infty \Rightarrow V\Pi_\infty = \frac{VA(T_\infty)}{1-e^{-\sigma T_\infty}}$$

$$V\Pi_{v-1} = \frac{\alpha(T_v)}{\sigma} \uparrow \frac{\alpha(T_\infty)}{\sigma} = V\Pi_\infty \le \frac{\alpha(0)}{\sigma}$$

Also in this case if we have $T_\infty > 0$*, then* $\alpha(T)$ *is constant for* $0 \le T \le T_\infty$*, strictly decreasing immediately afterwards.*

Proof

Part 2 is in the definitions and part 3 follows from the property that T_v *is the first time* $\alpha(T)$ *crosses below the level* $\sigma\Pi_v$*, where* $\sigma\Pi_v$ *is a strictly increasing positive sequence. Concerning the last part of 4 we note first that by the definitions we have* $\alpha(T) \ge \alpha(T_\infty)$ *in this interval. If* $\alpha(T)$ *is not constant then it will satisfy:*

$$VA(T_\infty) > \frac{\alpha(T_\infty)}{\sigma}(1-e^{-\sigma T_\infty}) \Rightarrow V\Pi_\infty = \frac{VA(T_\infty)}{1-e^{-\sigma T_\infty}} > \frac{\alpha(T_\infty)}{\sigma},$$

contradicting the limiting condition $V\Pi_\infty = \alpha(T_\infty)/\sigma$.

Classifying profitable equipment with respect to replacement properties, we will say that it is:

1. *Finitely* $N-$*replaceable,* if it has only N profitable replacements, for some $N \ge 0$. In particular we will say that it is *replaceable* if $N > 0$, *non-replaceable* if $N = 0$.
2. *Infinitely replaceable* or *disposable* if all replacements are profitable: $T_\infty \ge 0$.

   As indicated further below, in this case we will have essentially $T_\infty = 0$.
3. *Non-scrappable* if $T_0 = \infty$, *durable* if it is both non-scrappable and non-replaceable: $T_0 = \infty$ & $N = 0$.

2.2 Steady state replacements

Independently of the above limiting procedure, infinite replacements at equal time intervals can also be examined directly. In this case, if T is the uniform duration, the *steady state profit* can be separated in the variable part and in the transactions cost part, by writing:

$$\Pi(T)=\sum_{v=0}^{\infty}e^{-v\sigma T}A(T)=\frac{A(T)}{1-e^{-\sigma T}}=\frac{VA(T)}{1-e^{-\sigma T}}+\frac{A(0)}{1-e^{-\sigma T}}=V\Pi(T)+C(T)\,. \qquad \textbf{(8)}$$

$A(T)$ is the same as above since it involves maximizing one period profits for fixed T. Taking the derivative of the steady state variable profit part, we find:

$$V\Pi'(T)=\frac{\sigma e^{-\sigma T}}{1-e^{-\sigma T}}[\frac{\alpha(T)}{\sigma}-V\Pi(T)]\,. \qquad \textbf{(9)}$$

Clearly the steady state policy is easier to study and classify because it is determined by a single function related directly to the profit rate function. This may explain its popularity. We collect the main properties:

Lemma 2

1. *The steady state variable profit function increases, is constant, or decreases, according as its value $V\Pi(T)$ is smaller, equal to, or larger respectively, than the value $\alpha(T)/\sigma$.*
2. *Initially the function $V\Pi(T)$ lies between the function $\alpha(T)/\sigma$ and the constant $\alpha(0)/\sigma$. In particular, we have:*

$$V\Pi(0)=\frac{\alpha(0)}{\sigma} \quad \& \quad V\Pi'(0)=\frac{\alpha'(0)}{2\sigma} \qquad \textbf{(10)}$$

Part 1 is a consequence of the derivative formula, and the relations in part 2 are obtained by applying l' Hopital rule at $T=0$. We will say that the equipment is *steady state profitable* if it satisfies $V\Pi(0)>0$,[7] and then the *steady state duration* T^* is defined as the first time $V\Pi(T)$ starts dropping. In this case we call the equipment:

steady state, durable if $T^*=\infty$, *replaceable* if $0<T^*<\infty$, *disposable* if $T^*=0$.

