Ruin Theory Problems in Simple SDE Models with Large Deviation Asymptotics

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ABSTRACT

We examine hitting probability problems regarding the behavior of simple linear stochastic differential equations with exponential boundaries, related to problems arising in risk theory and asset and liability models in pension funds.

The first model we examine, in Chapter 2, is an Ornstein-Uhlenbeck (OU) process described by the Stochastic Differential Equation $dX_t = \mu X_t dt + \sigma dW_t$ with $X_0 = x_0$ given, where $\mu > 0$ and $\{W_t\}$ is standard Brownian motion. This model arises as a diffusion approximation of risk theory models in which the free reserves earn interest. The question posed then is that of determining the probability of hitting a lower deterministic boundary curve $v_0 e^{\beta t}$ and/or an upper boundary curve $u_0 e^{\alpha t}$ assuming that initially the free reserves lie between these values, i.e. $0 < v_0 < x_0 < u_0$ and that $\beta < \mu < \alpha$. Both the finite horizon "ruin probability problem" of determining the probability of hitting the boundary within a finite horizon, and the infinite horizon probability are examined. This problem may of course be formulated in terms of a second order PDE with curved (exponential) boundaries in the plane and solved numerically. (An alternative approach involving a time change argument is also discussed briefly in Chapter 2.) The main thrust of the analysis however involves Large Deviations techniques and in particular the Wentzell-Freidlin approach in order to obtain logarithmic asymptotics for the probability of hitting either the lower or the upper boundary. These low-noise asymptotics are valid when the variance σ is small and hence the event of hitting either boundary is rare. The exponential rate characterizing this probability is obtained by solving a variational problem which also gives the "path to ruin". We begin with a careful and detailed analysis of the finite horizon problem of hitting a lower boundary. The infinite horizon problem both for hitting the lower and the upper exponential boundary is treated using the transversality conditions approach of the calculus of variations. In addition, the OU process with a more general linear drift factor is examined, namely, the process resulting from the SDE $dX_t = (\mu X_t + r)dt + \sigma dW_t$ with the upper exponential boundary $u_0 e^{\alpha t}$ (with $0 < \mu < \alpha$).

We also consider, in the end of Chapter 2, the problem of two independent OU processes arising from the SDE's $dX_t = \alpha X_t dt + \sigma dW_t$, $dY_t = \beta Y_t dt + b dV_t$, $X_0 = x_0$, $Y_0 = y_0$ given. Also, $\{W_t\}$ and $\{V_t\}$ are independent standard Brownian motions. If $\alpha > \beta$ and $x_0 > y_0$ then, in the absence of noise, it would hold that $X_t > Y_t$ for all $t > 0$. We examine, again using the Wentzell-Freidlin approach, the probability that the two processes meet. The optimal paths followed by the two processes and the meeting time T is determined by solving a variational problem with trasversality conditions. Interestingly, the same model when a correlation is assumed between the two Brownian motions exhibits more complicated behavior if the correlation coefficient exceeds a certain threshold. This last case is discussed in chapter 4.

In Chapter 3 a corresponding problem involving a Geometric Brownian motion described by the SDE $dX_t = \mu X_t dt + \sigma X_t dW_t$ with $X_0 = x_0$ is examined, together with an upper and a lower exponential boundary. Again the Wentzell-Freidlin theory is used. In this case however, an exact solution is also possible, and therefore we are able to obtain an idea of the accuracy of the logarithmic asymptotics we propose. As expected, when the variance constant σ becomes smaller, the quality of the approximation improves. The case of two correlated Geometric Brownian motions is also discussed. These models are inspired by the Gerber and Shiu model of assets and liabilities in pension funds.

In Chapter 4, besides revisiting the problem of two Ornstein-Ulhenbeck processes in the presence of correlation, we also examine briefly OU processes with time-varying variance constant, arising from the SDE $dX_t = \mu X_t dt + \sigma(t) dW_t$. The hitting problem we examine has a lower exponential boundary and infinite horizon. The variational problem arising from the Wentzell-Freidlin method is tractable. However the equation giving the optimal hitting time may not have a unique solution. We solve an instance of this problem numerically in order to illustrate the approach.

1. INTRODUCTION

1.1 Ruin problems with compounding assets

Consider the following collective risk model: Claims are i.i.d. random variables $\{Y_i\}$, with distribution F on $\mathbb{R}^+$, and they occur according to an independent Poisson process with points $\{T_n\}$ and rate λ. We denote by $N(t) := \sum_{i=1}^{\infty} \mathbf{1}(T_i \le t)$ the corresponding counting process. Income from premiums comes at a constant rate c and the initial value of the free reserves is x_0. We assume further that free reserves accrue interest at a fixed rate β. If we denote by $Z_t := \sum_{i=1}^{N(t)} Y_i$, $t \ge 0$, the compound Poisson process describing the claim process then the free reserves can be described by the stochastic differential equation

$$dX_t \;=\; (\beta X_t + c)\,dt - dZ_t, \quad X_0 = x_0. \tag{1.1}$$

Along the above lines, Harrison [8] considered the following generalization of the classical model of collective risk theory. He assumed that the cumulative income of a firm is given by a process X with $X = \{X(t), t \ge 0\}$ be a stochastic process with stationary independent increments, finite variance and $x_0 = 0$. Then $Y(t)$ the assets of the firm at time t can be represented by a simple path-wise integral with respect to the income process X. He

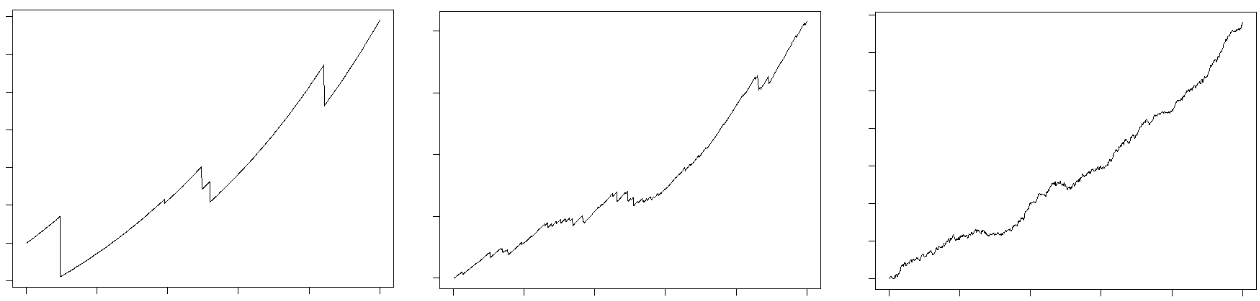

Fig. 1.1: Left: A sample path of the risk model. *Middle:* The same model with rescaled axes. *Right:* A sample path of the corresponding Ornstein-Uhlenbeck process driven by Brownian motion.

defined the corresponding assets process Y by

$$Y(t) = e^{\beta t} y + \int_0^t e^{\beta(t-s)} dZ(s), \quad t \geq 0, \tag{1.2}$$

with y positive level of initial assets and β positive interest rate. Harrison demonstrated that the Riemann-Stieltjes integral on the rigth side of (1.2) exists and is finite for all $t \geq 0$ and almost every sample path of Z. Thus the process is a small defined path-wise functional of the income process.

Typically $Z(t)$ may be a Lévy process with finite variation so that the stochastic integral in (1.2) may be defined pathwise. A model with $Z(t)$ being Brownian motion with drift would be natural as a diffusion approximation of such a model and this leads to the Ornstein-Uhlenbeck model we examine in detail in this thesis.

Models with compounding assets occur naturally in the study of pension funds as well Gerber and Shiu [4] have studied such models involving a pair of Geometric Brownian Motion processes with positive drift representing assets and liabilities over time and in this context ruin problems become relevant. With the notable exception of the Geometric Brownian Motion problems exact solutions are not possible and we will study these ruin problems related to these systems using Large Deviations techniques.

1.2 An overview of Large Deviation Results

Let $\mathcal{X}$ be a complete, separable metric space and $\mathcal{B}$ the Borel σ-field of its subsets. Let also $\{\mu_\epsilon\}$ be a family of probability measures on $(\mathcal{X}, \mathcal{B})$ and $I : \mathcal{X} \to [0, \infty]$ a (lower semicontinuous) function with values in the non-negative extended real numbers. Then, roughly speaking we say that the family of measures satisfies a *Large Deviation Principle* with *rate function* I if, as $\epsilon \to 0$, $\epsilon \log \mu_\epsilon(B) \approx -\inf_{x \in B} I(x)$. (The precise statement will be given presently.) Before giving the precise statement however we will state Cramér's theorem which will provide motivation for the definitions.

Theorem 1. *(Cramér). Suppose that $\{X_i\}$ are i.i.d. random variables with finite mean μ and moment generating function $M(\theta) := E[e^{\theta X_1}]$ (defined for all $\theta \in \mathbb{R}$ for which the expectation is finite). Then*

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left(\frac{1}{n} \sum_{i=1}^{n} X_i > x \right) \; = \; -I(x) \tag{1.3}$$

where the rate function *I is the Legendre-Fenchel transform of the cumulant function $\Lambda(\theta) := \log M(\theta)$ i.e.*

$$I(x) \; := \; \sup_{x \in \mathbb{R}} \{\theta x - \Lambda(\theta)\}. \tag{1.4}$$

Cramér's theorem dates from the 1930's and was a seminal result which provided the impetus for the initial work in Large Deviations Theory. Lundberg's exponent in the classical risk theory model plays precisely the same role as the rate function in the above theorem. For a comprehensive account of Risk Theory, both classical and modern, see [34].

Recall that a function f is *lower semicontinuous* at x iff, for every sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = x$, $\liminf_{n\to x} f(x_n) \geq I(x)$. A rate function $I : \mathcal{X} \to [0,\infty]$ is a lower semicontinuous function on $\mathcal{X}$ which implies that the level sets $\Psi_I(y) := \{x \in \mathcal{X} : I(x) \leq y\}$ are *closed subsets of* $\mathcal{X}$. A *good rate function* is one for which all the level sets $\Psi_I(y)$ are compact subsets of $\mathcal{X}$. The effective domain of the rate function I is the subset of $\mathcal{X}$, $\mathcal{D}_I := \{x : I(x) < \infty\}$ for which the rate function is finite.

The fact that the rate function I is lower-semicontinuous has as a consequence that the level sets of the form $\Psi(\alpha) := \{x : I(x) \leq \alpha\}$, $\alpha \in \mathbb{R}$ are *closed.* A rate function I is called *good* if all level sets $\Psi(\alpha)$ are *compact.* As usual, for any $\Gamma \subset \mathcal{X}$, $\bar{\Gamma}$ denotes the *closure* and Γ^o the *interior* of Γ.

With the above definition one may give a precise statement of the Large Deviation Principle (LDP):

Definition 2. *The family of measures on* $\{\mu_\epsilon\}$ *satisfies an LDP with rate function I if for all* $\Gamma \in \mathcal{B}$,

$$-\inf_{x\in\Gamma^o} I(x) \leq \liminf_{\epsilon\to 0} \epsilon \log \mu_\epsilon(\Gamma) \leq \limsup_{\epsilon\to 0} \epsilon \log \mu_\epsilon(\Gamma) \leq -\inf_{x\in\bar{\Gamma}} I(x). \tag{1.5}$$

Recall that a function $f : [0,T] \to \mathbb{R}$ is *absolutely continuous* if for all $\epsilon > 0$ there exists $\delta > 0$ such that, for all $n \in \mathbb{N}$, $0 < s_1 < t_1 < s_2 < t_2 < \cdots < s_n < t_n < T$ such that $\sum_{i=1}^n (t_i - s_i) < \delta$ implies $\sum_{i=1}^n |f(t_i) - f(s_i)| < \epsilon$. Clearly, an absolutely continuous function is continuous but the converse is not true. The set of all real, absolutely continuous functions on $[0,T]$ is denoted by $\mathcal{AC}[0,T]$.

A fundamental result in sample path Large Deviations theory is the following theorem due to Schilder [32]. Suppose that $\{W(t); t \in [0,1]\}$ is a Standard Brownian motion in $\mathbb{R}$ and define a family of processes $\{W_\epsilon(t); t \in [0,1]\}$ via $W_\epsilon(t)(t) := \sqrt{\epsilon}\, W(t)$ where $\epsilon > 0$.

Theorem 3 (Schilder)**.** *The family of measures* $\{\mu_\epsilon\}$ *induced by the family of processes* $\{W_\epsilon(t); t \in [0,1]\}$ *satisfies an LDP with good rate function*

$$I = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^1 f'(s)^2 ds & \text{if } f \in H_1 \\[2ex] +\infty & \text{otherwise} \end{cases}$$

where H_1 *is the Cameron-Martin space* $\{f \in \mathcal{AC}[0,T],\ f(0) = 0,\ \int_0^1 f'^2(s)ds < \infty\}$ *of absolutely continuous functions with square integrable derivatives.*

Wentzell-Freidlin theory generalizes this idea to Stochastic Differential Equations.

1.3 Wentzell-Freidlin theory

We begin with the following relatively simple problem. $\{X_t^\epsilon; t \in [0,1]\}$ is a family of real-valued diffusion processes defined on the same probability space, $(\Omega, \mathcal{F}, \mathbb{P})$ given by

$$X_t^\epsilon \;:=\; b(X_t^\epsilon)dt + \sqrt{\epsilon}dW_t, \qquad X_0^\epsilon = 0, \quad t \in [0,1], \;\; \epsilon \geq 0. \tag{1.6}$$

The function $b : \mathbb{R} \to \mathbb{R}$ is assumed to be uniformly Lipschitz-continuous, i.e. $|b(x) - b(y)| \leq B|x-y|$ for some $B > 0$ and all $x, y \in \mathbb{R}$. To simplify the exposition and the analysis to follow, the initial condition is assumed to be zero and the volatility term does not depend on the diffusion state. These restrictions will be later removed.

For each given $\epsilon > 0$ the stochastic differential equation in (1.6) has a unique solution which is a continuous function with probability 1 (see for instance [31, §5.2]). Let $C_0[0,1] := \{f : [0,1] \to \mathbb{R}, \text{s.t. } f(0) = 0,\ f \text{ continuous}\}$. Then $X_t^\epsilon \in C_0[0,1]$ and for each ϵ, it induces a probability measure $\widetilde{\mu}^\epsilon$ on $C_0[0,1]$.

Consider first the transformation $\Gamma : C_0[0,1] \to C_0[0,1]$ defined via $\phi = \Gamma(g)$ where

$$\phi(t) \;=\; g(t) + \int_0^t b(\phi(s))ds \quad \text{for } t \in [0,1] \quad \text{and } g \text{ continuous with } g(0) = 0. \tag{1.7}$$

The existence and uniqueness of a function $\phi \in C_0[0,1]$ satisfying (1.7) for a given function $g \in C_0[0,1]$ follows from the corresponding theorems on ordinary differential equations, given that b is uniformly Lipschitz continuous. Let μ^ϵ denote the measure induced on $C_0[0,1]$ by the Brownian motion $\{\sqrt{\epsilon}W_t; t \in [0,1]\}$ (where $\{W_t\}$ is Standard Brownian Motion). The measure $\tilde{\mu}_\epsilon$ on $C_0[0,1]$ can then be expressed as $\mu_\epsilon \circ \Gamma^{-1}$ where Γ^{-1} is the inverse map: If $\mathcal{B}(C_0[0,1])$ is the Borel σ-field of sets of continuous functions and $A \in \mathcal{B}(C_0[0,1])$ then $\Gamma^{-1}(A) = \{f \in C_0[0,1] : \Gamma(f) \in A\}$.

We note that Γ is an injective mapping: If g_i, $i = 1, 2$ are two different elements of $C_0[0,1]$ then since $\phi_i = \Gamma(g_i)$,

$$g_2(t) - g_1(t) \;=\; \phi_2(t) - \phi_1(t) - \int_0^t \left(b(\phi_2(s)) - b(\phi_1(s))\right) ds.$$

Since the left hand side in the above equation is not identically 0, then $\phi_2 - \phi_1$ cannot be identically zero.

Also, Γ is a continuous mapping. (The norm we use of course is the sup norm: $\|f\| = \sup_{t\in[0,1]} |f(t)|$). Then setting $\Delta(t) := |\phi_2(t) - \phi_1(t)|$ and $r(t) = |g_2(t) - g_1(t)|$ we have

$$\Delta(t) = \left| \int_0^t \left(b(\phi_2(s)) - b(\phi_1(s))\right) ds + g_2(t) - g_1(t) \right| \leq B \int_0^t \Delta(s)ds + r(t).$$

Then, as a result of Grownwal's inequality, if $r(t) \le \eta$,

$$\Delta(t) \le \eta\, e^{Bt}.$$

This establishes the continuity of the mapping Γ and hence, appealing to Schilder's theorem, we conclude that the measures induced on $C[0,T]$ by $\{X_t^\epsilon\}$ satisfy a Large Deviations Principle with good rate function

$$I(x) \;:=\; \frac{1}{2}\int_0^T \left(x'(t) - b(x(t))\right)^2 dt.$$

The precise arguments can be found in [3, §5.2] and [20] (chapter 2, theorem 2.25). For applications in Risk Theory see Asmussen and Steffensen [1].

1.4 Exit time of a diffusion from a deterministic boundary

The typical problem we examine in this thesis involves a family of SDE's parameterized by $\epsilon > 0$ which denotes the intensity of the noise factor

$$dX_t^\epsilon \;=\; \mu(X_t^\epsilon)dt + \sqrt{\epsilon}\,\sigma(X_t^\epsilon)dW_t, \qquad X_0^\epsilon = x_0 \tag{1.8}$$

together with the ODE ensuing when the noise factor is set to zero,

$$x'(t) \;=\; \mu(x(t)), \qquad x(0) = x_0, \tag{1.9}$$

and a second ODE,

$$u'(t) \;=\; \nu(u(t)), \qquad u(0) = u_0. \tag{1.10}$$

We assume that $\nu(x) \ge \mu(x)$ for all x and $x_0 < u_0$. Therefore $x(t) < u(t)$ for all $t \ge 0$ which means that the zero noise solution of (1.8) is always below the solution of (1.10). Of course, in the presence of noise, there is a positive probability that the solution of (1.8) exceeds that of (1.10). To be more specific, let $\tau_\epsilon \;=\; \inf\{t \ge 0 : X_t^\epsilon = u(t)\}$ (with $\tau_\epsilon = +\infty$ when the set is empty). (We will suppose of course that the functions μ, σ, and ν satisfy the usual Lipschitz continuity and rate of growth conditions to insure (strong) existence and uniqueness of the solutions.)

The main object of this study is the evaluation of the finite and infinite horizon "ruin probability" $\mathbb{P}(\tau_e < \infty)$ which, in general, is analytically intractable. Thus, with the exception of the Geometric Brownian motion case, i.e. when $m(x) = mx$, $\sigma(x) = \sigma x$ and $\nu(x) = \nu x$ where explicit closed form expressions can be obtained, as is shown in chapter 3, we resort to logarithmic asymptotics obtained by large deviation arguments.

More specifically, we apply the Wentzell-Freidlin technique (see for instance [7] or [3]) to obtain the value of

$$\alpha := -\lim_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(\tau_\epsilon < \infty\right) \,. \tag{1.11}$$

We will examine in detail the case of the Ornstein-Uhlenbeck process, corresponding to drift $\mu(x) = \mu x$ and volatility $\sigma(x) = \sigma$ with $\mu > 0$ as well as other linear SDE's.

2. LOW NOISE ASYMPTOTICS FOR THE ORNSTEIN-UHLENBECK PROCESS

In this chapter we examine an Ornstein-Uhlenbeck (OU) process with positive infinitesimal drift and consider the probability of hitting an upper or a lower exponential boundary. The problem is approached using the Wentzell-Freidlin theory for obtaining logarithmic asymptotics both for the finite and the infinite horizon problem. An OU process with an additional constant term in the drift is also examined. Interestingly, depending on the value of the constant drift, the variational problem from which the rate function is obtained, may not have a unique solution.

2.1 The Ornstein-Uhlenbeck SDE and the time to exit from a deterministic boundary

Consider the Ornstein-Uhlenbeck Stochastic Differential Equation (SDE)

$$dX_t = \mu X_t dt + \sigma dW_t, \qquad X_0 = x_0 \tag{2.1}$$

where $\mu > 0$. Note that its expectation increases exponentially with time according to $\mathbb{E}X_t = x_0 e^{\mu t}$, $t \geq 0$. Consider also the deterministic exponential function given by

$$V(t) = v_0 e^{\beta t} \quad \text{where} \quad 0 \leq \beta < \mu \quad \text{and} \quad 0 < v_0 < x_0. \tag{2.2}$$

Let

$$p(x_0, T) \;=\; \mathbb{P}(X_t > V(t);\ 0 \leq t \leq T) \tag{2.3}$$

denote the probability that the process $\{X_t\}$ stays above the exponential boundary $V(t)$. In this model $1 - p(x_0, T)$ may be thought of as a type of *ruin probability*. We are interested in evaluating $p(x_0, T)$ and the limiting probability $p(x_0) := \lim_{T\to\infty} p(x_0, T)$ for the process given in (2.1) with boundary given by (2.2). Due to the Markovian property of $\{X_t\}$, the "non-ruin probability" defined in (2.3) satisfies the PDE

$$\frac{1}{2}\sigma^2 f_{xx} + \mu x f_x + f_t \;=\; 0, \quad \text{in } D := \{(x,t) : 0 < t < T,\ x > v_0 e^{\beta t}\} \tag{2.4}$$

with boundary conditions $f(v_0 e^{\beta t}, t) = 0$ for $t \in [0, T]$ and $f(x, T) = 1$ for $x > v_0$.

We will not attempt to obtain an expression for the solution of (2.4) due to the difficulty introduced by the shape of the domain D. One may obtain numerical results for the ruin probability based on the above formulation. We will instead use Wentzell-Freidlin "low noise asymptotics" [7] in order to obtain a large deviations estimate for the probability that X_t crosses the path of $V(t)$ for some $t \in [0, T]$. There has of course been significant work on first-passage times in OU processes, in particular we mention L. Alili, P. Patie, and J.L. Pedersen [49].

2.2 A time-change approach to the Ornstein-Uhlenbeck ruin problem

Consider the two sided problem

$$dX_t \;=\; \mu X_t dt + \sigma dW_t, \quad X_0 = x_0$$

with an upper boundary given by the curve $U(t) := u_0 e^{\alpha t}$ and a lower boundary given by $V(t) := v_0 e^{\beta t}$. We assume that $0 < v_0 < x_0 < u_0$ and $0 < \beta < \mu < \alpha$. We are interested in the hitting time $T = \inf\{t \geq 0 : X_T \geq U(T) \text{ or } X_T \leq V(T)\}$. (Of course, if the set is empty, the hitting time is equal to $+\infty$ corresponding to the case where the process never exits from one of the two boundary curves.) The Ornstein-Uhlenbeck process has the solution

$$X_0 \;=\; x_0 e^{\mu t} + \sigma \int_0^t e^{\mu(t-s)} dW_s$$

The condition

$$V(t) < X_t < U(t)$$

is equivalent to $e^{-\mu t} V(t) < e^{-\mu t} X_t < e^{-\mu t} U(t)$ or

$$v_0 e^{-(\mu-\beta)t} < x_0 + \sigma \int_0^t e^{-s\mu} dW_s < u_0 e^{(\alpha-\mu)t}. \tag{2.5}$$

The stochastic integral $\xi(t) := \sigma \int_0^t e^{-s\mu} dW_s$ is a Gaussian process with independent intervals and variance function

$$\mathsf{Var}(\xi(t)) = \sigma^2 \int_0^t e^{-2\mu s} ds = \frac{\sigma^2}{2\mu}\left(1 - e^{-2\mu t}\right).$$

Note that the limit $\lim_{t\to\infty} \mathsf{Var}(\xi(t)) = \frac{\sigma^2}{2\mu}$ is finite. Consider the time change function $\tau(t)$ defined by

$$\tau(t) = \frac{\sigma^2}{2\mu}\left(1 - e^{-2\mu t}\right), \qquad t \in [0, \infty) \tag{2.6}$$

The inverse function (which necessarily exists since $\mathsf{Var}(\xi(t))$ is an increasing function) is

$$t(\tau) = -\frac{1}{2\mu} \log\left(1 - \frac{2\mu\tau}{\sigma^2}\right), \qquad \tau \in \left[0, \frac{\sigma^2}{2\mu}\right) \tag{2.7}$$

Applying this change of time to the double inequality (2.5) we obtain

$$v_0 e^{(\mu-\beta)\frac{1}{2\mu}\log\left(1-\frac{2\mu\tau}{\sigma^2}\right)} < x_0 + \sigma \int_0^{-\frac{1}{2\mu}\log\left(1-\frac{2\mu\tau}{\sigma^2}\right)} e^{-s\mu} dW_s < u_0 e^{-(\alpha-\mu)\frac{1}{2\mu}\log\left(1-\frac{2\mu\tau}{\sigma^2}\right)}, \qquad \tau \in \left[0, \frac{\sigma^2}{2\mu}\right).$$

However, $\tilde{W}_\tau := \sigma \int_0^{-\frac{1}{2\mu}\log\left(1-\frac{2\mu\tau}{\sigma^2}\right)} e^{-s\mu} dW_s$ is standard Brownian motion. (It can easily be seen that it is a continuous martingale with quadratic variation function $\langle \tilde{W} \rangle_\tau = \tau$.) Thus we have the equivalent problem

$$v_0 \left(1 - \frac{2\mu\tau}{\sigma^2}\right)^{\frac{\mu-\beta}{2\mu}} < x_0 + \tilde{W}_\tau < u_0 \left(1 - \frac{2\mu\tau}{\sigma^2}\right)^{-\frac{\alpha-\mu}{2\mu}}, \qquad \tau \in \left[0, \frac{\sigma^2}{2\mu}\right). \tag{2.8}$$

In general, the passage time – hitting probability problem associated with (2.8) must be solved numerically. Of course the time change transformation may have computational advantages. There is a great deal of work, both theoretical and applied, regarding passage times and hitting probabilities of Brownian motion with curving boundaries. In the special case where $\alpha = \beta = \mu$ an exact solution exists. In general we have not been able to obtain closed form expressions even with a single boundary even in the few cases where exact solutions are known, such as for a parabolic boundary: When $\beta = 0$ then the time-changed lower bound is $v_0\sqrt{1 - \frac{2\mu\tau}{\sigma^2}}$. While this is a parabolic boundary, the results that have obtained for this case, [37], [38], apply when it acts as an *upper* and not a lower boundary. Therefore, the exact solution in this case is not known, to the best of our knowledge.

A two–boundary case: $\alpha = \beta = \mu$. In that case (2.8) becomes

$$v_0 - x_0 < \tilde{W}_\tau < u_0 - x_0, \qquad \tau \in \left[0, \frac{\sigma^2}{2\mu}\right).$$

The exact probability of never exiting either boundary, can be obtained from the well known expression for the density of standard Brownian motion (starting at zero) with absorbing boundaries at a, b, $(a, b > 0)$. If $p(x,t)dx := \mathbb{P}\left(W_t \in (x, x+dx);\ -b < W_s < a,\ 0 \le s \le t\right)$, then, (see [23, p.222])

$$p(x,t) = \sum_{n=1}^{\infty} \frac{2}{a+b} \sin\left(\frac{n\pi b}{a+b}\right) e^{-\lambda_n t} \sin\left(n\pi \frac{x+b}{a+b}\right),$$
$$\text{where } \lambda_n = \frac{1}{2}\frac{n^2\pi^2}{(a+b)^2}, \quad n = 1, 2, \ldots.$$

Then

$$\mathbb{P}\left(-b < W_s < a, \quad \text{for } 0 \le s \le t\right) = \int_{-b}^{a} p(x,t)dx,$$

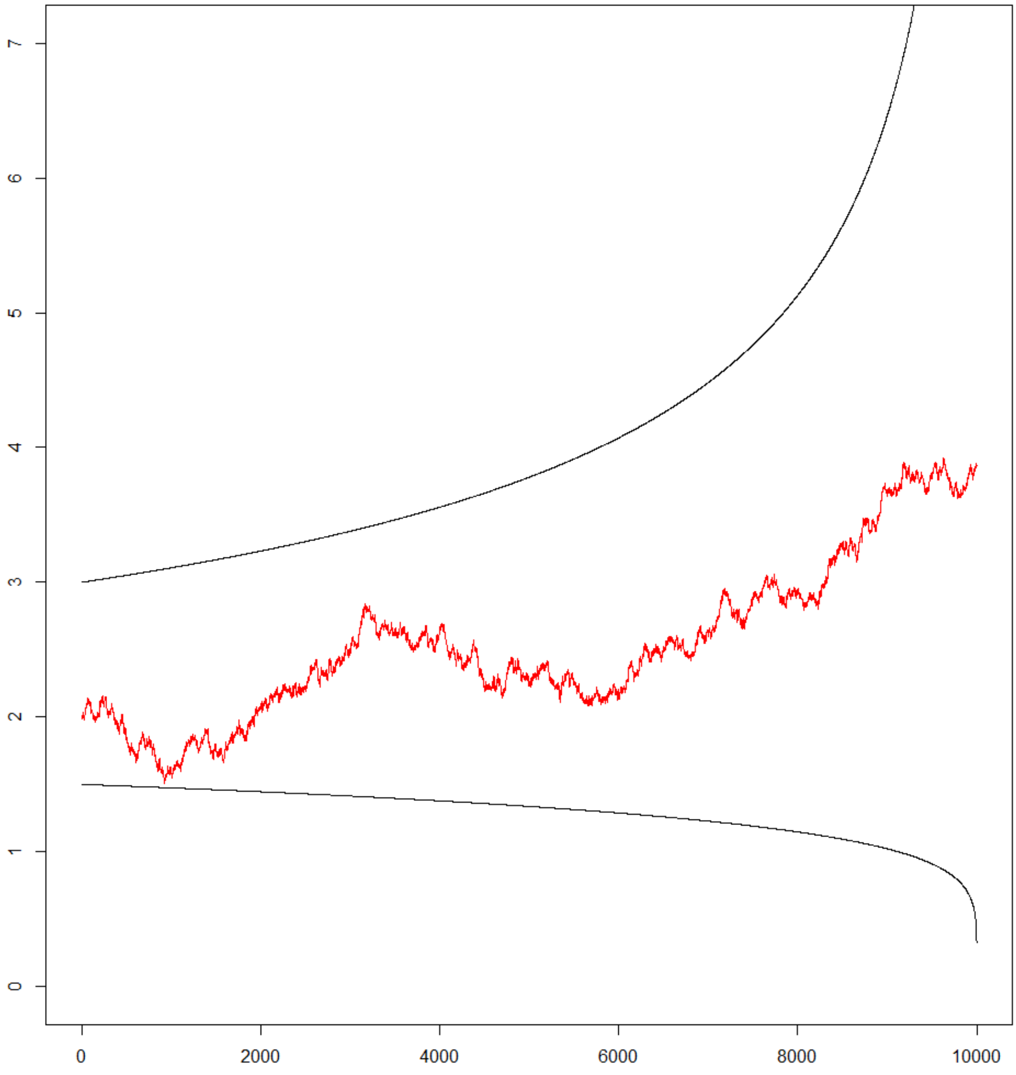

Fig. 2.1: Time-change in an Ornstein-Uhlenbeck ruin problem.

and in our case $-b = v_0 - x_0$, $a = u_0 - x_0$, $t = \frac{\sigma^2}{2\mu}$. Hence,

$$\mathbb{P}\left(-b < W_s < a,\ 0 \leq s \leq \frac{\sigma^2}{2\mu}\right) \tag{2.9}$$
$$= \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \exp\left(-\frac{(2k+1)^2\pi^2\sigma^2}{2(u_0-v_0)^2\mu}\right) \sin\frac{(2k+1)\pi(x_0-v_0)}{u_0-v_0}.$$

2.3 The Wentzell-Freidlin Framework - Finite Horizon Problem

To express the problem discussed in the previous section in the Wentzell-Freidlin framework we consider the family of processes $\{X_t^\epsilon\}$

$$dX_t^\epsilon = \mu X_t^\epsilon dt + \sqrt{\epsilon}\,\sigma\, dW_t, \quad X_0^\epsilon = x_0 \tag{2.10}$$

together with the deterministic process

$$\dot{x}(t) = \mu x(t), \quad x(0) = x_0.$$

Denote by $C[0,T]$ the set of continuous functions on $[0,T]$, and by $C_{x_0}[0,T]$ the set of all continuous functions $f : [0,T] \to \mathbb{R}$ with $f(0) = x_0$. Consider the transformation $F : C[0,T] \to C_{x_0}[0,T]$ defined by

$$f = F(g) \quad \text{with} \quad f(t) := \int_0^t \mu f(s) ds + \sigma g(t), \ \ t \in [0,T]. \tag{2.11}$$

Let f_i, denote the solution of (2.11) when the driving function is g_i, $i = 1, 2$. We may then establish the continuity of the map F by means of a Gronwall argument which shows that

$$\|f_1 - f_2\| \leq \sigma\, e^{\mu T} \|g_1 - g_2\|.$$

Theorem 5.6.7 of [3, p. 214] applies and therefore the solution of (2.10) satisfies a Large Deviation Principle with good rate function

$$I(f,T) := \begin{cases} \frac{1}{2}\int_0^T \left(f'(t) - \mu f(t)\right)^2 \sigma^{-2} dt & \text{if } f \in \mathcal{H}^1_{x_0} \\ +\infty & \text{otherwise} \end{cases} \tag{2.12}$$

where $\mathcal{H}^1_{x_0}(T) := \{f : [0,T] \to \mathbb{R} : f(t) = x_0 + \int_0^t \phi(s)ds\ ,\ t \in [0,T], \phi \in L^2[0,T]\}$ is the Cameron-Martin space of absolutely continuous functions with square integrable derivative with initial value $f(0) = x_0$.

