Valid Locally Uniform Edgeworth Expansions Under Weak Dependence and Sequences of Smooth Transformations
(Full Text pdf)

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Extended Abstract

1 Introduction

This paper is concerned with issue of the approximation of the distributions of a sequence of random vectors by sequences of Edgeworth measures *uniformly* with respect to a compact valued Euclidean parameter (locally uniform Edgeworth expansion). Our motivation resides on the fact that this could enable subsequent uniform approximations of analogous moments and their derivatives with respect to the aforementioned parameter. This in turn can facilitate the extraction of higher order asymptotic properties of estimators that are defined by the use of such moments. A prominent example is the indirect estimator defined by Gourieroux et al. [6] as a minimizer of a criterion involving the expectation of an auxiliary estimator. Analogous expansions have been studied by Bhattacharya and Ghosh [2] (see Theorem 3) in the iid case and Durbin [3] more generally.

We provide sufficient conditions for the *existence* of such an approximation in two cases. The first concerns the one where the random vectors are of the form of $\sqrt{n}$ times an arithmetic mean, the elements of which are members of a (possibly vector valued) stochastic process exhibiting weak dependence, in the spirit of Gotze and Hipp [5]. There, the authors validate the pointwise (w.r.t. the parameter) *formal* Edgeworth expansions. We essentially follow their line of reasoning, and by strengthening their conditions we establish the result ensuring that the relevant remainders are independent of the parameter. In the second case we assume that a locally uniform Edgeworth expansion is valid, and given a sequence of *polynomial* transformations for the random vector at hand, we provide sufficient conditions for an analogous expansion to exist for the *transformed* random vector. In this case our line of reasoning is close to the one in Skovgaard [8], but compared to this paper we utilize additional conditions concerning the *dependence* of the transformations on the parameter. Obviously the two cases can be combined for the establishment of valid locally uniform Edgeworth expansions in composite backgrounds.

2 First Case: Weak Dependence and Standardized Arithmetic Means

We denote with Θ a *compact* subset of $\mathbb{R}^p$ (w.r.t. the usual topology). The following assumption defines the form of the eligible stochastic processes for the results that follow.

Assumption A.1 *Let* $(\varepsilon_j)_{j\in\mathbb{N}}$ *be a sequence of* iid *random variables and* $g:\mathbb{R}^{\mathbb{N}}\times\Theta\to\mathbb{R}^k$ *sufficiently smooth*

$$X_j = g\left(\varepsilon_{j-i} : i\in\mathbb{Z},\theta\right), \qquad j\in\mathbb{Z}. \tag{1}$$

For $S_n(\theta)=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(X_i(\theta)-EX_0(\theta)\right)$, and $r=0,\dots,s$ let $\chi_{r,n}(t)$ be the cumulants of t^TS_n of order r, i.e.

$$\chi_{r,n}(t)=\left.\frac{d^r}{dx^r}\log E\exp\left(ixt^TS_n\right)\right|_{x=0}$$

Obviously $\chi_{r,n}$ depend on θ. Let $\Psi_{n,s}(t)$ be the *formal* Edgeworth measure of S_n of order $s-2$, $s\geq 3$, defined by its characteristic function $\widehat{\Psi}_{n,s}(t)=\exp\left(\chi_{2,s}\right)+\sum_{r=1}^{s-2}n^{-r/2}\widetilde{P}_{r,n}(t)$, where the functions $\widetilde{P}_{r,n}(t)$, $r=1,2,...$ satisfy the formal identity

$$\exp\left(\chi_{2,n}+\sum_{r=3}^{\infty}\frac{1}{r!}\tau^{r-2}n^{(r-2)/2}\chi_{r,n}(t)\right)=\exp\left(\chi_{2,s}\right)+\sum_{r=1}^{\infty}\tau^{r}\widetilde{P}_{r,n}(t)$$

and $\mathcal{B}_c$ the collection of convex Borel set of $\mathbb{R}^k$.

