THEORY AND POLICY IN MONETARY UNIONS: INDETERMINACY AND OPTIMAL CONTROL

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Theory and Policy in Monetary Unions: Indeterminacy and optimal control*

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Abstract

In this paper we study the effectiveness of the single interest rate policy rule in the context of a monetary union. Three different but complementary approaches are discussed. The first approach is based on determinacy and indeterminacy arguments; it shows the ineffectiveness of the common interest rate policy rule unless properly designed. In the second approach, we revisit the problem under the light of an infinite horizon optimal control scheme, designed so as to minimize the expected deviation of the economies of the member countries from a predescribed target. The optimal solution depends, not only on the characteristics of the member economies, but also on their whole histories. In the third approach, we abandon the assumption of an infinite horizon, and we ask the question of whether a single interest rate policy rule, may synchronize the states of the member countries of the union, in a preset time horizon. The behaviour of this solution is in line with the findings of the previous approaches, in that it shows that the optimal interest rate policy rule has to depend both on the characteristics of the member countries as well as on their histories within the preset time horizon, and not exclusively on their present states.

Keywords: Currency areas, Interest rate policy, Neo-Keynesian, Neo-Wicksellian economics, Optimal control, Backward stochastic difference equations.
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1 Introduction

The purpose of this paper is twofold. First, it aims to provide a review and a critical evaluation of the Neo-Wicksellian approach to monetary policy (Woodford 2003), and second to investigate possible applications of this approach to monetary unions. Woodford's work is a work of synthesis as it incorporates elements from both the Neoclassical macroeconomics and the Real Business Cycle Theory (inter-temporal optimization and rational expectations) and the New-Keynesian economics (imperfect competition and price rigidities), and its main characteristic is that it abstracts from the traditional focus on the nominal quantity of money. His aim is to provide a theoretical foundation for a rule-based approach to monetary policy (Woodford 2003, p.2), where this rule takes the form of an interest rate rule. This approach reminds Wicksell (1898), who long ago proposed a theory of monetary policy for a world without money (a pure credit economy), that revolved around the difference between the natural interest rate (determined by real factors such as the marginal productivity of capital) and the bank interest controlled by the central bank. According to Woodford (2003), Wicksell's approach is a particularly relevant one for the modern world, in which central banks adjust their operating targets in response to perceived risks of inflation without paying any particular attention to the evolution of monetary aggregates.

This is not the first time that the importance of monetary aggregate has been belittled in monetary theory and policy. Apart from Wicksell (1898), in the report of the Radcliffe Committee on the Working of the Monetary System (1959), that dominated the design of monetary policy during the sixties and the seventies, the control of the money supply is looked as an anachronism in economies with highly developed financial systems. In the view of the authors of the report, monetary policy works through rates of interest and the availability of credit, and therefore is better conducted by influencing these variables rather than the quantity of money (Kaldor 1982). In contrast to these views monetarists (Friedman 1968) emphasized the importance of the money supply as the key to controlling the price level, by arguing that there exists a stable function of money demand (a function of few identifiable variables) and by implication a stable velocity of circulation. But money demand turned out to be unstable and thus unable to serve as the basis for the conduct of monetary policy (Laidler 2005). Monetarism evolved in the seventies in the New Classical School. In this model, as expressed by the Lucas' (1972) island model, changes in money supply have real effects only if they are unanticipated. Therefore any policy that generates a systematic and predictable variation in the money supply will have no real effect (policy irrelevant hypothesis, as proposed by Sargent and Wallas (1975)), while the real business cycle (RBC) theorists adopted the view that monetary shocks do not affect the cycle.

Thus, the New Classical Macroeconomics and the Real Business Cycle theory failed to provide a theoretical background suitable for a theory of monetary policy. Central banks filled this theoretical vacuum by providing their own version of a model of monetary policy ("the standard monetary policy model" as Laidler (2005) calls it) in which the role of money supply is by-passed. This model consisted of an expectational IS curve, linking the interest rate with aggregate demand, a condition relating the rate of inflation to the gap between aggregate demand and a suitably defined equilibrium aggregate supply, and a central bank reaction function linking the interest rate controlled by monetary authorities to the inflation rate. This "standard monetary model" served as the basis of the New-Keynesian paradigm (Clarida, Galli, Geltler, 1999) and the Woodford's Neo-Wicksellian paradigm, which is considered as the definitive treatise of New-Keynesian approach to monetary policy (Walsh 2005).

In Woodford's Neo-Wicksellian paradigm the IS equation relates the output gap (which is the difference between actual and potential output, i.e., the output that would arise if prices and wages were perfectly flexible) on the expected future paths of the natural rate of interest and the real rate of interest. The output gap will be positive (there will be an excess demand) if the expected natural rate of interest increases faster than the real interest rate. The AS equation relates the inflationary rate on the expected future output gap (and on a push cost factor). These two equations describe the behaviour of an economy. In equilibrium, the output gap will be zero, the inflation rate will be zero, and the rates of increase of the natural interest rate and the real interest rate will be equal. A discrepancy between the real interest rate and the natural interest rate will create (by the IS equation) a positive output gap (in the case in which the natural interest rate increases faster than the real interest rate), and this positive output gap will increase the rate of inflation above zero (by the AS equation). However, these two equations contain no built-in stabilizers, and therefore there are no forces in the system able to restore equilibrium. Once the natural interest rate is above the real interest rate, the rate of inflation will tend to increase forever. The stabilizing factor of the system is provided by the central bank ,the behaviour of which is described by the monetary policy rule (MR) equation. The monetary policy rule equation states that the nominal rate of interest (controlled by the central bank) has to rise when the natural rate of interest is rising, and vice versa when the natural rate of interest is falling.

The effectiveness of the interest rate policy rule was investigated for the case of a single economy by Sargent and Wallace (1975), and more recently by Woodford (2003), who used the Neo-Wicksellian model for this purpose. In this paper, we investigate the same issue, that is the effectiveness of the interest rate policy rule, however, in the context of a monetary union. To this end, we extend Woodford's Neo-Wicksellian approach and derive a model for two coupled economies. Within the context of this formal model, we pose the question of whether an interest rate policy rule can stabilize the economies of the member countries at their national level.

We take three different but complementary approaches to this problem. The first approach is based on determinacy versus indeterminacy arguments as in Demopoulos, Yannacopoulos, Yannacopoulos and Warren (2009). We show that, unless properly designed and taking into account the economic aspects of the member countries of the union, a single interest rate policy rule may fail to provide determinacy for the whole system. In the second approach, we revisit this problem under the light of an infinite horizon optimal control scheme, designed so as to minimize the expected deviations of the economies of the member countries for a predescribed target, under a minimum cost of intervention. An optimal contrl is provided, which importantly depends on the characteristics of all economies as well as their whole histories. This reassures our concern on the effectiveness of a common interest rate policy rule. Finally, in the third approach, we abandon the framework of the infinite horizon, and ask the question of whether a single interest rate policy rule may synchronize the state of the economies in a preset finite horizon. This problem is adressed within the framework of controlled backward stochastic difference equations and an explicit analytic solution is provided. The behaviour of the solution is in line with the findings of the previous approaches, in that it shows that the optimal interest rate policy rule has to depend on the characteristics of both economies, and once more requires taking into account the whole history of the economies within the horizon, and not only their present states. To keep the paper as simple as possible, we present here only the results for a union comprising of two member states; however, the qualitative behaviour of the results persists for any finite number of member states.

