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Probabilistic models in Risk Theory

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0.1 Abstract

It is common practice for insurance companies to give dividends to their shareholders. Many papers have been written on dividends policies. It has been found that under some reasonable assumptions the optimal policy is to follow the so called constant barrier dividends policy also known in Risk theory as de Finetti model. However soon become apparent that this model is not "perfect" as questions and problems emerge from its application and that some "kind" of modifications are necessary.

In this spirit, this thesis extends the de Finetti model in order to include cases with barriers dividends policies which are modeled by diffusions. The approach is axiomatic and was motivated by the classical de Finetti model. We show that the de Finetti models with general (diffusion) barriers are well posed that is they exist and are unique, or in other words that there exist unique stochastic processes that evolve according to our conditions. When we say unique stochastic processes we mean up to the degree of indistinguishability.

We consider de Finetti models with one general barrier meaning that when the reserves of the insurance company reach a "particular" level, which also depends upon a diffusion process, then the company goes bankrupt. We also consider de Finetti models with two general barriers, that is when the reserves of the insurance company reach the level of the lower barrier, which also depends on a diffusion process, then the insurance company has the option to borrow money and continue it's function.

We derive differential equations with appropriate boundary conditions, the solution of which gives the quantities for which we are interesting. More specifically we find differential equations with appropriate boundary conditions, the solution of which gives the moments of the discounted dividends, the moments of the discounted financing, the Laplace transform of the time of ruin, the Laplace transform of the joint distribution of the time of ruin and the discounted dividends and the Laplace transform of the joint distribution of the discounted dividends and the discounted financing.

We apply the formulas in special cases and more specifically in cases where the reserves process follows a Brownian motion, a Geometric Brownian motion and an Ornstein–Uhlenbeck process (also see Gerber, H.U. and Shiu, E.S.W.([71],[72])).

Next we work on another important issue, which is the situation of insurance companies cooperation. We consider this issue from the perspective of a particular insurance company. We are interesting to look at parameters which are vital to the decisions of the company. Among these parameters very important role we consider to play the probability of survival in a particular cooperation and the shares that will be given to the shareholders during this cooperation. We find differential equations with appropriate boundary conditions the solution of which will give:

- The moments of the discounted dividends and the discounted financing.
- The Laplace transform of the joint distribution of the time of ruin and the discounted dividends.
- The Laplace transform of the discounted dividends.

- The Laplace transform of the time of ruin.
- The Survival probability for one of the two insurers.

We apply these results in two models:

(I) The Lundberg - de Finetti model.

(II) The de Finetti - de Finetti model.

We show how an insurance company can use the above results for policy making purposes. We also mention possible ways to extend the above considerations to various other models.

0.2 Notation

MOST COMMON NOTATION	DESCRIPTION
I. Sets and Spaces	
$(\Omega, \mathcal{F}, P)$	Probability space
$(\Omega, \mathcal{F}, F, P)$	a filtered complete probability space
R^+	Positive real numbers
R^n	the n–dimensional Euclidean space
N	the natural numbers
$R^{n\times m}$	the $n \times m$ matrices (real entries)
$C(U)$	the space of continuous functions from U into $\mathbb{R}$
$C^n(U)$	the space of continuous functions from U into $\mathbb{R}$ with continuous derivatives up to order n.
$C_0(U)$	the space of continuous functions from U into $\mathbb{R}$ with compact support
$C_0^n(U)$	the space of functions from U into $\mathbb{R}$ with compact support and continuous derivatives up to order n.
$C_b^n(U)$	the space of continuous, bounded functions from U into $\mathbb{R}$ with continuous, bounded derivatives up to order n.
S^V	see Definition 2.5.10
S^K	see Definition 2.5.11
S^M	see Definition 2.5.12
S^N	see Definition 2.5.13
$S^{V(\pm)}$	see Definition 2.5.18
$S^{K(\pm)}$	see Definition 2.5.19
S^L	see Definition 2.5.20

II. Functions	
θ_t	Right shift operator, see Definition 1.3.27
$\theta(x, a, b)$	see Definition 2.5.4
$\varphi(x, a, b)$	see Definition 2.5.4
$\xi(x, a, b)$	see Definition 2.5.4

III. Greek notation

δ	interest rate
μ	drift coefficient, see Definition 1.3.23
σ	diffusion coefficient, see Definition 1.3.23
ρ_{xa}	correlation of the Brownian motions B^x and B^a, see (2.1.4)
σ_{xa}^2	see Definition 2.5.3

IV. General notation

F_t^X	the σ-algebra generated by $\{X_s : s \leq t\}$
$f^+(\cdot)$	$\max(f(\cdot), 0)$, the positive part of the real function $f(\cdot)$
$f^-(\cdot)$	$\max(-f(\cdot), 0)$, the negative part of the real function $f(\cdot)$
$sign(x)$	$\begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$
$\mathcal{A}_x$	the generator of an Ito diffusion X (see definition 1.3.29)
$a.s.$	almost surely
$x \wedge y$	$\min(x, y)$ for $x, y \in \mathbb{R}$
$x \vee y$	$\max(x, y)$ for $x, y \in \mathbb{R}$
t	time
T	Ruin time

de Finetti model with one general reflecting barrier

$\mathcal{U}$	Dividends
U	Discounted Dividends, see (2.1.18)
$U(t)$	Discounted Dividends starting at time t (see definition 2.5.6)
$N(x, a, b, \lambda_1, \lambda_2)$	The Laplace transform of the joint distribution of the time of ruin and the discounted dividends, see (2.1.20)
$K(x, a, b, \lambda)$	The Laplace transform of the discounted dividends, see (2.1.21)
$M(x, a, b, \lambda)$	The Laplace transform of the time of ruin, see (2.1.22)
$V(x, a, b; n)$	The Moments of the discounted dividends, see (2.1.23)
$V(x, a, b)$	Equals with $V(x, a, b; 1)$ by definition.

de Finetti model with two general reflecting barriers

$\mathcal{U}^{(\pm)}$	We refer simultaneously to dividends and financing (whenever we use the symbol $\pm$ we use it with analogous logic)
$\mathcal{U}^{(+)}$	Dividends
$\mathcal{U}^{(-)}$	Financing
$U^{(\pm)}$	We refer simultaneously to the discounted dividends and the discounted financing
$U^{(+)}$	Discounted Dividends, see (2.1.24)
$U^{(+)}(t)$	Discounted Dividends starting at time t (see (2.5.22))
$U^{(-)}$	Discounted Financing, see (2.1.25)
$U^{(-)}(t)$	Discounted financing starting at time t (see (2.5.23))
$L(x, a, b, \lambda_1, \lambda_2)$	The Laplace transform of the joint distribution of the discounted dividends and the discounted financing, see (2.1.29)
$K^{(+)}(x, a, b, \lambda)$	The Laplace transform of the discounted dividends, see (2.1.30)
$K^{(-)}(x, a, b, \lambda)$	The Laplace transform of the discounted financing
$V^{(+)}(x, a, b; n)$	The Moments of the discounted dividends, see (2.1.31)
$V^{(+)}(x, a, b)$	Equals with $V^{(+)}(x, a, b; 1)$ by definition.
$V^{(-)}(x, a, b; n)$	The Moments of the discounted financing
$V^{(-)}(x, a, b)$	Equals with $V^{(-)}(x, a, b; 1)$ by definition.

Two Insurers (T.I.)

$\mathcal{M}(x, y, \lambda)$	The Laplace transform of the time of ruin see (4.1.10)

(T.I.)One reflecting barrier

$\mathcal{V}(x, y; n)$	The Moments of the discounted dividends, see (4.1.7)
$\mathcal{V}(x, y)$	Equals with $\mathcal{V}(x, y; 1)$ by definition.
$\mathcal{K}(x, y, \lambda)$	The Laplace transform of the discounted dividends, see (4.1.9)
$\mathcal{N}(x, y, \lambda_1, \lambda_2)$	The Laplace transform of the joint distribution of the time of ruin and the discounted dividends see (4.1.12).

(T.I.)Two reflecting barriers

$\mathcal{V}^{(+)}(x, y; n)$	The Moments of the discounted dividends see (4.1.6).
$\mathcal{V}^{(+)}(x, y)$	Equals with $\mathcal{V}^{(+)}(x, y; 1)$ by definition.
$\mathcal{V}^{(-)}(x, y; n)$	The Moments of the discounted financing
$\mathcal{V}^{(-)}(x, y)$	Equals with $\mathcal{V}^{(-)}(x, y; 1)$ by definition.
$\mathcal{K}^{(+)}(x, y, \lambda)$	The Laplace transform of the discounted dividends see (4.1.8)
$\mathcal{K}^{(-)}(x, y, \lambda)$	The Laplace transform of the discounted financing
$\mathcal{N}(x, y, \lambda_1, \lambda_2, \lambda_3)$	The Laplace transform of the joint distribution of the time of ruin , the discounted dividends and the discounted financing. (see (4.1.11))

Two Insurers (T.I.): de Finetti -de Finetti model

(T.I.)One reflecting barrier

$\mathcal{V}_i(x,y;n)$	The Moments of the discounted dividends, for the i - insurer $(i=1,2)$
$\mathcal{V}_i(x,y)$	Equals with $\mathcal{V}_i(x,y;1)$ by definition.
$\mathcal{K}_i(x,y,\lambda)$	The Laplace transform of the discounted dividends, for the i - insurer $(i=1,2)$

(T.I.)Two reflecting barriers

$\mathcal{V}_i^{(+)}(x,y;n)$	The Moments of the discounted dividends for the i - insurer $(i=1,2)$
$\mathcal{V}_i^{(+)}(x,y)$	Equals with $\mathcal{V}_i^{(+)}(x,y;1)$ by definition.
$\mathcal{V}_i^{(-)}(x,y;n)$	The Moments of the discounted financing for the i - insurer $(i=1,2)$
$\mathcal{V}_i^{(-)}(x,y)$	Equals with $\mathcal{V}_i^{(-)}(x,y;1)$ by definition.
$\mathcal{K}_i^{(+)}(x,y,\lambda)$	The Laplace transform of the discounted dividends for the i - insurer $(i=1,2)$
$\mathcal{K}_i^{(-)}(x,y,\lambda)$	The Laplace transform of the discounted financing for the i - insurer $(i=1,2)$

Stochastic Processes

γ_t	see Definition 2.5.1
h_t	see Definition 2.5.1
$\mathcal{Y}_t$	see Definition 2.5.1
$\mathcal{Z}_t$	see Definition 2.5.1
B_t	Brownian motion
B_t^x	Brownian motion driving the process X_t

Chapter 1

Risk Theory, Dividends, and the Stochastic Analysis perspective

1.1 Introduction

In this first chapter, collective risk theory is introduced together with the concept of dividends policy. We are more interested to consider the above concepts from a stochastic analysis perspective. We give the necessary notations, definitions and existing results.

Collective risk theory is concerned with the random fluctuations of the total assets of an insurance company. The so called risk process, models the time evolution of the reserves of an insurance company. The quantities that describe the risk process are: the initial capital, the premiums, the claims, the risk reserve, the economic environment and the reinsurance. For more information one can see texts on risk theory such as for example: T. Rolski, H. Schmidli, V. Schmidt and J. Teugels [160], Asmussen [6], Grandell [80], Daykin, T Pentikainen, M Pesonen [42] and Bühlmann [32] .

For each of the quantities of the risk process many choices are possible. As a consequence of this there are many models available in the literature. The most important of them which have been studied extensively are: The Sparre-Andersen model, (see for example Gerber [75], Albrecher [9], Cai[33]), the Markov-modulated risk model, (see for example Lu [125], Reinhard [154]) and the Cramer–Lundberg model (see for example Embrechts [56], Klüppelberg [114]). Next we will describe briefly the Cramer–Lundberg model.

We assume a complete probability space $(\Omega, \mathcal{F}, P)$. The classical Cramer–Lundberg model consists of the risk process $X = \{X_t : t \geq 0\}$ which models the reserves of an insurance company and which is described by a point process $N = \{N_t : t \geq 0\}$ with initial condition $N_0 = 0$ a.s. and mean $EN_t = \lambda t$ and a sequence $\{Y_k : k \in \mathbb{N}\}$ of independent and identically distributed random variables, with common distribution F, with $F(0) = 0$, mean μ, and variance σ^2. More specifically the risk process X is described by

$$X_t = x_0 + ct - \sum_{k=1}^{N_t} Y_k \tag{1.1.1}$$

where $c = \lambda\mu(1 + \theta)$ is the so called gross risk premium, θ is the so called safety loading factor and x_0 is the initial capital of the company.

Many modifications of the above model has been proposed. We will mention two of them which fall in the category of the so called perturbed risk models.

The first model is an extension of the reserves process by the addition of a diffusion component and is described by

$$X_t = x_0 + ct - \sum_{k=1}^{N_t} Y_k + \eta_1 B_t \tag{1.1.2}$$

where $B = \{B_t : t \geq 0\}$ is a standard Brownian motion and $\eta_1 > 0$.

The second model consists of the addition of an a-stable Lévy process instead of a Brownian motion, that is

$$X_t = x_0 + ct - \sum_{k=1}^{N_t} Y_k + \eta_2 Z_t \tag{1.1.3}$$

with $Z = \{Z_t : t \geq 0\}$ an a–stable Lévy process and $\eta_2 > 0$. For more information on perturbed risk models one can see for example Dufresne [50], Furrer [65], Wang [182], Schlegel [163], and Schmidli [167].

A central goal of risk theory is to obtain information on the ruin probability associated with a risk process, where the event of "ruin" is defined as the first time epoch where the reserves of the insurance company drop below the level 0. The ruin probability is therefore defined as

$$P\left(X_t < 0 \text{ for some } t \geq 0\right) \tag{1.1.4}$$

and the corresponding time to ruin, as

$$T := \inf\{t \geq 0 : X_t < 0\} \tag{1.1.5}$$

(with the understanding of the infimum of the empty set is $+\infty$). Many approaches are available to compute the ruin probability including exact solutions, numerical methods, approximations, bounds and inequalities, statistical methods and simulation (see Asmussen [6]).

In this thesis we will focus in diffusion approximation techniques. The classical Cramer–Lundberg model can be approximated (see Harrison [88]) by a Brownian motion with drift, that is we can model the risk process as

$$\begin{aligned} X_t &= \mu t + \sigma B_t \\ X_0 &= x_0 \text{ a.s.} \end{aligned} \tag{1.1.6}$$

where $B = \{B_t : t \geq 0\}$ is standard Brownian motion.

Diffusion approximations of the risk process are very popular because they can be applied to quite general models deviating from the usual restrictive assumptions. In particular distinguished are three types of approximations: Diffusion approximations (see Iglehart [97], Ruohonen [161], Schmidli [164],Asmussen [7]), Corrected Diffusion approximations (see Fuh [63], Siegmund [171]) and Lévy process approximations (see Furrer, Michna, and Weron [64]).

The approximation approach is a link between Risk theory and Stochastic Analysis. This connection gave a great boost on risk theory. The techniques of Stochastic Analysis are very important because they enable one to consider more complex and more realistic risk models. Some of the texts relevant to the methods and principles of stochastic analysis are Chung and Williams [40], Durrett [52], Klebaner [113], Oksendal [141], Revuz and Yor [156], Steele [174].

Stochastic Analysis and Risk theory make an excellent combination as one can see for example in the paper of Hipp[92]. The main advantage of the cooperation of Stochastic Analysis and Risk theory is that particularly difficult problems for risk theory can and already have been solved under the contexts of: Optimal stopping (see Jensen [106], Schottl [169], Ferenstein [58], Karpowicz [110], Bassan [20]) and Stochastic Control (see Ishikawa [101], Kushner [120]). Also extensive use of the methods of Stochastic Analysis have been applied in the field of optimal reinsurance (see Azuce [13]).

Continuing now with the classical Cramer–Lundberg model we mention that under the assumption that the premium income per unit time c is larger than the average amount claimed $\lambda\mu$ then the reserves in the Cramer-Lundberg model has positive first moment and has therefore the unrealistic property that it converges to infinity with probability one. In answer to this objection de Finetti [44] introduced the dividends barrier model, in which all surpluses above a given level are transferred to a beneficiary. This approach resembles more closely the "real" world but unfortunately has the drawback that under the barrier dividends model the risk process will down-cross the level zero with probability one.

In this dissertation we focus on dividends policies and for this reason we describe briefly in the next section the main components of the de Finetti model.

1.2 de Finetti model.

The de Finetti model is usually followed in practice by an insurance company and the main idea of it is that the insurance company pays some of its surplus to the shareholders as dividends, until ruin occurs, i.e. until the capital is negative for the first time. When ruin occurs, the company is bankrupt and no more dividends can be paid to the existing shareholders. Many papers have been written on dividends policies. It has been found that under some reasonable assumptions the optimal policy is that whenever the surplus goes above a "barrier" b the excess is immediately paid out as dividends and this policy is known as dividends barriers policy.

In the case of an insurance company which applies a dividends policy the model (1.1.1) is modified as

$$Z_t := X_t - \mathcal{U}_t \tag{1.2.1}$$

where with $\mathcal{U}_t$ we denote the accumulated dividends until time t. It is well known that the process $\{\mathcal{U}_t : t \geq 0\}$ is unique and is given a.s. by

$$\mathcal{U}_t = \sup_{0 \leq s \leq t} (X_s - b)^+ \tag{1.2.2}$$

(For a real number x we denote by $x^+ := \max(x, 0)$ its positive part and by $x^- := \max(-x, 0)$ its negative part.) Also

(I)

$$\{\mathcal{U}_t : t \geq 0\} \quad \text{is nondecreasing a.s.} \tag{1.2.3}$$

(II)

$$Z_t = X_t - \mathcal{U}_t \leq b_t \text{ for all } t \geq 0 \text{ a.s.} \tag{1.2.4}$$

(III) The process $\{\mathcal{U}_t; t \geq 0\}$ increases only when $Z_t = b_t$, i.e.

$$\int_0^t 1(Z_s < b) d\mathcal{U}_s = 0, \text{ for all time } t \geq 0 \text{ a.s.} \tag{1.2.5}$$

A variation of the above model is to allow for the possibility that when the reserve fund becomes zero then the company does not go bankrupt but has the possibility to be financed and continue its operation. This is the so called de Finetti model with two barriers of reflection. It can be described by the model:

$$Z_t := X_t - \mathcal{U}_t^{(+)} + \mathcal{U}_t^{(-)} \tag{1.2.6}$$

where with $\mathcal{U}_t^{(+)}$ we denote the accumulated dividends and with $\mathcal{U}_t^{(-)}$ we denote the accumulated financing of the company until time t. It is well known (see Harrison [89]) that the process

$\{(\mathcal{U}_t^{(+)}, \mathcal{U}_t^{(-)}); t \geq 0\}$ is unique and is given a.s. by

$$\mathcal{U}_t^{(+)} = \sup_{0 \leq s \leq t} \left(b - X_s - \mathcal{U}_s^{(-)}\right)^- \tag{1.2.7}$$

$$\mathcal{U}_t^{(-)} = \sup_{0 \leq s \leq t} \left(X_s - \mathcal{U}_s^{(+)}\right)^- \tag{1.2.8}$$

and also that

(I)

$$\{\mathcal{U}_t^{(+)} : t \geq 0\} \text{ and } \{\mathcal{U}_t^{(-)} : t \geq 0\} \text{ are nondecreasing a.s.} \tag{1.2.9}$$

(II)

$$0 \leq Z_t = X_t - \mathcal{U}_t^{(+)} + \mathcal{U}_t^{(-)} \leq b_t \text{ for all time } t \geq 0 \text{ a.s.} \tag{1.2.10}$$

(III) The process $\{\mathcal{U}_t^{(+)}; t \geq 0\}$ increases only when $Z_t = b_t$, i.e.

$$\int_0^t 1(Z_s < b) d\mathcal{U}_s^{(+)} = 0, \text{ for all time } t \geq 0 \text{ a.s.} \tag{1.2.11}$$

and the process $\{\mathcal{U}_t^{(-)}; t \geq 0\}$ *increases only when* $Z_t = 0$, i.e.

$$\int_0^t 1(Z_s > 0) d\mathcal{U}_s^{(-)} = 0, \text{ for all time } t \geq 0 \text{ a.s.} \tag{1.2.12}$$

In the mathematical finance and actuarial literature there is a good deal of work on dividend barrier models and the problem of finding an optimal policy for paying out dividends. In connection with a dividends policy we distinguish the papers as relevant to

(I) Optimal dividend payouts (see Paulsen [146], Gerber [71], Gerber and Shiu [72], Dickson and Waters [46], [47], Gerber and Shiu [73],[74]).

(II) Optimal reinsurance (see Azcue and Muler [12], Schmidli [165], Schmidli [166], Asmussen [8]).

(III) Optimal investment (see Hipp and Plum [93]).

(IV) Stochastic control (see Hipp and Taksar [94]).

In practice however the above model may fail to fit with all the requirements. That is, it might be the case that there exist requirements imposed on the company by the regulatory authorities, and it is possible that the insurance company will not be allowed to pay dividends according to the optimal scheme, and it will have to look for the optimal allowed dividend policy, or it might be the case that the company do not want to make the probability of bankruptcy in the near future unacceptably high and so it will probable alter its dividends policy.

In addition even if we suppose that the insurance company will be allowed to pay dividends according to the optimal scheme, which let us suppose that it is a constant dividends barriers policy then it might be the case that due to random fluctuations in the accuracy of the flow of information the barriers can not be determined exactly to have a "correct" constant deterministic value but rather the random fluctuations passed to the value of barriers and so in reality the barriers instead of being constant they evolve in some diffusion way.

In the dissertation we will try to handle the above situation by considering barriers strategies, with barriers that are some general diffusion processes. In the following section we will present the necessary theory, notions and notations in order to formulate our models.

1.3 Preliminaries.

We assume as given a complete probability space $(\Omega, \mathcal{F}, P)$. In addition we assume a given filtration $\{\mathcal{F}_t : t \geq 0\}$. A filtration $\mathbb{F} = \{F_t : t \geq 0\}$ is a non-decreasing family of sub-σ-algebras of $\mathbb{F}$. A stochastic process on (Ω, F, P) is a collection of $\mathbb{R}^d$–valued random variables $\{X_t : t \geq 0\}$, where $(d \geq 1)$. The functions $t \longrightarrow X_t(\omega)$ mapping $[0, \infty)$ into $\mathbb{R}^d$ are called the sample paths of the stochastic process X. The stochastic process X is said to be measurable if for every $A \in B(\mathbb{R}^d)$, the set $\{(t, \omega); X_t(\omega) \in A\}$ belongs to the product σ–field $B([0, \infty)) \otimes F$, where $B([0, \infty))$ and $B(\mathbb{R}^d)$ are the smallest $\sigma-$fields containing all the open sets of the topological spaces $[0, \infty)$ and $\mathbb{R}^d$ respectively. The stochastic process X is said to be adapted if $X_t \in F_t$ (that is, is F_t measurable) for each $t \geq 0$. The filtration generated by a process X is called the natural filtration, is defined by $F_t^X := \sigma\{X_s : s \leq t\}$ and is the smallest filtration for which the process X is adapted.

Definition 1.3.1 *A filtered complete probability space (Ω, F, F, P) is said to satisfy the usual hypotheses if*

1. *F_0 contains all the P-null sets of F.*
2. *The filtration F is right continuous, that is*

$$\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s, \qquad \forall t \geq 0$$

In order to avoid technical difficulties in this thesis we always assume that the usual hypotheses hold.

The risk process as a function for fixed $\omega \in \Omega$ evolves continuously until some claim occurs and makes it jump. This behavior is described with the notions of right continuous process and càdlàg process.

Definition 1.3.2 *A process $X = \{X_t : t \geq 0\}$ is called right continuous iff*

(I) The trajectories X_t are right continuous.

(II) The filtration F is right continuous, i.e.

$$\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s \text{ for } t \geq 0$$

Definition 1.3.3 *A stochastic process X is said to be càdlàg if it a.s. has sample paths which are right continuous, with left limits. We will denote the space of càdlàg processes with D. A stochastic process X is said to be càglàd if it a.s. has sample paths which are left continuous, with right limits. We will denote the space of càglàd processes with L.*

The notion of stopping time is central in risk theory.

Definition 1.3.4 *(Stopping Time) A random variable $T : \Omega \longrightarrow [0, \infty]$, is an $F-$stopping time iff $\{T \leq t\} \in \mathcal{F}_t$ for each $t \geq 0$.*

Usually there is the need of studying a process until some stopping time. For this reason we give now the definition of the so called *stopped process.*

Definition 1.3.5 *(Stopped Process) Let X be a stochastic process and let T be a random time. A process X^T is said to be the process X stopped at T if $X_t^T := X_{t\wedge T}$.*

If X is adapted and càdlàg and if T is a stopping time then the stopped process X^T which is given by

$$X_t^T := X_{t\wedge T} = X_t 1\{t < T\} + X_T 1\{t \geq T\} \tag{1.3.1}$$

is also adapted.

The following three definitions will be useful.

Definition 1.3.6 *Let σ denote a finite sequence of finite stopping times:*

$$0 = T_0 \leq T_1 \leq \ldots \leq T_k < \infty$$

The sequence σ is called a random partition.

Definition 1.3.7 *A sequence of random partitions σ_n,*

$$\sigma_n : T_0^n \leq T_1^n \leq \ldots \leq T_{k_n}^n$$

is said to tend to the identity if

(I)

$$\lim_n \sup_k T_k^n = \infty \qquad a.s.$$

(II)

$$\|\sigma_n\| := \sup_k |T_{k+1}^n - T_k^n| \qquad \text{converges to 0 as } n \to \infty \text{ a.s.}$$

Definition 1.3.8 *A sequence of processes $\{H^n\}_{n\geq 1}$ converges to a process H uniformly on compacts in probability if, for each $t > 0$ the $\sup_{0\leq s\leq t} |H_s^n - H_s|$ converges to 0 in probability.*

One often faces the need to describe that in some sense two stochastic processes say X and Y, are the "same". To this end the notion of indistinguishability is needed.

Definition 1.3.9 *Two stochastic processes X and Y are modifications of each other if $P\{X_t = Y_t\} = 1$, $(\forall t \geq 0)$. Two processes X and Y are indistinguishable if $P\{X_t = Y_t, \forall t \geq 0\} = 1$*

The next proposition is very useful.

Proposition 1.3.10 *Let X and Y two stochastic processes, with X a modification of Y. If X and Y have right continuous paths a.s., then X and Y are indistinguishable.*

Fundamental role in stochastic analysis and risk theory has the so called *martingale* property. An extension of the notion of martingale is the notion of local martingale. It arises from the need to study a process in a local mode.

Definition 1.3.11 *An $F-$martingale (resp. F-supermartingale, F-submartingale) $M = \{M_t : t \geq 0\}$ is a real valued process such that*

(I) M_t *is* F_t-*measurable.*

(II) $E(|M_t|) < \infty$ *for all* $t \geq 0$.

(III) If $s \leq t$*, then* $E(M_t|\mathcal{F}_s) = M_s$ P*–a.s. (resp.*$E(M_t|\mathcal{F}_s) \leq M_s$*, resp.* $E(M_t|\mathcal{F}_s) \geq M_s$*)*

Definition 1.3.12 *An adapted, càdlàg process* X *is a local martingale if there exists a sequence of increasing stopping times,* $\{T_n\}_{n\in N}$*, with* $\lim_{n\longrightarrow\infty} T_n = \infty$ *a.s. such that* $X_{t\wedge T_n}$ *is a martingale for each* $n \in N$*. Such a sequence* $\{T_n\}_{n\in N}$ *of stopping times is called a fundamental sequence.*

We note here that a (local) martingale stopped at a stopping time is still a (local) martingale. Useful results on martingales there are in many texts (see for example Neveu [136], Wall [181]).

One of the first martingales that have been studied and a process that plays a fundamental role in the stochastic analysis is the Brownian motion.

Definition 1.3.13 *Consider the filtered probability space* (Ω, F, P)*. An* F*–adapted process* $B = \{B_t : t \geq 0\}$ *taking values in* $\mathbb{R}^n$ *is called an n-dimensional Brownian motion if*

(I) Increments are independent of the past, that is for $0 \leq s < t < \infty$*,* $B_t - B_s$ *is independent of* F_s

(II) For $0 \leq s < t$*,* $B_t - B_s$ *is a Gaussian random variable with mean zero and covariance matrix* $(t-s)C$*, for a given, non-negative definite matrix* C*.*

A vast literature exists on Brownian motion and stochastic integration. See e.g. Karatzas and Schreve [109], Revuz and Yor [156], Yeh [189], Durrett [53], and Hida [90]).

The notion of stochastic integral which gives meaning to the so called Stochastic Differential Equations has a fundamental role in stochastic analysis (see Gikhman [78], Freedman [61], Ikeda [99], Gard [68], Sobczyk [173], Bass [17],[18], Cherny [38]). The classical definition of the integral is given with the use of an increasing process.

Definition 1.3.14 *Let* $A = \{A_t : t \geq 0\}$ *be a càdlàg process. The process* A *is an increasing process if the paths of* $A : t \longrightarrow A_t(\omega)$ *are non-decreasing for almost all* ω*. The process* A *is called a finite or bounded variation process (BV) if almost all of the paths of* A *are of finite variation on each compact interval of* $\mathbb{R}^+$*.*

Let A be an increasing process. It is well known from the theory of Lebesque-Stieljes integration that for a fixed ω the function $t \to A_t(\omega)$ induces a measure $\mu_A(\omega, ds)$ on R^+. For a bounded measurable process H the stochastic integral is defined by

$$\int_0^t H_s(\omega) dA_s(\omega) := \int_0^t H_s(\omega) \mu_A(\omega, ds)$$

If the process H is a.s. continuous then it holds

$$\int_0^t H_s dA_s = \lim_{n\to\infty} \sum_{t_k, t_{k+1} \in \pi_n} H_{s_k}(A_{t_{k+1}} - A_k)$$

where the convergence is a.s. and π_n is a sequence of partitions of $[0, t]$ with

$$\lim_{n\to\infty} \text{mesh}(\pi_n) = 0$$

where

$$\text{mesh}(\pi_n) := \sup_k |t_k - t_{k-1}|$$

and $t_k \leq s_k \leq t_{k+1}$. (see Protter [152]).

It soon became apparent that it would be very desirable to define the stochastic integral for other processes also. At first this goal was achieved using as integrator the Brownian motion, which is a.s. an infinite variation process. Later the stochastic integral was extended to have martingales, local martingales, and semimartingales as integrators.

Definition 1.3.15 *An adapted, càdlàg process X is a semimartingale if there exist processes M, A, with $M_0 = A_0 = 0$ such that*

$$X_t = X_0 + M_t + A_t \tag{1.3.2}$$

where M is a local martingale and A is a BV process.

In order to describe the stochastic integral with semimartingales as integrator we follow Protter[152] and we need the notions of stopping time σ-algebra, simple predictable process, and predictable process.

Definition 1.3.16 *Let T be a stopping time. The stopping time σ-algebra, denoted by F_T, is the smallest σ-algebra containing all adapted càdlàg processes sampled at T. That is*

$$\mathcal{F}_T := \sigma\{X_T; X \text{ is adapted càdlàg process}\}$$

Definition 1.3.17 *A process H is said to be simple predictable if H has a representation*

$$H_t = H_0 1_{\{0\}}(t) + \sum_{i=1}^{n} H_i 1_{(T_i, T_{i+1}]}(t) \tag{1.3.3}$$

where $0 = T_1 \leq \cdots \leq T_{n+1} < \infty$ is a finite sequence of stopping times, $H_i \in F_{T_i}$ with $|H_i| < \infty$ a.s., $0 \leq i \leq n$. We denote the collection of simple predictable processes with S.

It is well known that the space S is dense in L under the topology induced by the uniform convergence on compacts in probability.

Definition 1.3.18 *The predictable σ-algebra on $R^+ \times \Omega$, denoted by P, is he smallest σ-algebra making all càglàd processes measurable. We will denote with bP the bounded processes that are P measurable.*

Definition 1.3.19 *For a simple process $H \in S$ with representation as in (1.3.3) and X a càdlàg process the stochastic integral is defied to be the linear mapping $J_X : S \to D$ given by*

$$J_X(H) := \int_0^t H dX := H_0 X_0(t) + \sum_{i=1}^{n} H_i (X_t^{T_{i+1}} - X_t^{T_i}) \tag{1.3.4}$$

For a semimartingale X the mapping $J_X : S \to D$ is continuous when both spaces have the topology induced by the uniform convergence on compacts in probability. Thus the continuous linear mapping $J_X : S \to D$ can be extended to $J_X : L \to D$.

The stochastic integral with semimartingales as integrator extends to all predictable and locally bounded integrands, in a unique way. In general, the stochastic integral can be defined even in cases where the predictable process H is not locally bounded.

One of the first papers concerning the use of semimartingales in stochastic integration is the paper of Meyer [132]. For relevant references one can also see Dellacherie [45], Lenglart [122], Bichteler [23], Protter [153].

A very important special class of semimartingales is the so called Lévy processes. They have been used extensively in the context of Risk theory and a lot of papers have been written on them. For some recent and very interesting papers one can see for example in: Vandaele [180], Kluppelberg [115], Albin [3], Hainaut [87], Kassberger [111], Kostadinova [116], Jang [105], Morales [133], Riesner [157], Bollerslev [26], Xing [185], Ngwira [138], Irgens [100], Nakano [135], Zhang [193], Gerber [76].

Fundamental for the theory of semimartingales is the notion of quadratic covariation.

Definition 1.3.20 *Let X, Y semimartingales. The quadratic covariation of X, Y, denoted by $[X, Y] = \{[X, Y]_t : t \geq 0\}$, is defined to be the unique process that satisfies the following:*

(I)

$$[X, Y]_0 = X_0 Y_0 \tag{1.3.5}$$

(II)

$$\Delta[X, Y] = \Delta X \Delta Y \tag{1.3.6}$$

(III) If σ_n is a sequence of random partitions tending to the identity, then

$$[X, Y] = X_0 Y_0 + \lim_{n \to \infty} \sum_i \left(X^{T_{i+1}^n} - X^{T_i^n} \right) \left(Y^{T_{i+1}^n} - Y^{T_i^n} \right) \tag{1.3.7}$$

where convergence is uniformly on compacts in probability and σ_n is the sequence

$$0 = T_0^n \leq T_1^n \leq \cdots \leq T_i^n \leq \cdots \leq T_{k_n}^n$$

with T_i^n stopping times and where

$$\Delta X_t := X_t - X_{t-}$$

and

$$X_{t-} := \lim_{s \longrightarrow t} X_s \quad \textit{for } s < t.$$

Definition 1.3.21 *The **quadratic variation** of a semimartingale X, denoted by $[X, X] = \{[X, X]_t : t \geq 0\}$, is defined to be the unique càdlàg, increasing, adapted process that satisfies the following*

(I)

$$[X, X]_0 = X_0^2 \tag{1.3.8}$$

(II)

$$\Delta[X, X] = (\Delta X)^2 \tag{1.3.9}$$

(III) If σ_n is a sequence of random partitions tending to the identity, then

$$[X, X] = X_0^2 + \lim_{n \to \infty} \sum_i \left(X^{T_{i+1}^n} - X^{T_i^n} \right)^2 \tag{1.3.10}$$

where convergence is uniformly on compacts in probability and σ_n is the sequence

$$0 = T_0^n \leq T_1^n \leq \cdots \leq T_i^n \leq \cdots \leq T_{k_n}^n$$

with T_i^n stopping times.

Because the quadratic covariation $[X,Y]$ of the semimartingales X,Y is of bounded variation we can distinguish it in continuous and discontinuous part. For semimartingales X, Y the process $[X,Y]^c$ denotes the path-by-path continuous part of $[X,Y]$. We can then write

$$[X,Y]_t = [X,Y]_t^c + \sum_{0\leq s\leq t} (\Delta X_s)(\Delta Y_s) \tag{1.3.11}$$

where $X_{0-} = 0$ and $Y_{0-} = 0$.

One of the most important results in stochastic analysis is the so called Itô formula.

Theorem 1.3.22 *(Itô formula) Let $X = (X^1, \ldots, X^n)$ be a n–tuple of semimartingales and let $f : R^n \longrightarrow R$ have continuous second order partial derivatives. Then $f(X)$ is a semimartingale and the following formula holds:*

$$\begin{aligned} & f(X_t) - F(X_0) \qquad (1.3.12)\\ = \; & \sum_{i=1}^{n} \int_{0+}^{t} \frac{\partial f}{\partial x_i}(X_{s-})dX_s^i + \frac{1}{2}\sum_{1\leq i,j\leq n}\int_{0+}^{t}\frac{\partial^2 f}{\partial x_i \partial x_j}(X_{s-})d[X^i,X^j]_s^c + \\ & + \sum_{0<s\leq t}\left(f(X_s) - f(X_{s-}) - \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(X_{s-})\Delta X_s^i\right) \end{aligned}$$

The use of Itô formula for semimartingales is also very useful in the study of jump diffusions or Lévy Processes (see Applebaum [11], Bertoin [21], Jacod [104], Oksendal [142], Protter [152]).

As we saw in the previous section the classical Cramer-Lundberg model can be approximated by a Brownian motion with drift. This is an example that shows the important role that the so called Itô diffusion processes have in risk theory.

Definition 1.3.23 *(Itô diffusions)A stochastic process $X = \{X_t : t \geq 0\}$ is an Itô diffusion if it is adapted and can be expressed as the sum of an integral with respect to Brownian motion $B = \{B_t : t \geq 0\}$ and an integral with respect to time, that is*

$$X_t = X_0 + \int_0^t \sigma_s dB_s + \int_0^t \mu_s ds \tag{1.3.13}$$

where σ is a predictable B-integrable process and μ is predictable and (Lebesgue) integrable, that is

$$\int_0^t (\sigma_s^2 + |\mu_s|)ds < \infty$$

for each time $t > 0$.

In this dissertation we will be mostly concerned with stochastic differential equations (SDE) of the form

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t \tag{1.3.14}$$

where $X_t(\omega) : [0, \infty) \times \Omega \to \mathbb{R}^n$, $\mu(\cdot, \cdot) : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma(\cdot, \cdot) : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and B_t is an m–dimensional Brownian motion. The solution of the SDE (1.3.14) is an Itô diffusion. The function $\mu(\cdot, \cdot)$ is called the drift coefficient and the function $\sigma(\cdot, \cdot)$ the diffusion or volatility coefficient. If $\mu(t, X_t) = \mu(X_t)$ and $\sigma(t, X_t) = \sigma(X_t)$ then the Itô diffusion is called time-homogeneous.

In this dissertation we assume that the drift coefficient $\mu(\cdot, \cdot)$ and the diffusion coefficient $\sigma(\cdot, \cdot)$ always satisfy the following two conditions.

Condition 1.3.24 ***(Linear growth condition).*** *For every $x \in \mathbb{R}^n$ and every $t \in [0, \infty)$ it holds that*

$$\|\mu(t, x)\| + \|\sigma(t, x)\| \leq C(1 + \|x\|)$$

for some constant C, where $\|\mu(t, x)\|$ is the Euclidean norm and

$$\|\sigma(t, x)\| := \left(\sum_{i=1}^{n}\sum_{j=1}^{m}\sigma_{ij}^2(t, x)\right)^{1/2}$$

Condition 1.3.25 ***(Lipschitz condition)*** *For every $x, y \in \mathbb{R}^n$ and every $t \in [0, \infty)$ it holds that*

$$\|\mu(t, x) - \mu(t, y)\| + \|\sigma(t, x) - \sigma(t, y)\| \leq D\|x - y\|$$

for some constant D.

Under the above two conditions there is an unique stochastic process that satisfies the differential equation (1.3.14) (see Oksendal[141], Karatzas and Shreve [109]).

For the theory on diffusion processes one can see many texts, as for example the texts of Itô [103], Rogers [158], Stroock [175].

Next we will describe a very useful property of the Itô diffusions. This is the so called Markov property and we will define it with the aid of the canonical space and the right shift operator.

Definition 1.3.26 *The canonical space is the space $C[0, \infty)$, the set of all continuous functions $\omega : [0, \infty) \to \mathbb{R}$, with metric*

$$d(\omega_1, \omega_2) := \sum_{n=1}^{\infty}\frac{1}{2^n}\max_{0\leq t\leq n}(|\omega_1(t) - \omega_2(t)| \wedge 1)$$

where $a \wedge b := \min\{a, b\}$.

The canonical space under the metric d is a complete, separable metric space.

Definition 1.3.27 *(Right shift operator) The right shift operator* θ_t *on the canonical space* Ω *is given by* $\theta_t : \Omega \longrightarrow \Omega$ *and* $\theta_t(\omega) := \omega(t + .)$ *for all* $t \geq 0$.

Theorem 1.3.28 *(Markov property) Let* $f(\cdot)$ *be a bounded Borel function from* $\mathbb{R}^n$ *to* $\mathbb{R}$. *We say that a process* $X = \{X_t : t \geq 0\}$ *has the Markov property, if for* $t, h \geq 0$ *holds*

$$E^x[\theta_h f(X_t)|\mathcal{F}_t](\omega) = E^{X_t(\omega)}(f(X_h)) \qquad P^x - a.s. \tag{1.3.15}$$

where E^x *denotes the expectation with respect to the probability measure* P^x *which gives the distribution of the process* $\{X_t : t \geq 0\}$ *assuming* $X_0 = x$ *and* $E^{X_t(\omega)}(f(X_h))$ *means the function* $E^y(f(X_h))$ *evaluated at* $y = X_t(\omega)$.

A time-homogeneous Itô diffusion satisfies the Markov property (see Oksendal [141]). Generally speaking we say that a stochastic process has the Markov property if the evolution of the stochastic process after a deterministic time t does not depend on the evolution before t, given the value of the process at time t. (i.e. the "future" and "past" of the process are *conditionally independent* of each other given the "present"). A stochastic process that has the Markov property is called Markov process. For an in–depth treatment of Markov processes and the Markov property see the texts of Dynkin [54], Freidlin [62], Blumenthal [25], Chung [41] and Kurtz [119].

In connection with the diffusion processes comes the notion of generator operator. The generator is very useful in the study of diffusion processes.

Definition 1.3.29 *(Generator) Let* $X = \{X_t : t \geq 0\}$ *be an Itô diffusion in* $\mathbb{R}^n$. *The (infinitesimal) generator* A_x *of* X *is the operator defined by*

$$\mathcal{A}_x f(x) := \lim_{t \downarrow 0} \frac{E[f(X_t)] - f(x)}{t} \tag{1.3.16}$$

with $x \in \mathbb{R}^n$. *We also denote by* D_A *the set of functions for which the limit exists for all* $x \in \mathbb{R}^n$.

This operator can be thought of as the generator of the Markov semigroup associated with a Markov process X. (For very interesting results on generator operators and the semigroup theory see Hille [91], Krein[117] and Pazy [147]).

In this dissertation we often work on the function space $C_b^2(\mathbb{R}^n)$.

Definition 1.3.30 *The space $C_b^2(\mathbb{R}^n)$ is the space of functions which are continuous, bounded and have continuous bounded derivatives up to second order.*

Another important space is the function space $C_0^2(\mathbb{R}^n)$.

Definition 1.3.31 *The space $C_0^2(\mathbb{R}^n)$ is the space of functions which are continuous, have compact support, and continuous derivatives up to second order.*

The next theorem plays a key role in the formulation of the de Finetti models with general barriers.

Theorem 1.3.32 *Let X be the Itô diffusion*

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t \tag{1.3.17}$$

If $f(\cdot) \in C_0^2(\mathbb{R}^n)$ then $f(\cdot) \in D_A$ and

$$\mathcal{A}f(x) = \sum_i \mu_i(x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_i} \tag{1.3.18}$$

where σ^T is the converse matrix of σ.

Also if $f(\cdot) \in C_b^2(\mathbb{R}^n)$ and the drift $\mu_x(\cdot)$ and volatility $\sigma_x(\cdot)$ coefficients satisfy the linear growth condition (1.3.24) and the Lipschitz continuity condition (1.3.25), then $f(\cdot) \in D_A$ and the above formula holds.

One may consider the de Finetti model as a process reflected at a boundary (case with one reflected barrier) or as a process reflected at two boundaries (case with two reflected barriers). From this point of view it is appropriate to close this first chapter with the following theorem which also is needed for the proof of uniqueness in the de Finetti model with general barriers.

Theorem 1.3.33 *(The Skorohod Theorem, see Karatzas and Shreve [109]). Let $X = \{X_t; 0 \leq t < \infty\}$ be a continuous stochastic process. There exist a unique continuous stochastic process $K = \{K_t; 0 \leq t < \infty\}$ such that a.s.*

$$Y_t := X_t + K_t \geq 0 \qquad 0 \leq t < \infty \tag{1.3.19}$$

$$K_0 = 0 \textit{ and } K_t \textit{ is } \text{nondecreasing} \tag{1.3.20}$$

$$\int_0^\infty 1_{\{Y_s > 0\}} dK_s = 0 \tag{1.3.21}$$

The process K_t is given by

$$K_t = 0 \bigvee^{t} (-X_s), \qquad 0 \le t < \infty. \tag{1.3.22}$$

where

$$\bigvee^{t} (-X_s) := \sup_{0 \le s \le t} (-X_s).$$

The Skorohod Theorem can be used for the construction of the so called reflected processes. For example let us suppose a process $X = \{X_t; 0 \le t < \infty\}$ with state space $(0, \infty)$. In order from X to construct a process Y reflected at some boundary $b > 0$, that is a process Y with state space $(0, b)$, we can use the Skorohod Theorem and construct the process Y from the relation (1.3.19) with

$$K_t = 0 \bigvee^{t} (X_s - b), \qquad 0 \le t < \infty$$

Chapter 2

de Finetti model with general barriers

2.1 Introduction

Starting this chapter we owe to mention that the realization of this chapter has became possible thanks to the papers of Gerber, H.U. and Shiu, E.S.W., on the subject of risk models with dividends. Our inspiration stems from their work. Specially we want to mention the papers ([71] , [72]) in which we find most of the ideas of this chapter. We found their approach on this subject most fruitful and gave us the proper set up, helping us in our effort of introducing a generalization on risk models with dividends.

In the literature there are a lot of studies on the de Finetti model with constant dividend barriers policies. However we feel that it would resemble more closely the "real" world and the random environment in which evolves an insurance company, if we suppose that the barriers are subject to random effects as well. One should allow for fluctuations in the barriers in order to reflect possible fluctuations in the accuracy of information.

We want to extend the de Finetti model in order to include general barriers policies. With this we mean that we would like to treat a diffusion process $\mathbf{b} = \{b_t; t \geq 0\}$ as an upper barrier. We assume that the dynamics of the process $\mathbf{b}$ are described by the stochastic differential equation (SDE)

$$db_t = \mu_b(b_t)dt + \sigma_b(b_t)dB_t^b \tag{2.1.1}$$

with $\mu_b(b_t)$ denoting the drift and $\sigma_b(b_t)$ the diffusion component, and with initial condition $b_0 = b$ a.s.. The process $B^b = \{B_t^b; t \geq 0\}$ is standard Brownian motion.

We also assume a reserves process that evolves according to a diffusion model, that is it has dynamics

$$dX_t = \mu_x(X_t)dt + \sigma_x(X_t)dB_t^x \tag{2.1.2}$$

where $\mu_x(X_t)$ denotes the drift and $\sigma_x(X_t)$ the diffusion component and with initial condition for the process $\{X_t; t \geq 0\}$, $x_0 = x$ a.s.. The process $B^x = \{B_t^x; t \geq 0\}$ is again standard Brownian motion.

Finally we would also like to treat the lower barrier as a diffusion process $\mathbf{a} = \{a_t; t \geq 0\}$. We assume that the dynamics of the process $\mathbf{a}$ are described by the stochastic differential equation (SDE)

$$da_t = \mu_a(a_t)dt + \sigma_a(a_t)dB_t^a \tag{2.1.3}$$

where $\mu_a(a_t)$ denotes the drift and $\sigma_a(a_t)$ the diffusion coefficient and with initial condition for the process $\{a_t; t \geq 0\}$, $a_0 = a$ a.s.. The process $B^a = \{B_t^a; t \geq 0\}$ is a standard Brownian motion.

The three Brownian motions driving the above SDE's are not assumed independent. Instead, they are correlated and we denote these correlations by ρ. Thus

$$\begin{aligned} \rho_{xa}dt &:= \quad d[B^x, B^a]_t, \\ \rho_{xb}dt &:= \quad d[B^x, B^b]_t, \\ \rho_{ab}dt &:= \quad d[B^a, B^b]_t. \end{aligned} \tag{2.1.4}$$

The process $\mathbf{b}$ will play the role of a reflecting barrier for the reserves process X. When the reserves process is above the level of the process $\mathbf{b}$, dividends are going to be paid to the shareholders. We would also like to treat the process $\mathbf{a}$ as a lower barrier. We will consider two scenaria concerning this lower barrier. In the first scenario the process $\mathbf{a}$ will play the role of an absorbing barrier (we call this model de Finetti with one reflecting barrier), that is the insurance company will be ruined when the surplus process reaches the level of the process $\mathbf{a}$. In the second scenario the process $\mathbf{a}$ will play the role of a reflecting barrier (we call this model de Finetti with two reflecting barriers), that is when the surplus process reaches the level of $\mathbf{a}$ the insurance company has the option of borrowing money and continuing its operation.

In order for the above model to make sense it is necessary that the sample paths of the lower barrier process, $\mathbf{a} = \{a_t; 0 \leq t < \infty\}$, are with probability 1 below those of the upper barrier process $\mathbf{b} = \{b_t; 0 \leq t < \infty\}$. Thus we need to impose the following condition.

Condition 2.1.1 *We assume that*

$$a_t < b_t, \quad 0 \leq t < \infty \ a.s.. \tag{2.1.5}$$

The process $(\mathbf{b}-\mathbf{a})^{-1} = \{(b_t - a_t)^{-1}; 0 \leq t < \infty\}$ will play a critical role in our model. To ensure that this process is bounded we impose the following

Condition 2.1.2 *Through this chapter we will always assume that there exists* $\varepsilon > 0$ *such that, w.p. 1*

$$a_t + \varepsilon < b_t \quad 0 \leq t < \infty. \tag{2.1.6}$$

Of course condition 2.1.2 implies the condition 2.1.1.

The condition 2.1.2 we pose includes itself in a more general context of stochastic analysis which manifest itself under the context Stochastic Comparison theorems. For relevant results on this subject one can see Anderson [10], N. Ikeda, S. Watanabe [98], Yamada [186], O'Brien [31], L.I. Galcuk and M.H.A. Davis [67], Hajek [85], Yan [187], Mao [126], Ferreyra [59], [60], Tudor [178], Borkar [29], Geiss [70], Kroger [118], S. Peng and X. Zhu[148],Yang [188], Ding [48].

In the next proposition we mention conditions under which the condition 2.1.2 holds (see Karatzas and Shreve [109]).

Proposition 2.1.3 *Consider two processes,* $\mathbf{a} = \{a_t; t \geq 0\}$ *with* $a_0 = a$ *a.s., and* $\mathbf{b} = \{b_t; t \geq 0\}$ *with* $b_0 = b$ *a.s., which satisfy the SDE (2.1.3) and (2.1.1) respectively, with drift and diffusion coefficients that satisfy the linear growth condition (1.3.24) and the Lipschitz condition (1.3.25). We assume that*

(I) The coefficients $\mu_a(x)$, $\sigma_a(x)$, $\mu_b(x)$, $\sigma_b(x)$ *are continuous, real functions on* $\mathbb{R}$.

(II) $P\{B_t^a = B_t^b, \forall t \geq 0\} = 1$.

(III) $\sigma_a(x) = \sigma_b(x)$, *for each* $x \in \mathbb{R}$.

(IV) $\mu_a(x) < \mu_b(x)$, *for each* $x \in \mathbb{R}$.

(V) $a + \varepsilon < b$, *for some* $\varepsilon > 0$.

Then the condition 2.1.2 holds.

We now proceed with the definition of the De Finetti models with general barriers. Certainly the general model should be reducible to the classical de Finetti model in the case we would like to consider constant barriers.

Definition 2.1.4 ***de Finetti model with one general reflecting barrier*** (b) ***and one absorbing barrier*** (a)***.*** *Given three continuous stochastic processes* $X = \{X_t; 0 \leq t < \infty\}$*,* $\mathbf{a} = \{a_t; 0 \leq t < \infty\}$ *and* $\mathbf{b} = \{b_t; 0 \leq t < \infty\}$ *we will call a pair* $(Z,\mathcal{U})$ *of continuous stochastic processes a de Finetti model with general reflecting–absorbing barriers corresponding to the process* $(X, \mathbf{b}, \mathbf{a})$ *if and only if this pair of processes satisfies*

$$Z_t := X_t - \mathcal{U}_t, \quad 0 \leq t < \infty \tag{2.1.7}$$

$$a_t \leq Z_t \leq b_t, \quad 0 \leq t < \infty \ a.s. \tag{2.1.8}$$

$$\mathcal{U}_0 = 0 \ \text{and} \ \mathcal{U}_t \ \text{is nondecreasing a.s.} \tag{2.1.9}$$

$$\int_0^t 1_{\{Z_s < b_s\}} d\mathcal{U}_s = 0, \quad 0 \leq t < \infty a.s. \tag{2.1.10}$$

In the above definition of the de Finetti model with one general reflecting barrier we interpret the process X as the risk process, the process $\mathbf{a}$ as the lower barrier which is an absorbing barrier, the process $\mathbf{b}$ as the upper barrier which is a reflecting barrier, the process $\mathcal{U}$ as the accumulated dividends and the process Z as the modified risk process that is the process Z equals the risk process X minus the dividends process $\mathcal{U}$.

Definition 2.1.5 ***de Finetti model with two general reflecting barriers the processes*** $\mathbf{b}$ ***and*** $\mathbf{a}$*. Given three continuous stochastic processes* $X = \{X_t; 0 \leq t < \infty\}$*,* $\mathbf{a} = \{a_t; 0 \leq t < \infty\}$ *and* $\mathbf{b} = \{b_t; 0 \leq t < \infty\}$ *we will call a triple* $(Z,\mathcal{U}^{(+)},\mathcal{U}^{(-)})$ *a de Finetti model with two general reflecting barriers corresponding to the process* $(X, \mathbf{b}, \mathbf{a})$ *if and only if this triple of processes satisfies*

$$Z_t := X_t - \mathcal{U}_t^{(+)} + \mathcal{U}_t^{(-)}, \quad 0 \leq t < \infty \tag{2.1.11}$$

$$a_t \leq Z_t \leq b_t, \quad 0 \leq t < \infty \ a.s. \tag{2.1.12}$$

$$\mathcal{U}_0^{(+)} = 0 \ \text{and} \ \mathcal{U}_t^{(+)} \text{is nondecreasing a.s} \tag{2.1.13}$$

$$\mathcal{U}_0^{(-)} = 0 \ \text{and} \ \mathcal{U}_t^{(-)} \ \text{is nondecreasing a.s.} \tag{2.1.14}$$

$$\int_0^t 1_{\{Z_s < b_s\}} d\mathcal{U}_s^{(+)} = 0, \quad 0 \leq t < \infty \ a.s. \tag{2.1.15}$$

$$\int_0^t 1_{\{Z_s > a_s\}} d\mathcal{U}_s^{(-)} = 0, \quad 0 \leq t < \infty \ a.s. \tag{2.1.16}$$

In the above definition of the de Finetti model with two general reflecting barriers we interpret the process X as the risk process, the process $\mathbf{a}$ as the lower reflecting barrier, the process $\mathbf{b}$ as the upper reflecting barrier, the process $\mathcal{U}^{(+)}$ as the accumulated dividends, the process $\mathcal{U}^{(-)}$ as the accumulated financing and the process Z as the modified risk process that is the process Z equals the risk process X minus the dividends process $\mathcal{U}^{(+)}$ and plus the financing process $\mathcal{U}^{(-)}$.

Now that we have defined the general de Finetti models we can formulate our questions more clearly. We are interested in studying the discounted dividends and the discounted financing of an insurance company which adopts a dividends policy according to a de Finetti model with general barriers. Moreover we suppose that the reserves of the insurance company evolve in an economic environment in which there is some interest rate which we denote by δ.

With regards to a dividends policy which follows a de Finetti model with one general reflecting barrier, it is appropriate to consider that there is the possibility that the insurance company goes bankrupt. For this reason we must consider the time of ruin for the insurance company which we denote by T and which depends on the initial state (x, a, b) of the process $(X, \mathbf{a}, \mathbf{b})$ and is defined by

$$T := T(x, a, b) := \inf\{t > 0 : Z_t = a_t\} \tag{2.1.17}$$

Taking into account the economic environment, we are primary interested in the discounted dividends, denoted by U which are depending on the initial state (x, a, b) of the process $(X, \mathbf{a}, \mathbf{b})$ and given by

$$U := U_T := U(x, a, b) := \int_0^T e^{-\delta s} d\mathcal{U}_s \tag{2.1.18}$$

Here the following notation remark is in order.

Remark 2.1.6 *Let $f(U, T)$ be a function of the discounted dividends $U = U(x, a, b)$ and the time of ruin $T = T(x, a, b)$. The expected value $E(f(U, T))$ will depend on the initial state (x, a, b). In order to express this dependence we will use the notation $E^{(x,a,b)}$, that is we define*

$$E^{(x,a,b)}(f(U, T)) := E(f(U(x, a, b), T(x, a, b))) \tag{2.1.19}$$

The main quantities we are going to study in the de Finetti model with one general reflecting barrier are

- The *Laplace transform of the joint distribution of the time of ruin and the discounted dividends*, denoted $N(x, a, b, \lambda_1, \lambda_2)$ and given by

$$N(x, a, b, \lambda_1, \lambda_2) := E^{(x,a,b)}\left(e^{-\lambda_1 T - \lambda_2 U}\right) \tag{2.1.20}$$

- The *Laplace transform of the discounted dividends*, denoted $K(x, a, b, \lambda)$ and given by

$$K(x, a, b, \lambda) := E^{(x,a,b)}(e^{-\lambda U}) \tag{2.1.21}$$

- The *Laplace transform of the time of ruin,* denoted $M(x, a, b, \lambda)$ and given by

$$M(x, a, b, \lambda) := E^{(x,a,b)}(e^{-\lambda T}) \tag{2.1.22}$$

- The *moments of the discounted dividends,* denoted $V(x, a, b; n)$ and given by

$$V(x, a, b; n) := E^{(x,a,b)}(U^n) \tag{2.1.23}$$

Turning now our attention to the de Finetti model with two general barriers and taking into account the economic environment, we focus on the discounted dividends, denoted by $U^{(+)}$ which depend on the initial state (x, a, b) of the process $(X, \mathbf{a}, \mathbf{b})$ and given by

$$U^{(+)} := U^{(+)}(x, a, b) := \lim_{t\to\infty} U_t^{(+)} := \lim_{t\to\infty} \int_0^t e^{-\delta s} d\mathcal{U}_s^{(+)} = \int_0^\infty e^{-\delta s} d\mathcal{U}_s^{(+)} \tag{2.1.24}$$

and to the discounted financing, denoted by $U^{(-)}$ which depends on the initial state (x, a, b) of the process $(X, \mathbf{a}, \mathbf{b})$ and given by

$$U^{(-)} := U^{(-)}(x, a, b) := \lim_{t\to\infty} U_t^{(-)} := \lim_{t\to\infty} \int_0^t e^{-\delta s} d\mathcal{U}_s^{(-)} = \int_0^\infty e^{-\delta s} d\mathcal{U}_s^{(-)} \tag{2.1.25}$$

where

$$U_t^{(+)} := \int_0^t e^{-\delta s} d\mathcal{U}_s^{(+)} \tag{2.1.26}$$

$$U_t^{(-)} := \int_0^t e^{-\delta s} d\mathcal{U}_s^{(-)} \tag{2.1.27}$$

are the discounted dividends and the discounted financing until time t.

Here it is appropriate to make the following remark concerning notation.

Remark 2.1.7 *The expected value $E(f(U^{(+)}, U^{(-)}))$ will depend on the initial state (x, a, b). In order to express this dependence we will use the notation $E^{(x,a,b)}$, that is we define*

$$E^{(x,a,b)}(f(U^{(+)}, U^{(-)})) := E(f(U^{(+)}(x, a, b), U^{(-)}(x, a, b))) \tag{2.1.28}$$

We proceed to define

- The *Laplace transform of the joint distribution of the discounted dividends and the discounted financing,* denoted $L(x, a, b, \lambda_1, \lambda_2)$ and given by

$$L(x, a, b, \lambda_1, \lambda_2) := E^{(x,a,b)}(e^{-\lambda_1 U^{(+)} - \lambda_2 U^{(-)}}) \tag{2.1.29}$$

- The *Laplace transforms of the discounted dividends and the discounted financing,* denoted $K^{(+)}(x,a,b,\lambda)$ and $K^{(-)}(x,a,b,\lambda)$ respectively and given by

$$K^{(\pm)}(x,a,b,\lambda) := E^{(x,a,b)}(e^{-\lambda U^{(\pm)}}) \tag{2.1.30}$$

- The *Moments of the discounted dividends and the discounted financing* denoted $V^{(+)}(x,a,b;n)$ and $V^{(-)}(x,a,b;n)$ respectively and given by

$$V^{(\pm)}(x,a,b;n) := E^{(x,a,b)}\left(\left(U^{(\pm)}\right)^n\right) \tag{2.1.31}$$

We will denote by $\mathcal{A}_{(x,a,b)}$ the generator of the process $\{(X_t,a_t,b_t);t\geq 0\}$. It is well known that the generator coincides with the differential operator $\mathcal{L}_{(x,a,b)}$ given by

$$\begin{aligned}\mathcal{L}_{(x,a,b)} \quad := \quad & \frac{1}{2}\sigma_x^2(x)\frac{\partial^2}{\partial x^2}+\frac{1}{2}\sigma_a^2(a)\frac{\partial^2}{\partial a^2}+\frac{1}{2}\sigma_b^2(b)\frac{\partial^2}{\partial b^2} \\ & +\sigma_x(x)\sigma_a(a)\rho_{xa}\frac{\partial^2}{\partial x\partial a}+\sigma_x(x)\sigma_b(b)\rho_{xb}\frac{\partial^2}{\partial x\partial b}+\sigma_a(a)\sigma_b(b)\rho_{ab}\frac{\partial^2}{\partial a\partial b} \\ & +\mu_x(x)\frac{\partial}{\partial x}+\mu_a(a)\frac{\partial}{\partial a}+\mu_b(b)\frac{\partial}{\partial b}\end{aligned}$$

where $f(\cdot)\in C_b^2(\mathbb{R}^3)$ and the drift coefficients $\mu_x(\cdot),\mu_a(\cdot),\mu_b(\cdot)$ and volatility coefficients $\sigma_x(\cdot)$, $\sigma_a(\cdot)$, $\sigma_b(\cdot)$ satisfy the linear growth condition (1.3.24) and the Lipschitz continuity condition (1.3.25).

Up to this point we have constructed the "environment" in which will evolve the general de Finetti models and we have clarified what we are going to study. Now it is time to describe the course we are going to follow in the next sections.

After the definition of the de Finetti model with general barriers the first question which naturally arises is the question of existence and uniqueness, that is, if there exist processes which satisfy the requirements we pose and, if they do, whether they are unique or not. We answer these questions in the affirmative in the following section. This enables us to proceed with the consideration of the more general de Finetti models and with the study of the quantities we defined in this introduction.

In section 2.3 we show that these models have a property, which we call "scale property", which turns out to be very useful in order to formulate our results. In section 2.4 we state results that will help us to derive the boundary conditions of the differential equations we are going to derive. In section 2.5 we derive some useful results of the generator operator. Finally in section 2.6 we apply the previous results in order to formulate differential equations the solution of which will give us the quantities defined in (2.1.20)-(2.1.23) and (2.1.29)-(2.1.31).

2.2 The de Finetti models with general barriers are well defined.

As we mentioned in the introduction of this chapter the first question we must address is whether the classical de Finetti model with constant barriers can be extended to a model having as barriers diffusion processes while preserving the desirable properties (1.2.1)-(1.2.5) and (1.2.6)-(1.2.12). We will show that this is the case and that, in fact, there exist unique processes which satisfy the requirements of the definitions of the de Finetti models with general barriers, as stated in the Definition 2.1.4 and in the Definition 2.1.5. The uniqueness of these processes is actually easier to prove than the existence and so we proceed by assuming existence and establishing uniqueness.

2.2.1 Uniqueness

In this section we will prove that if there are exists a pair of processes $(Z, \mathcal{U})$ which satisfy the defining conditions (2.1.7)-(2.1.10) of the de Finetti model with one general reflecting barrier then this pair is unique. Similarly, we will prove that if there exists a triple of processes $(Z, \mathcal{U}^{(+)}, \mathcal{U}^{(-)})$ which satisfy the defining conditions (2.1.11)-(2.1.16) of the de Finetti model with two general reflecting barriers then this triple is unique.

We consider first the de Finetti model with one general reflecting barrier. The main idea of the proof of Proposition 2.2.1 is that if there exists a process that satisfies the conditions (2.1.7)-(2.1.10) of the de Finetti model with one general reflecting barrier then this process satisfies all the conditions of the Skorohod Theorem 1.3.33. Because of this we can first conclude the expression (2.2.1) and then conclude uniqueness for the de Finetti model with one general reflecting barrier by the uniqueness guaranteed by the Skorohod Theorem.

Proposition 2.2.1 *The dividends process $\mathcal{U} = \{\mathcal{U}_t; 0 \leq t < \infty\}$ is unique for the de Finetti model with one general reflecting barrier and is given a.s. by*

$$\mathcal{U}_t = 0 \bigvee^{t} (X_s - b_s) \tag{2.2.1}$$

Proof. Consider a process $(Z, \mathcal{U})$ which satisfies the conditions (2.1.7)-(2.1.10) of the Definition 2.1.4.

From the condition (2.1.10) in Definition 2.1.4 we conclude that for every time $0 \leq t < \infty$ holds a.s.

$$0 = \int_0^t 1_{\{Z_s < b_x\}} d\mathcal{U}_s = \int_0^t 1_{\{0 < b_s - Z_s\}} d\mathcal{U}_s = \int_0^t 1_{\{0 < b_s - X_s + \mathcal{U}_s\}} d\mathcal{U}_s \tag{2.2.2}$$

Defining the stopping time

$$T_a := \inf\{t > 0; Z_t = a_t\}$$

and the process

$$Y_t := \begin{cases} b_t - X_t + \mathcal{U}_t, & t \in [0, T_a] \\ 0, & t > T_a \end{cases}$$

relation (2.2.2) can be written as

$$\int_0^\infty 1_{\{Y_s > 0\}} d\mathcal{U}_s = 0 \tag{2.2.3}$$

From the above relation (2.2.3) we conclude that the dividends process $\mathcal{U}$ satisfies the condition (1.3.21) of the Skorohod Theorem 1.3.33.

Also from the condition (2.1.9) of the Definition 2.1.4 the dividends process $\mathcal{U}$ satisfies the condition (1.3.20) of the Skorohod Theorem 1.3.33.

Finally from the condition (2.1.8) we deduce that for every time $0 \le t < \infty$ holds a.s.

$$Z_t \le b_t \Longrightarrow b_t - X_t + \mathcal{U}_t \ge 0 \Longrightarrow Y_r \ge 0$$

and so the dividends process $\mathcal{U}$ satisfies the condition (1.3.19) of the Skorohod Theorem 1.3.33. Since the dividends process $\mathcal{U}$ corresponding to the process

$$\{b_t - X_t; \quad 0 \le t < \infty\}$$

satisfies all the requirements of the Skorohod Theorem 1.3.33 we conclude that the dividends process $\mathcal{U} = \{\mathcal{U}_t; \quad 0 \le t < \infty\}$ is unique for the process

$$\{b_t - X_t; \quad 0 \le t < \infty\}$$

and from the Skorohod Theorem 1.3.33 we conclude that the dividends process $\mathcal{U}$ is unique and given a.s. by

$$\mathcal{U}_t = 0 \bigvee^t (X_s - b_s)$$

■

Next we consider the de Finetti model with two general reflecting barriers. The main idea of the proof of proposition 2.2.2 is that if for a given financing process $\mathcal{U}^{(-)}$ there exists a dividends process $\mathcal{U}^{(+)}$ that satisfies the conditions of Definition 2.1.5, then this process satisfies all the conditions of the Skorohod Theorem 1.3.33. Because of this we can first conclude the expression (2.2.4) and second by the uniqueness result of the Skorohod Theorem 1.3.33 we conclude that for each financing process $\mathcal{U}^{(-)}$ there is a unique dividends process $\mathcal{U}^{(+)}$ such that the process $(\mathcal{U}^{(+)}, \mathcal{U}^{(-)})$ satisfy the defining conditions (2.1.11)-(2.1.16) we pose on the de Finetti model with two general reflecting barriers. After the proof of uniqueness of the dividends process $\mathcal{U}^{(+)}$ for each financing process $\mathcal{U}^{(-)}$ we repeat the same arguments to prove that for each dividends process $\mathcal{U}^{(+)}$ there is a unique financing process $\mathcal{U}^{(-)}$ that satisfies the conditions of definition 2.1.5.

Proposition 2.2.2 *The pair of processes* $(\mathcal{U}^{(+)},\mathcal{U}^{(-)}) = \{(\mathcal{U}_t^{(+)},\mathcal{U}_t^{(-)}), 0 \leq t < \infty\}$ *of the de Finetti model with two general reflecting barriers is unique and is given a.s. by*

$$\mathcal{U}_t^{(+)} = 0 \bigvee^{t} \left(X_s + \mathcal{U}_s^{(-)} - b_s \right), \quad 0 \leq t < \infty \tag{2.2.4}$$

$$\mathcal{U}_t^{(-)} = 0 \bigvee^{t} \left(\mathcal{U}_s^{(+)} - X_s + a_s \right), \quad 0 \leq t < \infty \tag{2.2.5}$$

Proof. We fix a financing process $\mathcal{U}^{(-)}$ and we consider a triple of processes $(Z,\mathcal{U}^{(+)},\mathcal{U}^{(-)})$ which satisfy the defining conditions (2.1.11)-(2.1.16).

From condition (2.1.15) we conclude that for every time $0 \leq t < \infty$ holds a.s.

$$0 = \int_0^t 1_{\{Z_s < b_x\}} d\mathcal{U}_s^{(+)} = \int_0^t 1_{\{b_s - Z_s > 0\}} d\mathcal{U}_s^{(+)} \tag{2.2.6}$$

If we define for every time $0 \leq t < \infty$ the process

$$Y_t^{(+)} := b_t - Z_t = b_t - X_t + \mathcal{U}_t^{(+)} - \mathcal{U}_t^{(-)}$$

then relation (2.2.6) can be written as

$$\int_0^t 1_{\{Y_s^{(+)} > 0\}} d\mathcal{U}_s^{(+)} = 0, \text{ for each } t \in [0, \infty).$$

Therefore taking limits

$$\int_0^\infty 1_{\{Y_s^{(+)} > 0\}} d\mathcal{U}_s^{(+)} = 0. \tag{2.2.7}$$

From the above relation (2.2.7) we conclude that the dividends process $\mathcal{U}^{(+)} = \{\mathcal{U}_t^{(+)}; 0 \leq t < \infty\}$ satisfies the condition (1.3.21) of the Skorohod Theorem 1.3.33.