Theorem 4. *In the above framework, if the lower boundary curve is $V(t) = v_0 e^{\beta t}$,*

$$\lim_{\epsilon\to 0} \epsilon \log \mathbb{P}\left(\min_{t\in[0,T]} X_t^\epsilon - V(t) \le 0\right) = -I_V(T). \tag{2.13}$$

The rate is given by

$$I_V(T) = \begin{cases} 2\mu\dfrac{\left(v_0 e^{\beta T} - x_0 e^{\mu T}\right)^2}{e^{2\mu T}-1} & \text{if } T \le t_V^o \\ \\ 2\mu\dfrac{\left(v_0 e^{\beta t_V^o} - x_0 e^{\mu t_V^o}\right)^2}{e^{2\mu t_V^o}-1} & \text{if } T > t_V^o \end{cases} \tag{2.14}$$

where t_V^o is the unique positive solution of the equation

$$\left(1 - \frac{\beta}{\mu}\right) e^{(\mu+\beta)t} + \frac{\beta}{\mu} e^{(\beta-\mu)t} = \frac{x_0}{v_0}. \tag{2.15}$$

Similarly, for the upper boundary curve $U(t) = u_0 e^{\alpha t}$,

$$\lim_{\epsilon\to 0} \epsilon \log \mathbb{P}\left(\max_{t\in[0,T]} X_t^\epsilon - U(t) \ge 0\right) = -I_U(T) \tag{2.16}$$

with

$$I_U(T) = \begin{cases} 2\mu\dfrac{\left(u_0 e^{\alpha T} - x_0 e^{\mu T}\right)^2}{e^{2\mu T}-1} & \text{if } T \le t_U^o \\ \\ 2\mu\dfrac{\left(u_0 e^{\beta t_U^o} - x_0 e^{\mu t_U^o}\right)^2}{e^{2\mu t_U^o}-1} & \text{if } T > t_U^o \end{cases} \tag{2.17}$$

where t_U^o is the unique positive solution of the equation

$$\frac{\alpha}{\mu} e^{(\alpha-\mu)t} - \left(\frac{\alpha}{\mu} - 1\right) e^{(\mu+\alpha)t} = \frac{x_0}{u_0}. \tag{2.18}$$

Proof. Part 1. We begin by fixing $t > 0$ and considering paths that start at x_0 at time 0 and end at $V(t) := v_0 e^{\beta t}$ at time t: Consider the set

$$\mathcal{H}^1_{x_0,V(t)} := \left\{h : [0,t] \to \mathbb{R} : h(s) = x_0 + \int_0^s \phi(u)du\ ,\ s \in [0,t],\ h(t) = V(t), \phi \in L^2[0,t]\right\}.$$

Then, for $\eta > 0$,

$$\lim_{\epsilon\to 0} \epsilon \log \mathbb{P}\left(\sup_{0\le s\le t} |X_s^\epsilon - h(s)| < \eta\right) = -J_*(t). \tag{2.19}$$

where $J_*(t)$ is the solution of the variational problem

$$J_*(t) = \inf\left\{J(x;t) : x \in \mathcal{H}^1_{x_0,V(t)}\right\} \tag{2.20}$$

where

$$J(x;t) = \int_0^t F(x,x',u)du, \qquad \text{and} \quad F(x,x',u) = \frac{1}{2\sigma^2}\left(x' - \mu x\right)^2. \tag{2.21}$$

$J(x;t)$ gives the rate function for a path that starts at x_0 and meets the lower boundary at the point $(t, v_0 e^{\beta t})$ i.e. satisfies the boundary conditions

$$x(0) = x_0, \qquad x(t) = v_0 e^{\beta t}. \tag{2.22}$$

The infimum in (2.20) is taken over all absolutely continuous functions on $[0,t]$ with derivative in L^2. The function $x \in \mathcal{H}^1_{x_0,V_t)}[0,t]$ that minimizes the integral defining the rate function is the solution of the Euler-Lagrange equation (e.g. see [26], [2])

$$F_x - \frac{d}{du}F_{x'} = 0 \tag{2.23}$$

and the boundary conditions (2.22). With the given form of F in (2.21) the Euler-Lagrange equation becomes

$$x''(u) = \mu^2 x(u) \tag{2.24}$$

which has the general solution

$$x(u) = c_1 e^{\mu u} + c_2 e^{-\mu u}. \tag{2.25}$$

The values of c_1, c_2 for which x satisfies the boundary conditions are given by the unique solution of the system

$$\begin{bmatrix} 1 & 1 \\ e^{\mu t} & e^{-\mu t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x_0 \\ v_0 e^{\beta t} \end{bmatrix}.$$

We obtain

$$c_1 = \frac{v_0 e^{\beta t} - x_0 e^{-\mu t}}{e^{\mu t} - e^{-\mu t}}, \qquad c_2 = -\frac{v_0 e^{\beta t} - x_0 e^{\mu t}}{e^{\mu t} - e^{-\mu t}}. \tag{2.26}$$

Thus (2.25) with the constants c_1, c_2 given by (2.26) give the optimal path

$$\begin{aligned} x(u) &= \frac{v_0 e^{\beta t}\left(e^{\mu u} - e^{-\mu u}\right) + x_0\left(e^{\mu(t-u)} - e^{-\mu(t-u)}\right)}{e^{\mu t} - e^{-\mu t}} \\ &= \frac{v_0 e^{\beta t}\sinh(\mu u) + x_0 \sinh(\mu(t-u))}{\sinh(\mu t)} \end{aligned} \tag{2.27}$$

From (2.25)

$$x'(u) - \mu x(u) = -2\mu c_2 e^{-\mu u}$$

and, from this together with (2.21),

$$J_*(t) \;=\; 4\mu^2 c_2^2 \int_0^t e^{-2\mu u} du \;=\; 2\mu c_2^2\left(1 - e^{-2\mu t}\right).$$

Taking into account the expression for c_2 we have

$$J_*(t) \;=\; 2\mu \frac{\left(v_0 e^{\beta t} - x_0 e^{\mu t}\right)^2}{e^{2\mu t} - 1}. \tag{2.28}$$

There remains to show that there is no path $x(u)$ with piece-wise continuous derivative which achieves a smaller value of the criterion, i.e. that the optimal solution does not have corners. To this end we consider the Erdeman corner conditions ([2, §2.5]). The first condition requires that $F_{x'}$ evaluated at the critical path be a continuous function of u. Since $F_{x'} = \frac{1}{\sigma^2}(x' - \mu x)$ and $x(u)$ is necessarily continuous, the first Erdeman condition implies the continuity of $x'(u)$ as well. Therefore, by virtue of the first Erdeman condition alone we may conclude that the optimal solution cannot have discontinuities in its derivative. For the sake of completeness we mention that the second Erdeman condition requires that $F - x'F_{x'}$ evaluated at the critical path be also a continuous function of u. Since $F - x'F_{x'} = -\frac{1}{2\sigma^2}((x')^2 - \mu^2 x^2)$ and because of the continuity of $x(u)$, this second condition by itself would allow the existence of corners at which the first derivative changes sign. (Such corners are of course precluded by the first condition.)

The solution we have found corresponds to a global minimum. To see this (c.f. Theorem 3.16 [2, p.45]) it suffices to note that, setting $F(x, x') := \frac{1}{2\sigma^2}(x' - \mu x)^2$, then F is convex on $\mathbb{R}^2$. Indeed, we can show that, for any $(x_0', x_0) \in \mathbb{R}^2$,

$$F(x, x') \geq F(x_0, x_0') + F_x(x_0, x_0')\,(x - x_0) + F_{x'}(x_0, x_0')\,(x' - x_0')$$

or

$$\frac{1}{2}(x' - \mu x)^2 \geq \frac{1}{2}(x_0' - \mu x_0)^2 - \mu\,(x_0' - \mu x_0)\;(x - x_0) + (x_0' - \mu x_0)\;(x' - x_0').$$

This last inequality is equivalent to

$$\frac{1}{2}(x' - \mu x)^2 \geq -\frac{1}{2}(x_0' - \mu x_0)^2 \;+\; (x' - \mu x)\,(x_0' - \mu x_0)$$

or

$$(x' - \mu x)^2 + (x_0' - \mu x_0)^2 \;-\; 2\,(x' - \mu x)\,(x_0' - \mu x_0) \geq 0$$

which is clearly true.

Part 2. In the first part we obtained the *fixed time* optimal solution under the boundary conditions (2.22). These conditions need to be supplemented with the additional *path inequality constraint*

$$x(u) \geq V(u) \quad \text{for all } u \in [0, t]. \tag{2.29}$$

In this part however we will solve the optimization problem

$$I(T) := \inf\{J(x, t) : 0 \leq t \leq T, x \in \mathcal{H}^1_{x_0, V(t)}, \text{ i.e. } x \text{ satisfies the conditions (2.22) }\}$$

with finite time horizon $t \in [0, T]$, ignoring the inequality path constraints (2.29). Clearly $I(T) = \inf_{t \in [0,T]} J_*(t)$.

We will next establish that $J_*(t)$ is strictly convex and has a global minimum. From (2.28)

$$J_*'(t) = \frac{4 v_0 \mu^2 e^{mut}\left(x_0 e^{\mu t} - v_0 e^{\beta t}\right)}{(e^{2\mu t} - 1)^2}\left[\left(1 - \frac{\beta}{\mu}\right)e^{(\beta+\mu)t} + \frac{\beta}{\mu}e^{(\beta-\mu)t} - \frac{x_0}{v_0}\right]. \tag{2.30}$$

Define the function

$$\phi_1(t) = \left(1 - \frac{\beta}{\mu}\right) e^{(\beta+\mu)t} + \frac{\beta}{\mu} e^{(\beta-\mu)t}, \quad t \geq 0. \tag{2.31}$$

Since $0 < \beta < \mu$ and $0 < v_0 < x_0$, $x_0 e^{\mu t} - v_0 e^{\beta t} > 0$ for all $t \geq 0$ and thus the sign of $J_*'(t)$ is that of $\phi_1(t) - \frac{x_0}{v_0}$. Note that $\phi_1'(t) = \frac{\mu-\beta}{\mu} e^{(\beta+\mu)t} [\mu + \beta(1 - e^{-2\mu t})] > 0$ for all $t \geq 0$ and thus ϕ_1 is an increasing function. Also, $\phi_1(0) = 1$, and $\frac{x_0}{v_0} > 1$, hence there exists $t^o > 0$ such that

$$\phi_1(t^o) = \frac{x_0}{v_0} > 1. \tag{2.32}$$

In view of the expression (2.30), $J_*'(t) < 0$ for $0 \leq t < t^o$, $J_*(t^o) = 0$ and $J_*'(t) > 0$ for $t > t^o$. Thus t^o is a point of global minimum for J_*. Given the definition of the function ϕ_1, t^o is the unique solution of (2.15). Figure 2.2 illustrates the behavior of the function $J_*(t)$.

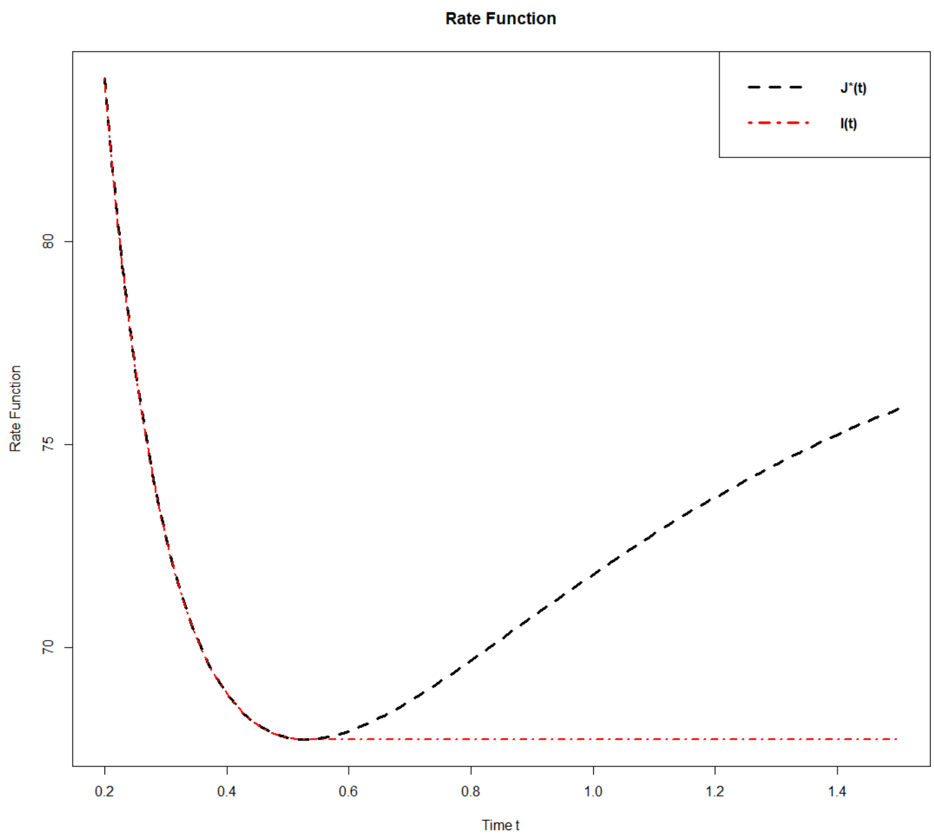

Fig. 2.2: The dotted black line denotes the function $J_*(t)$. The dotted red line denotes the rate function $I(t)$. Here $\mu = 2.5$, $\beta = 1.0$, $x_0 = 4$, $u_0 = 1$ and $t^o \approxeq 0.529$.

Then

$$I(T) = \inf_{t\in[0,T]} J_*(t) = \begin{cases} 2\mu\dfrac{\left(v_0e^{\beta T} - x_0e^{\mu T}\right)^2}{e^{2\mu T}-1} & \text{if } T \le t^o \\ \\ 2\mu\dfrac{\left(v_0e^{\beta t^o} - x_0e^{\mu t^o}\right)^2}{e^{2\mu t^o}-1} & \text{if } T > t^o \end{cases} \tag{2.33}$$

Part 3. We complete the proof by showing that the optimal rate given by (2.33) remains valid even after taking into account the path constraint (2.29). Define

$$J_{**}(t) = \inf\left\{J(x;t) : x \in \mathcal{H}^1_{x_0,V(t)}, x(u) \ge V(u) \text{ for } u \in [0,t].\right\} \tag{2.34}$$

In this part of the proof we will study the behavior of the optimal path $x(u)$ of (2.27) for all $u \ge 0$. Clearly $x(u) > 0$ for $0 \le u \le t$. By examining the numerator of (2.27) we see that, for $u > t$, $x(u) > 0$ iff

$$g(u) := \frac{\sinh(\mu u)}{\sinh(\mu(u-t))} - \frac{x_0}{v_0}e^{-\beta t} > 0.$$

We can easily see that $g'(u) < 0$ and hence g is strictly decreasing in (t,∞). Also, $\lim_{u\to t+} g(u) = +\infty$ and $\lim_{u\to\infty} g(u) = e^{\mu t} - \frac{x_0}{v_0}e^{-\beta t}$. Therefore there are two cases depending on whether the condition $e^{\mu t} - \frac{x_0}{v_0}e^{-\beta t} < 0$ or equivalently

$$e^{t(\mu+\beta)} < \frac{x_0}{v_0} \tag{2.35}$$

or, alternatively,

$$t < t_1 := \frac{1}{\mu+\beta}\log\frac{x_0}{v_0}. \tag{2.36}$$

If condition (2.35) holds then there exists a unique $u_0 \in (t,\infty)$ for which $g(u_0) = 0$ and therefore $x(u_0) = 0$. The derivative of x at u_0 is

$$x'(u_0) = \frac{\mu}{\sinh(\mu t)}\left(\cosh(\mu u) - x_0\cosh(\mu(u-t))\right).$$

However, by the definition of u_0, it also holds that

$$v_0e^{\beta t}\sinh(\mu u) = x_0\sinh(\mu(u-t)).$$

From these last two equations, together with the inequality $x_0 > v_0$ we can easily show that $x'(u_0) < 0$. Thus in the interval $[u_0,\infty)$, $x(u)$ satisfies the ODE (2.24) with initial conditions $x(u_0) = 0$, $x'(u_0) < 0$, and hence it must be negative in this whole interval. From the above analysis we conclude that when t is small enough to satisfy (2.35) then

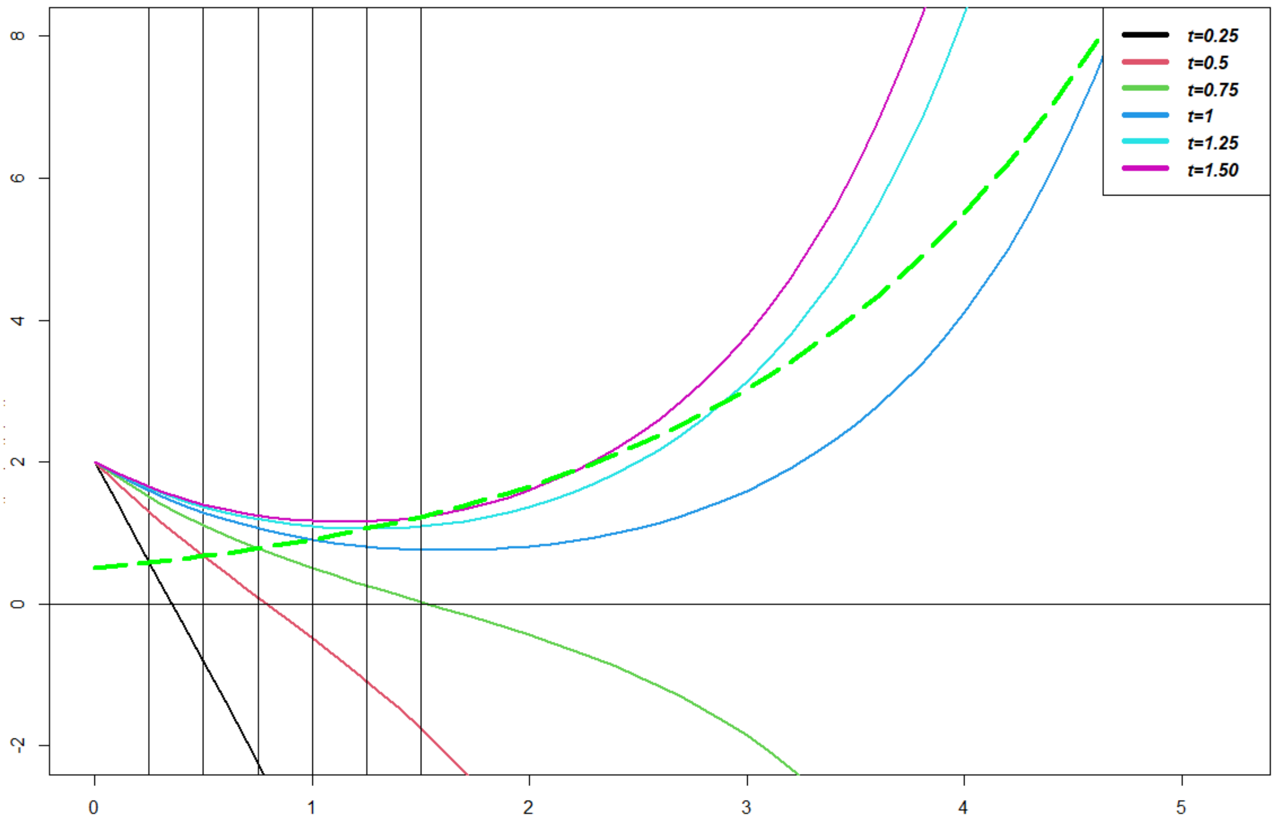

Fig. 2.3: Here $\mu = 1$, $\beta = 0.6$, $x_0 = 2$, $v_0 = 0.5$. The hitting times range from $t = 0.25$ to $t = 1.50$. Note that, for $t = 0.25$, 0.50, and 0.75 the path $x(u)$ eventually becomes negative, after hitting once $V(u)$, the dotted green line. In the rest of the cases the paths remain positive and intersect the dotted green line twice.

in the interval $[0, t)$, $x(u) > V(u)$ and hence the path inequalities constraints (2.29) are satisfied.

On the other hand, if $t \geq t_1$ then $x(u) > 0$ for all $u > 0$. Therefore, as a result of (2.24), $x''(u) > 0$ and the function x is strictly convex for $x \geq 0$. Figure 2.3 illustrates both cases. For $t = 0.25$, 0.5, and 0.75 (black, red, and green paths) the paths eventually become negative and intersect the dotted green line (i.e. $V(\cdot)$) once. In the rest of the cases the paths remain positive and intersect the dotted green line twice.

Since the curve $V(u) = v_0 e^{\beta u}$ is also convex there can be two points of intersection at most between them, or only one if the curve $x(u)$ is tangent to $v_0 e^{\beta u}$ at t. Thus if there are two intersection points, due the convexity of the curves and the fact that $x(0) = x_0 > v_0 = V(0)$, if $x'(t) < V'(t)$ then t is the first intersection point of the two curves, meaning that the path constraint $x(u) > V(u)$ is satisfied for all $0 \leq u < t$. Conversely, if $x'(t) > V'(t)$, then there exists $\tau(t)$ in the interval $(0, t)$ for which $x(\tau(t)) = V(\tau(t))$ and $x(u) < V(u)$ for $u \in (\tau(t), t)$. In this latter case, the inequality path constraint (2.29) is not satisfied. Thus the optimal path of Part 1 also satisfies the constraint (2.29) iff

$$x'(t) \;\leq\; V'(t). \tag{2.37}$$

From (2.27)

$$x'(t) \;=\; \mu\frac{v_0 e^{\beta t}\left(e^{\mu t}+e^{-\mu t}\right)-2x_0}{e^{\mu t}-e^{-\mu t}}. \tag{2.38}$$

Therefore (2.37) can be written as

$$\mu\frac{v_0 e^{\beta t}\left(e^{\mu t}+e^{-\mu t}\right)-2x_0}{e^{\mu t}-e^{-\mu t}} \;\leq\; v_0\beta e^{\beta t}.$$

or, equivalently, as

$$\left(1-\frac{\beta}{\mu}\right)e^{(\beta+\mu)t}+\left(1+\frac{\beta}{\mu}\right)e^{(\beta-\mu)t}\leq 2\frac{x_0}{v_0}. \tag{2.39}$$

If this condition is satisfied then the optimal path of part I also satisfies the constraint (2.29). Define the function

$$\phi_2(u) \;=\; \left(1-\frac{\beta}{\mu}\right)e^{(\beta+\mu)u}+\left(1+\frac{\beta}{\mu}\right)e^{(\beta-\mu)u}$$

It is easy to see that $\phi_2'(t) > 0$ and $\phi_2(0) = 2$. Hence, the equation $\phi_2(t) \;=\; 2\frac{x_0}{v_0}$ has a unique, positive solution, say t_2. Since the function $\phi_2(t)$ is increasing, if follows that

$$t < t_2 \tag{2.40}$$

is equivalent to condition (2.39).

Thus, when (2.40) is satisfied, the path given by (2.27) minimizes the functional $J(x,t)$ in (2.21) under the boundary conditions (2.22) and the path inequality constraints (2.29). Then

$$J_{**}(t) = J_*(t) \quad \text{when} \;\; t < t_2. \tag{2.41}$$

Figure 2.4 shows that for specific values of the parameters μ, β, x_0, v_0. For the values of the parameters in Figure 2.4 $t_0 = \frac{1}{2}\log 8 \approx 1.04$. Hence in the figure in the left the path $x(u)$ is decreasing and eventually becomes negative. There is a single intersection between the curves $x(u)$ and $V(u)$. On the other hand in the figure in the middle ($t = 2$) and in the right ($t = 3$) the path $x(u)$ is strictly convex, as is $V(u)$, and thus the two curves intersect in two points. For $t = 2$ the path $x(u)$ satisfies (2.39) and therefore (2.37) and (2.29) while for $t = 3$ it does not.

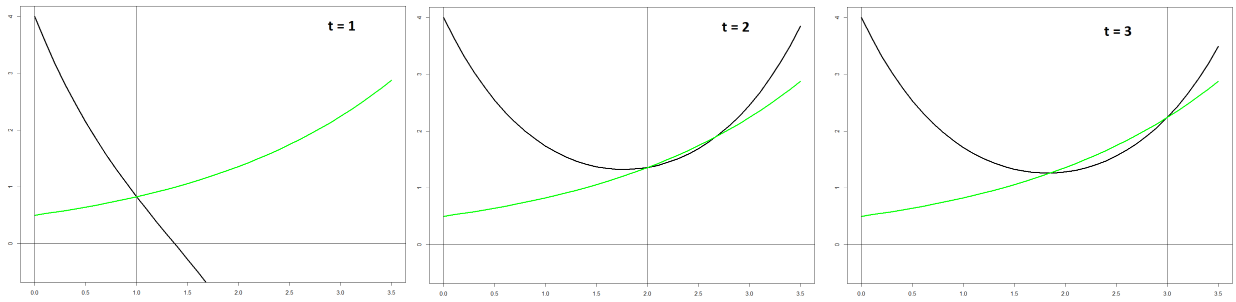

Fig. 2.4: Three cases. Here $\mu = 1$, $\beta = 1$, $x_0 = 4$, $v_0 = 0.5$. In the figure on the left the predetermined hitting time is $t = 1$. The optimal path $x(u)$, after meeting $V(u)$ at $t = 1$, keeps decreasing monotonically and eventually becomes negative. In the figure in the middle the predetermined hitting time is $t = 2$ and it is the first of the two points of intersection of the curves. The path is convex and $x(u) > 0$ for all u. In the figure on the right the predetermined hitting time is $t = 3$. The path is again convex and $x(u) > 0$. Here, another hitting time occurs before the predetermined hitting time at $t = 3$.

If (2.39) is not satisfied then t is *the second point of intersection of $x(u)$ with $V(u)$.* With $\tau(t)$ denoting the first point of intersection, so that $\tau(t) \le t$, the optimal path is given by

$$x_o(u) = \begin{cases} x(u) & \text{for} \quad 0 \le u \le \tau(t) \\ V(u) & \text{for} \quad \tau(t) < u \le t \end{cases} \qquad \text{or,} \;\; x_o(u) = \max(x(u), V(u)), \;\; 0 \le u \le t.$$

Figure 2.6 illustrates this situation. The corresponding optimal value of the criterion is

$$\begin{aligned} J_{**}(t) &= \frac{1}{2\sigma^2}\left(\int_0^{\tau(t)} (x'(u) - \mu x(u))^2 du + \int_{\tau(t)}^{t} (u_0 \beta e^{\beta u} - \mu u_0 e^{\beta u})^2 du\right) \\ &= J^*(\tau(t)) + \frac{u_0^2(\mu - \beta)^2}{4\beta\sigma^2}\left(e^{2\beta t} - e^{2\beta\tau(t)}\right) \quad \text{when} \;\; t > t_2. \end{aligned} \tag{2.42}$$

where J^* is the expression in (2.28).