The paper is organized as follows: In Section 2, we review the Neo-Wicksellian theory of stabilization policy for a single economy. In Section 3, we extend this theory to the case of a monetary union by deriving a mathematical model for two coupled economies. In Section 4, we present our first approach, based on determinacy versus inteterminacy arguments. In Section 5, we revisit this problem under the light of an infinite horizon optimal control scheme. In Section 6, we abandon the framework of infinite horizon and ask the question of whether a single interest rate policy rule may synchronize the state of two economies in a preset finite horizon. In Section 7, we conclude.

2 The Neo-Wicksellian theory of stabilization policy for a single economy

2.1 The analytical framework

The Neo-Wicksellian approach proposed by Woodford (2003) may be considered as a synthesis between the neoclassical long run analysis and the Keynesian short run analysis. It adopts the techniques of dynamic general equilibrium theory developed by the Real Business Cycle (RBC) analysis (intertemporal optimization and rational expectations) and elements from the Neo-Keynesian theory (short run price rigidities and imperfect competition). The RBC theory is used to define a long run dynamic equilibrium in the sense that it holds at every point of time, consistent with wage and price flexibility.

Woodford starts his discussion for monetary policy by assuming an economy the goods and financial markets of which are completely frictionless. In this economy, goods markets are perfectly competitive and clear continuously by the price mechanism. Given the assumption of frictionless financial markets, it is natural to suppose that no monetary assets are needed to facilitate transactions (a cashless economy). Money is used only as a unit of account. This is the setting which is commonly assumed in the RBC theory. He then extends this analysis to cover the case in which monetary frictions exist (as in the monetarist models), and therefore there is a basis for money demand for transaction purposes. He shows that the equlibrium relations obtained in this case are direct generalizations of the case of a cashless economy without frictions, and therefore the cashless analysis is a useful aproximation even for an economy where money is demanded for transaction purposes. In all these cases monetary policy affects the price level but not real economic activity. In order to discuss models in which monetary policy affects the level of real economic activity, he has to allow for nominal rigidities. This means that since monetary policy has small real effects in an environment with perfect wage and price flexibility nominal wage and price rigidities have to be taken into account.

A consequence of the sticky prices is that aggregate demand shocks affect output, provided that markets are imperfectly competitive. To see the importance of imperfect competition in this case consider a firm operating in a perfectly competitive market. This firm maximizes its profits by equating its marginal cost to the price given by the market. If aggregate demand declines, but price remains fixed, the firm has no incentive to reduce its output. A reduction of output means that the firms marginal cost will be lower than the price, which forms an incentive for the firm to expand its output. If aggregate demand increases, with fixed prices, an increase in the output means that the firms marginal cost will be higher than the price, which forms an incentive for the firm to reduce its output. Therefore under perfect competition, with fixed prices, changes in aggregate demand do not affect the level of output. By contrast if prices were fixed at a level higher than the marginal cost (as in the case of imperfectly competitive markets), then it is desirable for an individual firm to expand its output in the case in which aggregate demand expands, and reduce it in the opposite case.

The fact that markets are imperfectly competitive has an important welfare implication: the economy's output is below its social optimum . This is so because under monopolistic competition price setting agents maximize their profits by reducing output and therefore employment. This equilibrium is stable in the sense that economic agents have no intention to return the economy to its social optimum because this policy is inconsistent with profit maximization (Mankiw 1985). This unemployment is different from the Keynesian unemployment, which is caused by a lack in aggregate demand. This unemployment is the result of the monopolistic structure of the economy and cannot be treated by means of monetary policy. The policy implications derived from this analysis is that the reduction of unemployment cannot rest only on improving the flexibility of the labour markets. Improving competition among the firms is also necessary.

The assumption that prices are fixed in the short run requires an explanation. According to Mankiw (1985), firms set their prices in advance of the transaction period (an assumption consistent with the view that markets are imperfectly competitive); these prices remain fixed during the transaction period because their changes involve costs, such as printing new catalogues, informing salesmen on price changes etc. (small menu costs). Dreze (1999) offers a different explanation. According to him price rigidities are the result of uncertainty and market incompleteness. He argues that under imperfect competition firms setting their prices so as to equate marginal costs and marginal revenues are uncertain about the price elasticity of the demand for their products. Then under any kind of menu costs leading to finite price adjustments, uncertainty about demand elasticities leads risk averse firms to behave as if they faced a kinked demand curve (Dreze 1991, p.199). This feature is interpreted as incomplete market phenomenon. Firms cannot quote prices contingent on demand elasticities, which are not observable.

A dynamic model explaining finite price adjustments was proposed by Calvo (1983). In this model, at any given time period, a firm has a fixed probability θ to keep its price fixed during this period (and hence a probability $1-\theta$ that it may adjust). Calvo assumes that this probability is independent of the time that has elapsed since the last time the firm changed its price. The average time over which the price is fixed is given by the expression, $\frac{1}{1-\theta}$ (Clarida, Gali, Gertler 1999). This expression is equal to plus infinity when firms never revise their prices ($\theta = 1$) and equal to one in the absence of nominal price rigidity ($\theta = 0$).

2.2 Monetary policy in a Neo-Wickselian framework

We are now ready to describe the short run equilibrium model with nominal rigidities (Woodford 2003, Chapter 4). The Neo-Wicksellian model consists of three equations: an IS equation, an AS (the New-Keynesian Phillips curve), and an equation expressing the monetary policy rule (MR).

(a) The IS equation is the following:

$$x(t) = \mathbb{E}_t[x(t+1)] - z\,[i(t) - \mathbb{E}_t[\pi(t+1)] - r(t)]. \tag{1}$$

In this equation x denotes the output gap , i.e., the difference between the current output, and the equilibrium output, which is defined as the output which is consistent with perfect price flexibility. Thus, at equilibrium $x = 0$. The case of positive output gap may be identified as an excess demand for the current output, while the case of a negative output gap may be identified as an excess supply. The market interest rate is denoted by i , the natural interest rate by r, π is the inflation rate and $\mathbb{E}_t$ denotes the expectations conditioned on the history of the economy by time t. Thus, the expression $i(t) - \mathbb{E}_t[\pi(t+1)]$ denotes the real interest rate, and z the inter-temporal substitution in consumption.

This IS expression differs from the Hicksian IS, mainly because current output depends positively on the expected future output , and on the difference between the real rate of interest and the natural rate of interest. Because agents prefer to smooth consumption (this assumption rests on the principle of the inter-temporal marginal rate of substitution), expectations of higher future consumption (associated with higher expected future output) raises consumption in the present period, and thus output demand. If the real rate of interest is lower than the natural interest, the expression within the parenthesis will be positive implying that the present period output is higher than the expected future output. In equilibrium, output in the present period must be equal to the future expected output. This implies that real interest rate is equal to the natural interest rate, which is the Wicksellian equilibrium condition. But it is similar to the Neo-Keynesian IS curve (Clarida, Gali, Gertler, 1999) with the difference that Clarida Gali and Getler do not mention the natural interest rate.

If we iterate equation (1) forward we get the expression:

$$x(t) = \mathbb{E}_t\left[\sum_{\ell=1}^{\infty}(-z[i(t+\ell) - \pi(t+1+\ell) - r(t+\ell))\right] \tag{2}$$

This expression says that the output gap depends on the expected future paths of the natural rate of interest and the real rate of interest. The output gap will be positive (there will be an excess demand) if the expected natural interest rate increases faster than the expected real interest rate. And vice versa.