Also from the condition (2.1.13) the dividends process $\mathcal{U}^{(+)}$ satisfies the condition (1.3.20) of the Skorohod Theorem 1.3.33.

Finally from condition (2.1.12) we have that for every time $0 \leq t < \infty$

$$Z_t \leq b_t \Longrightarrow b_t - X_t + \mathcal{U}_t^{(+)} - \mathcal{U}_t^{(-)} \geq 0 \Longrightarrow Y_t^{(+)} \geq 0 \text{ a.s.}$$

and so the dividends process $\mathcal{U}^{(+)}$ satisfies the condition (1.3.19) of the Skorohod Theorem 1.3.33.

Since the dividends process $\mathcal{U}^{(+)} = \{\mathcal{U}_t^{(+)}; 0 \leq t < \infty\}$ corresponding to the process

$$\{b_t - X_t - \mathcal{U}_t^{(-)} \; ; \; 0 \leq t < \infty\}$$

satisfies all the requirements of the Skorohod Theorem 1.3.33 we conclude that the dividends process $\mathcal{U}^{(+)} = \{\mathcal{U}_t^{(+)}; 0 \leq t < \infty\}$ is unique for the process

$$\{b_t - X_t - \mathcal{U}_t^{(-)} \; ; \; 0 \leq t < \infty\}$$

and is given a.s. by

$$\mathcal{U}_t^{(+)} = 0 \bigvee^{t} \left(X_s + \mathcal{U}_s^{(-)} - b_s \right), \; 0 \le t < \infty$$

From the above we see that if there is another dividends process which together with the financing process $\mathcal{U}^{(-)}$ also satisfies the requirements of the de Finetti model then if must also satisfy the requirements of the Skorohod Theorem 1.3.33 for the same process $\{b_t - X_t - \mathcal{U}_t^{(-)}, 0 \le t < \infty\}$ and this is a contradiction. So we conclude that the dividends process $\mathcal{U}^{(+)}$ is unique for the financing process $\mathcal{U}^{(-)}$ and is given by (2.2.4). Finally we conclude that to a fixed financing process $\mathcal{U}^{(-)}$ corresponds a unique triple of processes $\left(Z,\mathcal{U}^{(+)},\mathcal{U}^{(-)}\right)$ which satisfy the defining conditions (2.1.11)-(2.1.16).

We turn now our attention to the financing process $\mathcal{U}^{(-)}$ and we will prove that for each dividends process $\mathcal{U}^{(+)}$ corresponds a unique financing process $\mathcal{U}^{(-)}$. In order to prove this we fix a dividends process $\mathcal{U}^{(+)}$ and we consider a triple of processes $(Z,\mathcal{U}^{(+)}, \mathcal{U}^{(-)})$ which satisfy the defining conditions (2.1.11)-(2.1.16).

From the condition (2.1.16) we conclude that for every time $0 \le t < \infty$ a.s. holds

$$0 = \int_0^t 1_{\{Z_s > a_x\}} d\mathcal{U}_s^{(-)} = \int_0^t 1_{\{Z_s - a_s > 0\}} d\mathcal{U}_s^{(-)} \tag{2.2.8}$$

If we define the process

$$Y_t^{(-)} := Z_t - a_t = X_t - \mathcal{U}_t^{(+)} + \mathcal{U}_t^{(-)} - a_t, \;\; 0 \le t < \infty$$

then relation (2.2.8) can be written as

$$\int_0^t 1_{\{Y_s^{(-)} > 0\}} d\mathcal{U}_s^{(-)} = 0, \;\; \text{for each } 0 \le t < \infty$$

So taking limit as $t \to \infty$

$$\int_0^\infty 1_{\{Y_s^{(-)} > 0\}} d\mathcal{U}_s^{(-)} = 0 \tag{2.2.9}$$

From relation (2.2.9) we conclude that the financing process $\mathcal{U}^{(-)} = \{\mathcal{U}_t^{(-)}; 0 \le t < \infty\}$ satisfies the condition (1.3.21) of the Skorohod Theorem 1.3.33.

Also from condition (2.1.14) the financing process $\mathcal{U}^{(-)} = \{\mathcal{U}_t^{(-)}; 0 \le t < \infty\}$ satisfies the condition (1.3.20) of the Skorohod Theorem 1.3.33.

Finally from the condition (2.1.12) we conclude that for every time $0 \le t < \infty$ a.s. holds

$$Z_t \ge a_t \Longrightarrow X_t - \mathcal{U}_t^{(+)} + \mathcal{U}_t^{(-)} - a_t \ge 0 \Longrightarrow Y_t^{(-)} \ge 0$$

and so the financing process $\mathcal{U}^{(-)} = \{\mathcal{U}_t^{(-)}; 0 \le t < \infty\}$ satisfies the condition (1.3.19) of the Skorohod Theorem 1.3.33.

Since the financing process $\mathcal{U}^{(-)} = \{\mathcal{U}_t^{(-)}; 0 \le t < \infty\}$ corresponding to the process

$$\{X_t - \mathcal{U}_t^{(+)} - a_t,\ \ 0 \le t < \infty\}$$

satisfies all the requirements of the Skorohod Theorem 1.3.33, we conclude that the financing process $\mathcal{U}^{(-)}$ is unique for the process

$$\{X_t - \mathcal{U}_t^{(+)} - a_t,\ \ 0 \le t < \infty\}$$

and is given a.s. by

$$\mathcal{U}_t^{(-)} = 0 \bigvee^{t} \left(\mathcal{U}_s^{(+)} - X_s + a_s\right),\ \ 0 \le t < \infty.$$

From the above we see that if there is another financing process which together with the dividends process $\mathcal{U}^{(+)}$ also satisfies the requirements of the de Finetti model with two general reflecting barriers then it must also satisfy the requirements of the Skorohod Theorem 1.3.33 for the same process $\{X_t - \mathcal{U}_t^{(+)} - a_t,\ 0 \le t < \infty\}$ and this is contradiction. So we conclude that the financing process $\mathcal{U}^{(-)}$ is unique for the dividends process $\mathcal{U}^{(+)}$ and is given by (2.2.4). Finally we conclude that to a fixed dividends process $\mathcal{U}^{(+)}$ corresponds an unique triple of processes $(Z, \mathcal{U}^{(+)}, \mathcal{U}^{(-)})$ which satisfy the defining conditions (2.1.11)-(2.1.16) we pose on the de Finetti model with two general reflecting barriers.

Let us suppose now that we have two triples of processes $(Z, \mathcal{U}^{(+)}, \mathcal{U}^{(-)})$ and $(\widetilde{Z}, \widetilde{\mathcal{U}}^{(+)}, \widetilde{\mathcal{U}}^{(-)})$ which satisfy the defining conditions (2.1.11)-(2.1.16). We will show that it holds

$$P\{(Z_t, \mathcal{U}_t^{(+)}, \mathcal{U}_t^{(-)}) = (\widetilde{Z_t}, \widetilde{\mathcal{U}_t}^{(+)}, \widetilde{\mathcal{U}_t}^{(-)}),\ \text{for each}\ \ 0 \le t < \infty\} = 1 \tag{2.2.10}$$

Let us suppose the contrary. Then because $\mathcal{U}_0^{(-)} = \widetilde{\mathcal{U}_0}^{(-)} = 0$ we can consider the first point in time in which these processes are different, that is we consider the stopping time τ defined by

$$\tau := \inf\{t > 0;\ \mathcal{U}_t^{(-)} > \widetilde{\mathcal{U}_t}^{(-)}\ \text{or}\ \mathcal{U}_t^{(-)} < \widetilde{\mathcal{U}_t}^{(-)}\} \tag{2.2.11}$$

We will show that

$$P\{\tau < \infty\} = 0 \tag{2.2.12}$$

By what we have found so far for the processes $(Z, \mathcal{U}^{(+)}, \mathcal{U}^{(-)})$ and $(\widetilde{Z}, \widetilde{\mathcal{U}}^{(+)}, \widetilde{\mathcal{U}}^{(-)})$ it holds that

$$\mathcal{U}_t^{(+)} = 0 \bigvee^{t} \left(X_s + \mathcal{U}_s^{(-)} - b_s\right),\ \ 0 \le t < \infty \tag{2.2.13}$$

$$\mathcal{U}_t^{(-)} = 0 \bigvee^{t} \left(\mathcal{U}_s^{(+)} - X_s + a_s\right),\ \ 0 \le t < \infty \tag{2.2.14}$$

$$\widetilde{\mathcal{U}}_t^{(+)} = 0 \bigvee^{t} \left(X_s + \widetilde{\mathcal{U}}_s^{(-)} - b_s\right),\ \ 0 \le t < \infty \tag{2.2.15}$$

$$\widetilde{\mathcal{U}}_t^{(-)} = 0 \bigvee^{t} \left(\widetilde{\mathcal{U}}_s^{(+)} - X_s + a_s\right),\ \ 0 \le t < \infty \tag{2.2.16}$$

On the event $\{\tau < \infty\}$ it holds

$$1_{[0,\tau)}(t)\mathcal{U}_t^{(-)} = 1_{[0,\tau)}(t)\widetilde{\mathcal{U}_t}^{(-)} \tag{2.2.17}$$

From the above relation (2.2.17) and the relations (2.2.13),(2.2.15) we conclude that also

$$1_{[0,\tau)}(t)\mathcal{U}_t^{(+)} = 1_{[0,\tau)}(t)\widetilde{\mathcal{U}_t}^{(+)} \tag{2.2.18}$$

By relation (2.2.17) we conclude that on the event $\{\mathcal{U}_\tau^{(-)} > \widetilde{\mathcal{U}_\tau}^{(-)}\}$ we must have

$$d\mathcal{U}_\tau^{(-)} > 0 \tag{2.2.19}$$

which implies that

$$P\{Z_\tau = a_\tau\} = 1 \tag{2.2.20}$$

because by the defining property (2.1.16) the process $\mathcal{U}^{(-)}$ is flat outside of the set $\{Z_t = a_t\}$.

On the other hand by (2.2.18) and (2.2.14),(2.2.16) we conclude that

$$d\mathcal{U}_\tau^{(+)} > 0 \tag{2.2.21}$$

which implies that

$$P\{Z_\tau = b_\tau\} = 1 \tag{2.2.22}$$

because by the defining property (2.1.15) the process $\mathcal{U}^{(+)}$ is flat outside the set $\{Z_t = b_t\}$. That is we have arrived at a contradiction and we conclude that

$$P\{\mathcal{U}_\tau^{(-)} > \widetilde{\mathcal{U}_\tau}^{(-)}\} = 0 \tag{2.2.23}$$

Working similarly we can also conclude that

$$P\{\mathcal{U}_\tau^{(-)} < \widetilde{\mathcal{U}_\tau}^{(-)}\} = 0 \tag{2.2.24}$$

and this finishes the proof. ■

Now that we have proved the uniqueness we turn our attention to the existence of the de Finetti model with general barriers.

2.2.2 Existence of the de Finetti model with general barriers.

In this subsection we prove the *existence* of the de Finetti model with general barriers. We start by first considering the de Finetti model with two general reflecting barriers, denoted $(Z,\mathcal{U}^{(+)},\mathcal{U}^{(-)})$ with defining properties (2.1.11)-(2.1.16).

Proposition 2.2.3 *There exists a de Finetti model with two general reflecting barriers,* $(Z,\mathcal{U}^{(+)},\mathcal{U}^{(-)})$ *with the properties (2.1.11)-(2.1.16).*

Proof. The idea behind the proof is the construction of the *De Finetti model with two general reflecting barriers* $(Z,\mathcal{U}^{(+)},\mathcal{U}^{(-)})$ as *a limiting process* of some suitable sequence of stochastic processes. After that, with the aid of the sequence of stochastic processes we will show that the limiting process $(Z,\mathcal{U}^{(+)},\mathcal{U}^{(-)})$ satisfies the defining *properties* (2.1.11)-(2.1.16) of the *De Finetti model with two general reflecting barriers.*

We split up the proof into several steps.

Step 1. (Construction of the process $(Z,\mathcal{U}^{(+)},\mathcal{U}^{(-)})$ as a limit of some suitable sequence of processes)

Step 1.(I) We *define* the families of processes $u^n = \{u^n_t; 0 \le t < \infty\}$, $l^n = \{l^n_t; 0 \le t < \infty\}$ with $n = 0, 1, 2, 3, ...$by:

$$u^0_t := 0 \quad , \quad t \ge 0 \tag{2.2.25}$$

$$l^0_t := 0 \quad , \quad t \ge 0 \tag{2.2.26}$$

$$u^n_t := 0 \bigvee^t \left(X_s + l^{n-1}_s - b_s\right) \quad , \quad 0 \le t < \infty \tag{2.2.27}$$

$$l^n_t := 0 \bigvee^t \left(u^{n-1}_s - X_s + a_s\right) \quad , \quad 0 \le t < \infty \tag{2.2.28}$$

Observe that the sequences of processes $\{u^n\}$ and $\{l^n\}$ are sequences of continuous processes and so in the following steps of the proof and when it is needed, in order to prove that a relation about these processes holds with probability one, it is enough to prove the relation for a *fixed* time $t \ge 0$, because if we prove that the relation holds for a *fixed* time $t \ge 0$ outside a set N_t with $P(N_t) = 0$ we can consider the set

$$N = \bigcup_{t \in Q \cap [0,\infty)} N_t$$

with $P(N) = 0$ and using the continuity of the processes u^n and l^n we can take the limit through the rationals $t \in Q \cap [0, \infty)$ and show that the relation holds for each $t \ge 0$ *a.s.*

Step 1.(II) *We will show that* the families of the processes $\{u^n\}$ and $\{l^n\}$ are **increasing** in n *for each fixed* time t. We will prove this by induction.

Let time $t \geq 0$ be fixed.

We *observe* that *a.s.*

$$u_t^1 = 0 \bigvee^t \left(X_s + l_s^0 - b_s\right) = 0 \bigvee^t (X_s + 0 - b_s) \geq 0 = u_t^0$$

and

$$l_t^1 = 0 \bigvee^t (u_s^0 - X_s + a_s) = 0 \bigvee^t (0 - X_s + a_s) \geq 0 = l_t^0$$

We *assume that* it holds

$$\begin{aligned} u_t^k &\geq u_t^{k-1} \\ l_t^k &\geq l_t^{k-1} \end{aligned}$$

for $k = 1, 2, ..., n$.

We conclude that *a.s.*

$$u_t^{n+1} = 0 \bigvee^t (X_s + l_s^n - b_s) \geq 0 \bigvee^t (X_s + l_s^{n-1} - b_s) = u_t^n$$

and

$$l_t^{n+1} = 0 \bigvee^t (u_s^n - X_s + a_s) \geq 0 \bigvee^t (u_s^{n-1} - X_s + a_s) = l_t^n$$

By the above relations we conclude that the families of the processes $\{u^n\}$ and $\{l^n\}$ are **increasing** in n *for each fix* time t.

Step 1.(III) We define the *stopping times*

$$\sigma_1 := \inf\{t > 0 : l_t^1 > l_t^0\} \tag{2.2.29}$$

$$\tau_n := \inf\{t > \sigma_n : u_t^n > u_t^{n-1}\} \quad , n = 1, 2, 3, .. \tag{2.2.30}$$

$$\sigma_{n+1} := \inf\{t > \tau_n : l_t^{n+1} > l_t^n\} \quad , n = 1, 2, 3, .. \tag{2.2.31}$$

The above are indeed stopping times. To see this let us suppose that it is given a *filtration* $F = \{\mathcal{F}_t : t \geq 0\}$ in which the process $\{(X_t, b_t, a_t) \ , 0 \leq t < \infty\}$ is adapted and with Q denoting the set of rational numbers we first observe that the processes $\{u^n\}$ and $\{l^n\}$ are adapted. Indeed let $\mathcal{O}$ be an open set of the $[0,\infty)$ then we have

$$\left\{\ u_t^1 \in \mathcal{O}\right\} = \bigcup_{\substack{0\leq s\leq t\\ s\in Q}} \{\max(0, X_s - b_s) \in \mathcal{O}\} \in \mathcal{F}_t$$

because $\{\max(0, X_s - b_s) \in \mathcal{O}\} \in \mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$.

Similarly

$$\left\{\ l_t^1 \in \mathcal{O}\right\} = \bigcup_{\substack{0\leq s\leq t\\ s\in Q}} \{\max(0, -X_s + a_s) \in \mathcal{O}\} \in \mathcal{F}_t$$

because $\{\max(0, -X_s + a_s) \in \mathcal{O}\} \in \mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$.

Suppose now that the processes $\{u_t^n\ ,\ 0 \leq t < \infty\}$ and $\{l_t^n\ ,\ 0 \leq t < \infty\}$ are adapted. Then we have

$$\left\{\ u_t^{n+1} \in \mathcal{O}\right\} = \bigcup_{\substack{0\leq s\leq t\\ s\in Q}} \{\max(0, (X_s + l_s^n - b_s)) \in \mathcal{O}\} \in \mathcal{F}_t$$

because $\{\max(0, (X_s + l_s^n - b_s)) \in \mathcal{O}\} \in \mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$.

Similarly we have

$$\left\{\ l_t^{n+1} \in \mathcal{O}\right\} = \bigcup_{\substack{0\leq s\leq t\\ s\in Q}} \{\max(0, (u_s^n - X_s + a_s)) \in \mathcal{O}\} \in \mathcal{F}_t$$

because $\{\max(0, (u_s^n - X_s + a_s)) \in \mathcal{O}\} \in \mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$. Thus we have proved that the processes $\{u^n\}$ and $\{l^n\}$ are adapted.

Next and taking into account the continuity of the process $l_t^1 := 0 \bigvee^t (-X_s + a_s)$ we conclude

$$\{\sigma_1 \leq t\} = \left\{l_t^1 > 0\right\} = \bigcup_{0\leq s\leq t} \left\{l_s^1 > 0\right\} = \bigcup_{\substack{0\leq s\leq t\\ s\in Q}} \left\{l_s^1 > 0\right\} \in \mathcal{F}_t$$

because $\left\{l_s^1 > 0\right\} \in \mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$.

Taking into account the continuity of the process $u_t^1 := 0 \bigvee^t (X_s - b_s)$ we conclude

$$\{\tau_1 \leq t\} = \bigcup_{\substack{0\leq s\leq t\\ s\in Q}} \left(\{\sigma_1 < s\} \cap \left\{u_s^1 > 0\right\}\right) \in \mathcal{F}_t$$

because $\{\sigma_1 < s\} \in \mathcal{F}_s$ and $\left\{u_s^1 > 0\right\} \in \mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$.

Assuming now that σ_n and τ_n are stopping times we conclude that

$$\{\sigma_{n+1} \leq t\} = \bigcup_{\substack{0\leq s\leq t\\ s\in Q}} \left(\{\tau_n < s\} \cap \left\{l_s^{n+1} > l_s^n\right\}\right) \in \mathcal{F}_t$$

because $\{\tau_n < s\} \in \mathcal{F}_s$ and $\left\{l_s^{n+1} > l_s^n\right\} \in \mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$.

Finally we conclude

$$\{\tau_{n+1} \leq t\} = \bigcup_{\substack{0 \leq s \leq t \\ s \in Q}} \left(\{\sigma_n < s\} \cap \left\{u_s^{n+1} > u_s^n\right\}\right) \in \mathcal{F}_t$$

because $\{\sigma_n < s\} \in \mathcal{F}_s$ and $\left\{u_s^{n+1} > u_s^n\right\} \in \mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$.

Next we will prove that $\tau_n \uparrow \infty$ and $\sigma_n \uparrow \infty$ $a.s.$ as $n \uparrow \infty$.

By the definition (2.2.29)-(2.2.31) we observe that it holds

$$\sigma_n < \tau_n < \sigma_{n+1} < \tau_{n+1} \tag{2.2.32}$$

for $n = 1, 2, 3, ...$ From (2.2.32) we conclude that the sequence $\{\sigma_n\}_{n \in N}$ of stopping times is increasing and also that the sequence $\{\tau_n\}_{n \in N}$ of stopping times is increasing.

Let time $t > 0$. The event $\{\sigma_{n+1} > t\}$ by definition implies

$$l_s^n = l_s^{n+1} \quad , 0 \leq s \leq t \tag{2.2.33}$$

The event $\{\sigma_{n+1} > t\}$ also implies the event $\{\tau_{n+1} > t\}$ and from the definition (2.2.30) we conclude

$$u_s^n = u_s^{n+1} \quad , 0 \leq s \leq t \tag{2.2.34}$$

The event $\{\sigma_{n+1} > t\}$ also implies the event $\{\sigma_{n+2} > t\}$ and from the definition (2.2.31) we conclude

$$l_s^{n+1} = l_s^{n+2} \quad , 0 \leq s \leq t \tag{2.2.35}$$

The event $\{\sigma_{n+1} > t\}$ also implies the event $\{\tau_{n+2} > t\}$ and from the definition (2.2.30) we conclude

$$u_s^{n+1} = u_s^{n+2} \quad , 0 \leq s \leq t \tag{2.2.36}$$

Continuing with the same logic we can conclude that the event $\{\sigma_{n+1} > t\}$ implies

$$l_s^m = l_s^n \quad , 0 \leq s \leq t \;\text{ and }\; m \geq n \tag{2.2.37}$$

$$u_s^m = u_s^n \quad , 0 \leq s \leq t \;\text{ and }\; m \geq n \tag{2.2.38}$$

and also that it holds

$$\{\sigma_{n+1} > t\} \subset \{\sigma_m > t\} \quad , m \geq n+1 \tag{2.2.39}$$

$$\{\sigma_{n+1} > t\} \subset \{\tau_m > t\} \quad , m \geq n+1 \tag{2.2.40}$$

Similarly we can conclude that the event $\{\tau_{n+1} > t\}$ implies

$$l_s^m = l_s^{n+1} \quad , 0 \leq s \leq t \text{ and } m \geq n+1 \tag{2.2.41}$$

$$u_s^m = u_s^n \quad , 0 \leq s \leq t \text{ and } m \geq n \tag{2.2.42}$$

and also that it holds

$$\{\tau_n > t\} \subset \{\tau_m > t\} \quad , m \geq n+1 \tag{2.2.43}$$

$$\{\tau_n > t\} \subset \{\sigma_m > t\} \quad , m \geq n+1 \tag{2.2.44}$$

Let us suppose that $\tau_n \uparrow t_1$ and $\sigma_n \uparrow t_2$ for some times $t_1, t_2 > 0$. If $t_1 < t_2$ then there exist a stopping time σ_n such as

$$\sigma_n > t_1 \tag{2.2.45}$$

But the above relation (2.2.45) in conjunction with (2.2.40) will lead to a contradiction. Similarly if $t_2 < t_1$ we conclude a contradiction. Hence $t_1 = t_2$.

We suppose now that $\tau_n \uparrow t$ and $\sigma_n \uparrow t$ for some time $t > 0$. First of all observe that by taking into account the condition (2.1.6) we have that it holds

$$a_t + \varepsilon < b_t \ , 0 \leq t < \infty \quad a.s. \tag{2.2.46}$$

for some $\varepsilon > 0$. Let us suppose that all the stochastic processes live on the probability space $(\Omega, \mathcal{F}, P)$ and let $\Omega_1 \subset \Omega$ the set in which the relation (2.2.46) holds. Let also $\Omega_2 \subset \Omega$ be the set in which the process $\{X_t \ , 0 \leq t < \infty\}$ is continuous, $\Omega_3 \subset \Omega$ be the set in which the process $\{b_t \ , 0 \leq t < \infty\}$ is continuous and $\Omega_4 \subset \Omega$ be the set in which the process $\{a_t \ , 0 \leq t < \infty\}$ is continuous. Then of course holds that $P(\Omega_i) = 1$ with $i = 1, 2, 3, 4$. We define the set $\widetilde{\Omega} = \Omega_1 \cap \Omega_2 \cap \Omega_3 \cap \Omega_4$ and it is obvious that it holds $P(\widetilde{\Omega}) = 1$. Let $\omega \in \widetilde{\Omega}$. Then the function $b_t(\omega)$, $0 \leq t < \infty$ is continuous and hence uniformly continuous on compact time intervals which implies that for any $\delta_1 > 0$ we can choose time $h_1 > 0$ such as that it holds

$$|b_{t_2}(\omega) - b_{s_1}(\omega)| < \delta_1 \tag{2.2.47}$$

for every times $s_1, t_2 \in (t - h_1, t)$. Combining the relations (2.2.46),(2.2.47) and choosing $\delta_1 < \varepsilon$ we conclude

$$b_{t_2}(\omega) - a_{s_1}(\omega) = (b_{t_2}(\omega) - b_{s_1}(\omega)) + (b_{s_1}(\omega) - a_{s_1}(\omega)) > -\delta_1 + \varepsilon$$

or equivalently

$$b_{t_2}(\omega) - a_{s_1}(\omega) > c \tag{2.2.48}$$

for some constant $c > 0$ which is defined by $c := \varepsilon - \delta_1$ and every times $s_1, t_2 \in (t - h_1, t)$.

Next observe that the function $X_t(\omega)$, $0 \le t < \infty$ is continuous and hence uniformly continuous on compact time intervals and so for any $\delta > 0$ we can choose time $h_2 > 0$ such as that it holds

$$|X_{t_2}(\omega) - X_{s_1}(\omega)| < \delta \tag{2.2.49}$$

for any times $s_1, t_2 \in (t - h_2, t)$.

In the rest of the proof we choose $h = \min(h_1, h_2)$ and $\delta < c$ where c is the constant of the relation (2.2.48) and of course for this choice the relations (2.2.48),(2.2.49) hold simultaneously.

The hypothesis we made, that is $\tau_n(\omega) \uparrow t$ and $\sigma_n(\omega) \uparrow t$ for some time $t > 0$, means that for any time $h > 0$ there is a $n_0 \in N$ such as that it holds

$$\tau_n(\omega), \sigma_n(\omega) \in (t - h, t) \quad , \text{for any } n \ge n_0 \tag{2.2.50}$$

The above is equivalent to saying that for any $n \ge n_0$ there are times $t_1, t_2 \in (t - h, t)$, $t_1 \le t_2$, such as

$$l_{t_1}^{n+1}(\omega) > l_{t_1}^{n}(\omega) \tag{2.2.51}$$

and

$$u_{t_2}^{n+1}(\omega) > u_{t_2}^{n}(\omega) \tag{2.2.52}$$

The relation (2.2.51) reflects the fact that $\sigma_{n+1}(\omega) \in (t - h, t)$ and the relation (2.2.52) reflects the fact that $\tau_{n+1}(\omega) \in (t - h, t)$ and the relation $t_1 \le t_2$ comes from $\sigma_{n+1}(\omega) < \tau_{n+1}(\omega)$ (see (2.2.32)).

From (2.2.51),(2.2.52) and taking into account the relations (2.2.27),(2.2.28) of the definition of the sequences of processes $\{u^n\}$ and $\{l^n\}$ and the fact that these processes are increasing in time t for fixed n, we conclude the representations

$$u_{t_2}^{n+1}(\omega) = X_{t_2}(\omega) + l_{t_2}^{n}(\omega) - b_{t_2}(\omega) \tag{2.2.53}$$

and

$$l_{t_1}^{n+1}(\omega) = u_{t_1}^{n}(\omega) - X_{t_1}(\omega) + a_{t_1}(\omega) \tag{2.2.54}$$

Because the process $\left\{l_t^{n+1} , 0 \le t < \infty\right\}$ is continuous assumes a maximum on the compact time interval $[0, t_2]$ and because of the relation (2.2.54) and the fact that the process $\left\{l_t^{n+1} , 0 \le t < \infty\right\}$ is increasing we conclude that there is a time s_1 in the time interval $[t_1, t_2]$ such as

$$l_{t_2}^{n+1}(\omega) = u_{s_1}^{n}(\omega) - X_{s_1}(\omega) + a_{s_1}(\omega) \tag{2.2.55}$$

Because the sequence of the processes $\{l^n\}$ is **increasing** in n *for each fixed* time t we also have

$$l^{n+1}_{t_2}(\omega) \geq l^n_{t_2}(\omega) \tag{2.2.56}$$

Combining the relations (2.2.55)-(2.2.56) we conclude

$$u^n_{s_1}(\omega) - X_{s_1}(\omega) + a_{s_1}(\omega) \geq l^n_{t_2}(\omega) \tag{2.2.57}$$

Taking into account the fact that the process $\{u^n_t \ , 0 \leq t < \infty\}$ is increasing with respect to time t and that $s_1 \leq t_2$ the relation (2.2.57) becomes

$$u^n_{t_2}(\omega) - X_{s_1}(\omega) + a_{s_1}(\omega) \geq l^n_{t_2}(\omega) \tag{2.2.58}$$

Combining the relations (2.2.52),(2.2.53) we conclude

$$X_{t_2}(\omega) + l^n_{t_2}(\omega) - b_{t_2}(\omega) > u^n_{t_2}(\omega) \tag{2.2.59}$$

Combining the relations (2.2.58),(2.2.59) we conclude

$$X_{t_2}(\omega) - X_{s_1}(\omega) > b_{t_2}(\omega) - a_{s_1}(\omega) \tag{2.2.60}$$

Thus taking into account (2.2.48) the relation (2.2.60) becomes

$$X_{t_2}(\omega) - X_{s_1}(\omega) > c \tag{2.2.61}$$

Considering the relation (2.2.49) with $\delta > 0$ such as $\delta < c$, where c is the constant of the relation (2.2.48), we find that

$$\delta > X_{t_2}(\omega) - X_{s_1}(\omega) > c \tag{2.2.62}$$

which is a contradiction. Hence we conclude that

$$\{\omega \in \Omega : \tau_n(\omega) \uparrow t \text{ and } \sigma_n(\omega) \uparrow t \text{ for some } t \in [0,\infty)\} \subset \left\{\omega \in \Omega : \omega \notin \widetilde{\Omega}\right\} \Longrightarrow$$

$$0 \leq P\left(\{\omega \in \Omega : \tau_n(\omega) \uparrow t \text{ and } \sigma_n(\omega) \uparrow t \text{ for some } t \in [0,\infty)\}\right) \leq P\left(\left\{\omega \in \Omega : \omega \notin \widetilde{\Omega}\right\}\right) = 0$$

and we have proved that $\tau_n \uparrow \infty$ and $\sigma_n \uparrow \infty$ $a.s.$ as $n \uparrow \infty$.

Step 1.(IV) By what we found on Step 1.(II) we easily conclude that there are processes $\mathcal{U}^{(+)} = \{\mathcal{U}^{(+)}_t; 0 \leq t < \infty\}$ and $\mathcal{U}^{(-)} = \{\mathcal{U}^{(-)}_t; 0 \leq t < \infty\}$ such that

$$u^n_t \uparrow \mathcal{U}^{(+)}_t \text{ and } l^n_t \uparrow \mathcal{U}^{(-)}_t \quad a.s. \text{ as } n \uparrow \infty \tag{2.2.63}$$

By what we found on Step 1.(III) we conclude that the limiting processes are finite on compact time intervals.

Step 1.(V) *In this step we will find the limiting processes.*

For $m \geq n$ we observe that *a.s.*

$$u_t^m = u_t^n \ \text{ if } \ \tau_n \leq t < \tau_{n+1} \tag{2.2.64}$$

and

$$l_t^m = l_t^n \ \text{ if } \ \sigma_n \leq t < \sigma_{n+1} \tag{2.2.65}$$

and so from the *relations* (2.2.64),(2.2.65) we conclude that the *limiting processes* are given *a.s.* by:

$$\mathcal{U}_t^{(+)} = u_t^n \ \text{ if } \ \tau_n \leq t < \tau_{n+1} \tag{2.2.66}$$

and

$$\mathcal{U}_t^{(-)} = l_t^n \ \text{ if } \ \sigma_n \leq t < \sigma_{n+1} \tag{2.2.67}$$

Observe that the process $\mathcal{U}^{(+)}$ is *a.s.* continuous on the intervals $[\tau_n, \tau_{n+1})$ with possible jumps on the times $\{\tau_n\}_{n\in N}$. But we have $\Delta\mathcal{U}_{\tau_n}^{(+)} := \mathcal{U}_{\tau_n}^{(+)} - \mathcal{U}_{\tau_n -}^{(+)} = u_{\tau_n}^n - u_{\tau_n -}^{n-1} = u_{\tau_n}^n - u_{\tau_n -}^n = 0$ (using relation (2.2.64) and also the *a.s.* continuity of the process u^n). Thus the process $\mathcal{U}^{(+)}$ is *a.s* continuous. Similarly we conclude that the process $\mathcal{U}^{(-)}$ is *a.s* continuous.

We define a process

$$Z_t^{(n,m)} := X_t - u_t^n + l_t^m \ , \text{ if } \tau_n \leq t < \tau_{n+1} \text{ and } \sigma_m \leq t < \sigma_{m+1} \tag{2.2.68}$$

and a *limiting process*

$$Z_t := \lim_{n,m \longrightarrow \infty} Z_t^{(n,m)} = X_t - \mathcal{U}_t^{(+)} + \mathcal{U}_t^{(-)} \tag{2.2.69}$$

which is given by

$$Z_t = X_t - u_t^n + l_t^m \ , \text{ if } \tau_n \leq t < \tau_{n+1} \text{ and } \sigma_m \leq t < \sigma_{m+1} \tag{2.2.70}$$

By relation (2.2.69) and the $a.s.$ continuity of the processes $\mathcal{U}^{(+)}$,$\mathcal{U}^{(-)}$ and X we conclude that the process Z is $a.s.$ continuous.

Step 2. We will show that the process $(Z,\mathcal{U}^{(+)},\mathcal{U}^{(-)})$ satisfies the requirements (2.1.11)-(2.1.16) of the *definition* of the *de Finetti model with two general reflecting barriers.*

Step 2.(I) It is obvious that the process $(Z,\mathcal{U}^{(+)},\mathcal{U}^{(-)})$ satisfies the *condition* (2.1.11) of the definition of the *de Finetti model with two general reflecting barriers.*

Step 2.(II) We will show that the process $(Z,\mathcal{U}^{(+)},\mathcal{U}^{(-)})$ satisfies the *condition* (2.1.12) of the definition of the *de Finetti model with two general reflecting barriers.*

Indeed from the *relations* (2.2.27),(2.2.28) we have that $a.s.$

$$u_t^n \geq X_t + l_t^{n-1} - b_t \quad , n = 1, 2, .. \tag{2.2.71}$$

and

$$l_t^m \geq u_t^{m-1} - X_t + a_t \quad , \quad m = 1, 2, .. \tag{2.2.72}$$

If $m \geq n$ then from the *relations* (2.2.72),(2.2.64),(2.2.65) we have $a.s.$

$$\left.\begin{array}{r} l_t^{m+1} \geq u_t^m - X_t + a_t \\ l_t^{m+1} = l_t^m \quad \text{on } \sigma_m \leq t < \sigma_{m+1} \\ u_t^m = u_t^n \quad \text{on } \ \tau_n \leq t < \tau_{n+1} \end{array}\right\} \Longrightarrow$$

$$l_t^m \geq u_t^n - X_t + a_t \Longrightarrow l_t^m - u_t^n + X_t \geq a_t \Longrightarrow Z_t^{(n,m)} \geq a_t$$

and

$$\left.\begin{array}{ll} u_t^{m+1} \geq X_t + l_t^m - b_t & \\ u_t^{m+1} = u_t^n & \text{on } \ \tau_n \leq t < \tau_{n+1} \end{array}\right\} \Longrightarrow$$

$$u_t^n \geq X_t + l_t^m - b_t \Longrightarrow l_t^m - u_t^n + X_t \leq b_t \Longrightarrow Z_t^{(n,m)} \leq b_t$$

By symmetry we conclude the same in the case $m < n$.

That is we found that $a.s.$

$$a_t \leq Z_t^{(n,m)} \leq b_t \quad , \qquad \text{if} \quad \tau_n \leq t < \tau_{n+1} \ \text{ and } \ \sigma_m \leq t < \sigma_{m+1}$$

and *taking limits* we conclude (2.1.12) that is $a.s.$

$$a_t \leq Z_t^{(n,m)} \leq b_t \Longrightarrow a_t \leq \lim_{n,m \longrightarrow \infty} Z_t^{(n,m)} \leq b_t \Longrightarrow a_t \leq Z_t \leq b_t.$$

Step 2.(III) We will show that the process $(Z, \mathcal{U}^{(+)}, \mathcal{U}^{(-)})$ satisfies the *conditions* (2.1.13),(2.1.14) of the definition of the *de Finetti model with two general reflecting barriers.*

Indeed it is easy one to see that for every fixed $n \in N$ the processes $u^n = \{u_t^n; 0 \leq t < \infty\}$ and $l^n = \{l_t^n; 0 \leq t < \infty\}$ are **nondecreasing** and so are the *limiting processes* $\mathcal{U}^{(+)}, \mathcal{U}^{(-)}$ and also we have $\mathcal{U}_0^{(+)} = 0$ and $\mathcal{U}_0^{(-)} = 0$ $a.s.$

Step 2.(IV) We will show that the process $(Z, \mathcal{U}^{(+)}, \mathcal{U}^{(-)})$ satisfies the conditions (2.1.15),(2.1.16) of the definition of the *de Finetti model with two general reflecting barriers.*

Indeed from (2.2.70) we have that $a.s.$

$$\int_0^t 1_{\{Z_s < b_s\}} d\mathcal{U}_s^{(+)} = \sum_{n=1}^{\infty} \int_{\tau_{n-1}}^{t \wedge \tau_n} 1_{\{X_s - u_s^n + \mathcal{U}_s^{(-)} < b_s\}} du_s^n = 0$$

because if $X_s - b_s + \mathcal{U}_s^{(-)} < u_s^n$ then $du_s^n = 0$ by the *Definition* (2.2.27) of the process u^n.

Also we have $a.s.$

$$\int_0^t 1_{\{Z_s > a_s\}} d\mathcal{U}_s^{(-)} = \sum_{n=1}^{\infty} \int_{\sigma_{n-1}}^{t \wedge \sigma_n} 1_{\{X_s - \mathcal{U}_s^{(+)} + l_s^n > a_s\}} dl_s^n = 0.$$

because if $l_s^n > a_s - X_s + \mathcal{U}_s^{(+)}$ then $dl_s^n = 0$ by the *Definition* (2.2.28) of the process l^n. ■

Working as above we can immediately conclude the existence of the de Finetti model with one general reflecting barrier by similar arguments and by taking in the proof of the previous proposition $l_t^n := 0$ for each $n \in \mathbb{N}$ and each time $t \in [0, \infty)$.

We have come to the end of this section having proved that the de Finetti models with general barriers are well defined. We move now into the next section in which we will derive a very useful property which we call the "scaling property". This property will help us to proceed with later calculations.

2.3 Scaling Property.

We start this section by posing a question. Let us suppose that we have the reserves process of an insurance company, which moves between two boundaries which are described by two general diffusion processes and dividends are paid to the shareholders according to the de Finetti model with general reflecting barriers. How will the discounted dividends be affected if we move up or down, by the same amount, the reserve process and the two boundaries processes? How will the discounted dividends be affected if we multiple the above processes by the same amount? In our effort to answer these questions we will derive a property which proves to be very useful. We call this property "scaling property".

More specifically we proceed by defining two auxiliary processes in a way as to express the changes induced in the original processes, that is the changes induced by adding a constant and the changes induced by multiplying by a constant.

Definition 2.3.1 *We define the following processes*

- *For a real number* $c \in (-\infty, \infty)$ *and for* $t \geq 0$

$$\left(\widehat{X}_t, \widehat{a}_t, \widehat{b}_t\right) := \left(X_t - c, a_t - c, b_t - c\right). \tag{2.3.1}$$

- *For a real number* $c > 0$ *and for* $t \geq 0$

$$\left(\widetilde{X}_t, \widetilde{a}_t, \widetilde{b}_t\right) := \left(X_t c^{-1}, a_t c^{-1}, b_t c^{-1}\right). \tag{2.3.2}$$

With these definitions, we proceed with the general idea of this section, which is to derive the discounted dividends and the discounted financing for the de Finetti model with general barriers of the above processes and to compare how these are related to the discounted dividends and the discounted financing for the de Finetti model of the initial processes. In order to accomplish this, we first will consider the de Finetti model with one general reflecting barrier and next the de Finetti model with two general reflecting barriers.

Starting with the de Finetti model with one general reflecting barrier we note that the *associated de Finetti models with one general reflecting barrier* for the processes $(\widetilde{X}, \widetilde{a}, \widetilde{b}) =$

$\{(\widetilde{X}_t, \widetilde{a}_t, \widetilde{b}_t); 0 \leq t < \infty\}$, $(\widehat{X}, \widehat{a}, \widehat{b}) = \{(\widehat{X}_t, \widehat{a}_t, \widehat{b}_t); 0 \leq t < \infty\}$ are described by

$$\widehat{\widetilde{\mathcal{U}}}_t = \sup_{0\leq s\leq t} \left(\widehat{\widetilde{X}}_s - \widehat{\widetilde{b}}_s\right)^+ \tag{2.3.3}$$

$$\widehat{\widetilde{Z}}_t := \widehat{\widetilde{X}}_t - \widehat{\widetilde{\mathcal{U}}}_t \tag{2.3.4}$$

$$\widehat{\widetilde{T}} := \inf\{t > 0 : \widehat{\widetilde{Z}}_t = \widehat{\widetilde{a}}_t\} \tag{2.3.5}$$

$$\widehat{\widetilde{\mathcal{U}}} := \int_0^{\widehat{\widetilde{T}}} e^{-\delta s} d\widehat{\widetilde{\mathcal{U}}}_s \tag{2.3.6}$$

Next we will consider the initial process $(X, \mathbf{a}, \mathbf{b})$ and the above two auxiliary processes $(\widetilde{X}, \widetilde{\mathbf{a}}, \widetilde{\mathbf{b}})$ and $(\widehat{X}, \widehat{\mathbf{a}}, \widehat{\mathbf{b}})$ and proceed to examine relations between the respective times of ruin and the respective dividends. We conclude the following lemma which proves to be very useful.

Lemma 2.3.2 *For the* de Finetti models with one general reflecting barrier *corresponding to the processes* $(X, \mathbf{a}, \mathbf{b}), (\widehat{X}, \widehat{\mathbf{a}}, \widehat{\mathbf{b}})$ *and* $(\widetilde{X}, \widetilde{\mathbf{a}}, \widetilde{\mathbf{b}})$ *it holds a.s. that*

(I)

$$\{\mathcal{U}_t; 0 \leq t < \infty\} = \{\widehat{\mathcal{U}}_t; 0 \leq t < \infty\} = \{c\widetilde{\mathcal{U}}_t; 0 \leq t < \infty\} \tag{2.3.7}$$

(II)

$$T = \widehat{T} = \widetilde{T} \tag{2.3.8}$$

(III)

$$U = \widehat{U} = c\widetilde{U} \tag{2.3.9}$$

Proof.

(I) Because the dividends processes $\mathcal{U}, \widehat{\mathcal{U}}$ and $\widetilde{\mathcal{U}}$ for the de Finetti models with one general reflecting barrier corresponding to the processes $(X, \mathbf{a}, \mathbf{b})$, $(\widehat{X}, \widehat{\mathbf{a}}, \widehat{\mathbf{b}})$ and $(\widetilde{X}, \widetilde{\mathbf{a}}, \widetilde{\mathbf{b}})$ respectively, are continuous processes, by using the Proposition 1.3.10 in order to prove the claim it is enough to prove that the dividends processes are modifications of each other. For this consider the fixed time $t \geq 0$. Taking into account Proposition 2.2.1, for the dividends processes $\mathcal{U}$ and $\widehat{\mathcal{U}}$ we can see that a.s.

$$\mathcal{U}_t = \sup_{0\leq s\leq t} (X_s - b_s)^+ = \sup_{0\leq s\leq t} [(X_s - c) - (b_s - c)]^+ = \sup_{0\leq s\leq t} (\widehat{X}_s - \widehat{b}_s)^+$$

hence

$$\mathcal{U}_t = \widehat{\mathcal{U}}_t.$$

Similarly for fixed time $t \geq 0$, for the dividends processes $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ we have that a.s.

$$\mathcal{U}_t = \sup_{0\leq s\leq t}(X_s - b_s)^+ = c\sup_{0\leq s\leq t}(X_s c^{-1} - b_s c^{-1})^+ = c\sup_{0\leq s\leq t}(\widetilde{X}_s - \widetilde{b}_s)^+ = c\,\widetilde{\mathcal{U}}_t$$

hence

$$\mathcal{U}_t = c\,\widetilde{\mathcal{U}}_t.$$

(II) Using relation (2.3.7) we conclude for the stopping times T and $\widehat{T}$ that a.s.

$$\begin{aligned} T &:= \inf\{t > 0 : Z_t = a_t\} = \inf\{t > 0 : X_t = \mathcal{U}_t + a_t\} \\ &= \inf\{t > 0 : X_t - c = \mathcal{U}_t + a_t - c\} = \inf\{t > 0 : \widehat{X}_t = \widehat{\mathcal{U}}_t + \widehat{a}_t\} \\ &= \inf\{t > 0 : \widehat{Z}_t = \widehat{a}_t\} = \widehat{T} \end{aligned}$$

Similarly for the stopping times T and $\widetilde{T}$ a.s. holds that

$$\begin{aligned} T &:= \inf\{t > 0 : Z_t = a_t\} = \inf\{t > 0 : X_t = \mathcal{U}_t + a_t\} \\ &= \inf\{t > 0 : X_t c^{-1} = \mathcal{U}_t c^{-1} + a_t c^{-1}\} = \inf\{t > 0 : \widetilde{X}_t = \widetilde{\mathcal{U}}_t + \widetilde{a}_t\} \\ &= \inf\{t > 0 : \widetilde{Z}_t = \widetilde{\mathcal{U}}_t\} = \widetilde{T} \end{aligned}$$

(III) Using the relations (2.3.7), (2.3.8) we find that for the discounted dividends U, $\widehat{U}$ and $\widetilde{U}$ a.s. holds that

$$U = \int_0^T e^{-\delta s} d\mathcal{U}_s = \int_0^{\widehat{T}} e^{-\delta s} d\widehat{\mathcal{U}_s} = \widehat{U}$$

and

$$U = \int_0^T e^{-\delta s} d\mathcal{U}_s = c\int_0^{\widetilde{T}} e^{-\delta s} d\widetilde{\mathcal{U}_s} = c\,\widetilde{U}$$

■

Next with the aid of Lemma 2.3.2 we are ready to prove the scaling property.

Proposition 2.3.3 *(Scaling property for the de Finetti model with one general reflecting barrier). For the moments of the discounted dividends $V(x, a, b; n)$, the Laplace transform of the discounted dividends $K(x, a, b, \lambda)$, the Laplace transform of the time of ruin $M(x, a, b, \lambda)$ and the Laplace transform of the joint distribution of the time of ruin and the discounted dividends $N(x, a, b, \lambda_1, \lambda_2)$ it holds that*

(I) For each real number $c \in (-\infty, \infty)$

$$\begin{aligned}
V(x,a,b;n) &= V(x-c,a-c,b-c;n) && (2.3.10)\\
K(x,a,b,\lambda) &= K(x-c,a-c,b-c,\lambda) && (2.3.11)\\
M(x,a,b,\lambda) &= M(x-c,a-c,b-c,\lambda) && (2.3.12)\\
N(x,a,b,\lambda_1,\lambda_2) &= N(x-c,a-c,b-c,\lambda_1,\lambda_2) && (2.3.13)
\end{aligned}$$

(II) For each real number $c > 0$

$$\begin{aligned}
V(x,a,b;n) &= c^n V(xc^{-1},ac^{-1},bc^{-1};n) && (2.3.14)\\
K(x,a,b,\lambda) &= K(xc^{-1},ac^{-1},bc^{-1},\lambda c) && (2.3.15)\\
M(x,a,b,\lambda) &= M(xc^{-1},ac^{-1},bc^{-1},\lambda) && (2.3.16)\\
N(x,a,b,\lambda_1,\lambda_2) &= N(xc^{-1},ac^{-1},bc^{-1},\lambda_1,\lambda_2 c) && (2.3.17)
\end{aligned}$$

Proof. The proposition is obvious if we take into account the relations (2.3.8),(2.3.9) and we will only prove here the properties (2.3.13) and (2.3.17), while the other statements are proved similar.

(I) From the relations (2.3.8) and (2.3.9) we conclude that

$$\begin{aligned}
N(x,a,b,\lambda_1,\lambda_2) &:= E^{(x,a,b)}(e^{-\lambda_1 T-\lambda_2 U})\\
&= E^{(x-c,a-c,b-c)}(e^{-\lambda_1 \widehat{T}-\lambda_2 \widehat{U}})\\
&= N(x-c,a-c,b-c,\lambda_1,\lambda_2).
\end{aligned}$$

(II) Similarly from the relations (2.3.8) and (2.3.9) we conclude that

$$\begin{aligned}
N(x,a,b,\lambda_1,\lambda_2) &:= E^{(x,a,b)}(e^{-\lambda_1 T-\lambda_2 U})\\
&= E^{(xc^{-1},ac^{-1},bc^{-1})}(e^{-\lambda_1 \widetilde{T}-\lambda_2 c\widetilde{U}})\\
&= N(xc^{-1},ac^{-1},bc^{-1},\lambda_1,\lambda_2 c)
\end{aligned}$$

■

The following remarks will be useful.

Remark 2.3.4 *For a function* $V(x,a,b;n)$ *which is* $C^1(\mathbb{R}^3)$ *and satisfy the scaling property (2.3.14) we can see by differentiating with respect of* c *that*

$$\left(x\frac{\partial}{\partial x}+a\frac{\partial}{\partial a}+b\frac{\partial}{\partial b}\right)V(xc^{-1},ac^{-1},bc^{-1};n) = ncV(xc^{-1},ac^{-1},bc^{-1};n) \qquad (2.3.18)$$

Remark 2.3.5 *For a function $K(x,a,b,\lambda)$ which is $C^1(\mathbb{R}^4)$ and satisfy the scaling property (2.3.15) we have that*

$$K(x,a,b,\lambda c^{-1}) = K(xc^{-1},ac^{-1},bc^{-1},\lambda) \tag{2.3.19}$$

and by differentiating with respect of c we conclude that

$$\lambda\frac{\partial}{\partial(\lambda c^{-1})}K(x,a,b,\lambda c^{-1}) = \left(x\frac{\partial}{\partial(xc^{-1})}+a\frac{\partial}{\partial(ac^{-1})}+b\frac{\partial}{\partial(bc^{-1})}\right)K(xc^{-1},ac^{-1},bc^{-1},\lambda) \tag{2.3.20}$$

Remark 2.3.6 *For a function $M(x,a,b,\lambda)$ which is $C^1(\mathbb{R}^4)$ and satisfy the scaling property (2.3.16) we can see by differentiating with respect of c that*

$$\left(x\frac{\partial}{\partial(xc^{-1})}+a\frac{\partial}{\partial(ac^{-1})}+b\frac{\partial}{\partial(bc^{-1})}\right)M(xc^{-1},ac^{-1},bc^{-1},\lambda) = 0 \tag{2.3.21}$$

Remark 2.3.7 *For a function $N(x,a,b,\lambda_1,\lambda_2)$ which is $C^1(\mathbb{R}^5)$ and satisfy the scaling property (2.3.17) we have that*

$$N(x,a,b,\lambda_1,\lambda_2 c^{-1}) = N(xc^{-1},ac^{-1},bc^{-1},\lambda_1,\lambda_2) \tag{2.3.22}$$

and by differentiating with respect of c we conclude that

$$\begin{aligned}&\lambda_2\frac{\partial}{\partial(\lambda_2 c^{-1})}N(x,a,b,\lambda_1,\lambda_2 c^{-1}) \qquad (2.3.23)\\ =&\left(x\frac{\partial}{\partial(xc^{-1})}+a\frac{\partial}{\partial(ac^{-1})}+b\frac{\partial}{\partial(bc^{-1})}\right)N(xc^{-1},ac^{-1},bc^{-1},\lambda_1,\lambda_2)\end{aligned}$$

Next we consider the de Finetti model with two general reflecting barriers. We proceed by following analogous arguments as above.

The *associated de Finetti models with two general reflecting barriers* for the processes $(\widetilde{X},\widetilde{\mathbf{a}},\widetilde{\mathbf{b}})$, $(\widehat{X},\widehat{\mathbf{a}},\widehat{\mathbf{b}})$ are described for every time $0 \le t < \infty$ by

$$\widehat{\widetilde{\mathcal{U}_t^{(-)}}} = \sup_{0\le s\le t}\left(\widehat{\widetilde{X_s}} - \widehat{\widetilde{\mathcal{U}_s^{(+)}}} - \widehat{\widetilde{a_s}}\right)^- \tag{2.3.24}$$

$$\widehat{\widetilde{\mathcal{U}_t^{(+)}}} = \sup_{0\le s\le t}\left(\widehat{\widetilde{b_s}} - \widehat{\widetilde{X_s}} - \widehat{\widetilde{\mathcal{U}_s^{(-)}}}\right)^- \tag{2.3.25}$$

$$\widehat{\widetilde{Z_t}} := \widehat{\widetilde{X_t}} - \widehat{\widetilde{\mathcal{U}_t^{(+)}}} + \widehat{\widetilde{\mathcal{U}_t^{(-)}}} \tag{2.3.26}$$

$$\widehat{\widetilde{U^{(+)}}} := \int_0^\infty e^{-\delta s}\, d\widehat{\widetilde{\mathcal{U}_s^{(+)}}} \tag{2.3.27}$$

$$\widehat{\widetilde{U^{(-)}}} := \int_0^\infty e^{-\delta s}\, d\widehat{\widetilde{\mathcal{U}_s^{(-)}}} \tag{2.3.28}$$

Proceeding now with the de Finetti model with two general reflecting barriers in an analogous manner as we did before in the de Finetti model with one general reflecting barrier, we first derive the following useful lemma.

Lemma 2.3.8 *For the* de Finetti models with two general reflecting barriers *corresponding to the processes* $(X, \mathbf{a}, \mathbf{b}), (\widehat{X}, \widehat{\mathbf{a}}, \widehat{\mathbf{b}})$ *and* $(\widetilde{X}, \widetilde{\mathbf{a}}, \widetilde{\mathbf{b}})$ *it holds a.s. that:*

(I)

$$\{\mathcal{U}_t^{(+)}; 0 \le t < \infty\} = \{\widehat{\mathcal{U}_t^{(+)}}; 0 \le t < \infty\} = \{c\, \widetilde{\mathcal{U}_t^{(+)}}; 0 \le t < \infty\} \tag{2.3.29}$$

(II)

$$\{\mathcal{U}_t^{(-)}; 0 \le t < \infty\} = \{\widehat{\mathcal{U}_t^{(-)}}; 0 \le t < \infty\} = \{c\, \widetilde{\mathcal{U}_t^{(-)}}; 0 \le t < \infty\} \tag{2.3.30}$$

(III)

$$U^{(+)} = \widehat{U^{(+)}} = c\, \widetilde{U^{(+)}} \tag{2.3.31}$$

$$U^{(-)} = \widehat{U^{(-)}} = c\, \widetilde{U^{(-)}} \tag{2.3.32}$$

Proof. For the process (X, a, b) as we have seen in the Proposition 2.2.2, the discounted financing and the discounted dividends are given from the relations (2.2.5) and (2.2.4) respectively, that is for every time $t \ge 0$ it holds a.s.:

$$\mathcal{U}_t^{(-)} = \sup_{0 \le s \le t} \left(X_s - \mathcal{U}_s^{(+)} - a_s \right)^- \tag{2.3.33}$$

$$\mathcal{U}_t^{(+)} = \sup_{0 \le s \le t} \left(b_s - X_s - \mathcal{U}_s^{(-)} \right)^- \tag{2.3.34}$$

Because the dividends processes $\mathcal{U}^{(+)}$, $\widehat{\mathcal{U}^{(+)}}$ and $\widetilde{\mathcal{U}^{(+)}}$ and the financing processes $\mathcal{U}^{(-)}$, $\widehat{\mathcal{U}^{(-)}}$ and $\widetilde{\mathcal{U}^{(-)}}$ for the de Finetti models with two general reflecting barriers corresponding to the processes (X, a, b), $(\widehat{X}, \widehat{\mathbf{a}}, \widehat{\mathbf{b}})$ and $(\widetilde{X}, \widetilde{\mathbf{a}}, \widetilde{\mathbf{b}})$ respectively, are continuous processes, by using the Proposition 1.3.10 in order to prove assertions (I) and (II) of the lemma it is enough to prove that the dividends processes and the financing processes are modifications of each other. For this we will consider a fixed time $t \ge 0$.

(I) From the relations (2.3.33) and (2.3.34) we conclude that for fixed time $t \ge 0$ it holds a.s.

$$\left.\begin{aligned} \mathcal{U}_t^{(-)} &= \sup_{0 \le s \le t} \left(X_s - \mathcal{U}_s^{(+)} - a_s \right)^- \\ \mathcal{U}_t^{(+)} &= \sup_{0 \le s \le t} \left(b_s - X_s - \mathcal{U}_s^{(-)} \right)^- \end{aligned}\right\} \Longrightarrow$$

$$\left.\begin{aligned} \mathcal{U}_t^{(-)} &= \sup_{0\le s\le t}\Big((X_s-c)-\mathcal{U}_s^{(+)}-(a_s-c)\Big)^- \\ \mathcal{U}_t^{(+)} &= \sup_{0\le s\le t}\Big((b_s-c)-(X_s-c)-\mathcal{U}_s^{(-)}\Big)^- \end{aligned}\right\}\Longrightarrow$$

$$\mathcal{U}_t^{(-)} = \sup_{0\le s\le t}\Big(\widehat{X_s}-\mathcal{U}_s^{(+)}-\widehat{a_s}\Big)^- \tag{2.3.35}$$

$$\mathcal{U}_t^{(+)} = \sup_{0\le s\le t}\Big(\widehat{b_s}-\widehat{X_s}-\mathcal{U}_s^{(-)}\Big)^- \tag{2.3.36}$$

Because the process $(\widehat{\mathcal{U}^{(+)}},\widehat{\mathcal{U}^{(-)}})$ is **unique** we conclude from relations (2.3.35), (2.3.36) and (2.3.24), (2.3.25) that for every time $t\ge 0$

$$\begin{aligned} \mathcal{U}_t^{(+)} &= \widehat{\mathcal{U}_t^{(+)}} \\ \mathcal{U}_t^{(-)} &= \widehat{\mathcal{U}_t^{(-)}} \qquad \text{a.s.} \end{aligned}$$

(II) From the relations (2.3.33), (2.3.34) we have that for every time $t\ge 0$

$$\left.\begin{aligned} \mathcal{U}_t^{(-)} &= \sup_{0\le s\le t}\Big(X_s-\mathcal{U}_s^{(+)}-a_s\Big)^- \\ \mathcal{U}_t^{(+)} &= \sup_{0\le s\le t}\Big(b_s-X_s-\mathcal{U}_s^{(-)}\Big)^- \end{aligned}\right\}\Longrightarrow$$

$$\left.\begin{aligned} c^{-1}\mathcal{U}_t^{(-)} &= \sup_{0\le s\le t}\Big(c^{-1}X_s-c^{-1}\mathcal{U}_s^{(+)}-c^{-1}a_s\Big)^- \\ c^{-1}\mathcal{U}_t^{(+)} &= \sup_{0\le s\le t}\Big(c^{-1}b_s-c^{-1}X_s-c^{-1}\mathcal{U}_s^{(-)}\Big)^- \end{aligned}\right\}\Longrightarrow$$

$$c^{-1}\mathcal{U}_t^{(-)} = \sup_{0\le s\le t}\Big(\widetilde{X_s}-c^{-1}\mathcal{U}_s^{(+)}-\widetilde{a_s}\Big)^- \tag{2.3.37}$$

$$c^{-1}\mathcal{U}_t^{(+)} = \sup_{0\le s\le t}\Big(\widetilde{b_s}-\widetilde{X_s}-c^{-1}\mathcal{U}_s^{(-)}\Big)^- \qquad \text{a.s.} \tag{2.3.38}$$

Because the process $(\widetilde{\mathcal{U}^{(+)}},\widetilde{\mathcal{U}^{(-)}})$ is unique we conclude from the relations (2.3.37), (2.3.38) and (2.3.24), (2.3.25) that for every time $t\ge 0$

$$\begin{aligned} c^{-1}\mathcal{U}_t^{(+)} &= \widetilde{\mathcal{U}_t^{(+)}} \\ c^{-1}\mathcal{U}_t^{(-)} &= \widetilde{\mathcal{U}_t^{(-)}} \qquad \text{a.s.} \end{aligned}$$

(III) Using the relations (2.3.29) and (2.3.30) we find that a.s. holds

$$U^{(\pm)} = \int_0^\infty e^{-\delta s} d\mathcal{U}_s^{(\pm)} = \int_0^\infty e^{-\delta s} d\widehat{\mathcal{U}_s^{(\pm)}} = \widehat{U^{(\pm)}}$$

and

$$U^{(\pm)} = \int_0^\infty e^{-\delta s} d\mathcal{U}_s^{(\pm)} = c \int_0^\infty e^{-\delta s} d\widetilde{\mathcal{U}_s^{(\pm)}} = c\, \widetilde{U^{(\pm)}}$$

■

Now we are ready to prove the scaling property for the de Finetti model with two general reflecting barriers.

Proposition 2.3.9 *(Scaling property for the de Finetti model with two general reflecting barriers).*

For the moments of the discounted dividends $V^{(+)}(x, a, b; n)$, the moments of the discounted financing $V^{(-)}(x, a, b; n)$, the Laplace transform of the discounted dividends $K^{(+)}(x, a, b, \lambda)$, the Laplace transform of the discounted financing $K^{(-)}(x, a, b, \lambda)$ and the Laplace transform of the joint distribution of the discounted dividends and the discounted financing $L(x, a, b, \lambda_1, \lambda_2)$ it holds that:

(I) For every real number $c \in (-\infty, \infty)$

$$V^{(\pm)}(x, a, b; n) = V^{(\pm)}(x - c, a - c, b - c; n) \tag{2.3.39}$$

$$K^{(\pm)}(x, a, b, \lambda) = K^{(\pm)}(x - c, a - c, b - c, \lambda) \tag{2.3.40}$$

$$L(x, a, b, \lambda_1, \lambda_2) = L(x - c, a - c, b - c, \lambda_1, \lambda_2) \tag{2.3.41}$$

(II) For every real number $c > 0$

$$V^{(\pm)}(x, a, b; n) = c^n V^{(\pm)}(xc^{-1}, ac^{-1}, bc^{-1}; n) \tag{2.3.42}$$

$$K^{(\pm)}(x, a, b, \lambda) = K^{(\pm)}(xc^{-1}, ac^{-1}, bc^{-1}, \lambda c) \tag{2.3.43}$$

$$L(x, a, b, \lambda_1, \lambda_2) = L(xc^{-1}, ac^{-1}, bc^{-1}, c\lambda_1, c\lambda_2) \tag{2.3.44}$$

Proof. The proposition is obvious if we take into account the relations (2.3.31), (2.3.32) and we will only prove here the properties (2.3.41) and (2.3.44), while the other statements are proved in a similar manner.

(I) From relations (2.3.31) and (2.3.32) we conclude that:

$$\begin{aligned} L(x,a,b,\lambda_1,\lambda_2) &:= E^{(x,a,b)}(e^{-\lambda_1 U^{(+)}-\lambda_2 U^{(-)}}) \\ &= E^{(x-c,a-c,b-c)}(e^{-\lambda_1 \widehat{U^{(+)}}-\lambda_2 \widehat{U^{(-)}}}) \\ &= L(x-c,a-c,b-c,\lambda_1,\lambda_2) \end{aligned}$$

(II) Similarly from relations (2.3.31) and (2.3.32) we conclude that

$$\begin{aligned} L(x,a,b,\lambda_1,\lambda_2) &:= E^{(x,a,b)}(e^{-\lambda_1 U^{(+)}-\lambda_2 U^{(-)}}) \\ &= E^{(xc^{-1},ac^{-1},bc^{-1})}(e^{-\lambda_1 c\widetilde{U^{(+)}}-\lambda_2 c\widetilde{U^{(-)}}}) \\ &= L(xc^{-1},ac^{-1},bc^{-1},c\lambda_1,c\lambda_2) \end{aligned}$$

■

The following remarks will be useful.

Remark 2.3.10 *For a function $L(x,a,b,\lambda_1,\lambda_2)$ which is $C^1(\mathbb{R}^5)$ and satisfy the scaling property (2.3.44) we have that*

$$L(x,a,b,\lambda_1 c^{-1},\lambda_2 c^{-1}) = L(xc^{-1},ac^{-1},bc^{-1},\lambda_1,\lambda_2) \tag{2.3.45}$$

and by differentiating with respect of c we conclude that

$$\begin{aligned} &\left(\lambda_1\frac{\partial}{\partial(\lambda_1 c^{-1})}+\lambda_2\frac{\partial}{\partial(\lambda_2 c^{-1})}\right)L(x,a,b,\lambda_1 c^{-1},\lambda_2 c^{-1}) \\ =\;&\left(x\frac{\partial}{\partial(xc^{-1})}+a\frac{\partial}{\partial(ac^{-1})}+b\frac{\partial}{\partial(bc^{-1})}\right)L(xc^{-1},ac^{-1},bc^{-1},\lambda_1,\lambda_2) \end{aligned} \tag{2.3.46}$$

Remark 2.3.11 *For a function $K^{(\pm)}(x,a,b,\lambda)$ which is $C^1(\mathbb{R}^4)$ and satisfy the scaling property (2.3.43) we have that*

$$K^{(\pm)}(x,a,b,\lambda c^{-1}) = K^{(\pm)}(xc^{-1},ac^{-1},bc^{-1},\lambda) \tag{2.3.47}$$

and by differentiating with respect of c we conclude that

$$\begin{aligned} &\lambda\frac{\partial}{\partial(\lambda c^{-1})}K^{(\pm)}(x,a,b,\lambda c^{-1}) \\ =\;&\left(x\frac{\partial}{\partial(xc^{-1})}+a\frac{\partial}{\partial(ac^{-1})}+b\frac{\partial}{\partial(bc^{-1})}\right)K^{(\pm)}(xc^{-1},ac^{-1},bc^{-1},\lambda) \end{aligned} \tag{2.3.48}$$

The scaling properties for the de Finetti models with general barriers which we establish in this section in the Proposition 2.3.3 and Proposition 2.3.9 will be very useful because we will use them in order to derive differential equations, the solution of which gives the quantities we are interested in. These quantities are:

(I) For the de Finetti model with one general reflecting barrier:

The moments of the discounted dividends, the Laplace transform of the discounted dividends, the Laplace transform of the time of ruin and the Laplace transform of the joint distribution of the time of ruin and the discounted dividends (see (2.1.20)-(2.1.23)).

(II) For the de Finetti model with two general reflecting barriers:

The moments of the discounted dividends and the discounted financing, the Laplace transform of the discounted dividends, the Laplace transform of the discounted financing and the Laplace transform of the joint distribution of the discounted dividends and the discounted financing (see (2.1.29)-(2.1.31)).

2.4 Boundary conditions.

We proceed in this section to find results that will help us later to find out the boundary conditions of the differential equations we are going to derive. The general idea is to consider two stochastic processes which we agree that they represent the evolution of the reserves of an insurance company in two scenarios. In the first scenario we consider the "original" reserve process which will start at some of the two boundaries and in the other scenario the reserve process will start a little above of the upper barrier or a little below the lower barrier. The important point is that the two stochastic processes are the same in the sense that they satisfy the same stochastic differential equation except from the fact that they have different initial conditions. We will try to find what relations might have the times of ruin, the dividends and the financing in the de Finetti models with general barriers for these two processes.

Before proceed with the main task of this section we want to mention that as we will see in next section, the quantities we are interesting for and which have been defined in (2.1.20)-(2.1.23) and (2.1.29)-(2.1.31) in the context of de Finetti models with general barriers, satisfy some differential equations. In the proposition that follows we derive relations that will be used in the derivation of boundary conditions for the differential equations.

We proceed now with the proposition by taking into account the following simplifying notation remark.

Remark 2.4.1 *For this section only, because in the two scenarios that we assume only the process X alters it's initial state, we drop the dependence from a and b. For example instead of writing $E^{(x,a,b)}$ we simply write E^x or instead of writing $U(x,a,b)$ we write $U(x)$.*

Proposition 2.4.2 ***(Boundary conditions).*** *Consider functions $f, g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, with*

$f, g \in C^1(\mathbb{R}^+ \times \mathbb{R}^+)$ such that

$$\begin{aligned} E^b\left(\frac{\partial}{\partial U} f\left(U,T\right)\right) &< \infty \\ E^b\left(\frac{\partial}{\partial U^{(+)}} g(U^{(+)},U^{(-)})\right) &< \infty \\ E^0\left(\frac{\partial}{\partial U^{(-)}} g(U^{(+)},U^{(-)})\right) &< \infty \end{aligned}$$

where T is the time of ruin and U is the discounted dividends in the de Finetti model with one general reflecting barrier and $U^{(+)}$ is the discounted dividends and $U^{(-)}$ is the discounted financing in the de Finetti model with two general reflecting barriers. Then the following hold

(I)

$$\lim_{\varepsilon\to 0} \frac{E^{b+\varepsilon}\left(f(U,T)\right) - E^b(f(U,T))}{\varepsilon} = E^b\left(\frac{\partial}{\partial U} f(U,T)\right) \tag{2.4.1}$$

(II)

$$E^0\left(f(U,T)\right) = f(0,0) \tag{2.4.2}$$

(III)

$$\lim_{\varepsilon\to 0} \frac{E^{b+\varepsilon}\left(g(U^{(+)},U^{(-)})\right) - E^b(g(U^{(+)},U^{(-)}))}{\varepsilon} = E^b\left(\frac{\partial}{\partial U^{(+)}} g(U^{(+)},U^{(-)})\right) \tag{2.4.3}$$

(IV)

$$\lim_{\varepsilon\to 0} \frac{E^{-\varepsilon}\left(g(U^{(+)},U^{(-)})\right) - E^0(g(U^{(+)},U^{(-)}))}{\varepsilon} = -E^0\left(\frac{\partial}{\partial U^{(-)}} g(U^{(+)},U^{(-)})\right) \tag{2.4.4}$$

Proof.