Question Given A.1, under what conditions

$$\sup_{\theta\in\Theta}\sup_{A\in\mathcal{B}_c}\left|P\left(S_n\left(\theta\right)\in A\right)-\Psi_{n,s}\left(\theta\right)\left(A\right)\right|=o\left(n^{\frac{s-2}{2}}\right) \tag{2}$$

The next assumption provides sufficient conditions so that the previous question is well-posed and has an affirmative answer. It essentially corresponds to a *uniform extension* of the analogous conditions (2)-(4) in Gotze and Hipp [5]. The proof of sufficiency follows naturally the line of proof of Theorem 1.1 of Gotze and Hipp [5], by establishing that due to assumption A.2 the terms appearing in the relevant bounds are independent of θ.

Assumption A.2 *Let the following conditions hold:*
-M ***(Existence of Moments)***

$$\sup_{\theta\in\Theta}E\left\|X_1\right\|^{s+1}\leq\beta_{s+1}$$

-WD ***(Weak Dependence)*** *There exist constants $K<\infty$ and $\alpha>0$ independent of θ such that for $m\geq1$,*

$$E\left\|g\left(\varepsilon_j:j\geq0,\theta\right)-g\left(\varepsilon_0,\ldots,\varepsilon_m,0,\ldots,\theta\right)\right\|\leq K\exp\left(-\alpha m\right)$$

-EL *There exist $K<\infty$, and $\alpha>0$, not depending on θ, such that,*

$$\sup_{\theta\in\Theta}\sum_{j=0}^{\infty}E\left(\left\|\frac{\partial}{\partial\varepsilon_0}X_j\right\|\right)<\frac{2K}{1-\exp\left(-\alpha\right)}$$

-CPD ***(Almost sure continuity of partial derivatives)*** *For $j\in\mathbb{Z}$ there exists $G_j\subset\mathbb{R}$, $P\left(G_j\right)=1$ independent of θ, such that for all $x_0\in G_j$, η, $\delta>0$ there exists $\tau>0$ independent of θ satisfying*

$$P\left\{\begin{array}{c} y\in\mathbb{R}^{\mathbb{Z}}:\forall x\in\mathbb{R},\left|x-x_0\right|<\tau,\frac{\partial}{\partial\varepsilon_0}X_j\text{ exists at the point }\left(y,x\right)^j\text{ and}\\ \sup_{\theta\in\Theta}\left|\frac{\partial}{\partial\varepsilon_0}X_j\left(\left(y,x\right)^j,\theta\right)-\frac{\partial}{\partial\varepsilon_0}X_j\left(\left(y,x_0\right)^j,\theta\right)\right|\leq\delta\end{array}\right\}\geq1-\eta$$

-NDD ***(Nondegenerate derivatives on a set of positive probability)*** *For some distinct $l_1,\ldots l_k\geq0$ independent of θ,*

$$\inf_{\theta\in\Theta}\det\left(\sum_{j=0}^{\infty}\frac{\partial}{\partial\varepsilon_{l_\nu}}X_j:\nu=1,\ldots,k\right)\neq0$$

-DENS ***(Absolute Continuity)*** *ε_0 admits a positive continuous density.*

Assumptions A.2.EL-DENS imply that for any θ densities exist for large enough n, thereby enable bounding integrals of a large class of real functions (including the indicators of convex Borel sets) with respect to differences between the associated densities, by integrals of the difference of derivatives (of sufficiently high order) of the Fourier transforms of the densities. The bound depends on the functions only through constants which are equal to 1 when these are indicators, hence it is uniform with respect to indicators. The latter integral is then uniformly over Θ bounded by analogous differences between derivatives of characteristic functions of $S_n(\theta)$ and an analogous sum (say $S'_n(\theta)$) comprised by appropriately truncating the X_i plus the one between derivatives of characteristic functions of $S'_n(\theta)$ and the ones of the formal Egdeworth ,easure. Then, assumptions A.2.M-WD imply that the last two integrals are of the appropriate order uniformly over Θ.