Next we will show that

$$t^o \; < \; t_2. \tag{2.43}$$

Indeed, using the definition of ϕ_1 and t^o,

$$\begin{aligned} \phi_2(t^o) &= \left(1 - \frac{\beta}{\mu}\right) e^{(\beta+\mu)t^o} + \left(1 + \frac{\beta}{\mu}\right) e^{(\beta-\mu)t^o} \\ &= \phi_1(t^o) \; + \; e^{(\beta-\mu)t^o} < \frac{x_0}{v_0} + 1 \; < \; 2\frac{x_0}{v_0} \; = \; \phi_2(t_2). \end{aligned}$$

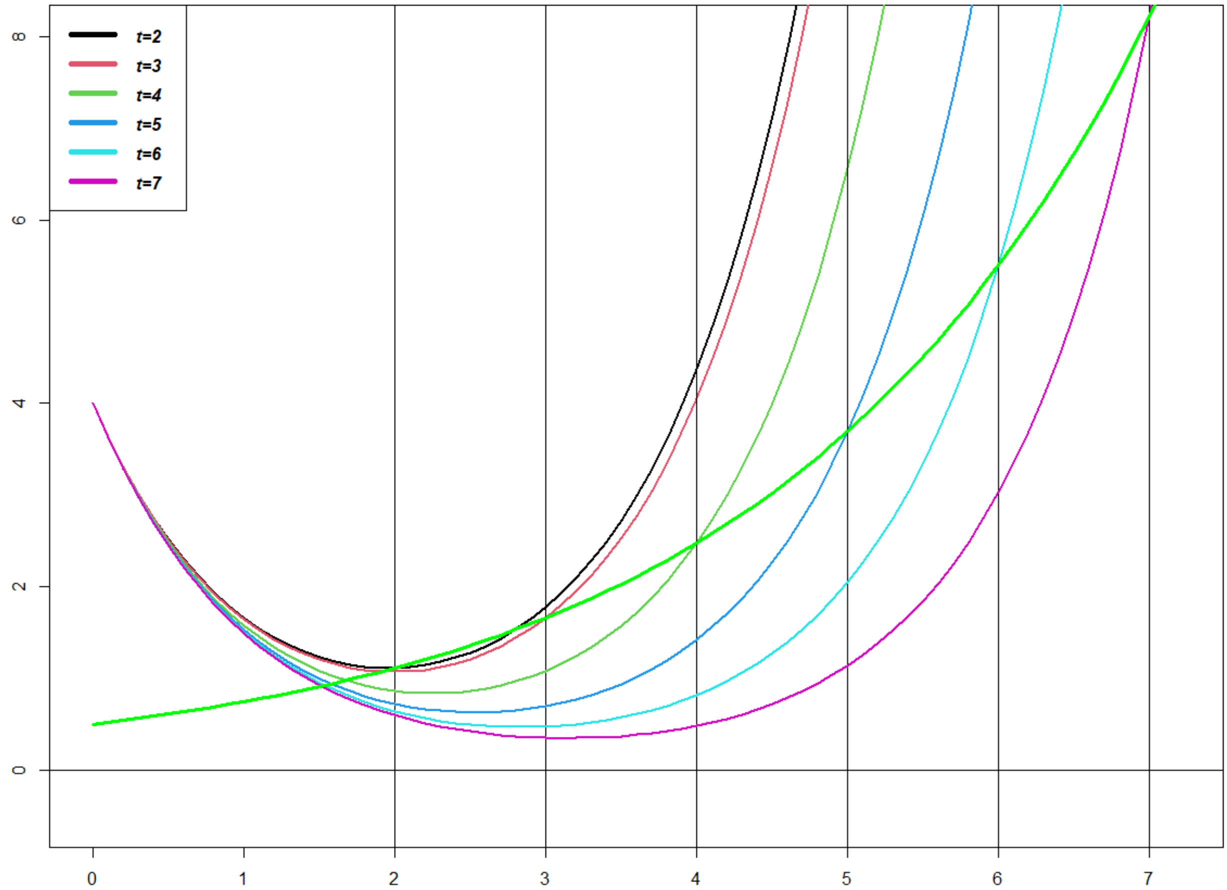

Fig. 2.5: Two cases. Here $\mu = 1$, $\beta = 0.4$, $x_0 = 4$, $v_0 = 0.5$. The hitting times range from $t = 2$ to $t = 7$. Note that, only for $t = 2$ the path $x(u)$ hits $V(u)$, the thick green line, for the *second time* at the hitting time, i.e. only for $t = 2$ does the condition $x'(t) < V'(t)$ hold.

where we have used the fact that $\beta - \mu < 0$ and that $x_0 > v_0$. Then (2.43) follows from the fact that ϕ_2 is increasing.

Then, the rate function in (2.13), defined as

$$I_V(T) \;:=\; \inf\left\{J(x;t) : x \in \mathcal{H}^1_{x_0,V(t)}, x(u) \geq V(u) \text{ for } 0 < u < t, \;\; 0 < t \leq T.\right\} \tag{2.44}$$

can be obtained as

$$I_V(T) \;:=\; \min_{t\in(0,T]} J_{**}(t). \tag{2.45}$$

If $T \leq t_2$ then $J_{**}(t) = J_*(t)$ and hence $I_V(T) = \min_{t\in(0,T]} J_*(t) = J_*(t^o \wedge T)$ due to the fact that J_* is strictly decreasing in $(0, t^o$ and strictly increasing in (t^o, ∞).

If $T > t_2$ then we can write $I_V(T) \;:=\; \min\left(\min_{t\in(0,t_2]} J_{**}(t), \min_{t\in(t_2,T]} J_{**}(t)\right)$. For every $t \in (t_2, T]$ there exists $\tau(t) < t$ for which, necessarily, $\tau(t) \in (0, t_2]$. Since $J_{**}(t) > J_*(\tau(t))$ for $t > t_2$, by virtue of (2.42), it follows that $\min_{t\in(t_2,T]} J_{**}(t) > \min_{t\in(0,t_2]} J_*(t)$.

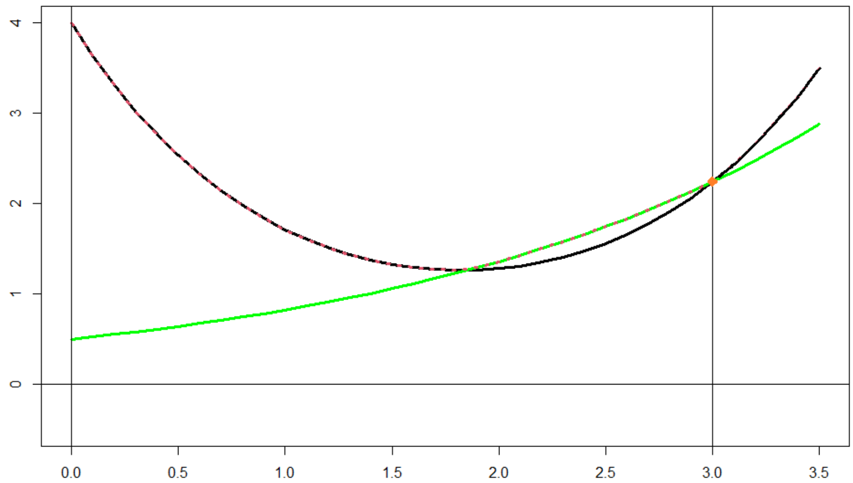

Fig. 2.6: The optimal solution for the problem with fixed time and path inequality constraint $x(u) \geq V(u)$. The red dotted line is the optimal path. It consists of a portion of the optimal path without constraints and then follows the constraint up to time t.

Therefore we conclude that $I_V(T)$ is also given by (2.28). This concludes the proof of the first part of Theorem 4. The proof of the second part, pertaining to the upper boundary curve is similar and will be omitted. □

2.3.1 The infinite horizon problem - lower bound

We now turn to the infinite horizon problem of obtaining a large deviations estimate for the probability $P(\inf_{t\geq 0} X_t - v_0 e^{\beta t} \leq 0)$ in the same context as that of the previous section. It is of course possible to solve first the finite horizon problem $P(\inf_{0\leq t\leq T} X_t - v_0 e^{\beta t} \leq 0)$ as we saw in the previous section and then minimize this probability over T. Instead of this we will use the standard *transversality conditions* approach of the Calculus of Variations in order to tackle in one step the infinite horizon problem. These are necessary conditions for optimality in variational problems with variable end-points.

$$\min \int_0^T F(x, x', t)dt,$$
$$\text{subject to the constraints } x(0) = x_0, \quad x(t) > V(t), \quad x(T) = V(T)$$
$$\text{with } F(x, x', t) = \frac{1}{2\sigma^2}\left(x' - \mu x\right)^2, \qquad V(t) = v_0 e^{\beta t}, \quad 0 < v_0 < x_0.$$

In the above, both the optimal path x and the horizon T are unknowns to be determined. Our approach to dealing with the *inequality path constraint,* $x(t) > V(t)$ for all $t \in [0, T)$ will be to initially ignore it and obtain an optimal hitting time T and an optimal path x_* minimizing the criterion $\int_0^T F(x, x', t)dt$ and satisfying the boundary conditions $x_*(0) = x_0$,

$x_*(T) = V(T)$. We then show that this optimal path satisfies the constraint $x_*(t) > V(t)$ for $0 \le t < T$.

The necessary conditions for a minimum in the problem *without the path inequality constraint* are

$$\text{Euler-Lagrange Equation:} \quad F_x - \frac{d}{dt}F_{x'} = 0, \tag{2.46}$$

$$\text{Boundary Conditions:} \quad x(0) = x_0, \quad x(T) = V(T), \tag{2.47}$$

$$\text{Transversality Condition:} \quad F + (V' - x')F_{x'} = 0 \quad \text{at } T. \tag{2.48}$$

Taking into account that $F_x = -\mu\sigma^{-2}(x' - \mu x)$, $F_{x'} = \sigma^{-2}(x' - \mu x)$, $\frac{d}{dt}F_{x'} = \sigma^{-2}(x'' - \mu x')$, the Euler-Lagrange equation becomes

$$F_x - \frac{d}{dt}F_{x'} = -\sigma^{-2}\left(x'' - \mu^2 x\right) = 0$$

and thus

$$x'' - \mu^2 x \; = \; 0. \tag{2.49}$$

This has the general solution

$$x(t) = C_1 e^{\mu t} + C_2 e^{-\mu t}. \tag{2.50}$$

Taking into account the boundary conditions (2.47), we obtain

$$x(0) \; = \; C_1 + C_2 \; = \; x_0, \tag{2.51}$$

$$x(T) \; = \; C_1 e^{\mu T} + C_2 e^{-\mu T} \; = \; v_0 e^{\beta T}. \tag{2.52}$$

Proof. The transversality condition (2.48):

$$\frac{1}{2\sigma^2}\left(x'(T) - \mu x(T)\right)^2 + \left(v_0\beta e^{\beta T} - x'(T)\right)\frac{1}{\sigma^2}\left(x'(T) - \mu x(T)\right) \; = \; 0$$

or

$$\left(x'(T) - \mu x(T)\right)\left(-x'(T) - \mu x(T) + 2v_0\beta e^{\beta T}\right) \; = \; 0. \tag{2.53}$$

Taking into account (2.50), it follows that $x'(T) - \mu x(T) = -2\mu C_2 e^{-\mu T}$ and hence, if the first factor of (2.53) were to vanish, this would imply that $C_2 = 0$. This in turn implies, in view of (2.50) and (2.51), that $x(T) = x_0 e^{\mu T} = v_0 e^{\beta T}$ which is impossible since $x_0 > v_0$ and $\mu > \beta$. Hence (2.53) implies

$$v_0 e^{\beta T} \; = \; \frac{\mu}{\beta} C_1 e^{\mu T}. \tag{2.54}$$

From (2.47) and (2.54) we obtain

$$\begin{aligned} C_1 + C_2 \; &= \; x_0 \\ C_1\left(1 - \frac{\mu}{\beta}\right) e^{\mu T} + C_2 e^{-\mu T} \; &= \; 0 \end{aligned}$$

whence it follows that

$$C_1 = \frac{x_0 e^{-\mu T}}{\left(\frac{\mu}{\beta}-1\right) e^{\mu T} + e^{-\mu T}}, \qquad C_2 = \frac{x_0 \left(\frac{\mu}{\beta}-1\right) e^{\mu T}}{\left(\frac{\mu}{\beta}-1\right) e^{\mu T} + e^{-\mu T}}. \tag{2.55}$$

(Note that, $\mu > \beta$ implies that the denominator in the above expressions never vanishes.) It remains to determine the time T where the two paths meet and this is obtained using (2.54) and (2.55) which gives

$$\left(\frac{\mu}{\beta}-1\right) e^{(\mu+\beta)T} + e^{-(\mu-\beta)T} = \frac{x_0}{v_0}\frac{\mu}{\beta}. \tag{2.56}$$

If $\psi(t) := \left(\frac{\mu}{\beta}-1\right) e^{(\mu+\beta)t} + e^{-(\mu-\beta)t}$ then it is easy to see that $\psi(0) = \frac{\mu}{\beta} < \frac{x_0}{v_0}\frac{\mu}{\beta}$ and $\psi'(t) = (\mu-\beta)\left(\left(1+\frac{\mu}{\beta}\right) e^{t(\mu+\beta)} - e^{-(\mu-\beta)t}\right) > 0$. Furthermore, $\psi(t) \to \infty$ as $t \to \infty$. Hence (2.56) has a unique solution $T > 0$.

The rate function then becomes

$$\frac{1}{2\sigma^2}\int_0^T (x'-\mu x)^2\, dt = \frac{1}{2\sigma^2}\int_0^T 4\mu^2 C_2^2 e^{-2\mu t} dt = \frac{\mu C_2^2}{\sigma^2}\left(1-e^{-2\mu T}\right)$$

or

$$I = \frac{x_0^2 \mu}{\sigma^2}\,\frac{1-e^{-2\mu T}}{\left(1+\frac{\beta}{\mu-\beta}\, e^{-2\mu T}\right)^2}. \tag{2.57}$$

Finally, the optimal path x_* hitting the lower bound is given by

$$x_*(t) = x_0 \frac{e^{-\mu(T-t)} + \left(\frac{\mu}{\beta}-1\right) e^{\mu(T-t)}}{e^{-\mu T} + \left(\frac{\mu}{\beta}-1\right) e^{\mu T}} \tag{2.58}$$

where T is the (unique) solution of equation (2.56).

Intuitively, the uniqueness of the solution of (2.56) makes sense. If T is very small the noise factor W_t must exhibit an extremely unlikely behavior in order for the OU process to drop to the level of the lower curve. So having more time available makes the rare event of hitting the lower boundary more likely. But if T is very large, because of the difference in the rates of the two processes, again hitting the lower boundary becomes extremely unlikely. In Chapter 4 we will study a linear SDE which the uniqueness of the solution does not necessarily hold. Also, in some cases, in the infinite horizon problem, an infimum may exist but no minimum. The rate function I is not "good" and compactness fails. In practical terms, the more time available the more likely it is that the noise term will cause the Stochastic Differential Equation to hit the deterministic boundary curve.

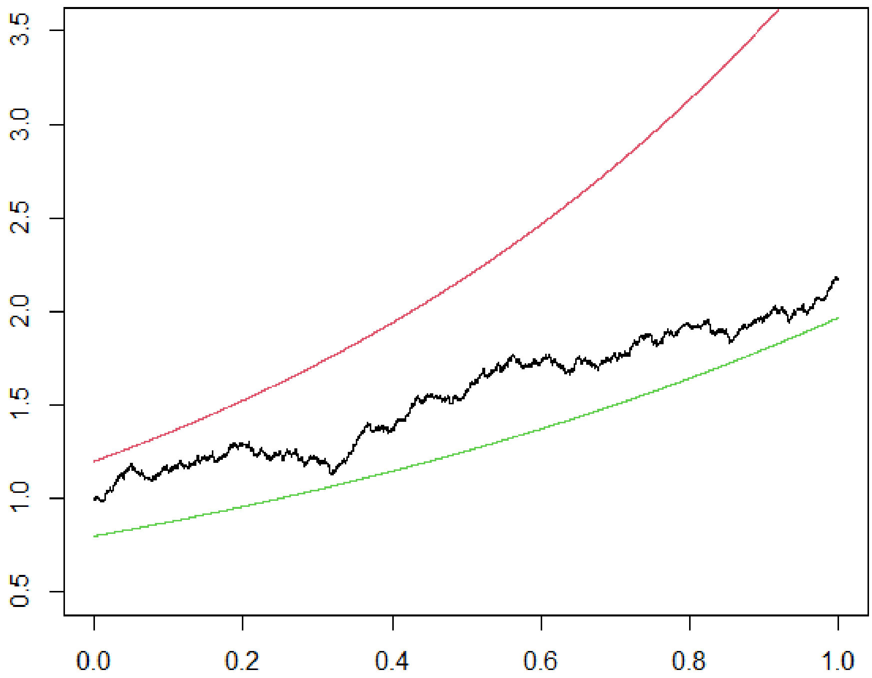

Fig. 2.7: An Ornstein-Uhlenbeck process evolving between an upper and a lower exponential bound.

It remains to show that the optimal path x_* for the problem without the inequality path constraint, satisfies these constraints as well, i.e. we will show that

$$x_*(t) > v_0 e^{\beta t}, \qquad 0 \le t < T. \tag{2.59}$$

Observe from (2.58) that $x_*(t) > 0$ for all $t \in [0, T]$ and, since it satisfies (2.49), $x_*''(t) > 0$ for all $t \in [0, T]$. Therefore x_* is a convex function of t and so is $V(t) = v_0 e^{\beta t}$. Since the two curves meet at T, they are either tangent to each other or they have exactly two points of intersection. If the condition

$$x_*'(T) \;\le\; V'(T) \;=\; v_0 \beta e^{\beta T} \tag{2.60}$$

holds then T is the first point where the path $x_*(t)$ hits the curve $V(t)$ and the path x_* satisfies the inequality constraints as well and hence gives the solution to our problem.. (If (2.60) holds as an equality then it is the only point where this happens.) From (2.58) we see that (2.60) is equivalent to

$$\frac{\mu x_0 \left(2 - \frac{\mu}{\beta}\right)}{\left(\frac{\mu}{\beta} - 1\right) e^{\mu T} + e^{-\mu T}} \;<\; v_0 \beta e^{\beta T}. \tag{2.61}$$

The above is also written as $\frac{\mu x_0}{v_0 \beta}\left(2 - \frac{\mu}{\beta}\right) \;<\; \left(\frac{\mu}{\beta} - 1\right) e^{(\mu+\beta)T} + e^{-(\mu-\beta)T}$ and using (2.56)

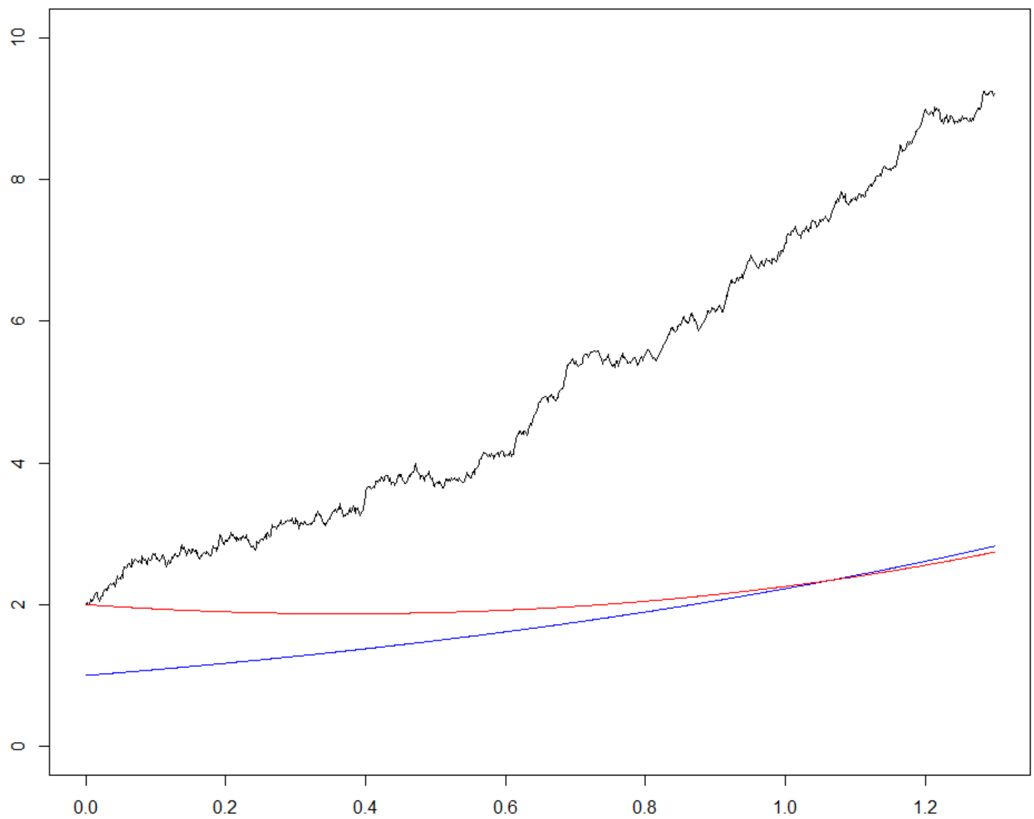

Fig. 2.8: The black line is a typical path of an OU process with $\mu = 1$, $\sigma = 1$ and starting point $x_0 = 2$. The blue curve is the lower exponential bound $v_0 e^{\beta t}$ with $v_0 = 1$ and $\beta = 0.8$. The meeting T obtained by solving numerically (2.56) is equal to 1.0621. Finally the red optimal (large deviation) path is obtained from (2.58)

this becomes $\frac{\mu x_0}{v_0 \beta}\left(2 - \frac{\mu}{\beta}\right) < \frac{x_0 \mu}{v_0 \beta}$ or $1 < \frac{\mu}{\beta}$ which holds by assumption. Thus (2.60) holds as a strict inequality and (2.58) is indeed the optimal path. □

2.3.2 The infinite horizon problem with an upper bound

This problem is similar to the lower bound treated in the previous section. We will discuss it very briefly. The optimization problem for the action functional now becomes

$$\min \int_0^T F(x, x', t)dt, \qquad x(0) = x_0,$$

subject to the constraints $x(0) = x_0$, $x(t) < U(t)$ for $0 \le t < T$, and $x(T) = U(T)$

with $F(x, x', t) = \frac{1}{2\sigma^2}\left(x' - \mu x\right)^2$, $\quad U(t) = u_0 e^{\alpha t}$, $\quad 0 < x_0 < u_0$.

The necessary conditions for the minimum are

$$\begin{aligned} \text{Euler-Lagrange Equation:} \quad & F_x - \frac{d}{dt}F_{x'} \;=\; 0, & (2.62)\\ \text{Boundary Conditions:} \quad & x(0) \;=\; x_0, \quad x(T) \;=\; U(T), & (2.63)\\ \text{Transversality Condition:} \quad & F + (U' - x')F_{x'} \;=\; 0 \quad \text{at } T. & (2.64) \end{aligned}$$

Proof. Again we first obtain the optimal solution without taking into account the inequality path constraint $x(t) < U(t)$ for $0 < t < T$. The solution method is the same as in section 2.3.1. In particular, the Euler-Lagrange equation again gives

$$x''(t) \;=\; \mu^2 x(t) \tag{2.65}$$

and the optimal path is again of the form

$$x(t) = C_1 e^{\mu t} + C_2 e^{-\mu t}. \tag{2.66}$$

The transversality condition (2.64), together with (2.65), gives

$$\frac{\mu}{\alpha} C_1 e^{\mu T} \;=\; u_0 e^{\alpha T}. \tag{2.67}$$

Together with the boundary conditions $C_1 + C_2 \;=\; x_0$, $C_1\, e^{\mu T} + C_2\, e^{-\mu T}$, this gives

$$C_1 \;=\; \frac{x_0 e^{-\mu T}}{\left(\frac{\mu}{\alpha} - 1\right)\, e^{\mu T} + e^{-\mu T}}, \qquad C_2 \;=\; \frac{x_0 \left(\frac{\mu}{\alpha} - 1\right)\, e^{\mu T}}{\left(\frac{\mu}{\alpha} - 1\right)\, e^{\mu T} + e^{-\mu T}}. \tag{2.68}$$

and the optimal hitting time is given by the unique solution of

$$\left(\frac{\alpha}{\mu} - 1\right)\, e^{(\mu+\alpha)T} \;-\; \frac{\alpha}{\mu}\, e^{(\alpha-\mu)T} + \frac{x_0}{u_0} \;=\; 0. \tag{2.69}$$

To establish that (2.69) has a unique solution define the function $\phi : [0,\infty) \to \mathbb{R}$ by

$$\phi(t) := \left(\frac{\alpha}{\mu} - 1\right)\, e^{(\mu+\alpha)t} \;-\; \frac{\alpha}{\mu} e^{(\alpha-\mu)t} + \frac{x_0}{u_0}. \tag{2.70}$$

It holds that $\phi(0) = \frac{x_0}{u_0} - 1 \;<\; 0$ and

$$\phi'(t) \;=\; \frac{\alpha - \mu}{\alpha}\, e^{t(\mu+\alpha)} \left(\mu + \alpha(1 - e^{-2\mu t})\right) > 0 \qquad \text{for all } t \geq 0.$$

Thus, ϕ is a strictly increasing function for which $\phi(t) \to +\infty$ when $t \to \infty$. Hence the equation $\phi(t) = 0$ has a unique solution, as claimed above. From (2.66) and (2.68), the critical path is

$$x(t) \;=\; x_0 \frac{e^{-\mu(T-t)} - \left(1 - \frac{\mu}{\alpha}\right) e^{\mu(T-t)}}{e^{-\mu T} - \left(1 - \frac{\mu}{\alpha}\right) e^{\mu T}}. \tag{2.71}$$

Consider the function $\psi(t) := e^{-\mu t} - \left(1 - \frac{\mu}{\alpha}\right) e^{\mu t}$, $t \geq 0$. $\psi(0) = \frac{\mu}{\alpha} > 0$ and it is easy to see that $\psi(t)$ is decreasing and goes to $-\infty$ as $t \to \infty$. There is therefore a unique root t_0 of the equation $\psi(t) = 0$ which is given by $t_0 = \frac{1}{2\mu} \log \frac{\alpha}{\alpha - \mu}$ and $\psi(t) > 0$ for $t \in [0, t_0)$. We next note that $\phi(t_0) = e^{\alpha - \mu}\left((\frac{\alpha}{\mu} - 1)e^{2\mu t_0} - \frac{\alpha}{\mu}\right) + \frac{x_0}{u_0} = \frac{x_0}{u_0} > 0$. This then shows, in view of the properties of ϕ (defined in (2.70)) that $T < t_0$ and therefore that both $\psi(t) > 0$ and $\psi(T - t) > 0$ when $t \in [0, T]$. This establishes that $x(t)$ defined in (2.71) is positive and, because of (2.65) also convex for $t \in [0, T]$.

Therefore there are precisely two intersection points between the optimal path $x(t)$ and the convex curve $u_0 e^{\alpha t}$. (Generally speaking there could be a single point of contact, assuming the two curves to be tangent to each other, we will see however that this is not the case.) It remains to show that

$$x'(T) > u_0 \alpha e^{\alpha T} \tag{2.72}$$

Using (2.71), (2.67), and (2.68) the above inequality is equivalent to

$$x_0 \mu \frac{2 - \frac{\mu}{\alpha}}{e^{-\mu T} - \left(1 - \frac{\mu}{\alpha}\right) e^{\mu T}} > \frac{x_0 \mu}{e^{-\mu T} - \left(1 - \frac{\mu}{\alpha}\right) e^{\mu T}}$$

which in turn is equivalent to $\alpha > \mu$ which holds by assumption. Therefore the critical path $x(t)$ satisfies the inequality $x(t) < U(t)$ as well, for all $t \in [0, T)$.

The corresponding value of the rate function is

$$I = \frac{x_0^2 \mu}{\sigma^2} \frac{1 - e^{-2\mu T}}{\left(1 + \frac{\alpha}{\mu - \alpha} e^{-2\mu T}\right)^2}. \tag{2.73}$$

and hence, on a practical note, the probability that the OU process reaches the upper boundary satisfies approximately

$$\log P(\sup_{t \geq 0} X_t - u_0 e^{\alpha t} > 0) \approx -I.$$

The quality of this approximation improves as σ becomes smaller. In fact the exact statement would be

$$\lim_{\sigma \to 0} \sigma^2 \log \mathbb{P}(\sup_{t \geq 0} X_t - u_0 e^{\alpha t} > 0) = x_0^2 \mu \frac{1 - e^{-2\mu T}}{\left(1 + \frac{\alpha}{\mu - \alpha} e^{-2\mu T}\right)^2}.$$

Note in particular that the value of T does not depend on σ as is clear from (2.69). Equation (2.73) gives the value of I in terms of T which is in terms determined by (2.69). Alternative expressions for the rate I, using (2.69) are, of course, possible. For instance,

$$I = \frac{\mu}{\sigma^2} \frac{\left(u_0 e^{(\alpha - \mu)T} - x_0\right)^2}{1 - e^{-2\mu T}} = \frac{\mu}{\sigma^2} u_0^2 \left(1 - \frac{\mu}{\alpha}\right)^2 e^{2\alpha T} \left(e^{2\mu T} - 1\right). \tag{2.74}$$

□

In Figures 2.9, 2.10, we consider the OU process $dX_t = X_t + dW_t$, with $X_0 = x_0$, (with the value of the parameters $\mu = 1$, $\sigma = 1$) and the lower and upper bounds $v(t) = 0.5e^{0.5t}$, $u(t) = 2e^{1.3t}$. (Thus $\alpha = 1.3$, $u_0 = 2$, $\beta = 0.5$ and $v_0 = 0.5$.) In Figure 2.8 the optimal value of T that corresponds to the solution of the optimization problems of sections 2.3.1 and 2.3.2 (equations (2.56) and (2.69)).