(I) We assume the de Finetti model with one general reflecting barrier and we consider two processes which we agree to represent the evolution of the reserves of an insurance company in two scenarios. The first scenario is that the insurance company starts with initial capital $b+\varepsilon$ and the second scenario is that the insurance company starts with initial capital b. In other words these stochastic processes are similar except that the first process has initial state $b+\varepsilon$ and the second process has initial state b. The first stochastic reserve process will give immediately amount ε on dividends and from there and on the two reserve processes will evolve in the same way (they become indistinguishable) and will give the same dividends. In mathematical language the above is expressed as

$$\mathcal{U}_t(b+\varepsilon) = \varepsilon + \mathcal{U}_t(b) \tag{2.4.5}$$

From the previous relation (2.4.5) we easily conclude that:

$$U(b+\varepsilon) = \varepsilon + U(b) \tag{2.4.6}$$

By taking expectations on the random variable $f(U,T)$ and using the above relation (2.4.6) we conclude

$$\begin{aligned} E^{b+\varepsilon}(f(U,T)) - E^{b}(f(U,T)) &= E(f(U(b+\varepsilon),T)) - E(f(U(b),T)) \\ &= E(f(\varepsilon+U(b),T)) - E(f(U(b),T)) \end{aligned}$$

and taking limits as ε tends to zero we have

$$\begin{aligned} & \lim_{\varepsilon\to 0}\frac{E^{b+\varepsilon}(f(U,T)) - E^{b}(f(U,T))}{\varepsilon} \\ = & \lim_{\varepsilon\to 0}\frac{E(f(\varepsilon+U(b),T)) - E(f(U(b),T))}{\varepsilon} \\ = & E(\lim_{\varepsilon\to 0}\frac{(f(\varepsilon+U(b),T)) - (f(U(b),T))}{\varepsilon}) \Longrightarrow \end{aligned}$$

$$\begin{aligned} & \lim_{\varepsilon\to 0}\frac{E^{b+\varepsilon}(f(U,T)) - E^{b}(f(U,T))}{\varepsilon} \\ = & E\left(\frac{\partial}{\partial U(b)} f(U(b),T)\right) \\ = & E^{b}\left(\frac{\partial}{\partial U} f(U,T)\right) \end{aligned}$$

(II) We assume the de Finetti model with one general reflecting barrier and that the reserves process has it's initial state at 0. Then the ruin is immediate and of course the dividends are zero, that is it holds that:

$$E^{0}\left(f(U,T)\right) = f(0,0) \tag{2.4.7}$$

(III) We assume the de Finetti model with two general reflecting barriers and let two processes which represent the reserves of an insurance company in the two scenarios and which are similar except that the first has initial state $b+\varepsilon$ and the second has initial state b. The first will give immediately amount ε on dividends and from there and on the two processes will be financing by the same amount and will give the same dividends, that is

$$\mathcal{U}_t^{(+)}(b+\varepsilon) = \varepsilon + \mathcal{U}_t^{(+)}(b) \tag{2.4.8}$$

$$\mathcal{U}_t^{(-)}(b+\varepsilon) = \mathcal{U}_t^{(-)}(b) \tag{2.4.9}$$

from which we conclude that:

$$U^{(+)}(b+\varepsilon) = \varepsilon + U^{(+)}(b) \tag{2.4.10}$$

$$U^{(-)}(b+\varepsilon) = U^{(-)}(b) \tag{2.4.11}$$

By taking expectations on the random variable $g(U^{(+)}, U^{(-)})$ we have that:

$$\begin{aligned} & E^{b+\varepsilon}(g(U^{(+)}, U^{(-)})) - E^{b}(g(U^{(+)}, U^{(-)})) \\ = \ & E(g(U^{(+)}(b+\varepsilon), U^{(-)}(b+\varepsilon))) - E(g(U^{(+)}(b), U^{(-)}(b))) = \\ = \ & E(g(\varepsilon + U^{(+)}(b), U^{(-)}(b))) - E(g(U^{(+)}(b), U^{(-)}(b))) \end{aligned}$$

Dividing by ε and taking limits as ε tends to zero we have:

$$\begin{aligned} & \lim_{\varepsilon \to 0} \frac{E^{b+\varepsilon}(g(U^{(+)}, U^{(-)})) - E^{b}(g(U^{(+)}, U^{(-)}))}{\varepsilon} \\ = \ & \lim_{\varepsilon \to 0} \frac{E(g(\varepsilon + U^{(+)}(b), U^{(-)}(b))) - E(g(U^{(+)}(b), U^{(-)}(b)))}{\varepsilon} \\ = \ & E\left(\lim_{\varepsilon \to 0} \frac{(g(\varepsilon + U^{(+)}(b), U^{(-)}(b))) - (g(U^{(+)}(b), U^{(-)}(b)))}{\varepsilon} \right) \Longrightarrow \end{aligned}$$

$$\begin{aligned} & \lim_{\varepsilon \to 0} \frac{E^{b+\varepsilon}(g(U^{(+)}, U^{(-)})) - E^{b}(g(U^{(+)}, U^{(-)}))}{\varepsilon} \\ = \ & E\left(\frac{\partial}{\partial U^{(+)}(b)} g(U^{(+)}(b), U^{(-)}(b)) \right) \\ = \ & E^{b}\left(\frac{\partial}{\partial U^{(+)}} g(U^{(+)}, U^{(-)}) \right) \end{aligned}$$

(IV) We assume the de Finetti model with two general reflecting barriers and let two stochastic processes which represent the reserves of an insurance company in the two scenarios and which are similar except that the first has initial state $-\varepsilon$ and the second has initial state 0. Then the first will be financing immediately by amount ε and from there and on the two processes will be financing by the same amount and will give the same dividends, that is

$$\mathcal{U}_t^{(-)}(-\varepsilon) = -\varepsilon + \mathcal{U}_t^{(-)}(0) \tag{2.4.12}$$

$$\mathcal{U}_t^{(+)}(-\varepsilon) = \mathcal{U}_t^{(+)}(0) \tag{2.4.13}$$

from which we conclude that:

$$U^{(-)}(-\varepsilon) = -\varepsilon + U^{(-)}(0) \tag{2.4.14}$$

$$U^{(+)}(-\varepsilon) = U^{(+)}(0) \tag{2.4.15}$$

By taking expectations on the random variable $g(U^{(+)}, U^{(-)})$ and using the above relations (2.4.14) and (2.4.15) we conclude that

$$\begin{aligned} & E^{-\varepsilon}(g(U^{(+)}, U^{(-)})) - E^{0}(g(U^{(+)}, U^{(-)})) \\ = \ & E(g(U^{(+)}(-\varepsilon), U^{(-)}(-\varepsilon))) - E(g(U^{(+)}(0), U^{(-)}(0))) \\ = \ & E(g(U^{(+)}(0), -\varepsilon + U^{(-)}(0)))) - E(g(U^{(+)}(0), U^{(-)}(0))) \end{aligned}$$

Dividing by ε and taking limits as ε tends to zero we have:

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{E^{-\varepsilon}(g(U^{(+)},U^{(-)}))-E^0(g(U^{(+)},U^{(-)}))}{\varepsilon} \\
= &\lim_{\varepsilon\to 0}\frac{E(g(U^{(+)}(0),-\varepsilon+U^{(-)}(0))))-E(g(U^{(+)}(0),U^{(-)}(0)))}{\varepsilon} \\
= &E\left(\lim_{\varepsilon\to 0}\frac{g(U^{(+)}(0),-\varepsilon+U^{(-)}(0)))-g(U^{(+)}(0),U^{(-)}(0))}{\varepsilon}\right) \Longrightarrow
\end{aligned}$$

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{E^{-\varepsilon}(g(U^{(+)},U^{(-)}))-E^0(g(U^{(+)},U^{(-)}))}{\varepsilon} \\
= &-E\left(\frac{\partial}{\partial U^{(-)}(0)}g(U^{(+)}(0),U^{(-)}(0))\right) \\
= &-E^0\left(\frac{\partial}{\partial U^{(-)}}g(U^{(+)},U^{(-)})\right)
\end{aligned}$$

■

2.5 Expressions for the generator.

In this section we will find expressions for the generator operator when it is applied to some particular functions. These expressions will be useful in Section 2.6.

First we will find expressions for the generator operator when it is applied to the quantities we are interesting in and have defined in the context of de Finetti models with general barriers (see (2.1.20)-(2.1.23) and (2.1.29)-(2.1.31)).

Second we will find expressions for the generator operator when it is applied to functions which belong to some special functions spaces which we will define in the Definitions 2.5.10-2.5.13 and 2.5.18-2.5.20.

Studying the de Finetti models with general barriers it turns out that we need to define some auxiliary processes.

Definition 2.5.1 *We define for $0 \le t < \infty$ the processes:*

$$\gamma_t := b_t - a_t \tag{2.5.1}$$

$$h_t := \gamma_t^{-1} = (b_t - a_t)^{-1} \tag{2.5.2}$$

$$\mathcal{Y}_t := (X_t - a_t)h_t \tag{2.5.3}$$

$$\mathcal{Z}_t := (\gamma_t, \mathcal{Y}_t) \tag{2.5.4}$$

Remark 2.5.2 *For reasons of simplicity and when there is no possibility of confusion we adopt the convention to write* μ_x*,* μ_a *and* μ_b *instead of* $\mu_x(x)$*,* $\mu_a(a)$ *and* $\mu_b(b)$ *and* σ_x*,* σ_a *and* σ_b *instead of* $\sigma_x(x)$*,* $\sigma_a(a)$ *and* $\sigma_b(b)$ *respectively.*

Definition 2.5.3 *We define*

$$\sigma_{xa}^2 := \sigma_{xa}^2(x,a) := \sigma_x^2 - 2\rho_{xa}\sigma_x\sigma_a + \sigma_a^2 \tag{2.5.5}$$

$$\sigma_{xb}^2 := \sigma_{xb}^2(x,b) := \sigma_x^2 - 2\rho_{xb}\sigma_x\sigma_b + \sigma_b^2 \tag{2.5.6}$$

$$\sigma_{ab}^2 := \sigma_{ab}^2(a,b) := \sigma_b^2 - 2\rho_{ab}\sigma_a\sigma_b + \sigma_b^2 \tag{2.5.7}$$

Definition 2.5.4 *We define*

$$\theta := \theta(x,a,b) := (b-a)\mu_x(x) + (x-b)\mu_a(a) + (a-x)\mu_b(b) \tag{2.5.8}$$

$$\begin{aligned}\phi &:= \varphi(x,a,b) := (b-x)(b-a)\sigma_{xa}^2(x,a) + (x-a)(x-b)\sigma_{ab}^2(a,b) + \\ &\quad +(a-x)(a-b)\sigma_{xb}^2(x,b)\end{aligned} \tag{2.5.9}$$

$$\begin{aligned}\xi &:= \xi(x,a,b) := (a+b-2x)\sigma_{ab}^2(a,b) - \\ &\quad -(a-b)(\sigma_{xa}^2(x,a) - \sigma_{xb}^2(x,b))\end{aligned} \tag{2.5.10}$$

However before proceed with the main task of this section we will make the following remark about the generators of the auxiliary processes γ, Y and Z. This remark will be useful later in this section.

Remark 2.5.5 *(I) By an application of the Itô rule (see Theorem 1.3.22) one can see that the process* $\gamma = \{\gamma_t; 0 \le t < \infty\}$ *has dynamics:*

$$d\gamma_t = (\mu_b - \mu_a)dt + (\sigma_b dB_t^b - \sigma_a dB_t^a) \tag{2.5.11}$$

or written in a matrix style form:

$$d\gamma_t = (\mu_b - \mu_a)dt + \begin{pmatrix} -\sigma a & \sigma_b \end{pmatrix} \cdot \begin{pmatrix} dB_t^a \\ dB_t^b \end{pmatrix} \tag{2.5.12}$$

and we easily conclude that the generator A_γ *of the process* γ *is given by:*

$$\mathcal{A}_\gamma = (\mu_b - \mu_a)\frac{\partial}{\partial\gamma} + \frac{1}{2}\sigma_{ab}^2\frac{\partial^2}{\partial\gamma^2} \tag{2.5.13}$$

for a function $f \in C_0^2(\mathbb{R})$ *or for a function* $f \in C_b^2(\mathbb{R})$*.*

(II) *Also by an application of the Itô rule (see Theorem 1.3.22) one can see that the process* $Y = \{Y_t; 0 \le t < \infty\}$ *has dynamics*

$$
\begin{aligned}
d\mathcal{Y}_t &= \{h_t(\mu_x - \mu_a) + (X_t - a_t)(-h_t^2(\mu_b - \mu_a) + h_t^3\sigma_{ab}^2) - \\
&\quad -h_t^2(\sigma_x\sigma_b\rho_{xb} - \sigma_x\sigma_a\rho_{xa} - \sigma_a\sigma_b\rho_{ab} + \sigma_a^2)\}dt + \\
&\quad h_t(\sigma_x dB_t^x - \sigma_a dB_t^a) - (X_t - a_t)h_t^2(\sigma_b dB_t^b - \sigma_a dB_t^a)
\end{aligned}
\tag{2.5.14}
$$

or written in a matrix style form:

$$
\begin{aligned}
d\mathcal{Y}_t &= \{h_t(\mu_x - \mu_a) + (X_t - a_t)(-h_t^2(\mu_b - \mu_a) + h_t^3\sigma_{ab}^2) - \\
&\quad -h_t^2(\sigma_x\sigma_b\rho_{xb} - \sigma_x\sigma_a\rho_{xa} - \sigma_a\sigma_b\rho_{ab} + \sigma_a^2)\}dt + \\
&\quad \begin{pmatrix} \sigma_x h_t & \sigma_a(X_t - a_t)h_t^2 - \sigma_a h_t & -\sigma_b(X_t - a_t)h_t^2 \end{pmatrix} \cdot \begin{pmatrix} dB_t^x \\ dB_t^a \\ dB_t^a \end{pmatrix}
\end{aligned}
\tag{2.5.15}
$$

and we easily conclude that the generator A_y *of the process* Y *is given by*

$$
\mathcal{A}_y = \left(\frac{\theta(x,a,b)}{(b-a)^2} - \frac{\xi(x,a,b)}{2(b-a)^3}\right)\frac{\partial}{\partial y} + \frac{1}{2}\frac{\varphi(x,a,b)}{(b-a)^4}\frac{\partial^2}{\partial y^2}
\tag{2.5.16}
$$

for a function $f \in C_0^2(\mathbb{R})$ *or for a function* $f \in C_b^2(\mathbb{R})$.

(III) *Finally again by an application of the Itô rule (see Theorem 1.3.22) one can see that the process* $Z = \{Z_t; 0 \le t < \infty\}$ *has dynamics:*

$$
\begin{aligned}
d\mathcal{Z}_t &:= \begin{pmatrix} d\gamma_t \\ d\mathcal{Y}_t \end{pmatrix} = \begin{pmatrix} \mu_b - \mu_a \\ \mu_y(t, \mathcal{Y}_t) \end{pmatrix} dt + \\
&\quad + \begin{pmatrix} \sigma_b dB_t^b - \sigma_a dB_t^a \\ h_t(\sigma_x dB_t^x - \sigma_a dB_t^a) - (X_t - a_t)h_t^2(\sigma_b dB_t^b - \sigma_a dB_t^a) \end{pmatrix}
\end{aligned}
\tag{2.5.17}
$$

or written in a matrix style form:

$$
\begin{aligned}
d\mathcal{Z}_t &:= \begin{pmatrix} d\gamma_t \\ d\mathcal{Y}_t \end{pmatrix} = \begin{pmatrix} \mu_b - \mu_a \\ \mu_y(t, \mathcal{Y}_t) \end{pmatrix} dt + \\
&\quad + \begin{pmatrix} 0 & -\sigma_a & \sigma_b \\ \sigma_x h_t & \sigma_a(X_t - a_t)h_t^2 - \sigma_a h_t & -\sigma_b(X_t - a_t)h_t^2 \end{pmatrix} \cdot \begin{pmatrix} dB_t^x \\ dB_t^a \\ dB_t^b \end{pmatrix}
\end{aligned}
\tag{2.5.18}
$$

with

$$
\begin{aligned}
\mu_y(t, \mathcal{Y}_t) &:= h_t(\mu_x - \mu_a) + (X_t - a_t)(-h_t^2(\mu_b - \mu_a) + h_t^3\sigma_{ab}^2) - \\
&\quad -h_t^2(\sigma_x\sigma_b\rho_{xb} - \sigma_x\sigma_a\rho_{xa} - \sigma_a\sigma_b\rho_{ab} + \sigma_a^2)
\end{aligned}
\tag{2.5.19}
$$

and we easily conclude that the generator A_z *of the process* Z *is given by*

$$\begin{aligned}\mathcal{A}_z &= (\mu_b - \mu_a)\frac{\partial}{\partial\gamma} + \frac{1}{2}\sigma^2_{ab}\frac{\partial^2}{\partial\gamma^2} + \\ &+ \frac{\xi(x,a,b)}{2(b-a)^2}\frac{\partial^2}{\partial\gamma\partial y} + \\ &+ \left(\frac{\theta(x,a,b)}{(b-a)^2} - \frac{\xi(x,a,b)}{2(b-a)^3}\right)\frac{\partial}{\partial y} + \frac{1}{2}\frac{\varphi(x,a,b)}{(b-a)^4}\frac{\partial^2}{\partial y^2}\end{aligned} \quad (2.5.20)$$

for a function $f \in C^2_0(\mathbb{R})$ *or for a function* $f \in C^2_b(\mathbb{R})$*.*

Also we will need the following definition.

Definition 2.5.6 *For the de Finetti model with one general reflecting barrier and time* t *smaller than the time of ruin* T*, that is* $t < T$ *we define the discounted dividends* $U(t)$ *from time* t *to the time of ruin* T *to be given by:*

$$U(t) := \int_t^T e^{-\delta s} d\mathcal{U}_s \quad (2.5.21)$$

Similarly for the de Finetti model with two general reflecting barriers we define the discounted dividends $U^{(+)}(t)$ *and the discounted financing* $U^{(-)}(t)$ *starting from the time* t *to be given by:*

$$U^{(+)}(t) := \int_t^\infty e^{-\delta s} d\mathcal{U}^{(+)}_s \quad (2.5.22)$$

$$U^{(-)}(t) := \int_t^\infty e^{-\delta s} d\mathcal{U}^{(-)}_s \quad (2.5.23)$$

Finally before we proceed to derive the main results of this section we make the following useful remark.

Remark 2.5.7 *(I) For the de Finetti model with one general reflecting barrier it holds that:*

$$\theta_t U^n = \begin{cases} e^{n\delta t}U^n(t), & \text{for } t \le T \\ 0, & \text{for } t > T \end{cases} \quad (2.5.24)$$

with $n = 1, 2, 3, \ldots$

Indeed by considering a sequence of partitions $\{t^m_k\}_{m\in\mathbb{N},k\in\mathbb{N}}$ *of* $[0,\infty)$ *such that:*

$$0 \le t^m_0 \le t^m_1 \le \cdots \le t^m_{k_m} < \infty$$

with

$$\lim_{m\to\infty} \sup_k t^m_k = \infty$$

and

$$||\pi_m|| := \sup_k |t^m_{k+1} - t^m_k| \to 0 \quad as\ m \to \infty$$

we can approximate the discounted dividends U by:

$$U = \int_0^T e^{-\delta s} d\mathcal{U}_s = \lim_{||\pi_m|| \to 0} \sum e^{-\delta t_i^m} 1_{\{t_i^m < T\}} \Delta \mathcal{U}_{t_i^m}$$

where $1_{\{.\}}$ is the indicator function. Applying the right shift operator to the above relation we conclude:

$$\begin{aligned} \theta_t U^n &= (\lim_{||\pi_m|| \to 0} \sum e^{-\delta t_i^m} 1_{\{t_i^m + t < T\}} \Delta \mathcal{U}_{t_i^m + t})^n = \\ &= e^{n\delta t} (\lim_{||\pi_m|| \to 0} \sum e^{-\delta (t_i^m + t)} 1_{\{t_i^m + t < T\}} \Delta \mathcal{U}_{t_i^m + t})^n \\ &= \begin{cases} e^{n\delta t} U^n(t), & for\ t \le T \\ 0, & for\ t > T \end{cases} \end{aligned}$$

Also it is easy to see that:

$$\theta_t T = \begin{cases} T - t & for\ t \le T \\ 0 & for\ t > T \end{cases} \tag{2.5.25}$$

(II) For the de Finetti model with two general reflecting barriers it holds that:

$$\theta_t (U^{(\pm)})^n = e^{n\delta t} \left(U^{(\pm)}(t) \right)^n \tag{2.5.26}$$

with $n = 1, 2, 3, \ldots$ and time $t \ge 0$.

Indeed working similarly as in (I) *we have that*

$$\begin{aligned} \theta_t (U^{(\pm)})^n &= \theta_t \left(\int_0^\infty e^{-\delta s} d\mathcal{U}_s^{(\pm)} \right)^n \\ &= \left(e^{\delta t} \int_t^\infty e^{-\delta s} d\mathcal{U}_s^{(\pm)} \right)^n = e^{n\delta t} \left(U^{(\pm)}(t) \right)^n \end{aligned}$$

Now we are ready to derive the main results. We start with the De Finetti model with one general reflecting barrier in the section 2.5.1 and we continue with the de Finetti model with two general reflecting barriers in the section 2.5.2.

2.5.1 Generator expressions for the one Reflecting barrier case.

We start this subsection by proving a useful proposition, concerning the generator operator applied to a regular function of the discounted dividends and the time of ruin in the context of de Finetti model with one general reflecting barrier.

Proposition 2.5.8 *Let $f(x,a,b) := E^{(x,a,b)}(g(U,T))$ with $g(\cdot) \in C_b^1(R^2)$. Then it holds that:*

$$\mathcal{A}_{(x,a,b)} f(x,a,b) = E^{(x,a,b)} \left(\frac{\partial}{\partial t} g(e^{\delta t} U(t), T-t)|_{t=0} \right) \tag{2.5.27}$$

Proof. Using the *relations* (2.5.24),(2.5.25) of the *Remark* (2.5.7) we have that:

$$\begin{aligned}
\mathcal{A}_{(x,a,b)} f(x,a,b) &= \lim_{t \to 0} \frac{E^{(x,a,b)}(f(X_t,a_t,b_t)) - f(x,a,b)}{t} = \\
&= \lim_{t \to 0} \frac{E^{(x,a,b)}\left(E^{(X_t,a_t,b_t)}(g(U,T))\right) - E^{(x,a,b)}(g(U,T))}{t} = \\
&= \lim_{t \to 0} \frac{E^{(x,a,b)}(E^{(x,a,b)}(\theta_t g(U,T)|\mathcal{F}_t)) - E^{(x,a,b)}(g(U,T))}{t} = \\
&= \lim_{t \to 0} \frac{E^{(x,a,b)}(g(\theta_t U, \theta_t T)) - E^{(x,a,b)}(g(U,T))}{t} = \\
&= \lim_{t \to 0} E^{(x,a,b)} \left(1_{\{T \geq t\}} \frac{g(e^{\delta t}U(t), T-t) - g(U,T)}{t} \right) + \\
&\quad + \lim_{t \to 0} E^{(x,a,b)} \left(1_{\{T < t\}} \frac{g(0,0) - g(U,T)}{t} \right) \\
&= E^{(x,a,b)} \left(1_{\{T \geq 0\}} \lim_{t \to 0} \frac{g(e^{\delta t}U(t), T-t) - g(U,T)}{t} \right) + \\
&\quad + E^{(x,a,b)} \left(1_{\{T \leq 0\}} \lim_{t \to 0} \frac{g(0,0) - g(U,T)}{t} \right)
\end{aligned}$$

where we have explicitly used the Markov property of the diffusion process $(X, \mathbf{a}, \mathbf{b})$ (see *Theorem* 1.3.28).

Because the process $(X, \mathbf{a}, \mathbf{b})$ is continuous we have that

$$T > 0 \quad a.s.$$

(that is the process X does not jump to the process $\mathbf{a}$ to be ruined and neither the reverse) and so the above relation becomes:

$$\mathcal{A}_{(x,a,b)} f(x,a,b) = E^{(x,a,b)} \left(\lim_{t \to 0} \frac{g(e^{\delta t}U(t), T-t) - g(U,T)}{t} \right) \Longrightarrow$$

$$\mathcal{A}_{(x,a,b)} f(x,a,b) = E^{(x,a,b)} \left(\frac{\partial}{\partial t} g(e^{\delta t} U(t), T-t)|_{t=0} \right)$$

■

Next we will use Proposition 2.5.8 in order to find relations with regards to the generator operator applied to the quantities we are interested in, that is for the Laplace transform of the discounted dividends $K(x,a,b,\lambda)$, the Laplace transform of the time of ruin $M(x,a,b,\lambda)$ and the Laplace transform of the joint distribution of the time of ruin and the discounted dividends $N(x,a,b,\lambda_1,\lambda_2)$, as they have defined at (2.1.20)–(2.1.22).

The following proposition clarifies the action of the operator on the aforementioned quantities.

Proposition 2.5.9 *(Relations with regards to the generator operator applied to the quantities (2.1.20)–(2.1.23) in the de Finetti model with one general reflecting barrier). It holds that:*

(i)

$$\mathcal{A}_{(x,a,b)} N(x,a,b,\lambda_1,\lambda_2) = \lambda_1 N(x,a,b,\lambda_1,\lambda_2) + \delta\lambda_2 \frac{\partial}{\partial\lambda_2} N(x,a,b,\lambda_1,\lambda_2) \tag{2.5.28}$$

where $\lambda_1, \lambda_2 > 0$.

(II)

$$\mathcal{A}_{(x,a,b)} K(x,a,b,\lambda) = \lambda\delta \frac{\partial}{\partial\lambda} K(x,a,b,\lambda) \tag{2.5.29}$$

where $\lambda > 0$.

(III)

$$\mathcal{A}_{(x,a,b)} M(x,a,b,\lambda) = \lambda M(x,a,b,\lambda) \tag{2.5.30}$$

where $\lambda > 0$.

Proof. The claims *(I)*,*(II)* and *(III)* follow immediately by applying Proposition 2.5.8 to the functions: $g_1(U,T) = e^{-\lambda_1 T - \lambda_2 U}$ with $\lambda_1, \lambda_2 > 0$, $g_2(U,T) = e^{-\lambda U}$ with $\lambda > 0$ and $g_3(U,T) = e^{-\lambda T}$ with $\lambda > 0$ respectively. ■

We turn our attention now to some special function spaces.

Definition 2.5.10

$$S^V := \{ f \in C_b^2(\mathbb{R}^3) \mid f \textit{ satisfy the scaling properties (2.3.10), (2.3.14)} \}$$

Definition 2.5.11

$$S^K := \{ f \in C_b^2(\mathbb{R}^4) \mid f \textit{ satisfy the scaling properties (2.3.11), (2.3.15)} \}$$

Definition 2.5.12

$$S^M := \{ f \in C_b^2(\mathbb{R}^4) \mid f \textit{ satisfy the scaling properties (2.3.12), (2.3.16)} \}$$

Definition 2.5.13

$$S^N := \{ f \in C_b^2(\mathbb{R}^5) \mid f \textit{ satisfy the scaling properties (2.3.13), (2.3.17)} \}$$

Next we find a set of expressions with regard of the generator operator applied to functions which belong to the above function spaces.

Lemma 2.5.14 *It holds that*

(I)

$$\mathcal{A}_{(x,a,b)}\widetilde{N}(x,a,b,\lambda_1,\lambda_2(b-a)^{-1}) = \mathcal{A}_y\widetilde{N}((x-a)(b-a)^{-1},0,1,\lambda_1,\lambda_2) \tag{2.5.31}$$

(II)

$$\mathcal{A}_{(x,a,b)}\widetilde{V}(x,a,b) = \mathcal{A}_z\gamma\widetilde{V}(y,0,1) \tag{2.5.32}$$

(III)

$$\mathcal{A}_{(x,a,b)}\widetilde{K}(x,a,b,(b-a)^{-1}\lambda) = \mathcal{A}_y\widetilde{K}((x-a)(b-a)^{-1},0,1,\lambda) \tag{2.5.33}$$

(IV)

$$\mathcal{A}_{(x,a,b)}\widetilde{M}(x,a,b,\lambda) = \mathcal{A}_y\widetilde{M}((x-a)(b-a)^{-1},0,1,\lambda) \tag{2.5.34}$$

Proof. The proof constitutes of two steps.

Step 1

We apply the scaling property (see Proposition 2.3.3) to the functions $\widetilde{V}(\cdot) \in S^V$, $\widetilde{K}(\cdot) \in S^K$, $\widetilde{M}(\cdot) \in S^M$ and $\widetilde{N}(\cdot) \in S^N$ and we easily conclude that:

$$\widetilde{N}(X_t,a_t,b_t,\lambda_1,h_t\lambda_2) = \widetilde{N}((X_t-a_t)h_t,0,1,\lambda_1,\lambda_2) \tag{2.5.35}$$

$$\widetilde{V}(X_t,a_t,b_t) = \widetilde{V}(X_t-a_t,0,b_t-a_t) = \gamma_t\widetilde{V}((X_t-a_t)h_t,0,1) \tag{2.5.36}$$

$$\widetilde{K}(X_t,a_t,b_t,h_t\lambda) = \widetilde{K}((X_t-a_t)h_t,0,1,\lambda) \tag{2.5.37}$$

$$\widetilde{M}(X_t,a_t,b_t,\lambda) = \widetilde{M}(X_t-a_t,0,b_t-a_t,\lambda) = \widetilde{M}((X_t-a_t)h_t,0,1,\lambda) \tag{2.5.38}$$

Step 2

We prove the relations (2.5.31)-(2.5.34) between the generator $A_{(x,a,b)}$ of the process $(X, \mathbf{a}, \mathbf{b})$ and the generators A_y of the process Y and A_z of the process Z.

(I) It holds that:

$$\begin{aligned}
&\mathcal{A}_{(x,a,b)}\widetilde{N}(x,a,b,\lambda_1,\lambda_2(b-a)^{-1})\\
=\ & \lim_{t\to 0}\frac{E(\widetilde{N}(X_t,a_t,b_t,\lambda_1,\lambda_2(b-a)^{-1})) - \widetilde{N}(x,a,b,\lambda_1,\lambda_2(b-a)^{-1})}{t}\\
=\ & \lim_{t\to 0}\frac{E(\widetilde{N}((X_t-a_t)(b_t-a_t)^{-1},0,1,\lambda_1,\lambda_2)) - \widetilde{N}((x-a)(b-a)^{-1},0,1,\lambda_1,\lambda_2)}{t}\\
=\ & \mathcal{A}_y\widetilde{N}((x-a)(b-a)^{-1},0,1,\lambda_1,\lambda_2) \Longrightarrow
\end{aligned}$$

$$\mathcal{A}_{(x,a,b)}\widetilde{N}(x,a,b,\lambda_1,\lambda_2(b-a)^{-1}) = \mathcal{A}_y\widetilde{N}((x-a)(b-a)^{-1},0,1,\lambda_1,\lambda_2)$$

(II)

$$\begin{aligned}
\mathcal{A}_{(x,a,b)}\widetilde{V}(x,a,b) &= \lim_{t\to 0}\frac{E(\widetilde{V}(X_t,a_t,b_t)) - \widetilde{V}(x,a,b)}{t}\\
&= \lim_{t\to 0}\frac{E(\gamma_t\widetilde{V}(Y_t,0,1)) - \gamma\widetilde{V}(y,0,1)}{t}\\
&= \mathcal{A}_z\gamma\widetilde{V}(y,0,1) \Longrightarrow
\end{aligned}$$

$$\mathcal{A}_{(x,a,b)}\widetilde{V}(x,a,b) = \mathcal{A}_z\gamma\widetilde{V}(y,0,1)$$

(III)

$$\begin{aligned}
&\mathcal{A}_{(x,a,b)}\widetilde{K}(x,a,b,(b-a)^{-1}\lambda)\\
=\ & \lim_{t\to 0}\frac{E(\widetilde{K}(X_t,a_t,b_t,(b-a)^{-1}\lambda)) - \widetilde{K}(x,a,b,(b-a)^{-1}\lambda)}{t}\\
=\ & \lim_{t\to 0}\frac{E(\widetilde{K}((X_t-a_t)(b_t-a_t)^{-1},0,1,\lambda)) - \widetilde{K}((x-a)(b-a)^{-1},0,1,\lambda)}{t}\\
=\ & \mathcal{A}_z\widetilde{K}((x-a)(b-a)^{-1},0,1,\lambda) \Longrightarrow
\end{aligned}$$

$$\mathcal{A}_{(x,a,b)}\widetilde{K}(x,a,b,(b-a)^{-1}\lambda) = \mathcal{A}_y\widetilde{K}((x-a)(b-a)^{-1},0,1,\lambda)$$

(IV)

$$\begin{aligned}
&\mathcal{A}_{(x,a,b)}\widetilde{M}(x,a,b,\lambda)\\
=\ & \lim_{t\to 0}\frac{E(\widetilde{M}(X_t,a_t,b_t,\lambda)) - \widetilde{M}(x,a,b,\lambda)}{t}\\
=\ & \lim_{t\to 0}\frac{E(\widetilde{M}((X_t-a_t)(b_t-a_t)^{-1},0,1,\lambda)) - \widetilde{M}((x-a)(b-a)^{-1},0,1,\lambda)}{t}\\
=\ & \mathcal{A}_z\widetilde{M}((x-a)(b-a)^{-1},0,1,\lambda) \Longrightarrow
\end{aligned}$$

$$\mathcal{A}_{(x,a,b)}\widetilde{M}(x,a,b,\lambda) = \mathcal{A}_y\widetilde{M}((x-a)(b-a)^{-1},0,1,\lambda)$$

■

Proposition 2.5.15 *(Relations with regard of the generator operator $\mathcal{A}_{(x,a,b)}$ of the process $(X, \mathbf{a}, \mathbf{b})$ when it is applied to functions which belong to the functions spaces S^V, S^K, S^M and S^N (see Definition 2.5.10-Definition 2.5.13)).*

Let functions $\widetilde{V}(\cdot) \in S^V$, $\widetilde{K}(\cdot) \in S^K$, $\widetilde{M}(\cdot) \in S^M$ and $\widetilde{N}(\cdot) \in S^N$. It holds that

(I)

$$\begin{aligned} &\mathcal{A}_{(x,a,b)} \widetilde{N}(x, a, b, \lambda_1, \lambda_2) = \\ = \ &\frac{\varphi(x, a, b)}{2(b-a)^2} \frac{\partial^2}{\partial x^2} \widetilde{N}(x, a, b, \lambda_1, \lambda_2) + \left(\frac{\theta(x, a, b)}{b-a} - \frac{\xi(x, a, b)}{2(b-a)^2} \right) \frac{\partial}{\partial x} \widetilde{N}(x, a, b, \lambda_1, \lambda_2) \end{aligned} \tag{2.5.39}$$

(II)

$$\begin{aligned} &\mathcal{A}_{(x,a,b)} \widetilde{K}(x, a, b, \lambda) \\ = \ &\frac{\varphi(x, a, b)}{2(b-a)^2} \frac{\partial^2}{\partial x^2} \widetilde{K}(x, a, b, \lambda) + \left(\frac{\theta(x, a, b)}{b-a} - \frac{\xi(x, a, b)}{2(b-a)^2} \right) \frac{\partial}{\partial x} \widetilde{K}(x, a, b, \lambda) \end{aligned} \tag{2.5.40}$$

(III)

$$\begin{aligned} &\mathcal{A}_{(x,a,b)} \widetilde{M}(x, a, b, \lambda) \\ = \ &\frac{\varphi(x, a, b)}{2(b-a)^2} \frac{\partial^2}{\partial x^2} \widetilde{M}(x, a, b, \lambda) + \left(\frac{\theta(x, a, b)}{b-a} - \frac{\xi(x, a, b)}{2(b-a)^2} \right) \frac{\partial}{\partial x} \widetilde{M}(x, a, b, \lambda) \end{aligned} \tag{2.5.41}$$

(IV)

$$\begin{aligned} \mathcal{A}_{(x,a,b)} \widetilde{V}(x, a, b; n) \ = \ &\frac{\mu_b(b) - \mu_a(a)}{b-a} \widetilde{V}(x, a, b; n) + \\ &+ \frac{\theta(x, a, b)}{b-a} \frac{\partial}{\partial x} \widetilde{V}(x, a, b; n) + \frac{\varphi(x, a, b)}{2(b-a)^2} \frac{\partial^2}{\partial x^2} \widetilde{V}(x, a, b; n) \end{aligned} \tag{2.5.42}$$

Proof. The general idea of the proof is to apply Remark 2.5.5 and Lemma 2.5.14 in order to find the expressions (2.5.39)-(2.5.42).

(I) By relation (2.5.31) in Lemma 2.5.14 we have that:

$$\mathcal{A}_{(x,a,b)} \widetilde{N}(x, a, b, \lambda_1, \lambda_2^*(b-a)^{-1}) = \mathcal{A}_y \widetilde{N}((x-a)(b-a)^{-1}, 0, 1, \lambda_1, \lambda_2^*) \tag{2.5.43}$$

By relation (2.5.16) in Remark 2.5.5 we have that:

$$\begin{aligned} &\mathcal{A}_y \widetilde{N}((x-a)(b-a)^{-1}, 0, 1, \lambda_1, \lambda_2^*) = \\ = \ &\left(\frac{\theta(x, a, b)}{(b-a)^2} - \frac{\xi(x, a, b)}{2(b-a)^3} \right) \frac{\partial}{\partial x} \widetilde{N}((x-a)(b-a)^{-1}, 0, 1, \lambda_1, \lambda_2^*) + \\ &+ \frac{1}{2} \frac{\varphi(x, a, b)}{(b-a)^4} \frac{\partial^2}{\partial x^2} \widetilde{N}((x-a)(b-a)^{-1}, 0, 1, \lambda_1, \lambda_2^*) \end{aligned} \tag{2.5.44}$$

and after some calculations relation (2.5.44) can be written as

$$\begin{aligned} &\mathcal{A}_y\widetilde{N}((x-a)(b-a)^{-1},0,1,\lambda_1,\lambda_2^*) = \\ = &\left(\frac{\theta(x,a,b)}{(b-a)} - \frac{\xi(x,a,b)}{2(b-a)^2}\right)\frac{\partial}{\partial x}\widetilde{N}(x,a,b,\lambda_1,\lambda_2^*(b-a)^{-1}) + \\ &+\frac{1}{2}\frac{\varphi(x,a,b)}{(b-a)^2}\frac{\partial^2}{\partial x^2}\widetilde{N}(x,a,b,\lambda_1,\lambda_2^*(b-a)^{-1}) \end{aligned} \tag{2.5.45}$$

By the relation (2.5.43) the above expression (2.5.45) is equivalent to

$$\begin{aligned} &\mathcal{A}_{(x,a,b)}\widetilde{N}(x,a,b,\lambda_1,\lambda_2^*(b-a)^{-1}) = \\ = &\left(\frac{\theta(x,a,b)}{(b-a)} - \frac{\xi(x,a,b)}{2(b-a)^2}\right)\frac{\partial}{\partial x}\widetilde{N}(x,a,b,\lambda_1,\lambda_2^*(b-a)^{-1}) + \\ &+\frac{1}{2}\frac{\varphi(x,a,b)}{(b-a)^2}\frac{\partial^2}{\partial x^2}\widetilde{N}(x,a,b,\lambda_1,\lambda_2^*(b-a)^{-1}) \end{aligned} \tag{2.5.46}$$

and setting $\lambda_2 := \lambda_2^*(b-a)^{-1}$ we conclude the result.

(II) As in (I) by setting $\lambda_1 = 0$ and $\lambda_2 = \lambda$.

(III) As in (I) by setting $\lambda_1 = \lambda$ and $\lambda_2 = 0$.

(IV) By relation (2.5.32) in Lemma 2.5.14 we have that:

$$\mathcal{A}_{(x,a,b)}\widetilde{V}(x,a,b;n) = \mathcal{A}_z\gamma\widetilde{V}(y,0,1;n) \tag{2.5.47}$$

By relation (2.5.20) in Remark 2.5.5 we have that:

$$\begin{aligned} \mathcal{A}_z\gamma\widetilde{V}(y,0,1;n) &= (\mu_b-\mu_a)\widetilde{V}(y,0,1;n) \\ &+\left(\frac{\theta(x,a,b)}{(b-a)} - \frac{\xi(x,a,b)}{2(b-a)^2}\right)\frac{\partial}{\partial y}\widetilde{V}(y,0,1;n) + \\ &+\frac{1}{2}\frac{\varphi(x,a,b)}{(b-a)^3}\frac{\partial^2}{\partial y^2}\widetilde{V}(y,0,1;n) + \frac{\xi(x,a,b)}{2(b-a)^2}\frac{\partial}{\partial y}\widetilde{V}(y,0,1;n) \end{aligned} \tag{2.5.48}$$

and simplifying we get

$$\begin{aligned} \mathcal{A}_z\gamma\widetilde{V}(y,0,1;n) &= (\mu_b-\mu_a)\widetilde{V}(y,0,1;n) \\ &+\frac{\theta(x,a,b)}{(b-a)}\frac{\partial}{\partial y}\widetilde{V}(y,0,1;n) + \frac{1}{2}\frac{\varphi(x,a,b)}{(b-a)^3}\frac{\partial^2}{\partial y^2}\widetilde{V}(y,0,1;n) \end{aligned} \tag{2.5.49}$$

Using relation (2.5.47) and simplifying relation (2.5.49) can be written as

$$\begin{aligned} \mathcal{A}_{(x,a,b)}\widetilde{V}(x,a,b;n) &= \frac{\mu_b-\mu_a}{b-a}\widetilde{V}(x,a,b;n) \\ &+\frac{\theta(x,a,b)}{(b-a)}\frac{\partial}{\partial x}\widetilde{V}(x,a,b;n) + \frac{1}{2}\frac{\varphi(x,a,b)}{(b-a)^2}\frac{\partial^2}{\partial x^2}\widetilde{V}(x,a,b;n) \end{aligned} \tag{2.5.50}$$

This concludes the proof. ■

In the next subsection we consider the de Finetti model with two general reflecting barriers and with the same logic as in this subsection we find analogous results.

2.5.2 Generator expressions for the two general reflecting barriers case.

We start this subsection by proving a proposition about the generator operator applied to a regular function of the discounted dividends and the discounting financing in the context of de Finetti model with two general reflecting barriers.

Proposition 2.5.16 *Let $f(x,a,b) := E^{(x,a,b)}(h(U^{(+)},U^{(-)}))$ with $h \in C_b^1(R^2)$. Then it holds that:*

$$\mathcal{A}_{(x,a,b)} f(x,a,b) = E^{(x,a,b)}\left(\frac{\partial}{\partial t} h(e^{\delta t}U^{(+)}(t), e^{\delta t}U^{(-)}(t))|_{t=0}\right) \tag{2.5.51}$$

Proof. Using *relation* (2.5.26) in *Remark* 2.5.7 and the *Markov property* of the diffusion process $(X,\mathbf{a},\mathbf{b})$ (see *Theorem* 1.3.28) we have that:

$$\begin{aligned}
& \mathcal{A}_{(x,a,b)} f(x,a,b) \\
= \;& \lim_{t\to 0} \frac{E^{(x,a,b)}(f(X_t,a_t,b_t)) - f(x,a,b)}{t} = \\
= \;& \lim_{t\to 0} \frac{E^{(x,a,b)}(E^{(X_t,a_t,b_t)}(h(U^{(+)},U^{(-)}))) - E^{(x,a,b)}(h(U^{(+)},U^{(-)}))}{t} = \\
= \;& \lim_{t\to 0} \frac{E^{(x,a,b)}(E^{(x,a,b)}(\theta_t h(U^{(+)},U^{(-)})|\mathcal{F}_t)) - E^{(x,a,b)}(h(U^{(+)},U^{(-)}))}{t} = \\
= \;& \lim_{t\to 0} \frac{E^{(x,a,b)}(E^{(x,a,b)}(h(\theta_t U^{(+)},\theta_t U^{(+)})|\mathcal{F}_t)) - E^{(x,a,b)}(h(U^{(+)},U^{(-)}))}{t} = \\
= \;& \lim_{t\to 0} \frac{E^{(x,a,b)}(E^{(x,a,b)}(h(e^{\delta t}U^{(+)}(t),e^{\delta t}U^{(-)}(t))|\mathcal{F}_t)) - E^{(x,a,b)}(h(U^{(+)},U^{(-)}))}{t} = \\
= \;& \lim_{t\to 0} \frac{E^{(x,a,b)}(h(e^{\delta t}U^{(+)}(t),e^{\delta t}U^{(-)}(t))) - E^{(x,a,b)}(h(U^{(+)},U^{(-)}))}{t} = \\
= \;& E^{(x,a,b)}\left(\lim_{t\to 0} \frac{h(e^{\delta t}U^{(+)}(t),e^{\delta t}U^{(-)}(t)) - h(U^{(+)},U^{(-)})}{t}\right) \Longrightarrow
\end{aligned}$$

$$\mathcal{A}_{(x,a,b)} f(x,a,b) = E^{(x,a,b)}\left(\frac{\partial}{\partial t} h(e^{\delta t}U^{(+)}(t), e^{\delta t}U^{(-)}(t))|_{t=0}\right)$$

■

Next we will use Proposition 2.5.16 in order to find relations with regards to the generator operator applied to functions we are interesting in, that is for the Laplace transforms of the discounted dividends $K^{(+)}(x,a,b,\lambda)$ and the discounted financing $K^{(-)}(x,a,b,\lambda)$ and the Laplace transform of the joint distribution of the discounted dividends and the discounted financing $L(x,a,b,\lambda_1,\lambda_2)$, as they were defined at (2.1.29)-(2.1.30).

We conclude the following proposition

Proposition 2.5.17 *(Relations with regard of the generator operator applied to the quantities (2.1.29)-(2.1.31) in the de Finetti model with two general reflecting barriers). It holds that*

(I)

$$\mathcal{A}_{(x,a,b)} L(x,a,b,\lambda_1,\lambda_2) = \delta \left(\lambda_1 \frac{\partial}{\partial \lambda_1} L(x,a,b,\lambda_1,\lambda_2) + \lambda_2 \frac{\partial}{\partial \lambda_2} L(x,a,b,\lambda_1,\lambda_2) \right) \qquad (2.5.52)$$

where $\lambda_1, \lambda_2 > 0$.

(II)

$$\mathcal{A}_{(x,a,b)} K^{(\pm)}(x,a,b,\lambda) = \lambda\delta \frac{\partial}{\partial \lambda} K^{(\pm)}(x,a,b,\lambda) \qquad (2.5.53)$$

where $\lambda > 0$.

Proof. The claims (I) and (II) follow immediately by applying Proposition 2.5.16 to the functions: (I) $h_1(U^{(+)}, U^{(-)}) := e^{-\lambda_1 U^{(+)} - \lambda_2 U^{(-)}}$ with $\lambda_1, \lambda_2 > 0$ and *(II)* $h_2(U^{(+)}, U^{(-)}) := e^{-\lambda U^{(+)}}$ with $\lambda > 0$ and $h_3(U^{(+)}, U^{(-)}) := e^{-\lambda U^{(-)}}$ with $\lambda > 0$. ■

We turn our attention now to some special function spaces.

Definition 2.5.18

$$S^{V^{(\pm)}} := \left\{ f \in C_b^2(\mathbb{R}^3) : f \textit{ satisfies the scaling properties } (2.3.39),(2.3.42) \right\}$$

Definition 2.5.19

$$S^{K^{(\pm)}} := \left\{ f \in C_b^2(\mathbb{R}^4) : f \textit{ satisfies the scaling properties } (2.3.40),(2.3.43) \right\}$$

Definition 2.5.20

$$S^{L} := \left\{ f \in C_b^2(\mathbb{R}^5) : f \textit{ satisfies the scaling properties } (2.3.41),\ (2.3.44) \right\}$$

Next we find a set of expressions with regard of the generator operator applied to functions which belong to the above function spaces.

Lemma 2.5.21 *It holds that*

(I)

$$\mathcal{A}_{(x,a,b)} \widetilde{L}(x,a,b,\lambda_1(b-a)^{-1}, \lambda_2(b-a)^{-1}) = \mathcal{A}_y \widetilde{L}((x-a)(b-a)^{-1}, 0, 1, \lambda_1, \lambda_2) \qquad (2.5.54)$$

(II)

$$\mathcal{A}_{(x,a,b)}\widetilde{V}^{(\pm)}(x,a,b) = \mathcal{A}_z\gamma\widetilde{V}^{(\pm)}(y,0,1) \tag{2.5.55}$$

(III)

$$\mathcal{A}_{(x,a,b)}\widetilde{K}^{(\pm)}(x,a,b,(b-a)^{-1}\lambda) = \mathcal{A}_y\widetilde{K}^{(\pm)}((x-a)(b-a)^{-1},0,1,\lambda) \tag{2.5.56}$$

Proof. The proof constitutes of two steps.

Step 1

We apply the scaling property (see Proposition 2.3.9) to the functions $\widetilde{V}^{(\pm)}(\cdot) \in S^{V^{(\pm)}}$, $\widetilde{K}^{(\pm)}(\cdot) \in S^{K^{(\pm)}}$ and $\widetilde{L}(\cdot) \in S^L$ and we easily conclude that

$$\begin{aligned}
\widetilde{L}(X_t,a_t,b_t,h_t\lambda_1,h_t\lambda_2) &= \widetilde{L}((X_t-a_t)h_t,0,1,\lambda_1,\lambda_2) && (2.5.57)\\
\widetilde{V}^{(\pm)}(X_t,a_t,b_t) &= \gamma_t\widetilde{V}^{(\pm)}((X_t-a_t)h_t,0,1) && (2.5.58)\\
\widetilde{K}^{(\pm)}(X_t,a_t,b_t,h_t\lambda) &= \widetilde{K}^{(\pm)}((X_t-a_t)h_t,0,1,\lambda) && (2.5.59)
\end{aligned}$$

Step 2

We prove the relations (2.5.54)-(2.5.56) between the generator $A_{(x,a,b)}$ of the process (X,a,b) and the generators A_y of the process Y and A_z of the process Z.

(I)

$$\begin{aligned}
&\mathcal{A}_{(x,a,b)}\widetilde{L}(x,a,b,\lambda_1\lambda_2(b-a)^{-1},\lambda_2(b-a)^{-1})\\
=\ &\lim_{t\to 0}\frac{E(\widetilde{L}(X_t,a_t,b_t,\lambda_1\lambda_2(b-a)^{-1},\lambda_2(b-a)^{-1}))-\widetilde{L}(x,a,b,\lambda_1\lambda_2(b-a)^{-1},\lambda_2(b-a)^{-1})}{t}\\
=\ &\lim_{t\to 0}\frac{E(\widetilde{L}((X_t-a_t)(b_t-a_t)^{-1},0,1,\lambda_1,\lambda_2))-\widetilde{L}((x-a)(b-a)^{-1},0,1,\lambda_1,\lambda_2)}{t}\\
=\ &\mathcal{A}_y\widetilde{L}((x-a)(b-a)^{-1},0,1,\lambda_1,\lambda_2) \Longrightarrow\\
&\mathcal{A}_{(x,a,b)}\widetilde{L}(x,a,b,\lambda_1\lambda_2(b-a)^{-1},\lambda_2(b-a)^{-1}) = \mathcal{A}_y\widetilde{L}((x-a)(b-a)^{-1},0,1,\lambda_1,\lambda_2)
\end{aligned}$$

(II)

$$\begin{aligned}
\mathcal{A}_{(x,a,b)}\widetilde{V}^{(\pm)}(x,a,b) &=\\
&= \lim_{t\to 0}\frac{E(\widetilde{V}^{(\pm)}(X_t,a_t,b_t))-\widetilde{V}^{(\pm)}(x,a,b)}{t}\\
&= \lim_{t\to 0}\frac{E(\gamma_t\widetilde{V}^{(\pm)}(Y_t,0,1))-\gamma\widetilde{V}^{(\pm)}(y,0,1)}{t}\\
&= \mathcal{A}_z\gamma\widetilde{V}^{(\pm)}(y,0,1) \Longrightarrow
\end{aligned}$$

$$\mathcal{A}_{(x,a,b)}\widetilde{V}^{(\pm)}(x,a,b) = \mathcal{A}_z\gamma\widetilde{V}^{(\pm)}(y,0,1)$$

(III)

$$\begin{aligned}
& \mathcal{A}_{(x,a,b)}\widetilde{K}^{(\pm)}(x,a,b,(b-a)^{-1}\lambda) \\
= & \lim_{t\to 0}\frac{E(\widetilde{K}^{(\pm)}(X_t,a_t,b_t,(b-a)^{-1}\lambda)) - \widetilde{K}^{(\pm)}(x,a,b,(b-a)^{-1}\lambda)}{t} \\
= & \lim_{t\to 0}\frac{E(\widetilde{K}^{(\pm)}((X_t-a_t)(b_t-a_t)^{-1},0,1,\lambda)) - \widetilde{K}^{(\pm)}((x-a)(b-a)^{-1},0,1,\lambda)}{t} \\
= & \mathcal{A}_z\widetilde{K}^{(\pm)}((x-a)(b-a)^{-1},0,1,\lambda) \Longrightarrow
\end{aligned}$$

$$\mathcal{A}_{(x,a,b)}\widetilde{K}^{(\pm)}(x,a,b,(b-a)^{-1}\lambda) = \mathcal{A}_y\widetilde{K}^{(\pm)}((x-a)(b-a)^{-1},0,1,\lambda)$$

■

Proposition 2.5.22 *(Relations with regard to the generator operator $A_{(x,a,b)}$ of the process $(X,\mathbf{a},\mathbf{b})$ when it is applied to functions which belong to the functions spaces $S^{V^{(\pm)}}$, $S^{K^{(\pm)}}$ and S^L (see Definition 2.5.18-Definition 2.5.20)).*

Let functions $\widetilde{V}^{(\pm)}(\cdot) \in S^{V^{(\pm)}}$, $\widetilde{K}^{(\pm)}(\cdot) \in S^{K^{(\pm)}}$ and $\widetilde{L}(\cdot) \in S^L$. It holds that

(I)

$$\begin{aligned}
& \mathcal{A}_{(x,a,b)}\widetilde{L}(x,a,b,\lambda_1,\lambda_2) = \\
= & \frac{\varphi(x,a,b)}{2(b-a)^2}\frac{\partial^2}{\partial x^2}\widetilde{L}(x,a,b,\lambda_1,\lambda_2) + \left(\frac{\theta(x,a,b)}{b-a} - \frac{\xi(x,a,b)}{2(b-a)^2}\right)\frac{\partial}{\partial x}\widetilde{L}(x,a,b,\lambda_1,\lambda_2)
\end{aligned} \tag{2.5.60}$$

(II)

$$\begin{aligned}
\mathcal{A}_{(x,a,b)}\widetilde{K}^{(\pm)}(x,a,b,\lambda) = & \frac{\varphi(x,a,b)}{2(b-a)^2}\frac{\partial^2}{\partial x^2}\widetilde{K}^{(\pm)}(x,a,b,\lambda) + \\
& + \left(\frac{\theta(x,a,b)}{b-a} - \frac{\xi(x,a,b)}{2(b-a)^2}\right)\frac{\partial}{\partial x}\widetilde{K}^{(\pm)}(x,a,b,\lambda)
\end{aligned} \tag{2.5.61}$$

(III)

$$\begin{aligned}
& \mathcal{A}_{(x,a,b)}\widetilde{V}^{(\pm)}(x,a,b;n) \\
= & \frac{\mu_b(b)-\mu_a(a)}{b-a}\widetilde{V}^{(\pm)}(x,a,b;n) + \\
& + \frac{\theta(x,a,b)}{b-a}\frac{\partial}{\partial x}\widetilde{V}^{(\pm)}(x,a,b;n) + \frac{\varphi(x,a,b)}{2(b-a)^2}\frac{\partial^2}{\partial x^2}\widetilde{V}^{(\pm)}(x,a,b;n)
\end{aligned} \tag{2.5.62}$$

Proof. The general idea of the proof is to apply Remark 2.5.5 and Lemma 2.5.21 in order to find the expressions (2.5.60)-(2.5.62).

(I) By relation (2.5.54) in Lemma 2.5.21 we have that

$$\mathcal{A}_{(x,a,b)}\widetilde{L}(x,a,b,\lambda_1^*(b-a)^{-1},\lambda_2^*(b-a)^{-1}) = \mathcal{A}_y\widetilde{L}((x-a)(b-a)^{-1},0,1,\lambda_1^*,\lambda_2^*) \quad (2.5.63)$$

By relation (2.5.16) in Remark 2.5.5 we have that:

$$\begin{aligned} &\mathcal{A}_y\widetilde{L}((x-a)(b-a)^{-1},0,1,\lambda_1^*,\lambda_2^*) = \quad (2.5.64)\\ = &\left(\frac{\theta(x,a,b)}{(b-a)^2} - \frac{\xi(x,a,b)}{2(b-a)^3}\right)\frac{\partial}{\partial x}\widetilde{L}((x-a)(b-a)^{-1},0,1,\lambda_1^*,\lambda_2^*) + \\ &+\frac{1}{2}\frac{\varphi(x,a,b)}{(b-a)^4}\frac{\partial^2}{\partial x^2}\widetilde{L}((x-a)(b-a)^{-1},0,1,\lambda_1^*,\lambda_2^*) \end{aligned}$$

After some calculations relation (2.5.64) can be written as

$$\begin{aligned} &\mathcal{A}_y\widetilde{L}((x-a)(b-a)^{-1},0,1,\lambda_1^*,\lambda_2^*) = \quad (2.5.65)\\ = &\left(\frac{\theta(x,a,b)}{(b-a)} - \frac{\xi(x,a,b)}{2(b-a)^2}\right)\frac{\partial}{\partial x}\widetilde{L}(x,a,b,\lambda_1(b-a)^{-1},\lambda_2^*(b-a)^{-1}) + \\ &+\frac{1}{2}\frac{\varphi(x,a,b)}{(b-a)^2}\frac{\partial^2}{\partial x^2}\widetilde{L}(x,a,b,\lambda_1^*(b-a)^{-1},\lambda_2^*(b-a)^{-1}) \end{aligned}$$

By relation (2.5.63) the expression (2.5.65) is equivalent to

$$\begin{aligned} &\mathcal{A}_{(x,a,b)}\widetilde{L}(x,a,b,\lambda_1^*(b-a)^{-1},\lambda_2^*(b-a)^{-1}) = \quad (2.5.66)\\ = &\left(\frac{\theta(x,a,b)}{(b-a)} - \frac{\xi(x,a,b)}{2(b-a)^2}\right)\frac{\partial}{\partial x}\widetilde{L}(x,a,b,\lambda_1^*,\lambda_2^*(b-a)^{-1}) + \\ &+\frac{1}{2}\frac{\varphi(x,a,b)}{(b-a)^2}\frac{\partial^2}{\partial x^2}\widetilde{L}(x,a,b,\lambda_1^*(b-a)^{-1},\lambda_2^*(b-a)^{-1}) \end{aligned}$$

and setting $\lambda_1 := \lambda_1^*(b-a)^{-1}$, $\lambda_2 := \lambda_2^*(b-a)^{-1}$ we conclude the result.

(II) As in (I) by considering $\lambda_1 = \lambda$ and $\lambda_2 = 0$.

(III) By relation (2.5.55) in Lemma 2.5.21 we have that

$$\mathcal{A}_{(x,a,b)}\widetilde{V}^{(\pm)}(x,a,b;n) = \mathcal{A}_z\gamma\widetilde{V}^{(\pm)}(y,0,1;n) \quad (2.5.67)$$

By relation (2.5.20) in Remark 2.5.5 we have that:

$$\begin{aligned}\mathcal{A}_z\gamma\widetilde{V}^{(\pm)}(y,0,1;n) &= (\mu_b-\mu_a)\widetilde{V}^{(\pm)}(y,0,1;n)+ \\ &+\left(\frac{\theta(x,a,b)}{(b-a)}-\frac{\xi(x,a,b)}{2(b-a)^2}\right)\frac{\partial}{\partial y}\widetilde{V}^{(\pm)}(y,0,1;n)+ \\ &+\frac{1}{2}\frac{\varphi(x,a,b)}{(b-a)^3}\frac{\partial^2}{\partial y^2}\widetilde{V}^{(\pm)}(y,0,1;n)+ \\ &+\frac{\xi(x,a,b)}{2(b-a)^2}\frac{\partial}{\partial y}\widetilde{V}^{(\pm)}(y,0,1;n)\end{aligned} \tag{2.5.68}$$

and using relation (2.5.67) and simplifying relation (2.5.68) can be written as

$$\begin{aligned}\mathcal{A}_{(x,a,b)}\widetilde{V}^{(\pm)}(x,a,b;n) &= \frac{\mu_b-\mu_a}{b-a}\widetilde{V}^{(\pm)}(x,a,b;n)+ \\ &+\frac{\theta(x,a,b)}{(b-a)}\frac{\partial}{\partial x}\widetilde{V}^{(\pm)}(x,a,b;n)+\frac{1}{2}\frac{\varphi(x,a,b)}{(b-a)^2}\frac{\partial^2}{\partial x^2}\widetilde{V}^{(\pm)}(x,a,b;n)\end{aligned} \tag{2.5.69}$$

This concludes the proof. ■

Proposition 2.5.22 is the last result for this section. We have now in our disposal all the necessary ingredients in order to derive the final result, which are the differential equations. We proceed now with the next section in which we will combine the results of the previous sections and conclude the final results.

2.6 Partial differential equations for the Laplace transforms and the moments.

As we mention in the previous section we are now in position to conclude the final results about the de Finetti model with general barriers. We will do that in this section and we start first with the de Finetti model with one general reflecting barrier.

2.6.1 PDEs for the de Finetti model with one general reflecting barrier.

In this subsection we provide propositions concerning the Laplace transform of the joint distribution of the time of ruin and the discounted dividends $N(x,a,b,\lambda_1,\lambda_2)$, the Laplace transform of the discounted dividends $K(x,a,b,\lambda)$, the Laplace transform of the time of ruin $M(x,a,b,\lambda)$ and the moments of the discounted dividends $V(x,a,b;n)$, as they have been defined at (2.1.20)-(2.1.23).

Proposition 2.6.1 *(The Laplace transform of the joint distribution of the time of ruin and the discounted dividends). Let the function* $N(x,a,b,\lambda_1,\lambda_2) \in C_b^2(\mathbb{R}^5)$ *satisfy the scaling properties (2.3.13),(2.3.17). If the function* $N(x,a,b,\lambda_1,\lambda_2)$ *solves the PDE*

$$\begin{aligned} & \frac{\varphi(x,a,b)}{2(b-a)^2}\frac{\partial^2}{\partial x^2}N(x,a,b,\lambda_1,\lambda_2) + \left(\frac{\theta(x,a,b)}{b-a} - \frac{\xi(x,a,b)}{2(b-a)^2}\right)\frac{\partial}{\partial x}N(x,a,b,\lambda_1,\lambda_2) \\ = & \lambda_1 N(x,a,b,\lambda_1,\lambda_2) + \delta\lambda_2\frac{\partial}{\partial\lambda_2}N(x,a,b,\lambda_1,\lambda_2) \end{aligned} \quad (2.6.1)$$

with boundary conditions:

$$N(a,a,b,\lambda_1,\lambda_2) = 1 \quad (2.6.2)$$

$$\frac{\partial}{\partial x}N(x,a,b,\lambda_1,\lambda_2)|_{x=b} = -\lambda_2 N(b,a,b,\lambda_1,\lambda_2) \quad (2.6.3)$$

then

$$N(x,a,b,\lambda_1,\lambda_2) = E^{(x,a,b)}(e^{-\lambda_1 T - \lambda_2 U}). \quad (2.6.4)$$

Proof. First we observe that because the function $N(x,a,b,\lambda_1,\lambda_2)$ is in $C_b^2(\mathbb{R}^5)$ and satisfy the scaling properties (2.3.13),(2.3.17) then by the relation (2.5.39) we must have

$$\begin{aligned} & \mathcal{L}_{(x,a,b)}N(x,a,b,\lambda_1,\lambda_2) \quad (2.6.5) \\ = & \mathcal{A}_{(x,a,b)}N(x,a,b,\lambda_1,\lambda_2) \\ = & \frac{\varphi(x,a,b)}{2(b-a)^2}\frac{\partial^2}{\partial x^2}N(x,a,b,\lambda_1,\lambda_2) + \left(\frac{\theta(x,a,b)}{b-a} - \frac{\xi(x,a,b)}{2(b-a)^2}\right)\frac{\partial}{\partial x}N(x,a,b,\lambda_1,\lambda_2) \end{aligned}$$

By relation (2.6.5) and the fact that the function $N(x,a,b,\lambda_1,\lambda_2)$ solves the PDE (2.6.1) we conclude

$$\mathcal{L}_{(x,a,b)}N(x,a,b,\lambda_1,\lambda_2) = \lambda_1 N(x,a,b,\lambda_1,\lambda_2) + \delta\lambda_2\frac{\partial}{\partial\lambda_2}N(x,a,b,\lambda_1,\lambda_2) \quad (2.6.6)$$

Applying the Itô formula to the process

$$e^{-\lambda_1(t\wedge T)-\lambda_2 U_{t\wedge T}}N\left(e^{-\delta(t\wedge T)}Z_{t\wedge T}, e^{-\delta(t\wedge T)}a_{t\wedge T}, e^{-\delta(t\wedge T)}b_{t\wedge T}, \lambda_1, \lambda_2\right)$$

taking expectations and using the condition (2.6.3) we have

$$\begin{aligned} & E^{(x,a,b)}\left(e^{-\lambda_1(t\wedge T)-\lambda_2 U_{t\wedge T}}N\left(e^{-\delta(t\wedge T)}Z_{t\wedge T}, e^{-\delta(t\wedge T)}a_{t\wedge T}, e^{-\delta(t\wedge T)}b_{t\wedge T}, \lambda_1, \lambda_2\right)\right) \\ = & N(x,a,b,\lambda_1,\lambda_2) + \\ & + E^{(x,a,b)}\left(\int_0^{t\wedge T} e^{-\lambda_1 s - \lambda_2 U_s}\left(\mathcal{L}_{(x,a,b)} - \lambda_1 - \right.\right. \\ & \left.\left. -\delta e^{-\delta s}\left(X_s\frac{\partial}{\partial\left(e^{-\delta s}z\right)} + a_s\frac{\partial}{\partial\left(e^{-\delta s}a\right)} + b_s\frac{\partial}{\partial\left(e^{-\delta s}b\right)}\right)\right)N\left(e^{-\delta s}Z_s, e^{-\delta s}a_s, e^{-\delta s}b_s, \lambda_1, \lambda_2\right)ds\right) + \\ & + \lambda_2 E^{(x,a,b)}\left(\int_0^{t\wedge T} e^{-\delta s}e^{-\lambda_1 s - \lambda_2 U_s}N\left(e^{-\delta s}Z_s, e^{-\delta s}a_s, e^{-\delta s}b_s, \lambda_1, \lambda_2\right)d\mathcal{U}_s\right) - \\ & - \lambda_2 E^{(x,a,b)}\left(\int_0^{t\wedge T} e^{-\lambda_1 s - \lambda_2 U_s}N\left(e^{-\delta s}Z_s, e^{-\delta s}a_s, e^{-\delta s}b_s, \lambda_1, \lambda_2\right)dU_s\right) \end{aligned} \quad (2.6.7)$$

By taking into account that by the relation (2.1.18) we have that

$$dU_t = e^{-\delta t} d\mathcal{U}_t \tag{2.6.8}$$

and also considering the relation (2.3.23) the above expression (2.6.7) becomes

$$\begin{aligned}
& E^{(x,a,b)}\left(e^{-\lambda_1(t\wedge T)-\lambda_2 U_{t\wedge T}} N\left(e^{-\delta(t\wedge T)} Z_{t\wedge T}, e^{-\delta(t\wedge T)} a_{t\wedge T}, e^{-\delta(t\wedge T)} b_{t\wedge T}, \lambda_1, \lambda_2\right)\right) \\
= \; & N(x,a,b,\lambda_1,\lambda_2) + \\
& + E^{(x,a,b)}\left(\int_0^{t\wedge T} e^{-\lambda_1 s-\lambda_2 U_s}\left(\left(\mathcal{L}_{(x,a,b)}-\lambda_1\right) N(e_s^{-\delta s} Z_s, e^{-\delta s} a_s, e^{-\delta s} b_s, \lambda_1, \lambda_2) - \right.\right. \\
& \left.\left. -\delta \lambda_2 e^{-\delta s} \frac{\partial}{\partial\left(\lambda_2 e^{-\delta s}\right)} N\left(Z_s, a_s, b_s, \lambda_1, \lambda_2 e^{-\delta s}\right)\right) ds\right)
\end{aligned}$$

Using the relation (2.3.22) we conclude

$$\begin{aligned}
& E^{(x,a,b)}\left(e^{-\lambda_1(t\wedge T)-\lambda_2 U_{t\wedge T}} N\left(e^{-\delta(t\wedge T)} Z_{t\wedge T}, e^{-\delta(t\wedge T)} a_{t\wedge T}, e^{-\delta(t\wedge T)} b_{t\wedge T}, \lambda_1, \lambda_2\right)\right) \\
= \; & N(x,a,b,\lambda_1,\lambda_2) + \\
& + E^{(x,a,b)}\left(\int_0^{t\wedge T} e^{-\lambda_1 s-\lambda_2 U_s}\left(\mathcal{L}_{(x,a,b)}-\lambda_1-\delta\lambda_2 e^{-\delta s}\frac{\partial}{\partial\left(\lambda_2 e^{-\delta s}\right)}\right) N\left(Z_s, a_s, b_s, \lambda_1, \lambda_2 e^{-\delta s}\right) ds\right)
\end{aligned}$$

Applying the PDE (2.6.6) with $\lambda_2 e^{-\delta s}$ instead of λ_2 the above simplifies to

$$E^{(x,a,b)}\left(e^{-\lambda_1(t\wedge T)-\lambda_2 U_{t\wedge T}} N\left(e_{t\wedge T}^{-\delta(t\wedge T)} Z_{t\wedge T}, e^{-\delta(t\wedge T)} a_{t\wedge T}, e^{-\delta(t\wedge T)} b_{t\wedge T}, \lambda_1, \lambda_2\right)\right) = N(x,a,b,\lambda_1,\lambda_2)$$

Taking limit as $t \to \infty$ (using the dominated convergence theorem as by hypotheses we have that $N(x,a,b,\lambda_1,\lambda_2) \in C_b^2(\mathbb{R}^5)$) and taking into account that

$$T := \inf\{t > 0 : Z_t = a_t\}$$

and the condition (2.6.2) we conclude:

$$N(x,a,b,\lambda_1,\lambda_2) = E^{(x,a,b)}\left(e^{-\lambda_1 T-\lambda_2 U}\right)$$

■

Working as for the proof of Proposition 2.6.1, setting first $\lambda_1 = 0$ and $\lambda_2 = \lambda$ and second $\lambda_1 = \lambda$ and $\lambda_2 = 0$ the following two propositions can be proved.