(b) The next step is to provide a link between the output gap, and the level of prices. This link is provided by the AS equation (New Keynesian Phillips curve), which is derived from the Calvo staggered price setting model (see above), and has the form:

$$\pi(t) = k\, x(t) + \beta\, \mathbb{E}_t[\pi(t+1)] + u(t) \tag{3}$$

This equation says that inflation depends positively on both the output gap and on the expected rate of inflation. In the AS equation, β is an intertermporal discounting factor, $u(t)$ is a cost push factor and k is related to the price stickiness. It gives the relative change in the rate of inflation when the output gap changes. Therefore, in the case in which firms never revise their prices (absolute price rigidity) the value of k approaches zero. Thus, k is decreasing in θ (in Calvo's model) which measures the degree of price rigidity.

If we iterate equation (3) forward we obtain the expression:

$$\pi(t) = \mathbb{E}_t \left[\sum_{\ell=1}^{\infty} \beta^{\ell} [kx(t+\ell) + u(t+\ell)] \right] \tag{4}$$

It says that the inflation rate depends on the expected future output gap and cost push factor $u(t)$.

(c) The model is closed by an equation describing the monetary rule (MR) which replaces the traditional LM equation. The nominal interest rate is taken as an instrument of the monetary policy. The monetary aggregate is bypassed. According to this rule (originally suggested by Wicksell), nominal interest rate has to rise when inflation is rising, and vice versa when inflation is declining. We will assume that the nominal interest rate responds to inflation according to the rule:

$$i(t) = \gamma\pi(t) + g(t) \tag{5}$$

where $\gamma > 1$. This rule is known as the Taylors principle. It says that in order to increase the real interest rate, the nominal interest rate must respond more than one for one to changes in inflation. Thus $\gamma > 1$ expresses the rate of growth on the real interest rate.

Equations (1) and (3) are supposed to describe the dynamics of the system. Solving these equations for the equilibrium paths of output and the nominal interest rate, under the assumption of zero inflation at all times (zero inflation is assumed to be the target set by monetary authorities), and given the natural interest rate (it is determined by real factors) we get:

$$i(t) = r(t)$$

which says that in equilibrium the market rate of interest , controlled by the central bank ,is equal to the natural interest rate. This condition may be termed as Wicksellian equilibrium , because it is consistent with zero output gap , and zero inflation rate. This equilibrium rests on the assumption that $\pi(t) = 0$ at all times. If the monetary authorities require a small positive rate of inflation $\pi^\star = \pi(t) > 0$, the equilibrium condition becomes:

$$i(t) = r(t) + \pi^\star$$

In what follows we will assume that $\pi(t) = 0$, an assumption which is more consistent with the original Wicksellian views.

This equilibrium can be disturbed either by a real disturbance affecting the natural rate of interest that it is not sufficiently offset by a change in the nominal interest rate or by a change in the nominal interest rate that is not justified by a change in the real factors of the economy or by a "cost push" affecting the position of the AS equation. Consider the first case first. An increase in the natural rate of interest above the market rate of interest, may create a positive output gap (an excess demand), leading to an increase in the inflation rate above its target. But the increase in the inflation rate reduces further the real interest rate, increasing thereby the gap between the natural interest rate and the real interest rate (the natural interest rate increases faster than the real interest rate). This gives a further stimulus to inflation. The same is true in the case of a "cost- push" inflation. If we assume that originally the output gap is unaffected, cost push inflation increases present inflation and inflationary expectations. Inflationary expectations will reduce the real interest rate below the natural rate setting in motion the Wicksellian cumulative process.

The question that arises is if there are any forces which bring the cumulative process to an end. In the original Wicksellian system the cumulative process plays the role of the equilibrating mechanism forcing the banks to eliminate the discrepancy between nominal rates and natural rates, and thus restoring equilibrium to the loan market (Patinkin 1962, p.429). In the Neo-Wicksellian system the equilibrating mechanism is provided by the monetary rule according to which monetary authorities raise the nominal interest rate above the inflation rate so as to equalize the real rate of interest with the natural interest rate. From the monetary rule equation we know that γ indicates the change in the real interest rate. Therefore, the restoration of equilibrium requires:

$$\gamma = r(t).$$

Thus, the interest rate policy rule plays the role of the equilibrating mechanism provided that the following two conditions are satisfied:

(i) Taylor's principle, which requires that the nominal interest rate should rise by more than the inflation rate, is satisfied, and

(ii) that the natural interest rate is positive.

If the first condition is not satisfied, the system may exhibit many undesirable equilibria. If the natural rate of interest is negative (Krugman, 1998) the monetary rule cannot be applied (monetary policy is ineffective), because the nominal interest rate cannot fall below zero. The situation is similar to that of the liquidity trap (Keynes 1936). The economy can be pulled off the liquidity trap by raising inflationary expectations or by expansionary fiscal policy. It should be noted that Wicksell (1935/1978, Vol. II, p. 197) did not ignore the phenomenon of the liquidity trap.

Woodford's work is considered as a major contribution to the theory of monetary policy as it is conducted today. However, some authors (Weber 2006) noted that the natural rate of interest is not readily computable from observable economic data and this may limit the applicability of the concept of the natural rate of interest to practical monetary policy.

3 A model for a monetary union

In the case of a single country the IS-AS system can be written in the form

$$\mathbb{E}_t[z(t+1)] = A\,z(t) \;+\; a\,(r(t) - i(t))$$

where $z(t) = (\pi(t), x(t))^T$ and the matrices of coefficients are

$$A = \begin{pmatrix} \beta^{-1} & -\beta^{-1}\,k \\ -\beta^{-1}\,\sigma & 1+\beta^{-1}\,k\,\sigma \end{pmatrix} \quad a = \begin{pmatrix} 0 & -\sigma \end{pmatrix}^T$$

as in Woodford (2003). If this system is in equilibrium (and therefore output gaps and inflation rates are zero) then $z(t) = 0$, and therefore,

$$r(t) = i(t)$$

which is the Wicksellian equilibrium condition.

Assume now that this equilibrium condition refers to country 1 and that this "country" is a currency area with the above characteristics. Assume further that country 2, the macroeconomic behaviour of which is described by its own IS-AS equations joins the currency area, which implies that country 2 has to accept the nominal interest rate $i(t)$ prevailing in 1. We have then two countries (or regions) coupled through a common interest rate policy $\hat{i}(t)$,

$$\begin{aligned}\mathbb{E}_t[z_1(t+1)] &= A_1\, z_1(t) + a_1\,(\hat{r}_1(t) - \hat{i}(t))\\ \mathbb{E}_t[z_2(t+1)] &= A_2\, z_2(t) + a_2\,(\hat{r}_2(t) - \hat{i}(t))\end{aligned}$$

where $z_i(t) = (x_i(t), \pi_i(t))$ and

$$A_i = \begin{pmatrix} \beta_i^{-1} & -\beta_i^{-1}\, k_i \\ -\beta_i^{-1}\sigma_i & 1+\beta_i^{-1}\, k_i\, \sigma_i \end{pmatrix},\ \ a_i = \begin{pmatrix} 0 \\ -\sigma_i \end{pmatrix},\ \ i = 1, 2$$

This is a system of stochastic dynamical systems which are coupled by a common interest rate policy rule.