Theorem 2.1 *If assumptions A.1 and A.2 are valid then 2 holds.*

3 Second Case: Polynomial Transformations

We then suppose that $(S_n(\theta))_{n\in\mathbb{N}}$ is a sequence of random elements *not necessarily* of the form described immediately after assumption A.1. Furthermore the distribution of $S_n(\theta)$ admits *a locally uniform Edgeworth expansion of order* $s-2$ with $\Psi_{n,s}(\theta)$ an Edgeworth distribution (*not necessarily the formal one*).[1]

Question Let $f_n : \mathbb{R}^q \to \mathbb{R}^p$. Find sufficient conditions for the validity of

$$\sup_{\theta\in\Theta}\sup_{A\in B_c^*}\left|P\left(f_n\left(S_n(\theta)\right)\in A\right)-\Psi^*_{n,s}(\theta)(A)\right| = o\left(n^{-\frac{s-2}{2}}\right) \tag{3}$$

where $\Psi^*_{n,s}(\theta)$ is an Edgeworth distribution of order $s-2$ ($s\geq 3$) on $\mathbb{R}^p$.

The following assumption restricts the examined f_n to polynomial functions with coefficients that could depend on θ.

Assumption A.3 *Let the following conditions hold:*
-POL $f_n(x,\theta) = \sum_{i=0}^{s-2}\frac{A_{i_n}(\theta)\left(x^{i+1}\right)}{n^{i/2}}$ *where* $A_{i_n} : \Theta\times\mathbb{R}^{q^{i+1}} \to \mathbb{R}^p$ *is* $(i+1)$-*linear* $\forall\theta\in\Theta$, $x^i = \left(\underbrace{x,...,x}_{i-times}\right)$, $A_{0_n}(\theta) = A_0(\theta)$, $\operatorname{rank} A_0(\theta) = p\ \forall\theta\in\Theta$,

[1] For the definition of the general form of an Edgeworth distribution see equations (3.7) and (3.8) of Magdalinos [7].

A_{i_n} *equicontinuous on* Θ, $\forall x^{i+1}$.
-EEQ *The* i^{th} *polynomial, say,* $\pi_i(z,\theta)$ *of* $\Psi_{n,s}(\theta)$ *is equicontinuous on* Θ $\forall z \in \mathbb{R}^q$, *for* $i = 1, \ldots, s-2$, *and if* $\Sigma(\theta)$ *denotes the variance matrix in the density of* $\Psi_{n,s}(\theta)$ *then it is continuous on* Θ *and positive definite.*

The following theorem provides the answer to the previous question.

Theorem 3.1 *Under assumption A.3 there exists an Edgeworth distribution* $\Psi^*_{n,s}(\theta)$ *for which equation 3 is valid, with polynomials that satisfy A.3.*EEQ. *Furthermore, if* K *is a* m*-linear real function on* $\mathbb{R}^p$ *then*

$$\sup_{\theta\in\Theta}\left|\int_{\mathbb{R}^p} K(x^m)\, d\Psi^*_{n,s}(\theta) - \int_{\mathbb{R}^q} K((f_n(x))^m)\, d\Psi_{n,s}(\theta)\right| = o\left(n^{\frac{s-2}{2}}\right)$$

The proof is given by the inductive establishment of an approximate "inverse" to f_n for any n, the use of the assumed Edgeworth expansion and appropriate transformations of integrals w.r.t. the latter measures. This theorem enables the establishment for analogous expansions for random elements that satisfy polynomial equations with immediate applications to M-estimators that satisfy sufficiently "smooth" foc′s with sufficiently high probability. The next result clarifies this.