Proposition 2.6.2 *(Laplace transform of the discounted dividends). Let the function* $K(x,a,b,\lambda) \in C_b^2(R^4)$ *satisfy the scaling properties (2.3.11), (2.3.15). If the function* $K(x,a,b,\lambda)$ *solves the PDE*

$$\begin{aligned}
& \frac{\varphi(x,a,b)}{2(b-a)^2}\frac{\partial^2}{\partial x^2}K(x,a,b,\lambda) + \left(\frac{\theta(x,a,b)}{b-a} - \frac{\xi(x,a,b)}{2(b-a)^2}\right)\frac{\partial}{\partial x}K(x,a,b,\lambda) \\
= \; & \lambda\delta\frac{\partial}{\partial\lambda}K(x,a,b,\lambda)
\end{aligned} \tag{2.6.9}$$

with boundary conditions:

$$K(a,a,b,\lambda) = 1 \quad (2.6.10)$$
$$\frac{\partial}{\partial x}K(x,a,b,\lambda)|_{x=b} = -\lambda K(b,a,b,\lambda) \quad (2.6.11)$$

then

$$K(x,a,b,\lambda) = E^{(x,a,b)}\left(e^{-\lambda U}\right) \quad (2.6.12)$$

Proposition 2.6.3 *(Laplace transform of the time of ruin). Let the function* $M(x,a,b,\lambda) \in C_b^2(\mathbb{R}^4)$ *satisfy the scaling properties (2.3.12), (2.3.16). If the function* $M(x,a,b,\lambda)$ *solves the ODE*

$$\frac{\varphi(x,a,b)}{2(b-a)^2}\frac{\partial^2}{\partial x^2}M(x,a,b,\lambda) + \left(\frac{\theta(x,a,b)}{b-a} - \frac{\xi(x,a,b)}{2(b-a)^2}\right)\frac{\partial}{\partial x}M(x,a,b,\lambda)$$
$$= \lambda M(x,a,b,\lambda) \quad (2.6.13)$$

with boundary conditions:

$$M(a,a,b,\lambda) = 1 \quad (2.6.14)$$
$$\frac{\partial}{\partial x}M(x,a,b,\lambda)|_{x=b} = 0 \quad (2.6.15)$$

then

$$M(x,a,b,\lambda) = E^{(x,a,b)}\left(e^{-\lambda T}\right) \quad (2.6.16)$$

The next proposition concerns with the moments of the discounted dividends.

Proposition 2.6.4 *(Moments of the discounted dividends). Let the functions* $V(x,a,b;n)$ *,* $n \in N$ *belonging to* $C_b^2(\mathbb{R}^3)$ *satisfy the scaling properties (2.3.10),(2.3.14). If the functions* $V(x,a,b;n)$ *,* $n \in N$ *solve the ODEs*

$$\frac{\varphi(x,a,b)}{2(b-a)^2}\frac{\partial^2}{\partial x^2}V(x,a,b;n) + \frac{\theta(x,a,b)}{b-a}\frac{\partial}{\partial x}V(x,a,b;n) +$$
$$+\left(\frac{\mu_b(b)-\mu_a(a)}{b-a} - n\delta\right)V(x,a,b;n)$$
$$= 0 \quad (2.6.17)$$

with boundary conditions:

$$V(a,a,b;n) = 0, \quad n = 1,2,\ldots \quad (2.6.18)$$
$$\frac{\partial}{\partial x}V(x,a,b;1)|_{x=b} = 1 \quad (2.6.19)$$
$$\frac{\partial}{\partial x}V(x,a,b;n)|_{x=b} = nV(b,a,b;n-1), \quad n = 2,3,\ldots \quad (2.6.20)$$

then

$$V(x,a,b;n) = E^{(x,a,b)}(U^n) \quad (2.6.21)$$

Proof. First we observe that because $V(x,a,b;n) \in C_b^2(\mathbb{R}^3)$ and satisfies the scaling properties (2.3.10), (2.3.14) then by the (2.5.42) we must have

$$\begin{aligned} & \mathcal{L}_{(x,a,b)} V(x,a,b;n) \\ = \; & \mathcal{A}_{(x,a,b)} V(x,a,b;n) \\ = \; & \frac{\mu_b(b)-\mu_a(a)}{b-a} V(x,a,b;n) + \frac{\theta(x,a,b)}{b-a}\frac{\partial}{\partial x} V(x,a,b;n) + \frac{\varphi(x,a,b)}{2(b-a)^2}\frac{\partial^2}{\partial x^2} V(x,a,b;n) \end{aligned} \tag{2.6.22}$$

By relation (2.6.22) and the fact that the function $V(x,a,b;n)$ solves the ODE (2.6.17) we conclude that

$$\left(\mathcal{L}_{(x,a,b)} - n\delta\right) V(x,a,b;n) = 0 \tag{2.6.23}$$

We consider the time instants h and t with $t \geq h$. Applying the Itô formula to the process

$$e^{-n\delta((t-h)\wedge T)} V\left(Z_{t\wedge T}, a_{t\wedge T}, b_{t\wedge T}; n\right)$$

taking conditional expectations and using the fact that the process $V(Z_h, a_h, b_h; n)$ is $\mathcal{F}_h$ measurable we have:

$$\begin{aligned} & E\left(e^{-n\delta((t-h)\wedge T)} V\left(Z_{t\wedge T}, a_{t\wedge T}, b_{t\wedge T}; n\right) |\mathcal{F}_h\right) \\ = \; & V\left(Z_h, a_h, b_h; n\right) + E\left(\left(\int_h^{t\wedge T} e^{-n\delta(s-h)} \left(\mathcal{L}_{(x,a,b)} - n\delta\right) V\left(Z_s, a_s, b_s; n\right) ds\right) |\mathcal{F}_h\right) - \\ & - E\left(\left(\int_h^{t\wedge T} e^{-n\delta(s-h)} \frac{\partial}{\partial z} V\left(Z_s, a_s, b_s; n\right) d\mathcal{U}_s\right) |\mathcal{F}_h\right) \end{aligned}$$

Using relation (2.6.23) and condition (2.6.18) the above becomes

$$V\left(Z_h, a_h, b_h; n\right) = E\left(\left(\int_h^{t\wedge T} e^{-n\delta(s-h)} \frac{\partial}{\partial z} V\left(Z_s, a_s, b_s; n\right) d\mathcal{U}_s\right) |\mathcal{F}_h\right) \tag{2.6.24}$$

Next we apply relation (2.6.24) with $n = 1$, taking into account that by relation (2.5.21) we have

$$dU(t) = -e^{-\delta t} d\mathcal{U}_t \tag{2.6.25}$$

and also considering relation (2.6.19) and taking the limit as $t \to \infty$ (using the monotone convergence theorem) we conclude that

$$E\left(e^{\delta h} U(h) |\mathcal{F}_h\right) = V\left(Z_h, a_h, b_h; 1\right) < \infty \tag{2.6.26}$$

We will use the method of induction. Suppose that for $n-1$ we have

$$E\left(e^{(n-1)\delta h} U^{n-1}(h) |\mathcal{F}_h\right) = V(Z_h, a_h, b_h; n-1) < \infty \tag{2.6.27}$$

Next by applying the relation (2.6.24) for n and taking into account the relations (2.6.20), (2.6.27), the fact that $\{s < T\} \in \mathcal{F}_s$ (because T is a stopping time) and interchange expectation and integral (using the Fubini theorem) because

$$\begin{aligned} & E\left(\int_h^{t\wedge T} e^{-n\delta(s-h)} V\left(Z_s, a_s, b_s; n-1\right) d\mathcal{U}_s | \mathcal{F}_h\right) \\ \leq \ & \sup_s V\left(Z_s, a_s, b_s; n-1\right) e^{n\delta h} E\left(U(h) | \mathcal{F}_h\right) < \infty \end{aligned}$$

we conclude

$$\begin{aligned} V\left(Z_h, a_h, b_h; n\right) &= nE\left(\int_h^{t\wedge T} e^{-n\delta(s-h)} V\left(Z_s, a_s, b_s; n-1\right) d\mathcal{U}_s | \mathcal{F}_h\right) \\ &= nE\left(\int_h^{t} 1_{\{s<T\}} e^{-n\delta(s-h)} E\left(e^{(n-1)\delta s} U^{n-1}(s) | \mathcal{F}_s\right) d\mathcal{U}_s | \mathcal{F}_h\right) \\ &= nE\left(\int_h^{t} e^{-n\delta(s-h)} E\left(1_{\{s<T\}} e^{(n-1)\delta s} U^{n-1}(s) | \mathcal{F}_s\right) d\mathcal{U}_s | \mathcal{F}_h\right) \\ &= n\int_h^{t} e^{-n\delta(s-h)} E\left(E\left(1_{\{s<T\}} e^{(n-1)\delta s} U^{n-1}(s) d\mathcal{U}_s | \mathcal{F}_s\right) | \mathcal{F}_h)\right) \\ &= n\int_h^{t} e^{-n\delta(s-h)} E\left(1_{\{s<T\}} e^{(n-1)\delta s} U^{n-1}(s) d\mathcal{U}_s | \mathcal{F}_h\right) \\ &= nE\left(e^{n\delta h} \int\limits_h^{t} 1_{\{s<T\}} e^{-\delta s} U^{n-1}(s) d\mathcal{U}_s | \mathcal{F}_h\right) \\ &= -E\left(e^{n\delta h} \int_h^{t\wedge T} nU^{n-1}(s) dU(s) | \mathcal{F}_h\right) \end{aligned}$$

Therefore

$$V\left(Z_h, a_h, b_h; n\right) = -E\left(e^{n\delta h} \int_h^{t\wedge T} nU^{n-1}(s) dU(s) | \mathcal{F}_h\right) \tag{2.6.28}$$

Taking the limit in relation (2.6.28) as $t \to \infty$ (using the monotone convergence theorem) we have

$$\begin{aligned} V\left(Z_h, a_h, b_h; n\right) &= -E\left(e^{n\delta h} \int_h^{T} nU^{n-1}(s) dU(s) | \mathcal{F}_h\right) \\ &= -E\left(e^{n\delta h} (U^n(s)|_h^T) | \mathcal{F}_h\right) \end{aligned}$$

Therefore

$$V(Z_h, a_h, b_h; n) = E(e^{n\delta h} U^n(h) | \mathcal{F}_h) \tag{2.6.29}$$

Finally by taking $h = 0$ in the relation (2.6.29) we conclude the relation (2.6.21). ■

Next adopting the same logic we conclude analogous results for the De Finetti model with two general reflecting barriers.

2.6.2 PDEs for the de Finetti model with two general reflecting barriers

In this subsection we provide propositions concerning the moments of the discounted dividends $V^{(+)}(x, a, b; n)$ and the discounted financing $V^{(-)}(x, a, b; n)$, the Laplace transform of the discounted dividends $K^{(+)}(x,a,b,\lambda)$ and the discounted financing $K^{(-)}(x,a,b,\lambda)$ and the Laplace transform of the joint distribution of the discounted dividends and the discounted financing $L(x,a,b,\lambda_1,\lambda_2)$, as they have been defined at (2.1.29)–(2.1.31).

Proposition 2.6.5 ***(Laplace transform of the joint distribution of the discounted dividends and the discounted financing).*** *Let the function $L(x,a,b,\lambda_1,\lambda_2) \in C_b^2(\mathbb{R}^5)$ satisfy the scaling properties (2.3.41),(2.3.44) and also*

$$L(x,a,b,0,0) = 1 \tag{2.6.30}$$

for any $(x,a,b) \in \mathbb{R}^3$. If the function $L(x,a,b,\lambda_1,\lambda_2)$ solves the PDE:

$$\begin{aligned} & \frac{\varphi(x,a,b)}{2(b-a)^2}\frac{\partial^2}{\partial x^2}L(x,a,b,\lambda_1,\lambda_2) + \left(\frac{\theta(x,a,b)}{b-a} - \frac{\xi(x,a,b)}{2(b-a)^2}\right)\frac{\partial}{\partial x}L(x,a,b,\lambda_1,\lambda_2) \\ = \;& \delta\left(\lambda_1\frac{\partial}{\partial\lambda_1}L(x,a,b,\lambda_1,\lambda_2) + \lambda_2\frac{\partial}{\partial\lambda_2}L(x,a,b,\lambda_1,\lambda_2)\right) \end{aligned} \tag{2.6.31}$$

with boundary conditions:

$$\frac{\partial}{\partial x}L(x,a,b,\lambda_1,\lambda_2)|_{x=a} = \lambda_2 L(a,a,b,\lambda_1,\lambda_2) \tag{2.6.32}$$

$$\frac{\partial}{\partial x}L(x,a,b,\lambda_1,\lambda_2)|_{x=b} = -\lambda_1 L(b,a,b,\lambda_1,\lambda_2) \tag{2.6.33}$$

then

$$L(x,a,b,\lambda_1,\lambda_2) = E^{(x,a,b)}(e^{-\lambda_1 U^{(+)} - \lambda_2 U^{(-)}}) \tag{2.6.34}$$

Proof. First we observe that because function $L(x,a,b,\lambda_1,\lambda_2)$ is in $C_b^2(\mathbb{R}^5)$ and satisfy the scaling properties (2.3.41), (2.3.44) then by relation (2.5.60) we must have

$$\begin{aligned} & \mathcal{L}_{(x,a,b)}L(x,a,b,\lambda_1,\lambda_2) \\ = \;& \mathcal{A}_{(x,a,b)}L(x,a,b,\lambda_1,\lambda_2) \\ = \;& \frac{\varphi(x,a,b)}{2(b-a)^2}\frac{\partial^2}{\partial x^2}L(x,a,b,\lambda_1,\lambda_2) + \left(\frac{\theta(x,a,b)}{b-a} - \frac{\xi(x,a,b)}{2(b-a)^2}\right)\frac{\partial}{\partial x}L(x,a,b,\lambda_1,\lambda_2) \end{aligned} \tag{2.6.35}$$

By the relation (2.6.35) and the fact that the function $L(x,a,b,\lambda_1,\lambda_2)$ solves the PDE (2.6.31) we conclude

$$\mathcal{L}_{(x,a,b)}L(x,a,b,\lambda_1,\lambda_2) = \lambda_1 L(x,a,b,\lambda_1,\lambda_2) + \delta\lambda_2\frac{\partial}{\partial\lambda_2}L(x,a,b,\lambda_1,\lambda_2) \tag{2.6.36}$$

We fix a number $n \in N$. Applying the Itô formula to the process

$$e^{-\lambda_1 U_{t\wedge n}^{(+)} - \lambda_2 U_{t\wedge n}^{(-)}} L\left(e^{-\delta(t\wedge n)}Z_{t\wedge n}, e^{-\delta(t\wedge n)}a_{t\wedge n}, e^{-\delta(t\wedge n)}b_{t\wedge n}, \lambda_1, \lambda_2\right)$$

taking expectations and using the conditions (2.6.32), (2.6.33) we have:

$$\begin{aligned}
& E^{(x,a,b)}(e^{-\lambda_1 U^{(+)}_{t\wedge n}-\lambda_2 U^{(-)}_{t\wedge n}} L(e^{-\delta(t\wedge n)}Z_{t\wedge n}, e^{-\delta(t\wedge n)}a_{t\wedge n}, e^{-\delta(t\wedge n)}b_{t\wedge n}, \lambda_1, \lambda_2)) \\
= \; & L(x,a,b,\lambda_1,\lambda_2) + \\
& +E^{(x,a,b)}\left(\int_0^{t\wedge n} e^{-\lambda_1 U^{(+)}_s-\lambda_2 U^{(-)}_s}\,(\mathcal{L}_{(x,a,b)}- \right.\\
& \left. -\delta e^{-\delta s}\left(Z_s\frac{\partial}{\partial\left(e^{-\delta s}z\right)}+a_s\frac{\partial}{\partial(e^{-\delta s}a)}+b_s\frac{\partial}{\partial(e^{-\delta s}b)}\right)\Big)\, L\left(e^{-\delta s}Z_s, e^{-\delta s}a_s, e^{-\delta s}b_s, \lambda_1, \lambda_2\right) ds\right) + \\
& +\lambda_1 E^{(x,a,b)}\left(\int_0^{t\wedge n} e^{-\delta s}e^{-\lambda_1 U^{(+)}_s-\lambda_2 U^{(-)}_s} L(e^{-\delta s}Z_s, e^{-\delta s}a_s, e^{-\delta s}b_s, \lambda_1, \lambda_2) d\mathcal{U}^{(+)}_s\right) - \\
& -\lambda_1 E^{(x,a,b)}\left(\int_0^{t\wedge n} e^{-\lambda_1 U^{(+)}_s-\lambda_2 U^{(-)}_s} L(e^{-\delta s}Z_s, e^{-\delta s}a_s, e^{-\delta s}b_s, \lambda_1, \lambda_2) dU^{(+)}_s\right) + \\
& +\lambda_2 E^{(x,a,b)}\left(\int_0^{t\wedge n} e^{-\delta s}e^{-\lambda_1 U^{(+)}_s-\lambda_2 U^{(-)}_s} L(e^{-\delta s}Z_s, e^{-\delta s}a_s, e^{-\delta s}b_s, \lambda_1, \lambda_2) d\mathcal{U}^{(-)}_s\right) - \\
& -\lambda_2 E^{(x,a,b)}\left(\int_0^{t\wedge n} e^{-\lambda_1 U^{(+)}_s-\lambda_2 U^{(-)}_s} L(e^{-\delta s}Z_s, e^{-\delta s}a_s, e^{-\delta s}b_s, \lambda_1, \lambda_2) dU^{(-)}_s\right)
\end{aligned} \tag{2.6.37}$$

Taking into account that by relations (2.1.26) and (2.1.27) we have that

$$\begin{aligned}
dU^{(+)}_t &= e^{-\delta t}d\mathcal{U}^{(+)}_t \\
dU^{(-)}_t &= e^{-\delta t}d\mathcal{U}^{(-)}_t
\end{aligned}$$

and also relation (2.3.46), expression (2.6.37) becomes

$$\begin{aligned}
& E^{(x,a,b)}(e^{-\lambda_1 U^{(+)}_{t\wedge n}-\lambda_2 U^{(-)}_{t\wedge n}} L(e^{-\delta(t\wedge n)}Z_{t\wedge n}, e^{-\delta(t\wedge n)}a_{t\wedge n}, e^{-\delta(t\wedge n)}b_{t\wedge n}, \lambda_1, \lambda_2)) \\
= \; & L(x,a,b,\lambda_1,\lambda_2) + \\
& +E^{(x,a,b)}\left(\int_0^{t\wedge n} e^{-\lambda_1 U^{(+)}_s-\lambda_2 U^{(-)}_s}\left(\mathcal{L}_{(x,a,b)} L(e^{-\delta s}_s Z_s, e^{-\delta s}a_s, e^{-\delta s}b_s, \lambda_1, \lambda_2)\right.\right. \\
& \left.\left. -\delta\left(\lambda_1 e^{-\delta s}\frac{\partial}{\partial(\lambda_1 e^{-\delta s})}+\lambda_2 e^{-\delta s}\frac{\partial}{\partial(\lambda_2 e^{-\delta s})}\right) L(Z_s, a_s, b_s, \lambda_1 e^{-\delta s}, \lambda_2 e^{-\delta s})\right) ds\right)
\end{aligned} \tag{2.6.38}$$

Applying relation (2.3.45) to expression (2.6.38) we conclude

$$\begin{aligned}
& E^{(x,a,b)}\left(e^{-\lambda_1 U^{(+)}_{t\wedge n}-\lambda_2 U^{(-)}_{t\wedge n}} L(e^{-\delta(t\wedge n)}Z_{t\wedge n}, e^{-\delta(t\wedge n)}a_{t\wedge n}, e^{-\delta(t\wedge n)}b_{t\wedge n}, \lambda_1, \lambda_2)\right) \\
= \; & L(x,a,b,\lambda_1,\lambda_2) + \\
& +E^{(x,a,b)}\left(\int\limits_0^{t\wedge n} e^{-\lambda_1 U^{(+)}_s-\lambda_2 U^{(-)}_s}\left(\mathcal{L}_{(x,a,b)} - \delta\lambda_1 e^{-\delta s}\frac{\partial}{\partial(\lambda_1 e^{-\delta s})} - \right.\right. \\
& \left.\left. -\delta\lambda_2 e^{-\delta s}\frac{\partial}{\partial\left(\lambda_2 e^{-\delta s}\right)}\right) L(Z_s, a_s, b_s, \lambda_1 e^{-\delta s}, \lambda_2 e^{-\delta s}) ds\right)
\end{aligned} \tag{2.6.39}$$

Applying the PDE (2.6.6) with $\lambda_1 e^{-\delta s}$ instead of λ_1 and $\lambda_2 e^{-\delta s}$ instead of λ_2 the above expression (2.6.39) simplifies to

$$E^{(x,a,b)}\left(e^{-\lambda_1 U^{(+)}_{t\wedge n}-\lambda_2 U^{(-)}_{t\wedge n}}L\left(e^{-\delta(t\wedge n)}Z_{t\wedge n}, e^{-\delta(t\wedge n)}a_{t\wedge n}, e^{-\delta(t\wedge n)}b_{t\wedge n}, \lambda_1, \lambda_2\right)\right) = L(x,a,b,\lambda_1,\lambda_2) \tag{2.6.40}$$

Applying relation (2.3.45) in expression (2.6.40) we have

$$L(x,a,b,\lambda_1,\lambda_2) = E^{(x,a,b)}\left(e^{-\lambda_1 U^{(+)}_{t\wedge n}-\lambda_2 U^{(-)}_{t\wedge n}}L\left(Z_{t\wedge n}, a_{t\wedge n}, b_{t\wedge n}, \lambda_1 e^{-\delta(t\wedge n)}, \lambda_2 e^{-\delta(t\wedge n)}\right)\right) \tag{2.6.41}$$

Taking limit as $t \to \infty$ (using the dominated convergence theorem as by hypotheses we have that $L(x,a,b,\lambda_1,\lambda_2) \in C_b^2(\mathbb{R}^5)$) the relation (2.6.41) becomes

$$L(x,a,b,\lambda_1,\lambda_2) = E^{(x,a,b)}\left(e^{-\lambda_1 U_n^{(+)}-\lambda_2 U_n^{(-)}}L(Z_n,a_n,b_n,0,0)\right) \tag{2.6.42}$$

Taking into account the condition (2.6.30) the above expression (2.6.42) simplifies to

$$L(x,a,b,\lambda_1,\lambda_2) = E^{(x,a,b)}\left(e^{-\lambda_1 U_n^{(+)}-\lambda_2 U_n^{(-)}}\right) \tag{2.6.43}$$

Finally taking limit as $n \to \infty$ we conclude:

$$L(x,a,b,\lambda_1,\lambda_2) = E^{(x,a,b)}\left(e^{-\lambda_1 U^{(+)}-\lambda_2 U^{(-)}}\right).$$

■

Working as for the proof of Proposition 2.6.5, setting first $\lambda_1 = 0$, $\lambda_2 = \lambda$ and second $\lambda_1 = \lambda$, $\lambda_2 = 0$ the following proposition can be proved.

Proposition 2.6.6 *(Laplace transforms of the discounted dividends and the discounted financing). Let the functions $K^{(+)}(x,a,b,\lambda)$ and $K^{(-)}(x,a,b,\lambda)$ belonging to $C_b^2(\mathbb{R}^4)$ which satisfy the scaling properties (2.3.40), (2.3.43). If the functions $K^{(+)}(x,a,b,\lambda)$ and $K^{(-)}(x,a,b,\lambda)$ solves the PDEs :*

$$\begin{aligned}&\frac{\varphi(x,a,b)}{2(b-a)^2}\frac{\partial^2}{\partial x^2}K^{(\pm)}(x,a,b,\lambda) + \left(\frac{\theta(x,a,b)}{b-a} - \frac{\xi(x,a,b)}{2(b-a)^2}\right)\frac{\partial}{\partial x}K^{(\pm)}(x,a,b,\lambda)\\ =\ &\lambda\delta\frac{\partial}{\partial\lambda}K^{(\pm)}(x,a,b,\lambda)\end{aligned} \tag{2.6.44}$$

with boundary conditions:

$$\frac{\partial}{\partial x}K^{(+)}(x,a,b,\lambda)|_{x=a} = 0 \tag{2.6.45}$$

$$\frac{\partial}{\partial x}K^{(+)}(x,a,b,\lambda)|_{x=b} = -\lambda K^{(+)}(b,a,b,\lambda) \tag{2.6.46}$$

$$\frac{\partial}{\partial x}K^{(-)}(x,a,b,\lambda)|_{x=b} = 0 \tag{2.6.47}$$

$$\frac{\partial}{\partial x}K^{(-)}(x,a,b,\lambda)|_{x=a} = \lambda K^{(-)}(a,a,b,\lambda) \tag{2.6.48}$$

then

$$K^{(+)}(x,a,b,\lambda) := E^{(x,a,b)}\left(e^{-\lambda U^{(+)}}\right) \quad (2.6.49)$$
$$K^{(-)}(x,a,b,\lambda) := E^{(x,a,b)}\left(e^{-\lambda U^{(-)}}\right) \quad (2.6.50)$$

The next proposition concerns with the moments of the discounted dividends and the discounted financing.

Proposition 2.6.7 *(Moments of the discounted dividends and the discounted financing). Let the functions $V^{(+)}(x,a,b;n)$ and $V^{(-)}(x,a,b;n)$, $n \in N$ belonging to $C_b^2(\mathbb{R}^3)$ satisfy the scaling properties (2.3.39), (2.3.42). If the functions $V^{(+)}(x,a,b;n)$ and $V^{(-)}(x,a,b;n)$ solve the ODEs :*

$$\begin{aligned} &\frac{\varphi(x,a,b)}{2(b-a)^2}\frac{\partial^2}{\partial x^2}V^{(\pm)}(x,a,b;n) + \frac{\theta(x,a,b)}{b-a}\frac{\partial}{\partial x}V^{(\pm)}(x,a,b;n) + \\ &+\left(\frac{\mu_b(b)-\mu_a(a)}{b-a} - n\delta\right)V^{(\pm)}(x,a,b;n) \\ = \;& 0 \end{aligned} \quad (2.6.51)$$

with boundary conditions:

$$\frac{\partial}{\partial x}V^{(+)}(x,a,b;n)|_{x=a} = 0, \quad n = 1,2,\ldots \quad (2.6.52)$$
$$\frac{\partial}{\partial x}V^{(+)}(x,a,b;1)|_{x=b} = 1 \quad (2.6.53)$$
$$\frac{\partial}{\partial x}V^{(+)}(x,a,b;n)|_{x=b} = nV^{(+)}(b,a,b;n-1), \quad n = 2,3,\ldots \quad (2.6.54)$$
$$\frac{\partial}{\partial x}V^{(-)}(x,a,b;n)|_{x=b} = 0, \quad n = 1,2,\ldots \quad (2.6.55)$$
$$\frac{\partial}{\partial x}V^{(-)}(x,a,b;1)|_{x=a} = -1 \quad (2.6.56)$$
$$\frac{\partial}{\partial x}V^{(-)}(x,a,b;n)|_{x=a} = nV^{(-)}(a,a,b;n-1), \quad n = 2,3,\ldots \quad (2.6.57)$$

then

$$V^{(+)}(x,a,b;n) = E^{(x,a,b)}\left((U^{(+)})^n\right) \quad (2.6.58)$$
$$V^{(-)}(x,a,b;n) = E^{(x,a,b)}((U^{(-)})^n) \quad (2.6.59)$$

Proof. We will prove the result only for the function $V^{(+)}(x,a,b;n)$ because the proof for the function $V^{(-)}(x,a,b;n)$ is similar. First we observe that because the function $V^{(+)}(x,a,b;n)$ is in $C_b^2(\mathbb{R}^3)$ and satisfies the scaling properties (2.3.39),(2.3.42) then by relation (2.5.62) we must have

$$\begin{aligned} & \mathcal{L}_{(x,a,b)} V^{(+)}(x,a,b;n) && (2.6.60)\\ = \; & \mathcal{A}_{(x,a,b)} V^{(+)}(x,a,b;n) \\ = \; & \frac{\mu_b(b)-\mu_a(a)}{b-a} V^{(+)}(x,a,b;n) + \frac{\theta(x,a,b)}{b-a}\frac{\partial}{\partial x} V^{(+)}(x,a,b;n) + \\ & + \frac{\varphi(x,a,b)}{2(b-a)^2}\frac{\partial^2}{\partial x^2} V^{(+)}(x,a,b;n) \end{aligned}$$

By the above relation (2.6.60) and the fact that the function $V^{(+)}(x,a,b;n)$ solves the *ODE* (2.6.51) we conclude

$$(\mathcal{L}_{(x,a,b)} - n\delta) V^{(+)}(x,a,b;n) = 0 \qquad (2.6.61)$$

We consider the time instants h and t with $t \geq h$. Applying the **Itô** formula to the process

$$e^{-n\delta(t-h)} V^{(+)}(Z_t, a_t, b_t; n)$$

taking conditional expectations, taking into account the condition (2.6.52) and using the fact that the process $V^{(+)}(Z_h, a_h, b_h; n)$ is $\mathcal{F}_h$ measurable we have:

$$\begin{aligned} & E\left(e^{-n\delta(t-h)} V^{(+)}(Z_t,a_t,b_t;n)|\mathcal{F}_h\right) \\ = \; & V^{(+)}(Z_h,a_h,b_h;n) + E\left(\left(\int_h^t e^{-n\delta(s-h)}\left(\mathcal{L}_{(x,a,b)} - n\delta\right) V^{(+)}(Z_s,a_s,b_s;n)ds\right)|\mathcal{F}_h\right) - \\ & - E\left(\left(\int_h^t e^{-n\delta(s-h)}\frac{\partial}{\partial z} V^{(+)}(Z_s,a_s,b_s;n) d\mathcal{U}_s^{(+)}\right)|\mathcal{F}_h\right) && (2.6.62) \end{aligned}$$

Using relation (2.6.61) expression (2.6.62) becomes :

$$V^{(+)}(Z_h,a_h,b_h;n) = E\left(\left(\int_h^t e^{-n\delta(s-h)}\frac{\partial}{\partial z} V^{(+)}(Z_s,a_s,b_s;n) d\mathcal{U}_s^{(+)}\right)|\mathcal{F}_h\right) \qquad (2.6.63)$$

Next we apply relation (2.6.63) with $n = 1$, taking into account that by the relation (2.5.22) we have

$$dU^{(+)}(t) = -e^{-\delta t} d\mathcal{U}_t^{(+)} \qquad (2.6.64)$$

and also considering the relation (2.6.53) and taking the limit as $t \to \infty$ (using the monotone convergence theorem) we conclude that

$$E(e^{\delta h} U^{(+)}(h)|\mathcal{F}_h) = V^{(+)}(Z_h,a_h,b_h;1) < \infty \qquad (2.6.65)$$

We will use the method of induction. Suppose that for $n-1$ we have

$$E\left(e^{(n-1)\delta h}(U^{(+)}(h))^{n-1}|\mathcal{F}_h\right) = V^{(+)}(Z_h,a_h,b_h;n-1) < \infty \qquad (2.6.66)$$

Next applying the relation (2.6.63) for n and taking into account the relations (2.6.54), (2.6.66) and interchange expectation and integral (using the Fubini theorem) because

$$\begin{aligned} & E\left(\int_h^t e^{-n\delta(s-h)} V^{(+)}\left(Z_s, a_s, b_s; n-1\right) d\mathcal{U}_s^{(+)} | \mathcal{F}_h\right) \\ \leq \quad & \sup_s V^{(+)}\left(Z_s, a_s, b_s; n-1\right) e^{n\delta h} E((U^{(+)}(h)) | \mathcal{F}_h) < \infty \end{aligned}$$

we conclude

$$\begin{aligned} V^{(+)}(Z_h, a_h, b_h; n) &= nE\left(\int_h^t e^{-n\delta(s-h)} V(Z_s, a_s, b_s; n-1) d\mathcal{U}_s^{(+)} | \mathcal{F}_h\right) \\ &= nE\left(\int_h^t e^{-n\delta(s-h)} E\left(e^{(n-1)\delta s}(U^{(+)}(s))^{n-1} | \mathcal{F}_s\right) d\mathcal{U}_s^{(+)} | \mathcal{F}_h\right) \\ &= n\int_h^t e^{-n\delta(s-h)} E(E(e^{(n-1)\delta s}(U^{(+)}(s))^{n-1} d\mathcal{U}_s^{(+)} | \mathcal{F}_s) | \mathcal{F}_h) \\ &= n\int_h^t e^{-n\delta(s-h)} E(e^{(n-1)\delta s}(U^{(+)}(s))^{n-1} d\mathcal{U}_s^{(+)} | \mathcal{F}_h) \\ &= nE\left(e^{n\delta h}\int_h^t e^{-\delta s}(U^{(+)}(s))^{n-1} d\mathcal{U}_s^{(+)} | \mathcal{F}_h\right) \\ &= -E\left(e^{n\delta h}\int_h^t n(U^{(+)}(s))^{n-1} dU^{(+)}(s) | \mathcal{F}_h\right) \end{aligned}$$

Therefore

$$V^{(+)}(Z_h, a_h, b_h; n) = -E\left(e^{n\delta h}\int_h^t n(U^{(+)}(s))^{n-1} dU^{(+)}(s) | \mathcal{F}_h\right)$$

Taking the limit in the above relation as $t \to \infty$ (using the monotone convergence theorem) we have

$$\begin{aligned} V^{(+)}(Z_h, a_h, b_h; n) &= -E\left(e^{n\delta h}\int_h^\infty n(U^{(+)}(s))^{n-1} dU^{(+)}(s) | \mathcal{F}_h\right) \\ &= -E\left(e^{n\delta h}((U^{(+)}(s))^n|_h^\infty) | \mathcal{F}_h\right) \end{aligned}$$

Therefore

$$V^{(+)}(Z_h, a_h, b_h; n) = E\left(e^{n\delta h}(U^{(+)}(h))^n | \mathcal{F}_h\right) \tag{2.6.67}$$

Finally by taking $h = 0$ in the expression (2.6.67) we conclude the relation (2.6.58). ■

With the end of the previous proposition we come to the end of this section and also at the end of this chapter. We accomplish our goal which was to extend the classical de Finetti model with constant barriers to the more general de Finetti model with general barriers. The PDEs of this section can be used in order one to find expectations about the discounted dividends, the discounted financing and the time of ruin in the context of de Finetti model with general barriers.

2.7 Conclusions.

We extended the de Finetti model in order to include cases with fluctuating barriers dividends policies which are modelled as diffusions. We made the extension in an axiomatic manner by posing particular properties which were motivated by the classical de Finetti model. We showed that the de Finetti models with general barriers are well defined that is there exist unique stochastic processes that evolve according to the conditions dictated by our axioms.

We considered de Finetti models with one general barrier meaning that when the reserves of the insurance company reach a "particular" level which depends upon a diffusion process then the company goes bankrupt. We also considered de Finetti models with two general barriers, that is when the reserves of the insurance company reach the level of the lower barrier, which also depends upon a diffusion process, then the insurance company has the option to borrow money and continue it's function.

We derived differential equations with appropriate boundary conditions, the solution of which gives the moments of the discounted dividends, the moments of the discounted financing, the Laplace transform of the time of ruin, the Laplace transform of the joint distribution of the time of ruin and the discounted dividends and the Laplace transform of the joint distribution of the discounted dividends and the discounted financing.

Chapter 3

Applications on de Finetti models.

3.1 Introduction.

We will devote this chapter to particular applications of the formulas of chapter 2. More specifically we want to consider three particular cases for the reserve process of an insurance company which are of great interest in the literature. These include the Ornstein-Uhlenbeck, the Geometric Brownian Motion and the Brownian motion process. We choose to apply the formulas of this section in an insurance model which follows a constant dividends barriers policy in order our results to be comparable to the results available in the literature on dividends barriers policy.

3.2 Ornstein-Uhlenbeck process.

The Ornstein–Uhlenbeck process is a fundamental process that plays very important role in financial and insurance mathematics. For example if one wants to study risk models and to consider also an investment of the reserves in the stock market then soon will find himself in the realm of the Ornstein–Uhlenbeck process. As a consequence a lot of papers have been written on this process. (see for example the papers of Eie[55], Jacobsen[66], Schobel[168], Barndor and Nielsen [15], [16], Ward[183], Gillespie [77], NG [137], Aquilina [5], Cai [34], Alili [4], Going [79], Patie [144], Shiga [170], Graversen [83], Chojnowska [39], Nicolato [139], Lladser [124], Bishwal [24], Simao [172])

In this subsection we suppose that the reserves $X = \{X_t : t \geq 0\}$ of an insurance company follow the Ornstein-Uhlenbeck (OU) process, with dynamics that are described by

$$dX_t = -aX_t dt + \sigma dB_t \tag{3.2.1}$$

with initial capital $X_0 = x$. If $\mathcal{A}$ is the generator of X then

$$\mathcal{A}f(x) = -ax\frac{\partial}{\partial x}f(x) + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}f(x)$$

for a function $f(\cdot) \in D_A$, (where D_A is the domain of the generator $\mathcal{A}$ (see Definition 1.3.29)).

The insurance company follows the de Finetti model with upper barrier the constant b, that is for each time $t \geq 0$

$$b_t = b$$

and lower barrier the zero constant, that is for each time $t \geq 0$

$$a_t = 0$$

We consider first the de Finetti model with one general reflecting barrier (1RB) and we find the expected value of discounted dividends, the Laplace transform of the discounted dividends and the Laplace transform of time of ruin. Next we consider the de Finetti model with two general reflecting barriers (2RB) and we find the expected discounted dividends and the expected discounted financing, the Laplace transforms of the discounted dividends and the discounted financing and the Laplace transform of the joint distribution of the discounted dividends and the discounted financing.

The following special function will play an important role for the calculations of this section.

Definition 3.2.1 *Definition of the function $H(\lambda_1, \lambda_2, x)$ which is known as confluent hypergeometric function of Kummer (first kind). The function $H(\lambda_1, \lambda_2, x)$ is defined by*

$$H(\lambda_1, \lambda_2, x) := \sum_{k=0}^{\infty} \frac{(\lambda_1)_k}{(\lambda_2)_k}\frac{x^k}{k!}$$

where $(\lambda_1)_k, (\lambda_2)_k$ are the rising factorials, that is

$$\begin{aligned}(\lambda_1)_k &= \frac{(\lambda_1 + k - 1)!}{(\lambda_1 - 1)!}\\ (\lambda_2)_k &= \frac{(\lambda_2 + k - 1)!}{(\lambda_2 - 1)!}\end{aligned}$$

3.2.1 Expected value of the discounted dividends. (OU-1RB)

By Proposition 2.6.4 the Expected value of the discounted dividends $V(x)$ is given by the solution of:

$$\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}V(x) - ax\frac{\partial}{\partial x}V(x) = \delta V(x) \tag{3.2.2}$$

with boundary conditions:

$$V(0) = 0 \tag{3.2.3}$$

$$\frac{\partial}{\partial x}V(x)|_{x=b} = 1 \tag{3.2.4}$$

The solution of (3.2.2)-(3.2.4) is

$$V(x) = \frac{xH\left(\frac{1}{2}+\frac{\delta}{2a}, \frac{3}{2}, \frac{ax^2}{\sigma^2}\right)3\sigma^2}{3\sigma^2 H\left(\frac{1}{2}+\frac{\delta}{2a}, \frac{3}{2}, \frac{ab^2}{\sigma^2}\right) + 4ab^2\left(\frac{1}{2}+\frac{\delta}{2a}\right)H\left(\frac{3}{2}+\frac{\delta}{2a}, \frac{5}{2}, \frac{ab^2}{\sigma^2}\right)} \tag{3.2.5}$$

3.2.2 The Laplace transform of the discounted dividends.(OU-1RB)

By Proposition 2.6.2 the Laplace transform of the discounted dividends $K(x,\lambda)$ is given by the solution of:

$$\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}K(x,\lambda) - ax\frac{\partial}{\partial x}K(x,\lambda) - \lambda\delta\frac{\partial}{\partial x}K(x,\lambda) = 0 \tag{3.2.6}$$

with boundary conditions:

$$K(0,\lambda) = 1 \tag{3.2.7}$$

$$\frac{\partial}{\partial x}K(x,\lambda)|_{x=b} = -\lambda K(b,\lambda) \tag{3.2.8}$$

In order to solve (3.2.6) with boundary conditions (3.2.7), (3.2.8) we consider the moment of the discounted dividends $V(x;k)$ which is given by

$$V(x;k) = E^x(U^k), \qquad k = 1,2,3,\ldots \tag{3.2.9}$$

We have that

$$K(x,\lambda) = 1 + \sum_{k=1}^{\infty}\frac{(-\lambda)^k}{k!}E^x(U^k) = 1 + \sum_{k=1}^{\infty}\frac{(-\lambda)^k}{k!}V(x;k) \tag{3.2.10}$$

Substituting (3.2.10) to (3.2.6) and comparing the coefficients of $(-\lambda)^k$ we have the differential equations:

$$\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}V(x;k) - ax\frac{\partial}{\partial x}V(x;k) - \delta k\frac{\partial}{\partial x}V(x;k) = 0, \qquad k = 1,2,3,\ldots \tag{3.2.11}$$

From condition $K(0,\lambda)=1$ we have that

$$V(0;k)=0, \qquad k=1,2,3,\dots \tag{3.2.12}$$

and from the condition $\frac{\partial}{\partial x}K(x,\lambda)|_{x=b}=-\lambda K(b,\lambda)$ we have that

$$\frac{\partial}{\partial x}V(x;1)|_{x=b}=1 \tag{3.2.13}$$

and

$$\frac{\partial}{\partial x}V(x;k)|_{x=b}=kV(b;k-1), \qquad k=2,3,4,\dots \tag{3.2.14}$$

The solutions of (3.2.11) are of the form:

$$V(x;k)=c_k(b)g_k(x), \qquad k=1,2,3,\dots \tag{3.2.15}$$

where

$$g_k(x)=xH\left(\frac{1}{2}+\frac{\delta k}{2a},\frac{3}{2},\frac{ax^2}{\sigma^2}\right), \qquad k=1,2,3,\dots$$

In order to find the $c_k(b)$ we have from (3.2.15), (3.2.13) and (3.2.14) that

$$c_1(b)=\frac{1}{\frac{\partial}{\partial x}g_1(x)|_{x=b}} \tag{3.2.16}$$

and

$$c_k(b)\left(\frac{\partial}{\partial x}g_k(x)|_{x=b}\right)=kc_{k-1}(b)g_{k-1}(b), \tag{3.2.17}$$

for $k=2,3,4,\dots$. So we find that

$$c_k(b)=k!\frac{g_1(b)\cdots g_{k-1}(b)}{\left(\frac{\partial}{\partial x}g_1(x)|_{x=b}\right)\cdots\left(\frac{\partial}{\partial x}g_{k-1}(x)|_{x=b}\right)\left(\frac{\partial}{\partial x}g_k(x)|_{x=b}\right)} \tag{3.2.18}$$

and the k–th moment $V(x;k)$ of the discounted dividends U about the origin is

$$V(x;k)=E^x(U^k)=k!\frac{g_1(b)\cdots g_{k-1}(b)g_k(x)}{\left(\frac{\partial}{\partial x}g_1(x)|_{x=b}\right)\cdots\left(\frac{\partial}{\partial x}g_{k-1}(x)|_{x=b}\right)\left(\frac{\partial}{\partial x}g_k(x)|_{x=b}\right)}, \tag{3.2.19}$$

$k=1,2,3,\dots$, and the Laplace transform $K(x,\lambda)$ of the discounted dividends U is

$$K(x,\lambda)=1+\sum_{k=1}^{\infty}(-\lambda)^k\frac{g_1(b)\cdots g_{k-1}(b)g_k(x)}{\left(\frac{\partial}{\partial x}g_1(x)|_{x=b}\right)\cdots\left(\frac{\partial}{\partial x}g_{k-1}(x)|_{x=b}\right)\left(\frac{\partial}{\partial x}g_k(x)|_{x=b}\right)}$$

3.2.3 The Laplace transform of the time of ruin. (OU-1RB)

By Proposition 2.6.3 the Laplace transform of time of ruin $M(x,\lambda)$ is given by the solution of:

$$\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}M(x,\lambda)-ax\frac{\partial}{\partial x}M(x,\lambda)-\lambda M(x,\lambda)=0 \tag{3.2.20}$$

with boundary conditions:

$$M(0,\lambda) = 1 \quad (3.2.21)$$

$$\frac{\partial}{\partial x} M(x,\lambda)|_{x=b} = 0 \quad (3.2.22)$$

The solution of (3.2.20)-(3.2.22) is

$$M(x,\lambda) = H\left(\frac{\lambda}{2a}, \frac{1}{2}, \frac{ax^2}{\sigma^2}\right) - xH\left(\frac{1}{2} + \frac{\lambda}{2a}, \frac{3}{2}, \frac{ax^2}{\sigma^2}\right) \widetilde{f}(a,b,\lambda,\sigma) \quad (3.2.23)$$

where

$$\widetilde{f}(a,b,\lambda,\sigma) := \frac{6\lambda b H\left(1 + \frac{\lambda}{2a}, \frac{3}{2}, \frac{ab^2}{\sigma^2}\right)}{\left(3\sigma^2 H\left(\frac{1}{2} + \frac{\lambda}{2a}, \frac{3}{2}, \frac{ab^2}{\sigma^2}\right) + 4ab^2\left(\frac{1}{2} + \frac{\lambda}{2a}\right) H\left(\frac{3}{2} + \frac{\lambda}{2a}, \frac{5}{2}, \frac{ab^2}{\sigma^2}\right)\right)}$$

Remark 3.2.2 *We found that the function $V(x) := E^x U$ is given by (3.2.5) and it is of the form $V(x) = \frac{g(x)}{g'(b)}$, with $g(x) = xH\left(\frac{1}{2} + \frac{\delta}{2a}, \frac{3}{2}, \frac{ax^2}{\sigma^2}\right)$. Let b^* be the value of the barrier b which gives the maximum dividends. Then the optimal barrier b^* can be found from the solution of equation*

$$g''(b^*) = 0 \quad (3.2.24)$$

The above equation is difficult to be solved analytically in general and has to be addressed using numerical methods.

Remark 3.2.3 *If the initial state is $x = b^*$ then it holds that*

$$V(b^*) = \frac{g(b^*)}{g'(b^*)}$$

which implies

$$V''(b^*) = 0 \quad (3.2.25)$$

Substituting (3.2.25) in (3.2.2) we find that

$$\frac{1}{2}\sigma^2 0 - ab^* 1 - \delta V(b^*) = 0$$

which leads to

$$V(b^*) = -\frac{ab^*}{\delta} \quad (3.2.26)$$

That is (3.2.26) gives the maximum value of the dividends when the initial state is $x = b^*$.

Remark 3.2.4 *The optimal barrier b^* is independent from the initial state of the process and for known values of the parameters can be calculated with numerical methods.*

3.2.4 Expected discounted dividends and expected discounted financing. (OU-2RB)

By Proposition 2.6.7 the expected discounted dividends $V^{(+)}(x)$ and expected discounted financing $V^{(-)}(x)$ are given by the solutions of:

$$\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}V^{(\pm)}(x) - ax\frac{\partial}{\partial x}V^{(\pm)}(x) - \delta V^{(\pm)}(x) = 0 \tag{3.2.27}$$

with boundary conditions:

$$\begin{aligned}\frac{\partial}{\partial x}V^{(+)}(x)|_{x=0} &= 0 \\ \frac{\partial}{\partial x}V^{(+)}(x)|_{x=b} &= 1 \\ \frac{\partial}{\partial x}V^{(-)}(x)|_{x=0} &= -1 \\ \frac{\partial}{\partial x}V^{(-)}(x)|_{x=b} &= 0\end{aligned} \tag{3.2.28}$$

The solutions of (3.2.27) with boundary conditions (3.2.28) are

$$V^{(+)}(x) = \frac{\sigma^2}{2\delta bH\left(1+\frac{\delta}{2a},\frac{3}{2},\frac{ab^2}{\sigma^2}\right)}H\left(\frac{\delta}{2a},\frac{1}{2},\frac{ax^2}{\sigma^2}\right) \tag{3.2.29}$$

$$V^{(-)}(x) = H\left(\frac{\delta}{2a},\frac{1}{2},\frac{ax^2}{\sigma^2}\right)\mathcal{Q}(a,b,\sigma,\delta) - xH\left(\frac{1}{2}+\frac{\delta}{2a},\frac{3}{2},\frac{ax^2}{\sigma^2}\right) \tag{3.2.30}$$

where

$$\mathcal{Q}(a,b,\sigma,\delta) := \frac{3\sigma^2H\left(\frac{1}{2}+\frac{\delta}{2a},\frac{3}{2},\frac{ab^2}{\sigma^2}\right)+4a\left(\frac{1}{2}+\frac{\delta}{2a}\right)b^2H\left(\frac{3}{2}+\frac{\delta}{2a},\frac{5}{2},\frac{ab^2}{\sigma^2}\right)}{6\delta bH\left(1+\frac{\delta}{2a},\frac{3}{2},\frac{ab^2}{\sigma^2}\right)}$$

3.2.5 The Laplace transforms of the discounted dividends and the discounted financing. (OU-2RB)

By Proposition 2.6.6 the Laplace transform of the discounted dividends $K^{(+)}(x,\lambda)$ is given by the solution of:

$$\frac{1}{2}\sigma^2\frac{\partial}{\partial x^2}K^{(+)}(x,\lambda) - ax\frac{\partial}{\partial x}K^{(+)}(x,\lambda) - \lambda\delta\frac{\partial}{\partial\lambda}K^{(+)}(x,\lambda) = 0 \tag{3.2.31}$$

with boundary conditions:

$$\begin{aligned}\frac{\partial}{\partial x}K^{(+)}(x,\lambda)|_{x=0} &= 0 \\ \frac{\partial}{\partial x}K^{(+)}(x,\lambda)|_{x=b} &= -\lambda K^{(+)}(b,\lambda)\end{aligned} \tag{3.2.32}$$

and the Laplace transform of the discounted financing $K^{(-)}(x,\lambda)$ is given by the solution of:

$$\frac{1}{2}\sigma^2\frac{\partial}{\partial x^2}K^{(-)}(x,\lambda)-ax\frac{\partial}{\partial x}K^{(-)}(x,\lambda)-\lambda\delta\frac{\partial}{\partial\lambda}K^{(-)}(x,\lambda)=0 \tag{3.2.33}$$

with boundary conditions:

$$\begin{aligned}\frac{\partial}{\partial x}K^{(-)}(x,\lambda)|_{x=b} &= 0 \\ \frac{\partial}{\partial x}K^{(-)}(x,\lambda)|_{x=0} &= \lambda K^{(-)}(0,\lambda)\end{aligned} \tag{3.2.34}$$

In order to solve (3.2.31) with boundary conditions (3.2.32) let

$$V^{(+)}(x;k)=E^x((U^{(+)})^k),\qquad k=1,2,3,\dots \tag{3.2.35}$$

Then we have

$$K^{(+)}(x,\lambda)=1+\sum_{k=1}^{\infty}\frac{(-\lambda)^k}{k!}E^x((U^{(+)})^k)=1+\sum_{k=1}^{\infty}\frac{(-\lambda)^k}{k!}V^{(+)}(x;k) \tag{3.2.36}$$

Substituting (3.2.36) to (3.2.31) and comparing the coefficients of $(-\lambda)^k$ we have the differential equations

$$\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}V^{(+)}(x;k)-ax\frac{\partial}{\partial x}V^{(+)}(x;k)-\delta k\frac{\partial}{\partial x}V^{(+)}(x;k)=0 \tag{3.2.37}$$

for $k=1,2,3,\dots$. From the condition $\frac{\partial}{\partial x}K^{(+)}(x,\lambda)|_{x=0}=0$ we have that

$$\frac{\partial}{\partial x}V^{(+)}(x;k)|_{x=0}=0,\qquad k=1,2,3,\dots \tag{3.2.38}$$

and from the condition $\frac{\partial}{\partial x}K^{(+)}(x,\lambda)|_{x=b}=-\lambda K^{(+)}(b,\lambda)$ it follows that

$$\frac{\partial}{\partial x}V^{(+)}(x;1)|_{x=b}=1 \tag{3.2.39}$$

and

$$\frac{\partial}{\partial x}V^{(+)}(x;k)|_{x=b}=kV^{(+)}(b;k-1),\qquad k=2,3,4,\dots \tag{3.2.40}$$

The solutions of (3.2.37) are of the form

$$V^{(+)}(x;k)=c_k(b)g_k(x),\qquad k=1,2,3,\dots \tag{3.2.41}$$

where

$$g_k(x)=H\left(\frac{\delta k}{2a},\frac{1}{2},\frac{ax^2}{\sigma^2}\right).$$

In order to find the $c_k(b)$ we have from (3.2.41), (3.2.39) and (3.2.40) that

$$c_1(b)=\frac{1}{\frac{\partial}{\partial x}g_1(x)|_{x=b}} \tag{3.2.42}$$

and

$$c_k(b)\left(\frac{\partial}{\partial x}g_k(x)|_{x=b}\right) = kc_{k-1}(b)g_{k-1}(b) \tag{3.2.43}$$

for $k = 2, 3, 4, \ldots$ .So we find that

$$c_k(b) = k!\frac{g_1(b)\cdots g_{k-1}(b)}{\left(\frac{\partial}{\partial x}g_1(x)|_{x=b}\right)\cdots\left(\frac{\partial}{\partial x}g_{k-1}(x)|_{x=b}\right)\left(\frac{\partial}{\partial x}g_k(x)|_{x=b}\right)} \tag{3.2.44}$$

and the k–th moment $V^{(+)}(x; k)$ of the discounted dividends $U^{(+)}$ about the origin is

$$V^{(+)}(x; k) = k!\frac{g_1(b)\cdots g_{k-1}(b)g_k(x)}{\left(\frac{\partial}{\partial x}g_1(x)|_{x=b}\right)\cdots\left(\frac{\partial}{\partial x}g_{k-1}(x)|_{x=b}\right)\left(\frac{\partial}{\partial x}g_k(x)|_{x=b}\right)} \tag{3.2.45}$$

for $k = 1, 2, 3, \ldots$. The Laplace transform $K^{(+)}(x, \lambda)$ of the discounted dividends $U^{(+)}$ is

$$K^{(+)}(x, \lambda) = 1 + \sum_{k=1}^{\infty}(-\lambda)^k\frac{g_1(b)\cdots g_{k-1}(b)g_k(x)}{\left(\frac{\partial}{\partial x}g_1(x)|_{x=b}\right)\cdots\left(\frac{\partial}{\partial x}g_{k-1}(x)|_{x=b}\right)\left(\frac{\partial}{\partial x}g_k(x)|_{x=b}\right)}$$

In order to solve (3.2.33) with boundary conditions (3.2.34) we proceed in analogous manner. Let

$$V^{(-)}(x; k) = E^x((U^{(-)})^k), \quad k = 1, 2, 3, \ldots \tag{3.2.46}$$

Then we have

$$K^{(-)}(x, \lambda) = 1 + \sum_{k=1}^{\infty}\frac{(-\lambda)^k}{k!}E^x((U^{(-)})^k) = 1 + \sum_{k=1}^{\infty}\frac{(-\lambda)^k}{k!}V^{(-)}(x; k) \tag{3.2.47}$$

Substituting (3.2.47) to (3.2.33) and comparing the coefficients of $(-\lambda)^k$ we have the differential equations

$$\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}V^{(-)}(x; k) - ax\frac{\partial}{\partial x}V^{(-)}(x; k) - \delta k\frac{\partial}{\partial x}V^{(-)}(x; k) = 0 \tag{3.2.48}$$

for $k = 1, 2, 3, \ldots$. From the condition

$$\frac{\partial}{\partial x}K^{(-)}(x, \lambda)|_{x=b} = 0$$

we have that

$$\frac{\partial}{\partial x}V^{(-)}(x; k)|_{x=b} = 0, \quad k = 1, 2, 3, \ldots \tag{3.2.49}$$

and from the condition

$$\frac{\partial}{\partial x}K^{(-)}(x, \lambda)|_{x=0} = -\lambda K^{(-)}(0, \lambda)$$

we obtain

$$\frac{\partial}{\partial x}V^{(-)}(x; 1)|_{x=0} = -1 \tag{3.2.50}$$

and

$$\frac{\partial}{\partial x}V^{(-)}(x; k)|_{x=0} = -kV^{(-)}(0; k-1) \quad k = 2, 3, 4, \ldots \tag{3.2.51}$$

The solutions of (3.2.48) are of the form

$$V^{(-)}(x;k) = c_k(0)m_k(x) \qquad k = 1,2,3,\dots \tag{3.2.52}$$

where

$$m_k(x) = H\left(\frac{\delta k}{2a}, \frac{1}{2}, \frac{ax^2}{\sigma^2}\right) - xH\left(\frac{1}{2} + \frac{\delta k}{2a}, \frac{3}{2}, \frac{ax^2}{\sigma^2}\right)\mathcal{S}(a,b,k,\delta,\sigma) \tag{3.2.53}$$

and

$$\mathcal{S}(a,b,k,\delta,\sigma) := \frac{6\delta bkH\left(1+\frac{\delta k}{2a}, \frac{3}{2}, \frac{ab^2}{\sigma^2}\right)}{3\sigma^2 H\left(\frac{1}{2}+\frac{\delta k}{2a}, \frac{3}{2}, \frac{ab^2}{\sigma^2}\right) + 4ab^2\left(\frac{1}{2}+\frac{\delta k}{2a}\right)H\left(\frac{3}{2}+\frac{\delta k}{2a}, \frac{5}{2}, \frac{ab^2}{\sigma^2}\right)}$$

In order to find the $c_k(0)$ we have from (3.2.52), (3.2.50) and (3.2.51) that

$$c_1(0) = \frac{-1}{\frac{\partial}{\partial x}m_1(x)|_{x=0}} \tag{3.2.54}$$

and

$$c_k(0)\left(\frac{\partial}{\partial x}m_k(x)|_{x=0}\right) = -kc_{k-1}(0)m_{k-1}(0) \qquad \text{for } k = 2,3,4,\dots \tag{3.2.55}$$

So we find that

$$c_k(0) = (-1)^k k! \frac{m_1(0)\cdots m_{k-1}(0)}{\left(\frac{\partial}{\partial x}m_1(x)|_{x=0}\right)\cdots\left(\frac{\partial}{\partial x}m_{k-1}(x)|_{x=0}\right)\left(\frac{\partial}{\partial x}m_k(x)|_{x=0}\right)} \tag{3.2.56}$$

and the k–th moment of $U^{(-)}$ about the origin is

$$\begin{aligned} V^{(-)}(x;k) &= (-1)^k k! \frac{m_1(0)\cdots m_{k-1}(0)m_k(x)}{\left(\frac{\partial}{\partial x}m_1(x)|_{x=0}\right)\cdots\left(\frac{\partial}{\partial x}m_{k-1}(x)|_{x=0}\right)\left(\frac{\partial}{\partial x}m_k(x)|_{x=0}\right)} \\ k &= 1,2,3,\dots \end{aligned} \tag{3.2.57}$$

and the Laplace transform of $U^{(-)}$ is

$$\begin{aligned} &K^{(-)}(x,\lambda) \\ &= 1 + \sum_{k=1}^{\infty} \lambda^k \frac{m_1(0)\cdots m_{k-1}(0)m_k(x)}{\left(\frac{\partial}{\partial x}m_1(x)|_{x=0}\right)\cdots\left(\frac{\partial}{\partial x}m_{k-1}(x)|_{x=0}\right)\left(\frac{\partial}{\partial x}m_k(x)|_{x=0}\right)} \end{aligned} \tag{3.2.58}$$

3.2.6 The Laplace transform of the joint distribution of the discounted dividends and the discounted financing. (OU-2RB)

By Proposition 2.6.5 the Laplace transform of the joint distribution of the discounted dividends and the discounted financing is given by the solution of:

$$\begin{aligned} &\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}N(x,\lambda_1,\lambda_2) - ax\frac{\partial}{\partial x}N(x,\lambda_1,\lambda_2) \\ &= \delta\left(\lambda_1\frac{\partial}{\partial\lambda_1}N(x,\lambda_1,\lambda_2) + \lambda_2\frac{\partial}{\partial\lambda_2}N(x,\lambda_1,\lambda_2)\right) \end{aligned} \tag{3.2.59}$$

with boundary conditions:

$$\frac{\partial}{\partial x}N(x,\lambda_1,\lambda_2)|_{x=0} = \lambda_2 N(0,\lambda_1,\lambda_2) \tag{3.2.60}$$

$$\frac{\partial}{\partial x}N(x,\lambda_1,\lambda_2)|_{x=b} = -\lambda_1 N(b,\lambda_1,\lambda_2) \tag{3.2.61}$$

In order to solve (3.2.59) with boundary conditions (3.2.60), (3.2.61) let

$$W_k(x,\lambda_1,\lambda_2) = E^x\left((-\lambda_1 U^{(+)} - \lambda_2 U^{(-)})^k\right) \tag{3.2.62}$$

$$k = 1,2,3\dots.$$

Then we have

$$N(x,\lambda_1,\lambda_2) = 1 + \sum_{k=1}^{\infty}\frac{E^x\left(\left(-\lambda_1 U^{(+)} - \lambda_2 U^{(-)}\right)^k\right)}{k!} = 1 + \sum_{k=1}^{\infty}\frac{W_k(x,\lambda_1,\lambda_2)}{k!} \tag{3.2.63}$$

Substituting (3.2.63) to (3.2.59) it follows that

$$\frac{1}{2}\sigma^2\sum_{k=1}^{\infty}\frac{W_k^{''}(x,\lambda_1,\lambda_2)}{k!} - ax\sum_{k=1}^{\infty}\frac{W_k^{'}(x,\lambda_1,\lambda_2)}{k!} =$$

$$= \delta\left(\lambda_1\sum_{k=1}^{\infty}\frac{E^x\left(\left(-U^{(+)}\right)\left(-\lambda_1 U^{(+)} - \lambda_2 U^{(-)}\right)^{k-1}\right)}{(k-1)!} + \right.$$
$$\left. +\lambda_2\sum_{k=1}^{\infty}\frac{E^x\left((-U^{(-)})(-\lambda_1 U^{(+)} - \lambda_2 U^{(-)})^{k-1}\right)}{(k-1)!}\right)$$

which implies

$$\frac{1}{2}\sigma^2\sum_{k=1}^{\infty}\frac{W_k^{''}(x,\lambda_1,\lambda_2)}{k!} - ax\sum_{k=1}^{\infty}\frac{W_k^{'}(x,\lambda_1,\lambda_2)}{k!}$$
$$= \delta\sum_{k=1}^{\infty}\frac{E^x\left((-\lambda_1 U^{(+)} - \lambda_2 U^{(-)})^k\right)}{(k-1)!} = \delta\sum_{k=1}^{\infty}\frac{W_k(x,\lambda_1,\lambda_2)}{(k-1)!} \Longrightarrow$$

$$\sum_{k=1}^{\infty}\frac{\frac{1}{2}\sigma^2 W_k^{''}(x,\lambda_1,\lambda_2) - axW_k^{'}(x,\lambda_1,\lambda_2) - \delta k W_k(x,\lambda_1,\lambda_2)}{k!} = 0 \tag{3.2.64}$$

and because (3.2.64) holds for every λ_1, λ_2 we conclude that

$$\frac{1}{2}\sigma^2 W_k^{''}(x,\lambda_1,\lambda_2) - axW_k^{'}(x,\lambda_1,\lambda_2) - \delta k W_k(x,\lambda_1,\lambda_2) = 0 \tag{3.2.65}$$

$$k = 1, 2, 3 \dots .$$

The solution of (3.2.65) is

$$W_k(x,\lambda_1,\lambda_2) = c_{k1}H(\left(\frac{\delta k}{2a},\frac{1}{2},\frac{ax^2}{\sigma^2}\right) + c_{k2}xH\left(\frac{1}{2}+\frac{\delta k}{2a},\frac{3}{2},\frac{ax^2}{\sigma^2}\right) \tag{3.2.66}$$

$$k = 1, 2, 3, \dots .$$

From the condition $\frac{\partial}{\partial x}N(x,\lambda_1,\lambda_2)|_{x=0} = \lambda_2 N(0,\lambda_1,\lambda_2)$ we have that :

$$W_1^{'}(0,\lambda_1,\lambda_2) = -1 \tag{3.2.67}$$

$$W_k^{'}(0,\lambda_1,\lambda_2) = -kW_{k-1}(0,\lambda_1,\lambda_2) \tag{3.2.68}$$

and from the condition $\frac{\partial}{\partial x}N(x,\lambda_1,\lambda_2)|_{x=b} = -\lambda_1 N(b,\lambda_1,\lambda_2)$ we have that :

$$W_1^{'}(b,\lambda_1,\lambda_2) = 1 \tag{3.2.69}$$

$$W_k^{'}(b,\lambda_1,\lambda_2) = kW_{k-1}(b,\lambda_1,\lambda_2) \tag{3.2.70}$$

From the relations (3.2.66), (3.2.67), (3.2.69) we find that

$$W_1(x,\lambda_1,\lambda_2) = \mathcal{T}(a,b,\sigma,\delta)H\left(\frac{\delta}{2a},\frac{1}{2},\frac{ax^2}{\sigma^2}\right) - xH\left(\frac{1}{2}+\frac{\delta k}{2a},\frac{3}{2},\frac{ax^2}{\sigma^2}\right) \tag{3.2.71}$$

where

$$\mathcal{T}(a,b,\sigma,\delta) := \frac{3\sigma^2\left(1+H(\frac{a+\delta}{2a},\frac{3}{2},\frac{ab^2}{\sigma^2})\right) + 2(a+\delta)b^2H\left(\frac{1}{2}(3+\frac{\delta}{a}),\frac{5}{2},\frac{ab^2}{\sigma^2}\right)}{6\delta bH\left(1+\frac{\delta}{2a},\frac{3}{2},\frac{ab^2}{\sigma^2}\right)}$$

From the relations (3.2.66), (3.2.68), (3.2.70) we find that

$$\begin{aligned} & W_k(x,\lambda_1,\lambda_2) \\ = \; & -kW_{k-1}(0,\lambda_1,\lambda_2)xH\left(\frac{1}{2}+\frac{\delta k}{2a},\frac{3}{2},\frac{ax^2}{\sigma^2}\right) + \\ & +\left(\mathcal{D}(a,b,\sigma,k,\delta)W_{k-1}(0,\lambda_1,\lambda_2) + \frac{3\sigma^2 V_{k-1}(b,\lambda_1,\lambda_2)}{6\delta bH\left(1+\frac{\delta k}{2a},\frac{3}{2},\frac{ab^2}{\sigma^2}\right)}\right)H\left(\frac{\delta k}{2a},\frac{1}{2},\frac{ax^2}{\sigma^2}\right) \end{aligned} \tag{3.2.72}$$

where

$$\mathcal{D}(a,b,\sigma,k,\delta) := \frac{\left(3\sigma^2H\left(\frac{a+\delta k}{2a},\frac{3}{2},\frac{ab^2}{\sigma^2}\right)\right) + 2(a+\delta k)b^2H\left(\frac{1}{2}(3+\frac{\delta k}{a}),\frac{5}{2},\frac{ab^2}{\sigma^2}\right)]}{6\delta bH\left(1+\frac{\delta k}{2a},\frac{3}{2},\frac{ab^2}{\sigma^2}\right)}$$

3.3 Geometric Brownian motion.

The Geometric Brownian motion is a fundamental process for stochastic analysis. Sooner or later anyone who involves with Risk theory will come across to this process. For this reason we consider as important the study of the properties of this process. Someone who is interesting in recent developments relevant to this process might see for example the papers of: Graversen [82], Donati [49], Dufresne [51], Ishiyama [102], Masamitsu [128], [129], Bhattacharya [22], Lefebvre [121], Carr [37], Matsumoto [130], [131], Marathe [127], Gushchin [84], Cai [35], Yor [190], Takatsuka [176].

In this subsection we suppose that the reserves $X = \{X_t : t \geq 0\}$ of an insurance company follow the Geometric Brownian motion (GBM) process, with dynamics that are described by

$$dX_t = \mu X_t dt + \sigma X_t dB_t \tag{3.3.1}$$

with $X_0 = x > 0$. If A is the generator of X then

$$\mathcal{A}f(x) = \mu x \frac{\partial}{\partial x} f(x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} f(x)$$

for a function $f(\cdot) \in D_A$, (where D_A is the domain of the generator A (see Definition 1.3.29)).

The insurance company follows the de Finetti model with upper barrier the constant b, that is for each time $t \geq 0$

$$b_t = b$$

and lower barrier the zero constant, that is for each time $t \geq 0$

$$a_t = 0$$

We consider first the de Finetti model with one general reflecting barrier (1RB) and we find the expected value of the discounted dividends, the Laplace transform of the discounted dividends, and the Laplace transform of the time of ruin. Next we consider the de Finetti model with two general reflecting barriers (2RB) and we find the expected discounted dividends and the expected discounted financing, the Laplace transforms of the discounted dividends and the discounted financing, and the Laplace transform of the joint distribution of the discounted dividends and the discounted financing (also see Gerber, H.U. and Shiu, E.S.W.([72])).

Before we start studying the case of one general reflecting barrier, we first define the quantities

$$s_{k,\delta} := \frac{\sigma^2 - 2\mu + \sqrt{(\sigma^2 - 2\mu) + 8\delta k \sigma^2}}{2\sigma^2} \tag{3.3.2}$$

$$r_{k,\delta} := \frac{\sigma^2 - 2\mu - \sqrt{(\sigma^2 - 2\mu) + 8\delta k \sigma^2}}{2\sigma^2} \tag{3.3.3}$$

for $k = 1, 2, \ldots$. We observe that $r_{k,\delta} \le 0 \le s_{k,\delta}$.