The following question is of interest:

> Can a common interest rate rule $\hat{i}$ stabilize these economies, even in the case where the shocks $\hat{r}_i$ may be assymetric? In other words, can an interest rate rule based on the macroeconomic figures of one country have a stabilizing effect on the other? If not, how can we design a policy rule that will have a stabilizing effect on both countries?

4 A first approach: Indeterminacy aspects

In this first approach to the problem we associate the well posedness and stability of the model with the concept of **determinacy** and the non well posedness with the concept of **indeterminacy**, following the approach of Woodford (2003). We remind the reader that the rational expectations (RE) model is determinate if given a neighbourghood $\mathcal{N}$ in the state space (π, x) of the model, by taking tight enough external perturbations $(r(t))$, then there exists a unique RE equilibrium of the model in $\mathcal{N}$. On the other hand, the model is indeterminate if within a given neigbourghood $\mathcal{N}$ of the state space there exists a denumerable infinity of RE equlibria. As Woodford (2003, Ch. 4, p. 253) puts it *"there exists an infinity of different possible equilibrium responses of the endogenous variables to real disturbances includng some in which the fluctuations in inflaion and output are disproportionately large relative to the change in fundamentals that has occured and some in which inflation and output vary in response to random events with no fundamental significance whatsoever."* Eventhough weak, indeterminacy gives a first (asymptotic in time) concept of non well posedness and of inability of stabilization to a selected equilibrium state, which importantly is quite straightforward to check by considering a simple criterion on the eigenvalues of the matrix defining the system.

Within this framework we are able to provide an answer to the first question posed above, which is a negative one as the following proposition shows.

Proposition 4.1 *Suppose that the two countries have identical features, i.e., $A_1 = A_2 = A$, $a_1 = a_2 = a$ but are subject to different external shocks $\hat{t}_1(r) \neq \hat{r}_2(t)$. Then, the dynamics of the divergence of the two countries $\hat{z}(t) = z_1(t) - z_2(t)$ are indeterminate.*

Proof: Subtract the equations for the dynamics of the two countries to obtain

$$\mathbb{E}[\hat{z}(t+1)] = A\,\hat{z}(t) + a\,(\hat{r}_1(t) - \hat{r}_2(t))$$

Observe that the control $\hat{i}(t)$ vanishes from the equation for the difference, so that there is no control effect of the interest rate rule on the divergence of the two countries. Since A has two eigenvalues inside the unit circle, the system for the difference is indeterminate, therefore the countries may respond badly to asymmetric shocks. An alternative proof may follow from the 4×4 matrix corresponding to the coupled system, which according to eigenvalue perturbation arguments has 4 eigenvalues, 2 of which (the ones corresponding to the block part of the matrix related to A_2 are inside the unit circle, thus leading to degeneracy of the full system. ■

This result shows that an *one size fits all* policy rule may be **inappropriate** for monetary policy conduct in a monetary union. If the policy rule is designed so that it brings one of the economies in the required state, it is not necessary that it has the same effect for the other economy eventhough the two economies are identical. In the case of an asymmetric shock, $r_1(t) \neq r_2(t)$, the indeterminacy of the system that provides the evolution of the difference of the states of the two economies, shows that in the long run there is an infinity of number of different possible equilibrium responses for the difference between the states of the two economies to real disturbances. Some of these responses may have undesirable effects, in the sense that they include some in which the fluctuations in inflation and output are disproportionally large relative to the change in fundamentals that has occured and some in which inflation and output vary in response to random events with no fundamental significance whatsoever. That means that all hope for convergence of the two economies to common targets is abandoned. Therefore, a common policy rule, of a feedback type, based only on the output characteristics of one of the economies cannot work as a stabilizing factor for both economies. Since there is always the chance to design a $\hat{i}_1(t)$ and $\hat{i}_2(t)$ such that the uncoupled economies are determinate, it is fair to say that the economies would be better off outside the monetary union, as the absence of a common interest rate rule would allow them to get as close as possible to their individual prefered targets.

This result is not valid only in the case of identical economies, but can be generalized to the case of economies with different characteristics, as the following Proposition shows.

Proposition 4.2 *Suppose that the policy rule $\hat{i}(t)$ is designed so that it stabilizes the dynamics of country 1, and it is only based on macroeconomic feature of this country. Then, it is not fit for stabilizing asymmetric shocks of these two countries, in the sense that the divergence dynamics $\hat{z}(t) = z_1(t) - z_2(t)$ are indeterminate.*

Proof: Assume, without loss of generality, that the policy rule is designed, keeping in mind country 1 so that $\hat{i}(t) = Cz_1$ where C is a suitably selected matrix. Let $\hat{z}(t) = z_1(t) - z_2(t)$ the divergence of the two countries. Since the policy rule is effective for country 1, z_1 is kept bounded for all times, no matter what the external shocks $\hat{r}_1$ are.

Subtract the two equations for z_1 and z_2 and rewrite them as

$$\mathbb{E}_t[\hat{z}(t+1)] = A_2\hat{z}(t) + \hat{R}(t) + (A_1 - A_2)\, z_1 + (a_1 - a_2)\hat{i}(t)$$

where $\hat{R}(t) = a_1\,\hat{r}_1(t) - a_2\,\hat{r}_2(t)$.

Under the assumption that $\hat{i}(t) = C\,z_1(t)$ and because of the effectiveness of the policy rule for country 1, we observe that the equation for the divergence of the two countries is of the general form

$$\mathbb{E}_t[\hat{z}(t+1)] = A_2\hat{z}(t) + \hat{R}'(t) \tag{6}$$

where

$$\hat{R}'(t) = a_1\,\hat{r}_1(t) - a_2\,\hat{r}_2(t) + (A_1 - A_2)\, z_1(a_1 - a_2)Cz_1(t).$$

Notice that $\hat{R}'(t)$ is bounded.

Furthermore, since the matrix A_2 has both eigenvalues inside the unit circle, system (6) is indeterminate, which means that we do not know what the behaviour of the divergence of the two countries will be when asymmetric shocks are applied. ■

Is there any hope of designing a common policy rule, such that both economies will be led to the desired states? Our only chance of the interest rate policy having a stabilizing effect on the dynamics would be to have $\hat{i} = C\hat{z}$ for a properly selected matrix C. However, this would imply an interest rate rule based on common measurements of the macroeconomic features of both countries rather than just on measurements of single countries features.

The design of such a policy rule is given in the next Proposition.

Proposition 4.3 *(a) Suppose that the policy rule is designed so that $\hat{i} = C_1 z_1(t) + C_2 z_2(t)$ and assume that C_1 and C_2 are such that the 4×4 matrix*

$$M = \begin{pmatrix} A_1 + a_1 C_1 & a_1 C_2 \\ a_2 C_1 & A_2 + a_2 C_2 \end{pmatrix}$$

has at least 2 eigenvalues outside the unit circle. Then this policy rule is effective for both countries.
(b) Such a policy rule exists.

Proof: If M satisfies the stated conditions, then the system is determinate. Therefore, the uniqueness of nonexplosive solution is guaranteed. Observe that if e.g. $C_2 = 0$, i.e., if the interest rate rule takes into account only the macroeconomic features of the first economy, then the sparse form of the matrix shows that the block part of the matrix which is associated with economy 2 will have 2 eigenvalues inside the unit matrix thus ruling out the possibility for determinacy. ■

The results of this section may be extended without much difficulty to the case of n coupled economies, however at the cost of more complicated linear algebra.