2.4 Ornstein-Uhlenbeck with a general linear drift

Here we consider the Ornstein-Uhlenbeck process with a more general drift. This is important since it arises as a diffusion approximation in the risk models with interest rates considered in chapter 1. Consider the SDE

$$dX_t = (\mu X_t + r)dt + \sigma dW_t, \quad X_0 = x_0.$$

The upper limit is $U(t) = u_0 e^{\alpha t}$. We assume that $u_0 > x_0$ and $\mu < \alpha$. In the deterministic limit, when $\sigma \to 0$, the solution X^0 satisfies the deterministic Differential Equation $\frac{d}{dt}X^0(t) = \mu X^0(t) + r$ which has the solution $X^0(t) = x_0 e^{\mu t} + \frac{r}{\mu}(e^{\mu t} - 1)$. To ensure that we remain in range of applicability of Large Deviation results we will need to ensure that the deterministic solution remains strictly below the upper bound, $U(t)$ for all $t \geq 0$. Let

$$\phi(t) := u_0 e^{\alpha t} - \left(x_0 + \frac{r}{\mu}\right) e^{\mu t} + \frac{r}{\mu}. \tag{2.75}$$

Then we must have

$$\inf_{t \geq 0} \phi(t) > 0. \tag{2.76}$$

We will make the additional assumption that

$$r \ < \ u_0(\alpha - \mu). \tag{2.77}$$

This assumption ensures that (2.76) holds. Indeed, $\phi(0) = u_0 - x_0 > 0$ and $\phi'(t) = e^{\mu t}\left[u_0 \alpha e^{(\alpha-\mu)t} - x_0\mu - r\right]$. Then,

$$u_0 \alpha e^{(\alpha-\mu)t} - x_0\mu - r \geq u_0\alpha - x_0\mu - r > u_0\alpha - x_0\alpha - r > 0$$

and hence (2.76) holds.

The action functional is

$$\frac{1}{2\sigma^2}\int_0^T (x' - \mu x - r)^2 \, du.$$

The Euler-Lagrange differential equation $F_x - \frac{d}{dt}F_{x'} = 0$ reduces to

$$x'' - \mu^2 x - \mu r = 0.$$

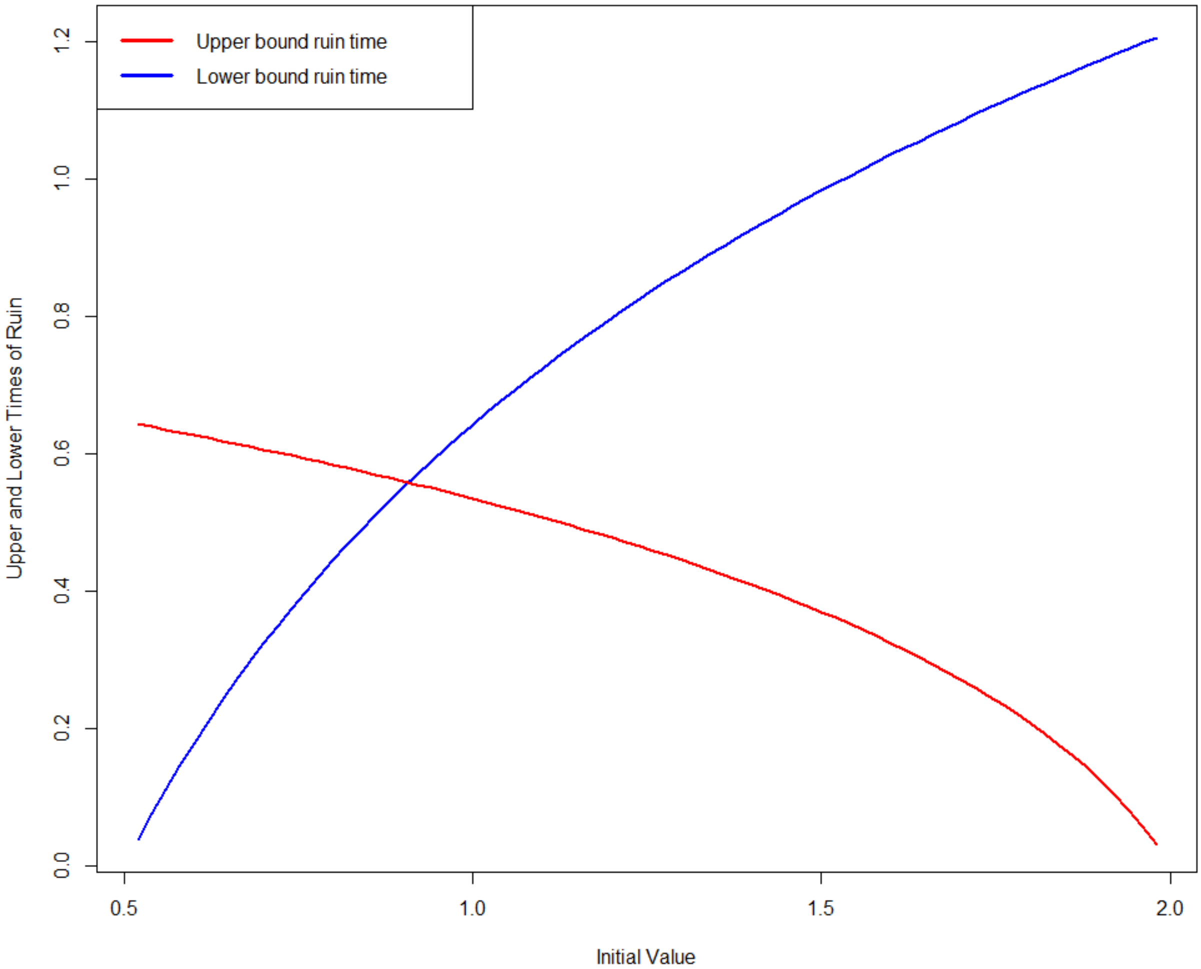

Fig. 2.9: The system under consideration is an OU process with $\mu = 1$, $\sigma = 1$ and initial position x_0. The red line is the "optimal hitting time" for the upper curve $u_0 e^{\alpha t}$ with $u_0 = 2$, $\alpha = 1.3$, i.e. the solution of (2.69). Note that this optimal time decreases to zero as x_0 increases to $u_0 = 2$. Respectively, the blue line is the corresponding "optimal hitting time" for the lower curve $v_0 e^{\beta t}$, $\beta = 0.5$, $v_0 = 0.5$, i.e. the solution of (2.56). In this case the optimal time increases as the distance of x_0 from v_0 increases.

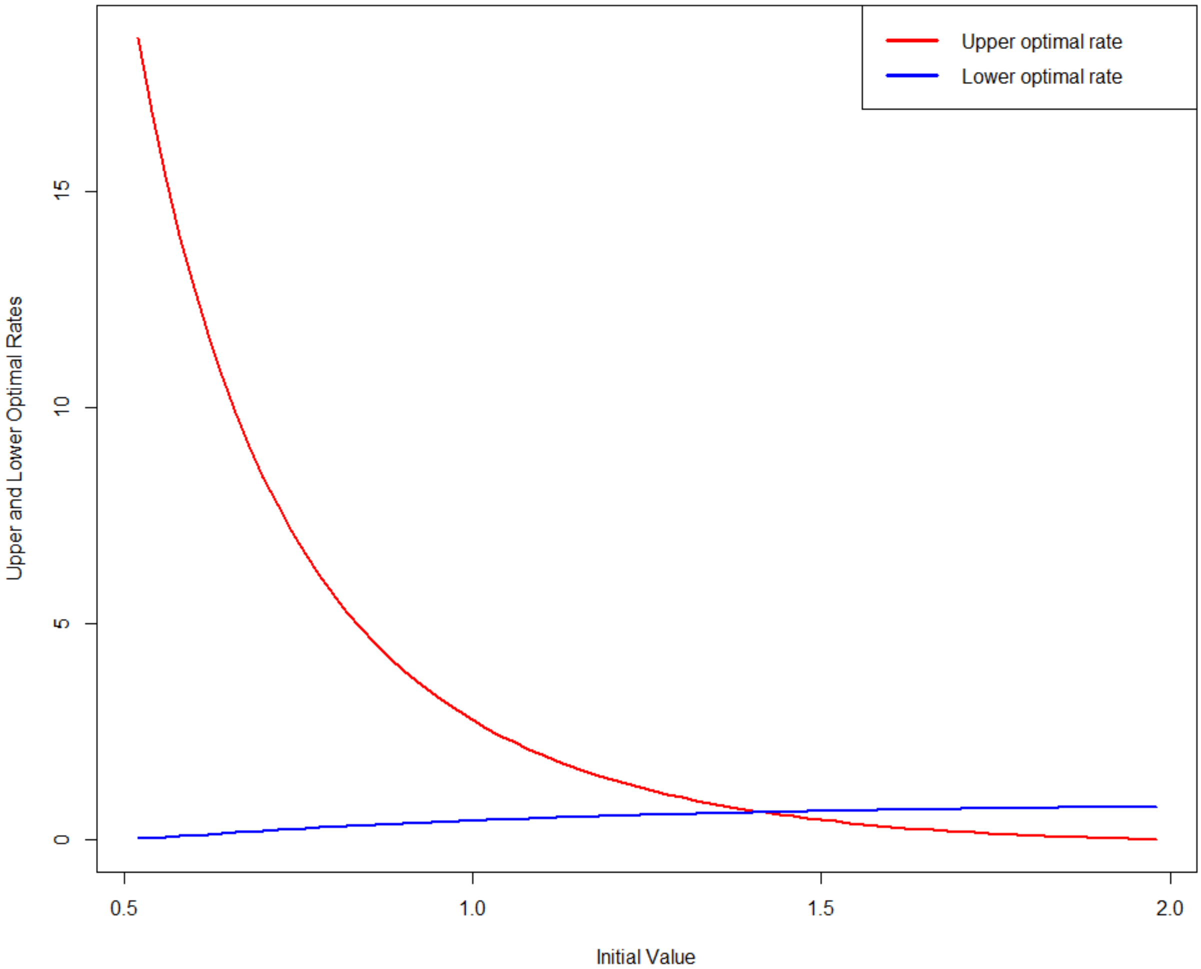

Fig. 2.10: The OU process and the upper and lower curves are as in (2.9). The red line is a plot of the optimal rate I for hitting the upper curve in the infinite horizon problem given by (2.73). Correspondingly, the blue line gives the plot of the optimal rate for hitting the lower curve, given by (2.57). The point of intersection of the two curves corresponds to the initial condition x_0 for which the exponential rate for the probability of hitting the upper curve is equal to that for the lower curve.

Its general solution is

$$x(t) = C_1 e^{\mu t} + C_2 e^{-\mu t} - \frac{r}{\mu}. \tag{2.78}$$

The boundary conditions are

$$x_0 = C_1 + C_2 - \frac{r}{\mu} \tag{2.79}$$

$$u_0 e^{\alpha T} = C_1 e^{\mu T} + C_2 e^{-\mu T} - \frac{r}{\mu}. \tag{2.80}$$

The transversality condition that must be satisfied by a critical path meeting the curve $U(t) := u_0 e^{\alpha t}$ at T is

$$F + (U'(T) - x'(T))F_{x'} = 0 \quad \text{or} \quad (x' - r - \mu x)\left(-x' - r - \mu x + 2u_0 e^{\alpha T}\right) = 0$$

which, using (2.78), reduces to

$$C_2\left(u_0 \alpha e^{\alpha T} - \mu C_1 e^{\mu T}\right) = 0. \tag{2.81}$$

The above equation leads to the examination of two cases:

Case 1. $C_2 = 0$. Using this value in (2.79), (2.80), and eliminating C_1 among them gives

$$u_0 e^{\alpha T} - \left(x_0 + \frac{r}{\mu}\right) e^{\mu T} + \frac{r}{\mu} = 0. \tag{2.82}$$

This equation corresponds to the requirement $\phi(T) = 0$ for the function defined in (2.75) which is impossible. Hence $C_2 = 0$ is impossible.

Case 2. $u_0 \alpha e^{\alpha T} - \mu C_1 e^{\mu T} = 0$. This, together with (2.80) gives

$$u_0 \left(1 - \frac{\alpha}{\mu}\right) e^{\alpha T} = C_2 e^{-\mu T} - \frac{r}{\mu}. \tag{2.83}$$

Using this, (2.79), (2.80), give

$$C_1 + C_2 = x_0 + \frac{r}{\mu} \tag{2.84}$$

$$C_1 e^{\mu T} + C_2 e^{-\mu T} = u_0 e^{\alpha T} + \frac{r}{\mu}. \tag{2.85}$$

The above system has the solution

$$C_1 = \frac{e^{-\mu T}\left(x_0 + \frac{r}{\mu}\right) - \left(u_0 e^{\alpha T} + \frac{r}{\mu}\right)}{e^{-\mu T} - e^{\mu T}}, \quad C_2 = \frac{u_0 e^{\alpha T} + \frac{r}{\mu} - e^{\mu T}\left(x_0 + \frac{r}{\mu}\right)}{e^{-\mu T} - e^{\mu T}}.$$

Using this, (2.83) reduces to

$$u_0\left(\frac{\alpha}{\mu}-1\right)e^{(\alpha+\mu)T}-u_0\frac{\alpha}{\mu}e^{(\alpha-\mu)T}-\frac{r}{\mu}e^{\mu T}+x_0+\frac{r}{\mu}=0. \tag{2.86}$$

Under Assumption (2.77) i.e. if the drift term r is either negative or, if positive, not too large the above equation has a unique solution which determines T.

Define

$$f(t)=u_0\left(\frac{\alpha}{\mu}-1\right)e^{t(\alpha+\mu)}-u_0\frac{\alpha}{\mu}e^{(\alpha-\mu)t}-\frac{r}{\mu}e^{\mu t}+x_0+\frac{r}{\mu}$$

$$f(0)=x_0-u_0<0.$$

Also $\lim_{t\to\infty} f(t)=+\infty$.

$$f'(t)=(\alpha+\mu)u_0\left(\frac{\alpha}{\mu}-1\right)e^{t(\alpha+\mu)}-u_0\frac{\alpha}{\mu}(\alpha-\mu)e^{(\alpha-\mu)t}-re^{\mu t}.$$

$$f'(0)=u_0(\alpha-\mu)-r.$$

Under the assumption $f'(0)>0$. We will show that the condition implies $f'(t)>0$ for all $t>0$.

$$e^{-\mu t}f'(t)=:g(t)=(\alpha+\mu)u_0\left(\frac{\alpha}{\mu}-1\right)e^{\alpha t}-u_0\frac{\alpha}{\mu}(\alpha-\mu)e^{(\alpha-2\mu)t}-r$$

$g(0)=f'(0)=u_0(\alpha-\mu)-r>0.$

$$g'(t)=\frac{\alpha}{\mu}e^{\alpha t}(\alpha-\mu)u_0\left(\alpha+\mu-(\alpha-2\mu)e^{-2\mu t}\right)>0 \quad \text{for all } t\geq 0.$$

This implies the uniqueness of the solution of (2.86).

Then

$$x'(T) \;=\; \mu\,\frac{-2(x_0+\frac{r}{\mu})+\left(u_0e^{\alpha T}+\frac{r}{\mu}\right)\left(e^{\mu T}+e^{-\mu T}\right)}{e^{\mu T}-e^{-\mu T}}. \tag{2.87}$$

The condition for this solution to satisfy the inequality constraints as well is

$$x'(T)>u_0\alpha e^{\alpha T}.$$

This is written as

$$\mu\frac{\left(u_0e^{\alpha T}+\frac{r}{\mu}\right)\left(e^{\mu T}+e^{-\mu T}\right)-2(x_0+\frac{r}{\mu})}{e^{\mu T}-e^{-\mu T}}>\alpha u_0e^{\alpha T}.$$

This is equivalent to

$$\frac{r}{\mu}\left(e^{\mu T}+e^{-\mu T}\right)+u_0e^{T(\alpha-\mu)}-2(x_0+\frac{r}{\mu})>u_0e^{\alpha T}\left[\left(\frac{\alpha}{\mu}-1\right)e^{\mu T}-\frac{\alpha}{\mu}e^{-\mu T}\right]\;=\;\frac{r}{\mu}(e^{\mu T}-1)-x_0$$

the last equation following from (2.86). Hence

$$\frac{r}{\mu}e^{-\mu T} + u_0 e^{(\alpha-\mu)T} > x_0 + \frac{r}{\mu}.$$

This inequality however is true because it is equivalent to $\phi(T) > 0$ for the function ϕ defined in (2.75), which is true.

The optimal path is in this case

$$x(t) = \frac{\left(x_0 + \frac{r}{\mu}\right)\sinh(\mu(T-t)) + \left(u_0 e^{\alpha T} + \frac{r}{\mu}\right)\sinh(\mu t)}{\sinh(\mu T)} - \frac{r}{\mu}.$$

The optimal rate can be obtained from the fact that $x'(t) - \mu x(t) - r = 2C_2 e^{\mu t}$ and hence

$$I = \frac{\mu}{\sigma^2}\int_0^T 4C_2^2 e^{\mu t} dt = \frac{\mu}{\sigma^2}\frac{\left(u_0 e^{(\alpha-\mu)T} - \frac{r}{\mu}\left(1 - e^{-\mu T}\right) - x_0\right)^2}{1 - e^{-2\mu T}}.$$

Note, of course, that when $r \to 0$ the above reduces to the value of I given in (2.74).

2.5 A Ruin Problem Involving Two Independent OU Processes

Here we generalize the problem examined in the previous section. The lower (or upper) deterministic exponential boundary now is also considered to be stochastic - in fact another, independent, OU process. We may thus study the following pair of SDE's

$$dX_t = \alpha X_t dt + \sigma dW_t, \qquad X_0 = x_0 \tag{2.88}$$

$$dY_t = \beta Y_t dt + b dB_t, \qquad Y_0 = y_0. \tag{2.89}$$

where $\beta < \alpha$ and $y_0 < x_0$. As a result of these inequalities, in the absence of noise, $(\sigma = b = 0)$ we would have $Y_t < X_t$ for all t. The presence of noise may cause the two curves to meet however. Again, an exact analysis does not give results in closed form and we obtain low noise logarithmic asymptotics in the Wentzell-Freidlin framework. Using again Theorem 5.6.7 of [3, p. 214] we obtain a two dimensional version of (2.12) for the action functional to be minimized:

$$I = \int_0^T F(x, x', y, y')dt, \qquad F = \frac{1}{2}\left[\frac{1}{\sigma^2}(x' - \alpha x)^2 + \frac{1}{b^2}(y' - \beta y)^2\right], \tag{2.90}$$

The boundary conditions $x(0) = x_0$, $y(0) = y_0$, and $x(T) = y(T)$.

We will again tackle the infinite horizon problem directly and solve the moving boundary variational problem using the appropriate transversality conditions. Thus the first order necessary conditions for an extremum are

$$F_x - \frac{d}{dt}F_{x'} = 0 \tag{2.91}$$

$$F_y - \frac{d}{dt}F_{y'} = 0 \tag{2.92}$$

$$x(T) = y(T) \tag{2.93}$$

$$F_{x'} + F_{y'} = 0 \quad \text{at } T, \tag{2.94}$$

$$F - x'F_{x'} - y'F_{y'} = 0 \quad \text{at } T. \tag{2.95}$$

We note that

$$F_x = -\frac{\alpha}{\sigma^2}(x' - \alpha x), \qquad F_y = -\frac{\beta}{b^2}(y' - \beta y), \qquad F_{x'} = \frac{1}{\sigma^2}(x' - \alpha x), \qquad F_{y'} = \frac{1}{b^2}(y' - \beta y).$$

The Euler-Lagrange equations give

$$x'' - \alpha^2 x = 0 \quad \text{and} \quad y'' - \beta^2 y = 0.$$

Thus,

$$x(t) = C_1 e^{\alpha t} + C_2 e^{-\alpha t}, \qquad y(t) = C_3 e^{\beta t} + C_4 e^{-\beta t}$$

with boundary conditions

$$C_1 + C_2 = x_0, \quad C_3 + C_4 = y_0, \text{ and } C_1 e^{\alpha T} + C_2 e^{-\alpha T} 0 = C_3 e^{\beta T} + C_4 e^{-\beta T}. \tag{2.96}$$

Also

$$x'(T) = \alpha C_1 e^{\alpha T} - \alpha C_2 e^{-\alpha T}, \quad x'(T) - \alpha x(T) = -2\alpha C_2 e^{-\alpha T}, \tag{2.97}$$

$$y'(T) = \beta C_3 e^{\beta T} - \beta C_4 e^{-\beta T}, \quad y'(T) - \beta y(T) = -2\beta C_4 e^{-\beta T}. \tag{2.98}$$

The first transversality condition is $F_{x'} + F_{y'} = 0$ or

$$\frac{1}{\sigma^2}(x'(T) - \alpha x(T)) + \frac{1}{b^2}(y'(T) - \beta y(T)) = 0 \tag{2.99}$$

or

$$\frac{\alpha}{\sigma^2}C_2 e^{-\alpha T} + \frac{\beta}{b^2}C_4 e^{-\beta T} = 0. \tag{2.100}$$

The second transversality condition:

$$x'F_{x'} + y'F_{y'} = (x' - \alpha x)F_{x'} + (y' - \beta y)F_{y'} + \alpha x F_{x'} + \beta y F_{y'} = 2F + \alpha x F_{x'} + \beta y F_{y'}$$

Then

$$\begin{aligned} x'F_{x'} + y'F_{y'} - F &= F + \alpha x F_{x'} + \beta y F_{y'} \\ &= \frac{1}{2}\left[\frac{1}{\sigma^2}(x' - \alpha x)^2 + \frac{1}{b^2}(y' - \beta y)^2\right] + \frac{\alpha x}{\sigma^2}(x' - \alpha x) + \frac{a y}{b^2}(y' - \beta y) \\ &= \frac{1}{2\sigma^2}(x' - \alpha x)(x' + \alpha x) + \frac{1}{2b^2}(y' - \beta y)(y' + \beta y) \ = \ 0. \end{aligned}$$

The above, in view of (2.99), becomes

$$(x'(T) - \alpha x(T))\ (x'(T) + \alpha x(T) - y'(T) - \beta y(T)) = 0.$$

If the first factor is zero then, in view of (2.99), we obtain

$$x'(T) - \alpha x(T) = 0, \quad y'(T) - \beta y(T) = 0.$$

In view of the fact that $x'(T) - \alpha x(T) = -2\alpha C_2 e^{-\alpha T}$ this translates into $C_2 = 0$ and similarly $y'(T) - \beta y(T) = -2\beta C_4 e^{-\beta T} = 0$ implies $C_4 = 0$. Hence $x(t) = x_0 e^{\alpha T}$, $y(t) = y_0 e^{\beta T}$, and $x(T) = y(T)$ implies that $x_0 e^{\alpha T} = y_0 e^{\beta T}$ or $e^{(\alpha-\beta)T} = \frac{y_0}{x_0}$. Since $\alpha - \beta > 0$ and $y_0/x_0 < 1$ it is impossible to find $T > 0$ which satisfies this last equation.

The alternative solution is

$$x'(T) + \alpha x(T) \ = \ y'(T) + \beta y(T). \tag{2.101}$$

Note that

$$x'(T) + \alpha x(T) = 2\alpha C_1 e^{\alpha T}, \quad y'(T) + \beta y(T) = 2\beta C_3 e^{\beta T}$$

and hence (2.101) gives

$$\alpha C_1 e^{\alpha T} \ = \ \beta C_3 e^{\beta T}. \tag{2.102}$$

Determination of the optimal path. Displays (2.96), (2.100), and (2.102) provide the following 5 equations to determine the 5 unknown quantities, C_i, $i = 1, \ldots, 4$, and T:

$$\begin{aligned} &C_1 + C_2 = x_0, \qquad C_3 + C_4 = y_0 \\ &C_1 e^{\alpha T} + C_2 e^{-\alpha T} = C_3 e^{\beta T} + C_4 e^{-\beta T} \\ &C_3 = \frac{\alpha}{\beta} e^{(\alpha-\beta)T}\, C_1, \qquad C_4 = -\frac{\alpha}{\beta}\frac{b^2}{\sigma^2} e^{-(\alpha-\beta)T}\, C_2 \end{aligned}$$

or

$$C_1 + C_2 \ = \ x_0 \tag{2.103}$$

$$\frac{\alpha}{\beta} e^{(\alpha-\beta)T}\, C_1 - \frac{\alpha}{\beta}\frac{b^2}{\sigma^2} e^{-(\alpha-\beta)T}\, C_2 \ = \ y_0 \tag{2.104}$$

$$C_1 e^{\alpha T} + C_2 e^{-\alpha T} \ = \ \frac{\alpha}{\beta} e^{\alpha T}\, C_1 - \frac{\alpha}{\beta}\frac{b^2}{\sigma^2} e^{-\alpha T}\, C_2 \tag{2.105}$$

$$C_3 \ = \ \frac{\alpha}{\beta} e^{(\alpha-\beta)T}\, C_1 \tag{2.106}$$

$$C_4 \ = \ -\frac{\alpha}{\beta}\frac{b^2}{\sigma^2} e^{-(\alpha-\beta)T}\, C_2 \tag{2.107}$$

From the above we may obtain the values of C_i, $i = 1, \ldots, 4$ in terms of T:

$$C_1 \ = \ x_0 \frac{\left(1 + \frac{\alpha}{\beta}\frac{b^2}{\sigma^2}\right)\, e^{-\alpha T}}{\left(1 + \frac{\alpha}{\beta}\frac{b^2}{\sigma^2}\right)\, e^{-\alpha T} + \left(\frac{\alpha}{\beta} - 1\right)\, e^{\alpha T}}, \quad C_2 \ = \ x_0 \frac{\left(\frac{\alpha}{\beta} - 1\right)\, e^{\alpha T}}{\left(1 + \frac{\alpha}{\beta}\frac{b^2}{\sigma^2}\right)\, e^{-\alpha T} + \left(\frac{\alpha}{\beta} - 1\right)\, e^{\alpha T}}. \tag{2.108}$$

and

$$C_3 \;=\; y_0\,\frac{\left(1+\frac{\beta}{\alpha}\frac{\sigma^2}{b^2}\right)\,e^{-\beta T}}{\left(1+\frac{\beta}{\alpha}\frac{\sigma^2}{b^2}\right)\,e^{-\beta T}+\left(\frac{\beta}{\alpha}-1\right)\,e^{\beta T}},\quad C_4 \;=\; y_0\,\frac{\left(\frac{\beta}{\alpha}-1\right)\,e^{\beta T}}{\left(1+\frac{\beta}{\alpha}\frac{\sigma^2}{b^2}\right)\,e^{-\beta T}+\left(\frac{\beta}{\alpha}-1\right)\,e^{\beta T}}. \tag{2.109}$$

From these we obtain the following expression for the critical path

$$x(t) \;=\; x_0\,\frac{\left(1+\frac{\alpha}{\beta}\frac{b^2}{\sigma^2}\right)\,e^{\alpha(t-T)}\;+\;\left(\frac{\alpha}{\beta}-1\right)\,e^{\alpha(T-t)}}{\left(1+\frac{\alpha}{\beta}\frac{b^2}{\sigma^2}\right)\,e^{-\alpha T}+\left(\frac{\alpha}{\beta}-1\right)\,e^{\alpha T}} \tag{2.110}$$

$$y(t) \;=\; y_0\,\frac{\left(1+\frac{\beta}{\alpha}\frac{\sigma^2}{b^2}\right)\,e^{\beta(t-T)}\;+\;\left(\frac{\beta}{\alpha}-1\right)\,e^{\beta(T-t)}}{\left(1+\frac{\beta}{\alpha}\frac{\sigma^2}{b^2}\right)\,e^{-\beta T}+\left(\frac{\beta}{\alpha}-1\right)\,e^{\beta T}} \tag{2.111}$$

Of course, there remains the task to determine the optimal meeting time T. From the above, when $t=T$ we have

$$\begin{aligned} x(T) \;&=\; x_0\,\frac{\alpha(b^2+\sigma^2)}{(\alpha-\beta)\sigma^2 e^{\alpha T}+(\beta\sigma^2+\alpha b^2)e^{-\alpha T}},\\ y(T) \;&=\; y_0\,\frac{\beta(b^2+\sigma^2)}{(\beta-\alpha)b^2 e^{\beta T}+(\beta\sigma^2+\alpha b^2)e^{-\beta T}}. \end{aligned}$$

At the meeting time T, $x(T)=y(T)$ and therefore

$$x_0\alpha\left[(\beta-\alpha)b^2 e^{\beta T}+(\beta\sigma^2+\alpha b^2)e^{-\beta T}\right] \;=\; y_0\beta\left[(\alpha-\beta)\sigma^2 e^{\alpha T}+(\beta\sigma^2+\alpha b^2)e^{-\alpha T}\right] \tag{2.112}$$

Determination of the meeting time T. We will show that the above equation determines uniquely T. To this end, define the function

$$f(t) \;:=\; (\alpha-\beta)\left[y_0\beta\sigma^2 e^{\alpha t}+x_0\alpha b^2 e^{\beta t}\right]+(\beta\sigma^2+\alpha b^2)\left[y_0\beta e^{-\alpha t}-x_0\alpha e^{-\beta t}\right],\qquad t\ge 0.$$

It holds that

$$\begin{aligned} f(0) \;&=\; (\alpha-\beta)\left[y_0\beta\sigma^2+x_0\alpha b^2\right]+(\beta\sigma^2+\alpha b^2)\left[y_0\beta-x_0\alpha\right]\\ &=\; \alpha\beta(\sigma^2+b^2)(y_0\beta-x_0\alpha)\;<\;0 \end{aligned}$$

and also $\lim_{t\to\infty} f(t)=+\infty$. Furthermore

$$f'(t) \;=\; (\alpha-\beta)\alpha\beta\left[y_0\sigma^2 e^{\alpha t}+x_0 b^2 e^{\beta t}\right]+(\beta\sigma^2+\alpha b^2)\alpha\beta\left[-y_0 e^{-\alpha t}+x_0 e^{-\beta t}\right]$$

Clearly $f'(t)\;>\;0$ for all $t\;\ge\;0$ since $\left[-y_0 e^{-\alpha t}+x_0 e^{-\beta t}\right]\;=\;e^{-\alpha t}\left[-y_0+x_0 e^{(\alpha-\beta)t}\right]\;>\;0$ because $\alpha>\beta$ and $x_0>y_0$.