3.3.1 Expected value of the discounted dividends. (GBM-1RB)

By Proposition 2.6.4 the expected value of the discounted dividends $V(x)$ is given by the solution of

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} V(x) + \mu x \frac{\partial}{\partial x} V(x) = \delta V(x) \tag{3.3.4}$$

with boundary conditions

$$V(a) = 0, \tag{3.3.5}$$

$$\frac{\partial}{\partial x} V(x)|_{x=b} = 1. \tag{3.3.6}$$

The solution of (3.3.4) is

$$V(x) = \frac{g(x)}{g'(b)} \tag{3.3.7}$$

where

$$g(x) := x^{s_{1,\delta}} - a^{s_{1,\delta} - r_{1,\delta}} x^{r_{k,\delta}} \tag{3.3.8}$$

3.3.2 The Laplace transform of the discounted dividends. (GBM-1RB)

By Proposition 2.6.2 the Laplace transform of the discounted dividends $K(x, \lambda)$ is given by the solution of

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} K(x,\lambda) + \mu x \frac{\partial}{\partial x} K(x,\lambda) - \lambda\delta \frac{\partial}{\partial x} K(x,\lambda) = 0 \tag{3.3.9}$$

with boundary conditions

$$K(a, \lambda) = 1 \tag{3.3.10}$$

$$\frac{\partial}{\partial x} K(x,\lambda)|_{x=b} = -\lambda K(b, \lambda) \tag{3.3.11}$$

Working as in section 3.2 we find that the k–th moment of the discounted dividends U,for $k = 1, 2, 3, \ldots$

$$V(x;k) = E^x(U^k) = k! \frac{g_1(b) \cdots g_{k-1}(b) g_k(x)}{\left(\frac{\partial}{\partial x} g_1(x)|_{x=b}\right) \cdots \left(\frac{\partial}{\partial x} g_{k-1}(x)|_{x=b}\right) \left(\frac{\partial}{\partial x} g_k(x)|_{x=b}\right)},$$

and the Laplace transform of the discounted dividends $K(x, \lambda)$ is given by

$$K(x,\lambda) = 1 + \sum_{k=1}^{\infty} (-\lambda)^k \frac{g_1(b) \cdots g_{k-1}(b) g_k(x)}{\left(\frac{\partial}{\partial x} g_1(x)|_{x=b}\right) \cdots \left(\frac{\partial}{\partial x} g_{k-1}(x)|_{x=b}\right) \left(\frac{\partial}{\partial x} g_k(x)|_{x=b}\right)} \tag{3.3.12}$$

with

$$g_k(x) := x^{s_{k,\delta}} - a^{s_{k,\delta} - r_{k,\delta}} x^{r_{k,\delta}} \tag{3.3.13}$$

3.3.3 The Laplace transform of the time of ruin. (GBM-1RB)

By Proposition 2.6.3 the Laplace transform of the time of ruin $M(x,\lambda)$ is given by the solution of:

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} M(x,\lambda) + \mu x \frac{\partial}{\partial x} M(x,\lambda) - \lambda M(x,\lambda) = 0 \tag{3.3.14}$$

with boundary conditions

$$M(a,\lambda) = 1 \tag{3.3.15}$$

$$\frac{\partial}{\partial x} M(x,\lambda)|_{x=b} = 0 \tag{3.3.16}$$

The solution of (3.3.14) with boundary conditions (3.3.15), (3.3.16) is

$$M(x,\lambda) = \frac{x^{s_{1,\lambda}} - \frac{s_{1,\lambda}}{r_{1,\lambda}} b^{s_{1,\lambda}-r_{1,\lambda}} x^{r_{1,\lambda}}}{a^{s_{1,\lambda}} - \frac{s_{1,\lambda}}{r_{1,\lambda}} b^{s_{1,\lambda}-r_{1,\lambda}} a^{r_{1,\lambda}}} \tag{3.3.17}$$

Remark 3.3.1 *(Maximum dividends barrier). Let b^* be the value of the barrier b which gives the maximum dividends $V(\cdot)$. Then the b^* can be found from the solution of equation $g''(b^*) = 0$, and from the solution of this equation we found that the optimal value for the barrier is*

$$b^* = a \left(\frac{r_{1,\delta}(r_{1,\delta}-1)}{s_{1,\delta}(s_{1,\delta}-1)} \right)^{\frac{1}{s_{1,\delta}-r_{1,\delta}}} \tag{3.3.18}$$

3.3.4 Expected discounted dividends and expected discounted financing. (GBM-2RB)

By Proposition 2.6.7 the expected discounted dividends $V^{(+)}(x)$ and expected discounted financing $V^{(-)}(x)$ are given by the solutions of:

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} V^{(\pm)}(x) + \mu x \frac{\partial}{\partial x} V^{(\pm)}(x) - \delta V^{(\pm)}(x) = 0 \tag{3.3.19}$$

with boundary conditions

$$\frac{\partial}{\partial x} V^{(+)}(x)|_{x=a} = 0 \tag{3.3.20}$$

$$\frac{\partial}{\partial x} V^{(+)}(x)|_{x=b} = 1 \tag{3.3.21}$$

$$\frac{\partial}{\partial x} V^{(-)}(x)|_{x=a} = -1 \tag{3.3.22}$$

$$\frac{\partial}{\partial x} V^{(-)}(x)|_{x=b} = 0 \tag{3.3.23}$$

The solutions of (3.3.19) with boundary conditions (3.3.20)-(3.3.23) are

$$V^{(+)}(x) = \frac{b}{a^{r_{1,\delta}}b^{s_{1,\delta}} - a^{s_{1,\delta}}b^{r_{1,\delta}}}\left(\frac{a^s x^{s_{1,\delta}}}{s_{1,\delta}} - \frac{a^{s_{1,\delta}} x^{r_{1,\delta}}}{r_{1,\delta}}\right) \tag{3.3.24}$$

$$V^{(-)}(x) = \frac{\alpha}{\alpha^{r_{1,\delta}}b^{s_{1,\delta}} - \alpha^{s_{1,\delta}}b^{r_{1,\delta}}}\left(\frac{b^{r_{1,\delta}} x^{s_{1,\delta}}}{s_{1,\delta}} - \frac{b^{s_{1,\delta}} x^{r_{1,\delta}}}{r_{1,\delta}}\right) \tag{3.3.25}$$

3.3.5 The Laplace transforms of the discounted dividends and the discounted financing. (GBM-2RB)

By Proposition 2.6.6 the Laplace transform of the discounted dividends $K^{(+)}(x,\lambda)$ is given by the solution of:

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial}{\partial x^2}K^{(+)}(x,\lambda) + \mu x\frac{\partial}{\partial x}K^{(+)}(x,\lambda) - \lambda\delta\frac{\partial}{\partial\lambda}K^{(+)}(x,\lambda) = 0 \tag{3.3.26}$$

with boundary conditions:

$$\frac{\partial}{\partial x}K^{(+)}(x,\lambda)|_{x=a} = 0 \tag{3.3.27}$$

$$\frac{\partial}{\partial x}K^{(+)}(x,\lambda)|_{x=b} = -\lambda K^{(+)}(b,\lambda) \tag{3.3.28}$$

and the Laplace transform of the discounted financing $K^{(-)}(x,\lambda)$ is given by the solution of

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial}{\partial x^2}K^{(-)}(x,\lambda) + \mu x\frac{\partial}{\partial x}K^{(-)}(x,\lambda) - \lambda\delta\frac{\partial}{\partial\lambda}K^{(-)}(x,\lambda) = 0 \tag{3.3.29}$$

with boundary conditions

$$\frac{\partial}{\partial x}K^{(-)}(x,\lambda)|_{x=b} = 0 \tag{3.3.30}$$

$$\frac{\partial}{\partial x}K^{(-)}(x,\lambda)|_{x=a} = \lambda K^{-}(a,\lambda) \tag{3.3.31}$$

Working as in section 3.2.5 we find that the solutions of (3.3.26) with boundary conditions (3.3.27), (3.3.28) and (3.3.29) with boundary conditions (3.3.30), (3.3.31) are:

- The k–th moment $V^{(+)}(x;k)$ of the discounted dividends $U^{(+)}$ about the origin is

$$\begin{aligned} V^{(+)}(x;k) &= E^x((U^{(+)})^k) \\ &= k!\frac{g_1(b)\cdots g_{k-1}(b)g_k(x)}{\left(\frac{\partial}{\partial x}g_1(x)|_{x=b}\right)\cdots\left(\frac{\partial}{\partial x}g_{k-1}(x)|_{x=b}\right)\left(\frac{\partial}{\partial x}g_k(x)|_{x=b}\right)} \end{aligned} \tag{3.3.32}$$

  for $k = 1, 2, 3, \ldots$

- The Laplace transform $K^{(+)}(x,\lambda)$ of the discounted dividends $U^{(+)}$ is

$$K^{(+)}(x,\lambda) = 1 + \sum_{k=1}^{\infty}(-\lambda)^k \frac{g_1(b)\cdots g_{k-1}(b)g_k(x)}{\left(\frac{\partial}{\partial x}g_1(x)|_{x=b}\right)\cdots\left(\frac{\partial}{\partial x}g_{k-1}(x)|_{x=b}\right)\left(\frac{\partial}{\partial x}g_k(x)|_{x=b}\right)} \tag{3.3.33}$$

where

$$g_k(x) := x^{s_{k,\delta}} - \frac{s_{k,\delta}}{r_{k,\delta}}a^{s_{k,\delta}-r_{k,\delta}}x^{r_{k,\delta}} \tag{3.3.34}$$

- The k-th moment $V^{(-)}(x;k)$ of the discounted financing $U^{(-)}$ about the origin is

$$\begin{aligned} V^{(-)}(x;k) &= E^x((U^{(-)})^k) \\ &= (-1)^k k! \frac{m_1(a)\cdots m_{k-1}(a)m_k(x)}{\left(\frac{\partial}{\partial x}m_1(x)|_{x=a}\right)\cdots\left(\frac{\partial}{\partial x}m_{k-1}(x)|_{x=a}\right)\left(\frac{\partial}{\partial x}m_k(x)|_{x=a}\right)} \end{aligned} \tag{3.3.35}$$

for $k = 1, 2, 3, \ldots$

- The Laplace transform $K^{(-)}(x,\lambda)$ of the discounted financing $U^{(-)}$ is:

$$\begin{aligned} K^{(-)}(x,\lambda) &= E^x\left(e^{-\lambda U^{(-)}}\right) \\ &= 1 + \sum_{k=1}^{\infty}\lambda^k \frac{m_1(a)\cdots m_{k-1}(a)m_k(x)}{\left(\frac{\partial}{\partial x}m_1(x)|_{x=a}\right)\cdots\left(\frac{\partial}{\partial x}m_{k-1}(x)|_{x=a}\right)\left(\frac{\partial}{\partial x}m_k(x)|_{x=a}\right)} \end{aligned} \tag{3.3.36}$$

where

$$m_k(x) := x^{s_{k,\delta}} - \frac{s_{k,\delta}}{r_{k,\delta}}b^{s_{k,\delta}-r_{k,\delta}}x^{r_{k,\delta}} \tag{3.3.37}$$

3.3.6 The Laplace transform of the joint distribution of the discounted dividends and the discounted financing. (GBM-2RB)

By Proposition 2.6.5 the Laplace transform of the joint distribution of the discounted dividends and the discounted financing $L(x,\lambda_1,\lambda_2)$ is given by the solution of:

$$\begin{aligned} &\frac{1}{2}\sigma^2x^2\frac{\partial^2}{\partial x^2}L(x,\lambda_1,\lambda_2) + \mu x\frac{\partial}{\partial x}L(x,\lambda_1,\lambda_2) \\ = \; &\delta\left(\lambda_1\frac{\partial}{\partial\lambda_1}L(x,\lambda_1,\lambda_2) + \lambda_2\frac{\partial}{\partial\lambda_2}L(x,\lambda_1,\lambda_2)\right) \end{aligned} \tag{3.3.38}$$

with boundary conditions:

$$\frac{\partial}{\partial x}L(x,\lambda_1,\lambda_2)|_{x=a} = \lambda_2 L(a,\lambda_1,\lambda_2) \tag{3.3.39}$$

$$\frac{\partial}{\partial x}L(x,\lambda_1,\lambda_2)|_{x=b} = -\lambda_1 L(b,\lambda_1,\lambda_2) \tag{3.3.40}$$

Working as in section 3.2.6 we find that the solution is

$$L(x,\lambda_1,\lambda_2)=1+\sum_{k=1}^{\infty}\frac{E^x\left(\left(-\lambda_1U^{(+)}-\lambda_2U^{(-)}\right)^k\right)}{k!}=1+\sum_{k=1}^{\infty}\frac{W_k(x,\lambda_1,\lambda_2)}{k!} \tag{3.3.41}$$

with

$$W_k(x,\lambda_1,\lambda_2)=E^x\left(\left(-\lambda_1U^{(+)}-\lambda_2U^{(-)}\right)^k\right),\qquad k=1,2,3,\ldots \tag{3.3.42}$$

$$W_1(x,\lambda_1,\lambda_2)=\frac{1}{a^{r_{1,\delta}}b^{s_{1,\delta}}-a^{s_{1,\delta}}b^{r_{1,\delta}}}\left(\frac{a^{r_{1,\delta}}b+ab^{r_{1,\delta}}}{s_{1,\delta}}x^{s_{1,\delta}}-\frac{a^{s_{1,\delta}}b+ab^{s_{1,\delta}}}{r_{1,\delta}}x^{r_{1,\delta}}\right) \tag{3.3.43}$$

$$\begin{aligned}&W_k(x,\lambda_1,\lambda_2)\\=&\left(\frac{ab^{r_{k,\delta}}W_{k-1}(a,\lambda_1,\lambda_2)+a^{r_{k,\delta}}bW_{k-1}(b,\lambda_1,\lambda_2)}{s_{k,\delta}}x^{s_{k,\delta}}-\right.\\&\left.-\frac{ab^{s_{k,\delta}}W_{k-1}(a,\lambda_1,\lambda_2)+a^{s_{k,\delta}}bW_{k-1}(b,\lambda_1,\lambda_2)}{r_{k,\delta}}x^{r_{k,\delta}}\right)\frac{k}{a^{r_{k,\delta}}b^{s_{k,\delta}}-a^{s_{k,\delta}}b^{r_{k,\delta}}}\end{aligned} \tag{3.3.44}$$

Remark 3.3.2 *Let b^* the value of the barrier that gives the maximum of the expected dividends $V^{(+)}(\cdot)$. Then from (3.3.24) we can find that the optimal value for the barrier is*

$$b^*=a\left(\frac{s_{1,\delta}-1}{r_{1,\delta}-1}\right)^{\frac{1}{r_{1,\delta}-s_{1,\delta}}}. \tag{3.3.45}$$

Remark 3.3.3 *If we consider the difference of the expected discounted dividends minus the expected discounted financing $V^+(\cdot)-V^-(\cdot)$ then if we denote by b^{**} the barrier that maximizes the above difference, we can see that the b^{**} can be found from the solution of the equation*

$$(s_{1,\delta}-1)a^{r_{1,\delta}}b^{s_{1,\delta}+1}-(s_{1,\delta}-r_{1,\delta})ab^{s_{1,\delta}+r_{1,\delta}}-(r_{1,\delta}-1)a^{s_{1,\delta}}b^{r_{1,\delta}+1}=0. \tag{3.3.46}$$

Remark 3.3.4 *The barrier which equalizes profits and loses is given from the solution of equation*

$$b\left(\frac{a^{r_{1,\delta}}x^{s_{1,\delta}}}{s_{1,\delta}}-\frac{a^{s_{1,\delta}}x^{r_{1,\delta}}}{r_{1,\delta}}\right)=a\left(\frac{b^{r_{1,\delta}}x^{s_{1,\delta}}}{s_{1,\delta}}-\frac{b^{s_{1,\delta}}x^{r_{1,\delta}}}{r_{1,\delta}}\right). \tag{3.3.47}$$

3.4 Brownian motion.

There is no need to mention the importance of Brownian motion in Stochastic Analysis. We only want to mention here for someone who is interesting for recent developments relevant to this process the papers of Graversen [81], Peskir [150], Promislow [151], Han [86], Kalashnikov [108], Huzak [96], Lifshits [123], DeBlassie [43], Bass [19], Palle [143], Uemura [179], Salminen [162], Najnudel [134], Hubalek [95], Wesołowski [184], Kager [107], Roslki [159], Taksar [177], Paulsen [145], Bai [14], Ren [155], Cai [36], Yuen [191], Gaier [69], Norberg [140], Pergamenshchikov [149].

In this section we suppose that the reserves $X = \{X_t : t \geq 0\}$ of an insurance company follow the Brownian motion (BM) process, with dynamics that are described by

$$dX_t = \mu dt + \sigma dB_t \tag{3.4.1}$$

with $X_0 = x$. If A is the generator of X_t then

$$\mathcal{A}f(x) = \mu\frac{\partial}{\partial x}f(x) + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}f(x) \tag{3.4.2}$$

for a function $f(\cdot) \in D_A$, (where D_A is the domain of the generator A (see Definition 1.3.29)).

The insurance company follows the de Finetti model with upper barrier the constant b, that is $b_t = b$ for all $t \geq 0$ and lower barrier the zero constant, that is $a_t = 0$ for all $t \geq 0$.

We consider first the de Finetti model with one general reflecting barrier (1RB) and we find the expected value of the discounted dividends, the Laplace transform of the discounted dividends and the Laplace transform of the time of ruin. Next we consider the de Finetti model with two general reflecting barriers (2RB) and we find the expected discounted dividends and the expected discounted financing, the Laplace transforms of the discounted dividends and the discounted financing and the Laplace transform of the joint distribution of the discounted dividends and the discounted financing.

We first define the following quantities

$$m_{k,\delta} := \frac{-\mu + \sqrt{\mu^2 + 2\delta k\sigma^2}}{\sigma^2} \tag{3.4.3}$$

$$l_{k,\delta} := \frac{-\mu - \sqrt{\mu^2 + 2\delta k\sigma^2}}{\sigma^2} \tag{3.4.4}$$

for $k = 1, 2, \ldots$. We observe that

$$l_{k,\delta} \leq 0 \leq m_{k,\delta}$$

3.4.1 Expected value of the discounted dividends. (BM-1RB)

By Proposition 2.6.4 the expected value of the discounted dividends is given by the solution of

$$\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}V(x) + \mu\frac{\partial}{\partial x}V(x) = \delta V(x) \tag{3.4.5}$$

with boundary conditions

$$V(0) = 0, \tag{3.4.6}$$

$$\frac{\partial}{\partial x}V(x)|_{x=b} = 1. \tag{3.4.7}$$

The solution of (3.4.5)-(3.4.7) is

$$V(x) = \frac{1}{l_{1,\delta}e^{l_{1,\delta}b} - m_{1,\delta}e^{m_{1,\delta}b}} \left(e^{l_{1,\delta}x} - e^{m_{1,\delta}x}\right) \tag{3.4.8}$$

(also see Gerber, H.U. and Shiu, E.S.W.([71])).

3.4.2 The Laplace transform of the discounted dividends. (BM-1RB)

By Proposition 2.6.2 the Laplace transform of the discounted dividends is given by the solution of

$$\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}K(x,\lambda) + \mu\frac{\partial}{\partial x}K(x,\lambda) - \lambda\delta\frac{\partial}{\partial x}K(x,\lambda) = 0 \tag{3.4.9}$$

with boundary conditions

$$K(0,\lambda) = 1 \tag{3.4.10}$$

$$\frac{\partial}{\partial x}K(x,\lambda)|_{x=b} = -\lambda K(b,\lambda) \tag{3.4.11}$$

Working as in section 3.2.2 we find the k–th moment of U about the origin is

$$V(x;k) = E^x(U^k) = k!\frac{g_1(b)\cdots g_{k-1}(b)g_k(x)}{\left(\frac{\partial}{\partial x}g_1(x)|_{x=b}\right)\cdots\left(\frac{\partial}{\partial x}g_{k-1}(x)|_{x=b}\right)\left(\frac{\partial}{\partial x}g_k(x)|_{x=b}\right)} \tag{3.4.12}$$

$$k = 1,2,3,\dots$$

and the Laplace transform of U is

$$\begin{aligned} K(x,\lambda) &= E^x\left(e^{-\lambda U}\right) = \\ &= 1 + \sum_{k=1}^{\infty}(-\lambda)^k\frac{g_1(b)\cdots g_{k-1}(b)g_k(x)}{\left(\frac{\partial}{\partial x}g_1(x)|_{x=b}\right)\cdots\left(\frac{\partial}{\partial x}g_{k-1}(x)|_{x=b}\right)\left(\frac{\partial}{\partial x}g_k(x)|_{x=b}\right)} \end{aligned} \tag{3.4.13}$$

where

$$g_k(x) := x^{l_{k,\delta}} - x^{m_{k,\delta}}$$

for $k = 1,2,3,\dots$

3.4.3 The Laplace transform of the time of ruin. (BM-1RB)

By *Proposition* 2.6.3 the **Laplace transform of the time of ruin** is given by the solution of:

$$\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}M(x,\lambda) + \mu\frac{\partial}{\partial x}M(x,\lambda) - \lambda M(x,\lambda) = 0 \tag{3.4.14}$$

with **boundary conditions**:

$$M(0,\lambda) = 1 \tag{3.4.15}$$

$$\frac{\partial}{\partial x}M(x,\lambda)|_{x=b} = 0 \tag{3.4.16}$$

The solution of (3.4.14)-(3.4.16) is

$$M(x,\lambda) = \frac{1}{m_{1,\lambda}e^{m_{1,\lambda}b} - l_{1,\lambda}e^{l_{1,\lambda}b}}\left(m_{1,\lambda}e^{m_{1,\lambda}b}e^{l_{1,\lambda}x} - l_{1,\lambda}e^{l_{1,\lambda}b}e^{m_{1,\lambda}x}\right) \tag{3.4.17}$$

(also see Gerber, H.U. and Shiu, E.S.W.([71])).

Remark 3.4.1 *(Maximum dividends barrier). Let b^* be the value of the barrier b which gives the maximum value for the dividends $V(\cdot)$. Then the b^* can be found from the solution of equation $g''(b^*) = 0$, and from the solution of this equation we found that the optimal value for the barrier is*

$$b^* = \frac{2\log\frac{-l_{1,\delta}}{m_{1,\delta}}}{m_{1,\delta} - l_{1,\delta}} \tag{3.4.18}$$

3.4.4 Expected discounted dividends and expected discounted financing. (BM-2RB)

By Proposition 2.6.7 the expected discounted dividends and the expected discounted financing are given by the solutions of:

$$\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}V^{(\pm)}(x) - ax\frac{\partial}{\partial x}V^{(\pm)}(x) - \delta V^{(\pm)}(x) = 0 \tag{3.4.19}$$

with boundary conditions:

$$\frac{\partial}{\partial x}V^{(+)}(x)|_{x=0} = 0 \tag{3.4.20}$$

$$\frac{\partial}{\partial x}V^{(+)}(x)|_{x=b} = 1 \tag{3.4.21}$$

$$\frac{\partial}{\partial x}V^{(-)}(x)|_{x=0} = -1 \tag{3.4.22}$$

$$\frac{\partial}{\partial x}V^{(-)}(x)|_{x=b} = 0 \tag{3.4.23}$$

The solutions of (3.4.19) with boundary conditions (3.4.20)-(3.4.23) are

$$V^{(+)}(x) = \frac{1}{e^{l_{1,\delta}b} - e^{m_{1,\delta}b}}\left(\frac{e^{l_{1,\delta}x}}{l_{1,\delta}} - \frac{e^{m_{1,\delta}x}}{m_{1,\delta}}\right) \tag{3.4.24}$$

$$V^{(-)}(x) = \frac{e^{l_{1,\delta}x}}{l_{1,\delta}\left(e^{(l_{1,\delta}-m_{1,\delta})b} - 1\right)} - \frac{e^{m_{1,\delta}x}}{m_{1,\delta}(1 - e^{(m_{1,\delta}-l_{1,\delta})b})} \tag{3.4.25}$$

3.4.5 The Laplace transforms of the discounted dividends and the discounted financing. (BM-2RB)

By Proposition 2.6.6 the Laplace transform of the discounted

$$\frac{\partial}{\partial x}K^{(+)}(x,\lambda)|_{x=0} = 0 \tag{3.4.26}$$

$$\frac{\partial}{\partial x}K^{(+)}(x,\lambda)|_{x=b} = -\lambda K^{(+)}(b,\lambda) \tag{3.4.27}$$

and the Laplace transform of the discounted financing is given by the solution of:

$$\frac{1}{2}\sigma^2\frac{\partial}{\partial x^2}K^{(-)}(x,\lambda) - ax\frac{\partial}{\partial x}K^{(-)}(x,\lambda) - \lambda\delta\frac{\partial}{\partial\lambda}K^{(-)}(x,\lambda) = 0 \tag{3.4.28}$$

with boundary conditions:

$$\frac{\partial}{\partial x}K^{(-)}(x,\lambda)|_{x=b} = 0 \tag{3.4.29}$$

$$\frac{\partial}{\partial x}K^{(-)}(x,\lambda)|_{x=0} = \lambda K^{(-)}(0,\lambda) \tag{3.4.30}$$

Working as in section 3.2.5 we find the k-th moment of $U^{(+)}$ about the origin is

$$\begin{aligned} V^{(+)}(x;k) &= E^x((U^{(+)})^k) \\ &= k!\frac{g_1(b)\cdots g_{k-1}(b)g_k(x)}{\left(\frac{\partial}{\partial x}g_1(x)|_{x=b}\right)\cdots\left(\frac{\partial}{\partial x}g_{k-1}(x)|_{x=b}\right)\left(\frac{\partial}{\partial x}g_k(x)|_{x=b}\right)} \end{aligned} \tag{3.4.31}$$

for $k = 1, 2, 3, \ldots$

The Laplace transform of $U^{(+)}$ is:

$$\begin{aligned} K^{(+)}(x,\lambda) &= E^x\left((U^{(+)})^k\right) \\ &= 1 + \sum_{k=1}^{\infty}(-\lambda)^k\frac{g_1(b)\cdots g_{k-1}(b)g_k(x)}{\left(\frac{\partial}{\partial x}g_1(x)|_{x=b}\right)\cdots\left(\frac{\partial}{\partial x}g_{k-1}(x)|_{x=b}\right)\left(\frac{\partial}{\partial x}g_k(x)|_{x=b}\right)} \end{aligned} \tag{3.4.32}$$

where

$$g_k(x) := e^{m_{k,\delta}x} - \frac{m_{k,\delta}}{l_{k,\delta}}e^{l_{k,\delta}x} \tag{3.4.33}$$

The kth moment of $U^{(-)}$ about the origin is

$$\begin{aligned} V^{(-)}(x;k) &= E^x((U^{(-)})^k) \\ &= (-1)^k k!\frac{m_1(0)\cdots m_{k-1}(0)m_k(x)}{\left(\frac{\partial}{\partial x}m_1(x)|_{x=0}\right)\cdots\left(\frac{\partial}{\partial x}m_{k-1}(x)|_{x=0}\right)\left(\frac{\partial}{\partial x}m_k(x)|_{x=0}\right)} \end{aligned} \tag{3.4.34}$$

for $k = 1, 2, 3, \ldots$

The Laplace transform of $U^{(-)}$ is:

$$\begin{aligned} K^{(-)}(x,\lambda) &= E^x((U^{(-)})^k) = \\ &= 1+\sum_{k=1}^{\infty}\lambda^k \frac{m_1(0)\cdots m_{k-1}(0)m_k(x)}{\left(\frac{\partial}{\partial x}m_1(x)|_{x=0}\right)\cdots\left(\frac{\partial}{\partial x}m_{k-1}(x)|_{x=0}\right)\left(\frac{\partial}{\partial x}m_k(x)|_{x=0}\right)} \end{aligned} \tag{3.4.35}$$

where

$$m_k(x) := e^{m_{k,\delta}x} - \frac{m_{k,\delta}}{l_{k,\delta}}e^{(m_{k,\delta}-l_{k,\delta})b}e^{l_{k,\delta}x} \tag{3.4.36}$$

3.4.6 The Laplace transform of the joint distribution of the discounted dividends and the discounted financing. (BM-2RB)

By Proposition 2.6.5 the Laplace transform of the joint distribution of the discounted dividends and the discounted financing is given by the solution of:

$$\begin{aligned} &\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}N(x,\lambda_1,\lambda_2) - ax\frac{\partial}{\partial x}N(x,\lambda_1,\lambda_2) \\ = \ &\delta\left(\lambda_1\frac{\partial}{\partial\lambda_1}N(x,\lambda_1,\lambda_2)+\lambda_2\frac{\partial}{\partial\lambda_2}N(x,\lambda_1,\lambda_2)\right) \end{aligned} \tag{3.4.37}$$

with boundary conditions

$$\frac{\partial}{\partial x}N(x,\lambda_1,\lambda_2)|_{x=0} = \lambda_2 N(0,\lambda_1,\lambda_2) \tag{3.4.38}$$

$$\frac{\partial}{\partial x}N(x,\lambda_1,\lambda_2)|_{x=b} = -\lambda_1 N(b,\lambda_1,\lambda_2) \tag{3.4.39}$$

With

$$W_k(x,\lambda_1,\lambda_2) = E^x(-\lambda_1 U^{(+)} - \lambda_2 U^{(-)})^k \tag{3.4.40}$$

$$k = 1,2,3\ldots.$$

Working as in section 3.2.6 we find

$$N(x,\lambda_1,\lambda_2) = 1+\sum_{k=1}^{\infty}\frac{E^x\left((-\lambda_1 U^{(+)}-\lambda_2 U^{(-)})^k\right)}{k!} = 1+\sum_{k=1}^{\infty}\frac{W_k(x,\lambda_1,\lambda_2)}{k!} \tag{3.4.41}$$

with

$$W_1(x,\lambda_1,\lambda_2) = \frac{1}{e^{bl_{1,\delta}}-e^{bm_{1,\delta}}}\left(\frac{1+e^{bm_{1,\delta}}}{l_{1,\delta}}e^{l_{1,\delta}x} - \frac{1+e^{bl_{1,\delta}}}{m_{1,\delta}}e^{m_{1,\delta}x}\right) \tag{3.4.42}$$

$$W_k(x,\lambda_1,\lambda_2) = \frac{k\left(e^{bm_{k,\delta}}W_{k-1}(0,\lambda_1,\lambda_2)+W_{k-1}(b,\lambda_1,\lambda_2)\right)}{e^{bl_{k,\delta}}-e^{bm_{k,\delta}}}\left(\frac{e^{l_{k,\delta}x}}{l_{k,\delta}} - \frac{e^{m_{k,\delta}x}}{m_{k,\delta}}\right) \tag{3.4.43}$$

Remark 3.4.2 *Let b^* the value of the barrier that gives the maximum of the expected dividends $V^{(+)}(\cdot)$. Then from (3.4.24) we find that*

$$b^* = \frac{\log\frac{-l_{1,\delta}}{m_{1,\delta}}}{m_{1,\delta}-l_{1,\delta}} \tag{3.4.44}$$

Remark 3.4.3 *If we consider the difference* $V^{+}(\cdot) - V^{-}(\cdot)$ *that is profits minus loses then if* b^{**} *is the barrier that maximizes the above difference, it can be found from the solution of equation*

$$m_{1,\delta} e^{b^{**} m_{1,\delta}} - l_{1,\delta} e^{b^{**} l_{1,\delta}} + (l_{1,\delta} - m_{1,\delta}) e^{b^{**}(m_{1,\delta} + l_{1,\delta})} = 0 \tag{3.4.45}$$

Remark 3.4.4 *The barrier which equalizes profits and loses is given from the solution of equation*

$$m_{1,\delta} e^{l_{1,\delta} x} - m_{1,\delta} e^{b m_{1,\delta} + l_{1,\delta} x} - l_{1,\delta} e^{m_{1,\delta} x} + l_{1,\delta} e^{b l_{1,\delta} + m_{1,\delta} x} = 0 \tag{3.4.46}$$

3.5 Conclusions.

We applied the formulas of chapter 2 in examples where the reserves process follows a Brownian motion, a Geometric Brownian morion and an Orstein-Uhlenbeck process.

We considered the de Finetti model with one general reflecting barrier (1RB) and we found the expected value of discounted dividends, the Laplace transform of the discounted dividends and the Laplace transform of time of ruin. We also considered the de Finetti model with two general reflecting barriers (2RB) and we found the expected discounted dividends and the expected discounted financing, the Laplace transforms of the discounted dividends and the discounted financing and the Laplace transform of the joint distribution of the discounted dividends and the discounted financing.

Chapter 4

Aspects on Insurance companies consortium.

4.1 Introduction.

In this chapter we deal with the situation of insurance companies cooperation. There are a lot of reasons as to way insurance companies want to cooperate with each other. One basic reason is to become more competitive to the general market. One way to achieve this is by sharing administrative expenses and common features and thus reducing in this way their running costs. Another way to become more competitive is by offering a lower premium via the tool of reinsurance. Relevant towards this direction are the papers of Aase [1] , Borch [27], [28] , Zafiropoulos, Y.D. and M.A. Zazanis [192].

Our interest is to look at the issue from the perspective of a particular insurance company. Obviously, a number of issues and questions arise and must be resolved by the insurance company in this kind of situation. What criteria must be used by the insurance company in order to collaborate with other insurance companies, choosing a set of companies among the total number of available insurance companies? Certainly the criteria should maximize it's prospects. Among many other questions, there are two critical questions that must be answered. The first concerns the probability to survive in a particular cooperation, that is not to go bankrupt and the second concerns the future prospects of the capital of the company.

As the insurance company will become part of a dynamic and renewable cooperation how this dynamic situation will influence its reserves as other companies depart from the cooperation because bankruptcy and new companies entering the insurance shape?

In this chapter we are interested in looking at two quantities which are very vital to all the decisions of the company. These are the probability of survival in a particular cooperation and with the shares that will be given to its shareholders during this cooperation.

In order to formulate our model we assume a complete probability space (Ω, F, P) as given. In addition we assume a given filtration $\{F_t : t \geq 0\}$. Let $\{(X_t, Y_t); t \geq 0\}$ be a two dimensional diffusion with SDE given by:

$$\begin{aligned} dX_t &= \mu_x(X_t)dt + \sigma_x(X_t)dB_t^x \\ dY_t &= \mu_y(Y_t)dt + \sigma_y(Y_t)dB_t^y \end{aligned}$$

where the drift coefficients $\mu_x(\cdot), \mu_y(\cdot)$ and the volatility coefficients $\sigma_x(\cdot), \sigma_y(\cdot)$ satisfy the conditions (1.3.24), (1.3.25).

Each of the two components of the process $\{(B_t^x, B_t^y); t \geq 0\}$ is standard Brownian motion and the correlation between them is given by:

$$\rho_{xy}dt := d\left[B^x, B^y\right]_t$$

We consider the process $X = \{X_t; t \geq 0\}$ as describing the reserves of the insurance company in which we are interesting for. In a two dimensional representation we will consider it as the horizontal movement and we will refer to this insurance company as the first insurer or as the X-insurer. We want to include in our study a practice which is usual in the insurance which is to be given dividends to the shareholders according to a strategy. In this chapter we consider a constant barrier strategy, that is when the process X is above from level b then dividends are paid to the shareholders. When the process reaches level 0 we will consider two cases:

(I) In the first case the company goes bankrupt and we will call this case as de Finetti model with one reflecting barrier. We denote by $\{\mathcal{U}_t; t \geq 0\}$ the dividends that are paid until ruin occurs.

(II) In the second case the company has the option to borrow money and continue it's function and we will call this case as de Finetti model with two reflecting barriers. We denote by $\{\mathcal{U}_t^{(-)}; t \geq 0\}$ the financing of the company and $\{\mathcal{U}_t^{(+)}; t \geq 0\}$ the dividends that are paid.

We consider a process Z given by

$$\begin{aligned} Z &= \{Z_t; t \geq 0\} \\ &:= \begin{cases} \{X_t - \mathcal{U}_t; t \geq 0\} & \text{for the de Finetti with one reflecting barrier} \\ \{X_t - \mathcal{U}_t^{(+)} + \mathcal{U}_t^{(-)}; t \geq 0\} & \text{for the de Finetti with two reflecting barriers} \end{cases} \end{aligned}$$

which represents the finances of the first insurer afterwards the abstraction of the dividends and the possible addition of the financing.

For the de Finetti models it is well known that the dividends and financing processes $\{\mathcal{U}_t; t \geq 0\}$, $\{\mathcal{U}_t^{(+)}; t \geq 0\}, \{\mathcal{U}_t^{(-)}; t \geq 0\}$ respectively, are unique processes and also that have the following properties

(i) The processes $\{\mathcal{U}_t; t \geq 0\}, \{\mathcal{U}_t^{(+)}; t \geq 0\}, \{\mathcal{U}_t^{(-)}; t \geq 0\}$ are nondecreasing.

(ii) It holds that

$$0 \leq Z_t = X_t - \mathcal{U}_t^{(+)} + \mathcal{U}_t^{(-)} \leq b$$

for the two reflecting barriers case and

$$Z_t = X_t - \mathcal{U}_t \leq b$$

for the one reflecting barrier case, for every time $t \geq 0$.

(iii) The processes $\{\mathcal{U}_t^{(+)}; t \geq 0\}$ and $\{\mathcal{U}_t; t \geq 0\}$ increase only when $Z_t = b$, i.e.

$$\int_0^t 1(Z_s < b) d\mathcal{U}_s^{(+)} = 0$$

and

$$\int_0^t 1(Z_s < b) d\mathcal{U}_s = 0$$

for all time $t \geq 0$.

(iv) The process $\{\mathcal{U}_t^{(-)}; t \geq 0\}$ increases only when $Z_t = 0$, i.e.

$$\int_0^t 1(Z_s > 0) d\mathcal{U}_s^{(-)} = 0$$

for all $t \geq 0$.

It is also well known that the dividends processes $\mathcal{U}$, $\mathcal{U}^{(+)}$ are given by:

$$\begin{aligned} \mathcal{U}_t &= \sup_{0 \leq s \leq t} (X_s - b) \vee 0 \\ \mathcal{U}_t^{(+)} &= \sup_{0 \leq s \leq t} \left(X_s - b + \mathcal{U}_s^{(-)} \right) \vee 0 \end{aligned}$$

and the financing process $\mathcal{U}^{(-)}$ by:

$$\mathcal{U}_t^{(-)} = \sup_{0\leq s\leq t}\left(\mathcal{U}_s^{(+)} - X_s\right) \vee 0$$

for time $t \geq 0$.

We would also like to consider another process $\{Y_t; t \geq 0\}$ as describing the reserves of another insurance company. In a two dimensional representation we will consider it as the vertical movement and we will refer to this insurance company as the second insurer or as the Y-insurer. We don't include dividend's policy for this insurance company. In other words this portfolio is described by the so called classical risk model with an absorbing barrier at 0 (we also call this model as Lundberg model).

The process $\{(Z_t, Y_t)\ ; t \geq 0\}$ is considering as the "total" reserve process of the group or else as the two members consortium of insurance companies. We are interested in the study of the cooperation until the time of ruin of some of the two companies. In order to study the problem we define two stopping times. The time of ruin for the first insurer (this has meaning only for the de Finetti with one reflecting barrier case), which depends on the initial state x of the process Z and is defined by

$$T^z := T^z(x) := \inf\{t > 0 : Z_t = 0\}$$

and the time of ruin for the second insurer, which depends on the initial state y of the process Y and is defined by

$$T^y := T^y(y) := \inf\{t > 0 : Y_t = 0\}$$

We assume that the insurance companies cooperation fails at the random time

$$T := T(x,y) := \begin{cases} T^z \wedge T^y & \text{for the de Finetti with one reflecting barrier} \\ T^y & \text{for the de Finetti with two reflecting barriers} \end{cases}$$

Assuming an interest-rate δ, the total discounted dividends and total discounted financing until some ruin occurs are given by

$$U \;:=\; U_T := U(x,y) := \int_0^T e^{-\delta s} d\mathcal{U}_s \tag{4.1.1}$$

$$U^{(+)} \;:=\; U_T^{(+)} := U^{(+)}(x,y) := \int_0^T e^{-\delta s} d\mathcal{U}_s^{(+)} \tag{4.1.2}$$

$$U^{(-)} \;:=\; U_T^{(-)} := U^{(-)}(x,y) := \int_0^T e^{-\delta s} d\mathcal{U}_s^{(-)} \tag{4.1.3}$$

Here the following notation remarks are in order.

Remark 4.1.1 *(de Finetti with one reflecting barrier). Consider a function $f(U,T)$ of the discounted dividends $U = U(x,y)$ and the time of ruin $T = T(x,y)$. The expected value $E(f(U,T))$ will depend on the initial state (x,y). In order to express this dependence we will use the notation $E^{(x,y)}$, that is we define*

$$E^{(x,y)}(f(U,T)) := E(f(U(x,y),T(x,y))) \tag{4.1.4}$$

Remark 4.1.2 *(de Finetti with two reflecting barriers). Consider a function $g(U^{(+)},U^{(-)},T)$ of the discounted dividends $U^{(+)} = U^{(+)}(x,y)$, the discounted financing $U^{(-)} = U^{(-)}(x,y)$ and the time of ruin $T = T(x,y)$. The expected value $E\left(g(U^{(+)},U^{(-)},T)\right)$ will depend on the initial state (x,y). In order to express this dependence we will use the notation $E^{(x,y)}$, that is we define*

$$E^{(x,y)}\left(g(U^{(+)},U^{(-)},T)\right) := E\left(g(U^{(+)}(x,y),U^{(-)}(x,y),T(x,y))\right) \tag{4.1.5}$$

Next we define the quantities which are the subject of our study, which are

- ▲ The Moments of the discounted dividends and the discounted financing. (Two reflecting barriers case).
$$\mathcal{V}^{(\pm)}(x,y;n) := E^{(x,y)}\left((U^{(\pm)})^n\right) \tag{4.1.6}$$

- • The Moments of the discounted dividends. (One reflecting barrier case).
$$\mathcal{V}(x,y;n) := E^{(x,y)}(U^n) \tag{4.1.7}$$

- ▲ The Laplace transforms of the discounted dividends and the discounted financing. (Two reflecting barriers case).
$$\mathcal{K}^{(\pm)}(x,y,\lambda) := E^{(x,y)}\left(e^{-\lambda U^{(\pm)}}\right) \tag{4.1.8}$$

- • The Laplace transforms of the discounted dividends. (One reflecting barrier case).
$$\mathcal{K}(x,y,\lambda) := E^{(x,y)}\left(e^{-\lambda U}\right) \tag{4.1.9}$$

- ▲ The Laplace transform of the time of ruin .
$$\mathcal{M}(x,y,\lambda) := E^{(x,y)}(e^{-\lambda T}) \tag{4.1.10}$$

- ▲ The Laplace transform of the joint distribution of the time of ruin, the discounted dividends and the discounted financing. (Two reflecting barriers case).
$$\mathcal{N}(x,y,\lambda_1,\lambda_2,\lambda_3) := E^{(x,y)}\left(e^{-\lambda_1 T-\lambda_2 U^{(+)}-\lambda_3 U^{(-)}}\right) \tag{4.1.11}$$

- The Laplace transform of the joint distribution of the time of ruin and the discounted dividends. (One reflecting barrier case).

$$\mathcal{N}(x,y,\lambda_1,\lambda_2) := E^{(x,y)}\left(e^{-\lambda_1 T-\lambda_2 U}\right) \tag{4.1.12}$$

For simplicity we write $V^{(\pm)}(x,y)$ for $V^{(\pm)}(x,y;1)$ and $V(x,y)$ for $V(x,y;1)$.

In order to study the quantities (4.1.6)-(4.1.12) we define the processes

$$\begin{array}{ll} \{V^{(\pm)}(X_t,Y_t);t\geq 0\}, & \{K^{(\pm)}(X_t,Y_t,\lambda);t\geq 0\}, \\ \{M(X_t,Y_t,\lambda);t\geq 0\} & \{N(X_t,Y_t,\lambda_1,\lambda_2,\lambda_3);t\geq 0\} \\ \{V(X_t,Y_t);t\geq 0\}, & \{K(X_t,Y_t,\lambda);t\geq 0\}, \\ \{N(X_t,Y_t,\lambda_1,\lambda_2);t\geq 0\} & \end{array} \tag{4.1.13}$$

We will denote by $A_{(x,y)}$ the generator of the process $\{(X_t,Y_t);t\geq 0\}$. It is well known that the generator coincides with the differential operator $L_{(x,y)}$

$$\mathcal{L}_{(x,y)} := \frac{\sigma_x^2(x)}{2}\frac{\partial^2}{\partial x^2}+\frac{\sigma_y^2(y)}{2}\frac{\partial^2}{\partial y^2}+\sigma_x(x)\sigma_y(y)\rho_{xy}\frac{\partial^2}{\partial x\partial y}+\mu_x(x)\frac{\partial}{\partial x}+\mu_y(y)\frac{\partial}{\partial y} \tag{4.1.14}$$

for a function $f(\cdot)\in C_b^2(R\times R)$ and for drift coefficients $\mu_x(\cdot),\mu_y(\cdot)$ and volatility coefficients $\sigma_x(\cdot),\sigma_y(\cdot)$ that satisfy the linear growth condition (1.3.24) and the Lipschitz continuity condition (1.3.25).

Remark 4.1.3 *When we want to consider the X-insurer alone, that is the case when there is no cooperation between the X-insurer and the Y-insurer (1- dimensional case), we write the functions in (4.1.6)-(4.1.12) as*

$$\mathcal{V}^{(\pm)}(x;n) := E^x((U^{(\pm)})^n) \tag{4.1.15}$$

$$\mathcal{V}(x;n) := E^x(U^n) \tag{4.1.16}$$

$$\mathcal{K}^{(\pm)}(x,\lambda) := E^x\left(e^{-\lambda U^{(\pm)}}\right) \tag{4.1.17}$$

$$\mathcal{K}(x,\lambda) := E^x(e^{-\lambda U}) \tag{4.1.18}$$

$$\mathcal{M}(x,\lambda) := E^x(e^{-\lambda T}) \tag{4.1.19}$$

$$\mathcal{N}(x,\lambda_1,\lambda_2,\lambda_3) := E^x\left(e^{-\lambda_1 T-\lambda_2 U^{(+)}-\lambda_3 U^{(-)}}\right) \tag{4.1.20}$$

$$\mathcal{N}(x,\lambda_1,\lambda_2) := E^x\left(e^{-\lambda_1 T-\lambda_2 U}\right) \tag{4.1.21}$$

In the following sections let the function $\theta_t:\Omega\longrightarrow\Omega$ denote the right shift operator defined by $\theta_t(\omega):=\omega(t+.)$ for all times $t\geq 0$.

After the introduction, in which we set the background, we proceed with the next section in which we will find expressions for the generator $\mathcal{A}_{(x,y)}$ of the process $\{(X_t,Y_t);t\geq 0\}$.

4.2 Expressions for the generator.

In this section we prove two useful propositions. The first proposition concerns the action of the generator operator to a regular function of the discounted dividends and the time of ruin for the de Finetti model with one reflecting barrier. The second proposition concerns the action of the generator operator to a regular function of the discounted dividends and the discounting financing for the de Finetti model with two reflecting barriers.

In order to prove these propositions we will need the discounted dividends $U(t), U^{(+)}(t)$ after time t and the discounted financing $U^{(-)}(t)$ after time t, that is:

$$U(t) := \int_t^T e^{-\delta s} d\mathcal{U}_s \tag{4.2.1}$$

$$U^{(+)}(t) := \int_t^\infty e^{-\delta s} d\mathcal{U}_s^{(+)} \tag{4.2.2}$$

$$U^{(-)}(t) := \int_t^\infty e^{-\delta s} d\mathcal{U}_s^{(-)} \tag{4.2.3}$$

With similar arguments as in the proof of the Remark 2.5.7 also is proving the next remark.

Remark 4.2.1 *1. It holds that*

$$\theta_t U^n = \begin{cases} e^{n\delta t} U^n(t) & \text{for } t \leq T \\ 0 & \text{for } t > T \end{cases} \tag{4.2.4}$$

with $n = 1, 2, 3, \ldots$

Also observe that:

$$\theta_t T = \begin{cases} T - t & \text{for } t \leq T \\ 0 & \text{for } t > T \end{cases} \tag{4.2.5}$$

2. It holds that:

$$\theta_t (U^{(\pm)})^n = \begin{cases} e^{n\delta t} \left(U^{(\pm)}(t)\right)^n & \text{for } t \leq T \\ 0 & \text{for } t > T \end{cases} \tag{4.2.6}$$

with $n = 1, 2, 3, \ldots$

Proposition 4.2.2 *Let* $f(x, y) := E^{(x,y)}(g(U, T))$ *with* $g \in C_b^1(R^2)$*. Then it holds that*

$$\mathcal{A}_{(x,y)} f(x, y) = E^{(x,y)} \left(\frac{\partial}{\partial t} g(e^{\delta t} U(t), T - t)|_{t=0} \right) \tag{4.2.7}$$

Proof. We have

$$\begin{aligned}
\mathcal{A}_{(x,y)} f(x,y) &= \lim_{t\to 0} \frac{E^{(x,y)}(f(X_t,Y_t)) - f(x,y)}{t} \\
&= \lim_{t\to 0} \frac{E^{(x,y)}(E^{(X_t,Y_y)}(g(U,T))) - E^{(x,y)}(g(U,T))}{t} \\
&= \lim_{t\to 0} \frac{E^{(x,y)}(E^{(x,y)}(\theta_t g(U,T)|\mathcal{F}_t)) - E^{(x,y)}(g(U,T))}{t} \\
&= \lim_{t\to 0} \frac{E^{(x,y)}(E^{(x,y)}(g(\theta_t U,\theta_t T)|\mathcal{F}_t)) - E^{(x,y)}(g(U,T))}{t} \\
&= \lim_{t\to 0} \frac{E^{(x,y)}(g(\theta_t U,\theta_t T)) - E^{(x,y)}(g(U,T))}{t} \\
&= \lim_{t\to 0} E^{(x,y)}\left(1_{\{T\geq t\}} \frac{g(e^{\delta t}U(t),T-t) - g(U,T)}{t}\right) + \\
&\quad + \lim_{t\to 0} E^{(x,y)}\left(1_{\{T<t\}} \frac{g(0,0) - g(U,T)}{t}\right) \\
&= E^{(x,y)}\left(1_{\{T\geq 0\}} \lim_{t\to 0} \frac{g(e^{\delta t}U(t),T-t) - g(U,T)}{t}\right) + \\
&\quad + E^{(x,y)}\left(1_{\{T\leq 0\}} \lim_{t\to 0} \frac{g(0,0) - g(U,T)}{t}\right)
\end{aligned}$$

where we have explicitly used the Markov property of the diffusion process (X,Y) (see Theorem 1.3.28).

Because the process (X,Y) is continuous we have that

$$T > 0 \quad a.s.$$

(that is the process X does not jump to 0 to be ruined and neither the process Y) and so the above relation becomes:

$$\mathcal{A}_{(x,y)} f(x,y) = E^{(x,y)}\left(\lim_{t\to 0} \frac{g(e^{\delta t}U(t),T-t) - g(U,T)}{t}\right) \Longrightarrow$$

$$\mathcal{A}_{(x,y)} f(x,y) = E^{(x,y)}\left(\frac{\partial}{\partial t} g(e^{\delta t}U(t),T-t)|_{t=0}\right)$$

■

We now proceed to prove the second proposition.

Proposition 4.2.3 *Let $f(x,y) := E^{(x,y)}(h(U^{(+)},U^{(-)}))$ with $h \in C_b^1(R^2)$. Then it holds that:*

$$\mathcal{A}_{(x,y)} f(x,y) = E^{(x,y)}\left(\frac{\partial}{\partial t} h(e^{\delta t}U^{(+)}(t), e^{\delta t}U^{(-)}(t))|_{t=0}\right) \tag{4.2.8}$$

Proof. We have

$$
\begin{aligned}
& \mathcal{A}_{(x,y)} f(x,y) \\
= \quad & \lim_{t\to 0} \frac{E^{(x,y)}(f(X_t,Y_t)) - f(x,y)}{t} \\
= \quad & \lim_{t\to 0} \frac{E^{(x,y)}(E^{(X_t,Y_t)}(h(U^{(+)},U^{(-)}))) - E^{(x,y)}(h(U^{(+)},U^{(-)}))}{t} \\
= \quad & \lim_{t\to 0} \frac{E^{(x,y)}(E^{(x,y)}(\theta_t h(U^{(+)},U^{(-)})|\mathcal{F}_t)) - E^{(x,y)}(h(U^{(+)},U^{(-)}))}{t} \\
= \quad & \lim_{t\to 0} \frac{E^{(x,y)}(E^{(x,y)}(h(\theta_t U^{(+)},\theta_t U^{(+)})|\mathcal{F}_t)) - E^{(x,y)}(h(U^{(+)},U^{(-)}))}{t} \\
= \quad & \lim_{t\to 0} \frac{E^{(x,y)}(h(\theta_t U^{(+)},\theta_t U^{(+)})) - E^{(x,y)}(h(U^{(+)},U^{(-)}))}{t} \\
= \quad & \lim_{t\to 0} E^{(x,y)}\left(1_{\{T\geq t\}}\frac{h(e^{\delta t}U^{(+)},e^{\delta t}U^{(-)}) - h(U^{(+)},U^{(-)})}{t}\right) + \\
& + \lim_{t\to 0} E^{(x,y)}\left(1_{\{T<t\}}\frac{h(0,0) - h(U^{(+)},U^{(-)})}{t}\right) \\
= \quad & E^{(x,y)}\left(1_{\{T\geq 0\}}\lim_{t\to 0}\frac{h(e^{\delta t}U^{(+)},e^{\delta t}U^{(-)}) - h(U^{(+)},U^{(-)})}{t}\right) + \\
& + E^{(x,y)}\left(1_{\{T\leq 0\}}\lim_{t\to 0}\frac{h(0,0) - h(U^{(+)},U^{(-)})}{t}\right)
\end{aligned}
$$

Because the process Y is continuous we have that

$$T > 0 \quad a.s.$$

(that is the process Y does not jump to 0 to be ruined) and so the above relation becomes:

$$\mathcal{A}_{(x,y)} f(x,y) = E^{(x,y)}\left(\lim_{t\to 0}\frac{h(e^{\delta t}U^{(+)},e^{\delta t}U^{(-)}) - h(U^{(+)},U^{(-)})}{t}\right) \Longrightarrow$$

$$\mathcal{A}_{(x,y)} f(x,y) = E^{(x,y)}\left(\frac{\partial}{\partial t} h(e^{\delta t}U^{(+)}(t),e^{\delta t}U^{(-)}(t))|_{t=0}\right)$$

■

If we suppose that the quantities (4.1.6)-(4.1.12) for which we are interested in can be found as a solution of some partial differential equations (PDE) then the above two propositions might give us a suggestion about these PDEs.

In the next section we derive a very useful property, which we call "scaling property", and which will help us to proceed with later calculations.

4.3 Scaling property.

As in the section 2.3 we start this section by setting a question. Let us suppose that we have the reserves process of an insurance company, X-insurer, which moves between two boundaries and dividends are paid to the shareholders according to the de Finetti model and we also have another insurance company, Y-insurer , which follows no dividend's policy or in other words follows the so called classical risk model. How the discounted dividends will be affected if we move up or down, by the same amount, the reserve process and the two boundaries of the X-insurer and the reserve process of the Y-insurer? How the discounted dividends will be affected if we consider some multiple by the same amount of the above processes and boundaries?

Working as in the section 2.3 we conclude that the model with the two insurers corporation satisfy analogous scaling properties as in chapter 2. The following propositions can be proved along the same lines as proposition 2.3.3 and proposition 2.3.9.

Proposition 4.3.1 *(Scaling property for the De Finetti model with one reflecting barrier). For the moments of the discounted dividends $V(x, y; n)$, the Laplace transform of the discounted dividends $K(x, y, \lambda)$, the Laplace transform of the time of ruin $M(x, y, \lambda)$ and the Laplace transform of the joint distribution of the time of ruin and the discounted dividends $N(x, y, \lambda_1, \lambda_2)$ it holds that:*

(I) For each real number $c \in (-\infty, \infty)$

$$\mathcal{V}(x, y; n) = \mathcal{V}(x - c, y - c; n) \tag{4.3.1}$$

$$\mathcal{K}(x, y, \lambda) = \mathcal{K}(x - c, y - c, \lambda) \tag{4.3.2}$$

$$\mathcal{M}(x, y, \lambda) = \mathcal{M}(x - c, y - c, \lambda) \tag{4.3.3}$$

$$\mathcal{N}(x, y, \lambda_1, \lambda_2) = \mathcal{N}(x - c, y - c, \lambda_1, \lambda_2) \tag{4.3.4}$$

(II) For each real number $c > 0$

$$\mathcal{V}(x, y; n) = c^n \mathcal{V}(xc^{-1}, yc^{-1}; n) \tag{4.3.5}$$

$$\mathcal{K}(x, y, \lambda) = \mathcal{K}(xc^{-1}, yc^{-1}, \lambda c) \tag{4.3.6}$$

$$\mathcal{M}(x, y, \lambda) = \mathcal{M}(xc^{-1}, yc^{-1}, \lambda) \tag{4.3.7}$$

$$\mathcal{N}(x, y, \lambda_1, \lambda_2) = \mathcal{N}(xc^{-1}, yc^{-1}, \lambda_1, \lambda_2 c) \tag{4.3.8}$$

Proposition 4.3.2 *(Scaling property for the De Finetti model with two reflecting barriers).*

For the moments of the discounted dividends $V^{(+)}(x, y; n)$, the moments of the discounted financing $V^{(-)}(x, y; n)$, the Laplace transform of the discounted dividends $K^{(+)}(x, y, \lambda)$, the

Laplace transform of the discounted financing $K^{(-)}(x,y,\lambda)$, the Laplace transform of the time of ruin $M(x,y,\lambda)$ and the Laplace transform of the joint distribution of the time of ruin, the discounted dividends and the discounted financing $N(x,a,b,\lambda_1,\lambda_2)$ holds that:

(I) For every real number $c \in (-\infty,\infty)$

$$\mathcal{V}^{(\pm)}(x,y;n) = \mathcal{V}^{(\pm)}(x-c,y-c;n) \tag{4.3.9}$$
$$\mathcal{K}^{(\pm)}(x,y,\lambda) = \mathcal{K}^{(\pm)}(x-c,y-c,\lambda) \tag{4.3.10}$$
$$\mathcal{M}(x,y,\lambda) = \mathcal{M}(x-c,y-c,\lambda) \tag{4.3.11}$$
$$\mathcal{N}(x,y,\lambda_1,\lambda_2,\lambda_3) = \mathcal{N}(x-c,y-c,b-c,\lambda_1,\lambda_2,\lambda_3) \tag{4.3.12}$$

(II) For every real number $c > 0$

$$\mathcal{V}^{(\pm)}(x,y;n) = c^n\mathcal{V}^{(\pm)}(xc^{-1},yc^{-1};n) \tag{4.3.13}$$
$$\mathcal{K}^{(\pm)}(x,y,\lambda) = \mathcal{K}^{(\pm)}(xc^{-1},yc^{-1},\lambda c) \tag{4.3.14}$$
$$\mathcal{M}(x,y,\lambda) = \mathcal{M}(xc^{-1},yc^{-1},\lambda) \tag{4.3.15}$$
$$\mathcal{N}(x,y,\lambda_1,\lambda_2,\lambda_3) = \mathcal{N}(xc^{-1},yc^{-1},\lambda_1,c\lambda_2,c\lambda_3) \tag{4.3.16}$$

The following remarks will be useful.

Remark 4.3.3 *For a function $N(x,y,\lambda_1,\lambda_2)$ which is $C^1(\mathbb{R}^4)$ and satisfy the scaling property (4.3.8) holds that*

$$\mathcal{N}(x,y,\lambda_1,\lambda_2c^{-1}) = \mathcal{N}(xc^{-1},yc^{-1},\lambda_1,\lambda_2) \tag{4.3.17}$$

and by differentiating with respect of c we conclude that it holds

$$\lambda_2\frac{\partial}{\partial(\lambda_2c^{-1})}\mathcal{N}(x,y,\lambda_1,\lambda_2c^{-1}) = \left(x\frac{\partial}{\partial(xc^{-1})} + y\frac{\partial}{\partial(yc^{-1})}\right)\mathcal{N}(xc^{-1},yc^{-1},\lambda_1,\lambda_2) \tag{4.3.18}$$

Remark 4.3.4 *For a function $N(x,y,\lambda_1,\lambda_2,\lambda_3)$ which is $C^1(\mathbb{R}^5)$ and satisfy the scaling property (4.3.16) holds that*

$$\mathcal{N}(x,y,\lambda_1,\lambda_2c^{-1},\lambda_3c^{-1}) = \mathcal{N}(xc^{-1},yc^{-1},\lambda_1,\lambda_2,\lambda_3) \tag{4.3.19}$$

and by differentiating with respect of c we conclude that it holds

$$\begin{aligned}&\left(\lambda_2\frac{\partial}{\partial(\lambda_2c^{-1})} + \lambda_3\frac{\partial}{\partial(\lambda_3c^{-1})}\right)\mathcal{N}(x,y,\lambda_1,\lambda_2c^{-1},\lambda_3c^{-1})\\ = &\left(x\frac{\partial}{\partial(xc^{-1})} + y\frac{\partial}{\partial(yc^{-1})}\right)\mathcal{N}(xc^{-1},yc^{-1},\lambda_1,\lambda_2,\lambda_3)\end{aligned} \tag{4.3.20}$$

We proceed now with the next section in which we derive the moments of the discounted dividends and the discounted financing as solutions of some partial differential equations.

4.4 Moments of the discounted dividends and the discounted financing.

We start this section with the following proposition which concerns with the moments of the discounted dividends and discounted financing in the case in which the X-insurer follows the de Finetti model with one reflecting barrier.

Proposition 4.4.1 *(Moments of the discounted dividends). Let functions $V(x, y; n)$, $n \in N$ belonging to $C_b^2(\mathbb{R}^2)$ which satisfy the scaling properties (4.3.1), (4.3.5). If the functions $V(x, y; n)$ solve the PDEs :*

$$(\mathcal{L}_{(x,y)} - n\delta)\mathcal{V}(x, y; n) = 0 \tag{4.4.1}$$

with boundary conditions:

$$\begin{aligned} \mathcal{V}(0, y; n) &= 0 && (4.4.2) \\ \frac{\partial}{\partial x}\mathcal{V}(x, y; 1)|_{x=b} &= 1 && (4.4.3) \\ \frac{\partial}{\partial x}\mathcal{V}(x, y; n)|_{x=b} &= n\mathcal{V}(b, y; n-1) \quad \textit{for} \quad n = 1, 2, 3, \ldots && (4.4.4) \\ \mathcal{V}(x, 0; n) &= 0 && (4.4.5) \\ \mathcal{V}(x, \infty; n) := & \lim_{y \to \infty} \mathcal{V}(x, y; n) = \mathcal{V}(x; n) && (4.4.6) \end{aligned}$$

then

$$\mathcal{V}(x, y; n) = E^{(x,y)}(U^n) \tag{4.4.7}$$

Proof. We consider the time instants h and t with $t \geq h$. Applying the Itô formula to the process

$$e^{-n\delta((t-h)\wedge T)}\mathcal{V}(Z_{t\wedge T}, Y_{t\wedge T}; n)$$

taking conditional expectations and using the fact that the process $V(Z_h, Y_h; n)$ is F_h measurable we have:

$$\begin{aligned} & E\left(e^{-n\delta((t-h)\wedge T)}\mathcal{V}(Z_{t\wedge T}, Y_{t\wedge T}; n)|\mathcal{F}_h\right) \\ = \ & \mathcal{V}(Z_h, Y_h; n) - E\left(\int_h^{t\wedge T} e^{-n\delta(s-h)}\frac{\partial}{\partial z}\mathcal{V}(Z_s, Y_s; n)d\mathcal{U}_s|\mathcal{F}_h\right) + \\ & + E\left(\int_h^{t\wedge T} e^{-n\delta(s-h)}(\mathcal{L}_{(x,a,b)} - n\delta)\mathcal{V}(Z_s, Y_s; n)ds|\mathcal{F}_h\right) \end{aligned}$$

Using the relation (4.4.1) and taking limit as $t \to \infty$ (using the dominated convergence theorem as by hypotheses we have that $V(x, y; n) \in C_b^2(\mathbb{R}^2)$) the above becomes :

$$\begin{aligned} & \mathcal{V}(Z_h, Y_h; n) \\ = \; & E\left(e^{-n\delta T} V(Z_T, Y_T; n)|\mathcal{F}_h\right) + E\left(\int_h^T e^{-n\delta(s-h)} \frac{\partial}{\partial z} \mathcal{V}(Z_s, Y_s; n) d\mathcal{U}_s|\mathcal{F}_h\right) \end{aligned} \tag{4.4.8}$$

We observe by taking into consideration the conditions (4.4.2), (4.4.5) that

$$\begin{aligned} & E(e^{-n\delta T} \mathcal{V}(Z_T, Y_T; n)|\mathcal{F}_h) \\ = \; & E(e^{-n\delta T} \mathcal{V}(Z_{T^z}, Y_{T^z}; n) 1\{T^z \leq T^y\}|\mathcal{F}_h) + E(e^{-n\delta T} \mathcal{V}(Z_{T^y}, Y_{T^y}; n) 1\{T^z > T^y\}|\mathcal{F}_h) \\ = \; & E(e^{-n\delta T} \mathcal{V}(0, Y_{T^z}; n) 1\{T^z \leq T^y\}|\mathcal{F}_h) + E(e^{-n\delta T} \mathcal{V}(Z_{T^y}, 0; n) 1\{T^z > T^y\}|\mathcal{F}_h) \\ = \; & 0 \end{aligned}$$

that is

$$E\left(e^{-n\delta T} \mathcal{V}(Z_T, Y_T; n)|\mathcal{F}_h\right) = 0 \tag{4.4.9}$$

Applying the relation (4.4.9) into the expression (4.4.8) we conclude

$$\mathcal{V}(Z_h, Y_h; n) = E\left(\int_h^T e^{-n\delta(s-h)} \frac{\partial}{\partial z} \mathcal{V}(Z_s, Y_s; n) d\mathcal{U}_s|\mathcal{F}_h\right) \tag{4.4.10}$$

for $n = 1, 2, 3, \ldots$ Next we apply the relation (4.4.10) with $n = 1$, taking into account that by the relation (4.2.1) we have

$$dU(t) = -e^{-\delta t} d\mathcal{U}_t \tag{4.4.11}$$

and also considering the relation (4.4.3) we conclude

$$\mathcal{V}(Z_h, Y_h; 1) = E(e^{\delta h} U(h)|\mathcal{F}_h) \tag{4.4.12}$$

We will use the method of induction. Suppose that for $n - 1$ we have

$$\mathcal{V}(Z_h, Y_h; n-1) = E\left(e^{(n-1)\delta h} U^{n-1}(h)|\mathcal{F}_h\right) \tag{4.4.13}$$

Next applying the relation (4.4.10) for n and taking into account the relations (4.4.4) and (4.4.13) we conclude

$$\begin{aligned}
\mathcal{V}(Z_h, Y_h; n) &= nE\left(\int_h^T e^{-n\delta(s-h)}\mathcal{V}(Z_s, Y_s; n-1)d\mathcal{U}_s|\mathcal{F}_h\right) \\
&= nE\left(\int_h^T e^{-n\delta(s-h)}E\left(e^{(n-1)\delta s}U^{n-1}(s)|\mathcal{F}_s\right)d\mathcal{U}_s|\mathcal{F}_h\right) \\
&= n\int_h^T e^{-n\delta(s-h)}E(E(e^{(n-1)\delta s}U^{n-1}(s)d\mathcal{U}_s|\mathcal{F}_s)|\mathcal{F}_h) \\
&= n\int_h^T e^{-n\delta(s-h)}E(e^{(n-1)\delta s}U^{n-1}(s)d\mathcal{U}_s|\mathcal{F}_h) \\
&= nE\left(e^{n\delta h}\int_h^T e^{-\delta s}U^{n-1}(s)d\mathcal{U}_s|\mathcal{F}_h\right) \\
&= -E\left(e^{n\delta h}\int_h^T nU^{n-1}(s)dU(s)|\mathcal{F}_h\right) \\
&= -E(e^{n\delta h}(U^n(s)|_h^T)|\mathcal{F}_h) \Longrightarrow
\end{aligned}$$

$$\mathcal{V}(Z_h, Y_h; n) = E(e^{n\delta h}U^n(h)|\mathcal{F}_h) \tag{4.4.14}$$

By taking $h = 0$ in the relation (4.4.14) we conclude the relation (4.4.7).

Finally the condition (4.4.6) arises by taking into account that :

$$\lim_{y\to\infty} T^y = \infty \qquad a.s.$$

which, with the use of the monotone convergence theorem, implies that

$$\begin{aligned}
&\lim_{y\to\infty}\mathcal{V}(x, y; n) \\
&= \lim_{y\to\infty}E\left(\left(\int_0^{T^z\wedge T^y} e^{-\delta s}d\mathcal{U}_s\right)^n\right) \\
&= E\left(\left(\int_0^{T^z} e^{-\delta s}d\mathcal{U}_s\right)^n\right) = \mathcal{V}(x; n)
\end{aligned}$$

■

Next we will consider the case in which the x–insurer follows the de Finetti model with two reflecting barriers.