5 A second approach: The construction of optimal policy rules using infinite horizon stochastic control

It is the aim of the present section to elaborate further on the construction of a common policy rule that allows us to bring both economies as close as possible to a desired equilibrium state.

The considerations in the previous section, provide a partial answer to the problem of design of a common policy rule that may lead both economies to a desired state. The answer to this problem is based upon the argument that the ability to do so is equivalent with the determinacy of the (backward) system, when enhanced with the policy rule $\hat{i} = C_1 z_1(t) + C_2 z_2(t)$. However, this is a result of the weak type, in the sense that it provides the family of policy rules that may in the asymptotic limit as $t \to \infty$ drive the monetary union to the desired state. There is no information as to how long this may take, or on how close to the target we may get in a finite horizon. Clearly, such information is very important when dealing with realistic situations of interest in the real world. Therefore, one may consider the question,

> Out of all the family of all possible interest rate rules, that may lead the system to the desired equilibrium state as $t \to \infty$, can we select one that has the best properties? An appropriate policy rule may be the one that minimizes some cost criterion, related to the "distance" of the realizations of the characteristics of the monetary union, from the desired states, not only asymptotically in time, but for all intermediate time instants.

This is the problem of defining optimal policy rules using an optimal control procedure. To simplify the exposition we consider first the problem of 2 coupled economies, so that it is easier to show the construction of the optimal policy rule, and then we give the general form of the optimal policy rule in the case of a monetary union consisting of n-coupled economies.

We write the model as

$$\mathbb{E}_t[Z(t+1)] = AZ(t) + B\, i(t) + C\, s(t) \tag{7}$$

where

$$Z(t) = (z_1(t), z_2(t))^T = (x_1(t), \pi_1(t), x_2(t), \pi_2)^T$$

in other words $Z_1(t) = x_1(t)$, $Z_2 = \pi_1(t)$, $Z_3(t) = x_2(t)$, $Z_4(t) = \pi_2(t)$ are the characteristic variables of the two economies. The matrix A determining the dynamics of the (un)coupled economies is

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

and the economies are coupled by a common interest rate rule $i(t)$. This commonly adopted interest rate affects in a possibly different fashion economy 1 and 2 and through its effect on the inflation rate π_i, $i = 1, 2$. This effect is modeled by the vector B defined by

$$B = C = (0, -\sigma_1, 0, -\sigma_2)^T$$

By $s(t)$ we denote the external shocks in the economy. As the model stands we consider that both economies are subject to symmetric shocks however this is by no means restrictive. We may easily consider within this general framework the case where the economies are subject to possibly asymmetric shocks.

We now consider the common interest rate $i(t)$ as a control variable which is to be chosen by the central bank so as to drive the members of the monetary union, as close as possible, to a desired state $Z^* = (Z_1^*, Z_2^*, Z_3^*, Z_4^*)$. For the sake of simplicity, we take the desired state to be a steady state of the economy, but this assumption is by no means restrictive, as we may easily expand the analysis to include targets which are time dependent.

The distance from this desired state will be quantified by a quadratic distance function (loss function). The central bank may attribute different weights on the divergence of the economies from the various components of the desired target Z^*, this weight is quantified by a 4×4, positive definite,symmetric matrix $W = \{W_{ij}\}$, $i, j = 1, \cdots, 4$. Therefore, the distance of the actual state of the economy at time t, from the desired state Z^* is expressed by the quadratic form

$$L_1(t) = \frac{1}{2} \sum_{i=1}^{4} \sum_{j=1}^{4} W_{ij}(Z_i(t) - Z_i^*)(Z_j(t) - Z_j^*)$$

If W is a diagonal matrix then this distance takes into account only deviations of each individual economy from the desired state set for it by the central authority. If W has non-diagonal terms also, then this distance takes into account cross deviations of the different economies from the desired states. For instance, the term W_{13} of the matrix W will quantify possibly simultaneous deviations of the outputs $x_i(t)$, $i = 1, 2$, of economies 1 and 2 from the desired output x_i^*, $i = 1, 2$. Similarly, the term W_{24} will quantify possible simultaneous deviations of inflation levels $\pi_i(t)$, $i = 1, 2$ of the economies 1 and 2 from the desired target inflation levels π_i^*, $i = 1, 2$.

It is conceivable (although this is not always true!) that if the central bank has unlimited power of intervention that it could bring the economies close enough to the desired levels. However, these interventions are costly, and this cost of intervention should be taken into account. To simplify the analysis, we assume that there is a desired level of intevention, quantified by a target interest rate i^*. The central bank at time t, may have to adopt a different interest rate $i(t)$ than this target, however, this done by a cost. This cost is assumed to be a function of the deviation of the actual interest rate $i(t)$ adopted by the bank and is quantified by the square of this deviation, meaning that either lower interest rates than i^* or higher interest rates than i^* are equally undesirable and therefore are penalized. Therefore, we define the function

$$L_0(t) = \frac{1}{2} W_0 (i(t) - i^*)^2$$

where W_0 is a positive constant, quantifying the importance given to the deviations from the desired target interest rate i^*.

The total cost at time t is then

$$L(t) = L_0(t) + L_1(t).$$

Since there is uncertainty in the economy, this cost is a random variable. We will choose to optimize a characteristic quantity of this random variable which is its expected value over all possible realizations of the economy. We also have to take into account the fact that this cost occurs at different time instants, so that it will have to be appropriately discounted by a discount factor ρ. The above discussion leads us to consider the following cost function

$$J = \mathbb{E}[\sum_{t=0}^{\infty} \rho^t \left(L_0(t) + L_1(t) \right)]$$

This function depends on the interest rate policy $\{i(t)\}$, $t = 0, 1, \dots$ adopted by the central bank. To make this dependence clear we will write $J = J(i)$. The central bank will adopt such an interest rate policy so as to minimize this cost, subject at all times to the dynamic constraints (7). The optimal interest rate is therefore $\hat{i}(t) = argmin\, J(i)$. This is a general form of quadratic optimization problem, where according to the choice of the matrix $W = (W_{ij})$, we may impose different targets for the central bank. In what follows, we provide a few examples of possible such targets, all of which fall within the general framework proposed here.

Example 5.1 If we choose $W_{ii} = R_i$, $i = 1, \cdots, 4$ and all other terms 0, i.e., if W is a diagonal matrix then we choose to lock all economies to a predescribed target. Of course, it is not necessary that the minimum of the functional L is equal to 0. Therefore, the controlled economy will result to an economy where each individual economy may stick close enough to a target, however allowing for deviations from this target. It is not also necessary that all economies will be in synchronized state.

Example 5.2 If we choose the control functional $L(t) = R_1 \left(x_1(t) - x_2(t) \right)^2 + R_2 \left(\pi_1(t) - \pi_2(t) \right)^2 + W_0 \left(i(t) - i^* \right)^2$ then the minimization of this functional will result to the two economies being as synchronized as possible. This control functional will be a special case of the quadratic functional in the form $W_{11} = W_{33} = R_1$, $W_{22} = W_{44} = R_2$, $W_{12} = W_{34} = -2$.