Determination of the rate I. Taking into account that $x'(t) - \alpha x(t) = -2\alpha C_2 e^{-\alpha t}$ and similarly $y'(t) - \beta y(t) = -2\beta C_4 e^{-\beta t}$ the rate function becomes

$$
\begin{aligned}
I &= \frac{1}{2\sigma^2}\int_0^T 4\alpha^2 C_2^2 e^{-2\alpha t}dt + \frac{1}{2b^2}\int_0^T 4\beta^2 C_4^2 e^{-2\beta t}dt = \frac{\alpha C_2^2}{\sigma^2}\left(1-e^{-2\alpha T}\right) + \frac{\beta C_4^2}{b^2}\left(1-e^{-2\beta T}\right) \\
&= \frac{\frac{\alpha}{\sigma^2}\left(1-e^{-2\alpha T}\right)x_0^2\left(\frac{\alpha}{\beta}-1\right)^2 e^{2\alpha T}}{\left[\left(1+\frac{\alpha}{\beta}\frac{b^2}{\sigma^2}\right)e^{-\alpha T}+\left(\frac{\alpha}{\beta}-1\right)e^{\alpha T}\right]^2} + \frac{\frac{\beta}{b^2}\left(1-e^{-2\beta T}\right)y_0^2\left(\frac{\beta}{\alpha}-1\right)^2 e^{2\beta T}}{\left[\left(1+\frac{\beta}{\alpha}\frac{\sigma^2}{b^2}\right)e^{-\beta T}+\left(\frac{\beta}{\alpha}-1\right)e^{\beta T}\right]^2}
\end{aligned}
$$

or equivalently

$$
I = \frac{\alpha(\alpha-\beta)^2\sigma^2 x_0^2\left(e^{2\alpha T}-1\right)}{\left[(\alpha-\beta)\sigma^2 e^{\alpha T}+(\beta\sigma^2+\alpha b^2)e^{-\alpha T}\right]^2} + \frac{\beta y_0^2 b^2\left(e^{2\beta T}-1\right)(\alpha-\beta)^2}{\left[(\alpha b^2+\beta\sigma^2)\,e^{-\beta T}+(\beta-\alpha)\,b^2\,e^{\beta T}\right]^2}. \quad (2.113)
$$

In particular, when $b = 0$ and $\alpha = \mu$ then the lower OU process becomes a deterministic lower bound and (2.113) reduces indeed to (2.57), as it should.

Again, as in the proof of Theorem 4 we will show that the solution obtained corresponds to a global minimum using the fact that $F : \mathbb{R}^4 \to \mathbb{R}$ is convex and appealing to Theorem 3.16 [2, p.45]. To establish the convexity of $F(x, x', y, y') := \frac{1}{2\sigma^2}(x'-\alpha x)^2 + \frac{1}{2b^2}(y'-\beta y)^2$ we note that, for any $(x_0, x_0', y_0, y_0') \in \mathbb{R}^4$,

$$
F(x,x',y,y') - F(x_0,x_0',y_0,y_0') \geq F_x^0\,(x-x_0) + F_{x'}^0\,(x'-x_0') + F_y^0\,(y-y_0) + F_{y'}^0\,(y'-y_0') \quad (2.114)
$$

where F_x^0 is shorthand for $F_x(x_0, x_0', y_0, y_0')$ and similarly for the other three such quantities. The above inequality is equivalent to

$$
\begin{aligned}
\frac{1}{2\sigma^2}(x'-\alpha x)^2 + \frac{1}{2b^2}(x'-\beta x)^2 - \frac{1}{2\sigma^2}(x_0'-\alpha x_0)^2 - \frac{1}{2b^2}(x_0'-\beta x_0)^2 \\
\geq -\frac{\alpha}{\sigma^2}(x_0'-\alpha x_0)\,(x-x_0) + \frac{1}{\sigma^2}(x_0'-\alpha x_0)\,(x'-x_0') \\
-\frac{\beta}{b^2}(y_0'-\beta y_0)\,(y-y_0) + \frac{1}{b^2}(y_0'-\beta y_0)\,(y'-y_0').
\end{aligned}
$$

Elementary algebraic manipulations can show the above inequality to be true and therefore establish inequality (2.114) which implies the convexity of F.

We may thus summarize the above long derivation as follows.

Theorem 5. *Consider the pair of Ornstein-Uhlenbeck SDE's depending on a parameter* $\epsilon > 0$

$$
\begin{aligned}
dX_t^\epsilon &= \alpha X_t^\epsilon dt + \sqrt{\epsilon}\sigma dW_t, \qquad X_0^\epsilon = x_0, \\
dY_t^\epsilon &= \beta Y_t^\epsilon dt + \sqrt{\epsilon} b dB_t, \qquad Y_0^\epsilon = y_0.
\end{aligned}
$$

Assume that $0 < y_0 < x_0$ *and* $0 < \beta < \alpha$*. Let* $T^\epsilon := \inf\{t \geq 0 : X_t^\epsilon = Y_t^\epsilon\}$ *(with the standard convention that* $T^\epsilon = +\infty$ *if the set is empty). Then*

$$\lim_{\epsilon \to 0} \epsilon \log P(T^\epsilon < \infty) \; = \; -I$$

where I *is given by (2.113). If this rare event occurs then the meeting path followed by the two processes, is given by (2.110), (2.111), and the meeting time* T *is the unique solution of (2.112).*

3. GEOMETRIC BROWNIAN MOTION

In this chapter an analysis of the problems we examined for the Ornstein-Uhlenbeck process is repeated for the Geometric Brownian motion. The techniques used and approach followed are analogous to that of the previous chapter. The reason for treating the Geometric Brownian motion in equal detail is, on one hand its great importance in applications but also the fact that in this case an analytic solution for the types of ruin problems we consider can be obtained. As a result, the accuracy and merit of the large deviation estimates we obtain may be gauged. This is carried out in this chapter.

3.1 The Finite Horizon Problem

Suppose that $\{X_t; t \geq 0\}$ is a Geometric Brownian motion satisfying the Stochastic Differential Equation

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \qquad X_0 = x_0 \text{ w.p. } 1. \tag{3.1}$$

As is well known this has the closed form solution

$$X_t = x_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}. \tag{3.2}$$

Let $u_0 > x_0$ and $\alpha > \mu$. Then the event $\{X_t \geq u_0 e^{at} \text{ for some } t \leq T\}$ is an event whose probability goes to 0 as $\sigma \to 0$. Our goal is to obtain low variance Wentzell-Freidlin asymptotics for this finite horizon hitting probability. For reasons of notational compatibility we introduce the *parameterized process*

$$dX_t^\epsilon = \mu X_t^\epsilon dt + \sqrt{\epsilon}\sigma X_t^\epsilon dW_t, \qquad X_0^\epsilon = x_0 \text{ w.p. } 1. \tag{3.3}$$

Theorem 6. *For the parameterized process* $\{X_t^\epsilon\}$,

$$\lim_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(\sup_{0 \leq t \leq T} \left(X_t^\epsilon - u_0 e^{\alpha t}\right) \geq 0\right) \;=\; -I(T). \tag{3.4}$$

The rate function $I(T)$ *is given by*

$$I(T) := \min_{0 \leq t \leq T} J_*(t) \tag{3.5}$$

where $J_*(t)$ is solution to the minimization problem

$$J_*(t) \;=\; \min\left\{J(x,t)\;:\; x\in\mathcal{H},\; x(0)=x_0,\; x(t)=u_0e^{\alpha t},\; x(s)<u_0e^{\alpha s},\; s\in[0,t)\right\}. \quad (3.6)$$

where $\mathcal{H}=\left\{h:[0,t]\to\mathbb{R} : h(s)=h(0)+\int_0^s\phi(u)du\;,\; s\in[0,t],\;\phi\in L^2[0,t]\right\}$ and $J(x,t)$ is the action functional

$$J(x,t) := \frac{1}{2}\int_0^t\left(\frac{x'(u)-\mu x(u)}{\sigma x(u)}\right)^2 du = \frac{1}{2\sigma^2}\int_0^t \left((\log x(u))'-\mu\right)^2 du. \quad (3.7)$$

This theorem is of course a consequence of the Wentzell-Freidlin theory. The minimizing path $x(t)$ can be easily obtained in this case either using the full machinery of the Euler-Lagrange differential equations, or simply by observing that the functional $J(x,t)$ is minimized when $(\log x)'$ is zero or equivalently when $\log x(s)=c$ for $s\in[0,t]$. This in turn implies that $x(s)=Ke^{ct}$ with $x(0)=x_0=K$ and $x(t)=x_0e^{ct}=u_0e^{\alpha t}$ whence we conclude that the function that minimizes the action functional under the boundary conditions is

$$x(t)=x_0e^{ct} \quad \text{where} \quad c=\alpha+\frac{1}{t}\log\frac{u_0}{x_0}. \quad (3.8)$$

It is easy to see that the above path satisfies the constraint $x(s)<u_0e^{\alpha s}$ for $s\in[0,t)$. The corresponding minimum action is then

$$J_*(t) \;=\; \frac{t}{2\sigma^2}\left(\alpha-\mu+\frac{1}{t}\log\frac{u_0}{x_0}\right)^2$$

or

$$J_*(t) \;=\; t\,\frac{(\alpha-\mu)^2}{2\sigma^2}+2\frac{(\alpha-\mu)\log\frac{u_0}{x_0}}{2\sigma^2}+\frac{1}{t}\,\frac{(\log\frac{u_0}{x_0})^2}{2\sigma^2}.$$

The value of t that minimizes the above expression is

$$t_{\min}=\frac{\log\frac{u_0}{x_0}}{\alpha-\mu}$$

and the corresponding minimum is

$$\frac{2(\alpha-\mu)\log\frac{u_0}{x_0}}{\sigma^2}.$$

Thus the rate function is

$$I(T)=\begin{cases}\frac{2(\alpha-\mu)\log\frac{u_0}{x_0}}{\sigma^2} & \text{if}\quad t_{\min}<T\\[2ex] \frac{T}{2\sigma^2}\left(\alpha-\mu+\frac{1}{T}\log\frac{u_0}{x_0}\right)^2 & \text{if}\quad t_{\min}\geq T\end{cases} \quad (3.9)$$

and, based on Theorem 6 we conclude that

$$-\log\mathbb{P}\left(\sup_{0\leq t\leq T}(X_t-u_0e^{at})\geq 0\right)\approx I(T). \quad (3.10)$$

The above approximation is satisfactory provided that σ is sufficiently small. We assess its quality in the next subsection taking advantage of the fact that an exact solution also exists in this situation.

3.2 The exact solution

Consider the GBM $X_t = x_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$ and the corresponding finite horizon hitting probability

$$p_T \;:=\; \mathbb{P}\left(\sup_{0\le t\le T} (X_t - u_0 e^{at}) \ge 0 \right)$$

where, as before $\alpha > \mu$ and $0 < x_0 < u_0$. Since the event $(X_t - u_0 e^{\alpha t}) \ge 0$ is the same as $X_t e^{-\alpha t} - u_0 \ge 0$, we will determine, equivalently the probability

$$\begin{aligned} p_T \;&=\; \mathbb{P}\left(\sup_{0\le t\le T} x_0 e^{(\mu - \frac{1}{2}\sigma^2 - a)t + \sigma W_t} \ge u_0 \right) \;=\; \mathbb{P}\left(\sup_{0\le t\le T} (\mu - \frac{1}{2}\sigma^2 - \alpha)t + \sigma W_t \ge \log \frac{u_0}{x_0} \right) \\ &= 1 - \Phi\left(\frac{\log\left(\frac{u_0}{x_0}\right) - (\mu - \alpha - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) \\ &\quad + e^{\frac{2}{\sigma^2}(\mu - \alpha - \frac{1}{2}\sigma^2)\log\left(\frac{u_0}{x_0}\right)} \Phi\left(\frac{-\log\left(\frac{u_0}{x_0}\right) - (\mu - \alpha - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) \end{aligned} \tag{3.11}$$

Here, $\Phi(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du$, the standard normal distribution function. The above exact formula for p_T allows us to evaluate the accuracy of the approximation (3.10). Figure 3.2 shows again $-\sigma^2 \log p_T$ together with the Wentzell-Freidlin asymptotic result when $\sigma \to 0$. One may see that approximation (3.10) may be considered satisfactory, provided that σ is small.

3.3 The Infinite Horizon Problem

The exact value of the infinite horizon hitting probability can be obtained from (3.11) by letting $T \to \infty$. This gives

$$\lim_{T\to\infty} p_T \;=:\; p_\infty \;=\; \exp\left(\frac{2}{\sigma^2}(\mu - \frac{1}{2}\sigma^2 - \alpha) \log \frac{u_0}{x_0} \right).$$

Returning to the parameterized version of the problem, concerning the family of processes $\{X_t^\epsilon\}$ defined in (3.3), the corresponding infinite horizon hitting probability is

$$p_\infty^\epsilon \;=\; \exp\left(\frac{2}{\epsilon\sigma^2}(\mu - \frac{1}{2}\epsilon\sigma^2 - \alpha) \log \frac{u_0}{x_0} \right)$$

and therefore

$$\lim_{\epsilon\to 0} \epsilon \log p_\infty^\epsilon \;=\; -\frac{2}{\sigma^2}(\alpha - \mu) \log \frac{u_0}{x_0}. \tag{3.12}$$

This, as we will see, is the same as the result obtained from Wentzell-Freidlin theory.

Theorem 7. *For the parameterized process* $\{X_t^\epsilon\}$,

$$\lim_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\sup_{t\geq 0}\left(X_t^\epsilon - u_0e^{\alpha t}\right)\geq 0\right) \;=\; -I(\infty) \tag{3.13}$$

where the rate function $I(\infty)$ *is the solution to the infinite horizon variational problem*

$$\inf\left\{J(x,T)\;:\;x\in\mathcal{H},\;x(s)<u_0e^{\alpha s}, 0\leq s<T,\;x(0)=x_0,\;x(T)=u_0e^{\alpha T}\right\} \tag{3.14}$$

where $J(x,t) := \frac{1}{2}\int_0^t\left((\log x(u))'-\mu\right)^2du$ *and* $\mathcal{H}$ *is again the Cameron-Martin space of absolutely continuous functions with square-integrable derivatives. In fact, the rate function for the infinite horizon problem is*

$$I(\infty)=2\frac{\alpha-\mu}{\sigma^2}\log\frac{u_0}{x_0}, \tag{3.15}$$

the optimal time horizon is

$$T=\frac{\log\frac{u_0}{x_0}}{\alpha-\mu}, \tag{3.16}$$

and the optimal path that achives the minimum *is*

$$x^*(t)=x_0e^{2\alpha-\mu t},\quad t\in[0,T]. \tag{3.17}$$

Figure 3.1 provides an illustration of the above result.

The optimization problem of Theorem 7 can of course be solved using the finite horizon analysis as a basis. However we prefer to use standard techniques of the calculus of variations for infinite horizon problems with the final value of the path constrained to lie on a prescribed curve using the *transversality conditions*

$$\begin{aligned}&\min\int_0^T F(x,x',t)dt,\qquad\text{with boundary conditions } x(0)=x_0,\text{ and } x(T)=u(T)\\ &\text{with } F(x,x',t)=\frac{1}{2\sigma^2}\left(\frac{x'}{x}-\mu\right)^2.\end{aligned} \tag{3.18}$$

In the above $u(t)=u_0e^{\alpha t}$ is a given boundary curve with $x_0<u_0$ and x is a $C^1[0,\infty)$ function which minimizes the "action" integral given the boundary conditions in (3.18). The conditions for a minimum is

$$F_x-\frac{d}{dt}F_{x'}=0 \tag{3.19}$$

$$x(0)=x_0\quad\text{and}\quad x(T)=u(T) \tag{3.20}$$

$$F+(u'-x')F_{x'}=0\quad\text{at } T. \tag{3.21}$$

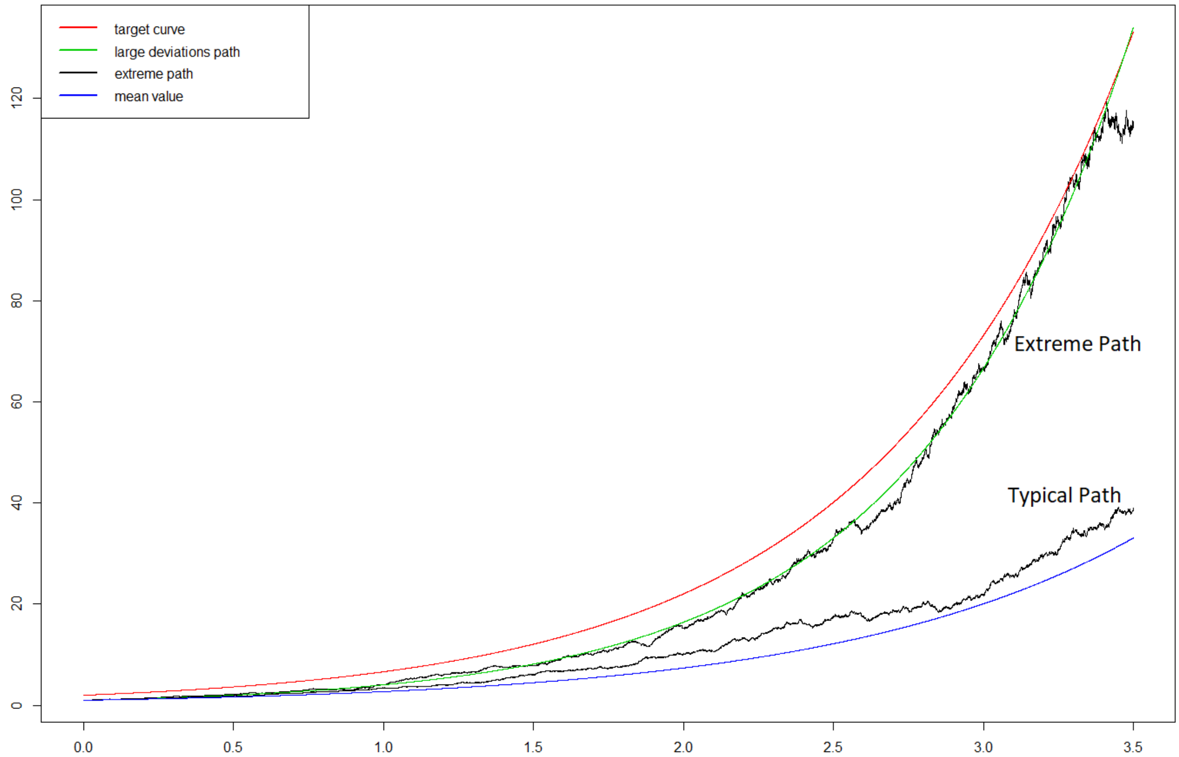

Fig. 3.1: Simulated sample path for $\alpha = 1, x_0 = 1, u_0 = 2$ and $\sigma = 0.15$. The red curve is the exponential target curve $u_0 e^{\alpha t}$. The green curve is optimal path predicted by Large Deviations theory and given by $x^*(t) = x_0 e^{(2\alpha-\mu)t} t$. Both a typical path and an extreme path of the Geometric Brownian motion are displayed. The extreme path was generated by simulating a large number of paths ($\approx 10^5$) and selecting one that hit the target, i.e. reached the red curve. As expected it follows closely the green curve. The smaller the variance the smaller the probability of hitting the target and the closer the agreement with the theoretical path.

The first equation is the Euler-Lagrange DE of the Calculus of Variations. Equation (3.21) is known as the *transversality condition* resulting from the fact that the end time T is not fixed but is itself to be chosen optimally, under the restriction that $x(T) = u(T)$. Then $F_x = \frac{\mu x'}{\sigma^2 x^2} - \frac{(x')^2}{\sigma^2 x^3}$, $F_{x'} = \frac{x'}{\sigma^2 x^2} - \frac{\mu}{\sigma^2 x}$, and $\frac{d}{dt} F_{x'} = \frac{x''}{\sigma^2 x^2} - 2\frac{(x')^2}{\sigma^2 x^3} + \mu \frac{x'}{\sigma^2 x^2}$ and the Euler-Lagrange equation (3.19) becomes

$$\frac{2}{x^3}\left((x')^2 - x'' x\right) = 0$$

or equivalently

$$\frac{x'}{x} = \frac{x''}{x'} \quad \Leftrightarrow \quad (\log x')' - (\log x)' = 0 \quad \Leftrightarrow \quad \log x' - \log x = c_1 \quad \Leftrightarrow \quad \frac{x'}{x} = \gamma.$$

Hence

$$x(t) = x_0 e^{\gamma t}. \tag{3.22}$$

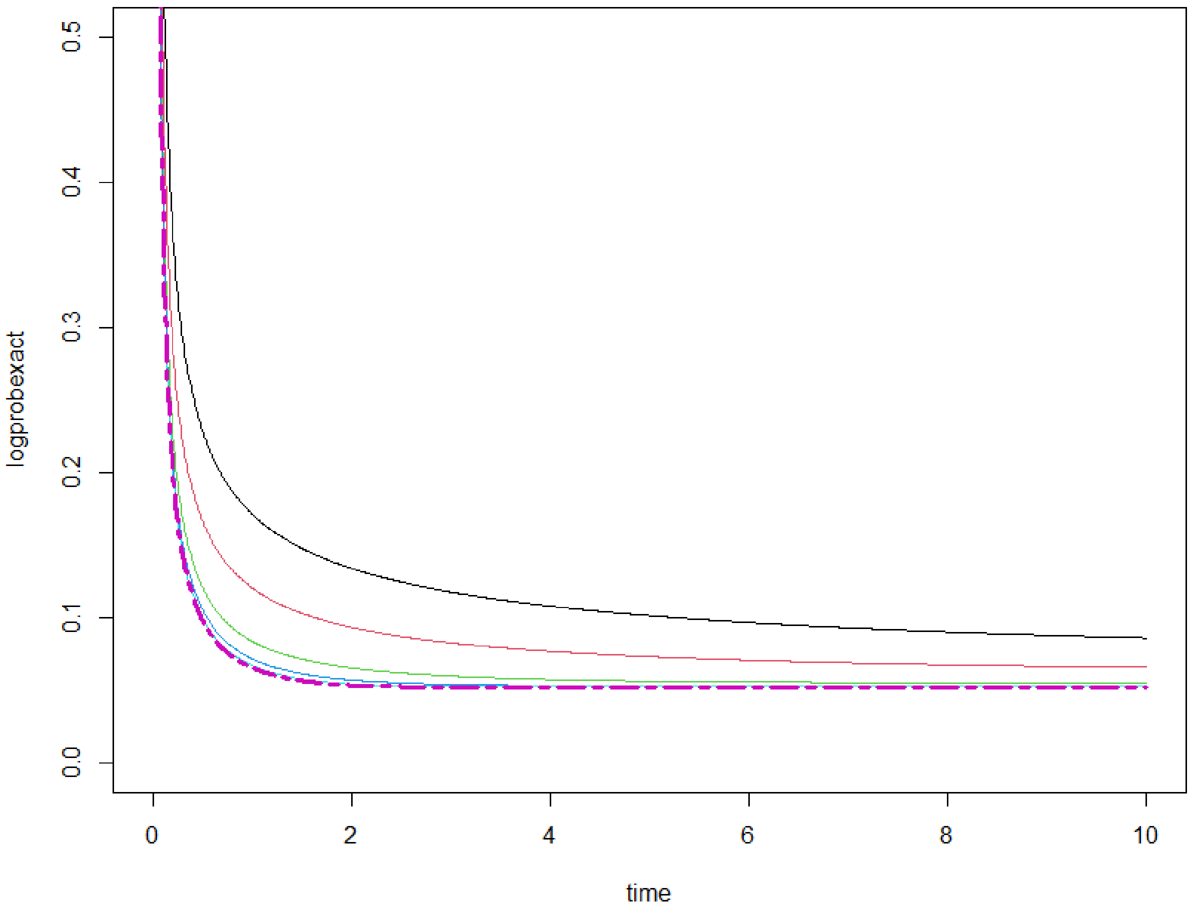

Fig. 3.2: Logarithm of Hitting Probability and Comparison with the Wentzell-Freidlin low variance limit

The transversality condition (3.21) reduces to

$$\left(\frac{x'(T)}{x(T)} - \mu\right)\left(\frac{x'(T)}{x(T)} - \mu + \left(u_0 \alpha e^{\alpha T} - x'(T)\right)\frac{2}{x(T)}\right) = 0$$

and taking into account (3.22) we obtain either $\mu = \gamma$ or

$$\gamma - \mu + 2\alpha \frac{u_0}{x_0} e^{(\alpha-\gamma)T} - 2\gamma = 0$$

or

$$2\alpha \frac{u_0}{x_0} e^{(\alpha-\gamma)T} = \mu + \gamma. \tag{3.23}$$

Equation (3.20) gives $x_0 e^{\gamma T} = u_0 e^{\alpha T}$ and therefore

$$e^{(\alpha-\gamma)T} = \frac{x_0}{u_0}. \tag{3.24}$$

From (3.23) and (3.24) we have

$$\gamma = 2\alpha - \mu \tag{3.25}$$

Once the variational problem for a fixed time horizon is solved we may then find the value T of the "most likely meeting point" for the two curves by optimizing over T. Alternatively we may use (and have done so) the transversality conditions approach of the Calculus of Variations. The solution of the variational process that minimizes the action functional I and satisfies the boundary conditions yields the optimal path is $x_t = x_0 e^{(2\alpha-\mu)t}$ and the rate function

$$I = 2\frac{\alpha - \mu}{\sigma^2} \log \frac{u_0}{x_0} \text{ and } T = \frac{\log \frac{u_0}{x_0}}{\alpha - \mu}.$$

It is worth pointing out that, in this case, a closed form analytic expression can also be obtained. The solution of the SDE is $X_t^\epsilon = x_0 e^{(\mu - \frac{1}{2}\epsilon\sigma^2)t + \sqrt{\epsilon}\sigma W_t}$ and one may show that

$$\lim_{\epsilon \to 0} \epsilon \log P\left(\sup_{t \geq 0}(X_t^\epsilon - u_0 e^{\alpha t}) \geq 0\right) = -\frac{2}{\sigma^2}(\alpha - \mu)\log\frac{u_0}{x_0}.$$

The exact solution agrees with the Wentzell-Freidlin asymptotic result. In Figure 3.1 the extreme path was selected by simulating a large number of paths and picking the largest among them.

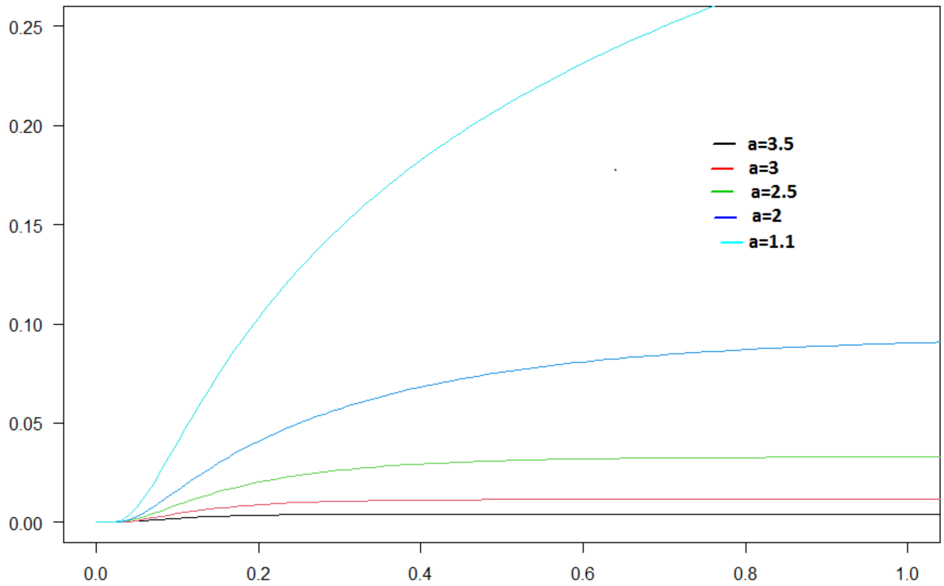

Fig. 3.3: Probability of hitting the upper boundary as a function of time horizon based on the exact solution (3.11). Here $\sigma = 0.5$, $x_0 = 1$, $u_0 = 1.3$, $\mu = 1$. The function is plotted for $\alpha = 1.1, 2, 2.5, 3, 3.5$.

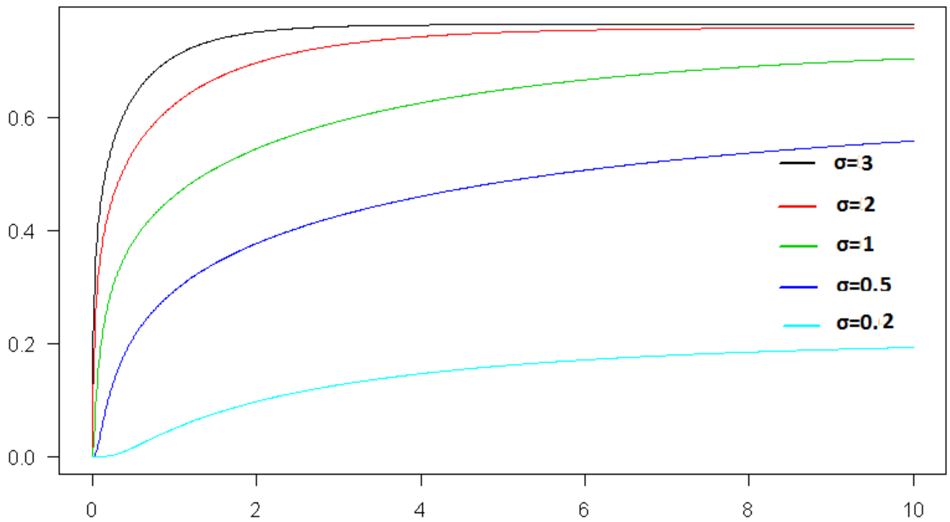

Fig. 3.4: Probability of hitting the upper boundary as a function of time horizon based on the exact solution (3.11). Here $x_0 = 1$, $u_0 = 1.3$, $\mu = 1$ $\alpha = 1.1$. The function is plotted for $\sigma = 0.2, 0.5, 1, 2, 3$.