Comparing now the two policies, we find:

Proposition

Ignoring transactions costs for new equipment, we have:

1. *The profitability condition for steady state replacement and transitory replace-*

ment policies is the same: $\alpha(0) > 0$ *.*

2. *Assuming profitability, we distinguish the following cases:*
   - A. *If the equipment is steady state durable then it is also scrapping durable:* $T^* = T_0 = \infty$ and $V\Pi^* = V\Pi_0 = VA(\infty)$, with $N = 0$.
     In this case the profit rate function satisfies: $\alpha(0) \le \alpha(T)$ *for all* T *.*
   - B. *If the equipment is steady state replaceable, then it is also scrapping finite* $N-$*replaceable for some* $N \ge 0$*. In this case the function* $\alpha(T)$ *initially rises strictly above* $\alpha(0)$*, and we have:*
     $$0 < \tau^* \le T^* < T_N \;\; and \;\; \frac{\alpha(0)}{\sigma} < \{V\Pi^*, V\Pi_N\} < \frac{\bar{\alpha}}{\sigma},$$
     where τ^* *is the time of first maximum of* $\alpha(T)$ *and* $\bar{\alpha}$ *is the global maximum of* $\alpha(T)$*. We note that the equipment may be non-replaceable:* $N = 0$*, non-scrappable:* $T_0 = \infty$*, even scrapping durable.*
   - C. *If the equipment is steady state disposable, then in general it is also scrapping disposable, with:*
     $$T^* = T_\infty = 0 \;\; and \; V\Pi^* = V\Pi_\infty = \frac{\alpha(0)}{\sigma}$$

Proof

Part 1 is a consequence of the formula for $V\Pi(0) = \alpha(0)/\sigma$ *in Lemma 2.2. For part 2 we note the following:*

A. *By assumption* $V\Pi(T)$ *is increasing and then* $\alpha(T)$ *has no zeroes because Lemma 2 implies:* $\alpha(T) \ge \sigma V\Pi(T) \ge \sigma V\Pi(0) = \alpha(0) > 0$*. Here belongs also the case where* $V\Pi(T)$ *and hence also* $\alpha(T)$ *are constant throughout.*

B. *Starting from the level* $\alpha(0)/\sigma$*, the function* $V\Pi(T)$ *increases in the interval* $0 \le T \le T^*$*, to a level strictly above* $\alpha(0)/\sigma$ *and decreases immediately after, at least initially (see also the remark bellow). By Lemma 2,* $\alpha(T)/\sigma$ *will lie above it while* $V\Pi(T)$ *increases and drop back crossing the level* $V\Pi^*$ *at* T^**. We conclude that it must have a maximum at some* $\tau^* \le T^*$*. Concerning transitory replacement policy we note that if it is* $N-$*replaceable then* $\sigma\Pi_{N-1} < \alpha(0) \le \sigma V\Pi^*$*, and hence* $\alpha(T)$ *will cross the level* $\sigma\Pi_{N-1}$ *after* T^**, giving* $T_N > T^*$*. Finally the bounds on the total profits follow from Lemma 1.4. In particular we have:*
$$V\Pi_N = VA(T_N) + e^{-\sigma T_v} V\Pi_{N-1} < \frac{\bar{\alpha}}{\sigma}(1 - e^{-\sigma T_N}) + e^{-\sigma T_N}\frac{\alpha(0)}{\sigma} < \frac{\bar{\alpha}}{\sigma}$$