To solve this problem we will employ a convex duality technique. To take into account the dynamic constraints (7) we will use a set of dynamic Lagrange multipliers $\{\phi(t)\} = \{\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t)\}$, $t = 0, 1, \cdots$. As the dynamic constraints are random variables, this nececitates that the Lagrange multipliers are random variables as well. We will use the augmented Lagrangian

$$\begin{aligned} \mathcal{L} &= \mathbb{E}[\sum_{t=0}^{\infty} \rho^t \left(L_0(t) + L_1(t) \right)] \\ &+ \sum_{j=1}^{4} \sum_{t=0}^{\infty} \rho^t \mathbb{E} \left[\phi_j(t) \left(\mathbb{E}_t[Z_j(t+1)] - \sum_{\ell=1}^{4} A_{j,\ell} Z_\ell(t) - (Bi(t))_j - (Cs(t))_j \right) \right] \end{aligned}$$

where $(B\,i(t))_j$ and $(C\,s(t))_j$ denote the j component of the vectors $B\,i(t)$ and $C\,s(t)$ respectively.

Assuming that we are able to interchange infinite summation with expectation (an assumption which is justified by standard technical conditions) and using the "tower property" of conditional expectation according to which $\mathbb{E}_s[\mathbb{E}_t[X]] = \mathbb{E}_{\min(s,t)}[X]$, (recall that $\mathbb{E}_0[X] = \mathbb{E}[X]$), for any random variable X, the augmented Lagrangian $\mathcal{L}$ can be simplified to

$$\begin{aligned}\mathcal{L} &= \mathbb{E}[\sum_{t=0}^{\infty} \rho^t \left(L_0(t) + L_1(t)\right)] \\ &+ \sum_{j=1}^{4}\sum_{t=0}^{\infty} \rho^t \mathbb{E}\left[\phi_j(t)\left(Z_j(t+1) - \sum_{\ell=1}^{4} A_{j,\ell} Z_\ell(t) - (Bi(t))_j - (Cs(t))_j\right)\right]\end{aligned}$$

We first take the first order conditions with respect to the control variables $\{i(t)\}$. Then

$$\frac{\partial \mathcal{L}}{\partial i(t)} = W_0(i(t) - i^*) + \sigma_1 \phi_2(t) + \sigma_2\,\phi_4(t) = 0$$

therefore the optimal interest rate policy is determined by the Lagrange multipliers $\phi_2(t)$ and $\phi_4(t)$, via the rule

$$i(t) = i^* - \frac{\sigma_1}{W_0}\,\phi_2(t) - \frac{\sigma_2}{W_0}\,\phi_4(t). \tag{8}$$

We next take the first order conditions with respect to the state variables $\{Z(t)\}$. These will provide a dynamic system for the determination of the Lagrange multipliers. We will take the first order conditions with respect to the state variables componentwise. This yields

$$\frac{\partial \mathcal{L}}{\partial Z_j(t)} = \sum_{\ell=1}^{4} W_{j,\ell}(Z_\ell(t) - Z^*_\ell) + \rho^{-1}\phi_j(t-1) + \sum_{\ell=1}^{4} A_{\ell,j}\phi_\ell(t) = 0, \;\; j = 1, \cdots, 4.$$

This is a set of random difference equations for the (dynamic) Lagrange multipliers which has to be complemented with a boundary condition for $\phi(-1)$. We set $\phi(-1)_j = 0$ for all $j = 1, \cdots, 4$. This can be considered as an analogue of the usual transversality conditions needed for infinite horizon optimal control problems. There is still some work to be done on the adjoint system. We shift time by 1, for symmetry reasons, and consider the important issue of measurability of the Lagrange multipliers in terms of the information structure (filtration) generated by the shocks on the economy. In other words, since the dynamic Lagrange multipliers $\{\phi(t)\}$ can be interpreted as the actions needed to be adopted by the central bank to direct the coupled economies to the desired state, we have to insist on the plausible assumption that these actions will have to be decided solely by knowledge of the states of the economy up until this time. To ensure that we have to rewrite the equation for the Lagrange multipliers as

$$\sum_{\ell=1}^{4} W_{j,\ell}\mathbb{E}_t[(Z_\ell(t+1) - Z^*_\ell)] + \rho^{-1}\phi_j(t) + \sum_{\ell=1}^{4} \mathbb{E}_t[A_{\ell,j}\phi_\ell(t+1)] = 0, \;\; j = 1, \cdots, 4$$

This set of equations will have to be solved as a system with the equations for $Z(t)$, in which i is substituted by the expression (9). This system is then

$$\begin{aligned}\mathbb{E}_t[Z_j(t+1)] &= \sum_{\ell=1}^{4} A_{j,\ell} Z_\ell(t) + B_j\left(i^* - \frac{\sigma_1}{W_0}\,\phi_2(t) - \frac{\sigma_2}{W_0}\,\phi_4(t)\right) + C_j s(t) \\ \sum_{\ell=1}^{4} \mathbb{E}_t[A_{\ell,j}\phi_\ell(t+1)] &+ \sum_{\ell=1}^{4} W_{j,\ell}\mathbb{E}_t[(Z_\ell(t+1) - Z^*_\ell)] = -\rho^{-1}\phi_j(t)\end{aligned}$$

for $j = 1, \cdots, 4$. This is a coupled system the solution $(\hat{Z}(t), \hat{\phi}(t)$ of which will yield the optimal interest rate policy by the rule

$$\hat{i}(t) = i^* - \frac{\sigma_1}{W_0}\,\hat{\phi}_2(t) - \frac{\sigma_2}{W_0}\,\hat{\phi}_4(t). \tag{9}$$

Observe that this optimal policy implicitly depends on the states of the two coupled economies since the solution $\hat{\phi}(t)$ is the solution of a coupled system involving both Z and ϕ. Furthermore, the optimal policy will need in principle knowledge of the whole history of the economies and the shocks. To make this point clear let us rewrite the adjoint equation in matrix form as

$$\mathbb{E}_t[A^T \phi(t+1)] = \rho^{-1}\phi(t) + R(t)$$

where the vector $R(t) = (R_1(t), R_2(t), R_3(t), R_4)^T$ has components that reflect the state of the two economies by

$$R_j(t) = -\sum_{\ell=1}^{4} W_{j,\ell}\mathbb{E}_t[(Z_\ell(t+1) - Z_\ell^*)], \quad j = 1, \cdots, 4.$$

This equation can be solved by a formal iteration argument, after multiplying both sides from the left by the inverse of the adjoint matrix $(A^T)^{-1}$. The invertibility of the matrix A^T can be guaranteed by standard linear algebra arguments. The solution can be seen to depend on the whole history of the economies up to time t, rather than just on the state of the economy at time t (we omit the exact solution for the sake of brevity). Therefore, the optimal policy may not be considered as a simple local feedback control procedure, adapting the policy rule so as to drive the system to the desired state, but rather as a nonlocal version of a feedback control procedure.

The two important lessons one takes from this approach are in our opinion the following:

1. The optimal policy rule has to be designed taking into account the macroeconomic features of both countries.

2. The optimal policy rule cannot be local in time, i.e., take into account at time t only the states of the economies at time t. The history of the states of economies, as well as the shocks that have taken place, need to be taken into account.

The first of these points is a further assertion of our intuition concering the scepticism for "one size fits all" monetary policy rules as developed by the asymptotic analysis of the previous section. The second point brings to the surface the important issue that history matters, one may not design the optimal policy rule simply by sticking to the last observation concerning the economies but has to keep track of the whole paths of these economies, as they play an important role concerning the future evolution.

One could comment upon the practical implementation of this policy rule.