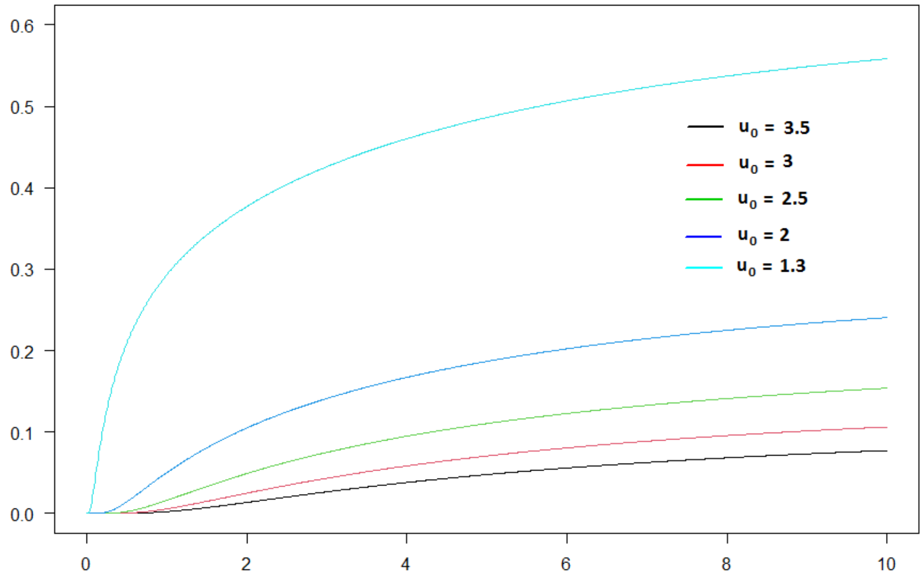

Fig. 3.5: Probability of hitting the upper boundary as a function of time horizon based on the exact solution (3.11). Here $x_0 = 1$, $\alpha = 1.1$, $\mu = 1$ $\sigma = 0.5$. The function is plotted for $u_0 = 1.3,\ 2,\ 2.5,\ 3,\ 3.5$.

3.4 Two Correlated Geometric Brownian Motions

Suppose that W_t, V_t, are independent standard Brownian motions and $\rho \in [-1, 1]$. Set $B_t = \rho W_t + \sqrt{1-\rho^2}\, V_t$. Then (W_t, B_t) are correlated Brownian motions with correlation ρ. Consider now the processes

$$\begin{aligned} dX_t &= \alpha X_t dt + \sigma X_t dW_t, \quad X_0 = x_0, \\ dY_t &= \beta Y_t dt + b Y_t dB_t, \quad Y_0 = y_0, . \end{aligned}$$

We will assume that $\alpha > \beta$ and $x_0 > y_0 > 0$. Thus, in the absence of noise one would have $X_t > Y_y$ for all $t > 0$. In the presence of noise however the probability that $X_T = Y_T$ for some $T > 0$ is non-zero. The second equation can be written equivalently as

$$dY_t = \beta Y_t dt + \rho b Y_t dW_t + \sqrt{1-\rho^2} b Y_t dV_t.$$

Heuristically, we have

$$dW_t = \frac{dX_t - \alpha X_t dt}{\sigma X_t}$$

and

$$dV_t = \frac{1}{bY_t\sqrt{1-\rho^2}} \left(dY_t - \beta Y_t dt - \rho b Y_t \frac{dX_t - \alpha X_t dt}{\sigma X_t} \right).$$

and thus we obtain the following action functional to be minimized:

$$I = \frac{1}{2} \int_0^T \left(\frac{x' - \alpha x}{x\sigma} \right)^2 + \frac{1}{1-\rho^2} \left(\frac{y' - \beta y}{yb} - \rho \frac{x' - \alpha x}{x\sigma} \right)^2 dt \qquad (3.26)$$

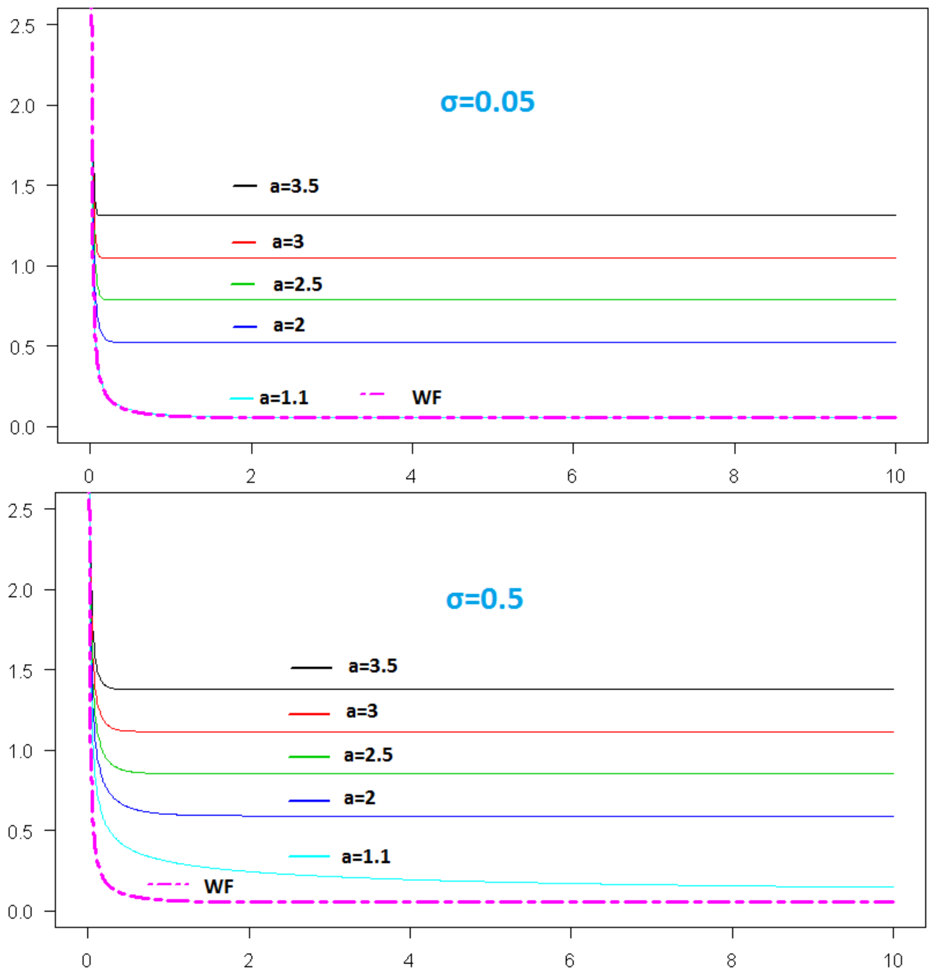

Fig. 3.6: $-log$ Probability of hitting the upper boundary based on the exact solution (3.11). Here $x_0 = 1$, $u_0 = 1.3$, $\mu = 1$. The upper graph was obtained for $\sigma = 0.05$ while the lower for $\sigma = 0.5$. The magenta dotted line gives the value of (the exponent of) the Wentzell-Freidlin approximation.

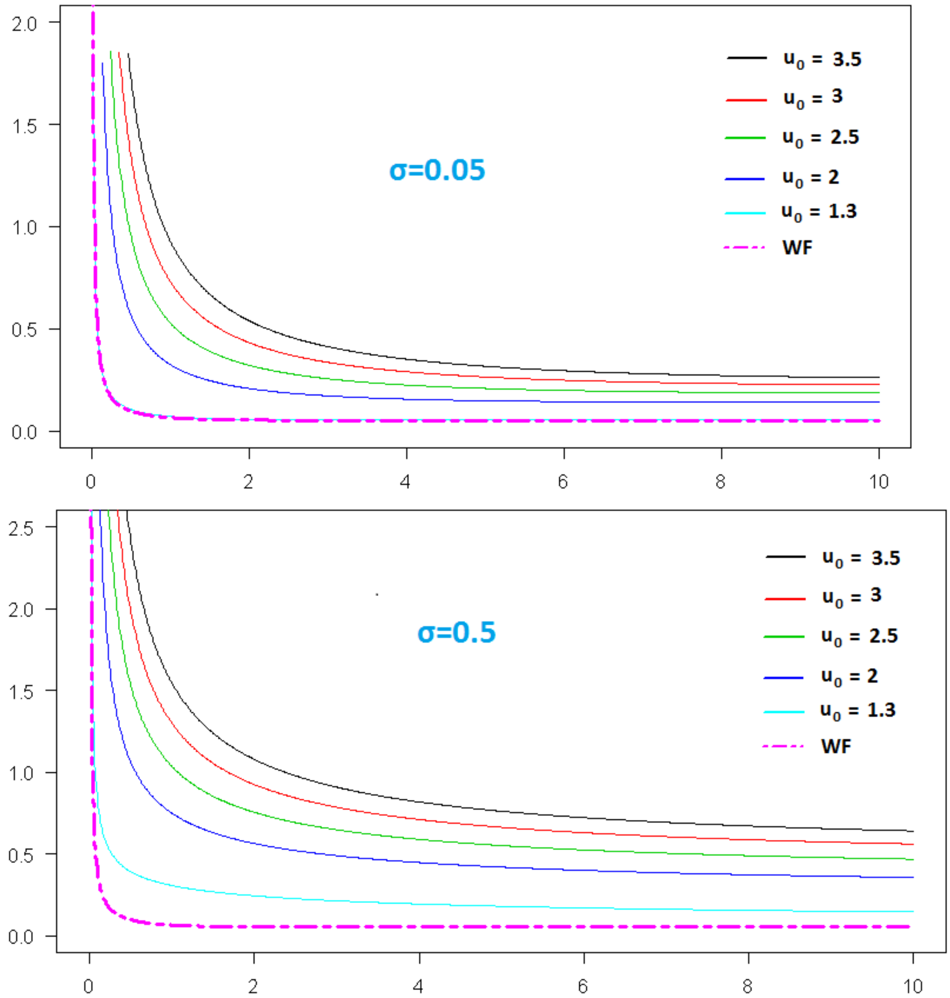

Fig. 3.7: $-log$ Probability of hitting the upper boundary based on the exact solution (3.11). Here $x_0 = 1$, $\mu = 1$, $\alpha = 1.3$. The upper graph was obtained for $\sigma = 0.05$ while the lower for $\sigma = 0.5$. The magenta dotted line gives the value of (the exponent of) the Wentzell-Freidlin approximation.

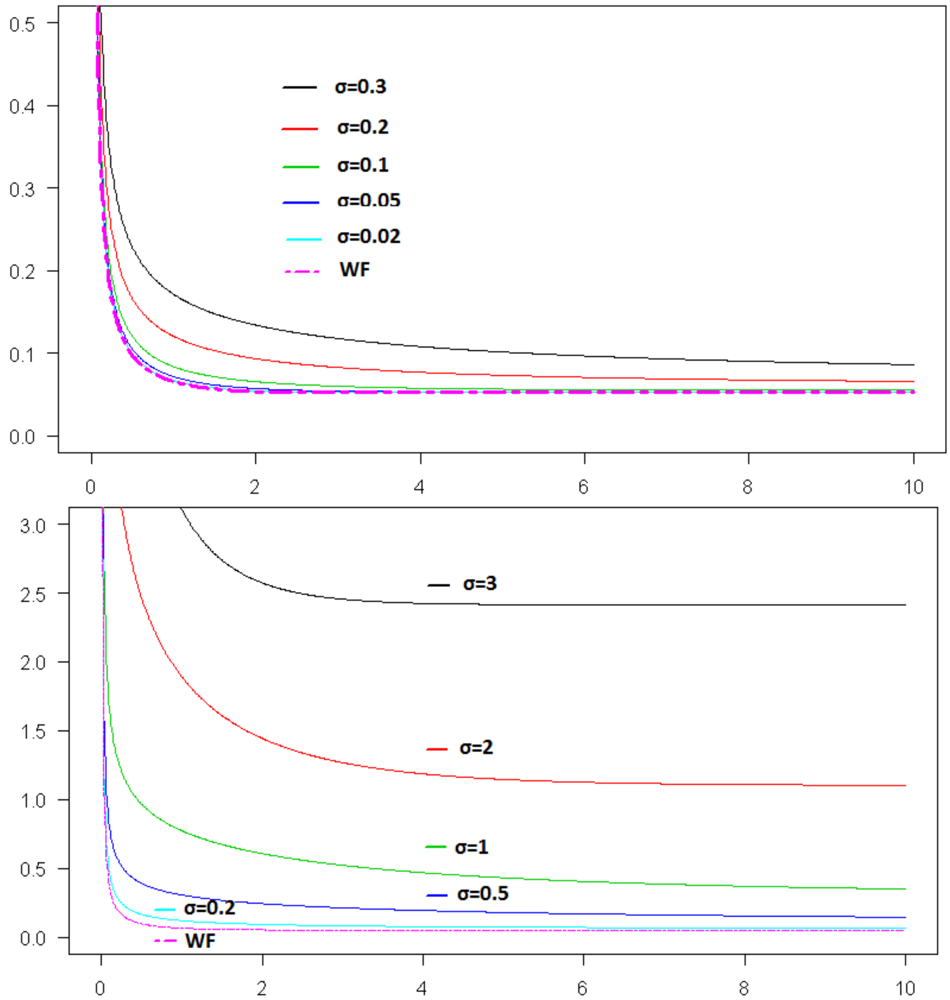

Fig. 3.8: $-log$ Probability of hitting the upper boundary based on the exact solution (3.11). Here $x_0 = 1$, $u_0 = 1.3$, $\mu = 1$, $\alpha = 1.1$. The magenta dotted line gives the value of (the exponent of) the Wentzell-Freidlin approximation.

This of course can be justified by appealing to the multidimensional version of (2.12) as we have already seen. Set

$$F = \frac{1}{2\sigma^2}\left(\frac{x'}{x}-\alpha\right)^2 + \frac{1}{2(1-\rho^2)}\left(\frac{1}{b}\left(\frac{y'}{y}-\beta\right)-\frac{\rho}{\sigma}\left(\frac{x'}{x}-\alpha\right)\right)^2 \tag{3.27}$$

The conditions for minimum are

$$F_x - \frac{d}{dt}F_{x'} = 0 \tag{3.28}$$

$$F_y - \frac{d}{dt}F_{y'} = 0 \tag{3.29}$$

$$x(T) = y(T) \tag{3.30}$$

$$F_{x'} + F_{y'} = 0 \quad \text{at } T, \tag{3.31}$$

$$F - x'F_{x'} - y'F_{y'} = 0 \quad \text{at } T. \tag{3.32}$$

Rewrite (3.27) as

$$F = \frac{1}{2b^2\sigma^2(1-\rho^2)}\left[b^2\left(\frac{x'}{x}-\alpha\right)^2 + \sigma^2\left(\frac{y'}{y}-\beta\right)^2 - 2\rho b\sigma\left(\frac{x'}{x}-\alpha\right)\left(\frac{y'}{y}-\beta\right)\right]. \tag{3.33}$$

From the above equation we obtain

$$\begin{aligned} F_x &= \frac{1}{2b^2\sigma^2(1-\rho^2)}\frac{2x'}{x^2}\left[-b^2\left(\frac{x'}{x}-\alpha\right)+\rho b\sigma\left(\frac{y'}{y}-\beta\right)\right] \\ F_y &= \frac{1}{2b^2\sigma^2(1-\rho^2)}\frac{2y'}{y^2}\left[-\sigma^2\left(\frac{y'}{y}-\beta\right)+\rho b\sigma\left(\frac{x'}{x}-\alpha\right)\right] \end{aligned}$$

$$\begin{aligned} F_{x'} &= \frac{1}{2b^2\sigma^2(1-\rho^2)}\frac{2}{x}\left[b^2\left(\frac{x'}{x}-\alpha\right)-\rho b\sigma\left(\frac{y'}{y}-\beta\right)\right] \\ F_{y'} &= \frac{1}{2b^2\sigma^2(1-\rho^2)}\frac{2}{y}\left[\sigma^2\left(\frac{y'}{y}-\beta\right)-\rho b\sigma\left(\frac{x'}{x}-\alpha\right)\right] \end{aligned}$$

Then (3.28) becomes

$$\begin{aligned} &\frac{x'}{x^2}\left[-b^2\left(\frac{x'}{x}-\alpha\right)+\rho b\sigma\left(\frac{y'}{y}-\beta\right)\right] + \frac{x'}{x^2}\left[b^2\left(\frac{x'}{x}-\alpha\right)-\rho b\sigma\left(\frac{y'}{y}-\beta\right)\right] \\ &+\frac{1}{x}\left[b^2\left(\frac{x''}{x}-\left(\frac{x'}{x}\right)^2\right)-\rho b\sigma\left(\frac{y''}{y}-\left(\frac{y'}{y}\right)^2\right)\right] = 0 \end{aligned}$$

which, since $x > 0$, gives

$$b^2(\log x)'' - \rho b\sigma(\log y)'' = 0$$

Similarly (3.29) gives

$$\sigma^2(\log y)'' - \rho b\sigma(\log x)'' = 0$$

These equations together imply that $(\log x)'' = (\log y)'' = 0$ whence we obtain $\frac{x'}{x} = c_1$ and $\frac{y'}{y} = c_2$ for arbitrary c_1, c_2 and hence

$$x(t) = x_0 e^{c_1 t}, \qquad y(t) = y_0 e^{c_2 t}. \tag{3.34}$$

Condition (3.30) gives

$$x_0 e^{c_1 T} = y_0 e^{c_2 T}. \tag{3.35}$$

Taking into account that $\frac{x'}{x} = c_1$ and similarly $\frac{y'}{y} = c_2$, condition (3.31) gives

$$\frac{1}{x_0 e^{c_1 T}} \left[b^2 (c_1 - \alpha) - \rho b\sigma (c_2 - \beta)\right] + \frac{1}{y_0 e^{c_2 T}} \left[\sigma^2 (c_2 - \beta) - \rho b\sigma (c_1 - \alpha)\right] = 0$$

Setting $u_1 = c_1 - \alpha$, $u_2 = c_2 - \beta$, we rewrite the above $b^2 u_1 - \rho b\sigma u_2 + \sigma^2 u_2 - \rho b\sigma u_1 = 0$. This gives

$$u_2 = \lambda u_1 \quad \text{with} \quad \lambda = \frac{b}{\sigma}\frac{\rho\sigma - b}{\sigma - \rho b} \tag{3.36}$$

Finally, from (3.32),

$$b^2 u_1^2 + \sigma^2 u_2^2 - 2\rho b\sigma u_1 u_2 - 2c_1\left[b^2 u_1 - \rho b\sigma u_2\right] - 2c_2\left[\sigma^2 u_2 - \rho b\sigma u_1\right] = 0$$

or

$$-u_1^2\left[b^2 + \sigma^2\lambda^2 - 2\rho b\sigma\lambda\right] + 2u_1\left[-\alpha b^2 + \beta\rho b\sigma - \lambda\beta\sigma^2 + \lambda\alpha b\sigma\rho\right] = 0.$$

Besides the solution $u_1 = 0$ which means ($c_1 = \alpha$), we obtain

$$u_1 = -\frac{2}{b^2 + \sigma^2\lambda^2 - 2\rho b\sigma\lambda}\left(\alpha b^2 + a\lambda\sigma^2 - \rho b\sigma(a + \lambda\alpha)\right).$$

The denominator can be written as

$$\begin{aligned} b^2 + \sigma^2\lambda^2 - 2\rho b\sigma\lambda &= b^2 + \sigma^2\left(\frac{\rho b\sigma - b^2}{\sigma^2 - \rho b\sigma}\right)^2 - 2\rho b\sigma\frac{\rho b\sigma - b^2}{\sigma^2 - \rho b\sigma} \\ &= \frac{b^2(1-\rho^2)}{(\sigma - \rho b)^2}\left[\sigma^2 + b^2 - 2\rho b\sigma\right] \end{aligned} \tag{3.37}$$

The numerator is

$$\begin{aligned} \alpha b^2 + a\lambda\sigma^2 - \rho b\sigma(a + \lambda\alpha) &= \alpha b^2 + a\sigma^2\frac{b}{\sigma}\frac{\rho\sigma - b}{\sigma - \rho b} - \rho b\sigma\left(a + \frac{b}{\sigma}\alpha\frac{\rho\sigma - b}{\sigma - \rho b}\right) \\ &= (\alpha - \beta)\frac{\sigma b^2(1-\rho^2)}{\sigma - \rho b}. \end{aligned} \tag{3.38}$$

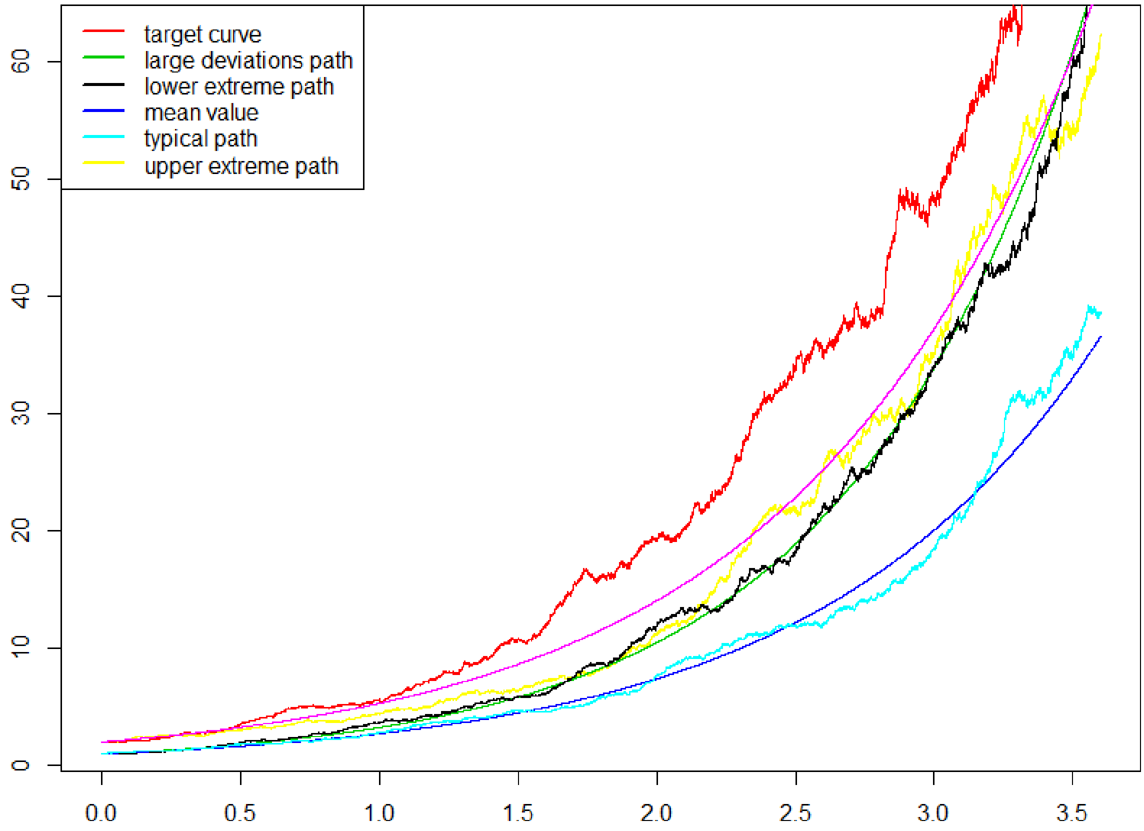

Fig. 3.9: Two independent Geometric Brownian Motions.

The above expression simplifies to

$$u_1 = 2(\beta - \alpha)\frac{\sigma(\sigma - \rho b)}{\sigma^2 + b^2 - 2\rho b\sigma}, \qquad u_2 = 2(\beta - \alpha)\frac{b(\rho\sigma - b)}{\sigma^2 + b^2 - 2\rho b\sigma}. \tag{3.39}$$

From (3.33) and (3.39), together with the definition of u_1, u_2,

$$F = \frac{1}{2b^2\sigma^2(1-\rho^2)}\left[b^2u_1^2 + \sigma^2u_2^2 - 2\rho b\sigma u_1u_2\right] = \frac{2(\beta-\alpha)^2}{\sigma^2 + b^2 - 2\rho b\sigma}. \tag{3.40}$$

Thus, since

$$T = \frac{1}{\alpha - \beta}\log\left(\frac{x_0}{y_0}\right),$$

the optimal rate is

$$I = \frac{2(\alpha-\beta)\log\left(\frac{x_0}{y_0}\right)}{\sigma^2 + b^2 - 2\rho b\sigma}. \tag{3.41}$$

Exact analysis for two correlated Brownian motions

An exact analysis is again possible here. Suppose

$$X_t^\epsilon = x_0 e^{\left(\alpha - \frac{1}{2}\sigma_\epsilon^2\right)t + \sigma_\epsilon W_t}, \quad Y_t^\epsilon = y_0 e^{\left(\beta - \frac{1}{2}b_\epsilon^2\right)t + b_\epsilon B_t},$$

are two families of Geometric Brownian Motions, indexed by a positive parameter ϵ. We will assume that $\sigma_\epsilon = \sigma\sqrt{\epsilon}$ and, similarly, $b_\epsilon = b\sqrt{\epsilon}$. Assuming that $\alpha > \beta$ and $x_0 > y_0$ and that $\{W_t\}$, $\{B_t\}$ are standard Brownian motions with correlation ρ as in section 3.4, we are interested in obtaining an expression for the probability

$$\mathbb{P}(T_\epsilon < \infty) \quad \text{where } T_\epsilon = \inf\{t > 0 : Y_t^\epsilon > X_t^\epsilon\}. \tag{3.42}$$

The condition $Y_t^\epsilon > X_t^\epsilon$ is equivalent to

$$\left(\alpha - \beta + \frac{1}{2}(b_\epsilon^2 - \sigma_\epsilon^2)\right) t + \sigma_\epsilon W_t - b_\epsilon B_t < \log \frac{y_0}{x_0}.$$

Set $\log \frac{y_0}{x_0} = -u$, $\gamma_\epsilon := \alpha - \beta + \frac{1}{2}(b_\epsilon^2 - \sigma_\epsilon^2)$ and $\theta_\epsilon := \sqrt{\sigma_\epsilon^2 + b_\epsilon^2 - 2\rho b_\epsilon \sigma_\epsilon}$. If $\{\tilde{W}_t\}$ is standard Brownian motion, then (3.42) becomes

$$\mathbb{P}(T_\epsilon < \infty) = \mathbb{P}\left(\inf_{t \geq 0}(\gamma_\epsilon t + \theta_\epsilon \tilde{W}_t) < -u\right). \tag{3.43}$$

Since $\alpha > \beta$, when ϵ is sufficiently small, $\gamma_\epsilon > 0$ regardless of the values of σ and b. Therefore (see [23]) (3.43) becomes

$$\mathbb{P}(T_\epsilon < \infty) = e^{-u\frac{2\gamma_\epsilon}{\theta_\epsilon^2}} = e^{\log \frac{y_0}{x_0} \frac{2(\alpha-\beta)+(b_\epsilon^2-\sigma_\epsilon^2)}{\sigma_\epsilon^2+b_\epsilon^2-2\rho b_\epsilon \sigma_\epsilon}}.$$

It therefore follows that

$$\lim_{\epsilon \to 0} \epsilon \log \mathbb{P}(T_\epsilon < \infty) = \log \frac{y_0}{x_0} \lim_{\epsilon \to 0} \frac{2(\alpha - \beta) + (b_\epsilon^2 - \sigma_\epsilon^2)}{\epsilon^{-1}\left(\sigma_\epsilon^2 + b_\epsilon^2 - 2\rho b_\epsilon \sigma_\epsilon\right)} = \log \frac{y_0}{x_0} \frac{2(\alpha - \beta)}{\sigma^2 + b^2 - 2\rho b \sigma}.$$

This result of course agrees with (3.41).

4. MODELS EXHIBITING MORE COMPLEX BEHAVIOR

Here we discuss a number of models examining more complex behavior. First we revisit the problem of two OU processes with ordered drift constants and initial conditions. However we now assume that the driving brownian motions are not independent. This introduces additional difficulties in the solution of the optimization problem. More importantly, however the existence of the solution requires an additional condition involving the correlation coefficient between the two brownian motions, ρ. If ρ is too close to 1 then the two processes will move essentially in unison and, roughly speaking, the lower process will never be able to catch up with the upper process.

In section 4.2 we study an OU process with variance that has a general time varying form and, perhaps surprisingly, we are able to obtain a solution essentially in closed form.

Finally, in section 4.3 we examine the same type of problem in connection to the SDE $dX_t = rdt + \sigma X_t dW_t$ as well as the general linear SDE $dX_t = (r + \mu X_t)dt + (b + \sigma X_t)dW_t$. While these seem similar to the models already examined in the previous chapters, their behavior is more complex and the exact solution of the resulting variational problems is more challenging. In particular, depending on the value of the parameters we may not have uniqueness of the solution or, in the infinite horizon case a solution may not exist.