C. *By assumption, the function* $\sigma V\Pi(T)$ *will be initially strictly decreasing and by lemma 2.1* $\alpha(T)$ *will be even lower, in particular bellow* $\alpha(0)$*. If it remains so, i.e. if* $\alpha(0)$ *is the maximal value, then all replacement profits will stay strictly below* $\alpha(0)$*, and the equipment is infinitely replaceable. By Lemma 1.4* $\alpha(T)$ *will be constant in the interval* $0 \le T \le T_\infty$*, falling immediately afterwards. Since* $\alpha(T)$ *is in fact initially strictly decreasing it follows that* $T_\infty = 0$*. If however* $\alpha(T)$ *rises eventually above* $\alpha(0)$*, we would still obtain the same conclusion if some replacement profit level* $\sigma\Pi_v$ *meets the initial decreasing section of* $\alpha(T)$*. Otherwise we would need more specifications as indicated in the consideration of equipment of type D given bellow.*

Remark.
The case where $V\Pi(T)$ has an initial segment of constancy decreasing immediately afterwards requires special treatment. According to the definition the steady state duration T^* is defined as the time of first drop, i.e. at the right end of the interval, because this is the correct assignment if we obtain it as the limit when the transactions cost goes to zero. Hence the equipment should be called steady state replaceable. Actually we have excluded it from the treatment in B above where we assumed that the first maximum of $V\Pi(T)$ is strictly higher than $V\Pi(0)$. Of course, the steady state duration in this case is indifferent to any value in the interval of constancy. Actually this type of equipment behaves more like the disposable type in C because it is infinitely replaceable with $T^* = T_\infty > 0$. We note that this is the only case where we obtain $T_\infty > 0$, in the limiting procedure of transitory replacements.

The above allow us to classify equipment according to the properties of $\alpha(T)$. For convenience we will consider *regular* equipment in the sense that $\alpha(T)$ is not very complex. In particular, as shown in Figure 1, we will assume that $\alpha(T)$ has at most two monotone sections, not necessarily strict. Actually, these monotonicity properties are relevant only until T_0. Thus we distinguish the following types:

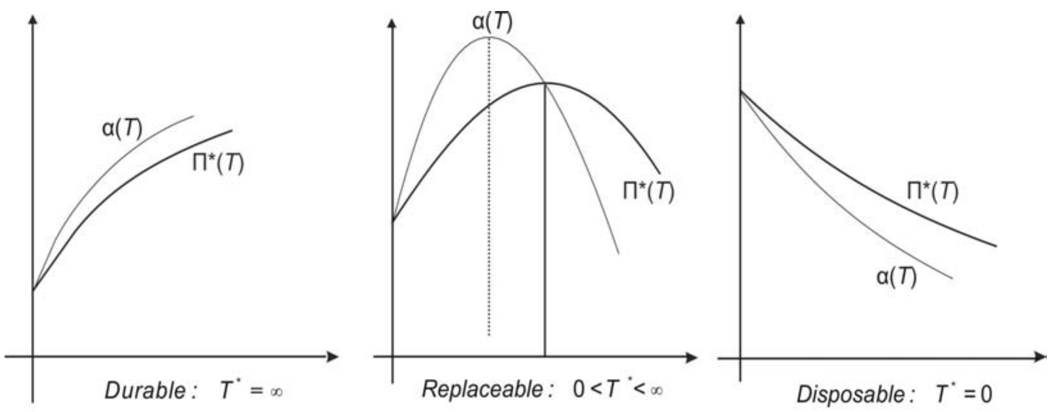

Figure 1

Type A: *$\alpha(T) \geq \alpha(0)$. The equipment will be scrapping durable. Also steady state durable if it is monotone increasing, otherwise it can be steady state replaceable.*

Type B: *$\alpha(T)$ at first rises strictly above $\alpha(0)$ but eventually drops strictly bellow. The equipment will be scrapping finitely replaceable and steady state replaceable.*

Type C: *$\alpha(T) \leq \alpha(0)$. It will be steady state disposable and scrapping disposable.*

Type D: *At first it drops strictly below $\alpha(0)$ but eventually rises strictly above. It will be steady state disposable. Also it will be scrapping disposable unless $A(T_0) > \alpha(0)/\sigma$ in which case it will be non-replaceable.*

In figure 1 we show the first three types.