Theorem 3.2 *Suppose that:*
-POLFOC $M_n(\theta)$ *satisfies* $0_{p\times 1} = \sum_{i=0}^{s-2} \frac{1}{n^{i/2}} \sum_{j=0}^{i+1} C_{ij_n}(\theta)\left(M_n(\theta)^j, S_n(\theta)^{i+1-j}\right) + R_n(\theta)$ *with probability* $1 - o\left(n^{-\frac{s-2}{2}}\right)$ *independent of* θ *where* $C_{ij_n} : \Theta \times \mathbb{R}^{q^{i+1}} \to \mathbb{R}^p$ *is* $(i+1)$*-linear* $\forall\theta \in \Theta$, $C_{00_n}(\theta), C_{01_n}(\theta)$ *are independent of* n *and have rank* p $\forall\theta \in \Theta$, C_{ij_n} *are equicontinuous on* Θ, $\forall x^{i+1}$,
-LUE $S_n(\theta)$ *admits a locally uniform Edgeworth expansion that satisfies assumption A.3.*EEQ,
-UAT $\sup_{\theta\in\Theta} P\left(\|M_n(\theta)\| > C\ln^{1/2} n\right) = o\left(n^{-\frac{s-2}{2}}\right)$ *for some* $C > 0$ *independent of* θ,
-USR $\sup_{\theta\in\Theta} P(\|R_n(\theta)\| > \gamma_n) = o\left(n^{-\frac{s-2}{2}}\right)$ *for some real sequence* $\gamma_n = o\left(n^{-\frac{s-2}{2}}\right)$ *independent of* θ.
Then $M_n(\theta)$ *admits a locally uniform Edgeworth expansion that satisfies assumption A.3.*EEQ.

Its proof is given via the use of theorem 3.1 via the establishment of an approximate (uniformly over Θ) equality of S_n and $f_n(M_n)$ for an inductively constructed f_n that satisfies the assumptions of theorem 3.1.

We then construct an example involving stationary GARCH processes and use the previous results in order to validate locally uniform Edgeworth expansions for various GMM-type and Indirect estimators in this context. Notice that in the scope of similar examples involving strictly stationary processes smoothness for the coefficients of the polynomials of the resulting Edgeworth measures can be established via analogous smoothness conditions for $C_{ij_n}(\theta)$ and the moments of $S_n(\theta)$ due to remark (2.12) of Gotze and Hipp [4]. This along with an appeal to dominated convergence implies that the moment approximations of the aforementioned estimators emerging from these expansions are also smooth.

4 Further Research

A question for future research concerns the issue of establishing Edgeworth type expansions (see Magdalinos [7]) when θ lies in the boundary of the parameter space.

References

[1] Arvanitis S. and A. Demos (2011), "Stochastic Expansions and Moment Approximations for Three Indirect Estimators", working paper, Dept. of Economics, AUEB.

[2] Bhattacharya R.N. , and J.K. Ghosh (1978),"On the validity of the formal Edgeworth expansion", *The Annals of Statistics,* Vol. 6, No. 2, pp. 434-451.

[3] Durbin, J. (1980), "Approximations for densities of sufficient statistics", *Biometrika* 67, pp. 311–333.

[4] Gotze F., and C. Hipp (1983), "Asymptotic expansions for sums of weakly dependent random vectors", *Zeitschrift für Wahrscheinlichkeitstheorie und verwandie Gebiete*, 64, pp. 211-239.

[5] Gotze F., and C. Hipp (1994), "Asymptotic distribution of statistics in time series", *The Annals of Statistics*, 22, pp. 2062-2088.

[6] Gourieroux C., A. Monfort, and E. Renault (1993), "Indirect Inference", *Journal of Applied Econometrics*, Volume 8 Issue 1, pp. 85-118.

[7] Magdalinos, M.A. (1992), "Stochastic Expansions and Asymptotic Approximations", *Econometric Theory*, Vol. 8, No. 3, pp. 343-367.

[8] Skovgaard I.M. (1981), "Transformation of an Edgeworth expansion by a sequence of smooth functions", *Scandinavian Journal* of Statistics 8, 207-217.