4.1 *Two Correlated OU Processes*

Suppose that W_t, V_t, are independent standard Brownian motions and $\rho \in [-1, 1]$. Set $B_t = \rho W_t + \sqrt{1-\rho^2}\, V_t$. Then (W_t, B_t) are correlated Brownian motions with correlation ρ. Consider now the processes

$$\begin{aligned} dX_t &= \alpha X_t dt + \sigma dW_t, \\ dY_t &= \beta Y_t dt + b dB_t. \end{aligned}$$

The second equation can be written equivalently as

$$dY_t = \beta Y_t dt + \rho b dW_t + \sqrt{1-\rho^2} b dV_t$$

Using the same heuristic derivation as the one that led to (3.26) we have

$$dW_t = \frac{dX_t - \alpha X_t dt}{\sigma}$$

and

$$dV_t = \frac{1}{b\sqrt{1-\rho^2}} \left(dY_t - \beta Y_t dt - \rho b \frac{dX_t - \alpha X_t dt}{\sigma} \right).$$

We thus obtain the following action functional to be minimized

$$I = \frac{1}{2} \int_0^T \left(\frac{x' - \alpha x}{\sigma} \right)^2 + \frac{1}{1-\rho^2} \left(\frac{y' - \beta y}{b} - \rho \frac{x' - \mu x}{\sigma} \right)^2 dt \tag{4.1}$$

The conditions for minimum are

$$F_x - \frac{d}{dt} F_{x'} = 0 \tag{4.2}$$

$$F_y - \frac{d}{dt} F_{y'} = 0 \tag{4.3}$$

$$x(T) = y(T) \tag{4.4}$$

$$F_{x'} + F_{y'} = 0 \quad \text{at } T, \tag{4.5}$$

$$F - x' F_{x'} - y' F_{y'} = 0 \quad \text{at } T. \tag{4.6}$$

An alternative expression for F is

$$F = \frac{1}{2(1-\rho^2)\sigma^2 b^2} \left[b^2 f^2 + \sigma^2 g^2 - 2\rho b \sigma f g \right], \quad f := x' - \alpha x, \quad g := y' - \beta y.$$

Omitting the factor $\frac{1}{2(1-\rho^2)\sigma^2 b^2}$ we have

$$F_x = 2\alpha b^2 \left[-f + \rho \frac{\sigma}{b} g \right], \quad F_y = 2\beta \sigma^2 \left[-g + \rho \frac{b}{\sigma} f \right]$$

$$F_{x'} = 2b^2 \left[f - \rho \frac{\sigma}{b} g \right], \quad F_{y'} = 2\sigma^2 \left[g - \rho \frac{b}{\sigma} f \right]$$

$$\frac{d}{dt} F_{x'} = 2b^2 \left[f' - \rho \frac{\sigma}{b} g' \right], \quad \frac{d}{dt} F_{y'} = 2\sigma^2 \left[g' - \rho \frac{b}{\sigma} f' \right].$$

Set

$$p(t) = f(t) - \rho \frac{\sigma}{b} g(t), \qquad q(t) = g(t) - \rho \frac{b}{\sigma} f(t).$$

Then the Euler-Lagrange equations become

$$F_x - \frac{d}{dt} F_{x'} = 0 \ \text{ or } \ p'(t) + \alpha p(t) = 0$$

$$F_y - \frac{d}{dt} F_{y'} = 0 \ \text{ or } \ q'(t) + \beta q(t) = 0$$

The condition $F_{x'} + F_{y'} = 0$ at T becomes

$$b^2 p(T) + \sigma^2 q(T) = 0 \quad \text{or} \quad q(T) = -\frac{b^2}{\sigma^2} p(T). \tag{4.7}$$

By the definition of f and g, $f(T) = x'(T) - \alpha x(T)$ or $x'(T) = f(T) + \alpha x(T)$ and similarly $y'(T) = g(T) + \beta y(T)$. Thus (all expressions are evaluated at T) taking into account that $x = y$

$$x'F_{x'}+y'F_{y'} = 2(f+\alpha x)b^2p+2(g+\beta y)\sigma^2 q = 2b^2p\,(f+\alpha x-g-\beta x) = 2b^2p\,(f-g+x(\alpha-\beta))$$

$$\begin{aligned} F &= b^2f^2+\sigma^2g^2-2\rho b\sigma fg \;=\; b^2\left(f^2+\frac{\sigma^2}{b^2}g^2-\rho\frac{\sigma}{b}fg-\rho\frac{\sigma}{b}fg\right) \\ &= b^2f\left(f-\rho\frac{\sigma}{b}g\right)+b\sigma g\left(\frac{\sigma}{b}g-\rho f\right) \;=\; b^2f\left(f-\rho\frac{\sigma}{b}g\right)+\sigma^2 g\left(g-\rho\frac{b}{\sigma}f\right) \\ &= b^2fp+\sigma^2gq \;=\; b^2fp-b^2gp \;=\; b^2p(f-g). \end{aligned}$$

Thus $F - x'Fx' - y'F_{y'} = 0$ becomes

$$b^2p(f-g) = 2b^2p\,(f-g+x(\alpha-\beta))$$

or

$$p(f-g+2x(\alpha-\beta)) = 0.$$

From the above we obtain two sets of conditions: The first is

$$p(T) = q(T) = 0 \quad \text{or equivalently} \quad f(T)-\rho\frac{\sigma}{b}g(T) = 0, \;\; g(T)-\rho\frac{b}{\sigma}f(T) = 0, \;\; \text{or} \;\; f(T) = g(T)$$

or

$$x'(T)-\alpha x(T) = 0, \quad y'(T)-\beta y(T) = 0. \tag{4.8}$$

The second is

$$f-g+2x(\alpha-\beta) = 0, \;\; \text{or} \;\; x'(T)-\alpha x(T)-y'(T)+\beta y(T)+x(T)(\alpha-\beta)+y(T)(\alpha-\beta) = 0$$

(we have used again $x(T) = y(T)$). One possible way of expressing this is

$$x'(T)+\alpha x(T) \;=\; y'(T)+\beta y(T). \tag{4.9}$$

The Euler-Lagrange equations give

$$p(t) = p_0e^{-\alpha t}, \qquad q(t) = q_0e^{-\beta t}. \tag{4.10}$$

Then

$$\begin{aligned} f(t)-\rho\frac{\sigma}{b}g(t) &= p(t) \\ -\rho\frac{b}{\sigma}f(t)+g(t) &= q(t) \end{aligned}$$

$$\begin{bmatrix} 1 & -\rho\frac{\sigma}{b} \\ -\rho\frac{b}{\sigma} & 1 \end{bmatrix} \begin{bmatrix} f(t) \\ g(t) \end{bmatrix} = \begin{bmatrix} p(t) \\ q(t) \end{bmatrix}$$

whence

$$f(t) = \frac{1}{1-\rho^2}\left(p(t)+\rho\frac{\sigma}{b}q(t)\right), \tag{4.11}$$

$$g(t) = \frac{1}{1-\rho^2}\left(q(t)+\rho\frac{b}{\sigma}p(t)\right). \tag{4.12}$$

Since $f(t) = x'(t) - \alpha x(t)$ and $g(t) = y'(t) - \beta y(t)$,

$$x(t) = x_0 e^{\alpha t} + \int_0^t e^{\alpha(t-s)} f(s)ds,$$

$$y(t) = y_0 e^{\beta t} + \int_0^t e^{\beta(t-s)} g(s)ds.$$

Substituting we have

$$x(t) = x_0 e^{\alpha t} + \frac{e^{\alpha t}}{1-\rho^2}\left(\frac{p_0}{2\alpha}(1-e^{-2\alpha t}) + \frac{\rho\sigma}{b}\frac{q_0}{\alpha+\beta}(1-e^{-(\alpha+\beta)t})\right), \tag{4.13}$$

$$y(t) = y_0 e^{\beta t} + \frac{e^{\beta t}}{1-\rho^2}\left(\frac{q_0}{2\beta}(1-e^{-2\beta t}) + \frac{\rho b}{\sigma}\frac{p_0}{\alpha+\beta}(1-e^{-(\alpha+\beta)t})\right). \tag{4.14}$$

We need to determine the unknown quantities p_0, q_0, and the meeting time T.

$$F_{x'} + F_{y'} = 0 \text{ at } T \text{ gives: } q_0 e^{-\beta T} = -\left(\frac{b}{\sigma}\right)^2 p_0 e^{-\alpha T}. \tag{4.15}$$

$$F + x'F_{x'} + y'F_{y'} = 0 \text{ at } T \text{ gives: } f(T) - g(T) + 2[\alpha x(T) - \beta y(T)] = 0. \tag{4.16}$$

From (4.11), (4.12),

$$f(T) = \frac{1}{1-\rho^2}\left(p_0 e^{-\alpha T} + \rho\frac{\sigma}{b}q_0 e^{-\beta T}\right), \quad g(T) = \frac{1}{1-\rho^2}\left(q_0 e^{-\beta T} + \rho\frac{b}{\sigma}p_0 e^{-\alpha T}\right).$$

and

$$f(T) - g(T) = \frac{1}{b\sigma(1-\rho^2)}\left(p_0 e^{-\alpha T}(b\sigma - \rho b^2) + q_0 e^{-\beta T}(\rho\sigma^2 - b\sigma)\right).$$

From (4.13), (4.14),

$$\begin{aligned}
\alpha x(T) - \beta y(T) &= \alpha x_0 e^{\alpha T} - \beta y_0 e^{\beta T} + \frac{1}{1-\rho^2}\left(p_0 e^{-\alpha T}\frac{e^{2\alpha T}-1}{2} + \frac{\rho\sigma\alpha}{b}q_0 e^{\alpha T}\frac{1-e^{-(\alpha+\beta)T}}{\alpha+\beta}\right) \\
&\quad - \frac{1}{1-\rho^2}\left(q_0 e^{-\beta T}\frac{e^{2\beta T}-1}{2} + \frac{\rho b\beta}{\sigma}p_0 e^{\beta T}\frac{1-e^{-(\alpha+\beta)T}}{\alpha+\beta}\right) \\
&= x_0\alpha e^{\alpha T} - y_0\beta e^{\beta T} \\
&+ \frac{1}{2(1-\rho^2)}\left(p_0 e^{-\alpha T}(e^{2\alpha T}-1) - q_0 e^{-\beta T}(e^{2\beta T}-1) + \frac{2\rho\sigma}{b}\frac{e^{(\alpha+\beta)T}-1}{\alpha+\beta}\left(\alpha q_0 e^{-\beta T} - \beta\left(\frac{b}{\sigma}\right)^2 p_0 e^{-\alpha T}\right)\right) \\
&= x_0\alpha e^{\alpha T} - y_0\beta e^{\beta T} \\
&+ \frac{1}{2(1-\rho^2)}\left(p_0 e^{-\alpha T}(e^{2\alpha T}-1) - q_0 e^{-\beta T}(e^{2\beta T}-1) + \frac{2\rho\sigma}{b}(e^{(\alpha+\beta)T}-1)q_0 e^{-\beta T}\right)
\end{aligned}$$

or

$$\begin{aligned}\alpha x(T)-\beta y(T) &= x_0\alpha e^{\alpha T}-y_0\beta e^{\beta T}\\ &+\frac{1}{2(1-\rho^2)}\left(p_0e^{-\alpha T}\left(e^{2\alpha T}-1\right)+q_0e^{-\beta T}\left(\frac{2\rho\sigma}{b}\left(e^{(\alpha+\beta)T}-1\right)-\left(e^{2\beta T}-1\right)\right)\right).\end{aligned}$$

Then (4.16) gives

$$\begin{aligned}&p_0e^{-\alpha T}\left(1-\rho\frac{b}{\sigma}\right)+q_0e^{-\beta T}\left(\rho\frac{\sigma}{b}-1\right)+p_0e^{-\alpha T}\left(e^{2\alpha T}-1\right)\\ &+q_0e^{-\beta T}\left(\frac{2\rho\sigma}{b}\left(e^{(\alpha+\beta)T}-1\right)-\left(e^{2\beta T}-1\right)\right)\\ &= -2(1-\rho^2)\left(x_0\alpha e^{\alpha T}-y_0\beta e^{\beta T}\right)\end{aligned}$$

$$\begin{aligned}&p_0e^{-\alpha T}\left(-\rho\frac{b}{\sigma}+e^{2\alpha T}\right)+q_0e^{-\beta T}\left(\rho\frac{\sigma}{b}+\frac{2\rho\sigma}{b}\left(e^{(\alpha+\beta)T}-1\right)-e^{2\beta T}\right)\\ &= -2(1-\rho^2)\left(x_0\alpha e^{\alpha T}-y_0\beta e^{\beta T}\right)\end{aligned}$$

$$\begin{bmatrix} -\rho\frac{b}{\sigma}+e^{2\alpha T} & \rho\frac{\sigma}{b}+\frac{2\rho\sigma}{b}\left(e^{(\alpha+\beta)T}-1\right)-e^{2\beta T}\\ 1 & \left(\frac{\sigma}{b}\right)^2\end{bmatrix}\begin{bmatrix} p_0e^{-\alpha T}\\ q_0e^{-\beta T}\end{bmatrix} = \begin{bmatrix} -2(1-\rho^2)\left(x_0\alpha e^{\alpha T}-y_0\beta e^{\beta T}\right)\\ 0\end{bmatrix}. \quad (4.17)$$

$$\Delta = e^{2\beta T}+\left(\frac{\sigma}{b}\right)^2e^{2\alpha T}-2\rho\frac{\sigma}{b}e^{(\alpha+\beta)T}.$$

$$p_0e^{-\alpha T} = -\frac{2}{\Delta}\left(\frac{\sigma}{b}\right)^2(1-\rho^2)\left(x_0\alpha e^{\alpha T}-y_0\beta e^{\beta T}\right) \quad (4.18)$$

$$q_0e^{-\beta T} = \frac{2}{\Delta}(1-\rho^2)\left(x_0\alpha e^{\alpha T}-y_0\beta e^{\beta T}\right) \quad (4.19)$$

$$x(T) = x_0e^{\alpha T}+\frac{1}{1-\rho^2}\left(\frac{p_0e^{-\alpha T}}{2\alpha}(e^{2\alpha T}-1)+\frac{\rho\sigma}{b}\frac{q_0e^{-\beta T}}{\alpha+\beta}(e^{(\alpha+\beta)T}-1)\right), \quad (4.20)$$

$$y(T) = y_0e^{\beta T}+\frac{1}{1-\rho^2}\left(\frac{q_0e^{-\beta T}}{2\beta}(e^{2\beta T}-1)+\frac{\rho b}{\sigma}\frac{p_0e^{-\alpha T}}{\alpha+\beta}(e^{(\alpha+\beta)T}-1)\right). \quad (4.21)$$

$$x(T) = x_0e^{\alpha T}+2\frac{x_0\alpha e^{\alpha T}-y_0\beta e^{\beta T}}{\Delta}\left(-\left(\frac{\sigma}{b}\right)^2\frac{e^{2\alpha T}-1}{2\alpha}+\frac{\rho\sigma}{b}\frac{e^{(\alpha+\beta)T}-1}{\alpha+\beta}\right), \quad (4.22)$$

$$y(T) = y_0e^{\beta T}+2\frac{x_0\alpha e^{\alpha T}-y_0\beta e^{\beta T}}{\Delta}\left(\frac{e^{2\beta T}-1}{2\beta}-\frac{\rho b}{\sigma}\left(\frac{\sigma}{b}\right)^2\frac{e^{(\alpha+\beta)T}-1}{\alpha+\beta}\right). \quad (4.23)$$

$$\left(x_0 e^{\alpha T} - y_0 e^{\beta T}\right) \Delta \;=$$
$$\left(x_0 \alpha e^{\alpha T} - y_0 \beta e^{\beta T}\right) \left(-\left(\frac{\sigma}{b}\right)^2 \frac{e^{2\alpha T}-1}{\alpha} + \frac{2\rho\sigma}{b} \frac{e^{(\alpha+\beta)T}-1}{\alpha+\beta} - \frac{e^{2\beta T}-1}{\beta} + \frac{2\rho b}{\sigma} \left(\frac{\sigma}{b}\right)^2 \frac{e^{(\alpha+\beta)T}-1}{\alpha+\beta}\right)$$

$$\left(x_0 e^{\alpha T} - y_0 e^{\beta T}\right) \Delta \;+$$
$$\left(x_0 \alpha e^{\alpha T} - y_0 \beta e^{\beta T}\right) \left(-\left(\frac{\sigma}{b}\right)^2 \frac{e^{2\alpha T}-1}{\alpha} + \frac{4\rho\sigma}{b} \frac{e^{(\alpha+\beta)T}-1}{\alpha+\beta} - \frac{e^{2\beta T}-1}{\beta}\right) \;=\; 0$$

$$\left(x_0 e^{\alpha T} - y_0 e^{\beta T}\right) \left(e^{2\beta T} + \left(\frac{\sigma}{b}\right)^2 e^{2\alpha T} - 2\rho\frac{\sigma}{b} e^{(\alpha+\beta)T}\right) \;+$$
$$\left(x_0 \alpha e^{\alpha T} - y_0 \beta e^{\beta T}\right) \left(-\left(\frac{\sigma}{b}\right)^2 \frac{e^{2\alpha T}-1}{\alpha} + \frac{4\rho\sigma}{b} \frac{e^{(\alpha+\beta)T}-1}{\alpha+\beta} - \frac{e^{2\beta T}-1}{\beta}\right) \;=\; 0$$

This gives

$$x_0 \alpha e^{\alpha T} \left(b^2(\beta-\alpha)e^{2\beta T} + 2\rho\sigma b \frac{e^{(\alpha+\beta)T}}{\alpha+\beta}(\alpha-\beta)\beta + \alpha b^2 + \beta\sigma^2 - 4\rho b\sigma \frac{\alpha\beta}{\alpha+\beta}\right)$$
$$=\; y_0 \beta e^{\beta T} \left(\sigma^2(\alpha-\beta)e^{2\alpha T} + 2\rho\sigma b \frac{e^{(\alpha+\beta)T}}{\alpha+\beta}(\beta-\alpha)\alpha + \alpha b^2 + \beta\sigma^2 - 4\rho b\sigma \frac{\alpha\beta}{\alpha+\beta}\right)$$

$$x_0 \alpha \left(b^2(\beta-\alpha)e^{\beta T} + 2\rho\sigma b \frac{e^{\alpha T}}{\alpha+\beta}(\alpha-\beta)\beta + \left(\alpha b^2 + \beta\sigma^2 - 4\rho b\sigma \frac{\alpha\beta}{\alpha+\beta}\right) e^{-\beta T}\right)$$
$$=\; y_0 \beta \left(\sigma^2(\alpha-\beta)e^{\alpha T} + 2\rho\sigma b \frac{e^{\beta T}}{\alpha+\beta}(\beta-\alpha)\alpha + \left(\alpha b^2 + \beta\sigma^2 - 4\rho b\sigma \frac{\alpha\beta}{\alpha+\beta}\right) e^{-\alpha T}\right)$$

To determine the value or values of T that satisfy the above equation we need to examine the function $f : (0,\infty) \to \mathbb{R}$ defined below and determine its roots:

$$\begin{aligned} f(t) \;=\; & e^{\beta t}\alpha(\alpha-\beta)b^2 \left(-x_0 + y_0 2\rho\frac{\sigma}{b}\frac{\beta}{\alpha+\beta}\right) + e^{\alpha t}\beta(\alpha-\beta)\sigma^2 \left(-y_0 + x_0 2\rho\frac{b}{\sigma}\frac{\alpha}{\alpha+\beta}\right) \\ & + \left(\alpha b^2 + \beta\sigma^2 - 4\rho b\sigma\frac{\alpha\beta}{\alpha+\beta}\right)\left(x_0\alpha e^{-\beta t} - y_0\beta e^{-\alpha t}\right). \end{aligned} \tag{4.24}$$

A moment's examination reveals that the roots of the equation $f(t) = t$ depend on three parameters, namely $y_0/x_0 \in (0,1)$, $\beta/\alpha \in (0,1)$, and $\sigma^2/b^2 \in (0,\infty)$. In the sequel, to simplify the problem we will examine the case $b = \sigma$.

The special case $b = \sigma$

Let us first examine this special case which will enable us to reach some interesting conclusions regarding the effect of the correlation between the two processes. We will examine the existence of solutions of $f(t) = 0$ where f is the function defined in (4.24). Here

$$\begin{aligned} f(t) &= (\alpha-\beta)\sigma^2\left(\alpha e^{\beta t}\left(-x_0+2\rho y_0\frac{\beta}{\alpha+\beta}\right)+\beta e^{\alpha t}\left(-y_0+2\rho x_0\frac{\alpha}{\alpha+\beta}\right)\right) \\ &\quad +\sigma^2\left(\alpha+\beta-4\rho\frac{\alpha\beta}{\alpha+\beta}\right)\left(\alpha x_0 e^{-\beta t}-\beta y_0 e^{-\alpha t}\right). \end{aligned} \tag{4.25}$$

We assume of course that $x_0 > y_0$ and $\alpha > \beta$ and hence the term on the second line of (4.25) is positive for all $t \geq 0$. Also,

$$f(0) = 2\alpha\beta\sigma^2\left(1-\rho\right)\left(x_0-y_0\right) > 0. \tag{4.26}$$

The condition for $\lim_{t\to\infty} f(t) = -\infty$ is $-y_0 + 2\rho x_0 \frac{\alpha}{\alpha+\beta} < 0$ or equivalently

$$\rho < \frac{y_0}{x_0}\frac{\alpha+\beta}{2\alpha}. \tag{4.27}$$

Now set $g(t) := e^{\alpha t} f'(t)(\alpha\beta\sigma^2)^{-1}$. Then

$$\begin{aligned} g(t) &= (\alpha-\beta)\left(e^{\beta t}\left(-x_0+2\rho y_0\frac{\beta}{\alpha+\beta}\right)+e^{\alpha t}\left(-y_0+2\rho x_0\frac{\alpha}{\alpha+\beta}\right)\right) \\ &\quad +\left(\alpha+\beta-4\rho\frac{\alpha\beta}{\alpha+\beta}\right)\left(-x_0 e^{-\beta t}+y_0 e^{-\alpha t}\right), \end{aligned}$$

Then we can see that $g(0) = -2(1-\rho)\left[x_0\alpha - y_0\beta\right] < 0$. Define next the function $h(t) := e^{-(\alpha-\beta)t} g'(t)(\alpha-\beta)^{-1}$.

$$\begin{aligned} h(t) &= (\alpha+\beta)e^{2\beta t}\left(-x_0+2\rho y_0\frac{\beta}{\alpha+\beta}\right)+2\alpha e^{(\alpha+\beta)t}\left(-y_0+2\rho x_0\frac{\alpha}{\alpha+\beta}\right) \\ &\quad -x_0\left(\alpha+\beta-4\rho\frac{\alpha\beta}{\alpha+\beta}\right). \end{aligned}$$

Finally,

$$h(0) = 2x_0(\alpha+\beta)\left(\frac{2\alpha}{\alpha+\beta}\rho-1\right)+2y_0\beta\left(\rho-\frac{\alpha}{\beta}\right) < 0.$$

Also,

$$\frac{h'(t)e^{-2\beta t}}{2(\alpha+\beta)} = \beta\left(-x_0+2\rho y_0\frac{\beta}{\alpha+\beta}\right)+\alpha e^{(\alpha-\beta)t}\left(-y_0+2\rho x_0\frac{\alpha}{\alpha+\beta}\right) < 0.$$

This last inequality is a consequence of (4.27). Thus, the last two inequalities imply that $h(t) < 0$ for all t. This in turn implies that $g'(t) < 0$ for all t and, since $g(0) < 0$,

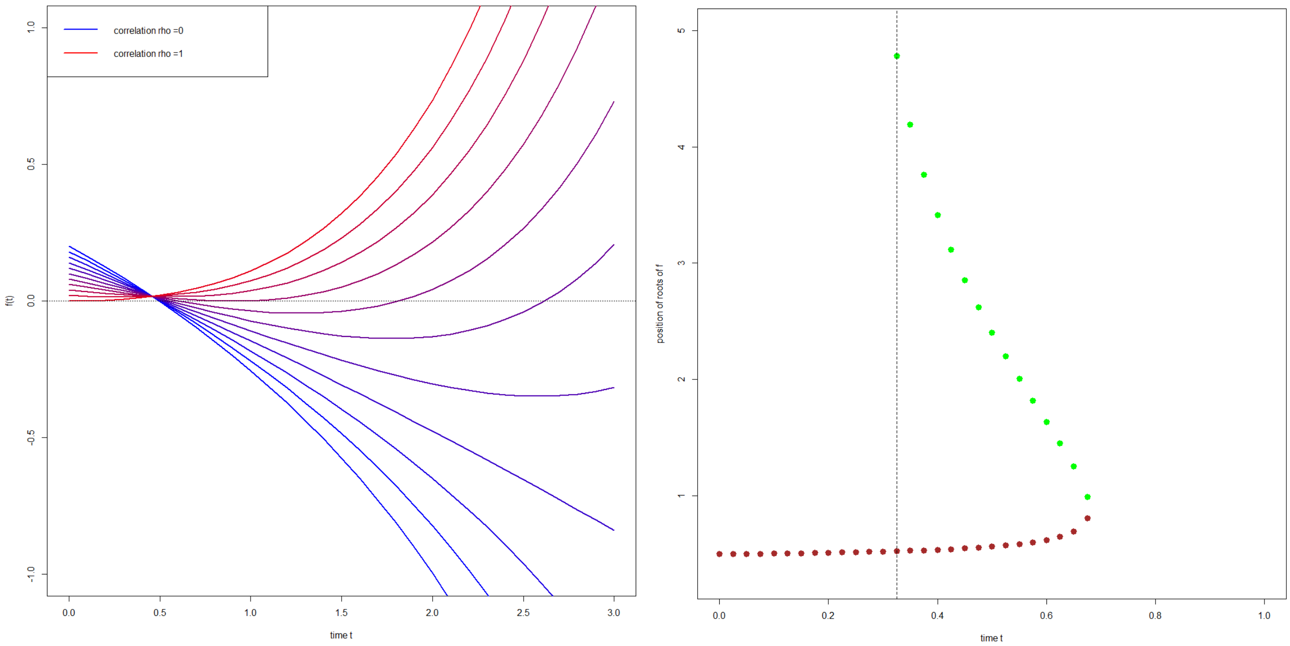

Fig. 4.1: Two correlated OU processes. The figure on the left plots $f(t)$ defined in (4.25) for ρ ranging from 0 (blue line) to 1 (red line). The values of the other parameters are $\alpha = 1$, $x_0 = 1$, $\beta = 0.2$, $y_0 = 0.5$. In the figure on the right the roots of the equation $f(t) = 0$ are plotted as a function of ρ. Note that for $\rho \in [0, 0.325]$ f has a single root (brown point). At $\rho \approx 0.325$ a second root of f appears (green points). The equation $f(t) = 0$ has two roots when ρ belongs to the (approximate) interval $[0.325, 0.7]$. Finally, for $\rho > 0.7$ no solutions exist.

that $g(t) < 0$ for all t. Hence $f(0) > 0$, f is strictly decreasing and $f(t) \to -\infty$ as $t \to \infty$. Therefore it follows that $f(t) = 0$ has a single root, T, the meeting time, provided that condition (4.27) holds. The need for *condition (4.27)* should not surprise us. If ρ is sufficiently close to 1 then the two processes X_t and Y_t will move nearly in unison and hence Y_t, starting from below, will not be able to catch up with X_t.

4.2 An Ornstein-Uhlenbeck Process with Time-Varying Variance

Here we examine a family of OU processes with time varying variance. It turns out that a closed form solution of the variational problem arising from the Wentzell-Freidlin approach to determining the hitting probability of a lower exponential boundary is possible.

Theorem 8. *Consider the family of SDE's*

$$dX_t^\epsilon \ = \ \mu X_t^\epsilon dt + \sigma(t) dW_t, \qquad X_0^\epsilon = x_0.$$

(We assume that $\sigma(t) > 0$ for all $t \geq 0$.) Let also $V(t) = v_0 e^{\beta t}$ with $0 < v_0 < x_0$ and $0 < \beta < \mu$ and $T^\epsilon = \inf\{t \geq 0 : X_t^\epsilon - v_0 e^{\beta t} \leq 0\}$. Then

$$\lim_{\epsilon \to 0} \epsilon \log P(T^\epsilon < \infty) = -I$$

where I is given in (4.44)

The proof as well as the determination of the optimal path is given below. We have the optimization problem

$$\begin{aligned} &\min \int_0^T F(x, x', t)dt, \quad x(0) = x_0, \text{ and } x(T) = V(T) \\ &\text{with } F(x, x', t) = \frac{1}{2}\sigma(t)^{-2}\left(x' - \mu x\right)^2, \quad V(t) = v_0 e^{\beta t}, \quad v_0 < x_0. \end{aligned}$$

The conditions for a minimum is

$$F_x - \frac{d}{dt} F_{x'} = 0 \tag{4.28}$$

$$x(T) = V(T) \tag{4.29}$$

$$F + (V' - x')F_{x'} = 0 \quad \text{at } T. \tag{4.30}$$

Assume $\beta < \mu$. In this case

$$\begin{aligned} F_x &= -\mu\sigma^{-2}\left(x' - \mu x\right) \\ F_{x'} &= \sigma^{-2}\left(x' - \mu x\right) \\ \frac{d}{dt} F_{x'} &= \sigma^{-2}\left(x'' - \mu x'\right) - 2\sigma'(t)\sigma(t)^{-3}\left(x' - \mu x\right). \end{aligned}$$

The Euler-Lagrange equation can be written as

$$-\alpha\sigma^{-2}\left(x' - \mu x\right) - \sigma^{-2}\left(x'' - \mu x\right) + 2\sigma'\sigma^{-3}\left(x' - \mu x\right) = 0$$

or, equivalently,

$$x'' - \mu^2 x = \frac{2\sigma'}{\sigma}\left(x' - \mu x\right). \tag{4.31}$$

In order to solve this equation we define the function

$$y(t) = e^{-\mu t} x(t). \tag{4.32}$$

Then

$$\begin{aligned} y' &= \left(x' - \mu x\right) e^{-\mu t} \\ y'' &= \left(x'' - 2\mu x' + \mu^2 x\right) e^{-\mu t} = \left(x'' - \mu^2 x + 2\mu^2 x - 2\mu x'\right) e^{-\mu t} \end{aligned}$$

Hence, from the above,

$$\begin{aligned} x' - \mu x &= y' e^{\mu t} \\ x'' - \mu^2 x &= y'' e^{\mu t} + 2\mu(x' - \mu x) \;=\; y'' e^{\mu t} + 2\mu y' e^{\mu t} \end{aligned}$$

Substituting into (4.31) we obtain

$$y'' e^{\mu t} + 2\mu y' e^{\mu t} = \frac{2\sigma'}{\sigma} y' e^{\mu t}$$

or, setting $z = y'$,

$$z' + 2\left(\mu - (\log \sigma)'\right) z = 0.$$

This is a first order linear differential equation which can be easily solved to obtain

$$z(t) = C_1 e^{-2\mu t} \sigma^2(t)$$

and hence

$$y(t) = C_1 \int_0^t e^{-2\mu s} \sigma^2(s) ds + C_2. \tag{4.33}$$

Thus, from (4.32),

$$x(t) = C_1 e^{\mu t} \int_0^t e^{-2\mu s} \sigma^2(s) ds + C_2 e^{\mu t}. \tag{4.34}$$

Differentiating we obtain

$$x'(t) = C_1 \mu e^{\mu t} \int_0^t e^{-2\mu s} \sigma^2(s) ds + C_1 e^{-\mu t} \sigma^2(t) + C_2 \mu e^{\mu t}. \tag{4.35}$$

From the initial value of $x(0)$ and the transversality condition we obtain

$$x_0 \;=\; C_2 \tag{4.36}$$

$$v_0 e^{\beta T} \;=\; C_1 e^{\mu T} \int_0^T e^{-2\mu s} \sigma^2(s) ds + C_2 e^{\mu T} \tag{4.37}$$

We also have the relationship $F + (V' - x')F_{x'} = 0$ (evaluated at time T) which gives

$$\frac{1}{2} \frac{(x' - \mu x)^2}{\sigma^2} + \left(\beta v_0 e^{\beta T} - x'\right) \frac{(x' - \mu x)}{\sigma^2} = 0$$

or

$$\frac{x' - \mu x}{\sigma^2} \left(2\beta v_0 e^{\beta T} - x' - ax\right) = 0 \tag{4.38}$$

Noting that

$$x'(t) - \mu x(t) = C_1 e^{-\mu t} \sigma^2(t) \tag{4.39}$$

(4.38) becomes

$$C_1 e^{-\alpha T} \left(2\beta v_0 e^{\beta T} - 2C_1 \mu e^{\mu T} \int_0^T e^{-2\mu s} \sigma^2(s) ds - C_1 e^{-\mu T} \sigma^2(T) - 2C_2 \mu e^{\mu T}\right) \;=\; 0. \tag{4.40}$$

Assuming that $C_1 \neq 0$ and taking into account (4.37) the above equation becomes

$$2\beta v_0 e^{\beta T} = C_1 e^{-\mu T} \sigma^2(T) + 2\mu v_0 e^{\beta T}.$$

which gives

$$C_1 = -\frac{2(\mu-\beta)v_0}{\sigma^2(T)}\, e^{(\mu+\beta)T}. \tag{4.41}$$

From equations (4.36), (4.37), we obtain

$$C_1 \int_0^T e^{-2\mu s}\sigma^2(s)ds = v_0 e^{(\beta-\mu)T} - x_0$$

and thus, taking into account (4.41),

$$\frac{2(\beta-\mu)v_0}{\sigma^2(T)} \int_0^T \sigma^2 e^{-2\mu s} ds = v_0 e^{(\beta-\mu)T} - x_0.$$

Thus T is determined by the solution of the equation

$$\int_0^T e^{-2\mu s}\sigma^2(s)ds \;=\; \frac{\sigma^2(T)e^{-2\mu T}}{2(\mu-\beta)} \left(\frac{x_0}{v_0} e^{(\mu-\beta)T} - 1\right). \tag{4.42}$$

One can check that the above equation, when σ is constant, reduces to (2.56).