Proposition 4.4.2 *(Moments of the discounted dividends and the discounted financing). Let functions $V^{(+)}(x,y;n)$ and $V^{(-)}(x,y;n)$, $n \in N$ belonging in $C_b^2(R^2)$ which satisfy the scaling properties (4.3.9), (4.3.13). If the functions $V^{(+)}(x,y;n)$ and $V^{(-)}(x,y;n)$ solve the PDEs :*

$$(\mathcal{L}_{(x,y)} - n\delta)\mathcal{V}^{(\pm)}(x,y;n) = 0 \tag{4.4.15}$$

with boundary conditions:

$$\begin{aligned}
\frac{\partial}{\partial x}\mathcal{V}^{(+)}(x,y;n)|_{x=0} &= 0 && (4.4.16)\\
\frac{\partial}{\partial x}\mathcal{V}^{(+)}(x,y;1)|_{x=b} &= 1 && (4.4.17)\\
\frac{\partial}{\partial x}\mathcal{V}^{(+)}(x,y;n)|_{x=b} &= n\mathcal{V}^{(+)}(b,y;n-1) \quad for \quad n=1,2,3,\ldots && (4.4.18)\\
\mathcal{V}^{(+)}(x,0;n) &= 0 && (4.4.19)\\
\mathcal{V}^{(+)}(x,\infty;n) := & \lim_{y\to\infty}\mathcal{V}^{(+)}(x,y;n) = \mathcal{V}^{(+)}(x;n) && (4.4.20)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial x}\mathcal{V}^{(-)}(x,y;n)|_{x=0} &= -n\mathcal{V}^{(-)}(0,y;n) && (4.4.21)\\
\frac{\partial}{\partial x}\mathcal{V}^{(-)}(x,y;n)|_{x=b} &= 0 && (4.4.22)\\
\mathcal{V}^{(-)}(x,0;n) &= 0 && (4.4.23)\\
\mathcal{V}^{(-)}(x,\infty;n) := & \lim_{y\to\infty}\mathcal{V}^{(-)}(x,y;n) = \mathcal{V}^{(-)}(x;n) && (4.4.24)
\end{aligned}$$

then

$$\begin{aligned}
\mathcal{V}^{(+)}(x,y;n) &= E^{(x,y)}((U^{(+)})^n) && (4.4.25)\\
\mathcal{V}^{(-)}(x,y;n) &= E^{(x,y)}((U^{(-)})^n) && (4.4.26)
\end{aligned}$$

Proof. We will prove the result only for the function $V^{(+)}(x,y;n)$ because the proof for the function $V^{(-)}(x,y;n)$ is similar. We consider the time instants h and t with $t \geq h$. Applying the Itô formula to the process

$$e^{-n\delta((t-h)\wedge T^y)}\mathcal{V}^{(+)}(Z_t,Y_t;n)$$

taking conditional expectations, taking into account the condition (4.4.16) and using the fact that the process $V^{(+)}(Z_h,Y_h;n)$ is F_h measurable we have:

$$\begin{aligned}
& E(e^{-n\delta((t-h)\wedge T^y)}\mathcal{V}^{(+)}(Z_t,Y_t;n)|\mathcal{F}_h)\\
= \; & \mathcal{V}^{(+)}(Z_h,Y_h;n) - E\left(\int_h^{t\wedge T^y} e^{-n\delta(s-h)}\frac{\partial}{\partial z}\mathcal{V}^{(+)}(Z_s,Y_s;n)d\mathcal{U}_s^{(+)}|\mathcal{F}_h\right) +\\
& +E\left(\int_h^{t\wedge T^y} e^{-n\delta(s-h)}(\mathcal{L}_{(x,y)} - n\delta)\mathcal{V}^{(+)}(Z_s,Y_s;n)ds|\mathcal{F}_h\right) \qquad (4.4.27)
\end{aligned}$$

Using the relation (4.4.15) and taking limit as $t \to \infty$ (using the dominated convergence theorem as by hypotheses we have that $V^{(+)}(x, y; n) \in C_b^2(R^2)$) the above expression (4.4.27) becomes

$$\mathcal{V}^{(+)}(Z_h, Y_h; n) = E(e^{-n\delta T^y}\mathcal{V}^{(+)}(Z_{T^y}, Y_{T^y}; n)|\mathcal{F}_h) + \\ +E\left(\int_h^{T^y} e^{-n\delta(s-h)}\frac{\partial}{\partial z}\mathcal{V}^{(+)}(Z_s, Y_s; n)d\mathcal{U}_s^{(+)}|\mathcal{F}_h\right) \tag{4.4.28}$$

We observe by taking into consideration the condition (4.4.19) that

$$E(e^{-n\delta T^y}\mathcal{V}^{(+)}(Z_{T^y}, Y_{T^y}; n)|\mathcal{F}_h) = E(e^{-n\delta T^y}\mathcal{V}^{(+)}(Z_{T^y}, 0; n)|\mathcal{F}_h) = 0 \tag{4.4.29}$$

Applying the relation (4.4.29) into the expression (4.4.28) we conclude

$$\mathcal{V}^{(+)}(Z_h, Y_h; n) = E\left(\int_h^{T^y} e^{-n\delta(s-h)}\frac{\partial}{\partial z}\mathcal{V}^{(+)}(Z_s, Y_s; n)d\mathcal{U}_s|\mathcal{F}_h\right) \tag{4.4.30}$$

for $n = 1, 2, 3, \ldots$ Next we apply the relation (4.4.30) with $n = 1$, taking into account that by the relation (4.2.2) we have

$$dU^{(+)}(t) = -e^{-\delta t}d\mathcal{U}_t^{(+)} \tag{4.4.31}$$

and also considering the relation (4.4.17) we conclude

$$\mathcal{V}^{(+)}(Z_h, Y_h; 1) = E(e^{\delta h}U^{(+)}(h)|\mathcal{F}_h) \tag{4.4.32}$$

We will use the method of induction. Suppose that for $n - 1$ we have

$$\mathcal{V}^{(+)}(Z_h, Y_h; n-1) = E(e^{(n-1)\delta h}(U^{(+)}(h))^{n-1}|\mathcal{F}_h) \tag{4.4.33}$$

Next applying the relation (4.4.30) for n and taking into account the relations (4.4.18) and (4.4.33), (4.4.31) we conclude

$$\begin{aligned}
\mathcal{V}^{(+)}(Z_h, Y_h; n) &= nE\left(\int_h^{T^y} e^{-n\delta(s-h)}\mathcal{V}^{(+)}(Z_s, Y_s; n-1)d\mathcal{U}_s^{(+)}|\mathcal{F}_h\right) \\
&= nE\left(\int_h^{T^y} e^{-n\delta(s-h)}E(e^{(n-1)\delta s}(U^{(+)}(s))^{n-1}|\mathcal{F}_s))d\mathcal{U}_s^{(+)}|\mathcal{F}_h\right) \\
&= n\int_h^{T^y} e^{-n\delta(s-h)}E(E(e^{(n-1)\delta s}(U^{(+)}(s))^{n-1}d\mathcal{U}_s^{(+)}|\mathcal{F}_s)|\mathcal{F}_h) \\
&= n\int_h^{T^y} e^{-n\delta(s-h)}E(e^{(n-1)\delta s}(U^{(+)}(s))^{n-1}d\mathcal{U}_s^{(+)}|\mathcal{F}_h)
\end{aligned}$$

$$
\begin{aligned}
&= nE\left(e^{n\delta h}\int_h^{T^y} e^{-\delta s}(U^{(+)}(s))^{n-1}d\mathcal{U}_s^{(+)}|\mathcal{F}_h\right) \\
&= -E\left(e^{n\delta h}\int_h^{T^y} n(U^{(+)}(s))^{n-1}dU^{(+)}(s)|\mathcal{F}_h\right) \\
&= -E(e^{n\delta h}((U^{(+)}(s))^n|_h^{T^y})|\mathcal{F}_h) \Longrightarrow
\end{aligned}
$$

$$\mathcal{V}^{(+)}(Z_h, Y_h; n) = E(e^{n\delta h}(U^{(+)}(h))^n|\mathcal{F}_h) \tag{4.4.34}$$

By taking $h = 0$ in the relation (4.4.34) we conclude the relation (4.4.25).

Finally the condition (4.4.20) arises by taking into account that :

$$\lim_{y\to\infty} T^y = \infty \quad \text{a.s.}$$

which, with the use of the monotone convergence theorem, implies that

$$
\begin{aligned}
\lim_{y\to\infty} \mathcal{V}^{(+)}(x, y; n) &= \lim_{y\to\infty} E\left(\int_0^{T^y} e^{-\delta s}d\mathcal{U}_s\right)^n \\
&= E\left(\int_0^{\infty} e^{-\delta s}d\mathcal{U}_s\right)^n = \mathcal{V}^{(+)}(x; n)
\end{aligned}
$$

■

Remark 4.4.3 *The functions $V(x;n)$, $V^{(+)}(x;n)$ and $V^{(-)}(x;n)$ can be founded from the PDEs (4.4.1) and (4.4.15) respectively by taking $\mu_y(y) = 0$ and $\sigma_y(y) = 0$, and boundary conditions only the (4.4.2),(4.4.4) and (4.4.16),(4.4.18) and (4.4.21), (4.4.22) respectively that is regarding the second insurer still during time.*

Remark 4.4.4 *The uniqueness and existence of solutions of the PDEs (4.4.1) and (4.4.15) subject for example to mixed boundary conditions (4.4.2)-(4.4.6), (4.4.16)-(4.4.20) and (4.4.21)-(4.4.24) respectively depends upon the coefficients of the differential operator $L_{(x,y)}$. For the case of constant coefficients one can consult for example W. Boyce and R. Diprima [30] and L.C. Evans [57].*

Next we consider an example in which we apply the Proposition 4.4.1.

Example 1 *Let us suppose the two dimensional diffusion $\{(X_t, Y_t); t \geq 0\}$ which has dynamics that are described by:*

$$
\begin{aligned}
dX_t &= \mu_1 dt + \sigma_1 dB_t^x \\
dY_t &= \mu_2 dt + \sigma_2 dB_t^y
\end{aligned}
$$

The correlation between the Brownian motions B^x, B^y them is given by:

$$\rho dt := d\left[B^x, B^y\right]_t$$

We assume X to represent the reserves of an X-insurance company which follows a dividends policy with one reflecting barrier and Y the reserves of an Y-insurance company which follows no dividends policy. We want to find the expected value of the discounted dividends for the X-insurance company.

By Proposition 4.4.1 the expected value of the discounted dividends is given by the solution of

$$\frac{\sigma_1^2}{2}\mathcal{V}_{xx}(x,y) + \frac{\sigma_2^2}{2}\mathcal{V}_{yy}(x,y) + \sigma_1\sigma_2\rho\mathcal{V}_{xy}(x,y) + \mu_1\mathcal{V}_x(x,y) + \mu_2\mathcal{V}_y(x,y) = \delta\mathcal{V}(x,y) \qquad (4.4.35)$$

with boundary conditions (4.4.2)-(4.4.6). We assume the transform:

$$\overline{\mathcal{V}}(x,s) = \int_0^\infty e^{-sy}\mathcal{V}(x,y)dy \qquad (4.4.36)$$

Applying (4.4.36) to (4.4.35) we have:

$$\frac{1}{2}\overline{\mathcal{V}}_{xx}(x,s)\sigma_1^2 + \left(\mu_1 + s\rho\sigma_1\sigma_2\right)\overline{\mathcal{V}}_x(x,s) + \left(\frac{1}{2}s^2\sigma_2^2 + s\mu_2 - \delta\right)\overline{\mathcal{V}}(x,s) = 0 \qquad (4.4.37)$$

and the boundary conditions (4.4.2),(4.4.4) become:

$$\overline{\mathcal{V}}(0,s) = 0 \qquad (4.4.38)$$

$$\overline{\mathcal{V}}_x(b,v) = \frac{1}{s} \qquad (4.4.39)$$

The solution of (4.4.37)-(4.4.39) is :

$$\overline{\mathcal{V}}(x,s) = 2\exp\left(\frac{\left(A(s)+2B(s)\right)(b-x)}{2\sigma_1^2}\right) \times \frac{\left(-1+\exp\left(\frac{A(s)x}{\sigma_1^2}\right)\right)\sigma_1^2}{\left(A(s)+2B(s)\right)+\left(A(s)-2B(s)\right)\exp\left(\frac{A(s)b}{\sigma_1^2}\right)}\frac{1}{s} \qquad (4.4.40)$$

where :

$$A(s) = 2\sqrt{\left(\rho^2-1\right)\sigma_1^2\sigma_2^2 s^2 + 2\sigma_1\left(\rho\mu_1\sigma_2 - \mu_2\sigma_1\right)s + \left(\mu_1^2 + 2\delta\sigma_1^2\right)} \qquad (4.4.41)$$

$$B(s) = s\rho\sigma_1\sigma_2 + \mu_1 \qquad (4.4.42)$$

Remark 4.4.5 *From the relation (4.4.40) we see that :*

$$\lim_{s\to 0} s\overline{\mathcal{V}}(x,s) = \frac{1}{r_2 e^{r_2 b} - r_1 e^{r_1 b}}\left(e^{r_2 x} - e^{r_1 x}\right) \qquad (4.4.43)$$

where

$$\begin{aligned} r_1 &= \frac{-\mu_1 + \sqrt{\mu_1^2 + 2\delta\sigma_1^2}}{\sigma_1^2} \\ r_2 &= \frac{-\mu_1 - \sqrt{\mu_1^2 + 2\delta\sigma_1^2}}{\sigma_1^2} \end{aligned}$$

a result which we already have found before and which is what was expected as:

$$\lim_{s\to 0} s\overline{\mathcal{V}}(x,s) = \lim_{y\to\infty} \mathcal{V}(x,y) = \mathcal{V}(x)$$

by the condition (4.4.6).

4.5 The Laplace transform of the joint distribution of the time of ruin, the discounted dividends and the discounted financing.

We consider first the case in which the X-insurer follows the de Finetti model with one reflecting barrier.

Proposition 4.5.1 *(The Laplace transform of the joint distribution of the time of ruin and the discounted dividends). Let function $N(x,y,\lambda_1,\lambda_2) \in C_b^2(\mathbb{R}^4)$ which satisfy the scaling properties (4.3.4), (4.3.8). If the function $N(x,y,\lambda_1,\lambda_2)$ solves the PDE*

$$\mathcal{L}_{(x,y)}\mathcal{N}(x,y,\lambda_1,\lambda_2) = \lambda_1\mathcal{N}(x,y,\lambda_1,\lambda_2) + \delta\lambda_2\frac{\partial}{\partial\lambda_2}\mathcal{N}(x,y,\lambda_1,\lambda_2) \tag{4.5.1}$$

with boundary conditions:

$$\begin{aligned} \mathcal{N}(0,y,\lambda_1,\lambda_2) &= 1 && (4.5.2)\\ \frac{\partial}{\partial x}\mathcal{N}(x,y,\lambda_1,\lambda_2)|_{x=b} &= -\lambda_2\mathcal{N}(x,y,\lambda_1,\lambda_2) && (4.5.3)\\ \mathcal{N}(x,0,\lambda_1,\lambda_2) &= 1 && (4.5.4)\\ \mathcal{N}(x,\infty,\lambda_1,\lambda_2) &:= \lim_{y\to\infty}\mathcal{N}(x,y,\lambda_1,\lambda_2) = \mathcal{N}(x,\lambda_1,\lambda_2) && (4.5.5) \end{aligned}$$

then

$$\mathcal{N}(x,y,\lambda_1,\lambda_2) = E^{(x,y)}\left(e^{-\lambda_1 T - \lambda_2 U}\right) \tag{4.5.6}$$

Proof. Applying the Itô formula to the process

$$e^{-\lambda_1(t\wedge T)-\lambda_2 U_{t\wedge T}}\mathcal{N}(e^{-\delta(t\wedge T)}Z_{t\wedge T}, e^{-\delta(t\wedge T)}Y_{t\wedge T}, \lambda_1, \lambda_2)$$

taking expectations and using the condition (4.5.3) we have:

$$\begin{aligned} & E\left(e^{-\lambda_1(t\wedge T)-\lambda_2 U_{t\wedge T}}\mathcal{N}(e^{-\delta(t\wedge T)}Z_{t\wedge T}, e^{-\delta(t\wedge T)}Y_{t\wedge T}, \lambda_1, \lambda_2)\right) \\ = \; & \mathcal{N}(x,y,\lambda_1,\lambda_2) + \\ & +E\left(\int_0^{t\wedge T} e^{-\lambda_1 s-\lambda_2 U_s}(\mathcal{L}_{(x,y)} - \lambda_1 - \right. \\ & \left. -\delta e^{-\delta s}\left(X_s\frac{\partial}{\partial\left(e^{-\delta s}z\right)} + Y_s\frac{\partial}{\partial(e^{-\delta s}y)}\right)\right)\mathcal{N}(e^{-\delta s}Z_s, e^{-\delta s}Y_s, \lambda_1, \lambda_2)ds\Bigg) + \\ & +\lambda_2 E\left(\int_0^{t\wedge T} e^{-\delta s}e^{-\lambda_1 s-\lambda_2 U_s}\mathcal{N}(e^{-\delta s}Z_s, e^{-\delta s}Y_s, \lambda_1, \lambda_2)d\mathcal{U}_s\right) - \\ & -\lambda_2 E\left(\int_0^{t\wedge T} e^{-\lambda_1 s-\lambda_2 U_s}\mathcal{N}(e^{-\delta s}Z_s, e^{-\delta s}Y_s, \lambda_1, \lambda_2)dU_s\right) \end{aligned}$$

Taking into account that by the relation (4.1.1) we have

$$dU_t = e^{-\delta t}d\mathcal{U}_t \tag{4.5.7}$$

and also considering the relation (4.3.18) we conclude

$$\begin{aligned} & E(e^{-\lambda_1(t\wedge T)-\lambda_2 U_{t\wedge T}}\mathcal{N}(e^{-\delta(t\wedge T)}Z_{t\wedge T}, e^{-\delta(t\wedge T)}Y_{t\wedge T}, \lambda_1, \lambda_2)) \\ = \; & \mathcal{N}(x,y,\lambda_1,\lambda_2) + \\ & +E\left(\int_0^{t\wedge T} e^{-\lambda_1 s-\lambda_2 U_s}\left((\mathcal{L}_{(x,y)} - \lambda_1)\mathcal{N}(e_s^{-\delta s}Z_s, e^{-\delta s}Y_s, \lambda_1, \lambda_2) - \right.\right. \\ & \left.\left. -\delta\lambda_2 e^{-\delta s}\frac{\partial}{\partial(\lambda_2 e^{-\delta s})}\mathcal{N}(Z_s, Y_s, \lambda_2 e^{-\delta s})\right)ds\right) \end{aligned} \tag{4.5.8}$$

Applying the relation (4.3.17) in the above expression (4.5.8) we conclude

$$\begin{aligned} & E(e^{-\lambda_1(t\wedge T)-\lambda_2 U_{t\wedge T}}\mathcal{N}(e^{-\delta(t\wedge T)}Z_{t\wedge T}, e^{-\delta(t\wedge T)}Y_{t\wedge T}, \lambda_1, \lambda_2)) \\ = \; & \mathcal{N}(x,y,\lambda_1,\lambda_2) + \\ & +E\left(\int_0^{t\wedge T} e^{-\lambda_1 s-\lambda_2 U_s}\left(\mathcal{L}_{(x,y)} - \lambda_1 - \delta\lambda_2 e^{-\delta s}\frac{\partial}{\partial(\lambda_2 e^{-\delta s})}\right)\mathcal{N}(Z_s, Y_s, \lambda_1, \lambda_2 e^{-\delta s})ds\right) \end{aligned} \tag{4.5.9}$$

Applying the PDE (4.5.1) with $\lambda_2 e^{-\delta s}$ instead of λ_2 the above expression (4.5.9) simplifies to

$$E(e^{-\lambda_1(t\wedge T)-\lambda_2 U_{t\wedge T}}\mathcal{N}(e^{-\delta(t\wedge T)}Z_{t\wedge T}, e^{-\delta(t\wedge T)}Y_{t\wedge T}, \lambda_1, \lambda_2)) = \mathcal{N}(x,y,\lambda_1,\lambda_2) \tag{4.5.10}$$

Taking limit as $t \to \infty$ (using the dominated convergence theorem as by hypotheses we have that $N(x,y,\lambda_1,\lambda_2) \in C_b^2(R^4)$) and taking into account the conditions (4.5.2), (4.5.4) we conclude:

$$\mathcal{N}(x,y,\lambda_1,\lambda_2) = E\left(e^{-\lambda_1 T-\lambda_2 U}\right)$$

Finally the condition (4.5.5) arises by taking into account that

$$\lim_{y\to\infty} T^y = \infty \quad \text{a.s.}$$

which implies that

$$\lim_{y\to\infty} T = \lim_{y\to\infty} T^z \wedge T^y = T^z \quad \text{a.s.}$$

and with the use of the bounded convergence theorem we conclude

$$\begin{aligned} & \lim_{y\to\infty} \mathcal{N}(x, y, \lambda_1, \lambda_2) \\ = & \lim_{y\to\infty} E(e^{-\lambda_1 T - \lambda_2 U}) \\ = & E(e^{-\lambda_1 T^z - \lambda_2 U}) = \mathcal{N}(x, \lambda_1, \lambda_2) \end{aligned}$$

■

We consider next the case in which the X-insurer follows the de Finetti model with two reflecting barriers.

Proposition 4.5.2 *(Laplace transform of the joint distribution of the time of ruin, the discounted dividends and the discounted financing). Consider the function $N(x, y, \lambda_1, \lambda_2, \lambda_2) \in C_b^2(R^5)$ which satisfy the scaling properties (4.3.12), (4.3.16). If the function $N(x, y, \lambda_1, \lambda_2, \lambda_3)$ solves the PDE :*

$$\begin{aligned} \mathcal{L}_{(x,y)}\mathcal{N}(x, y, \lambda_1, \lambda_2, \lambda_3) &= \lambda_1 \mathcal{N}(x, y, \lambda_1, \lambda_2, \lambda_3) + \delta\lambda_2 \frac{\partial}{\partial \lambda_2}\mathcal{N}(x, y, \lambda_1, \lambda_2, \lambda_3) + \\ & + \delta\lambda_3 \frac{\partial}{\partial \lambda_3}\mathcal{N}(x, y, \lambda_1, \lambda_2, \lambda_3) \end{aligned} \tag{4.5.11}$$

with boundary conditions

$$\frac{\partial}{\partial x}\mathcal{N}(x, y, \lambda_1, \lambda_2, \lambda_3)|_{x=0} = \lambda_3 \mathcal{N}(0, y, \lambda_1, \lambda_2, \lambda_3) \tag{4.5.12}$$

$$\frac{\partial}{\partial x}\mathcal{N}(x, y, \lambda_1, \lambda_2, \lambda_3)|_{x=b} = -\lambda_2 \mathcal{N}(b, y, \lambda_1, \lambda_2, \lambda_3) \tag{4.5.13}$$

$$\mathcal{N}(x, 0, \lambda_1, \lambda_2, \lambda_3) = 1 \tag{4.5.14}$$

$$\begin{aligned} \mathcal{N}(x, \infty, \lambda_1, \lambda_2, \lambda_3) &:= \lim_{y\to\infty} \mathcal{N}(x, y, \lambda_1, \lambda_2, \lambda_3) \\ &= 0 \end{aligned} \tag{4.5.15}$$

then

$$\mathcal{N}(x, y, \lambda_1, \lambda_2, \lambda_3) := E^{(x,y)}(e^{-\lambda_1 T - \lambda_2 U^{(+)} - \lambda_3 U^{(-)}}) \tag{4.5.16}$$

Proof. Applying the Itô formula to the process

$$e^{-\lambda_1 (t\wedge T^y) - \lambda_2 U^{(+)}_{t\wedge T^y} - \lambda_3 U^{(-)}_{t\wedge T^y}} \mathcal{N}(e^{-\delta(t\wedge T^y)} Z_{t\wedge T^y}, e^{-\delta(t\wedge T^y)} Y_{t\wedge T^y}, \lambda_1, \lambda_2, \lambda_3)$$

taking expectations and using the conditions (4.5.12), (4.5.13) we have:

$$\begin{aligned}
& E\left(e^{-\lambda_1(t\wedge T^y)-\lambda_2 U^{(+)}_{t\wedge T^y}-\lambda_3 U^{(-)}_{t\wedge T^y}}\mathcal{N}(e^{-\delta(t\wedge T^y)}Z_{t\wedge T^y}, e^{-\delta(t\wedge T^y)}Y_{t\wedge T^y},\lambda_1,\lambda_2,\lambda_3)\right) \\
= \; & \mathcal{N}(x,y,\lambda_1,\lambda_2,\lambda_3)+ \\
& +E\left(\int_0^{t\wedge T^y} e^{-\lambda_1 s-\lambda_2 U_s^{(+)}-\lambda_3 U_s^{(-)}}(\mathcal{L}_{(x,y)}-\lambda_1- \right. \\
& \left. -\delta e^{-\delta s}\left(X_s\frac{\partial}{\partial(e^{-\delta s}z)}+Y_s\frac{\partial}{\partial(e^{-\delta s}y)}\right)\right)\mathcal{N}(e^{-\delta s}Z_s, e^{-\delta s}Y_s,\lambda_1,\lambda_2,\lambda_3)ds\Bigg)+ \\
& +\lambda_2 E\left(\int_0^{t\wedge T^y} e^{-\delta s}e^{-\lambda_1 s-\lambda_2 U_s^{(+)}-\lambda_3 U_s^{(-)}}\mathcal{N}(e^{-\delta s}Z_s, e^{-\delta s}Y_s,\lambda_1,\lambda_2,\lambda_3)d\mathcal{U}_s^{(+)}\right)- \\
& -\lambda_2 E\left(\int_0^{t\wedge n} e^{-\lambda_1 s-\lambda_2 U_s^{(+)}-\lambda_3 U_s^{(-)}}\mathcal{N}(e^{-\delta s}Z_s, e^{-\delta s}Y_s,\lambda_1,\lambda_2,\lambda_3)dU_s^{(+)}\right)+ \\
& +\lambda_3 E\left(\int_0^{t\wedge T^y} e^{-\delta s}e^{-\lambda_1 s-\lambda_2 U_s^{(+)}-\lambda_3 U_s^{(-)}}\mathcal{N}(e^{-\delta s}Z_s, e^{-\delta s}Y_s,\lambda_1,\lambda_2,\lambda_3)d\mathcal{U}_s^{(-)}\right)- \\
& -\lambda_3 E\left(\int_0^{t\wedge T^y} e^{-\lambda_1 s-\lambda_2 U_s^{(+)}-\lambda_3 U_s^{(-)}}\mathcal{N}(e^{-\delta s}Z_s, e^{-\delta s}Y_s,\lambda_1,\lambda_2,\lambda_3)dU_s^{(-)}\right)
\end{aligned} \tag{4.5.17}$$

Taking into account that by the relations (4.1.2) and (4.1.3) we have that

$$\begin{aligned}
dU_t^{(+)} &= e^{-\delta t}d\mathcal{U}_t^{(+)} \\
dU_t^{(-)} &= e^{-\delta t}d\mathcal{U}_t^{(-)}
\end{aligned}$$

and also the considering relation (4.3.20), the expression (4.5.17) becomes

$$\begin{aligned}
& E\left(e^{-\lambda_1(t\wedge T^y)-\lambda_2 U^{(+)}_{t\wedge T^y}-\lambda_3 U^{(-)}_{t\wedge T^y}}\mathcal{N}(e^{-\delta(t\wedge T^y)}Z_{t\wedge T^y}, e^{-\delta(t\wedge T^y)}Y_{t\wedge T^y},\lambda_1,\lambda_2,\lambda_3)\right) \\
= \; & \mathcal{N}(x,y,\lambda_1,\lambda_2,\lambda_3)+ \\
& +E\left(\int_0^{t\wedge T^y}(e^{-\lambda_1 s-\lambda_2 U_s^{(+)}-\lambda_3 U_s^{(-)}}\left((\mathcal{L}_{(x,y)}-\lambda_1)\mathcal{N}(e_s^{-\delta s}Z_s, e^{-\delta s}Y_s,\lambda_1,\lambda_2,\lambda_3)- \right.\right. \\
& \left.\left. -\delta\left(\lambda_2 e^{-\delta s}\frac{\partial}{\partial(\lambda_2 e^{-\delta s})}+\lambda_3 e^{-\delta s}\frac{\partial}{\partial(\lambda_3 e^{-\delta s})}\right)\mathcal{N}(Z_s,Y_s,\lambda_1,\lambda_2 e^{-\delta s},\lambda_3 e^{-\delta s})\right)ds\right)
\end{aligned} \tag{4.5.18}$$

Applying the relation (4.3.19) in the expression (4.5.18) we conclude

$$
\begin{aligned}
& E(e^{-\lambda_1(t\wedge T^y)-\lambda_2 U^{(+)}_{t\wedge T^y}-\lambda_3 U^{(-)}_{t\wedge T^y}}\mathcal{N}(e^{-\delta(t\wedge T^y)}Z_{t\wedge T^y}, e^{-\delta(t\wedge T^y)}Y_{t\wedge T^y}, \lambda_1, \lambda_2, \lambda_3)) \\
= \; & \mathcal{N}(x, y, \lambda_1, \lambda_2, \lambda_3) + \\
& +E\left(\int_0^{t\wedge T^y} e^{-\lambda_1 s-\lambda_2 U^{(+)}_s-\lambda_3 U^{(-)}_s}(\mathcal{L}_{(x,y)} - \lambda_1 - \right. \\
& \left. -\delta\left(\lambda_2 e^{-\delta s}\frac{\partial}{\partial(\lambda_2 e^{-\delta s})} + \lambda_3 e^{-\delta s}\frac{\partial}{\partial(\lambda_3 e^{-\delta s})}\right)\right)\mathcal{N}(Z_s, Y_s, \lambda_1, \lambda_2 e^{-\delta s}, \lambda_3 e^{-\delta s})ds\Bigg)
\end{aligned}
\tag{4.5.19}
$$

Applying the PDE (4.5.11) with $\lambda_2 e^{-\delta s}$ instead of λ_2 and $\lambda_3 e^{-\delta s}$ instead of λ_3 the above expression (4.5.19) simplifies to

$$
\begin{aligned}
& E\left(e^{-\lambda_1(t\wedge T^y)-\lambda_2 U^{(+)}_{t\wedge T^y}-\lambda_3 U^{(-)}_{t\wedge T^y}}\mathcal{N}(e^{-\delta(t\wedge n)}Z_{t\wedge T^y}, e^{-\delta(t\wedge n)}Y_{t\wedge T^y}, \lambda_1, \lambda_2, \lambda_3)\right) \\
= \; & \mathcal{N}(x, y, \lambda_1, \lambda_2, \lambda_3)
\end{aligned}
\tag{4.5.20}
$$

Taking limit as $t \to \infty$ (using the dominated convergence theorem as by hypotheses we have that $N(x, y, \lambda_1, \lambda_2, \lambda_3) \in C^2_b(\mathbb{R}^5)$) the relation (4.5.20) becomes

$$
\mathcal{N}(x, y, \lambda_1, \lambda_2, \lambda_3) = E\left(e^{-\lambda_1 T^y-\lambda_2 U^{(+)}_{T^y}-\lambda_3 U^{(-)}_{T^y}}\mathcal{N}(e^{-\delta T^y}Z_{T^y}, e^{-\delta T^y}Y_{T^y}, \lambda_1, \lambda_2, \lambda_3)\right) \tag{4.5.21}
$$

Taking into account the condition (4.5.14) we conclude:

$$
\mathcal{N}(x, y, \lambda_1, \lambda_2, \lambda_3) = E\left(e^{-\lambda_1 T^y-\lambda_2 U^{(+)}_{T^y}-\lambda_3 U^{(-)}_{T^y}}\right) \tag{4.5.22}
$$

Finally the condition (4.5.15) arises by taking into account that

$$
\lim_{y\to\infty} T^y = \infty \qquad a.s.
$$

which, with the use of the bounded convergence theorem, implies that

$$
\lim_{y\to\infty}\mathcal{N}(x, y, \lambda_1, \lambda_2, \lambda_3) = \lim_{y\to\infty} E\left(e^{-\lambda_1 T^y-\lambda_2 U^{(+)}_{T^y}-\lambda_3 U^{(-)}_{T^y}}\right) = 0
$$

■

Next we turn our attention to the Laplace transform of the discounted dividends and the discounted financing.

4.6 The Laplace transform of the discounted dividends and the discounted financing.

Following the proof of Proposition 4.5.1 with $\lambda_1 = 0$ and $\lambda_2 = \lambda$ we may also prove the next proposition.

Proposition 4.6.1 *(Laplace transform of the discounted dividends-1 reflecting barrier). Consider the function $K(x,y,\lambda) \in C_b^2(R^3)$ which satisfy the scaling properties (4.3.2), (4.3.6). If the function $K(x,y,\lambda)$ solve the PDE :*

$$\mathcal{L}_{(x,y)}\mathcal{K}(x,y,\lambda) = \lambda\delta\frac{\partial}{\partial\lambda}\mathcal{K}(x,y,\lambda) \tag{4.6.1}$$

with boundary conditions

$$\begin{aligned} \mathcal{K}(0,y,\lambda) &= 1 && (4.6.2)\\ \frac{\partial}{\partial x}\mathcal{K}(x,y,\lambda)|_{x=b} &= -\lambda\mathcal{K}(b,y,\lambda) && (4.6.3)\\ \mathcal{K}(x,0,\lambda) &= 1 && (4.6.4)\\ \mathcal{K}(x,\infty,\lambda) &:= \lim_{y\to\infty}\mathcal{K}(x,y,\lambda) = \mathcal{K}(x,\lambda) && (4.6.5) \end{aligned}$$

then

$$\mathcal{K}(x,y,\lambda) = E^{(x,y)}(e^{-\lambda U}) \tag{4.6.6}$$

Following the proof of Proposition 4.5.2 first with $\lambda_1 = 0$, $\lambda_2 = \lambda$, $\lambda_3 = 0$ and second with $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = \lambda$ we may also prove the next proposition.

Proposition 4.6.2 *(Laplace transforms of the discounted dividends and the discounted financing-2 reflecting barriers). Consider the functions $K^{(+)}(x,y,\lambda)$ and $K^{(-)}(x,y,\lambda)$ belonging in$C_b^2(R^3)$ which satisfy the scaling properties (4.3.10), (4.3.14). If the functions $K^{(+)}(x,y,\lambda)$ and $K^{(-)}(x,y,\lambda)$ solves the PDEs*

$$\mathcal{L}_{(x,y)}\mathcal{K}^{(\pm)}(x,y,\lambda) = \lambda\delta\frac{\partial}{\partial\lambda}\mathcal{K}^{(\pm)}(x,y,\lambda) \tag{4.6.7}$$

with boundary conditions:

$$\begin{aligned} \frac{\partial}{\partial x}\mathcal{K}^{(+)}(x,y,\lambda)|_{x=0} &= 0 && (4.6.8)\\ \frac{\partial}{\partial x}\mathcal{K}^{(+)}(x,y,\lambda)|_{x=b} &= -\lambda\mathcal{K}^{(+)}(b,y,\lambda) && (4.6.9)\\ \mathcal{K}^{(+)}(x,0,\lambda) &= 1 && (4.6.10)\\ \mathcal{K}^{(+)}(x,\infty,\lambda) &:= \lim_{y\to\infty}\mathcal{K}^{(+)}(x,y,\lambda) = \mathcal{K}^{(+)}(x,\lambda) && (4.6.11) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial x}\mathcal{K}^{(-)}(x,y,\lambda)|_{x=0} &= \lambda\mathcal{K}^{(-)}(0,y,\lambda) && (4.6.12)\\ \frac{\partial}{\partial x}\mathcal{K}^{(-)}(x,y,\lambda)|_{x=b} &= 0 && (4.6.13)\\ \mathcal{K}^{(-)}(x,0,\lambda) &= 1 && (4.6.14)\\ \mathcal{K}^{(-)}(x,\infty,\lambda) &:= \lim_{y\to\infty}\mathcal{K}^{(-)}(x,y,\lambda) = \mathcal{K}^{(-)}(x,\lambda) && (4.6.15) \end{aligned}$$

then

$$\begin{aligned} \mathcal{K}^{(+)}(x,y,\lambda) &= E^{(x,y)}(e^{-\lambda U^{(+)}}) \quad (4.6.16)\\ \mathcal{K}^{(-)}(x,y,\lambda) &= E^{(x,y)}(e^{-\lambda U^{(-)}}) \quad (4.6.17) \end{aligned}$$

We continue with the next section in which we find the Laplace transform of the time of ruin.

4.7 The Laplace transform of time of ruin.

Following the proof of Proposition 4.5.1 with $\lambda_1 = \lambda$ and $\lambda_2 = 0$ we may also prove the next proposition.

Proposition 4.7.1 *(Laplace transform of the time of ruin-1 reflecting barrier). Consider the function $M(x,y,\lambda) \in C_b^2(\mathbb{R}^3)$ which satisfy the scaling properties (4.3.3), (4.3.7). If the function $M(x,y,\lambda)$ solves the PDE :*

$$\mathcal{L}_{(x,y)}\mathcal{M}(x,y,\lambda) = \lambda\mathcal{M}(x,y,\lambda) \quad (4.7.1)$$

with boundary conditions:

$$\begin{aligned} \mathcal{M}(0,y,\lambda) &= 1 \quad (4.7.2)\\ \frac{\partial}{\partial x}\mathcal{M}(x,y,\lambda)|_{x=b} &= 0 \quad (4.7.3)\\ \mathcal{M}(x,0,\lambda) &= 1 \quad (4.7.4)\\ \mathcal{M}(x,\infty,\lambda) &:= \lim_{y\to\infty}\mathcal{M}(x,y,\lambda) = \mathcal{M}(x,\lambda) \quad (4.7.5) \end{aligned}$$

then

$$\mathcal{M}(x,y,\lambda) = E^{(x,y)}(e^{-\lambda T}) \quad (4.7.6)$$

Following the proof of Proposition 4.5.2 with $\lambda_1 = \lambda$, $\lambda_2 = 0$ and $\lambda_3 = 0$ we may also prove the next proposition.

Proposition 4.7.2 *(Laplace transform of the time of ruin-2 reflecting barriers). Consider the function $M(x,y,\lambda) \in C_b^2(\mathbb{R}^3)$ which satisfy the scaling properties (4.3.11),(4.3.15). If the function $M(x,y,\lambda)$ solves the PDE :*

$$\mathcal{L}_{(x,y)}\mathcal{M}(x,y,\lambda) = \lambda\mathcal{M}(x,y,\lambda) \quad (4.7.7)$$

with boundary conditions:

$$\frac{\partial}{\partial x}\mathcal{M}(x,y,\lambda)|_{x=0} = 0 \tag{4.7.8}$$

$$\frac{\partial}{\partial x}\mathcal{M}(x,y,\lambda)|_{x=b} = 0 \tag{4.7.9}$$

$$\mathcal{M}(x,0,\lambda) = 1 \tag{4.7.10}$$

$$\mathcal{M}(x,\infty,\lambda) := \lim_{y\to\infty}\mathcal{M}(x,y,\lambda) = 0 \tag{4.7.11}$$

then

$$\mathcal{M}(x,y,\lambda) = E^{(x,y)}(e^{-\lambda T}) \tag{4.7.12}$$

4.8 Survival probability for one of the two insurers.

We are interested in the probability of survival of one of the two insurers, that is for the probability $P(T^y < T^z)$. To this direction we prove the next proposition.

Proposition 4.8.1 *Consider the function $P(x,y) \in C_b^2(\mathbb{R}^+ \times \mathbb{R}^+)$. If the function $P(x,y)$ solves the PDE :*

$$\mathcal{L}_{(x,y)}\mathcal{P}(x,y) = 0 \tag{4.8.1}$$

with boundary conditions

$$P(0,y) = 0 \tag{4.8.2}$$

$$P(x,0) = 1 \tag{4.8.3}$$

$$\frac{\partial}{\partial x}P(x,y)|_{x=b} = 0 \tag{4.8.4}$$

$$\lim_{y\to\infty}P(x,y) = 0 \tag{4.8.5}$$

then

$$P(x,y) = P(T^y < T^z) \tag{4.8.6}$$

Proof. Applying the Itô formula to the process $P(Z_{t\wedge T^z\wedge T^y}, Y_{t\wedge T^z\wedge T^y})$, taking expectations and using the condition (4.8.4) we have:

$$E(P(Z_{t\wedge T^z\wedge T^y}, Y_{t\wedge T^z\wedge T^y})) = P(x,y) + E\left(\int_0^{t\wedge T^z\wedge T^y} \mathcal{L}_{(x,y)}P(Z_s,Y_s)ds\right)$$

By the conditions (4.8.1), (4.8.2), (4.8.3) and taking limit as $t \to \infty$ (using the dominated convergence theorem as by hypotheses we have that $P(x,y) \in C_b^2(\mathbb{R}^+ \times \mathbb{R}^+)$) the above becomes

$$\begin{aligned} P(x,y) &= E(P(Z_{T^z \wedge T^y}, Y_{T^z \wedge T^y})) \\ &= E(P(Z_{T^z}, Y_{T^z})1\{T^z \leq T^y\}) + E(P(Z_{T^y}, Y_{T^y})1\{T^z > T^y\}) \\ &= E(P(0, Y_{T^z})1\{T^z \leq T^y\}) + E(P(Z_{T^y}, 0)1\{T^z > T^y\}) \\ &= E(0 \cdot 1\{T^z \leq T^y\}) + E(1 \cdot 1\{T^z > T^y\}) \end{aligned}$$

so that

$$P(x,y) = P(T^y < T^z)$$

It is also obvious that it must hold the condition (4.8.5) which arises from :

$$\lim_{y \to \infty} T^y = \infty \quad \text{a.s.}$$

■

Next we consider an example of Proposition 4.8.1.

Example 2 *We consider the two dimensional diffusion $\{(X_t, Y_t); t \geq 0\}$ of the example 1, with X to represent the reserves of the X-insurance company which follows a dividends policy with one reflecting barrier and Y the reserves of the Y-insurance company which follows no dividends policy. We want to find the probability of survival of the X-insurance company. We will consider two cases: (I) There are no correlations ($\rho = 0$) and (II) There are correlations ($\rho \neq 0$).*

(I) Case with $\rho = 0$.

By (4.8.1) the probability of survival of the X-insurer is given by the solution of the PDE

$$\frac{\sigma_1^2}{2} P_{xx} + \frac{\sigma_2^2}{2} P_{yy} + \mu_1 P_x + \mu_2 P_y = 0 \tag{4.8.7}$$

subject to the boundary conditions:

$$\begin{aligned} P(0,y) &= 0 && (4.8.8) \\ P(x,0) &= 1 && (4.8.9) \\ \frac{\partial}{\partial x} P(x,y)|_{x=b} &= 0 && (4.8.10) \\ \lim_{y \to \infty} P(x,y) &= 0 && (4.8.11) \end{aligned}$$

We consider a solution of the form $P(x,y) = u(x)w(y)$. Substituting into (4.8.7) we have

$$\frac{\sigma_1^2}{2} u_{xx}(x)w(y) + \frac{\sigma_2^2}{2} u(x)w_{yy}(y) + \mu_1 u_x(x)w(y) + \mu_2 u(x)w_y(y) = 0.$$

Dividing through with $u(x)w(y)$ we obtain, after separating variables,

$$\frac{\sigma_1^2}{2}\frac{u_{xx}(x)}{u(x)}+\mu_1\frac{u_x(x)}{u(x)}=\lambda=-\frac{\sigma_2^2}{2}\frac{w_{yy}(y)}{w(y)}-\mu_2\frac{w_y(y)}{w(y)}. \tag{4.8.12}$$

The general solution of

$$\frac{\sigma_1^2}{2}\frac{u_{xx}(x)}{u(x)}+\mu_1\frac{u_x(x)}{u(x)}=\lambda$$

is then

$$u(x)=c_1\exp\left(x\left(-\mu_1-\sqrt{\mu_1^2+2\lambda\sigma_1^2}\right)\sigma_1^{-2}\right)+c_2\exp\left(x\left(-\mu_1+\sqrt{\mu_1^2+2\lambda\sigma_1^2}\right)\sigma_1^{-2}\right)$$

where c_1, c_2 arbitrary constants. We observe that if $\lambda\geq-\frac{\mu_1^2}{2\sigma_1^2}$ then the only solution is the trivial zero solution. So in order to have non-trivial solutions we must consider:

$$\lambda<-\frac{\mu_1^2}{2\sigma_1^2} \tag{4.8.13}$$

and then the solution becomes

$$u(x)=\sin(x\varphi_1(\lambda))\exp(-x\mu_1/\sigma_1^2)$$

with

$$\varphi_i(\lambda):=\sigma_i^{-2}\sqrt{(-1)^i\mu_i^2-2\lambda\sigma_i^2},\qquad i=1,2.$$

In view of the condition $u'(b)=0$ we also have

$$\sigma_1^2\varphi_1(\lambda)\cos\left(b\varphi_1(\lambda)\right)-\mu_1\sin\left(b\varphi_1(\lambda)\right)=0\Longrightarrow$$

$$\tan\left(b\varphi_1(\lambda)\right)=\frac{\sigma_1^2\varphi_1(\lambda)}{\mu_1} \tag{4.8.14}$$

Consider the equation

$$\tan\left(b\nu\right)=\frac{\sigma_1^2\nu}{\mu_1} \tag{4.8.15}$$

This is a transcendental equation and can be solved numerically. Let $\{\nu_n; n\in\mathbb{Z}\}$, be the set of its solutions. We consider only strictly positive solutions since from (4.8.13) we conclude $\varphi_1(\cdot)>0$. Also holds that

$$0<\nu_1<\nu_2<\cdots$$

Asymptotically the solutions can be approximated by

$$\nu_n\approx\frac{\left(n+\frac{1}{2}\right)\pi}{b}$$

From the set of solutions we obtain the eigenvalues λ_n *via*

$$\lambda_n = -\frac{\nu_n^2\sigma_1^4 + \mu_1^2}{2\sigma_1^2}$$

for $n = 1, 2, 3, \ldots$

We now turn to the second equation obtained from (4.8.12), namely

$$-\frac{\sigma_2^2}{2}\frac{w_{yy}(y)}{w(y)} - \mu_2\frac{w_y(y)}{w(y)} = \lambda_n.$$

Its general solution is

$$\begin{aligned} w(y) &= c_1 \exp\left(y\left(-\mu_2 - \sqrt{\mu_2^2 - 2\lambda_n\sigma_2^2}\right)\sigma_2^{-2}\right) \\ &\quad + c_2 \exp\left(y\left(-\mu_2 + \sqrt{\mu_2^2 - 2\lambda_n\sigma_2^2}\right)\sigma_2^{-2}\right) \end{aligned}$$

Because of condition (4.8.11) the solution is

$$w(y) = c_1 \exp\left(y\left(-\mu_2 - \sqrt{\mu_2^2 - 2\lambda_n\sigma_2^2}\right)\sigma_2^{-2}\right)$$

Thus (4.8.7) has solutions of the form

$$P(x,y) = c_n \sin(x\nu_n)\exp(-x\mu_1\sigma_1^{-2})\exp(-y\left(\mu_2\sigma_2^{-2} + \varphi_2(\lambda_n)\right))$$

Using the relationship

$$\varphi_2(\lambda_n) = \sigma_2^{-2}\sqrt{\mu_2^2 + \frac{\left(\nu_n^2\sigma_1^4 + \mu_1^2\right)\sigma_2^2}{\sigma_1^2}}$$

and the principle of superposition the solution becomes:

$$\begin{aligned} P(x,y) &= \sum_{n=1}^{\infty} c_n \sin(x\nu_n)\exp\left(-x\mu_1\sigma_1^{-2}\right) \\ &\quad \times \exp\left(-y\left(\mu_2\sigma_2^{-2} + \sigma_2^{-2}\sqrt{\mu_2^2 + \frac{\left(\nu_n^2\sigma_1^4 + \mu_1^2\right)\sigma_2^2}{\sigma_1^2}}\right)\right) \end{aligned} \tag{4.8.16}$$

Finally we will find the coefficients c_n *so that the solution satisfies the condition (4.8.9). The eigenfunctions* $u_n(x)$ *are orthogonal with respect to the weight function*

$$q(x) := \frac{2}{\sigma_1^2}e^{2x\mu_1\sigma_1^{-2}}$$

and thus, from the requirement $P(x,0)=1$ *we have*

$$\sum_{n=1}^{\infty} c_n \sin(x\nu_n)\exp(-x\mu_1\sigma_1^{-2}) = 1. \tag{4.8.17}$$

From the relation (4.8.17) we find that the coefficients c_n *are given by*

$$c_n = \frac{\int_0^b \sin(x\nu_n) exp(x\mu_1\sigma_1^{-2})dx}{\int_0^b \sin^2(x\nu_n)dx}$$

which gives

$$c_n = \frac{\nu_n\sigma_1^4 + e^{\frac{b\mu_1}{\sigma_1^2}}\left(\sin(b\nu_n)\mu_1 - \cos(b\nu_n)\nu_n\sigma_1^2\right)\sigma_1^2}{\frac{b}{2}\left(1-\frac{\sin(2b\nu_n)}{2b\nu_n}\right)\left(\nu_n^2\sigma_1^4+\mu_1^2\right)}$$

and by taking into account the relation (4.8.14) we conclude

$$c_n = \frac{\nu_n\sigma_1^4}{\frac{b}{2}\left(1-\frac{\sin(2b\nu_n)}{2b\nu_n}\right)\left(\nu_n^2\sigma_1^4+\mu_1^2\right)} \qquad n = 1, 2, 3, \ldots.$$

(II) Case with $\rho \neq 0$.

By (4.8.1) the probability of survival of the X-insurer is given by the solution of the PDE:

$$\frac{\sigma_1^2}{2}P_{xx} + \frac{\sigma_2^2}{2}P_{yy} + \rho\sigma_1\sigma_2 P_{xy} + \mu_1 P_x + \mu_2 P_y = 0 \tag{4.8.18}$$

Subject to the boundary conditions:

$$P(0,y) = 0 \tag{4.8.19}$$
$$P(x,0) = 1 \tag{4.8.20}$$
$$\frac{\partial}{\partial x}P(x,y)|_{x=b} = 0 \tag{4.8.21}$$
$$\lim_{y\to\infty} P(x,y) = 0 \tag{4.8.22}$$

Using the transformation:

$$\begin{aligned} s &= x \\ t &= x - \frac{\sigma_1}{\rho\sigma_2}y \end{aligned}$$

and the conditions (4.8.19) and (4.8.22) we find a solution of the form:

$$\begin{aligned} &P(x,y) \\ = \ &ce^{-\frac{x\mu_1}{\sigma_1^2}}\sin(x\varphi_1(\lambda))\exp\left(-\frac{\rho^2}{(1-\rho^2)\sigma_1^2}\left(x - y\frac{\sigma_1}{\rho\sigma_2}\right)R(\mu_1,\sigma_1,\mu_2,\sigma_2,\rho,\lambda)\right) \end{aligned} \tag{4.8.23}$$

where

$$\begin{aligned} & R(\mu_1,\sigma_1,\mu_2,\sigma_2,\rho,\lambda) \\ := & \left(\mu_1-\mu_2\frac{\sigma_1}{\rho\sigma_2}-sign(\rho)\sqrt{\left(\mu_1-\mu_2\frac{\sigma_1}{\rho\sigma_2}\right)^2-2\lambda\left(1-\rho^2\right)\sigma_1^2\rho^{-2}}\right)\end{aligned}$$

In order the solution to satisfy the condition (4.8.21) we must have:

$$\tan(b\nu_n)=\frac{\left(1-\rho^2\right)\nu_n\sigma_1^2\sigma_2}{\mu_1\sigma_2-\rho\mu_2\sigma_1-\rho\sqrt{\left(\mu_2\sigma_1-\rho\mu_1\sigma_2\right)^2-2\lambda\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}} \tag{4.8.24}$$

for $n=1,2,3,\ldots$.

The transcendental equation (4.8.24) can be solved numerically and for large values of n *we observe that*

$$\lim_{\lambda\to-\infty}\frac{\left(1-\rho^2\right)\nu_n\sigma_1^2\sigma_2}{\mu_1\sigma_2-\rho\mu_2\sigma_1-\rho\sqrt{\left(\mu_2\sigma_1-\rho\mu_1\sigma_2\right)^2-2\lambda\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}}=\sqrt{\frac{1-\rho^2}{\rho^2}}$$

that is for large values of n *the* ν_n *can be approximated by the solution of* $\tan(b\nu_n)=\sqrt{\frac{1-\rho^2}{\rho^2}}$ *which gives*

$$\nu_n=\frac{1}{b}\left(\arctan\left(\sqrt{\frac{1-\rho^2}{\rho^2}}\right)+n\pi\right)$$

and the eigenvalues vi

$$\lambda_n=-\frac{\nu_n^2\sigma_1^4+\mu_1^2}{2\sigma_1^2}$$

Thus the solution of (4.8.24) if of the form:

$$\begin{aligned} P(x,y) &= c_n\sin(x\nu_n)\exp(-x\mu_1\sigma_1^{-2}) \\ &\times\exp\left(-\left(x-y\frac{\sigma_1}{\rho\sigma_2}\right)\frac{\mu_1-\mu_2\frac{\sigma_1}{\rho\sigma_2}}{(\rho^{-2}-1)\sigma_1^2}\left(1-sign(\rho)\sqrt{1-\frac{2\lambda(\rho^{-2}-1)\sigma_1^2}{\left(\mu_1-\mu_2\frac{\sigma_1}{\rho\sigma_2}\right)^2}}\right)\right)\end{aligned}$$

By the principle of superposition the solution becomes:

$$\begin{aligned} P(x,y) &= \sum_{n=1}^{\infty}c_n\sin(x\nu_n)\exp(-x\mu_1\sigma_1^{-2}) \\ &\times\exp\left(-\left(x-y\frac{\sigma_1}{\rho\sigma_2}\right)\frac{(\mu_1-\mu_2\frac{\sigma_1}{\rho\sigma_2})}{(\rho^{-2}-1)\sigma_1^2}\left(1-sign(\rho)\sqrt{1-\frac{2\lambda(\rho^{-2}-1)\sigma_1^2}{\left(\mu_1-\mu_2\frac{\sigma_1}{\rho\sigma_2}\right)^2}}\right)\right)\end{aligned}$$

Unfortunately we are not able to repeat the last step as we did on the previous pde (case with $\rho=0$*) and find the coefficients* c_n *in order the solution to satisfy the condition (4.8.20), because we can not find a weight function such that the eigenfunctions* $u_n(x)$ *to be orthogonal with respect to the weight function.*

4.9 Some Variations and Extensions.

In this section we want to mention some possibilities for variations and extensions of the model we have proposed.

(I) Dividends also for the second insurer.

We can assume a model in which the second insurer (Y-insurer) gives also dividends if his reserves go beyond an upper level, but he is ruined if his reserves reaches the zero level, that is he is following one reflecting barrier policy. We can again consider for the first insurer (X-insurer) dividends policies in two cases: (a) To follow one reflecting barrier policy and (b) To follow two reflecting barriers policy. We start by considering the first case.

(a) One reflecting barrier case.

Let U_1, U_2 to be the total discounted dividends for the first and the second insurer. With analogous manner as in the previous sections one can conclude the following propositions:

- Expected value of the discounted dividends.

Proposition 4.9.1 *Let $n \in \mathbb{N}$ and functions $V_1(x, y; n)$ and $V_2(x, y; n)$ belonging in $C_b^2(\mathbb{R}^2)$ which satisfy the scaling properties (4.3.1), (4.3.5). If the functions $V_i(x, y; n)$, $i = 1, 2$ solve the PDEs :*

$$(\mathcal{L}_{(x,y)} - n\delta)\mathcal{V}_i(x, y; n) = 0, \quad i = 1, 2$$

with boundary conditions

$$\begin{aligned}
\mathcal{V}_1(0, y; n) &= 0, \quad n = 1, 2, 3, \ldots \\
\frac{\partial}{\partial x}\mathcal{V}_1(x, y; 1)|_{x=b_1} &= 1 \\
\frac{\partial}{\partial x}\mathcal{V}_1(x, y; n)|_{x=b_1} &= n\mathcal{V}_1(b_1, y; n-1), \quad n = 2, 3, \ldots \\
\mathcal{V}_1(x, 0; n) &= 0, \quad n = 1, 2, 3, \ldots \\
\frac{\partial}{\partial y}\mathcal{V}_1(x, y; n)|_{y=b_2} &= 0, \quad n = 1, 2, 3, \ldots
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{V}_2(x,0;n) &= 0, \quad n=1,2,3,\dots \\
\frac{\partial}{\partial y}\mathcal{V}_2(x,y;1)|_{y=b_2} &= 1 \\
\frac{\partial}{\partial y}\mathcal{V}_2(x,y;n)|_{y=b_2} &= n\mathcal{V}_2(x,b_2;n-1) \quad , \quad n=2,3,\dots \\
\mathcal{V}_2(0,y;n) &= 0, \quad n=1,2,3,\dots \\
\frac{\partial}{\partial x}\mathcal{V}_2(x,y;n)|_{x=b_1} &= 0, \quad n=1,2,3,\dots
\end{aligned}$$

then

$$\mathcal{V}_i(x,y;n) := E^{(x,y)}(U_i^n)$$

$i=1,2$ *and* $n \in \mathbb{N}$.

The total expected discounted dividends for the insurance as a total (that is considering the two insurers establishing one company) are:

$$\mathcal{V}(x,y) = \mathcal{V}_1(x,y) + \mathcal{V}_2(x,y)$$

where $V_i(x,y) := V_i(x,y;1)$, $i=1,2$.

- The Laplace transform of the discounted dividends.

Proposition 4.9.2 *Consider the functions* $K_1(x,y,\lambda)$ *and* $K_2(x,y,\lambda)$ *belonging in* $C_b^2(\mathbb{R}^3)$ *which satisfy the scaling properties (4.3.2), (4.3.6). If the functions* $K_i(x,y,\lambda)$, $i=1,2$ *solve the PDEs :*

$$\mathcal{L}_{(x,y)}\mathcal{K}_i(x,y,\lambda) = \lambda\delta\frac{\partial}{\partial\lambda}\mathcal{K}_i(x,y,\lambda) \quad , \ i=1,2$$

with boundary conditions

$$\begin{aligned}
\mathcal{K}_1(0,y,\lambda) &= 1 \\
\frac{\partial}{\partial x}\mathcal{K}_1(x,y,\lambda)|_{x=b_1} &= -\lambda\mathcal{K}_1(b_1,y,\lambda) \\
\mathcal{K}_1(x,0,\lambda) &= 1 \\
\frac{\partial}{\partial y}\mathcal{K}_1(x,y,\lambda)|_{y=b_2} &= 0
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{K}_2(x,0,\lambda) &= 1 \\
\frac{\partial}{\partial y}\mathcal{K}_2(x,y,\lambda)|_{y=b_2} &= -\lambda\mathcal{K}_2(x,b_2,\lambda) \\
\mathcal{K}_2(0,y,\lambda) &= 1 \\
\frac{\partial}{\partial x}\mathcal{K}_2(x,y,\lambda)|_{x=b_1} &= 0
\end{aligned}$$

then

$$\mathcal{K}_i(x, y, \lambda) := E^{(x,y)}(e^{-\lambda U_i})$$

$i = 1, 2$.

- The Laplace transform of time of ruin.

Proposition 4.9.3 *Consider the function $M(x, y, \lambda) \in C_b^2(\mathbb{R}^3)$ which satisfies the scaling properties (4.3.3),(4.3.7). If the function $M(x, y, \lambda)$ solves the PDE :*

$$\mathcal{A}_{(x,y)}\mathcal{M}(x, y, \lambda) = \lambda \mathcal{M}(x, y, \lambda)$$

with boundary conditions

$$\begin{aligned}
\mathcal{M}(0, y, \lambda) &= 1 \\
\frac{\partial}{\partial x}\mathcal{M}(x, y, \lambda)|_{x=b_1} &= 0 \\
\mathcal{M}(x, 0, \lambda) &= 1 \\
\frac{\partial}{\partial y}\mathcal{M}(x, y, \lambda)|_{y=b_2} &= 0
\end{aligned}$$

then

$$\mathcal{M}(x, y, \lambda) := E^{(x,y)}(e^{-\lambda T})$$

- The Laplace transform of the joint distribution of the time of ruin and the discounted dividends.

Proposition 4.9.4 *Consider the functions $N_1(x, y, \lambda_1, \lambda_2)$ and $N_2(x, y, \lambda_1, \lambda_2)$ belonging in $C_b^2(\mathbb{R}^4)$ which satisfy the scaling properties (4.3.4), (4.3.8). If the functions $N_i(x, y, \lambda_1, \lambda_2)$, $i = 1, 2$ solve the PDEs :*

$$\mathcal{L}_{(x,y)}\mathcal{N}_i(x, y, \lambda_1, \lambda_2) = \lambda_1 \mathcal{N}_i(x, y, \lambda_1, \lambda_2) + \delta\lambda_2 \frac{\partial}{\partial \lambda_2}\mathcal{N}_i(x, y, \lambda_1, \lambda_2) \quad , i = 1, 2$$

with boundary conditions

$$\begin{aligned}
\mathcal{N}_1(0, y, \lambda_1, \lambda_2) &= 1 \\
\frac{\partial}{\partial x}\mathcal{N}_1(x, y, \lambda_1, \lambda_2)|_{x=b_1} &= -\lambda_2 \mathcal{N}_1(x, y, \lambda_1, \lambda_2) \\
\mathcal{N}_1(x, 0, \lambda_1, \lambda_2) &= 1 \\
\frac{\partial}{\partial y}\mathcal{N}_1(x, y, \lambda_1, \lambda_2)|_{y=b_2} &= 0
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{N}_2(x,0,\lambda_1,\lambda_2) &= 1\\
\frac{\partial}{\partial y}\mathcal{N}_2(x,y,\lambda_1,\lambda_2)|_{y=b_2} &= -\lambda_2\mathcal{N}_2(x,y,\lambda_1,\lambda_2)\\
\mathcal{N}_2(0,y,\lambda_1,\lambda_2) &= 1\\
\frac{\partial}{\partial x}\mathcal{N}_2(x,y,\lambda_1,\lambda_2)|_{x=b_1} &= 0
\end{aligned}$$

then

$$\mathcal{N}_i(x,y,\lambda_1,\lambda_2) := E^{(x,y)}(e^{-\lambda_1 T-\lambda_2 U_i})$$

$i = 1, 2.$

- Survival Probability.

Proposition 4.9.5 *Consider the function $P(x,y) \in C_b^2(\mathbb{R}^+ \times \mathbb{R}^+)$. If the function $P(x,y)$ solves the PDE :*

$$\mathcal{L}_{(x,y)}\mathcal{P}(x,y) = 0$$

with boundary conditions

$$\begin{aligned}
P(0,y) &= 0\\
P(x,0) &= 1\\
\frac{\partial}{\partial x}P(x,y)|_{x=b_1} &= 0\\
\frac{\partial}{\partial y}P(x,y)|_{y=b_2} &= 0
\end{aligned}$$

then

$$P(x,y) = P(T^{z_2} < T^{z_1})$$

(b) Two reflecting barriers case.

Let $U_1^{(+)}, U_2^{(+)}$ to be the total discounted dividends for the first and the second insurer and $U_1^{(-)}$ to be the total discounted financing for the first insurer. With arguments similar to the previous sections one can conclude the following proposition.

Proposition 4.9.6 *Let $n \in \mathbb{N}$ and functions $V_1^{(+)}(x,y;n)$, $V_1^{(-)}(x,y;n)$ and $V_2^{(+)}(x,y;n)$ belonging in $C_b^2(\mathbb{R}^2)$ which satisfy the scaling properties (4.3.9),(4.3.13). If the functions $V_1^{(+)}(x,y;n)$, $V_1^{(-)}(x,y;n)$ and $V_2^{(+)}(x,y;n)$ solve the PDEs :*

$$\begin{aligned}
(\mathcal{L}_{(x,y)} - n\delta)\mathcal{V}_1^{(\pm)}(x,y;n) &= 0\\
(\mathcal{L}_{(x,y)} - n\delta)\mathcal{V}_2^{(+)}(x,y;n) &= 0
\end{aligned}$$

with boundary conditions

$$\begin{aligned}
\mathcal{V}_1^{(+)}(0,y;n) &= 0, \quad n=1,2,3,\ldots \\
\frac{\partial}{\partial x}\mathcal{V}_1^{(+)}(x,y;1)|_{x=b_1} &= 1 \\
\frac{\partial}{\partial x}\mathcal{V}_1^{(+)}(x,y;n)|_{x=b_1} &= n\mathcal{V}_1^{(+)}(b_1,y;n-1) \quad , \quad n=2,3,\ldots \\
\mathcal{V}_1^{(+)}(x,0;n) &= 0, \quad n=1,2,3,\ldots \\
\frac{\partial}{\partial y}\mathcal{V}_1^{(+)}(x,y;n)|_{y=b_2} &= 0, \quad n=1,2,3,\ldots
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial x}\mathcal{V}_1^{(-)}(x,y;n)|_{x=0} &= -n\mathcal{V}_1^{(-)}(0,y;n-1) \\
\frac{\partial}{\partial x}\mathcal{V}_1^{(-)}(x,y;n)|_{x=b_1} &= 0 \\
\mathcal{V}_1^{(-)}(x,0;n) &= 0 \\
\frac{\partial}{\partial y}\mathcal{V}_1^{(-)}(x,y;n)|_{x=b_2} &= 0 \\
n &= 1,2,3,\ldots
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{V}_2^{(+)}(x,0;n) &= 0 \ , \quad n=1,2,3,\ldots \\
\frac{\partial}{\partial y}\mathcal{V}_2^{(+)}(x,y;1)|_{y=b_2} &= 1 \\
\frac{\partial}{\partial y}\mathcal{V}_2^{(+)}(x,y;n)|_{y=b_2} &= n\mathcal{V}_2^{(+)}(x,b_2;n-1) \quad , \quad n=2,3,\ldots \\
\mathcal{V}_2^{(+)}(0,y;n) &= 0 \quad , \quad n=1,2,3,\ldots \\
\frac{\partial}{\partial x}\mathcal{V}_2^{(+)}(x,y;n)|_{x=b_1} &= 0 \quad , \quad n=1,2,3,\ldots
\end{aligned}$$

then

$$\begin{aligned}
\mathcal{V}_1^{(+)}(x,y;n) &= E^{(x,y)}((U_1^{(+)})^n) \\
\mathcal{V}_1^{(-)}(x,y;n) &= E^{(x,y)}((U_1^{(-)})^n) \\
\mathcal{V}_2^{(+)}(x,y;n) &= E^{(x,y)}((U_2^{(+)})^n)
\end{aligned}$$

(II) General Barriers.

One can consider general barriers policies and try to study the model in this set up.

(III) n dimensions.

It is interesting to extend the above models in n-dimensions. For example let us consider a model with n-insurers all of which give dividends and they follow the one reflecting barrier dividends policy. We are interested in the total expected discounted dividends until ruin occurs.

Let U_i to be the total discounted dividends for the $i = 1, \dots, n$ insurer. Working as in the previous sections one can conclude the following proposition.

Proposition 4.9.7 *Let functions $V_i(x_1, x_2, \dots, x_n)$, $i = 1, 2 \dots, n$ belonging in $C_b^2(\mathbb{R}^n)$ which satisfy the scaling properties*

(I)

$$\mathcal{V}_i(x_1, x_2, \dots, x_n) = \mathcal{V}_i(x_1 - c, x_2 - c, \dots, x_n - c)$$

for each real number $c \in (-\infty, \infty)$ and $i = 1, 2 \dots, n$.

(II)

$$\mathcal{V}_i(x_1, x_2, \dots x_n) = c\mathcal{V}_i(c^{-1}x_1, c^{-1}x_2, \dots, c^{-1}x_n)$$

for each real number $c > 0$ and $i = 1, 2 \dots, n$.

If the functions $V_i(x_1, x_2, \dots, x_n)$, $i = 1, 2 \dots, n$ solve the PDEs :

$$(\mathcal{L}_{(x_1, x_2, \dots, x_n)} - \delta)\mathcal{V}_i(x_1, x_2, \dots x_n) = 0 \quad , \ i = 1, 2, \dots, n$$

with boundary conditions

$$\begin{aligned}
\mathcal{V}_i(x_1, x_2, \dots, x_{j-1}, 0, x_{j+1} \dots, x_n) &= 0 \quad , \ i, j = 1, 2, \dots, n \\
\frac{\partial}{\partial x_i}\mathcal{V}_i(x_1, x_2, \dots, x_{i-1}, x_i, x_{i+1} \dots, x_n)|_{x_i = b_i} &= 1 \quad , \ i = 1, 2, \dots, n. \\
\frac{\partial}{\partial x_j}\mathcal{V}_i(x_1, x_2, \dots, x_{j-1}, x_j, x_{j+1} \dots, x_n)|_{x_j = bj} &= 0 \quad , \ i \neq j, \ \ i, j = 1, 2, \dots, n
\end{aligned}$$

then

$$\mathcal{V}_i(x_1, x_2, \dots x_n) := E^{(x_1, x_2, \dots, x_n)}(U_i) \quad , \ i = 1, 2 \dots, n$$

The total expected discounted dividends for the insurance as a total (that is considering the n insurers establishing one company) are:

$$\mathcal{V}(x_1, x_2, \dots x_n) = \sum_{i=1}^{n} \mathcal{V}_i(x_1, x_2, \dots x_n)$$

4.10 Conclusions.

In this chapter we worked on the situation of insurance companies cooperation. We examined the issue from the perspective of a particular insurance company. We were interested to look at parameters which are vital to the decisions of the company. Among these parameters very important role we consider to play the probability of survival in a particular cooperation and the shares that will be given to the shareholders during this cooperation. We found differential equations with appropriate boundary conditions the solution of which will give:

- The moments of the discounted dividends and the discounted financing.
- The Laplace transform of the joint distribution of the time of ruin and the discounted dividends.
- The Laplace transform of the discounted dividends.
- The Laplace transform of the time of ruin.
- The Survival probability for one of the two insurers.

We also mentioned possible ways to extend the above considerations to various other models.

Chapter 5

Applications on Insurance companies consortium.

5.1 Introduction.

In this chapter we apply the formulas of chapter 4 in the case of two insurance companies cooperation. We consider a process $X = \{X_t; t \geq 0\}$ as describing the reserves of an insurance company and we will refer to this insurance company as the first insurer or as the X-insurer. We would also like to consider another process $\{Y_t; t \geq 0\}$ as describing the reserves of another insurance company and we will refer to this insurance company as the second insurer or as the Y-insurer.

When the insurance company (first insurer or second insurer) follows dividends policy according to the de Finetti model we will refer to this insurance as de Finetti model. When the insurance company follows no dividends policy we will refer to this insurance as Lundberg model.

We consider two insurance companies in cooperation and we study two models.

(I)

$$\text{The de Finetti - Lundberg model.} \tag{5.1.1}$$

(II)

$$\text{The de Finetti - de Finetti model.} \tag{5.1.2}$$

That is, in the case (I) we consider that the first insurer follows the de Finetti model and the second insurer follows the Lundberg model and in the case (II) we consider that the first insurer follows the de Finetti model and the second insurer follows the de Finetti model.

In section 5.4 we study the situation in which an insurer, say X, has to choose to collaborate with one out of n other insurers, say Y_i, $i = 1, \ldots, n$. We concern with the problem of how the X-insurer can make the best choice in terms of maximizing his survival probability or in terms of maximizing his expected discounted dividends. We provide a method on how he can construct a policy, that is to rank his available choices from the best choice to the worst choice.

5.2 One Reflecting barrier.

In this section we study the models (5.1.1)-(5.1.2) considering that if an insurance company (first insurer or second insurer) follows the de Finetti model then it follows a dividends policy with a constant upper reflecting barrier and a constant lower absorbing barrier which is the zero constant.