2.3 Transactions cost

Introducing now transactions cost we will examine only the case where it is small negative: $A(0) < 0 \Rightarrow S(0) < P$. In the steady state policy we have the correction to the profit given by the cost term:

$$C(T) = \frac{A(0)}{1 - e^{-\sigma T}}. \quad \textbf{(11)}$$

It is monotonically increasing in T, from $C(0) = -\infty$ to $C(\infty) = 0$. Thus, as $A(0)$ starts decreasing from zero to negative values, the steady state duration starts increasing and the steady state profit starts falling. The effect is similar for the transitory replacement policy, except that T_0 is not affected. Also, since $\alpha(T)$ remains the same we may have an increase in the number of profitable replacements because $\sigma\Pi_v$ will be lower.

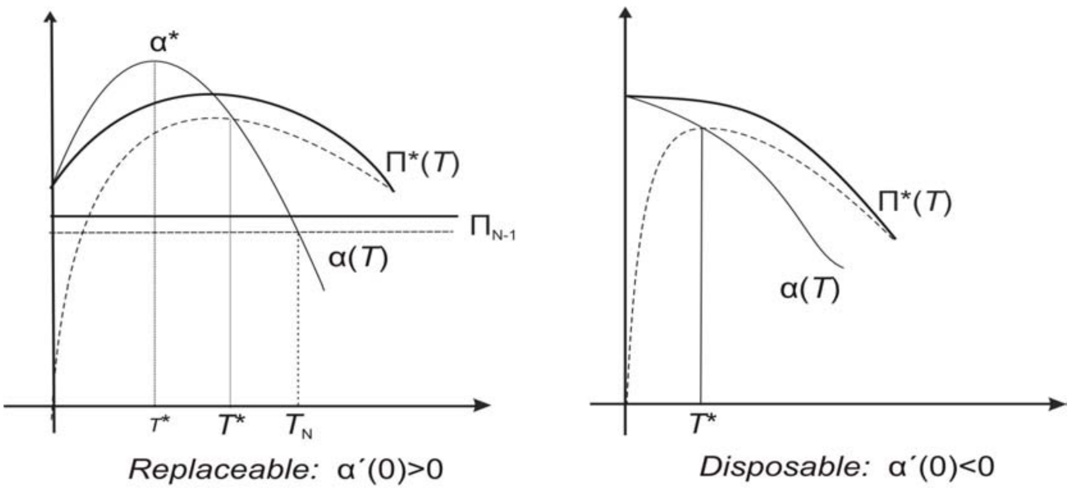

Figure 2

For small transaction costs we can estimate these effects directly, as follows:

Corollary
As $C = A(0)$ starts decreasing from the zero value, the profit and the duration are affected as follows:

1. For steady state replaceable equipment: $d\Pi^* = C^*$ and $dT^* = \dfrac{\sigma C^*}{\alpha'(T^*)}$, *where:*

$$C^* = \frac{A(0)}{1 - e^{-\sigma T^*}} = A(0)(1 + e^{-\sigma T^*} + e^{-2\sigma T^*} + \cdots)$$

2. For finitely replaceable equipment: $d\Pi_v = C_v$, $C_0 = A(0)$ *and* $dT_v = \dfrac{\sigma C_{v-1}}{\alpha'(T_v)}$,

$dT_0 = 0$, *where:*

$$C_v = A(0) + e^{-\sigma T_v} C_{v-1} = A(0)[1 + e^{-\sigma T_v} + \cdots + e^{-\sigma(T_v + \cdots + T_1)}].$$

Proof.
Concerning the profits it is a direct consequence of the envelope theorem applied to the functions
$\Pi(T) = A(T)/(1 - e^{-\sigma T})$, $\Pi_v(T) = A(T) + e^{-\sigma T}\Pi_{v-1}$, where $A(T) = VA(T) + A(0)$
Concerning the durations we use in addition the defining relations:

$$\alpha(T) = \sigma\Pi(T),\ \alpha(T) = \sigma\Pi_{v-1}.$$

The profit reduction per period is more pronounced in the steady state policy. We note that by their definition we have: $\alpha'(T^*) < 0$, $\alpha'(T_v) < 0$.