First of all, eventhough the system of equations for (Z, ϕ) is fairly complicated to be solved analytically, at least for general cost functions, its numerical treatment is feasible and can be achieved, using modern day computing power and techniques, in real time. Therefore, in principle the central bank may, using econometric data, compute numerically the exact optimal policy using the approach described in this section. However, this optimal policy is no longer as elegant and easy to communicate to the general public as the Taylor rule.

A second comment, is associated with robustness issues. Our analysis is based on a model for the evolution of the states of the economy. How robust are the results we obtain with respect to the model? Such issues are of great importance and can be treated using for instance techniques from the theory of robust control.

A third comment, is related to the complicated nature of the optimal policy determined by the solution of this optimal control problem. It is conceivable, that a simpler policy rule, akin to the Taylor rule might be preferable, either to speed up calculations or to simplify the life of practitioners, or even to make it easier for economists to assess the sensitivities of the economy or the policy to the shocks. One can then reconsider the optimal control problem we stated in this section, over a subset $\mathcal{U}_0$ of the policies, representing the class of simpler policy rules, e.g., the set of generalized Taylor rules. Although it is possible to solve this problem, however, we must always keep in mind that such a solution will necessarily be suboptimal, in the sense that there exists a policy rule, the one presented in this section, which is outside this simple class, and which performs better. Therefore, in using simpler rules we sacrifice, in our knowledge, performance for the sake of simplicity.

Our approach is inspired by that of Giannoni and Woodford (2002) (see Woodford 2003 for details) who study linear quadratic control problems for rational expectations models of this type. Eventhough these authors do not consider the problems of coupled economies in monetary unions and the effectiveness of a common policy rule in their work, they provide a general mathematical framework upon which our analysis is based.

The above discussion leads us to the general conclusion:

Proposition 5.1 *The optimal interest rate rule will be of the form*

$$i(t) = i^* + \sigma_1\, \phi_2(t) + \sigma_2\, \phi_4(t)$$

where in general $\phi_2(t)$ and $\phi_4(t)$ depend on the past states of both economies. Therefore, a common interest rate rule will only be effective if it takes into account the particular macroeconomic features of **both** *economies. Any other interest rate rule, will necessarily be suboptimal.*

The optimal rule we have provided in principle needs the whole history of the states of the economy, from time $i = 1$ to $i = t$. In certain cases, we may provide a recursive formula, using only lag one variables for the optimal policy rule, but this is not necessarily always the case.

6 A third approach: Synchronization in finite horizon using controlled backward stochastic difference equations

We close our analysis of the subject by a discussion of an alternative approach, in which we propose the construction of a common optimal policy rule, designed in such a way as to synchronize the two economies within a predescribed finite horizon T.

In this model, we assume that economy 1 represents a monetary union with a common interest rate policy i, who admits a second economy, economy 2. Economy 2, will have to accept the common interest rate i, which will be selected in such a way as to bring the economy 2 into an identical state with economy 1, by the end of a time period T. This is the problem of synchronization, according to which we select the interest rate policy in such a manner so as to ensure that $z_1(t) = z_2(t)$, after the passage of a sufficient time T.

In this model, economy 1 (the monetary union) can be thought of as a driving mechanism whereas economy 2 responds to that driving mechanism. There may be discrepancies between the driver and the reponse, but by the proper design of i, one can eliminate these discrepancies and synchronize the states of the two systems.

Define the divergence of the two economies as $e(t) = z_1(t) - z_2(t)$. We may now obtain a dynamic scheme for the evolution of this divergence as

$$\mathbb{E}_t[e(t+1)] = A_2 e(t) + (A_1 - A_2) z_1(t) + (C_1 - C_2)\, i(t) \tag{10}$$

To simplify the problem slightly, we assume that during the initial stage of the entrance of economy 2 into the union, this economy has an interest rate $i(t)$, which may differ from the one employed in the union, and after synchronization both economies adopt the common rule. This simplifying assumption allows us to consider equation (10)for the divergence, as a stand alone equation, in which the state of economy 1, $z_1(t)$ is given an an external source term. In the case where the common interest rate rule is adopted since the very begining we have to consider equation (10), coupled with the equation for $z_1(t)$. This, while feasible, will complicate the exposition and is not pursued here.

One approach to the solution of this problem is via an adaptive feedback control mechanism. One may then consider the choice of $i(t)$ in terms of an adaptive feedback rule, depending on the divergence $e(t)$ via a dynamic feedback controller $\xi(t)$ following the dynamics

$$\xi(t+1) = L\xi(t) + e(t)$$

so that $i(t) = i^* + G\,\xi(t)$. By stability arguments one may choose the matrices L and G so as to drive the system (10) asymptotically in time to the state $e = 0$. However, such arguments, cannot guarantee the convergence of the two economies in finite time horizon, which is an important practical problem.

In order to address the problem of convergence of the economies in finite time horizon, we adopt a different approach, that will guarantee the convergence of the two economies within a predetermined and finite time horizon T. This approach is based on an optimal control problem, for a properly selected backward stochastic difference equation.

We will consider (10) as a dynamical system in finite horizon T, but with predescribed final condition $e(T) = 0$. This final condition guarantees the convergence between the two economies in finite horizon. Equation (10) assumes the form of a backward stochastic difference equation. It can be shown (see, e.g., Yannacopoulos 2010) that it may assume the equivalent formulation

$$e(t+1) = A_2 e(t) + (A_2 - A_1)z_1(t) + (C_2 - C_1)i(t) + w(t)\,\Delta M(t)$$

where $\Delta M(t) = M(t+1) - M(t)$ is a martingale difference, and $w(t)$ is a predictable process (i.e. a process which at time t can be prescribed by the history of both economies, the union and the candidate economy, up to time $t-1$). In this new formulation, we have a pair of unknown processes $(e(t), w(t))$, whereas the characteristics of the martingale difference are determined by the nature of the uncertainty in the economy, i.e., by the nature of the shocks. A wide class of rational expectation models can be expressed via this alternative formulation, leading to the ability of utilization of advanced mathematical tools for the quantitative and qualitative treatment of such problems (see Yannacopoulos 2008, for a detailed treatment of rational expectations models, in continuous time, using this approach).

Therefore, the problem of convergence of the two economies can be written in the form of the controlled backward stochastic difference equation

$$\begin{aligned} e(t+1) &= A_2 e(t) + (A_2 - A_1)z_1(t) + (C_2 - C_1)i(t) + w(t)\,\Delta M(t) && (11)\\ e(T) &= 0 && (12) \end{aligned}$$

where now the unknowns are the pair $(e(t), w(t))$, $z_1(t)$ is the state of the economy 1, assumed to be an external signal, whereas the policy rule $i(t)$ is assumed to be a control procedure, to be chosen so as to guarantee this final condition, while satisfying a suitably selected cost criterion, quantifying allowable divergence from predescribed targets or the cost of economic policy.

The problem is to specify the control procedure $i(t)$. As there are obvious limitations to the policy interventions that the central bank may take, we will adopt once more an optimal control approach, according to which we choose $i(t)$ so as to minimize the quadratic functional

$$J(\{i\}) = \frac{1}{2}\langle G(e(0) - e^*(0), e(0) - e^*(0)\rangle + \frac{1}{2}\sum_{t=0}^{T-1} \rho^t i^2(t)$$

where $e^*(0)$ is the "true" initial deviation of the two economies, and G is a diagonal matrix with positive entries.