5.2.1 Survival Probability for the de Finetti - de Finetti model.

By Proposition 4.9.5 in order to find the survival probability for the first insurer (X-insurer) in a de Finetti - de Finetti model we have to solve the pde:

$$\frac{\sigma_1^2}{2}P_{xx}(x,y) + \frac{\sigma_2^2}{2}P_{yy}(x,y) + \mu_1 P_x(x,y) + \mu_2 P_y(x,y) = 0 \tag{5.2.1}$$

Subject to the boundary conditions:

$$P(0,y) = 0 \tag{5.2.2}$$

$$P(x,0) = 1 \tag{5.2.3}$$

$$\frac{\partial}{\partial x}P(x,y)|_{x=b_1} = 0 \tag{5.2.4}$$

$$\frac{\partial}{\partial y}P(x,y)|_{y=b_2} = 0 \tag{5.2.5}$$

By (5.A.31)-(5.A.33), (5.A.1)-(5.A.5) and the principle of superposition the solution is:

$$\begin{aligned} P(x,y) &= \sum_{n=0}^{\infty} \exp\left(\frac{-x\mu_1}{\sigma_1^2}\right) \sin\left(\frac{x\sqrt{-\mu_1^2 - 2\lambda_n\sigma_1^2}}{\sigma_1^2}\right) \\ &\times \left(c_n \exp\left(\frac{y\left(-\mu_2 - \sqrt{\mu_2^2 - 2\lambda_n\sigma_2^2}\right)}{\sigma_2^2}\right) + d_n \exp\left(\frac{y\left(-\mu_2 + \sqrt{\mu_2^2 - 2\lambda_n\sigma_2^2}\right)}{\sigma_2^2}\right)\right) \end{aligned}$$

where λ_n are calculated by the solution of (5.A.3) with b replaced by b_1 and for large values are approximated by (5.A.4). By the condition (5.2.5) we have:

$$\frac{\partial}{\partial y}P(x,y)|_{x=b_2} = 0 \Longrightarrow$$

$$d_n\left(\sqrt{\mu_2^2-2\lambda_n\sigma_2^2}-\mu_2\right)\exp\left(\frac{2b_2\sqrt{\mu_2^2-2\lambda_n\sigma_2^2}}{\sigma_2^2}\right)-c_n\left(\mu_2+\sqrt{\mu_2^2-2\lambda_n\sigma_2^2}\right) \quad = \quad 0 \tag{5.2.6}$$

Finally by the condition (5.2.3) and taking into account that the eigenfunctions $u_n(x)$ are orthogonal with respect to the weight function

$$q(x):=\frac{2}{\sigma_1^2}\exp\left(\frac{2x\mu_1}{\sigma_1^2}\right)$$

we have $P(x,0)=1$

$$\sum_{n=0}^{\infty}(c_n+d_n)\exp\left(\frac{-x\mu_1}{\sigma_1^2}\right)\sin\left(\frac{x\sqrt{-\mu_1^2-2\lambda_n\sigma_1^2}}{\sigma_1^2}\right)=1$$

which gives

$$c_n+d_n \quad = \quad \frac{\int\limits_0^b \exp\left(\frac{-x\mu_1}{\sigma_1^2}\right)\sin\left(\frac{x\sqrt{-\mu_1^2-2\lambda_n\sigma_1^2}}{\sigma_1^2}\right)q(x)dx}{\int\limits_0^b\left(\exp\left(\frac{-x\mu_1}{\sigma_1^2}\right)\sin\left(\frac{x\sqrt{-\mu_1^2-2\lambda_n\sigma_1^2}}{\sigma_1^2}\right)\right)^2 q(x)dx}\Longrightarrow$$

$$c_n+d_n \quad = \quad \frac{\int\limits_0^b \exp\left(\frac{x\mu_1}{\sigma_1^2}\right)\sin\left(\frac{x\sqrt{-\mu_1^2-2\lambda_n\sigma_1^2}}{\sigma_1^2}\right)dx}{\int\limits_0^b\left(\sin\left(\frac{x\sqrt{-\mu_1^2-2\lambda_n\sigma_1^2}}{\sigma_1^2}\right)\right)^2 dx}\Longrightarrow$$

$$c_n+d_n=-\frac{2\left(\sqrt{-\mu_1^2-2\lambda_n\sigma_1^2}\,(1-A_1(\mu_1,\sigma_1,b,\lambda_n))+A_2(\mu_1,\sigma_1,b,\lambda_n)\right)}{A_3(\mu_1,\sigma_1,b,\lambda_n)} \tag{5.2.7}$$

where

$$A_1(\mu_1,\sigma_1,b,\lambda_n) \quad := \quad \exp\left(\frac{b\mu_1}{\sigma_1^2}\right)\cos\left(\frac{b\sqrt{-\mu_1^2-2\lambda_n\sigma_1^2}}{\sigma_1^2}\right)$$

$$A_2(\mu_1,\sigma_1,b,\lambda_n) \quad : \quad =\exp\left(\frac{b\mu_1}{\sigma_1^2}\right)\sin\left(\frac{b\sqrt{-\mu_1^2-2\lambda_n\sigma_1^2}}{\sigma_1^2}\right)\mu_1$$

$$A_3(\mu_1,\sigma_1,b,\lambda_n) \quad : \quad =2b\lambda_n-\frac{\lambda_n\sigma_1^2}{\sqrt{-\mu_1^2-2\lambda_n\sigma_1^2}}\sin\left(\frac{2b\sqrt{-\mu_1^2-2\lambda_n\sigma_1^2}}{\sigma_1^2}\right)$$

5.2.2 Expected Discounted Dividends in the case of the De Finetti -Lundberg model.

By Proposition 4.4.1 in order to find the Expected Discounted Dividends for the first insurer (X-insurer) in a de Finetti - Lundberg model we have to solve the pde

$$\frac{\sigma_1^2}{2}\mathcal{V}_{xx}(x,y) + \frac{\sigma_2^2}{2}\mathcal{V}_{yy}(x,y) + \mu_1\mathcal{V}_x(x,y) + \mu_2\mathcal{V}_y(x,y) = \delta\mathcal{V}(x,y) \tag{5.2.8}$$

subject to the boundary conditions:

$$\begin{aligned} \mathcal{V}(0,y) &= 0 && (5.2.9)\\ \frac{\partial}{\partial x}\mathcal{V}(x,y)|_{x=b_1} &= 1 && (5.2.10)\\ \mathcal{V}(x,0) &= 0 && (5.2.11)\\ \mathcal{V}(x,\infty) &:= \lim_{y\to\infty}\mathcal{V}(x,y) = \mathcal{V}(x) && (5.2.12) \end{aligned}$$

- Step 1.

  We consider the ODE

$$\frac{\sigma_1^2}{2}\mathcal{V}_{xx}(x) + \mu_1\mathcal{V}_x(x) = \delta\mathcal{V}(x) \tag{5.2.13}$$

with boundary conditions:

$$\begin{aligned} \mathcal{V}(0) &= 0 && (5.2.14)\\ \frac{\partial}{\partial x}\mathcal{V}(x)|_{x=b} &= 1 && (5.2.15) \end{aligned}$$

  The solution of (5.2.13)–(5.2.15) is

$$\mathcal{V}(x) = \exp\left(\frac{(b-x)\left(\mu_1 + \sqrt{\mu_1^2 + 2\delta\sigma_1^2}\right)}{\sigma_1^2}\right)\frac{\widetilde{r}_1(x;\mu_1,\sigma_1,0,1,\delta)}{\widetilde{r}_2(b;\mu_1,\sigma_1,0,1,\delta)} \tag{5.2.16}$$

where $\widetilde{r}_1(x;\mu_1,\sigma_1,0,1,\delta)$ and $\widetilde{r}_2(b;\mu_1,\sigma_1,0,1,\delta)$ are given by (5.A.10) and (5.A.11) respectively.

- Step 2.

  Now it is enough to solve the pde (5.2.8) with boundary conditions:

$$\begin{aligned} \widehat{\mathcal{V}}(0,y) &= 0 && (5.2.17)\\ \frac{\partial}{\partial x}\widehat{\mathcal{V}}(x,y)|_{x=b} &= 0 && (5.2.18)\\ \widehat{\mathcal{V}}(x,0) &= -\mathcal{V}(x) && (5.2.19)\\ \widehat{\mathcal{V}}(x,\infty) &:= \lim_{y\to\infty}\widehat{\mathcal{V}}(x,y) = 0 && (5.2.20) \end{aligned}$$

By (5.A.31)-(5.A.33), (5.A.1)-(5.A.5), (5.2.20) and the principle of superposition the solution is:

$$\begin{aligned}\widehat{\mathcal{V}}(x,y) &= \sum_{n=0}^{\infty} c_n \exp\left(\frac{-x\mu_1}{\sigma_1^2}\right) \sin\left(\frac{x\sqrt{-\mu_1^2-2\lambda_n\sigma_1^2}}{\sigma_1^2}\right) \\ &\quad \times \exp\left(\frac{y\left(-\mu_2-\sqrt{\mu_2^2-2(\lambda_n-\delta)\sigma_2^2}\right)}{\sigma_2^2}\right)\end{aligned}$$

where the eigenvalues λ_n are given by (5.A.3), (5.A.4). Finally we will find the coefficients c_n in order the solution to satisfy the condition (5.2.19). We have:

$$\begin{aligned}\widehat{\mathcal{V}}(x,0) &= -\mathcal{V}(x) \Longrightarrow \\ \sum_{n=0}^{\infty} c_n \exp\left(\frac{-x\mu_1}{\sigma_1^2}\right) \sin\left(\frac{x\sqrt{-\mu_1^2-2\lambda_n\sigma_1^2}}{\sigma_1^2}\right) &= -\mathcal{V}(x) \Longrightarrow\end{aligned}$$

$$c_n = -\frac{\int\limits_0^b V(x)\exp\left(\frac{x\mu_1}{\sigma_1^2}\right)\sin\left(\frac{x\sqrt{-\mu_1^2-2\lambda_n\sigma_1^2}}{\sigma_1^2}\right)dx}{\int\limits_0^b \left(\sin\left(\frac{x\sqrt{-\mu_1^2-2\lambda_n\sigma_1^2}}{\sigma_1^2}\right)\right)^2 dx}$$

Thus the solution of the initial problem (5.2.8)-(5.2.12) is:

$$\mathcal{V}(x,y) = \widehat{\mathcal{V}}(x,y) + \mathcal{V}(x) \Longrightarrow$$

$$\begin{aligned}\mathcal{V}(x,y) &= \sum_{n=0}^{\infty} c_n \exp\left(\frac{-x\mu_1}{\sigma_1^2}\right) \sin\left(\frac{x\sqrt{-\mu_1^2-2\lambda_n\sigma_1^2}}{\sigma_1^2}\right) \\ &\quad \times \exp\left(\frac{y\left(-\mu_2-\sqrt{\mu_2^2-2(\lambda_n-\delta)\sigma_2^2}\right)}{\sigma_2^2}\right) + \mathcal{V}(x)\end{aligned}$$

Remark 5.2.1 *The quantity $V(x,y)-V(x)$ can be interpreted as losses on dividends for the first insurer (X-insurer) due to the presence of the second insurer (Y-insurer or Lundberg), that is the X-insurer could have earned $V(x)-V(x,y)$ more dividends if he has not cooperated with the Y-insurer.*

5.2.3 Expected Discounted Dividends for the de Finetti - de Finetti model.

By Proposition 4.9.1 in order to find the Expected Discounted Dividends for the first insurer (X-insurer) in a de Finetti - de Finetti model we have to solve the PDE:

$$\frac{\sigma_1^2}{2}\mathcal{V}_{1xx}(x,y)+\frac{\sigma_2^2}{2}\mathcal{V}_{1yy}(x,y)+\mu_1\mathcal{V}_{1x}(x,y)+\mu_2\mathcal{V}_{1y}(x,y)=\delta\mathcal{V}_1(x,y) \tag{5.2.21}$$

subject to the boundary conditions:

$$\mathcal{V}_1(0,y) = 0 \tag{5.2.22}$$
$$\frac{\partial}{\partial x}\mathcal{V}_1(x,y)|_{x=b_1} = 1 \tag{5.2.23}$$
$$\mathcal{V}_1(x,0) = 0 \tag{5.2.24}$$
$$\frac{\partial}{\partial y}\mathcal{V}_1(x,y)|_{y=b_2} = 0 \tag{5.2.25}$$

By (5.A.31)-(5.A.33), (5.A.1)-(5.A.5), (5.A.7)-(5.A.9), (5.2.21) and the principle of superposition the solution is:

$$\begin{aligned}\mathcal{V}_1(x,y) &= \sum_{n=0}^{\infty}\exp\left(\frac{(b_1-x)\left(\mu_1+\sqrt{\mu_1^2+2\lambda_n\sigma_1^2}\right)}{\sigma_1^2}\right)\frac{\widetilde{r}_1(x;\mu_1,\sigma_1,\lambda_n,0,0)}{\widetilde{r}_2(b_1;\mu_1,\sigma_1,\lambda_n,0,0)}\\ &\times\ \exp\left(\frac{-y\mu_2}{\sigma_2^2}\right)\sin\left(\frac{y\sqrt{-\mu_2^2-2(\lambda_n-\delta)\sigma_2^2}}{\sigma_2^2}\right)\end{aligned}$$

where $\widetilde{r}_1(x;\mu_1,\sigma_1,\lambda_n,0,0)$ and $\widetilde{r}_2(b_1;\mu_1,\sigma_1,\lambda_n,0,0)$ are given by (5.A.10) and (5.A.11) respectively and the eigenvalues λ_n are calculated by the solution of (5.A.3) (with b replaced by b_2, μ_1 by μ_2, σ_1 by σ_2) and for large values of n are approximated by (5.A.4).

Working similarly we can find that the expected discounted dividends for the second insurer (Y-insurer) are

$$\begin{aligned}\mathcal{V}_2(x,y) &= \sum_{n=0}^{\infty}\exp\left(\frac{(b_2-y)\left(\mu_2+\sqrt{\mu_2^2+2\lambda_n\sigma_2^2}\right)}{\sigma_2^2}\right)\frac{\widetilde{r}_1(y;\mu_2,\sigma_2,\lambda_n,0,0)}{\widetilde{r}_2(b_2;\mu_2,\sigma_2,\lambda_n,0,0)}\times\\ &\times\ \exp\left(\frac{-x\mu_1}{\sigma_1^2}\right)\sin\left(\frac{x\sqrt{-\mu_1^2-2(\lambda_n-\delta)\sigma_1^2}}{\sigma_1^2}\right)\end{aligned}$$

where $\widetilde{r}_1(y;\mu_2,\sigma_2,\lambda_n,0,0)$ and $\widetilde{r}_2(b_2;\mu_2,\sigma_2,\lambda_n,0,0)$ are given by (5.A.10) and (5.A.11) respectively and the eigenvalues λ_n are calculated by the solution of (5.A.3) (with b replaced by b_1) and for large values of n are approximated by (5.A.4).

5.2.4 Moments of the Discounted Dividends in the case of the de Finetti - Lundberg model.

By Proposition 4.4.1 in order to find the Expected Discounted Dividends for the first insurer (X-insurer) in a de Finetti - Lundberg model we have to solve the PDE:

$$\frac{\sigma_1^2}{2}\mathcal{V}_{xx}(x,y;k)+\frac{\sigma_2^2}{2}\mathcal{V}_{yy}(x,y;k)+\mu_1\mathcal{V}_x(x,y;k)+\mu_2\mathcal{V}_y(x,y;k)=k\delta\mathcal{V}(x,y;k) \tag{5.2.26}$$

subject to the boundary conditions:

$$\mathcal{V}(0,y;k) = 0 \tag{5.2.27}$$

$$\frac{\partial}{\partial x}\mathcal{V}(x,y;k)|_{x=b} = k\mathcal{V}(b,y;k-1) \tag{5.2.28}$$

$$\mathcal{V}(x,0;k) = 0 \tag{5.2.29}$$

$$\mathcal{V}(x,\infty;k) := \lim_{y\to\infty}\mathcal{V}(x,y;k)=\mathcal{V}(x;k) \tag{5.2.30}$$

- Step 1.

We consider the ODE

$$\frac{\sigma_1^2}{2}\mathcal{V}_{xx}(x;k)+\mu_1\mathcal{V}_x(x;k)=k\delta\mathcal{V}(x;k) \tag{5.2.31}$$

with boundary conditions

$$\mathcal{V}(0;k) = 0 \tag{5.2.32}$$

$$\frac{\partial}{\partial x}\mathcal{V}(x;k)|_{x=b} = k\mathcal{V}(b;k-1) \tag{5.2.33}$$

The solution of (5.2.31)-(5.2.33) is

$$\mathcal{V}(x;k)=\exp\left(\frac{(b-x)\left(\mu_1+\sqrt{\mu_1^2+2k\delta\sigma_1^2}\right)}{\sigma_1^2}\right)\frac{\widetilde{r}_1(x;\mu_1,\sigma_1,0,k,\delta)}{\widetilde{r}_2(b;\mu_1,\sigma_1,0,k,\delta)}k\mathcal{V}(b;k-1)$$

where $\widetilde{r}_1(x;\mu_1,\sigma_1,0,k,\delta)$ and $\widetilde{r}_2(b;\mu_1,\sigma_1,0,k,\delta)$ are given by (5.A.10) and (5.A.11) respectively.

- Step 2.

In this step we will solve the pde (5.2.26) with boundary conditions

$$\mathcal{W}(0,y;k) = 0 \tag{5.2.34}$$

$$\frac{\partial}{\partial x}\mathcal{W}(x,y;k)|_{x=b} = k\widehat{\mathcal{V}}(b,y;k-1) \tag{5.2.35}$$

$$\mathcal{W}(x,\infty;k) := \lim_{y\to\infty}\mathcal{W}(x,y;k)=0 \tag{5.2.36}$$

By (5.A.31)-(5.A.33) and (5.2.34)- (5.2.36) we consider a solution of the form:

$$\begin{aligned}
&\mathcal{W}(x,y;k)\\
=\ &\sum_{n=0}^{\infty} c^*_{k,n}\left(-\exp\left(\frac{x\left(-\mu_1-\sqrt{\mu_1^2+2\sigma_1^2\lambda^*_{k,n}}\right)}{\sigma_1^2}\right)+\right.\\
&\left.+\exp\left(\frac{x\left(-\mu_1+\sqrt{\mu_1^2+2\sigma_1^2\lambda^*_{k,n}}\right)}{\sigma_1^2}\right)\right)\exp\left(\frac{y\left(-\mu_2-\sqrt{\mu_2^2-2(\lambda^*_{k,n}-k\delta)\sigma_2^2}\right)}{\sigma_2^2}\right)
\end{aligned}$$

where $\lambda^*_{k,n}$ are the separation of variables constants. Now because of the condition (5.2.35) we take

$$\lambda^*_{k,n}=\lambda_{k-1,n}+\delta$$

and so we have

$$\frac{\partial}{\partial x}\mathcal{W}(x,y;k)|_{x=b}=k\widehat{\mathcal{V}}(b,y;k-1)\Longrightarrow$$

which after some algebra leads to

$$c^*_{k,n}=kc_{k-1,n}\frac{\sin\left(\frac{b\sqrt{-\mu_1^2-2\lambda_n\sigma_1^2}}{\sigma_1^2}\right)}{\widetilde{\Lambda}(b;\mu_1,\sigma_1,\lambda_{k-1,n},\delta)}\sigma_1^2$$

where

$$\begin{aligned}
&\widetilde{\Lambda}(b;\mu_1,\sigma_1,\lambda_{k-1,n},\delta)\\
:=\ &2\cosh\left(\frac{b\sqrt{\mu_1^2+2\sigma_1^2(\lambda_{k-1,n}+\delta)}}{\sigma_1^2}\right)\sqrt{\mu_1^2+2\sigma_1^2(\lambda_{k-1,n}+\delta)}-\\
&-2\sinh\left(\frac{b\sqrt{\mu_1^2+2\sigma_1^2(\lambda_{k-1,n}+\delta)}}{\sigma_1^2}\right)\mu_1
\end{aligned}$$

- Step 3.

Now it is enough to solve the pde (5.2.26) with boundary conditions

$$\begin{aligned}
\widehat{\mathcal{V}}(0,y;k) &= 0 && (5.2.37)\\
\frac{\partial}{\partial x}\widehat{\mathcal{V}}(x,y;k)|_{x=b} &= 0 && (5.2.38)\\
\widehat{\mathcal{V}}(x,0;k) &= -\mathcal{V}(x;k)-\mathcal{W}(x,0;k) && (5.2.39)\\
\widehat{\mathcal{V}}(x,\infty;k) &:= \lim_{y\to\infty}\widehat{\mathcal{V}}(x,y;k)=0 && (5.2.40)
\end{aligned}$$

By (5.A.31)-(5.A.33) , (5.A.1)-(5.A.5) , (5.2.37), (5.2.40) and the principle of superposition the solution is

$$\begin{aligned}\widehat{\mathcal{V}}(x,y;k) &= \sum_{n=0}^{\infty} c_{k,n} \exp\left(\frac{-x\mu_1}{\sigma_1^2}\right) \sin\left(\frac{x\sqrt{-\mu_1^2-2\lambda_{k,n}\sigma_1^2}}{\sigma_1^2}\right) \\ &\quad \times \exp\left(\frac{y\left(-\mu_2-\sqrt{\mu_2^2-2(\lambda_{k,n}-k\delta)\sigma_2^2}\right)}{\sigma_2^2}\right)\end{aligned}$$

where the eigenvalues $\lambda_{k,n}$ are calculated by the solution of (5.A.3) and for large values of n are approximated by (5.A.4). Finally we will find the coefficients $c_{k,n}$ in order the solution to satisfy the condition (5.2.39). We have

$$\begin{aligned}\widehat{\mathcal{V}}(x,0;k) &= -\mathcal{V}(x;k)-\mathcal{W}(x,0;k) \Longrightarrow \\ \sum_{n=0}^{\infty} c_{k,n} \exp\left(\frac{-x\mu_1}{\sigma_1^2}\right) \sin\left(\frac{x\sqrt{-\mu_1^2-2\lambda_{k,n}\sigma_1^2}}{\sigma_1^2}\right) &= -\mathcal{V}^{+}(x;k)-\mathcal{W}^{+}(x,0;k) \Longrightarrow\end{aligned}$$

$$c_{k,n} = -\frac{\int_0^b \left(\mathcal{V}(x)+\mathcal{W}(x,0;k)\right) \exp\left(\frac{x\mu_1}{\sigma_1^2}\right) \sin\left(\frac{x\sqrt{-\mu_1^2-2\lambda_{k,n}\sigma_1^2}}{\sigma_1^2}\right) dx}{\int_0^b \left(\sin\left(\frac{x\sqrt{-\mu_1^2-2\lambda_{k,n}\sigma_1^2}}{\sigma_1^2}\right)\right)^2 dx}$$

Thus the solution of the initial problem (5.2.26)-(5.2.30) is

$$\mathcal{V}(x,y;k) = \widehat{\mathcal{V}}(x,y;k)+\mathcal{W}(x,y;k)+\mathcal{V}(x;k)$$

5.2.5 Moments of the Discounted Dividends for the de Finetti - de Finetti model.

By Proposition 4.9.1 in order to find the Expected Discounted Moments for the first insurer (X-insurer) in a de Finetti - de Finetti model we have to solve the PDE

$$\frac{\sigma_1^2}{2}\mathcal{V}_{1xx}(x,y;k)+\frac{\sigma_2^2}{2}\mathcal{V}_{1yy}(x,y;k)+\mu_1\mathcal{V}_{1x}(x,y;k)+\mu_2\mathcal{V}_{1y}(x,y;k) = k\delta\mathcal{V}_1(x,y;k) \tag{5.2.41}$$

subject to the boundary conditions

$$\mathcal{V}_1(0,y;k) = 0 \tag{5.2.42}$$

$$\frac{\partial}{\partial x}\mathcal{V}_1(x,y;k)|_{x=b_1} = k\mathcal{V}_1(b_1,y;k-1) \tag{5.2.43}$$

$$\mathcal{V}_1(x,0;k) = 0 \tag{5.2.44}$$

$$\frac{\partial}{\partial y}\mathcal{V}_1(x,y;k)|_{y=b_2} = 0 \tag{5.2.45}$$

By (5.A.31)-(5.A.33) and (5.2.42)- (5.2.44) the solution is:

$$\mathcal{V}_1(x,y;k)$$
$$= \sum_{n=0}^{\infty} c_{k,n} \times \exp\left(\frac{-y\mu_2}{\sigma_2^2}\right) \sin\left(\frac{y\sqrt{-\mu_2^2 - 2(\lambda_{k,n} - k\delta)\sigma_2^2}}{\sigma_2^2}\right) \times$$
$$\times \left(-\exp\left(\frac{x\left(-\mu_1 - \sqrt{\mu_1^2 + 2\sigma_1^2\lambda_{k,n}}\right)}{\sigma_1^2}\right) + \exp\left(\frac{x\left(\sqrt{\mu_1^2 + 2\sigma_1^2\lambda_{k,n}} - \mu_1\right)}{\sigma_1^2}\right)\right)$$

where the eigenvalues $\lambda_{k,n}$ are calculated by the solution of (5.A.3) (with b replaced by b_2 , μ_1 by μ_2, σ_1 by σ_2) and for large values of n are approximated by (5.A.4). By the condition (5.2.43) and using (5.A.6) we have

$$\frac{\partial}{\partial x}\mathcal{V}_1(x,y;k)|_{x=b_1} = k\mathcal{V}_1(b_1,y;k-1)$$

which give

$$c_{k,n}\exp\left(-\frac{b_1\mu_1}{\sigma_1^2}\right)\frac{\widetilde{\gamma}_2(b_1;\mu_1,\sigma_1,\lambda_{k,n})}{\sigma_1^2} = kc_{k-1,n}\widetilde{\gamma}_1(b_1;\mu_1,\sigma_1,\lambda_{k-1,n})$$

or

$$c_{k,n} = kc_{k-1,n}\exp\left(\frac{b_1\mu_1}{\sigma_1^2}\right)\frac{\widetilde{\gamma}_1(b_1;\mu_1,\sigma_1,\lambda_{k-1,n})}{\widetilde{\gamma}_2(b_1;\mu_1,\sigma_1,\lambda_{k,n})}\sigma_1^2$$

where

$$\widetilde{\gamma}_1(b_1;\mu_1,\sigma_1,\lambda_{k-1,n}) \quad : \quad = -\exp\left(\frac{b_1\left(-\mu_1 - \sqrt{\mu_1^2 + 2\sigma_1^2\lambda_{k-1,n}}\right)}{\sigma_1^2}\right) + $$
$$+\exp\left(\frac{b_1\left(\sqrt{\mu_1^2 + 2\sigma_1^2\lambda_{k-1,n}} - \mu_1\right)}{\sigma_1^2}\right) \tag{5.2.46}$$

$$\widetilde{\gamma}_2(b_1;\mu_1,\sigma_1,\lambda_{k,n}) \quad : \quad = 2\cosh\left(\frac{b_1\sqrt{\mu_1^2 + 2\sigma_1^2\lambda_{k,n}}}{\sigma_1^2}\right)\sqrt{\mu_1^2 + 2\sigma_1^2\lambda_{k,n}} -$$
$$-2\sinh\left(\frac{b_1\sqrt{\mu_1^2 + 2\sigma_1^2\lambda_{k,n}}}{\sigma_1^2}\right)\mu_1 \tag{5.2.47}$$

Similar by the Proposition 4.9.1 in order to find the Expected Discounted Moments for the second insurer (Y-insurer) in a de Finetti - de Finetti model we have to solve the pde:

$$\frac{\sigma_1^2}{2}\mathcal{V}_{2xx}(x,y;k) + \frac{\sigma_2^2}{2}\mathcal{V}_{2yy}(x,y;k) + \mu_1\mathcal{V}_{2x}(x,y;k) + \mu_2\mathcal{V}_{2y}(x,y;k) = k\delta\mathcal{V}_2(x,y;k) \tag{5.2.48}$$

subject to the boundary conditions:

$$\mathcal{V}_2(x,0;k) = 0 \quad (5.2.49)$$

$$\frac{\partial}{\partial y}\mathcal{V}_2(x,y;k)|_{y=b_2} = k\mathcal{V}_2(x,b_2;k-1) \quad (5.2.50)$$

$$\mathcal{V}_2(0,y;k) = 0 \quad (5.2.51)$$

$$\frac{\partial}{\partial x}\mathcal{V}_2(x,y;k)|_{x=b_1} = 0 \quad (5.2.52)$$

Working as before we find that the discounted moments for the second insurer (Y-insurer) in a de Finetti - de Finetti model are given by

$$\begin{aligned}
&\mathcal{V}_2(x,y;k) \\
= &\sum_{n=0}^{\infty} c_{k,n} \exp\left(\frac{-x\mu_1}{\sigma_1^2}\right) \sin\left(\frac{x\sqrt{-\mu_1^2 - 2(\lambda_{k,n} - k\delta)\sigma_1^2}}{\sigma_1^2}\right) \\
&\times \left(-\exp\left(\frac{y\left(-\mu_2 - \sqrt{\mu_2^2 + 2\sigma_2^2\lambda_{k,n}}\right)}{\sigma_2^2}\right) + \exp\left(\frac{y\left(-\mu_2 + \sqrt{\mu_2^2 + 2\sigma_2^2\lambda_{k,n}}\right)}{\sigma_2^2}\right)\right)
\end{aligned}$$

with

$$c_{k,n} = kc_{k-1,n} \exp\left(\frac{b_2\mu_2}{\sigma_2^2}\right) \frac{\widetilde{\gamma}_1(b_2;\mu_2,\sigma_2,\lambda_{k-1,n})}{\widetilde{\gamma}_2(b_2;\mu_2,\sigma_2,\lambda_{k,n})}\sigma_2^2$$

where $\widetilde{\gamma}_1(b_2;\mu_2,\sigma_2,\lambda_{k-1,n})$ and $\widetilde{\gamma}_2(b_2;\mu_2,\sigma_2,\lambda_{k,n})$ are given by (5.2.46),(5.2.47) respectively and the eigenvalues $\lambda_{k,n}$ are calculated by the solution of (5.A.3) (with b replaced by b_1) and for large values of n are approximated by (5.A.4).

5.2.6 The Laplace Transform of the discounted dividends.

The Laplace transform of the discounted dividends can be found from:

$$\mathcal{K}(x,y,\lambda) = 1 + \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\mathcal{V}(x,y;k)$$

5.2.7 The Laplace transform of the time of ruin for the De Finetti - Lundeberg model.

By Proposition 4.7.1 in order to find the Laplace transform of the time of ruin in a de Finetti - Lundberg model we have to solve the pde:

$$\frac{\sigma_1^2}{2}\mathcal{M}_{xx}(x,y,\lambda) + \frac{\sigma_2^2}{2}\mathcal{M}_{yy}(x,y,\lambda) + \mu_1\mathcal{M}_x(x,y,\lambda) + \mu_2\mathcal{M}_y(x,y,\lambda) = \lambda\mathcal{M}(x,y,\lambda) \quad (5.2.53)$$

subject to the boundary conditions

$$\mathcal{M}(0,y,\lambda) = 1 \tag{5.2.54}$$

$$\frac{\partial}{\partial x}\mathcal{M}(x,y,\lambda)|_{x=b} = 0 \tag{5.2.55}$$

$$\mathcal{M}(x,0,\lambda) = 1 \tag{5.2.56}$$

$$\mathcal{M}(x,\infty,\lambda) := \lim_{y\to\infty}\mathcal{M}(x,y,\lambda) = \mathcal{M}(x,\lambda) \tag{5.2.57}$$

- Step 1.

We consider the ODE

$$\frac{\sigma_1^2}{2}\mathcal{M}_{xx}(x,\lambda) + \mu_1\mathcal{M}_x(x,\lambda) = \lambda\mathcal{M}(x,\lambda) \tag{5.2.58}$$

with boundary conditions

$$\mathcal{M}(0,\lambda) = 1 \tag{5.2.59}$$

$$\frac{\partial}{\partial x}\mathcal{M}(x,\lambda)|_{x=b} = 0 \tag{5.2.60}$$

The solution of (5.2.58)-(5.2.60) is

$$\mathcal{M}(x,\lambda) = \frac{\exp\left(-\frac{x\mu_1}{\sigma_1^2}\right)\left(\sqrt{\mu_1^2+2\lambda\sigma_1^2}+\mu_1\tanh\left(\frac{(b-x)\sqrt{\mu_1^2+2\lambda\sigma_1^2}}{\sigma_1^2}\right)\right)}{\sqrt{\mu_1^2+2\lambda\sigma_1^2}-\mu_1\tanh\left(\frac{b\sqrt{\mu_1^2+2\lambda\sigma_1^2}}{\sigma_1^2}\right)} \tag{5.2.61}$$

- Step 2.

Now it is enough to solve the pde (5.2.53) with boundary conditions

$$\widehat{\mathcal{M}}(0,y,\lambda) = 0 \tag{5.2.62}$$

$$\frac{\partial}{\partial x}\widehat{\mathcal{M}}(x,y,\lambda)|_{x=b} = 0 \tag{5.2.63}$$

$$\widehat{\mathcal{M}}(x,0,\lambda) = 1-\mathcal{M}(x,\lambda) \tag{5.2.64}$$

$$\widehat{\mathcal{M}}(x,\infty,\lambda) := \lim_{y\to\infty}\widehat{\mathcal{M}}(x,y,\lambda) = 0 \tag{5.2.65}$$

By (5.A.31)-(5.A.33), (5.A.1)-(5.A.5), (5.2.65) and the principle of superposition the solution is:

$$\begin{aligned}
&\widehat{\mathcal{M}}(x,y,\lambda)\\
=\ &\sum_{n=0}^{\infty} c_n \exp\left(\frac{-x\mu_1}{\sigma_1^2}\right)\sin\left(\frac{x\sqrt{-\mu_1^2-2\lambda_n^*\sigma_1^2}}{\sigma_1^2}\right)\\
&\times\exp\left(\frac{y\left(-\mu_2-\sqrt{\mu_2^2-2(\lambda_n^*-\lambda)\sigma_2^2}\right)}{\sigma_2^2}\right)
\end{aligned}$$

and the eigenvalues λ_n^* are calculated by the solution of (5.A.3) (with b replaced by b_1) and for large values of n are approximated by (5.A.4). Finally we will find the coefficients c_n in order the solution to satisfy the condition (5.2.64). We have

$$\begin{aligned}
\widehat{\mathcal{M}}(x,0,\lambda) &= 1-\mathcal{M}(x,\lambda) \Longrightarrow\\
\sum_{n=0}^{\infty} c_n \exp\left(\frac{-x\mu_1}{\sigma_1^2}\right)\sin\left(\frac{x\sqrt{-\mu_1^2-2\lambda_n^*\sigma_1^2}}{\sigma_1^2}\right) &= 1-\mathcal{M}(x,\lambda) \Longrightarrow
\end{aligned}$$

$$c_n = \frac{\int\limits_0^b (1-\mathcal{M}(x,\lambda))\exp\left(\frac{x\mu_1}{\sigma_1^2}\right)\sin\left(\frac{x\sqrt{-\mu_1^2-2\lambda_n^*\sigma_1^2}}{\sigma_1^2}\right)dx}{\int\limits_0^b \left(\sin\left(\frac{x\sqrt{-\mu_1^2-2\lambda_n^*\sigma_1^2}}{\sigma_1^2}\right)\right)^2 dx}$$

Thus the solution of the initial problem (5.2.8)-(5.2.12) is

$$\mathcal{M}(x,y,\lambda) = \widehat{\mathcal{M}}(x,y,\lambda) + \mathcal{M}(x,\lambda) \Longrightarrow$$

$$\begin{aligned}
\mathcal{M}(x,y,\lambda) =\ &\sum_{n=0}^{\infty} c_n \exp\left(\frac{-x\mu_1}{\sigma_1^2}\right)\sin\left(\frac{x\sqrt{-\mu_1^2-2\lambda_n^*\sigma_1^2}}{\sigma_1^2}\right)\\
&\times\exp\left(\frac{y\left(-\mu_2-\sqrt{\mu_2^2-2(\lambda_n^*-\lambda)\sigma_2^2}\right)}{\sigma_2^2}\right)+\mathcal{M}(x,\lambda)
\end{aligned}$$

5.2.8 The Laplace transform of the time of ruin for the de Finetti - de Finetti model.

By Proposition 4.9.3 in order to find the Laplace transform of the time of ruin in a de Finetti - de Finetti model we have to solve the pde:

$$\frac{\sigma_1^2}{2}\mathcal{M}_{xx}(x,y,\lambda)+\frac{\sigma_2^2}{2}\mathcal{M}_{yy}(x,y,\lambda)+\mu_1\mathcal{M}_x(x,y,\lambda)+\mu_2\mathcal{M}_y(x,y,\lambda)=\lambda\mathcal{M}(x,y,\lambda) \tag{5.2.66}$$

subject to the boundary conditions

$$\mathcal{M}(0,y,\lambda) = 1 \tag{5.2.67}$$
$$\frac{\partial}{\partial x}\mathcal{M}(x,y,\lambda)|_{x=b_1} = 0 \tag{5.2.68}$$
$$\mathcal{M}(x,0,\lambda) = 1 \tag{5.2.69}$$
$$\frac{\partial}{\partial y}\mathcal{M}(x,y,\lambda)|_{y=b_2} = 0 \tag{5.2.70}$$

It is enough to solve the pde (5.2.66) with boundary conditions

$$\widehat{\mathcal{M}}(0,y,\lambda) = 0 \tag{5.2.71}$$
$$\frac{\partial}{\partial x}\widehat{\mathcal{M}}(x,y,\lambda)|_{x=b_1} = 0 \tag{5.2.72}$$
$$\widehat{\mathcal{M}}(x,0,\lambda) = 1-\mathcal{M}(x,\lambda) \tag{5.2.73}$$
$$\frac{\partial}{\partial y}\widehat{\mathcal{M}}(x,y,\lambda)|_{y=b_2} = 0 \tag{5.2.74}$$

where $M(x,\lambda)$ is the solution of (5.2.58)-(5.2.60) and is given by (5.2.61). By (5.A.31)-(5.A.33), (5.A.1)-(5.A.5), (5.2.74) and the principle of superposition the solution is:

$$\begin{aligned}&\widehat{\mathcal{M}}(x,y,\lambda)\\ =\;&\sum_{n=0}^{\infty}c_n\exp\left(\frac{-x\mu_1}{\sigma_1^2}\right)\sin\left(\frac{x\sqrt{-\mu_1^2-2\lambda_n^*\sigma_1^2}}{\sigma_1^2}\right)\\ &\times\exp\left(-\frac{\sqrt{\mu_2^2-2\sigma_2^2\left(\lambda_n^*+\lambda\right)}\,(y-2b_2)+y\mu_2}{\sigma_2^2}\right)\\ &\times\frac{\left(\sqrt{\mu_2^2-2\sigma_2^2\left(\lambda_n^*+\lambda\right)}\left(1+\widetilde{B}\right)+\left(-1+\widetilde{B}\right)\mu_2\right)}{\mu_2+\sqrt{\mu_2^2-2\sigma_2^2\left(\lambda_n^*+\lambda\right)}}\end{aligned}$$

where

$$\widetilde{B}:=\widetilde{B}(y,\mu_2,\sigma_2,b_2,\lambda,\lambda_n^*):=\exp\left(\frac{2\,(y-b_2)\sqrt{\mu_2^2-2\sigma_2^2\left(\lambda_n^*+\lambda\right)}}{\sigma_2^2}\right)$$

and the eigenvalues λ_n^* are calculated by the solution of (5.A.3) (with b replaced by b_1) and for large values of n are approximated by (5.A.4). Finally we will find the coefficients c_n in order the solution to satisfy the condition (5.2.73). We have:

$$\widehat{\mathcal{M}}(x,0,\lambda)=1-\mathcal{M}(x,\lambda)\Longrightarrow$$

$$\sum_{n=0}^{\infty} c_n \exp\left(\frac{-x\mu_1}{\sigma_1^2}\right) \sin\left(\frac{x\sqrt{-\mu_1^2 - 2\lambda_n^* \sigma_1^2}}{\sigma_1^2}\right) \widetilde{\Delta}(\mu_2, \sigma_2, b_2, \lambda, \lambda_n^*) = 1 - M(x, \lambda) \Longrightarrow$$

$$c_n = \frac{\int\limits_0^b (1 - M(x,\lambda)) \exp\left(\frac{x\mu_1}{\sigma_1^2}\right) \sin\left(\frac{x\sqrt{-\mu_1^2 - 2\lambda_n^*\sigma_1^2}}{\sigma_1^2}\right) dx}{\widetilde{\Delta}(\mu_2, \sigma_2, b_2, \lambda, \lambda_n^*) \int\limits_0^b \left(\sin\left(\frac{x\sqrt{-\mu_1^2 - 2\lambda_n^*\sigma_1^2}}{\sigma_1^2}\right)\right)^2 dx}$$

where

$$\widetilde{\Gamma} \;:=\; \widetilde{\Gamma}(\mu_2, \sigma_2, b_2, \lambda, \lambda_n^*) := \exp\left(\frac{2b_2\sqrt{\mu_2^2 - 2\sigma_2^2(\lambda_n^* + \lambda)}}{\sigma_2^2}\right)$$

$$\widetilde{\Delta} \;:=\; \widetilde{\Delta}(\mu_2, \sigma_2, b_2, \lambda, \lambda_n^*) := \frac{\sqrt{\mu_2^2 - 2\sigma_2^2(\lambda_n^* + \lambda)}\left(1 + \widetilde{\Gamma}\right) + \left(1 - \widetilde{\Gamma}\right)\mu_2}{\mu_2 + \sqrt{\mu_2^2 - 2\sigma_2^2(\lambda_n^* + \lambda)}}$$

Thus the solution of our original problem (5.2.8)-(5.2.12) is

$$\mathcal{M}(x, y, \lambda) = \widehat{\mathcal{M}}(x, y, \lambda) + \mathcal{M}(x, \lambda)$$

Next we proceed with the two reflecting barriers case.

5.3 Two Reflecting barriers.

In this section we study the models (5.1.1)-(5.1.2) considering that if an insurance company (first insurer or second insurer) follows the de Finetti model then it follows a dividends policy with a constant upper reflecting barrier and a constant lower reflecting barrier which is the zero constant.

5.3.1 Expected Discounted Dividends and Financing for the de Finetti - Lundberg model.

- Discounted Dividends.

By Proposition 4.4.2 in order to find the expected discounting dividends for the first insurer (X-insurer) in a de Finetti - Lundberg model we have to solve the pde:

$$\frac{\sigma_1^2}{2}\mathcal{V}_{xx}^{(+)}(x,y) + \frac{\sigma_2^2}{2}\mathcal{V}_{yy}^{(+)}(x,y) + \mu_1\mathcal{V}_x^{(+)}(x,y) + \mu_2\mathcal{V}_y^{(+)}(x,y) = \delta\mathcal{V}^{(+)}(x,y) \tag{5.3.1}$$

subject to the boundary conditions:

$$\frac{\partial}{\partial x}\mathcal{V}^{(+)}(x,y)|_{x=0} = 0 \tag{5.3.2}$$
$$\frac{\partial}{\partial x}\mathcal{V}^{(+)}(x,y)|_{x=b_1} = 1 \tag{5.3.3}$$
$$\mathcal{V}^{(+)}(x,0) = 0 \tag{5.3.4}$$
$$\mathcal{V}^{(+)}(x,\infty) := \lim_{y\to\infty}\mathcal{V}^{(+)}(x,y) = \mathcal{V}^{(+)}(x) \tag{5.3.5}$$

- Step 1.

We consider the ODE
$$\frac{\sigma_1^2}{2}\mathcal{V}^{(+)}_{xx}(x) + \mu_1\mathcal{V}^{(+)}_x(x) = \delta\mathcal{V}^{(+)}(x) \tag{5.3.6}$$
with boundary conditions
$$\frac{\partial}{\partial x}\mathcal{V}^{(+)}(x)|_{x=0} = 0 \tag{5.3.7}$$
$$\frac{\partial}{\partial x}\mathcal{V}^{(+)}(x)|_{x=b} = 1 \tag{5.3.8}$$
The solution of (5.3.6)-(5.3.8) is
$$\mathcal{V}^{(+)}(x) = \frac{\exp\left(\frac{(b-x)\mu_1}{\sigma_1^2}\right)\left(\sqrt{\mu_1^2+2\delta\sigma_1^2}\cosh\left(\frac{x\sqrt{\mu_1^2+2\delta\sigma_1^2}}{\sigma_1^2}\right)+\sinh\left(\frac{x\sqrt{\mu_1^2+2\delta\sigma_1^2}}{\sigma_1^2}\right)\mu_1\right)}{2\delta\sinh\left(\frac{b\sqrt{\mu_1^2+2\delta\sigma_1^2}}{\sigma_1^2}\right)} \tag{5.3.9}$$

- Step 2.

Now it is enough to solve the pde (5.3.1) with boundary conditions
$$\frac{\partial}{\partial x}\widehat{\mathcal{V}^{(+)}}(x,y)|_{x=0} = 0 \tag{5.3.10}$$
$$\frac{\partial}{\partial x}\widehat{\mathcal{V}^{(+)}}(x,y)|_{x=b} = 0 \tag{5.3.11}$$
$$\widehat{\mathcal{V}^{(+)}}(x,0) = -\mathcal{V}^{(+)}(x) \tag{5.3.12}$$
$$\widehat{\mathcal{V}^{(+)}}(x,\infty) := \lim_{y\to\infty}\widehat{\mathcal{V}^{(+)}}(x,y) = 0 \tag{5.3.13}$$
By (5.A.31)–(5.A.33), (5.A.12)–(5.A.15), (5.3.13) and the principle of superposition the solution is
$$\begin{aligned}\widehat{\mathcal{V}^{(+)}}(x,y) &= \sum_{n=0}^{\infty} c_n^{(+)}\exp\left(\frac{y\left(-\mu_2-\sqrt{\mu_2^2-2\left(\lambda_n-\delta\right)\sigma_2^2}\right)}{\sigma_2^2}\right)\times \\ &\times\exp\left(-\frac{x\mu_1}{\sigma_1^2}\right)\left(\cos\left(\frac{n\pi x}{b}\right)+\frac{b\mu_1}{n\pi\sigma_1^2}\sin\left(\frac{n\pi x}{b}\right)\right)\end{aligned}$$

where by (5.A.14) we have

$$\lambda_n = -\frac{n^2\pi^2\sigma_1^4 + b^2\mu_1^2}{2b^2\sigma_1^2} + \delta$$

Finally we will find the coefficients $c_n^{(+)}$ in order the solution to satisfy the condition (5.3.12). We have

$$\begin{aligned}
\widehat{\mathcal{V}^{(+)}}(x,0) &= -\mathcal{V}^{(+)}(x) \Longrightarrow \\
\sum_{n=0}^{\infty} c_n^{(+)} \exp\left(-\frac{x\mu_1}{\sigma_1^2}\right)\left(\cos\left(\frac{n\pi x}{b}\right) + \frac{b\mu_1}{n\pi\sigma_1^2}\sin\left(\frac{n\pi x}{b}\right)\right) &= -\mathcal{V}^{(+)}(x) \Longrightarrow
\end{aligned}$$

$$c_n^{(+)} = -\frac{\int\limits_0^b \mathcal{V}^{(+)}(x)\exp\left(\frac{x\mu_1}{\sigma_1^2}\right)\left(\cos\left(\frac{n\pi x}{b}\right) + \frac{b\mu_1}{n\pi\sigma_1^2}\sin\left(\frac{n\pi x}{b}\right)\right)dx}{\int\limits_0^b \left(\cos\left(\frac{n\pi x}{b}\right) + \frac{b\mu_1}{n\pi\sigma_1^2}\sin\left(\frac{n\pi x}{b}\right)\right)^2 dx}$$

Thus the solution of the initial problem (5.3.1)-(5.3.5) is

$$\mathcal{V}^{(+)}(x,y) = \widehat{\mathcal{V}^{(+)}}(x,y) + \mathcal{V}^{(+)}(x) \Longrightarrow$$

$$\begin{aligned}
\mathcal{V}^{(+)}(x,y) &= \sum_{n=0}^{\infty} c_n^{(+)} \exp\left(\frac{y\left(-\mu_2 - \sqrt{\mu_2^2 - 2\left(\lambda_n - \delta\right)\sigma_2^2}\right)}{\sigma_2^2}\right) \times \\
&\quad \times \exp\left(-\frac{x\mu_1}{\sigma_1^2}\right)\left(\cos\left(\frac{n\pi x}{b}\right) + \frac{b\mu_1}{n\pi\sigma_1^2}\sin\left(\frac{n\pi x}{b}\right)\right) + \mathcal{V}^{(+)}(x)
\end{aligned}$$

- Discounted Financing.

By Proposition 4.4.2 in order to find the expected discounted financing for the first insurer (X-insurer) in a de Finetti - Lundberg model we have to solve the PDE

$$\frac{\sigma_1^2}{2}\mathcal{V}_{xx}^{(-)}(x,y) + \frac{\sigma_2^2}{2}\mathcal{V}_{yy}^{(-)}(x,y) + \mu_1\mathcal{V}_x^{(-)}(x,y) + \mu_2\mathcal{V}_y^{(-)}(x,y) = \delta\mathcal{V}^{(-)}(x,y) \tag{5.3.14}$$

subject to the boundary conditions

$$\begin{aligned}
\frac{\partial}{\partial x}\mathcal{V}^{(-)}(x,y)|_{x=0} &= -1 && (5.3.15)\\
\frac{\partial}{\partial x}\mathcal{V}^{(-)}(x,y)|_{x=b} &= 0 && (5.3.16)\\
\mathcal{V}^{(-)}(x,0) &= 0 && (5.3.17)\\
\mathcal{V}^{(-)}(x,\infty) &:= \lim_{y\to\infty}\mathcal{V}^{(-)}(x,y) = \mathcal{V}^{(-)}(x) && (5.3.18)
\end{aligned}$$

- Step 1.

We consider the ODE

$$\frac{\sigma_1^2}{2}\mathcal{V}_{xx}^{(-)}(x) + \mu_1 \mathcal{V}_x^{(-)}(x) = \delta \mathcal{V}^{(-)}(x) \tag{5.3.19}$$

with boundary conditions

$$\frac{\partial}{\partial x}\mathcal{V}^{(-)}(x)|_{x=0} = -1 \tag{5.3.20}$$

$$\frac{\partial}{\partial x}\mathcal{V}^{(-)}(x)|_{x=b} = 0 \tag{5.3.21}$$

The solution of (5.3.19)-(5.3.21) is

$$\mathcal{V}^{(-)}(x) = \frac{\psi_1(x;\mu_1,\sigma_1,b,0,1,\delta)}{\psi_2(\mu_1,\sigma_1,b,0,1,\delta)} \exp\left(\frac{x\left(\sqrt{\mu_1^2+2\delta\sigma_1^2}-\mu_1\right)}{\sigma_1^2}\right) \tag{5.3.22}$$

where $\psi_1(x;\mu_1,\sigma_1,b,0,1,\delta)$ and $\psi_2(\mu_1,\sigma_1,b,0,1,\delta)$ are given by (5.A.24) and (5.A.25) respectively.

Step 2.

Now it is enough to solve the pde (5.3.14) with boundary conditions

$$\frac{\partial}{\partial x}\widehat{\mathcal{V}^{(-)}}(x,y)|_{x=0} = 0 \tag{5.3.23}$$

$$\frac{\partial}{\partial x}\widehat{\mathcal{V}^{(-)}}(x,y)|_{x=b} = 0 \tag{5.3.24}$$

$$\widehat{\mathcal{V}^{(-)}}(x,0) = -\mathcal{V}^{(-)}(x) \tag{5.3.25}$$

$$\widehat{\mathcal{V}^{(-)}}(x,\infty) := \lim_{y\to\infty}\widehat{\mathcal{V}^{(-)}}(x,y) = 0 \tag{5.3.26}$$

By (5.A.31)–(5.A.33), (5.A.12)–(5.A.15), (5.3.26) and the principle of superposition the solution is:

$$\widehat{\mathcal{V}^{(-)}}(x,y) = \sum_{n=0}^{\infty} c_n^{(-)} \exp\left(\frac{y\left(-\mu_2-\sqrt{\mu_2^2-2\left(\lambda_n-\delta\right)\sigma_2^2}\right)}{\sigma_2^2}\right) \times$$
$$\times \exp\left(-\frac{x\mu_1}{\sigma_1^2}\right)\left(\cos\left(\frac{n\pi x}{b}\right)+\frac{b\mu_1}{n\pi\sigma_1^2}\sin\left(\frac{n\pi x}{b}\right)\right)$$

where by (5.A.14) we have:

$$\lambda_n = -\frac{n^2\pi^2\sigma_1^4+b^2\mu_1^2}{2b^2\sigma_1^2}+\delta$$

Finally we will find the coefficients $c_n^{(-)}$ in order the solution to satisfy the condition (5.3.25). We have

$$
\begin{aligned}
\widehat{\mathcal{V}^{(-)}}(x,0) &= -\mathcal{V}^{(-)}(x) \Longrightarrow \\
\sum_{n=0}^{\infty} c_n^{(-)} \exp\left(-\frac{x\mu_1}{\sigma_1^2}\right)\left(\cos\left(\frac{n\pi x}{b}\right) + \frac{b\mu_1}{n\pi\sigma_1^2}\sin\left(\frac{n\pi x}{b}\right)\right) &= -\mathcal{V}^{(-)}(x) \Longrightarrow
\end{aligned}
$$

$$
c_n^{(-)} = -\frac{\int\limits_0^b V^{(-)}(x)\exp\left(\frac{x\mu_1}{\sigma_1^2}\right)\left(\cos\left(\frac{n\pi x}{b}\right) + \frac{b\mu_1}{n\pi\sigma_1^2}\sin\left(\frac{n\pi x}{b}\right)\right)dx}{\int\limits_0^b \left(\cos\left(\frac{n\pi x}{b}\right) + \frac{b\mu_1}{n\pi\sigma_1^2}\sin\left(\frac{n\pi x}{b}\right)\right)^2 dx}
$$

Thus the solution of the initial problem (5.3.14)-(5.3.18) is

$$
\mathcal{V}^{(-)}(x,y) = \widehat{\mathcal{V}^{(-)}}(x,y) + \mathcal{V}^{(-)}(x) \Longrightarrow
$$

$$
\begin{aligned}
\mathcal{V}^{(-)}(x,y) &= \sum_{n=0}^{\infty} c_n^{(-)} \exp\left(\frac{y\left(-\mu_2 - \sqrt{\mu_2^2 - 2(\lambda_n - \delta)\sigma_2^2}\right)}{\sigma_2^2}\right) \\
&\times \exp\left(-\frac{x\mu_1}{\sigma_1^2}\right)\left(\cos\left(\frac{n\pi x}{b}\right) + \frac{b\mu_1}{n\pi\sigma_1^2}\sin\left(\frac{n\pi x}{b}\right)\right) + \mathcal{V}^{(-)}(x)
\end{aligned}
$$

5.3.2 Expected Discounted Dividends and Financing for the De Finetti -de Finetti model.

- Discounted Dividends.

By Proposition 4.9.6 in order to find the expected discounted dividends for the first insurer (X-insurer) in a de Finetti - de Finetti model we have to solve the PDE

$$
\frac{\sigma_1^2}{2}\mathcal{V}_{1xx}^{(+)}(x,y) + \frac{\sigma_2^2}{2}\mathcal{V}_{1yy}^{(+)}(x,y) + \mu_1\mathcal{V}_{1x}^{(+)}(x,y) + \mu_2\mathcal{V}_{1y}^{(+)}(x,y) = \delta\mathcal{V}_1^{(+)}(x,y) \tag{5.3.27}
$$

subject to the boundary conditions

$$
\frac{\partial}{\partial x}\mathcal{V}_1^{(+)}(x,y)|_{x=0} = 0 \tag{5.3.28}
$$

$$
\frac{\partial}{\partial x}\mathcal{V}_1^{(+)}(x,y)|_{x=b_1} = 1 \tag{5.3.29}
$$

$$
\mathcal{V}_1^{(+)}(x,0) = 0 \tag{5.3.30}
$$

$$
\frac{\partial}{\partial y}\mathcal{V}_1^{(+)}(x,y)|_{y=b_2} = 0 \tag{5.3.31}
$$

By (5.A.31)–(5.A.33), (5.A.16)–(5.A.18), (5.A.1)–(5.A.5), (5.3.27) and the principle of superposition the solution is

$$\begin{aligned}\mathcal{V}_1^{(+)}(x,y) &= \sum_{n=0}^{\infty} \frac{\widetilde{\omega}_1(x;\mu_1,\sigma_1,\lambda_n,0,0)}{\widetilde{\omega}_2(b_1;\mu_1,\sigma_1,\lambda_n,0,0)} \exp\left(\frac{(b_1-x)\,\mu_1}{\sigma_1^2}\right) \\ &\quad \times \exp\left(\frac{-y\mu_2}{\sigma_2^2}\right) \sin\left(\frac{y\sqrt{-\mu_2^2+2(\lambda_n-\delta)\sigma_2^2}}{\sigma_2^2}\right)\end{aligned}$$

where $\widetilde{\omega}_1(x;\mu_1,\sigma_1,\lambda_n,0,0)$ and $\widetilde{\omega}_2(b_1;\mu_1,\sigma_1,\lambda_n,0,0)$ are given by (5.A.19) and (5.A.20) respectively.

The eigenvalues λ_n are calculated by the solution of (5.A.3) (with b replaced by b_2 , μ_1 by μ_2 , σ_1 by σ_2) and for large values of n are approximated by (5.A.4).

Similarly in order to find the discounting dividends for the second insurer (Y-insurer), by the Proposition 4.9.6 we must solve the PDE

$$\frac{\sigma_1^2}{2}\mathcal{V}_{2xx}^{(+)}(x,y) + \frac{\sigma_2^2}{2}\mathcal{V}_{2yy}^{(+)}(x,y) + \mu_1\mathcal{V}_{2x}^{(+)}(x,y) + \mu_2\mathcal{V}_{2y}^{(+)}(x,y) = \delta\mathcal{V}_2^{(+)}(x,y) \tag{5.3.32}$$

subject to the boundary conditions

$$\frac{\partial}{\partial x}\mathcal{V}_2^{(+)}(x,y)|_{x=0} = 0 \tag{5.3.33}$$

$$\frac{\partial}{\partial x}\mathcal{V}_2^{(+)}(x,y)|_{x=b_1} = 0 \tag{5.3.34}$$

$$\mathcal{V}_2^{(+)}(x,0) = 0 \tag{5.3.35}$$

$$\frac{\partial}{\partial y}\mathcal{V}_2^{(+)}(x,y)|_{y=b_2} = 1 \tag{5.3.36}$$

By (5.A.31)–(5.A.33), (5.A.12)–(5.A.15), (5.A.7)–(5.A.9) and the principle of superposition the solution is:

$$\begin{aligned}\mathcal{V}_2^{(+)}(x,y) &= \sum_{n=0}^{\infty} \exp\left(-\frac{x\mu_1}{\sigma_1^2}\right)\left(\cos\left(\frac{n\pi x}{b_1}\right) + \frac{b_1\mu_1}{n\pi\sigma_1^2}\sin\left(\frac{n\pi x}{b_1}\right)\right) \\ &\times \quad \exp\left(-\frac{(y-b_2)\left(\mu_2+\sqrt{\mu_2^2-2\left(\lambda_n-\delta\right)\sigma_2^2}\right)}{\sigma_2^2}\right) \frac{\widetilde{r}_1(y,\mu_2,\sigma_2,\lambda_n,\delta)}{\widetilde{r}_2(\mu_2,\sigma_2,b_2,\lambda_n,\delta)}\end{aligned}$$

where $\widetilde{r}_1(y;\mu_2,\sigma_2,\lambda_n,1,\delta)$ and $\widetilde{r}_2(b_2;\mu_2,\sigma_2,\lambda_n,1,\delta)$ are given by (5.A.10) and (5.A.11) respectively and by (5.A.14) we have

$$\lambda_n = -\frac{n^2\pi^2\sigma_1^4 + b^2\mu_1^2}{2b^2\sigma_1^2} + \delta$$

- Discounted Financing.

By Proposition 4.9.6 in order to find the expected discounted financing for the first insurer (X-insurer) in a de Finetti - de Finetti model we have to solve the PDE:

$$\frac{\sigma_1^2}{2}\mathcal{V}^{(-)}_{1xx}(x,y)+\frac{\sigma_2^2}{2}\mathcal{V}^{(-)}_{1yy}(x,y)+\mu_1\mathcal{V}^{(-)}_{1x}(x,y)+\mu_2\mathcal{V}^{(-)}_{1y}(x,y)=\delta\mathcal{V}^{(-)}_{1}(x,y) \tag{5.3.37}$$

subject to the boundary conditions

$$\frac{\partial}{\partial x}\mathcal{V}^{(-)}_1(x,y)|_{x=0} = -1 \tag{5.3.38}$$

$$\frac{\partial}{\partial x}\mathcal{V}^{(-)}_1(x,y)|_{x=b_1} = 0 \tag{5.3.39}$$

$$\mathcal{V}^{(-)}_1(x,0) = 0 \tag{5.3.40}$$

$$\frac{\partial}{\partial y}\mathcal{V}^{(-)}_1(x,y)|_{y=b_2} = 0 \tag{5.3.41}$$

By (5.A.31)-(5.A.33), (5.A.21)-(5.A.23), (5.A.1)-(5.A.5) and the principle of superposition the solution is:

$$\begin{aligned}\mathcal{V}^{(-)}_1(x,y) &= \sum_{n=0}^{\infty}\exp\left(\frac{-y\mu_2}{\sigma_2^2}\right)\sin\left(\frac{y\sqrt{-\mu_2^2+2(\lambda_n-\delta)\sigma_2^2}}{\sigma_2^2}\right)\times\\ &\times\exp\left(\frac{x\left(\sqrt{\mu_1^2+2\lambda_n\sigma_1^2}-\mu_1\right)}{\sigma_1^2}\right)\frac{\psi_1(x;\mu_1,\sigma_1,b_1,\lambda_n,0,0)}{\psi_2(\mu_1,\sigma_1,b_1,\lambda_n,0,0)}\end{aligned}$$

where $\psi_1(x;\mu_1,\sigma_1,b_1,\lambda_n,0,0)$ and $\psi_2(\mu_1,\sigma_1,b_1,\lambda_n,0,0)$ are given by (5.A.24) and (5.A.25) respectively. The eigenvalues λ_n are calculated by the solution of (5.A.3) (with b replaced by b_2 , μ_1 by μ_2 , σ_1 by σ_2) and for large values of n are approximated by (5.A.4).

5.3.3 Moments of the Discounted Dividends and Financing for the de Finetti - Lundberg model.

- Discounted Dividends.

By Proposition 4.4.2 in order to find the moments of the discounted dividends for the first insurer (X-insurer) in a de Finetti - Lundberg model we have to solve the PDE

$$\frac{\sigma_1^2}{2}\mathcal{V}^{(+)}_{xx}(x,y;k)+\frac{\sigma_2^2}{2}\mathcal{V}^{(+)}_{yy}(x,y;k)+\mu_1\mathcal{V}^{(+)}_{x}(x,y;k)+\mu_2\mathcal{V}^{(+)}_{y}(x,y;k)=k\delta\mathcal{V}^{(+)}(x,y;k) \tag{5.3.42}$$

subject to the boundary conditions

$$\frac{\partial}{\partial x}\mathcal{V}^{(+)}(x,y;k)|_{x=0} = 0 \tag{5.3.43}$$

$$\frac{\partial}{\partial x}\mathcal{V}^{(+)}(x,y;k)|_{x=b} = k\mathcal{V}^{(+)}(b,y;k-1) \tag{5.3.44}$$

$$\mathcal{V}^{(+)}(x,0;k) = 0 \tag{5.3.45}$$

$$\mathcal{V}^{(+)}(x,\infty;k) := \lim_{y\to\infty}\mathcal{V}^{(+)}(x,y;k)=\mathcal{V}^{(+)}(x;k) \tag{5.3.46}$$

- Step 1.

We consider the ODE

$$\frac{\sigma_1^2}{2}\mathcal{V}_{xx}^{(+)}(x;k) + \mu_1 \mathcal{V}_x^{(+)}(x;k) = k\delta \mathcal{V}^{(+)}(x;k) \tag{5.3.47}$$

with boundary conditions:

$$\frac{\partial}{\partial x}\mathcal{V}^{(+)}(x;k)|_{x=0} = 0 \tag{5.3.48}$$

$$\frac{\partial}{\partial x}\mathcal{V}^{(+)}(x;k)|_{x=b} = k\mathcal{V}^{(+)}(b;k-1) \tag{5.3.49}$$

The solution of (5.3.47)–(5.3.49) is

$$\mathcal{V}^{(+)}(x;k) = \exp\left(\frac{(b-x)\left(\mu_1 + \sqrt{\mu_1^2 + 2k\delta\sigma_1^2}\right)}{\sigma_1^2}\right)\frac{\widetilde{r}_1(x;\mu_1,\sigma_1,0,k,\delta)}{\widetilde{r}_2(b;\mu_1,\sigma_1,0,k,\delta)}k\mathcal{V}^{(+)}(b;k-1)$$

where $\widetilde{r}_1(x;\mu_1,\sigma_1,0,k,\delta)$ and $\widetilde{r}_2(b;\mu_1,\sigma_1,0,k,\delta)$ are given by (5.A.10) and (5.A.11) respectively.

- Step 2.

In this step we will solve the pde (5.3.42) with boundary conditions

$$\frac{\partial}{\partial x}\mathcal{W}^{(+)}(x,y;k)|_{x=0} = 0 \tag{5.3.50}$$

$$\frac{\partial}{\partial x}\mathcal{W}^{(+)}(x,y;k)|_{x=b} = k\widehat{\mathcal{V}^{(+)}}(b,y;k-1) \tag{5.3.51}$$

$$\mathcal{W}^{(+)}(x,\infty;k) := \lim_{y\to\infty}\mathcal{W}^{(+)}(x,y;k) = 0 \tag{5.3.52}$$

By (5.A.31)-(5.A.33) and (5.3.50)- (5.3.52) we consider a solution of the form

$$\begin{aligned}&\mathcal{W}^{(+)}(x,y;k)\\ &= \sum_{n=0}^{\infty} c_{k,n}^{*,(+)}\widehat{\alpha}(x,\mu_1,\sigma_1,\lambda_{k,n}^*)\exp\left(\frac{-x\mu_1}{\sigma_1^2}\right)\exp\left(\frac{y\left(-\mu_2 - \sqrt{\mu_2^2 - 2(\lambda_{k,n}^* - k\delta)\sigma_2^2}\right)}{\sigma_2^2}\right)\end{aligned}$$

where

$$\begin{aligned}\widehat{\alpha}(x,\mu_1,\sigma_1,\lambda_{k,n}^*) &:= \sin\left(\frac{x\sqrt{-\mu_1^2 - 2\lambda_{k,n}^*\sigma_1^2}}{\sigma_1^2}\right) + \\ &\quad + \frac{\sqrt{-\mu_1^2 - 2\lambda_{k,n}^*\sigma_1^2}}{\mu_1}\cos\left(\frac{x\sqrt{-\mu_1^2 - 2\lambda_{k,n}^*\sigma_1^2}}{\sigma_1^2}\right)\end{aligned}$$

and $\lambda^*_{k,n}$ are the constants of the separation of variables. Now because of the condition (5.3.51) we take

$$\lambda^*_{k,n} = \lambda_{k-1,n} + \delta$$

and so we have

$$\frac{\partial}{\partial x}\mathcal{W}^{(+)}(x,y;k)|_{x=b} = k\widehat{\mathcal{V}^{(+)}}(b,y;k-1) \Longrightarrow$$

$$\begin{aligned} & c^{*,(+)}_{k,n}\frac{2\exp\left(-\frac{b\mu_1}{\sigma_1^2}\right)(\lambda_{k-1,n}+\delta)\sin\left(\frac{b\sqrt{-\mu_1^2-2(\lambda_{k-1,n}+\delta)\sigma_1^2}}{\sigma_1^2}\right)}{\mu_1} \\ = \quad & kc^{(+)}_{k-1,n}\frac{\exp\left(-\frac{b\mu_1}{\sigma_1^2}\right)\left(n\pi\cos\left(\frac{n\pi b}{b}\right)\sigma_1^2 + b\sin\left(\frac{n\pi b}{b}\right)\mu_1\right)}{n\pi\sigma_1^2} \Longrightarrow \end{aligned}$$

$$c^{*,(+)}_{k,n} = kc^{(+)}_{k-1,n}\frac{(-1)^n b\mu_1}{2n\pi\sigma_1^2(\lambda_{k-1,n}+\delta)\sin\left(\frac{b\sqrt{-\mu_1^2-2(\lambda_{k-1,n}+\delta)\sigma_1^2}}{\sigma_1^2}\right)}$$

- Step 3.

Now it is enough to solve the pde (5.3.42) with boundary conditions

$$\frac{\partial}{\partial x}\widehat{\mathcal{V}^{(+)}}(x,y;k)|_{x=0} = 0 \quad (5.3.53)$$

$$\frac{\partial}{\partial x}\widehat{\mathcal{V}^{(+)}}(x,y;k)|_{x=b} = 0 \quad (5.3.54)$$

$$\widehat{\mathcal{V}^{(+)}}(x,0;k) = -\mathcal{V}^{(+)}(x;k) - \mathcal{W}^{(+)}(x,0;k) \quad (5.3.55)$$

$$\widehat{\mathcal{V}^{(+)}}(x,\infty;k) := \lim_{y\to\infty}\widehat{\mathcal{V}^{(+)}}(x,y;k) = 0 \quad (5.3.56)$$

By (5.A.31)-(5.A.33), (5.A.12)-(5.A.15), (5.3.53), (5.3.56) and the principle of superposition the solution is

$$\begin{aligned}\widehat{\mathcal{V}^{(+)}}(x,y;k) = & \sum_{n=0}^{\infty} c^{(+)}_{k,n}\exp\left(\frac{y\left(-\mu_2-\sqrt{\mu^2-2(\lambda_{k,n}-k\delta)\sigma_2^2}\right)}{\sigma_2^2}\right) \\ & \times\exp\left(-\frac{x\mu_1}{\sigma_1^2}\right)\left(\cos\left(\frac{n\pi x}{b}\right)+\frac{b\mu_1}{n\pi\sigma_1^2}\sin\left(\frac{n\pi x}{b}\right)\right)\end{aligned}$$

where the eigenvalues $\lambda_{k,n}$ are given by (5.A.14). Finally we will find the coefficients $c^{(+)}_{k,n}$ in order the solution to satisfy the condition (5.3.55). We have

$$\widehat{\mathcal{V}^{(+)}}(x,0;k) = -\mathcal{V}^{(+)}(x;k) - \mathcal{W}^{(+)}(x,0;k)$$

which leads to

$$\sum_{n=0}^{\infty} c_{k,n}^{(+)} \exp\left(-\frac{x\mu_1}{\sigma_1^2}\right)\left(\cos\left(\frac{n\pi x}{b}\right)+\frac{b\mu_1}{n\pi\sigma_1^2}\sin\left(\frac{n\pi x}{b}\right)\right) = -V^{(+)}(x;k) - W^{(+)}(x,0;k)$$

or

$$c_{k,n}^{(+)} = -\frac{\int_0^b (\mathcal{V}^{(+)}(x) + \mathcal{W}^{(+)}(x,0;k)) \exp\left(\frac{x\mu_1}{\sigma_1^2}\right)\left(\cos\left(\frac{n\pi x}{b}\right)+\frac{b\mu_1}{n\pi\sigma_1^2}\sin\left(\frac{n\pi x}{b}\right)\right) dx}{\int_0^b \left(\cos\left(\frac{n\pi x}{b}\right)+\frac{b\mu_1}{n\pi\sigma_1^2}\sin\left(\frac{n\pi x}{b}\right)\right)^2 dx}$$

Thus the solution of the initial problem (5.3.42)-(5.3.46) is:

$$\mathcal{V}^{(+)}(x,y;k) = \widehat{\mathcal{V}^{(+)}}(x,y;k) + \mathcal{W}^{(+)}(x,y;k) + \mathcal{V}^{(+)}(x;k)$$

- Discounted Financing.