3. An example

In Bitros and Flytzanis (2004) we examined a problem of determining the optimal policies for a one period model, by using techniques of optimal control. Considering now the problem of multi-period replacements we will examine the simplest version of this model for which we can compute directly all the relevant quantities. The specifications are as follows:

$$A(T) = \max \int_0^T e^{-\sigma t} q(S)dt + e^{-\sigma T} S - P \quad \text{with} \quad \dot{S} = -wS,\ q = rS^{\varepsilon},\ P = S_0 \qquad \textbf{(12)}$$

We consider only the case where the capital stock downgrades: $w > 0$. This will exclude equipment of **type A**. The operating revenue q decreases at the rate εw, which is smaller or larger than w, depending on ε. We compute the functions:

$$S = S_0 e^{-wT},\ q = rS_0 e^{-\varepsilon w T},\ Q = \frac{rS_0}{\varepsilon w + \sigma}[1 - e^{-(\varepsilon w + \sigma)T}] \qquad \textbf{(13)}$$

$$A(T) = \frac{rS_0}{\varepsilon w + \sigma}[1 - e^{-(\varepsilon w + \sigma)T}] + S_0 e^{-(w+\sigma)T} - S_0 \Rightarrow A(\infty) = \frac{rS_0}{\varepsilon w + \sigma} - S_0 \qquad \textbf{(14)}$$

$$\alpha(T) = e^{\sigma T} A'(T) = S_0[re^{-\varepsilon w T} - (w + \sigma)e^{-wT}] \quad \Rightarrow \alpha(0) = [r - (w + \sigma)]\ S_0 \qquad \textbf{(15)}$$

$$\alpha'(T) = -w[\varepsilon r e^{-\varepsilon w T} - (w + \sigma)e^{-sT}]S_0 \qquad \Rightarrow \alpha'(0) = -w[\varepsilon r - (w + \sigma)]S_0 \qquad \textbf{(16)}$$

The equipment is profitable if:

$$\alpha(0) > 0 \Rightarrow r > w + \sigma \quad \textbf{(17)}$$

We distinguish the following cases:

$\varepsilon = 1$*: Capital stock and services deteriorate at the same rate.* The equipment is of ***Type C***, disposable for both types of policies.

$\varepsilon > 1$*: Services deteriorate faster than capital stock.* The equipment is of ***Type B,*** i.e. steady state replaceable and finitely replaceable scrappable, with scrapping duration:

$$T_0 = \frac{1}{(\varepsilon - 1)w} \ln \frac{r}{w + \sigma} \quad \textbf{(18)}$$

$\varepsilon < 1$ *: Services deteriorate slower than capital stock.* This is the usual case and we examine it in more detail. Expressing the conditions in terms of the discount rates, we find two critical values:

$$\alpha(0) = 0 \Rightarrow \sigma_0 = r - w\,,\ \alpha'(0) = 0 \Rightarrow \sigma_\infty = \varepsilon r - w\,, \quad \textbf{(19)}$$

with the following properties:

$\sigma > \sigma_0$: The equipment is non-profitable

$\sigma_\infty > \sigma$: The equipment is of ***Type C***, i.e., disposable for both types of policies.

$\sigma_0 > \sigma > \sigma_\infty$: The equipment is of ***Type B***, i.e. steady state replaceable and finitely N – replaceable, with:

$$\begin{aligned} &\alpha'(0) = 0 \Rightarrow \tau^* = \frac{1}{(1-\varepsilon)w} \ln \frac{(w+\sigma)}{\varepsilon r} < T^* < T_N \\ &\frac{\alpha(0)}{\sigma} = \frac{r-(w+\sigma)}{\sigma} < \{\Pi^*, \Pi_N\} < \frac{\bar{\alpha}}{\sigma}. \end{aligned} \quad \textbf{(20)}$$

Moreover, with respect to the policy of transitory replacements, we note first that the equipment is non-scrappable, with:

$$T_0 = \infty,\ \Pi_0 = A(\infty) = \left(\frac{r}{\varepsilon w + \sigma} - 1 \right) S_0 \quad \textbf{(21)}$$

Proceeding, we can determine further critical values for the discount rate within this interval, of increasing replaceability. In particular, for $N = 1$, the critical value:

$$\alpha(0) = \sigma\Pi_0 \Rightarrow \sigma_1 = \varepsilon r - \varepsilon w \,, \tag{22}$$

distinguishes the following sub-cases:

$\sigma_0 > \sigma > \sigma_1$: The equipment is non-replaceable and hence scrapping durable.