From this last equation we determine T. Having determined T, the critical path is given as

$$x(t) \;=\; x_0 e^{\mu t} - \frac{2(\mu-\beta)v_0}{\sigma^2(T)}\, e^{(\mu+\beta)T+\mu t} \int_0^t \sigma^2(s)e^{-2\mu s}ds, \qquad t\in[0,T] \tag{4.43}$$

Finally the rate function is equal to

$$\begin{aligned} I \;&=\; \int_0^T \frac{(x'-\mu x)^2}{2\sigma^2(t)}dt \;=\; \frac{C_1^2}{2}\int_0^T e^{-2\mu t}\sigma^2(t)dt \\ &=\; \frac{(\mu-\beta)v_0 e^{2(\mu+\beta)T}}{\sigma^2(T)} \left(x_0 - v_0 e^{(\beta-\mu)T}\right). \end{aligned} \tag{4.44}$$

We illustrate the above derivation with a numerical example presented in Figure 4.3. Equation (4.42) is solved numerically, and of course when the time dependence of the variance $\sigma^2(t)$ is arbitrary there is no guarantee that it has a unique solution. Figure 4.2 illustrates this point. The solid black curve plots the function

$$f(t) := \int_0^t e^{-2\mu s}\sigma^2(s)ds \;-\; \frac{\sigma^2(t)e^{-2\mu t}}{2(\mu-\beta)} \left(\frac{x_0}{v_0}e^{(\mu-\beta)t} - 1\right).$$

With the given form of the function $\sigma^2(t)$ and the given values of the parameters (see Figure 4.2) there are three roots of this equation, namely $T = 0.224$ (red vertical line), $T = 0.4194$ (green vertical line), and $T = 0.7639$ (blue vertical line).

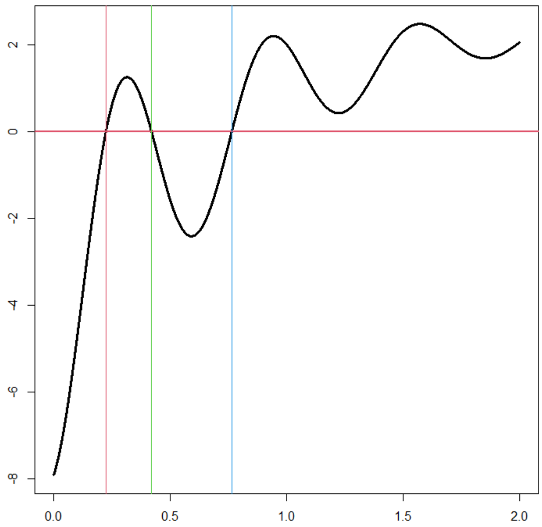

Fig. 4.2: Here $\mu = 1$, $\beta = 0.5$, $x_0 = 1.8$, $v_0 = 1$, and $\sigma^2(t) = 5 + 4.9\cos(20t)$. There are three candidates for the optimal solution $T = 0.224$ (red vertical line), $T = 0.4194$ (green vertical line), and $T = 0.7639$ (blue vertical line).

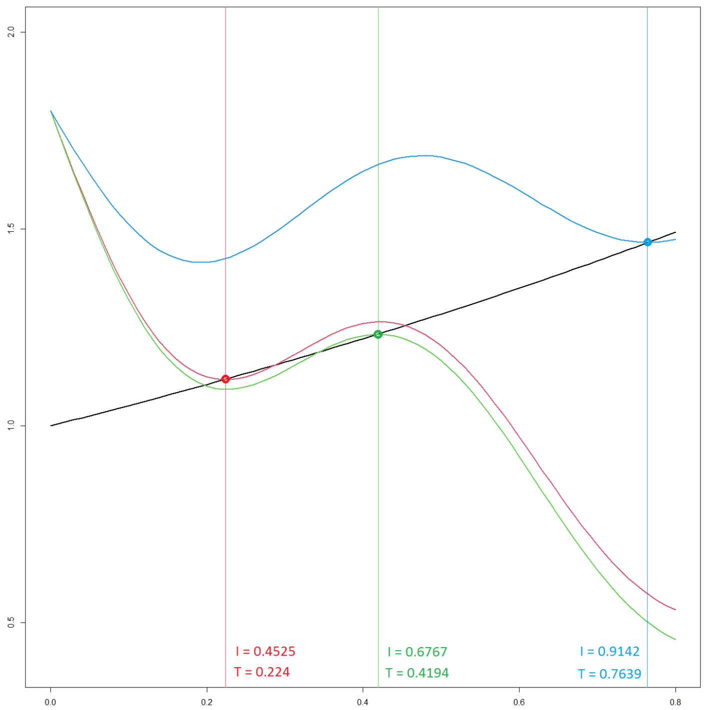

Fig. 4.3: Again, with $\mu = 1$, $\beta = 0.5$, $x_0 = 1.8$, $v_0 = 1$, and $\sigma^2(t) = 5 + 4.9\cos(20t)$ the three candidates for the optimal solution. The red path corresponds to the minimum.

To each of these values of T there corresponds a value of the action functional $I = 0.4525$,

$I = 0.6767$, and $I = 0.9142$ respectively. Thus the value of T that corresponds to the minimum is the first value $T = 0.224$. Figure (4.3) shows the three candidate optimal paths. The red path corresponds to the minimum value of I. In this particular case, the red path also satisfies the constraint $x(t) > v_0 e^{\beta t}$ for all $t \in [0, T)$. Note however that the paths are not necessarily convex and that the simple arguments of Chapter 2 cannot be applied in general here.

4.3 A Linear SDE

Here we examine the same type of problem in connection to the linear SDE which has a more complex behavior and the exact solution of the resulting variational problem is more challenging. Consider the linear SDE

$$dX_t = rdt + \sigma X_t dW_t, \qquad X_0 = x_0 \text{ w.p. } 1.$$

Suppose that since

$$dW_t = \frac{dX_t - rdt}{\sigma X_t}$$

our objective is to find the function $x : [0, T] \to \mathbb{R}$ which minimizes the functional

$$I(T) = \frac{1}{2\sigma^2} \int_0^T \left(\frac{\dot{x} - r}{x} \right)^2 dt \tag{4.45}$$

under the constraints $x(0) = x_0$, $x(T) = x_T := u_0 e^{\gamma T}$. After we do this we will find $\min_{T \in [0, T_f]} I(T)$. The corresponding Euler equation is

$$F_x - \frac{d}{dt} F_{\dot{x}} = 0$$

Since $F_x = -\frac{(\dot{x}-r)^2}{x^3}$, $F_{\dot{x}} = \frac{\dot{x}-r}{x^2}$, and $\frac{d}{dt} F_{\dot{x}} = \frac{\ddot{x}}{x^2} - 2\frac{\dot{x}-r}{x^3}\dot{x}$, the Euler equation becomes

$$-\frac{(\dot{x} - r)^2}{x^3} - \frac{\ddot{x}}{x^2} + 2\frac{(\dot{x} - r)}{x^3}\dot{x} = 0$$

or equivalently

$$x\ddot{x} - \dot{x}^2 + r^2 = 0. \tag{4.46}$$

This is a second order DE which does not contain the independent variable, t, and thus we may reduce its order by one if we treat t as the dependent variable and x as the independent. $t'(x) = \dot{x}^{-1}$ and

$$t''(x) = \frac{d}{dx} \frac{1}{\frac{dx}{dt}} = -\frac{1}{\left(\frac{dx}{dt}\right)^2} \frac{d}{dx} \frac{dx}{dt} = -\frac{1}{\left(\frac{dx}{dt}\right)^2} \left(\frac{d}{dt} \frac{dx}{dt} \right) \frac{dt}{dx} = -\frac{1}{\dot{x}^3} \ddot{x}$$

whence we obtain $t'' = -t'^3\ddot{x}$. Substituting in (4.46) we have $-xt'^{-3}t'' - t'^{-2} + r^2 = 0$ or

$$t'' + \frac{1}{x}t' - \frac{r^2}{x}t'^3 = 0. \tag{4.47}$$

Setting

$$y = t' \tag{4.48}$$

(4.47) becomes

$$y' + \frac{1}{x}y - \frac{r^2}{x}y^3 = 0. \tag{4.49}$$

This is a first order Bernoulli DE and thus can be linearized by means of the change of variables

$$u = y^{-2} \tag{4.50}$$

which gives $u' = -2y^{-3}y'$ and hence $-\frac{1}{2}y^3u' + \frac{1}{x}y - \frac{r^2}{x}y^3 = 0$ or

$$u' - \frac{2}{x}u + \frac{2r^2}{x} = 0. \tag{4.51}$$

This, at long last, is a linear first order DE which can be integrated if we multiply with the integrating factor $e^{-\int \frac{2}{x}dx} = e^{-2\log x} = x^{-2}$. We thus obtain $x^{-2}u' - 2x^{-3}u + 2r^2x^{-3} = 0$ or

$$(x^{-2}u)' = -2r^2x^{-3}$$

which can be integrated directly to give $x^{-2}u = r^2x^{-2} + C^2$ or

$$u = r^2 + C^2x^2. \tag{4.52}$$

Now we can start reversing this long process. From (4.50) we have

$$y = \frac{1}{\sqrt{r^2 + C^2x^2}}$$

and from (4.48)

$$t = \int \frac{1}{\sqrt{r^2 + C^2x^2}}dx = \frac{1}{C}\sinh^{-1}(\frac{xC}{r}) - K$$

whence $x(t) = \frac{r}{C}\sinh(C(t+K))$. We may write

$$x(t) = \frac{r}{C}\sinh(Ct + K) \tag{4.53}$$

with $K > 0$ as we will soon see.

Setting $t = 0$ we obtain

$$\frac{x_0C}{r} = \sinh(CK). \tag{4.54}$$

Set $CK = w$, $CT = z$, $a = \frac{x_0}{rT}$, $b = \frac{x_T}{rT}$. (Note that $a < b$.) Then (4.53), (4.54) become

$$za = \sinh w \tag{4.55}$$
$$zb = \sinh(w + z) \tag{4.56}$$

and hence

$$z = \sinh^{-1}(zb) - \sinh^{-1}(za).$$

Taking into account that $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$ and that $\cosh \sinh^{-1} x = \sqrt{(1+x^2)}$ we have

$$\sinh z = zb\sqrt{1+(za)^2} - za\sqrt{1+(zb)^2}. \tag{4.57}$$

The above equation has a single positive solution z^* to which there corresponds a unique $w^* = \sinh^{-1}(z^* a)$. Thus the unknown constants are

$$C = \frac{z^*}{T}, \qquad K = \frac{w^*}{z^*}T. \tag{4.58}$$

Substituting into (4.53) we have

$$x(t) = \frac{rT}{z^*}\sinh(z^*\frac{t}{T} + w^*). \tag{4.59}$$

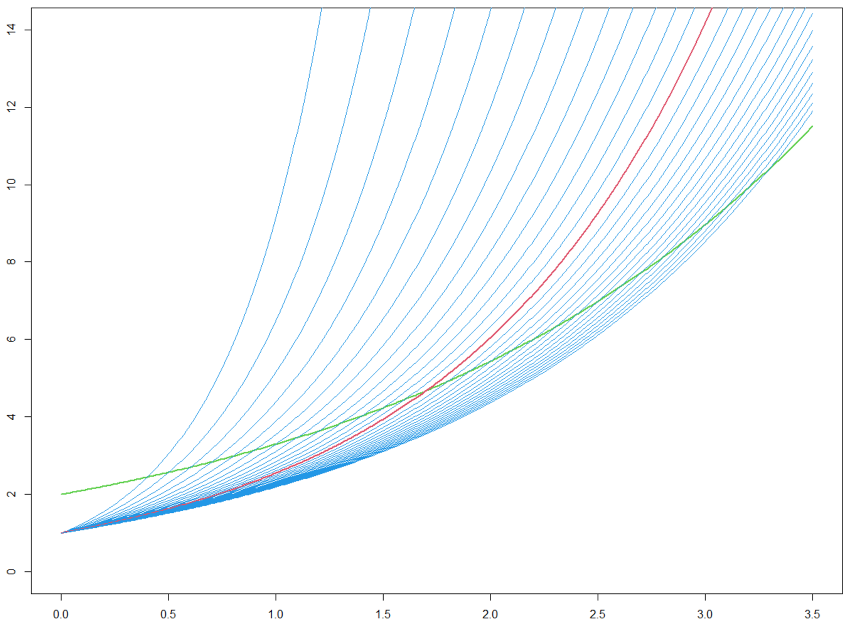

Fig. 4.4: The heavy green line corresponds to the upper boundary $2 \cdot e^{0.5\,t}$. The light blue lines are optimal paths for various meeting times T. The red line corresponds to the minimum value of the action functional, as shown in Figure 4.4.

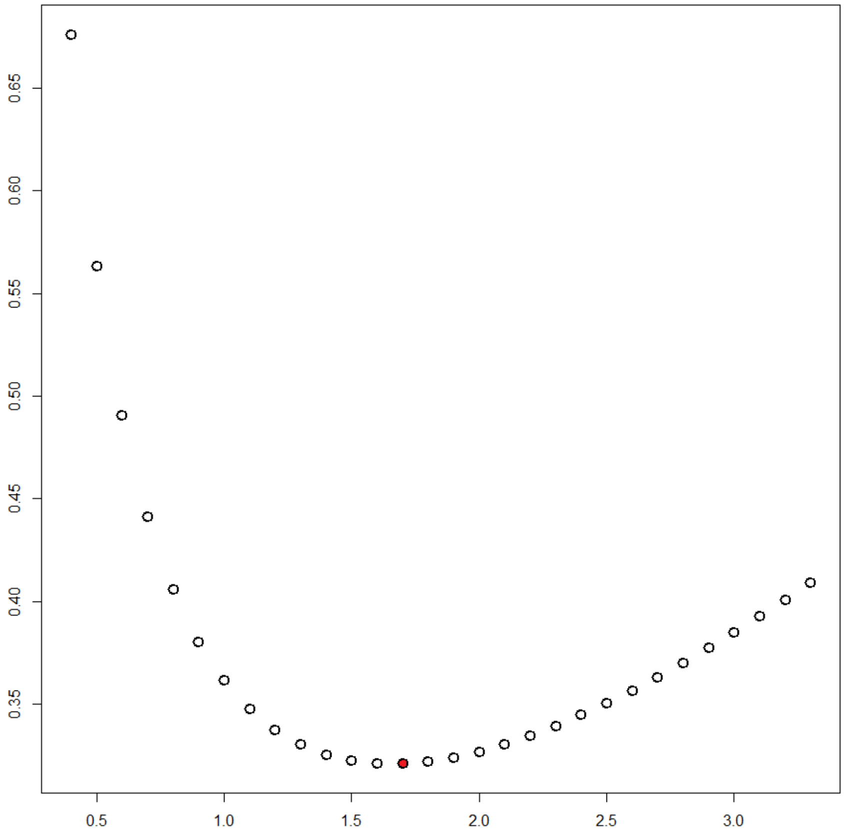

Fig. 4.5: The value of the action functional for various values of T. $T = 1.7$ corresponds to the minimum.

The corresponding value of the rate functional is

$$I(T) = \frac{1}{2\sigma^2}\int_0^T \left(\frac{r\cosh\left(z^*\frac{t}{T}+w^*\right)-r}{r\frac{T}{z^*}\sinh\left(z^*\frac{t}{T}+w^*\right)} \right)^2 dt$$

$$= \frac{z^*}{2\sigma^2 T}\int_{w^*}^{w^*+z^*} \left(\frac{\cosh(s)-1}{\sinh(s)} \right)^2 ds = \frac{z^*}{2\sigma^2 T}\int_{w^*}^{w^*+z^*} \tanh^2(s) ds \quad (4.60)$$

$$= \frac{z^*}{2\sigma^2 T}\left(z^* - 2\tanh\frac{w^*+z^*}{2} + 2\tanh\frac{w^*}{2} \right). \quad (4.61)$$

Linear SDE - Infinite Horizon Problem

We have the optimization problem

$$\min \int_0^T F(x,x',t)dt, \quad x(0)=x_0, \text{ and } x(T)=R(T)$$
$$\text{with } F(x,x',t) = \frac{1}{2\sigma^2}\left(\frac{x'-r}{x}\right)^2, \quad R(t)=u_0+\alpha t, \quad x_0<u_0.$$

The conditions for a minimum is

$$F_x - \frac{d}{dt}F_{x'} = 0 \tag{4.62}$$
$$x(T) = R(T) \tag{4.63}$$
$$F + (R'-x')F_{x'} = 0 \quad \text{at } T. \tag{4.64}$$

In this case

$$F_x = -2\frac{(x'-r)^2}{x^3}$$
$$F_{x'} = 2\frac{x'-r}{x^2}$$
$$\frac{d}{dt}F_{x'} = 2\frac{x''}{x^2} - 4\frac{(x')^2}{x^3} + 2\frac{x'r}{x^3}.$$

The Euler-Lagrange equation (4.62) becomes

$$F_x - \frac{d}{dt}F_{x'} = \frac{2}{x^3}\left((x')^2 - x''x\right) = 0$$

or equivalently

$$x\ddot{x} - \dot{x}^2 + r^2 = 0$$

Thus

$$x(t) = \frac{r}{C}\sinh(Ct+K). \tag{4.65}$$

From transversality (4.63) we have

$$\frac{r}{C}\sinh(C(T+K)) = u_0 + \alpha T$$

and from (4.64)

$$(x'(T)-r)\,(x'(T)-r+2(\alpha - x'(T))) = 0 \tag{4.66}$$

The case $x'(T)=r$ implies $r\cosh(CT+K)=r$ whence it follows that $CT+K=0$. This however is inconsistent with (4.66). Hence

$$x'(T) = 2\alpha - r$$

or

$$\cosh(CT+K) = 2\frac{\alpha}{r} - 1.$$

Since $\alpha > 1$ there exists a unique $\gamma > 0$ such that

$$\cosh(\gamma) = 2\frac{\alpha}{r} - 1, \qquad \text{and } CT+K = \gamma. \tag{4.67}$$

We thus obtain the following system of three equations which will allow us to obtain the three unknown constants C, K, T.

$$\sinh(K) = C\frac{x_0}{r}, \tag{4.68}$$

$$\sinh(\gamma) = C\frac{u_0}{r} + \frac{\alpha}{r}CT, \tag{4.69}$$

$$CT + K = \gamma. \tag{4.70}$$

K is obtained as the solution of

$$\sinh(K) - \frac{x_0}{u_0}\frac{\alpha}{r}K + \frac{x_0}{u_0}\left(\sinh\gamma - \frac{\alpha}{r}\gamma\right) = 0. \tag{4.71}$$

and

$$C = \frac{r}{x_0}\sinh(K), \quad T = \frac{\gamma - K}{C}. \tag{4.72}$$

Depending on the values of the parameters r, u_0, x_0 and $\gamma = \cosh^{-1}(2\frac{\alpha}{r} - 1)$, equation (4.71) may have no positive solution or two positive solutions.

4.4 The general linear SDE with constant coefficients

Consider the SDE

$$dX_t = (r + \mu X_t)dt + (b + \sigma X_t)dW_t. \tag{4.73}$$

Arguing as before we have

$$dW_t = \frac{dX_t - (r + \mu X_t)dt}{b + \sigma X_t}$$

which gives

$$I(t) = \frac{1}{2}\int_0^t \left(\frac{x' - r - \mu x}{b + \sigma x}\right)^2 du\,. \tag{4.74}$$

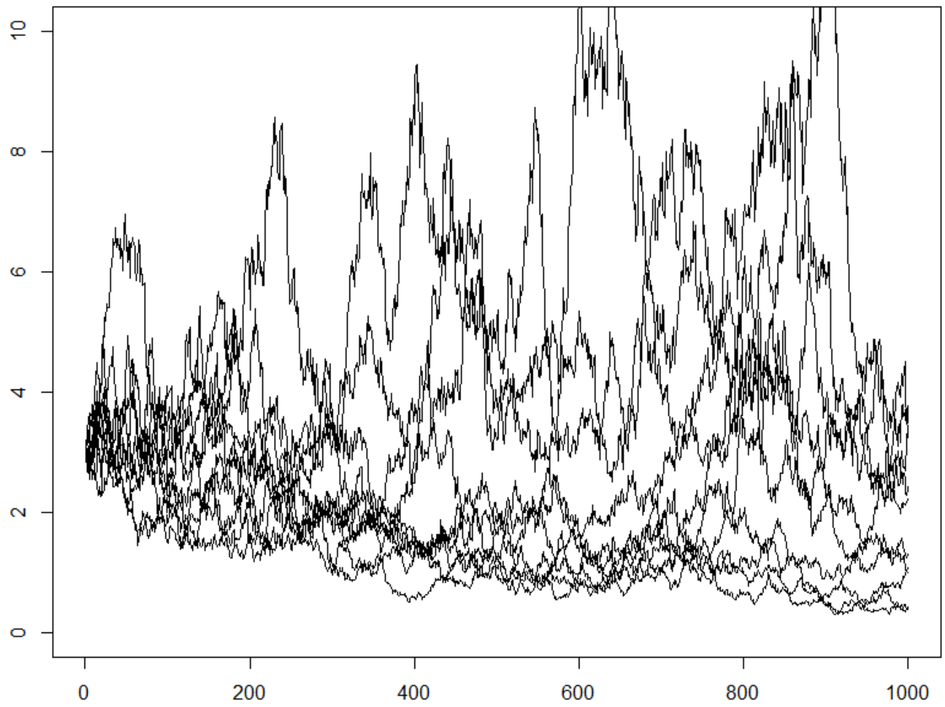

Fig. 4.6: Sample paths of $dX_t = rdt + \sigma X_t dW_t$. The mean is $EX_t = x_0 + rt$ but the linear increase in not evident due to high variability.

Set $F = \frac{1}{2}\left(\frac{x'-r-\mu x}{b+\sigma x}\right)^2$. Then

$$\begin{aligned} F_x &= \frac{x'-r-\mu x}{b+\sigma x}\left(-\frac{\mu}{b+\sigma x} - \frac{x'-r-\mu x}{(b+\sigma x)^2}\sigma\right) \\ F_{x'} &= \frac{x'-r-\mu x}{(b+\sigma x)^2} \\ \frac{d}{du}F_{x'} &= \frac{x''-\mu x'}{(b+\sigma x)^2} - 2\sigma x' \frac{x'-r-\mu x}{(b+\sigma x)^3}. \end{aligned}$$

The Euler equation is $F_x - \frac{d}{du}F_{x'} = 0$ or equivalently

$$-\mu(x'-r-\mu x)(b+\sigma x) - (x'-r-\mu x)^2\sigma - (x''-\mu x')(b+\sigma x) + 2\sigma x'(x'-r-\mu x) = 0 \quad (4.75)$$

Changing the roles between the independent variable, t, and the dependent variable, x we have

$$x' = \frac{1}{t'}, \qquad x'' = -\frac{t''}{t'^3}$$

and substituting into (4.75) we have

$$-\mu\left(\frac{1}{t'} - r - \mu x\right)(b+\sigma x) - \left(\frac{1}{t'} - r - \mu x\right)^2 \sigma - \left(-\frac{t''}{t'^3} - \mu\frac{1}{t'}\right)(b+\sigma x) + 2\sigma\frac{1}{t'}\left(\frac{1}{t'} - r - \mu x\right) = 0.$$

Setting $y = t'$ we reduce the order of the differential equation by one and obtain

$$-\mu(y^2-(r+\mu x)y^3)(b+\sigma x)-(1-(r+\mu x)y)^2 y\sigma+(y'+\mu y^2)(b+\sigma x)+2\sigma y(1-y(r+\mu x)) = 0.$$

which simplifies into

$$y' + \frac{\sigma}{b+\sigma x}y + (\mu b - \sigma r)\frac{r+\mu x}{b+\sigma x}y^3 = 0. \tag{4.76}$$

This last equation is a Bernoulli differential equation which can be integrated using the transformation $u = y^{-2}$. We then obtain

$$u' - \frac{2\sigma}{b+\sigma x}u + 2(\sigma r - \mu b)\frac{r+\mu x}{b+\sigma x} = 0. \tag{4.77}$$

In order to solve this linear differential equation we multiply with the integrating factor $(b+\sigma x)^{-2}$ and obtain

$$\left(u(b+\sigma x)^{-2}\right)' = 2(\mu b - \sigma r)\frac{r+\mu x}{(b+\sigma x)^3}$$

which we integrate to obtain

$$u(b+\sigma x)^{-2} = 2(\mu b - \sigma r)\left(\frac{\mu b - r\sigma}{2\sigma^2}(b+\sigma x)^{-2} - \frac{\mu}{\sigma^2}(b+\sigma x)^{-1}\right) + C^2$$

or

$$u = l^2 - \frac{2\mu l}{\sigma}(b+\sigma x) + C^2(b+\sigma x)^2 \tag{4.78}$$

where

$$l = \frac{\mu b - r\sigma}{\sigma}.$$

Note that we assume the integration constant to be positive. By "completing the square" we rewrite (4.78) as

$$u = l^2\left(1 - \frac{\mu^2}{\sigma^2 C^2}\right) + \left(C(b+\sigma x) - \frac{\mu l}{\sigma C}\right)^2 \tag{4.79}$$

and thus

$$y = \frac{1}{\sqrt{l^2\left(1-\frac{\mu^2}{\sigma^2C^2}\right) + \left(C(b+\sigma x) - \frac{\mu l}{\sigma C}\right)^2}}$$

which gives

$$t = \int \frac{dx}{\sqrt{l^2\left(1-\frac{\mu^2}{\sigma^2C^2}\right) + \left(C(b+\sigma x) - \frac{\mu l}{\sigma C}\right)^2}}.$$

Assume that

$$C > \frac{\mu}{\sigma}.$$

Then, setting

$$\xi = C(b + \sigma x) - \frac{\mu l}{\sigma C}, \qquad \lambda^2 = l^2 \left(1 - \frac{\mu^2}{\sigma^2 C^2}\right),$$

we have

$$t = \frac{1}{C\sigma} \int \frac{d\xi}{\sqrt{\lambda^2 + \xi^2}} \tag{4.80}$$

which gives

$$t = \frac{1}{C\sigma} \sinh^{-1} \frac{\xi}{\lambda} - K \tag{4.81}$$

with $K > 0$. Thus

$$x(t) = \frac{\lambda}{C\sigma} \sinh C\sigma(t + K) + \frac{\mu l}{\sigma^2 C^2} - \frac{b}{\sigma}. \tag{4.82}$$

Note that when $\mu = b = 0$, $\lambda = l = r$ the above expression becomes

$$x(t) = \frac{r}{\sigma C} \sinh \sigma C(t + K)$$

which essentially agrees with our previous result.

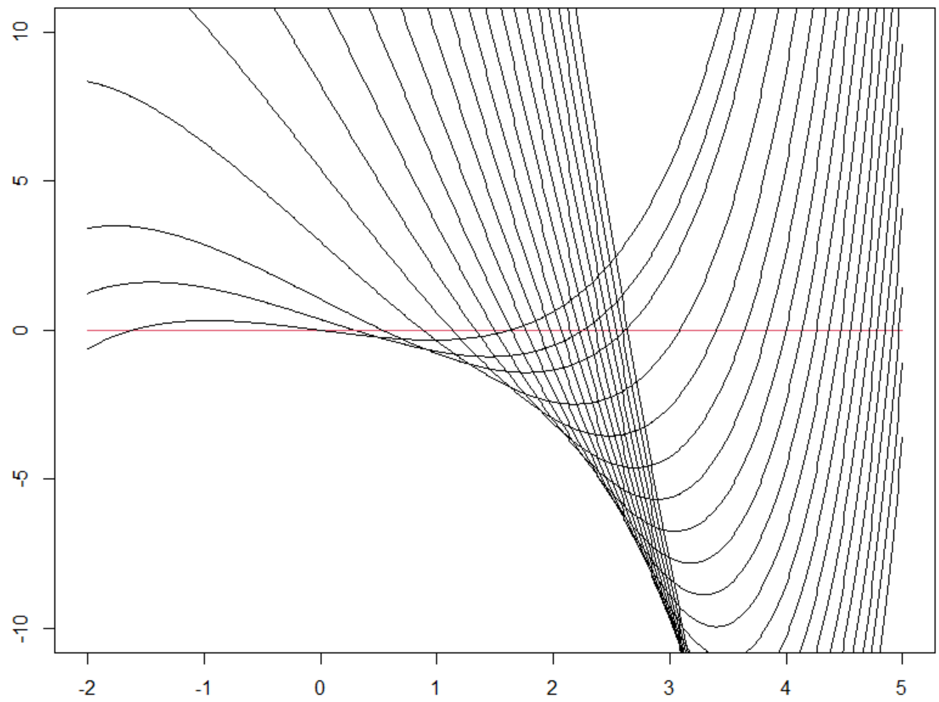

Fig. 4.7: The right hand side of (4.71) with multiple solutions.

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