By Proposition 4.4.2 in order to find the moments of the discounted financing for the first insurer (X-insurer) in a de Finetti - Lundberg model we have to solve the PDE

$$\frac{\sigma_1^2}{2}\mathcal{V}_{xx}^{(-)}(x,y;k) + \frac{\sigma_2^2}{2}\mathcal{V}_{yy}^{(-)}(x,y;k) + \mu_1\mathcal{V}_x^{(-)}(x,y;k) + \mu_2\mathcal{V}_y^{(-)}(x,y;k) = k\delta\mathcal{V}^{(-)}(x,y;k) \quad (5.3.57)$$

subject to the boundary conditions

$$\frac{\partial}{\partial x}\mathcal{V}^{(-)}(x,y;k)|_{x=0} = -k\mathcal{V}^{(-)}(0,y;k-1) \quad (5.3.58)$$

$$\frac{\partial}{\partial x}\mathcal{V}^{(-)}(x,y;k)|_{x=b} = 0 \quad (5.3.59)$$

$$\mathcal{V}^{(-)}(x,0;k) = 0 \quad (5.3.60)$$

$$\mathcal{V}^{(-)}(x,\infty;k) := \lim_{y\to\infty}\mathcal{V}^{(-)}(x,y;k) = \mathcal{V}^{(-)}(x;k) \quad (5.3.61)$$

- Step 1.

We consider the ODE

$$\frac{\sigma_1^2}{2}\mathcal{V}_{xx}^{(-)}(x;k) + \mu_1\mathcal{V}_x^{(-)}(x;k) = k\delta\mathcal{V}^{(-)}(x;k) \quad (5.3.62)$$

with boundary conditions

$$\frac{\partial}{\partial x}\mathcal{V}^{(-)}(x;k)|_{x=b} = 0 \quad (5.3.63)$$

$$\frac{\partial}{\partial x}\mathcal{V}^{(-)}(x;k)|_{x=0} = -k\mathcal{V}^{(-)}(0;k-1) \quad (5.3.64)$$

The solution of (5.3.62)-(5.3.64) is

$$\mathcal{V}^{(-)}(x;k) = \exp\left(\frac{x\left(\sqrt{\mu_1^2+2k\delta\sigma_1^2}-\mu_1\right)}{\sigma_1^2}\right)\frac{\psi_1(x;\mu_1,\sigma_1,b,0,k,\delta)}{\psi_2(\mu_1,\sigma_1,b,0,k,\delta)}k\mathcal{V}^{(-)}(0;k-1)$$

where $\psi_1(x;\mu_1,\sigma_1,b,0,k,\delta)$ and $\psi_2(\mu_1,\sigma_1,b,0,k,\delta)$ are given by (5.A.24) and (5.A.25) respectively.

- Step 2.

In this step we will solve the PDE (5.3.57) with boundary conditions

$$\frac{\partial}{\partial x}\mathcal{W}^{(-)}(x,y;k)|_{x=0} = -k\widehat{\mathcal{V}^{(-)}}(0,y;k-1) \tag{5.3.65}$$

$$\frac{\partial}{\partial x}\mathcal{W}^{(-)}(x,y;k)|_{x=b} = 0 \tag{5.3.66}$$

$$\mathcal{W}^{(-)}(x,\infty;k) := \lim_{y\to\infty}\mathcal{W}^{(-)}(x,y;k)=0 \tag{5.3.67}$$

By (5.A.31)-(5.A.33) and (5.3.50)- (5.3.52) we consider a solution of the form

$$\begin{aligned}&\mathcal{W}^{(+)}(x,y;k)\\ =& \sum_{n=0}^{\infty}c_{k,n}^{*,(-)}\exp\left(\frac{-x\mu_1}{\sigma_1^2}\right)\exp\left(\frac{y\left(-\mu_2-\sqrt{\mu_2^2-2(\lambda_{k,n}^*-k\delta)\sigma_2^2}\right)}{\sigma_2^2}\right)\\ &\times\left(\widetilde{\Theta}\cos\left(\frac{x\sqrt{-\mu_1^2-2\lambda_{k,n}^*\sigma_1^2}}{\sigma_1^2}\right)+\sin\left(\frac{x\sqrt{-\mu_1^2-2\lambda_{k,n}^*\sigma_1^2}}{\sigma_1^2}\right)\right)\end{aligned}$$

where

$$\widetilde{\Theta} := \widetilde{\Theta}(\mu_1,\sigma_1,b,\lambda_{k,n}^*) := \frac{\sqrt{-\mu_1^2-2\lambda_{k,n}^*\sigma_1^2}-\mu_1\tan\left(\frac{b\sqrt{-\mu_1^2-2\lambda_{k,n}^*\sigma_1^2}}{\sigma_1^2}\right)}{\mu_1+\sqrt{-\mu_1^2-2\lambda_{k,n}^*\sigma_1^2}\tan\left(\frac{b\sqrt{-\mu_1^2-2\lambda_{k,n}^*\sigma_1^2}}{\sigma_1^2}\right)}$$

and $\lambda_{k,n}^*$ are the separation of variables constants. Now because of the condition (5.3.65) we take

$$\lambda_{k,n}^* = \lambda_{k-1,n}+\delta$$

and so we have

$$\frac{\partial}{\partial x}\mathcal{W}^{(-)}(x,y;k)|_{x=0} = -k\widehat{\mathcal{V}^{(-)}}(0,y;k-1)$$

which leads to

$$c_{k,n}^{*,(-)} = kc_{k-1,n}^{(-)}\frac{\mu_1+\sqrt{-\mu_1^2-2(\lambda_{k-1,n}+\delta)\sigma_1^2}\tan\left(\frac{b\sqrt{-\mu_1^2-2(\lambda_{k-1,n}+\delta)\sigma_1^2}}{\sigma_1^2}\right)}{2(\lambda_{k-1,n}+\delta)\tan\left(\frac{b\sqrt{-\mu_1^2-2(\lambda_{k-1,n}+\delta)\sigma_1^2}}{\sigma_1^2}\right)}$$

- Step 3.

Now it is enough to solve the PDE (5.3.57) with boundary conditions

$$\begin{aligned}
\frac{\partial}{\partial x}\widehat{\mathcal{V}^{(-)}}(x,y;k)|_{x=0} &= 0 && (5.3.68)\\
\frac{\partial}{\partial x}\widehat{\mathcal{V}^{(-)}}(x,y;k)|_{x=b} &= 0 && (5.3.69)\\
\widehat{\mathcal{V}^{(+)}}(x,0;k) &= -\mathcal{V}^{(-)}(x;k) - \mathcal{W}^{(-)}(x,0;k) && (5.3.70)\\
\widehat{\mathcal{V}^{(-)}}(x,\infty;k) &:= \lim_{y\to\infty}\widehat{\mathcal{V}^{(-)}}(x,y;k) = 0 && (5.3.71)
\end{aligned}$$

By (5.A.31)–(5.A.33), (5.A.12)–(5.A.15), (5.3.68), (5.3.71) and the principle of superposition the solution is

$$\begin{aligned}
\widehat{\mathcal{V}^{(-)}}(x,y;k) &= \sum_{n=0}^{\infty} c_{k,n}^{(-)} \exp\left(\frac{y\left(-\mu_2 - \sqrt{\mu_2^2 - 2(\lambda_{k,n} - k\delta)\sigma_2^2}\right)}{\sigma_2^2}\right)\\
&\quad \times \exp\left(-\frac{x\mu_1}{\sigma_1^2}\right)\left(\cos\left(\frac{n\pi x}{b}\right) + \frac{b\mu_1}{n\pi\sigma_1^2}\sin\left(\frac{n\pi x}{b}\right)\right)
\end{aligned}$$

where the eigenvalues $\lambda_{k,n}$ are given by (5.A.14). Finally we will find the coefficients $c_{k,n}^{(-)}$ in order the solution to satisfy the condition (5.3.70). We have

$$\widehat{\mathcal{V}^{(-)}}(x,0;k) = -\mathcal{V}^{(-)}(x;k) - \mathcal{W}^{(-)}(x,0;k)$$

which gives

$$\sum_{n=0}^{\infty} c_{k,n}^{(-)} e^{-\frac{x\mu_1}{\sigma_1^2}}\left(\cos\left(\frac{n\pi x}{b}\right) + \frac{b\mu_1}{n\pi\sigma_1^2}\sin\left(\frac{n\pi x}{b}\right)\right) = -\mathcal{V}^{(-)}(x;k) - \mathcal{W}^{(-)}(x,0;k)$$

or

$$c_{k,n}^{(-)} = -\frac{\int_0^b \left(V^{(-)}(x) + W^{(-)}(x,0;k)\right)\exp\left(\frac{x\mu_1}{\sigma_1^2}\right)\left(\cos\left(\frac{n\pi x}{b}\right) + \frac{b\mu_1}{n\pi\sigma_1^2}\sin\left(\frac{n\pi x}{b}\right)\right)dx}{\int_0^b \left(\cos\left(\frac{n\pi x}{b}\right) + \frac{b\mu_1}{n\pi\sigma_1^2}\sin\left(\frac{n\pi x}{b}\right)\right)^2 dx}.$$

Thus the solution of the initial problem (5.3.57)-(5.3.61) is

$$\mathcal{V}^{(-)}(x,y;k) = \widehat{\mathcal{V}^{(-)}}(x,y;k) + \mathcal{W}^{(-)}(x,y;k) + \mathcal{V}^{(-)}(x;k)$$

5.3.4 Moments of the Discounted Dividends and Financing for the de Finetti - de Finetti model.

- Discounted Moment of Dividends.

By Proposition 4.9.6 in order to find the moments of the discounted dividends for the first insurer (X-insurer) in a de Finetti - de Finetti model we have to solve the pde:

$$\frac{\sigma_1^2}{2}\mathcal{V}_{1xx}^{(+)}(x,y;k)+\frac{\sigma_2^2}{2}\mathcal{V}_{1yy}^{(+)}(x,y;k)+\mu_1\mathcal{V}_{1x}^{(+)}(x,y;k)+\mu_2\mathcal{V}_{1y}^{(+)}(x,y;k)=k\delta\mathcal{V}_1^{(+)}(x,y;k) \quad (5.3.72)$$

subject to the boundary conditions

$$\frac{\partial}{\partial x}\mathcal{V}_1^{(+)}(x,y;k)|_{x=0} = 0 \quad (5.3.73)$$

$$\frac{\partial}{\partial x}\mathcal{V}_1^{(+)}(x,y;k)|_{x=b_1} = k\mathcal{V}_1^{(+)}(b_1,y;k-1) \quad (5.3.74)$$

$$\mathcal{V}_1^{(+)}(x,0;k) = 0 \quad (5.3.75)$$

$$\frac{\partial}{\partial y}\mathcal{V}_1^{(+)}(x,y;k)|_{y=b_2} = 0 \quad (5.3.76)$$

By (5.A.31)-(5.A.33), (5.A.1)-(5.A.5), (5.3.27) and the principle of superposition the solution is

$$\begin{aligned}\mathcal{V}_1^{(+)}(x,y;k) &= \sum_{n=0}^{\infty} c_{k,n}^{(+)} \exp\left(\frac{-y\mu_2}{\sigma_2^2}\right)\sin\left(\frac{y\sqrt{-\mu_2^2+2(\lambda_{k,n}-k\delta)\sigma_2^2}}{\sigma_2^2}\right)\\ &\times\exp\left(-\frac{x\left(\mu_1+\sqrt{\mu_1^2+2\sigma_1^2\lambda_{k,n}}\right)}{\sigma_1^2}\right)\frac{q(x;\mu_1,\sigma_1,\lambda_{k,n})}{\mu_1+\sqrt{\mu_1^2+2\sigma_1^2\lambda_{k,n}}}\end{aligned}$$

where

$$\begin{aligned}q(x;\mu_1,\sigma_1,\lambda_{k,n}) &:= \left(-1+\exp\left(\frac{2x\sqrt{\mu_1^2+2\sigma_1^2\lambda_{k,n}}}{\sigma_1^2}\right)\right)\mu_1\\ &+\sqrt{\mu_1^2+2\sigma_1^2\lambda_{k,n}}\left(1+\exp\left(\frac{2x\sqrt{\mu_1^2+2\sigma_1^2\lambda_{k,n}}}{\sigma_1^2}\right)\right)\end{aligned}$$

and the eigenvalues $\lambda_{k,n}$ are calculated by the solution of (5.A.3) (with b replaced by b_2 , μ_1 by μ_2, σ_1 by σ_2) and for large values of n are approximated by (5.A.4). By the condition (5.3.74) and using (5.A.6) we have

$$\frac{\partial}{\partial x}\mathcal{V}_1^{(+)}(x,y;k)|_{x=b_1} = k\mathcal{V}_1^{(+)}(b_1,y;k-1)$$

which leads to

$$\begin{aligned} c_{k,n}^{(+)} &= kc_{k-1,n}^{(+)} \exp\left(-\frac{b_1\left(\mu_1+\sqrt{\mu_1^2+2\sigma_1^2\lambda_{k-1,n}}\right)}{\sigma_1^2}\right) \times \\ &\times \widetilde{\varepsilon}(b_1;\mu_1,\sigma_1,\lambda_{k-1,n},\lambda_{k,n}) \frac{\mu_1+\sqrt{\mu_1^2+2\sigma_1^2\lambda_{k,n}}}{\mu_1+\sqrt{\mu_1^2+2\sigma_1^2\lambda_{k-1,n}}} \end{aligned}$$

where

$$\begin{aligned} &\widetilde{\varepsilon}(b_1;\mu_1,\sigma_1,\lambda_{k-1,n},\lambda_{k,n}) \\ := &\frac{q(b_1;\mu_1,\sigma_1,\lambda_{k-1,n})}{2\exp\left(-\frac{b_1\left(\mu_1+\sqrt{\mu_1^2+2\sigma_1^2\lambda_{k,n}}\right)}{\sigma_1^2}\right)\left(-1+\exp\left(\frac{2b_1\sqrt{\mu_1^2+2\sigma_1^2\lambda_{k,n}}}{\sigma_1^2}\right)\right)\lambda_{k,n}} \end{aligned}$$

Similarly in order to find the discounted dividends for the second insurer (Y-insurer) we would like to solve the PDE

$$\frac{\sigma_1^2}{2}\mathcal{V}_{2xx}^{(+)}(x,y)+\frac{\sigma_2^2}{2}\mathcal{V}_{2yy}^{(+)}(x,y)+\mu_1\mathcal{V}_{2x}^{(+)}(x,y)+\mu_2\mathcal{V}_{2y}^{(+)}(x,y)=\delta\mathcal{V}_2^{(+)}(x,y) \tag{5.3.77}$$

subject to the boundary conditions

$$\begin{aligned} \frac{\partial}{\partial x}\mathcal{V}_2^{(+)}(x,y;k)|_{x=0} &= 0 &(5.3.78)\\ \frac{\partial}{\partial x}\mathcal{V}_2^{(+)}(x,y;k)|_{x=b_1} &= 0 &(5.3.79)\\ \mathcal{V}_2^{(+)}(x,0;k) &= 0 &(5.3.80)\\ \frac{\partial}{\partial y}\mathcal{V}_2^{(+)}(x,y;k)|_{y=b_2} &= k\mathcal{V}_2^{(+)}(x,b_2;k-1) &(5.3.81) \end{aligned}$$

By (5.A.31)–(5.A.33), (5.A.12)–(5.A.15) and the principle of superposition the solution is

$$\begin{aligned} &\mathcal{V}_2^{(+)}(x,y;k) \\ = &\sum_{n=0}^{\infty} c_{k,n}^{+}\exp\left(-\frac{x\mu_1}{\sigma_1^2}\right)\left(\cos\left(\frac{n\pi x}{b}\right)+\frac{b\mu_1}{n\pi\sigma_1^2}\sin\left(\frac{n\pi x}{b}\right)\right) \times \\ &\times\left(-\exp\left(\frac{y\left(-\mu_2-\sqrt{\mu_2^2+2\sigma_2^2\lambda_{k,n}}\right)}{\sigma_2^2}\right)+\exp\left(\frac{y\left(-\mu_2+\sqrt{\mu_2^2+2\sigma_2^2\lambda_{k,n}}\right)}{\sigma_2^2}\right)\right) \end{aligned}$$

where by (5.A.14) we have

$$\lambda_{k,n}=-\frac{n^2\pi^2\sigma_1^4+b^2\mu_1^2}{2b^2\sigma_1^2}-k\delta$$

By condition (5.3.81) we have

$$\frac{\partial}{\partial y}\mathcal{V}_2^{(+)}(x,y;k)|_{y=b_2} = k\mathcal{V}_2^{(+)}(x,b_2;k-1)$$

from which follows that

$$c_{k,n}^{(+)}\exp\left(-\frac{b_2\mu_2}{\sigma_2^2}\right)\frac{\widetilde{\Upsilon}_2(\mu_2,\sigma_2,b_2,\lambda_{k,n})}{\sigma_2^2} = kc_{k-1,n}^{(+)}\widetilde{\Upsilon}_1(\mu_2,\sigma_2,b_2,\lambda_{k-1,n})$$

or

$$c_{k,n}^{(+)} = kc_{k-1,n}^{(+)}\exp\left(\frac{b_2\mu_2}{\sigma_2^2}\right)\frac{\widetilde{\Upsilon}_1(\mu_2,\sigma_2,b_2,\lambda_{k-1,n})}{\widetilde{\Upsilon}_2(\mu_2,\sigma_2,b_2,\lambda_{k,n})}\sigma_2^2$$

where

$$\begin{aligned}&\widetilde{\Upsilon}_1(\mu_2,\sigma_2,b_2,\lambda_{k-1,n})\\ :=\ & -\exp\left(\frac{b_2\left(-\mu_2-\sqrt{\mu_2^2+2\sigma_2^2\lambda_{k-1,n}}\right)}{\sigma_2^2}\right)+\exp\left(\frac{b_2\left(-\mu_2+\sqrt{\mu_2^2+2\sigma_2^2\lambda_{k-1,n}}\right)}{\sigma_2^2}\right)\\ &\widetilde{\Upsilon}_2(\mu_2,\sigma_2,b_2,\lambda_{k,n})\\ :\ & = 2\cosh\left(\frac{b_2\sqrt{\mu_2^2+2\sigma_2^2\lambda_{k,n}}}{\sigma_2^2}\right)\sqrt{\mu_2^2+2\sigma_2^2\lambda_{k,n}}-2\sinh\left(\frac{b_2\sqrt{\mu_2^2+2\sigma_2^2\lambda_{k,n}}}{\sigma_2^2}\right)\mu_2\end{aligned}$$

- Discounted Financing.

By Proposition 4.9.6 in order to find the moments of the discounted financing for the first insurer (X-insurer) in a de Finetti - de Finetti model we have to solve the PDE

$$\frac{\sigma_1^2}{2}\mathcal{V}_{1xx}^{(-)}(x,y;k)+\frac{\sigma_2^2}{2}\mathcal{V}_{1yy}^{(-)}(x,y;k)+\mu_1\mathcal{V}_{1x}^{(-)}(x,y;k)+\mu_2\mathcal{V}_{1y}^{(-)}(x,y;k) = k\delta\mathcal{V}_1^{(-)}(x,y;k) \quad (5.3.82)$$

subject to the boundary conditions

$$\frac{\partial}{\partial x}\mathcal{V}_1^{(-)}(x,y;k)|_{x=0} = -k\mathcal{V}_1^{(-)}(0,y;k-1) \quad (5.3.83)$$

$$\frac{\partial}{\partial x}\mathcal{V}_1^{(-)}(x,y;k)|_{x=b_1} = 0 \quad (5.3.84)$$

$$\mathcal{V}_1^{(-)}(x,0;k) = 0 \quad (5.3.85)$$

$$\frac{\partial}{\partial y}\mathcal{V}_1^{(-)}(x,y;k)|_{x=b_2} = 0 \quad (5.3.86)$$

By (5.A.31)–(5.A.33), (5.A.1)–(5.A.5) and the principle of superposition the solution is

$$\begin{aligned}\mathcal{V}_1^{(-)}(x,y;k) &= \sum_{n=0}^{\infty}c_{k,n}^{(-)}\exp\left(\frac{-y\mu_2}{\sigma_2^2}\right)\sin\left(\frac{y\sqrt{-\mu_2^2+2(\lambda_{k,n}-k\delta)\sigma_2^2}}{\sigma_2^2}\right)\times\\ &\times\exp\left(\frac{x\left(\sqrt{\mu_1^2+2\sigma_1^2\lambda_{k,n}}-\mu_1\right)}{\sigma_1^2}\right)\frac{\widetilde{\zeta}(x;\mu_1,\sigma_1,b_1,\lambda_{k,n})}{\mu_1+\sqrt{\mu_1^2+2\sigma_1^2\lambda_{k,n}}}\end{aligned}$$

where

$$\widetilde{\zeta}(x;\mu_1,\sigma_1,b_1,\lambda_{k,n}) := \left(1-\exp\left(\frac{2\left(b_1-x\right)\sqrt{\mu_1^2+2\sigma_1^2\lambda_{k,n}}}{\sigma_1^2}\right)\right)\mu_1 + \\ +\sqrt{\mu_1^2+2\sigma_1^2\lambda_{k,n}}\left(1+\exp\left(\frac{2\left(b_1-x\right)\sqrt{\mu_1^2+2\sigma_1^2\lambda_{k,n}}}{\sigma_1^2}\right)\right)$$

and the eigenvalues $\lambda_{k,n}$ are calculated by the solution of (5.A.3) (with b replaced by b_2, μ_1 by μ_2, σ_1 by σ_2) and for large values of n are approximated by (5.A.4). By the condition (5.3.83) and using (5.A.6) we have

$$\frac{\partial}{\partial x}\mathcal{V}_1^{(-)}(x,y;k)|_{x=0} = -k\mathcal{V}_1^{(-)}(0,y;k-1)$$

hence we obtain

$$c_{k,n}^{(-)}\frac{2\left(1-\exp\left(\frac{2b_1\sqrt{\mu_1^2+2\sigma_1^2\lambda_{k,n}}}{\sigma_1^2}\right)\right)\lambda_{k,n}}{\mu_1+\sqrt{\mu_1^2+2\sigma_1^2\lambda_{k,n}}} = -kc_{k-1,n}^{(-)}\frac{\widetilde{\zeta}(0;\mu_1,\sigma_1,b_1,\lambda_{k-1,n})}{\mu_1+\sqrt{\mu_1^2+2\sigma_1^2\lambda_{k-1,n}}} \Longrightarrow$$

$$c_{k,n}^{(-)} = -kc_{k-1,n}^{(-)}\frac{\mu_1+\sqrt{\mu_1^2+2\sigma_1^2\lambda_{k,n}}}{\mu_1+\sqrt{\mu_1^2+2\sigma_1^2\lambda_{k-1,n}}}\frac{\widetilde{\zeta}(0;\mu_1,\sigma_1,b_1,\lambda_{k-1,n})}{2\left(1-\exp\left(\frac{2b_1\sqrt{\mu_1^2+2\sigma_1^2\lambda_{k,n}}}{\sigma_1^2}\right)\right)\lambda_{k,n}}$$

5.4 Policy Making.

We start this section with a question. We suppose that an insurer, say X, has to choose to collaborate with one out of n other insurers, say Y_i, $i = 1, ..., n$. How the *X-insurer* can make the best choice in terms of *maximizing* his *survival probability* or in terms of *maximizing* his *expected discounted dividends*? He wants to be able to conclude that the cooperation with, say the Y_4-*insurer,* is the cooperation that gives him the maximum *expected discounted dividends* from all the other cooperations. Also if the best cooperation is not possible then he wants to be able to move to the next best cooperation, for example he wants to be in position to conclude that the cooperation, say with the Y_8-*insurer* gives him the maximum *expected discounted dividends* excluding the Y_4-*insurer.* How he can construct a policy, that is to *rank his available choices* from the *best choice* to the *worst choice*?

The results we find in the previous sections can be used in order to answer these questions and help an insurance company to construct a *policy.*

Definition 5.4.1 *Let us suppose that we have a* X-insurer *who has to choose to collaborate with one out of* n *other* insurers $Y_i, i = 1, ..., n$. *We will call* best policy *for the* X-insurer *with respect of his* survival probability*, and denote it by* $\pi^P(X)$, *the vector*

$$\pi^P(X) := \Big(\; Y_{(1)}, \quad Y_{(2)}, \quad .. \quad Y_{(i)} \quad Y_{(i+1)} \quad ... \quad Y_{(n)} \; \Big)$$

with the property

$$P(x, y_{(i)}) \geq P(x, y_{(j)}) \quad , \; for \quad i < j, \quad i, j = 1, 2, ..., n.$$

where $P(x, y_{(i)})$ *is the* survival probability *for the* X-insurer *when he collaborates with the* $\mathrm{Y}_{(i)}$-insurer.

We will call best policy *for the* X-insurer *with respect of his* expected discounted dividends*, and denote it by* $\pi^V(X)$, *the vector*

$$\pi^V(X) := \Big(\; Y_{(1)}, \quad Y_{(2)}, \quad .. \quad Y_{(i)} \quad Y_{(i+1)} \quad ... \quad Y_{(n)} \; \Big)$$

with the property

$$V(x, y_{(i)}) \geq V(x, y_{(j)}) \quad , \; for \quad i < j, \quad i, j = 1, 2, ..., n.$$

where $V(x, y_{(i)})$ *is the* expected discounted dividends *for the* X-insurer *when he collaborates with the* $\mathrm{Y}_{(i)}$-insurer.

In order to proceed with the analysis of how to construct a policy, it is preferable in this section to show the dependency of all the parameters of the problem and thus we write the *survival probability* $P(x, y)$ as

$$P(x, y) \equiv P((x, \mu_x, \sigma_x; b), (y, \mu_y, \sigma_y))$$

and the *expected discounted dividends* $V(x, y)$ as

$$V(x, y) \equiv V((x, \mu_x, \sigma_x; b), (y, \mu_y, \sigma_y))$$

We think that it would be better to present the method of constructing a policy for an insurance company with the aid of an example.

Example 3 *Let us consider an insurer which we call* X-insurer *which follows a* dividends policy *with* one reflecting barrier *at* $b = 8$ *and* one absorbing barrier *at* 0*. He has* initial capital $x = 5$ *and his* reserves *are described by a diffusion process with* drift coefficient $\mu_x = 1$ *and* volatility coefficient $\sigma_x = 3$.

Suppose that the X-insurer *has to choose to collaborate with one out of ten other insurance companies, each one of them follows the* Lundberg model *with* initial capital, drift *and* volatility coefficients *as in the following table*

Y-insurer	y	μ_y	σ_y	$P((5,1,3;8),(y,\mu_y,\sigma_y))$	$V((5,1,3;8),(y,\mu_y,\sigma_y))$
1	*2*	*1*	*2*	*0.326795*	*8.67359*
2	*2*	*2*	*5*	*0.690534*	*4.0308*
3	*3*	*1*	*7*	*0.797428*	*2.74389*
4	*1*	*1*	*4*	*0.844774*	*2.06148*
5	*4*	*3*	*4*	*0.206494*	*10.1494*
6	*2*	*3*	*6*	*0.693488*	*3.96526*
7	*5*	*2*	*4*	*0.245622*	*9.70484*
8	*3*	*2*	*4*	*0.43425*	*7.30096*
9	*2*	*4*	*3*	*0.165838*	*10.6374*
10	*2*	*3*	*5*	*0.598884*	*5.16358*

How to choose the best company to collaborate in terms of maximizing his survival probability *and in terms of* maximizing his expected discounted dividends *?*

In order to find the best choice in terms of maximizing the survival probability *we will make use of the results of the example 2 in section 4.8. We make the calculations and present the* *results, that is the respective* survival probabilities, *in the fifth column of the above table. We conclude that the best choice for the* X-insurer *in terms of* maximizing *his* survival probability *is to collaborate with the* Y-insurer numbered with 4*. If this is not possible then his next best choice is the* Y-insurer numbered with 3*. In this way the* X-insurer *has ordered the possible collaborations according to the* maximization of his survival probability *and thus he has constructed his policy for choosing the best partner, which is*

$$\pi^P(X) = \begin{pmatrix} Y_4, & Y_3, & Y_6, & Y_2, & Y_{10}, & Y_8, & Y_1, & Y_7, & Y_5, & Y_9 \end{pmatrix}$$

In the following figures (Figure 3.1-Figure 3.6) we plot the survival probability *as a function of one or two of the parameters y, μ_y and σ_y while having all the other parameters fixed.*

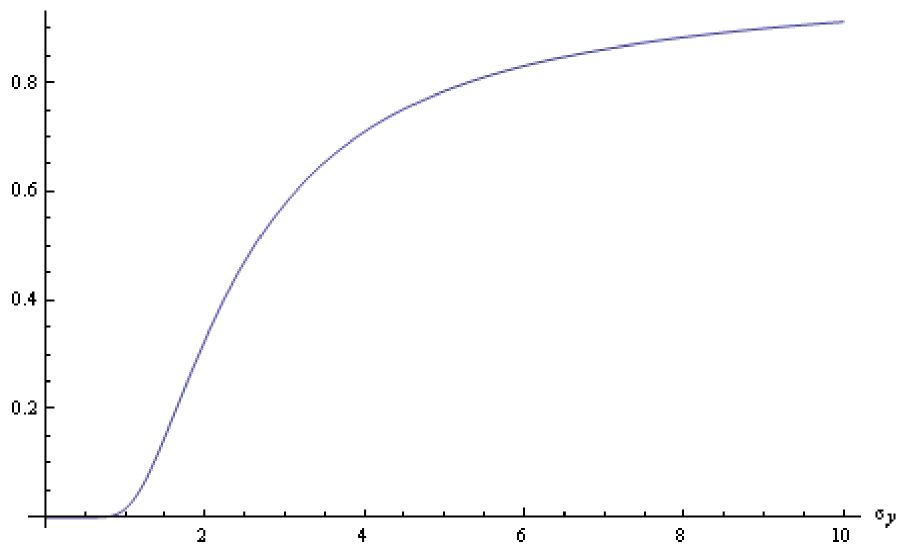

Figure 3.1: Survival Probability $P((5,1,3;8),(2,1,\sigma_y))$

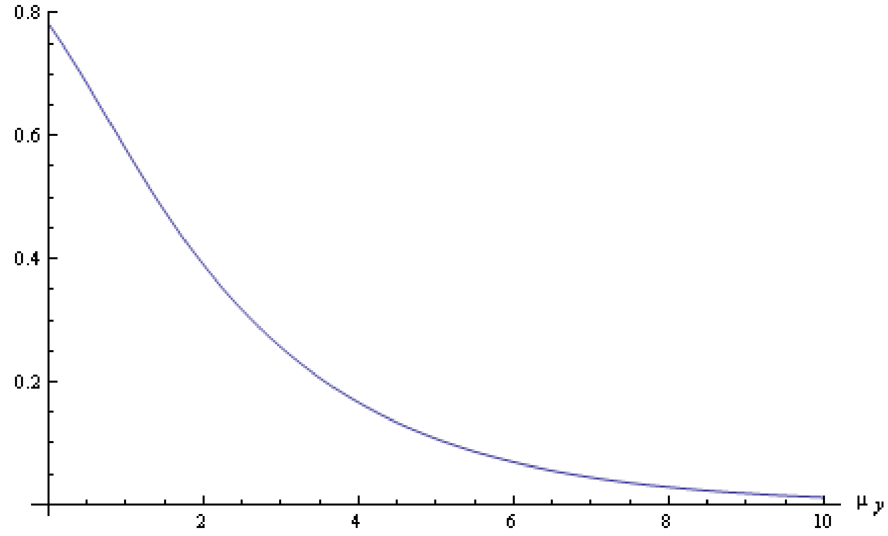

Figure 3.2: Survival Probability $P((5,1,3;8),(2,\mu_y,3))$

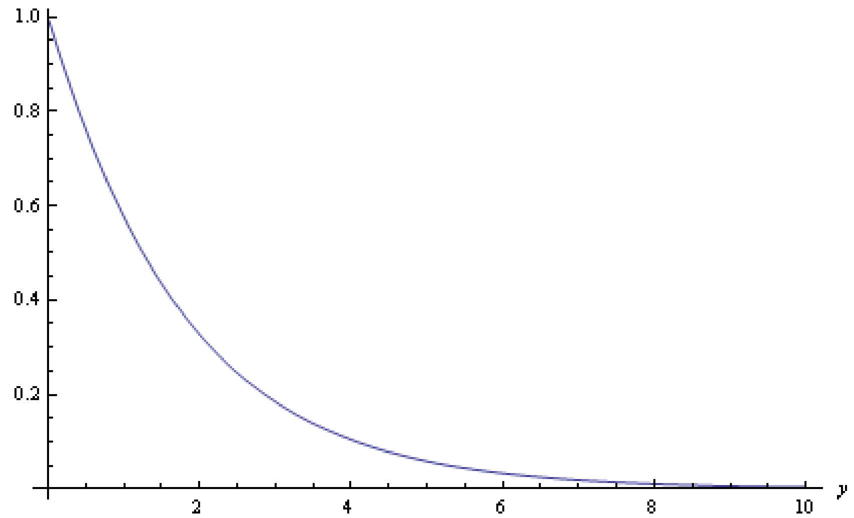

Figure 3.3: Survival Probability $P((5,1,3;8),(y,1,2))$

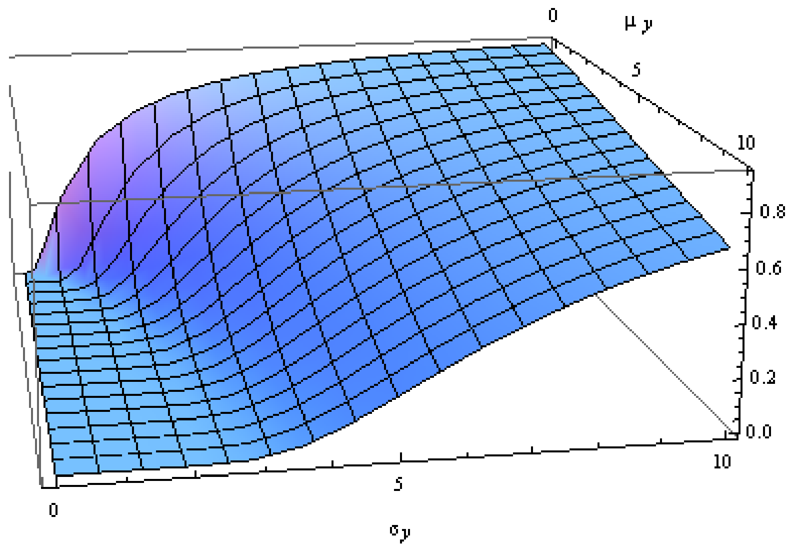

Figure 3.4: Survival Probability $P((5,1,3;8),(2,\mu_y,\sigma_y))$

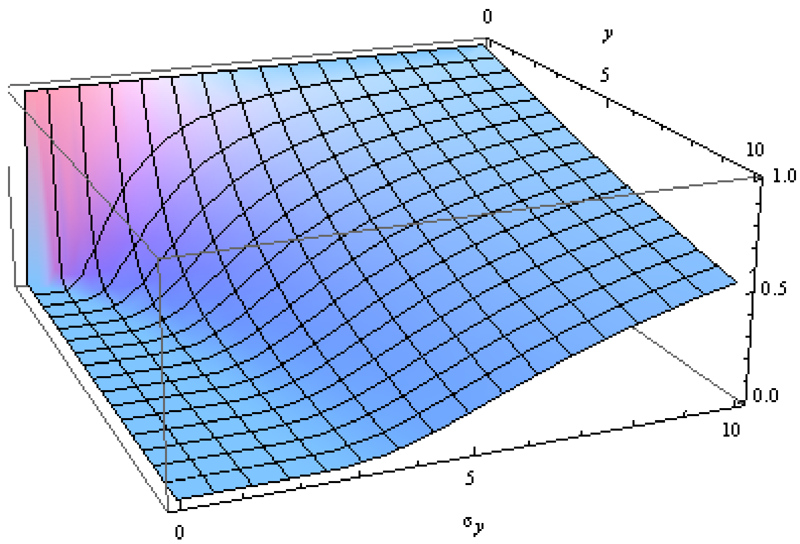

Figure 3.5: Survival Probability $P((5,1,3;8),(y,2,\sigma_y))$

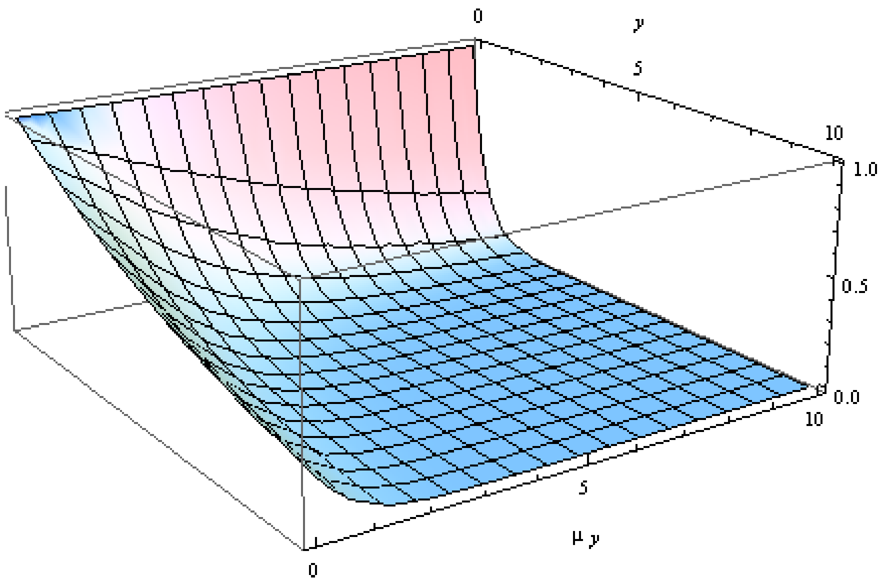

Figure 3.6: Survival Probability $P((5, 1, 3; 8), (y, \mu_y, 3))$

Next in order to find the best choice for the X-insurer *in terms of* maximizing *his* expected discounted dividends *we consider the section 5.2.2. We make the calculations and present the results, that is the respective* expected discounted dividends, *in the sixth column of the above table. We conclude that the best choice for the* X-insurer *in terms of* maximizing *his* expected discounted dividends *is to collaborate with the* Y-insurer numbered with 9*. If this is not possible then his* next best choice *is the* Y-insurer numbered with 5*. In this way the* X-insurer *has ordered the possible collaborations according to the* maximization *of his* expected discounted dividends *and thus he has constructed his policy for choosing the best partner, which is*

$$\pi^V(X) = \begin{pmatrix} Y_9, & Y_5, & Y_7, & Y_1, & Y_8, & Y_{10}, & Y_2, & Y_6, & Y_3, & Y_4 \end{pmatrix}$$

In the following figures (Figure 3.7-Figure 3.12) we plot the expected discounted dividends *as a function of one or two of the parameters y, μ_y and σ_y while having all the other parameters fixed.*

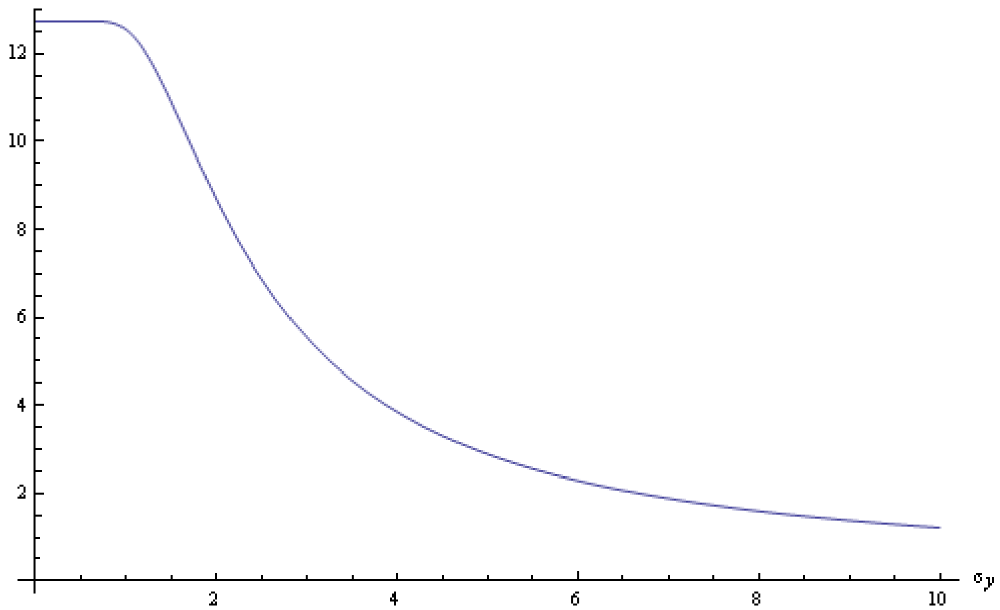

Figure 3.7: Expected Discounted Dividends $V((5,1,3;8),(2,1,\sigma_y))$

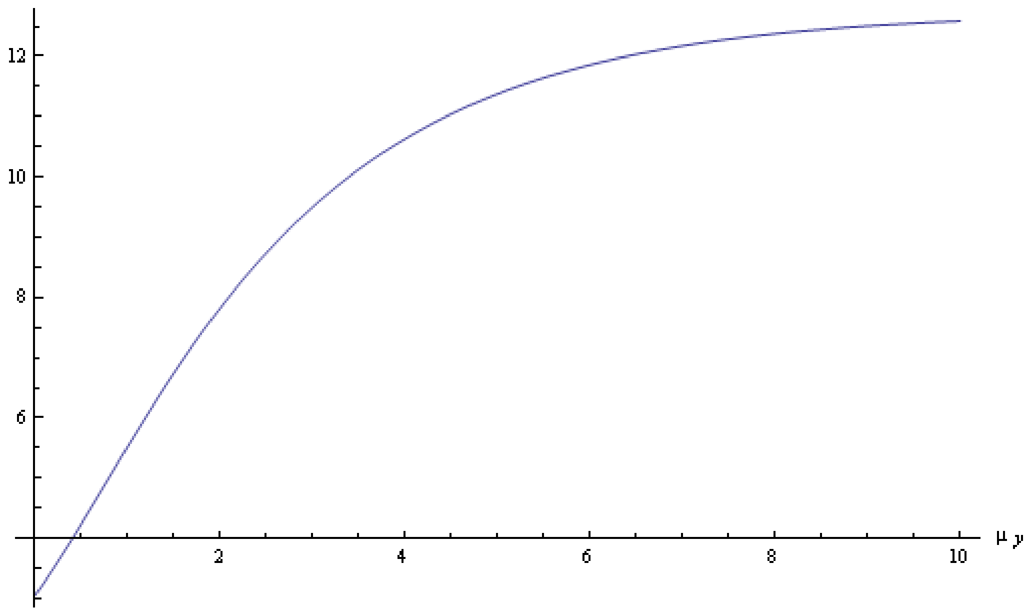

Figure 3.8: Expected Discounted Dividends $V((5,1,3;8),(2,\mu_y,3))$

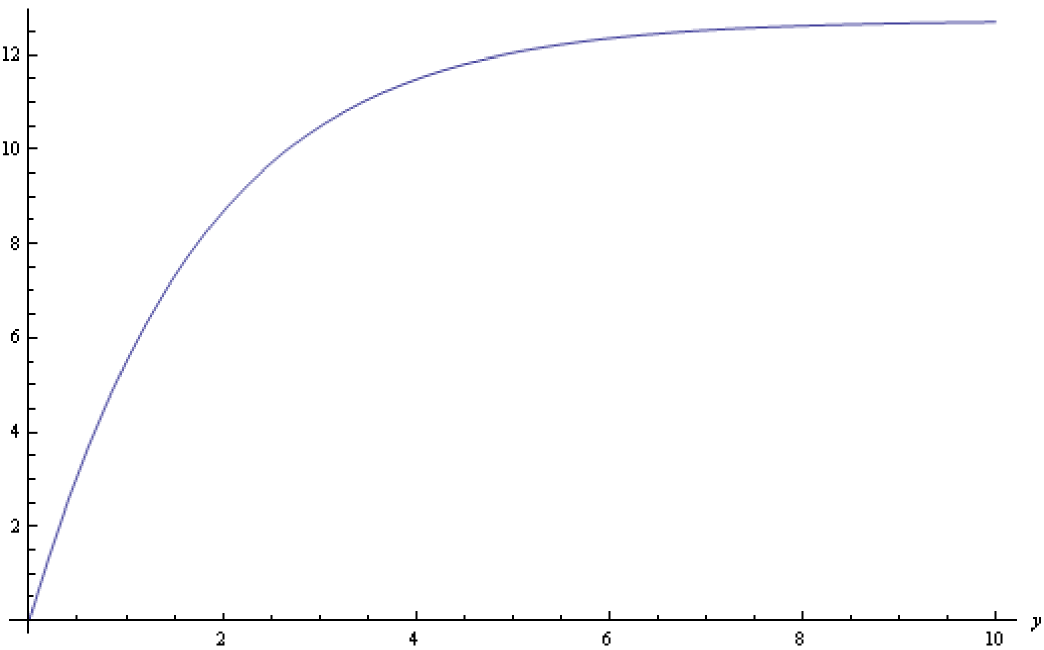

Figure 3.9: Expected Discounted Dividends $V((5,1,3;8),(y,1,2))$

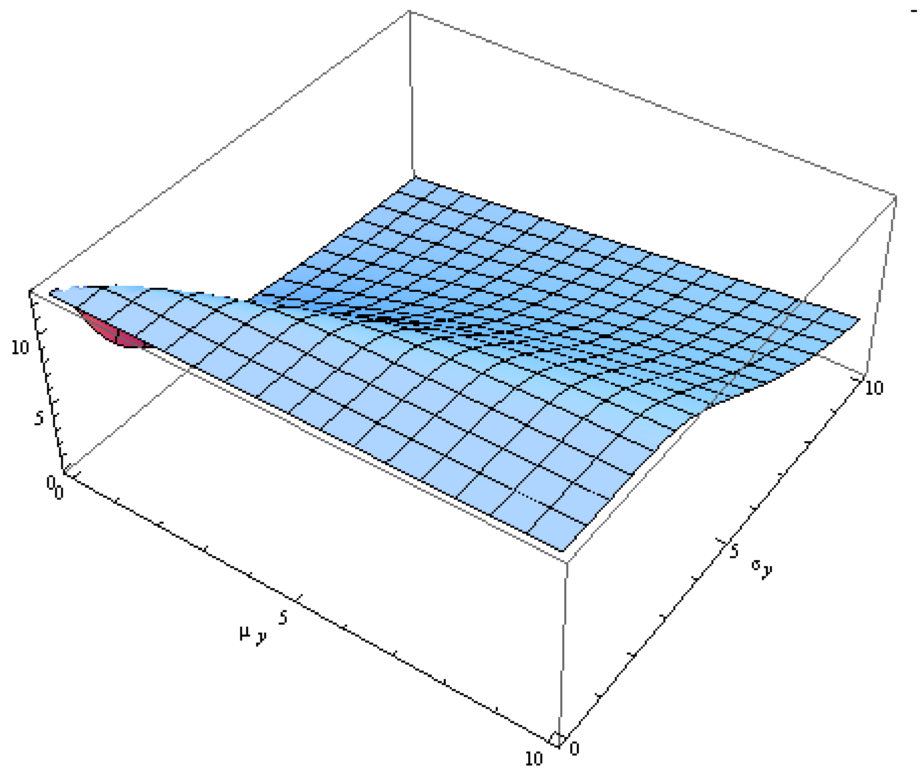

Figure 3.10: Expected Discounted Dividends $V((5,1,3;8),(2,\mu_y,\sigma_y))$

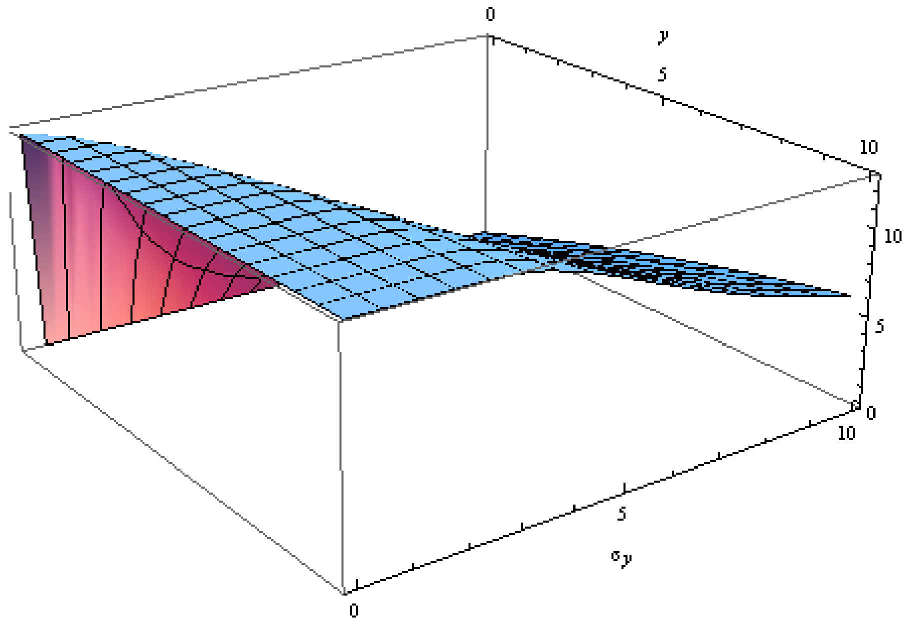

Figure 3.11: Expected Discounted Dividends $V((5,1,3;8),(y,2,\sigma_y))$

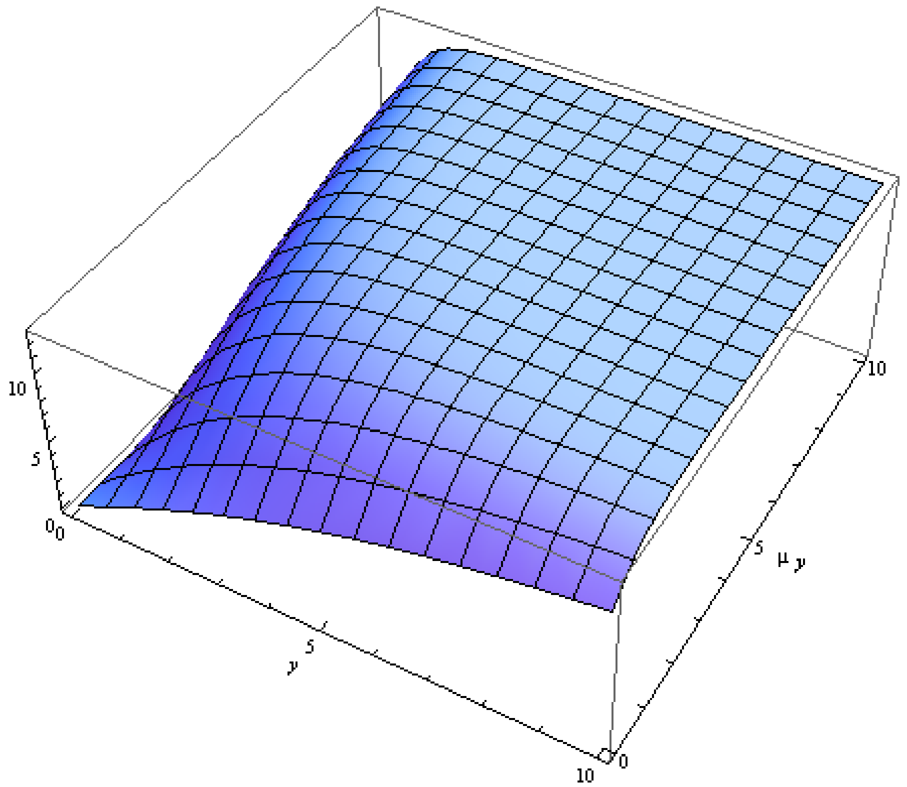

Figure 3.12: Expected Discounted Dividends $V((5,1,3;8),(y,\mu_y,3))$

From the above example we observe that the policy of an insurance company who wants to maximize the survival probability is different from the policy of maximizing the expected discounted dividends. We see that in this example the policy of maximizing the survival probability is the opposite from the policy of maximizing the expected discounted dividends.

5.5 Conclusions.

We applied the formulas of chapter 4 in the case of two insurance companies cooperation. We considered two models:

- The de Finetti - Lundberg model.
- The de Finetti - de Finetti model.

We found the moments of the discounted dividends and the discounted financing and the Laplace transform of the time of ruin. We also found the survival probability for the first insurer (X-insurer) in a de Finetti - de Finetti model.

We showed how an insurance company can use the formulas of chapter 4 for policy making purposes.

5.A Appendix of chapter 5.

In applying the formulas we found it is needed to be solved particular types of differential equations. For this reason we summarize first some results on some particular types of differential equations and boundary conditions in order to reference to them later.

- ODE-1.

The ode :

$$\frac{\sigma_1^2}{2} Q_{xx}(x) + \mu_1 Q_x(x) \mp (\lambda_k - k\delta) Q(x) = 0 \tag{5.A.1}$$

with boundary conditions:

$$\begin{aligned} Q(0) &= 0 \\ Q_x(b) &= 0 \end{aligned} \tag{5.A.2}$$

has general solution :

$$\begin{aligned} Q(x;\mu_1,\sigma_1,\lambda,b) &= c_1 \exp\left(\frac{x\left(-\mu_1 - \sqrt{\mu_1^2 \pm 2(\lambda_k - k\delta)\sigma_1^2}\right)}{\sigma_1^2}\right) + \\ &\quad + c_2 \exp\left(\frac{x\left(\sqrt{\mu_1^2 \pm 2(\lambda_k - k\delta)\sigma_1^2} - \mu_1\right)}{\sigma_1^2}\right) \end{aligned}$$

Because of the boundary condition $Q(0) = 0$ the solution becomes :

$$Q(x;\mu_1,\sigma_1,\lambda) = \exp\left(\frac{-x\mu_1}{\sigma_1^2}\right) \sin\left(\frac{x\sqrt{-\mu_1^2 \mp 2(\lambda_k - k\delta)\sigma_1^2}}{\sigma_1^2}\right)$$

In order the above solution to be in the real numbers without to be trivial it must hold that

$$\mp(\lambda_k - k\delta) > \frac{\mu_1^2}{2\sigma_1^2}$$

By the condition $Q_x(b) = 0$ we have

$$\tan\left(\frac{b\sqrt{-\mu_1^2 \mp 2(\lambda_k - k\delta)\sigma_1^2}}{\sigma_1^2}\right) = \frac{\sqrt{-\mu_1^2 \mp 2(\lambda_k - k\delta)\sigma_1^2}}{\mu_1} \tag{5.A.3}$$

The above transcendental equation can be solved numerically and for large values of n the eigenvalues are can be approximated by

$$\mp(\lambda_{k,n} - k\delta) \approx \frac{(2n+1)^2\pi^2\sigma_1^4 + 4b^2\mu_1^2}{8b^2\sigma_1^2} \tag{5.A.4}$$

So the general solution of (5.A.1) is

$$Q(x;\mu_1,\sigma_1,\lambda,b)=\sum_{n=0}^{\infty}c_{k,n}\exp\left(\frac{-x\mu_1}{\sigma_1^2}\right)\sin\left(\frac{b\sqrt{-\mu_1^2\mp 2\left(\lambda_{k,n}-k\delta\right)\sigma_1^2}}{\sigma_1^2}\right) \tag{5.A.5}$$

Remark 5.A.1 *From (5.A.3) we see that* $\forall\ k_1,k_2\in R$ *it holds that:*

$$\lambda_{k_1,n}-k_1\delta=\lambda_{k_2,n}-k_2\delta \tag{5.A.6}$$

- ODE-2.

The ODE

$$\frac{\sigma_1^2}{2}R_{xx}(x)+\mu_1R_x(x)\mp(\lambda_k-k\delta)R(x)=0 \tag{5.A.7}$$

with boundary conditions

$$\begin{aligned}R(0)&=0 \\ R_x(b)&=f(b)\end{aligned} \tag{5.A.8}$$

has solution

$$\begin{aligned}&R(x;\mu_1,\sigma_1,\lambda,b)\\ =\ &\exp\left(\frac{(b-x)\left(\mu_1+\sqrt{\mu_1^2\pm 2\left(\lambda_k-k\delta\right)\sigma_1^2}\right)}{\sigma_1^2}\right)\frac{\widetilde{r}_1(x;\mu_1,\sigma_1,\lambda_k,k,\delta)}{\widetilde{r}_2(b;\mu_1,\sigma_1,\lambda_k,k,\delta)}f(b)\end{aligned} \tag{5.A.9}$$

where

$$\widetilde{r}_1(x;\mu_1,\sigma_1,\lambda_k,k,\delta) := \left(-1+\exp\left(\frac{2x\sqrt{\mu_1^2\pm 2\left(\lambda_k-k\delta\right)\sigma_1^2}}{\sigma_1^2}\right)\right)\sigma_1^2 \tag{5.A.10}$$

$$\begin{aligned}\widetilde{r}_2(b;\mu_1,\sigma_1,\lambda_k,k,\delta) :=\ &\sqrt{\mu_1^2\pm 2\left(\lambda_k-k\delta\right)\sigma_1^2}\left(1+\exp\left(\frac{2b\sqrt{\mu_1^2\pm 2\left(\lambda_k-k\delta\right)\sigma_1^2}}{\sigma_1^2}\right)\right)+\\ &+\left(1-\exp\left(\frac{2b\sqrt{\mu_1^2\pm 2\left(\lambda_k-k\delta\right)\sigma_1^2}}{\sigma_1^2}\right)\right)\mu_1\end{aligned} \tag{5.A.11}$$

- ODE-3.

The ODE

$$\frac{\sigma_1^2}{2}F_{xx}(x)+\mu_1F_x(x)\mp(\lambda_k-k\delta)F(x)=0 \tag{5.A.12}$$

with boundary conditions

$$\begin{aligned} F_x(0) &= 0 \\ F_x(b) &= 0 \end{aligned} \tag{5.A.13}$$

has general solution

$$\begin{aligned} & F(x;\mu_1,\sigma_1,\lambda) \\ = \;& \exp\left(\frac{-x\mu_1}{\sigma_1^2}\right)\left(c\cos\left(\frac{x\sqrt{-\mu_1^2\mp 2(\lambda_k-k\delta)\sigma_1^2}}{\sigma_1^2}\right)+d\sin\left(\frac{x\sqrt{-\mu_1^2\mp 2(\lambda_k-k\delta)\sigma_1^2}}{\sigma_1^2}\right)\right) \end{aligned}$$

By the condition $F_x(0)=0$ we have

$$\begin{aligned} & F(x;\mu_1,\sigma_1,\lambda,b) \\ = \;& c\exp\left(\frac{-x\mu_1}{\sigma_1^2}\right)\left(\cos\left(\frac{x\sqrt{-\mu_1^2\mp 2(\lambda_k-k\delta)\sigma_1^2}}{\sigma_1^2}\right)+\frac{\mu_1\sin\left(\frac{x\sqrt{-\mu_1^2\mp 2(\lambda_k-k\delta)\sigma_1^2}}{\sigma_1^2}\right)}{\sqrt{-\mu_1^2\mp 2(\lambda_k-k\delta)\sigma_1^2}}\right) \end{aligned}$$

By the condition $F_x(b)=0$ we have $F_x(b)=0$ which implies

$$\frac{2\exp\left(-\frac{b\mu_1}{\sigma_1^2}\right)(\lambda_k-k\delta)c\sin\left(\frac{b\sqrt{-\mu_1^2\mp 2(\lambda_k-k\delta)\sigma_1^2}}{\sigma_1^2}\right)}{\sqrt{-\mu_1^2\mp 2(\lambda_k-k\delta)\sigma_1^2}}=0$$

or

$$\sin\left(\frac{b\sqrt{-\mu_1^2\mp 2(\lambda_k-k\delta)\sigma_1^2}}{\sigma_1^2}\right)=0$$

whence we obtain

$$\mp(\lambda_k-k\delta)=\frac{n^2\pi^2\sigma_1^4+b^2\mu_1^2}{2b^2\sigma_1^2} \tag{5.A.14}$$

and by the principle of superposition the solution becomes

$$F(x;\mu_1,\sigma_1,\lambda,b)=\sum_{n=0}^{\infty}c_n\exp\left(-\frac{x\mu_1}{\sigma_1^2}\right)\left(\cos\left(\frac{n\pi x}{b}\right)+\frac{b\mu_1}{n\pi\sigma_1^2}\sin\left(\frac{n\pi x}{b}\right)\right) \tag{5.A.15}$$

- ODE-4.

$$\frac{\sigma_1^2}{2}G_{xx}(x)+\mu_1 G_x(x)\mp(\lambda_k-k\delta)G(x)=0 \tag{5.A.16}$$

with boundary conditions

$$\begin{aligned} G_x(0) &= 0 \\ G_x(b) &= f(b) \end{aligned} \tag{5.A.17}$$

has solution

$$G(x;\mu_1,\sigma_1,\lambda_k,b,k,\delta) = \exp\left(\frac{(b-x)\mu_1}{\sigma_1^2}\right)\frac{\widetilde{\omega}_1(x;\mu_1,\sigma_1,\lambda_k,b,k,\delta)}{\widetilde{\omega}_2(b;\mu_1,\sigma_1,\lambda_k,k,\delta)}f(b) \tag{5.A.18}$$

where

$$\begin{aligned}\widetilde{\omega}_1(x;\mu_1,\sigma_1,\lambda_k,b,k,\delta) &:= \sqrt{\mu_1^2 \pm 2\left(\lambda_k - k\delta\right)\sigma_1^2}\cosh\left(\frac{x\sqrt{\mu_1^2 \pm 2\left(\lambda_k - k\delta\right)\sigma_1^2}}{\sigma_1^2}\right) + \\ &\quad + \sinh\left(\frac{x\sqrt{\mu_1^2 \pm 2\left(\lambda_k - k\delta\right)\sigma_1^2}}{\sigma_1^2}\right)\mu_1 \qquad (5.A.19)\\ \widetilde{\omega}_2(b;\mu_1,\sigma_1,\lambda_k,k,\delta) &:= 2(\lambda_k - k\delta)\sinh\left(\frac{b\sqrt{\mu_1^2 \pm 2\left(\lambda_k - k\delta\right)\sigma_1^2}}{\sigma_1^2}\right) \qquad (5.A.20)\end{aligned}$$

- ODE-5.

$$\frac{\sigma_1^2}{2}H_{xx}(x) + \mu_1 H_x(x) \mp (\lambda_k - k\delta)H(x) = 0 \tag{5.A.21}$$

with boundary conditions

$$\begin{aligned} H_x(0) &= -f(0) \qquad (5.A.22)\\ H_x(b) &= 0\end{aligned}$$

has solution

$$\begin{aligned} &H(x;\mu_1,\sigma_1,\lambda_k,b,k,\delta) \qquad (5.A.23)\\ = \;& \exp\left(x\left(\frac{\sqrt{\mu_1^2 \pm 2\left(\lambda_k - k\delta\right)\sigma_1^2} - \mu_1}{\sigma_1^2}\right)\right)\frac{\psi_1(x;\mu_1,\sigma_1,b,\lambda_k,k,\delta)}{\psi_2(\mu_1,\sigma_1,b,\lambda_n,k,\delta)}f(0)\end{aligned}$$

where

$$\begin{aligned}\psi_1(x;\mu_1,\sigma_1,b,\lambda_k,k,\delta) &:= \mu_1^2 + \mu_1\sqrt{\mu_1^2 \pm 2\left(\lambda_k - k\delta\right)\sigma_1^2} - \qquad (5.A.24)\\ &\quad - \left(1 + \exp(2(b-x)\frac{\sqrt{\mu_1^2 \pm 2\left(\lambda_k - k\delta\right)\sigma_1^2}}{\sigma_1^2}\right)(k\delta - \lambda_k)\,\sigma_1^2\\ \psi_2(\mu_1,\sigma_1,b,\lambda_n,k,\delta) &: = \left(1 - \exp(2b\frac{\sqrt{\mu_1^2 \pm 2\left(\lambda_k - k\delta\right)\sigma_1^2}}{\sigma_1^2})\right) \times\\ &\quad \times (k\delta - \lambda_k)\left(\mu_1 + \sqrt{\mu_1^2 \pm 2\left(\lambda_k - k\delta\right)\sigma_1^2}\right) \qquad (5.A.25)\end{aligned}$$

- ODE-6.

$$\frac{\sigma_1^2}{2}J_{xx}(x)+\mu_1 J_x(x)\mp(\lambda_k-k\delta)J(x)=0 \tag{5.A.26}$$

with boundary conditions

$$\begin{aligned} J(0) &= f(0) \\ J_x(b) &= 0 \end{aligned} \tag{5.A.27}$$

has solution

$$J(x;\mu_1,\sigma_1,\lambda_k,b)=\exp\left(-\frac{x\mu_1}{\sigma_1^2}\right)\frac{Q_1(x;\mu_1,\sigma_1,b,\lambda_k,\delta,k)}{Q_2(\mu_1,\sigma_1,b,\lambda_k,\delta,k)}f(0) \tag{5.A.28}$$

where

$$\begin{aligned} & Q_1(x;\mu_1,\sigma_1,b,\lambda_k,\delta,k) && (5.A.29)\\ := \; & \sqrt{\mu_1^2\pm 2(\lambda_k-k\delta)\sigma_1^2}+\mu_1\tanh\left(\frac{(b-x)\sqrt{\mu_1^2\pm 2(\lambda_k-k\delta)\sigma_1^2}}{\sigma_1^2}\right) \\ & Q_2(\mu_1,\sigma_1,b,\lambda_k,\delta,k) && (5.A.30)\\ := \; & \sqrt{\mu_1^2\pm 2(\lambda_k-k\delta)\sigma_1^2}-\mu_1\tanh\left(\frac{b\sqrt{\mu_1^2\pm 2(\lambda_k-k\delta)\sigma_1^2}}{\sigma_1^2}\right) \end{aligned}$$

- PDE-1

We consider the PDE

$$\frac{\sigma_1^2}{2}P_{xx}(x,y)+\frac{\sigma_2^2}{2}P_{yy}(x,y)+\mu_1 P_x(x,y)+\mu_2 P_y(x,y)=k\delta P(x,y) \tag{5.A.31}$$

We consider a solution of the form $P(x,y)=u(x)w(y)$. Substituting into (5.A.31) we have

$$\begin{aligned} \frac{\sigma_1^2}{2}u_{xx}(x)w(y)+\frac{\sigma_2^2}{2}u(x)w_{yy}(y)+\mu_1 u_x(x)w(y)+\mu_2 u(x)w_y(y) &= k\delta \Longrightarrow \\ \frac{\sigma_1^2}{2}\frac{u_{xx}(x)}{u(x)}+\frac{\sigma_2^2}{2}\frac{w_{yy}(y)}{w(y)}+\mu_1\frac{u_x(x)}{u(x)}+\mu_2\frac{w_y(y)}{w(y)} &= k\delta \Longrightarrow \end{aligned}$$

$$\frac{\sigma_1^2}{2}\frac{u_{xx}(x)}{u(x)}+\mu_1\frac{u_x(x)}{u(x)}=\lambda_k=-\frac{\sigma_2^2}{2}\frac{w_{yy}(y)}{w(y)}-\mu_2\frac{w_y(y)}{w(y)}+k\delta$$

and we conclude the ODE's

$$\frac{\sigma_1^2}{2}\frac{u_{xx}(x)}{u(x)}+\mu_1\frac{u_x(x)}{u(x)}=\lambda_k \tag{5.A.32}$$

$$\frac{\sigma_2^2}{2}\frac{w_{yy}(y)}{w(y)}+\mu_2\frac{w_y(y)}{w(y)}=k\delta-\lambda_k \tag{5.A.33}$$

Now we are ready to proceed and we will start with the one reflecting barrier case and continue with the two reflecting barriers case.

Chapter 6

Conclusions and Further Research

In this thesis we extended the de Finetti model in order to include barriers dividends policies with barriers that are diffusions. We made the extension in axiomatic manner by posing particular properties which was motivated by the classical de Finetti model. We showed that the de Finetti models with general barriers are well defined that is they are exists and are unique, or to say it in other words that there are exist unique stochastic processes that evolve according to our conditions. When we say unique stochastic processes we mean up to the degree of indistinguishability.

We considered de Finetti models with one general barrier meaning that when the reserves of the insurance company reach a "particular" level which depends upon a diffusion process then the company goes bankrupt. We also considered de Finetti models with two general barriers, that is when the reserves of the insurance company reach the level of the lower barrier, which also depends upon a diffusion process, then the insurance company has the option to borrow money and continue its operation.

We derived differential equations with appropriate boundary conditions, the solution of which gives the quantities for which we are interesting. More specifically we found differential equations with appropriate boundary conditions, the solution of which gives the moments of the discounted dividends, the discounted financing, the Laplace transform of the time of ruin, the Laplace transform of the joint distribution of the time of ruin and the discounted dividends and the Laplace transform of the joint distribution of the discounted dividends and the discounted financing.

We applied the formulas in special cases and more specifically in cases where the reserves process follows a Brownian motion, a Geometric Brownian motion and an Orstein-Uhlenbeck process.

Next we worked on another important issue, which is the situation of insurance companies cooperation. We considered this issue from the perspective of a particular insurance company. We were interesting to look at parameters which are vital to the decisions of the company. Among these parameters very important role we consider to play the probability of survival in a particular cooperation and the shares that will be given to the shareholders during this cooperation. We found differential equations with appropriate boundary conditions the solution of which will give:

- The moments of the discounted dividends and the discounted financing.
- The Laplace transform of the joint distribution of the time of ruin and the discounted dividends.
- The Laplace transform of the discounted dividends.
- The Laplace transform of the time of ruin.
- The Survival probability for one of the two insurers.

We applied the formulas we found in two models:

(I) The de Finetti - Lundberg model.

(II) The de Finetti - de Finetti model.

We showed how an insurance company can use the above results for policy making purposes. We also mentioned possible ways to extend the above considerations to various other models.

An interesting point for future research is to consider a de Finetti model with barriers that are continuous diffusions and reserve process which is Levy process. Also one could consider barriers that are semimartingales and an intermediate step to this direction is to consider first Levy processes as barriers. Another possibility is the inclusion of more economic aspects into the model as for example the case of investing the reserves in the stock market.

Also point for further research is the consideration of insurance companies cooperation in more general context, for example considering an insurer which follows the de Finetti model with general barriers. One could then try to extend the model in n-dimensions.

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