$\sigma_1 > \sigma > \sigma_\infty$: The equipment is replaceable.

For a numerical example, we consider equipment with the following technical characteristics:

$$\{r = 0.2, w = 0.1, \varepsilon = 0.4\} \Rightarrow \{\sigma_0 = 0.1, \sigma_1 = 0.04, \sigma_\infty = -0.02\}$$

It is of ***Type B***, i.e. finitely replaceable, if profitable. In particular, it is: Non-profitable if $\sigma > 0.1$, non-replaceable if $\sigma > 0.4$, replaceable for smaller values of σ. We compute it for two values of σ.

1. $\sigma = 0.06 > \sigma_1 \Rightarrow \alpha(0)/\sigma = 0.67,\ \bar{\alpha}/\sigma = 1.25,\ \tau^* = 8$

T^*, Π^*	T_0^*, Π_0^*	T_1, Π_1	T_2, Π_2	T_3, Π_3
21 1.1	∞ 1	25 1.1	22 1.1	21 1.1

$\Rightarrow N = 0,\ \Pi_0 = 1$

2. $\sigma = 0.03 < \sigma_1 \Rightarrow \alpha(0)/\sigma = 2.33,\ \bar{\alpha}/\sigma = 2.89,\ \tau^* = 11.5$

T^*, Π^*	T_0^*, Π_0^*	T_1^*, Π_1^*	T_2^*, Π_2^*	T_3, Π_3
14 2.8	∞ 1.8	29 2.3	22 2.5	19 2.6

$\Rightarrow N = 2,\ \Pi_2 = 2.5$

We indicate with asterisks the profitable replacements in the sense of initially replaceable as used in this work. The last profitable replacement is given by the first time the profit crosses the level $\alpha(0)/\sigma$. Thus in the first example the steady state policy realizes total profit $\Pi^* = 1.1$ with replacement duration $T^* = 21$, while the transitory policy total profit $\Pi_0 = 1$ is realized without any replacements, because the equipment is scrapping durable. The second replacement, without being profitable by our definition, is eventually profitable recovering the remaining part of the profit: $1.1 - 1 = 0.1$, with replacement duration $T_1 = 25$. In the second calculation, we half the discount rate and the equipment becomes $2-$replaceable. Now the steady state policy total profit $\Pi^* = 2.8$ is realized with replacement duration $T^* = 14$, while the transitory replacement policy total profit $\Pi_2 = 2.5$ is realized with two replacements of duration $T_2 = 22$ and $T_1 = 29$, and then $T_0 = \infty$. The 3rd replacement is not profitable by our definition.

4. Some implications

We summarize the basic differences between the two approaches to replacement, the steady state replacement policy and the transitory replacement policy.

1. Concerning the profit horizon, in the steady state case it is taken invariably as infinite, while in the transitory case it is determined by the parameters of the equipment and of the market. This in itself is an important finding because by adopting, for example, in capital budgeting the transitory approach to replacement we can allow endogenously for the influence of profit horizon on the selection of projects.
2. As indicated in the example, the steady state policy binds the operator to undertaking an infinite number of replacements, often with small profit in each consecutive replacement period. This implies in turn that re-investment opportunities last forever. So the steady state policy is overly restrictive because it precludes abandonment or scrapping at any time. On the contrary, the transitory replacement policy may allow the owner to realize almost the same total profit as under the steady state policy, but with few or even without any replacements, depending on the parameters: $\{r, w, \varepsilon, \sigma\}$.
3. The steady state policy predicts consistently shorter replacement durations than does the transitory replacement policy, with sometimes very small profits per replacement period. In some cases this difference may be extreme, as some steady state replaceable equipment may be classified as durable non-replaceable in the context of transitory replacement policy.
4. As the parameters change, in the case of steady state replacements the economy adjusts gradually, e.g. by changing the replacement period. In the case of transitory replacement policy, except for this smooth change, we have also sudden changes when the parameters cross certain critical values where an additional replacement policy becomes profitable or ceases to be so. Thus at some parameter values, e.g. the interest rate, we would observe a burst or a slump in the demand for replacement investment much like the “spikes” discovered in recent years by