A comment is due here. Of course one would in principle prefer to start at $e(0) = e^*(0)$ and design the policy rule in such a way so that $e(T) = 0$. This reduces to the problem of exact controllability of the backward stochastic difference equation (11). However, such a policy may be rather costly, in terms of i, for both economies, resulting to the analogue of a Pyrrhic victory. Our proposed approach is to find the "cheapest" such strategy, that is the one minimizing the functional $J(\{i\})$ which while guaranteeing that $e(T) = 0$, will deviate from the actual initial position $e^*(0)$ by a quantity $e(0)-e^*(0)$, bounded by the minimum value of J, J^*. In fact if G is chosen as λI, where I is the unit matrix and λ is a positive quantity then $||e(0) - e^*(0)|| \leq \lambda^{-1}J^*$. Choosing G (or equivalently λ) accordingly we may make this deviation as small as possible, but never exactly 0. This can be interpreted in that we may reach the final state $e(T) = 0$, under the minimum cost, if we start from a nearby initial state $e(0)$ and not exactly the actual state $e^*(0)$.

The above discussion provides two important insights:
(a) It provides an estimate of the initial deviation of two countries so that one may expect that given a time horizon T and a properly selected interest rate policy, we can guarantee convergence of the two economies. If the initial state of the economy 2 is such that $||e(0) - e^*(0)|| > \lambda^{-1}J^*$ then this country may not enter the union, hoping for positive convergence results. Of course, the results depend among others, to the time of grace T, given to economy 2.
(b) Given that the initial convergence criterion $||e(0) - e^*(0)|| \leq \lambda^{-1}J^*$ is satisfied, a transfer mechanism should be used, at time $t = 0$ only, to shift $e^*(0)$ the actual difference of the two economies to $e(0)$. This transfer mechanism, which act only at $t = 0$, may be considered as part of the control procedure, an initial stage, before turning on the optimal interest rule $i(t)$, that will drive the system in the cheapest possible manner to the final convergence at T.

We now give a proposition regarding the construction of the optimal policy rule i.

Proposition 6.1 *The optimal policy rule is given by $i(t) = \rho^{-t}\bar{C}^T\psi(t)$ where $\psi(t)$ is the solution of the forward backward coupled stochastic difference equation*

$$\begin{aligned} e(t+1) &= A_2 e(t) + (A_2 - A_1)z_1(t) + (C_2 - C_1)i(t) + w(t)\,\Delta M(t) \\ \psi(t-1) &= A_2^T\,\psi(t) \\ e(T) &= 0 \\ A_2^T\psi(0) &= G^T\,(e(0) - e^*(0)) \end{aligned} \tag{13}$$

Proof: To prove this we consider the augmented Lagrangian

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\langle G(e(0) - e^*(0)), e(0) - e^*(0)\rangle + \frac{1}{2}\sum_{t=0}^{T-1}\rho^t\, i(t)^2 \\ &+ \sum_{t=0}^{T-1}\psi(t)\left(e(t+1) - A_2 e(t) - f(t) - \bar{C}i(t) - w(t)\,\Delta M(t)\right) \end{aligned}$$

where $\psi(t)$ are the random Lagrange multipliers for the dynamic constraints and in $f(t)$ we collect all terms in the RHS of the difference equation, which do not depend on $e(t)$ and on the control (i.e. the shock terms as well as the "signal" terms depending on $z_1(t)$). The first order conditions with respect to $i(t)$ give the form of control law, given in the statement of the proposition. The first order conditions with respect to $e(t)$ give the adjoint equation for $\psi(t)$. The first order conditions with respect to $e(0)$ give the initial condition. Substituting the control law in the equation for $e(t)$ we get the first part of the forward-backward system. The details are left to the reader. ■

Then the policy rule i can be obtained as long as the forward backward system which gives $\psi(t)$

is solved. One sees immediately that $\psi(t) = (A_2)^{-t}\psi(0)$, so the knowledge of $\psi(t)$ requires knowledge of $\psi(0)$, but one knows $\psi(0)$ only if $e(0)$ is given. However, we only know beforehand $e(T)$ and unless one solves the equation for $e(t)$ with this final condition, there is no way to determine $e(0)$. Therefore, the equations for $e(t)$ and $\psi(t)$ are coupled through the initial conditions, so one must know the whole process $e(t)$ for all t, so as to obtain $e(0)$ and through that obtain $\psi(t)$ and the control process. So, this problem is no longer solvable by a local in time control process, and the whole history of the economy is required once more in order to specify the optimal control policy.

The explicit construction of the optimal policy is provided in the next proposition.

Proposition 6.2 *The solution for $e(t)$ and $\psi(t)$ is given as $e(t) = P(t)\,e(t) + h(t)$ where*
(a) the matrix $P(t)$ solves the matrix equation

$$\begin{aligned} P(t+1) &= A_2 P(t) A_2^T + \hat{C} A_2^T \\ P(T) &= 0 \end{aligned} \tag{14}$$

with

$$\hat{C} = \rho^{-t}\,(\sigma_1 - \sigma_2)^2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

(b) the process $h(t)$ solves the backward equation

$$\begin{aligned} \mathbb{E}_t[h(t+1)] &= A_2 h(t) + f(t), \\ h(T) &= 0 \end{aligned}$$

with $f(t)$ defined as in the proof of the previous proposition, while,
(c) $\psi(t)$ solves $\psi(t-1) = A_2^T \psi(t)$ with initial condition such that

$$((G^T)^{-1} A_2^T - P(0))\,\psi(0) = h(0) - e^*(0).$$

Proof: We look for solutions using the ansatz $e(t) = P(t)\,\psi(t) + h(t)$ where $P(t)$ and $h(t)$ are as yet unspecified. Since we want $e(T) = 0$ this is guaranteed as long as $P(T) = 0$ and $h(T) = 0$. We now substitute this ansatz in the equation for $e(t)$ noting that $\psi(t)$ has to satisfy the equation $\psi(t-1) = A_2^T\psi(t)$. Choosing $h(t)$ such that

$$h(t+1) = A_2 h(t) + f(t) + w(t)\,\Delta M(t)$$

or equivalently such that

$$\mathbb{E}_t[h(t+1)] = A_2 h(t) + f(t)$$

it is seen immediately that $e(t)$ in the form $e(t) = P(t)\,\psi(t) + h(t)$ solve the equation for $e(t+1)$ as long as $P(t)$ solves the matrix equation (14). It remains to obtain the right initial condition for $\psi(0)$. To this end, set $t = 0$ to get that $e(0) = P(0)\,\psi(0) + h(0)$. Comparing with the initial condition as given in the previous proposition we obtain the stated result. ∎

7 Concluding remarks

In this work, we propose an extension of the Neo-Wicksellian monetary theory and policy to monetary unions. Using arguments from economic theory, empirical findings (Moons and Van Poeck, 2008) as well as a formal model for a monetary union we support the thesis that a single monetary policy rule may not be able to fully stabilize output gaps of the member economies, unless properly designed. Some results on the construction of such a monetary rule are presented, using three different but complementary approaches: A very weak approach based on determinacy arguments, an approach based on infinite horizon optimal control theory, and an approach guaranteeing convergence of the economies in finite and pre-prescribed time horizon T, based on controlled backward stochastic difference equations. Our findings, using the three different approaches, support the thesis that the design of an effective monetary policy rule must take into account the characteristics of all the economies in the union, using properly selected weighting schemes as well as, the full history of the economies and not simply their final states.

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