researchers studying investment at the plant level.

5. How the technical characteristics of the equipment combine with the structure of the market to determine the value of parameter ε is of critical importance for differentiating between the two types of replacement policies. In particular, in the simple case $\varepsilon = 1$, the equipment is disposable and the two replacement policies become indiscernible, whereas in the usual case where $\varepsilon < 1$, the value of productive services deteriorates slower than the value of equipment and the two policies may lead to substantially different economic implications at both the firm and the economy levels.[8]

5. Conclusions

Past studies of the optimal lifetime of assets have assumed that the duration of the reinvestment process is determined by the owner of the asset on the basis of his perception on how long the investment opportunity remains profitable. As a result, whereas researchers in the field of capital budgeting have emphasized the so-called "abandonment" model, in which the owner's profit horizon is limited to a single investment cycle, economic theorists have adopted the so-called "steady state" replacement model in which reinvestments take place indefinitely at equal time intervals. Our analysis showed that these two polar classes of models constitute subcases of a more general model in which the owner's profit horizon is chosen jointly along with the other decision variables. Moreover, comparisons between our model with that of steady state replacement revealed that there are considerable differences. In particular, we found that: i) the two models lead to different estimates concerning the profit horizon, the duration of replacements, the timing of abandonment or scrapping, and the impact of productive capacity and market structure on service lives, as these are determined by various parameters, ii) even though the steady state replacement policy may result in higher total profit, it does so at great expense in flexibility for the planner, because the replacements are built into the model from the beginning, and iii) the transitory replacement policy seems more realistic in that the replacements are undertaken only if forced on the planner by decreasing profits.

Clearly then, it is important to investigate which of the two models is closer to actual practice, since as indicated above they lead to very different predictions concerning the dependence of reinvestment policies on the market parameters.

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Endnotes

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[1] Users of equipment may be either owners or lessees. The difference between the two is that when the owner decides to stop operations he scraps his equipment, whereas the lessee is obliged to replace it. This leads to two distinct sets of replacement policies, one with terminal scrapping and the other with terminal replacement. We examine only the problem with terminal scrapping faced by owners.

[2] Clearly, this is an extreme assumption because in the great majority of actual cases technological breakthroughs do not render older equipment worthless at the time of their appearance. However, since modelling the case where the breakthrough reduces the earning power and the salvage value of the equipment by a certain percentage would be quite straightforward, we chose this specification in order to keep the analysis as simple as possible.

[3] Profitability does not necessarily imply operating profits. It may have operating losses balanced by capital gains, e.g. as happens with antiques, or more generally when we have *upgrading*.

[4] For convenience, in this and in all profitability or replaceability conditions that follow we will be considering only the generic case. Thus we ignore the special case where we may have profitable equipment starting with $A'(0) = 0 \Rightarrow \alpha(0) = 0$

[5] We assume that replacement is equivalent to scrapping the old equipment and buying new, i.e. we have ignored incentives in the form of discounts for replacements, except maybe for a fixed discount independent of the state of the equipment.

[6] If $A(T)$ is not continuously differentiable then, the inequalities for period durations and period profits may not be strict in the special non-generic case where the replacement falls on a point of abrupt change of the profit rate.

[7] Generically, i.e. excluding the special case $\alpha(0) = 0$, this is equivalent to requiring that the first maximum of $V\Pi(T)$ is strictly positive.

[8] For an account of the implications that arise in this regard see Bitros (2005).

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