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Autoregressive Conditional

Heteroscedasticity Model Selection

By

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ΔΙΑΤΡΙΒΗ

Που υποβλήθηκε στο Τμήμα Στατιστικής
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DEDICATION

To my parents, Adriana and Antonis
and to my brother, Angelos.

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I am very grateful my supervisor Professor Evdokia Xekalaki, whose experience and knowledge led me to the completion of my PhD thesis.

VITA

In 1993, I entered the Department of Statistics in the Athens University of Economics and Business (AUEB) where I received my first Degree in Statistics in 1997. In 1998, I received my MSc in Econometrics in the University of Essex. In 1998 I started working on my PhD Thesis. Since 1996, I have been employed as a statistician within both the public and private sector. Since 2001, I am adjunct Lecturer at the Department of Statistics of AUEB, where I teach Applied Econometrics, Applications of Time Series Analysis, Data Analysis and Statistical Packages to the undergraduate and postgraduate students of the department. Through my research I managed to have several publications in international journals and presentations in scientific conferences in topics such as applied and theoretical statistics, time series forecasting and financial econometrics.

ABSTRACT

Autoregressive Conditional Heteroscedasticity (ARCH) models have successfully been employed in order to predict asset return volatility. Predicting volatility is of great importance in pricing financial derivatives, selecting portfolios, measuring and managing investment risk more accurately.

Most of the methods used in the ARCH literature for selecting the appropriate model are based on evaluating the ability of the models to describe the data. In this thesis, the approach taken is based on evaluating the ability of the models to predict the conditional variance rather than on the ability of the models to describe the data. Based on a standardized prediction error criterion (SPEC), a model selection algorithm is developed. According to this algorithm, the ARCH model with the lowest sum of squared standardized forecasting errors as judged by the value of the ratio of two correlated gamma variables is selected for predicting future volatility. The proposed model selection method allows the use of a virtually different model for prediction at each of a sequence of points in time.

A number of evaluation criteria are used to examine whether the SPEC model selection procedure has a satisfactory performance in selecting that model that generates "better" volatility predictions. Moreover, we consider assessing model performance through computing real and simulated option prices based on the volatility forecasts of the underlying asset returns, devising trading rules to trade options on a daily basis and comparing the resulting profits. The results show that traders using the SPEC algorithm for deciding which model's forecasts to use at any given point in time achieve the highest profits.

Finally, a multi-model selection procedure is proposed, which leads to the selection of the model with the lowest sum of squared standardized one-step-ahead prediction errors. The form of the exact distribution of the test statistic is explicitly derived and the procedure is illustrated in the case of three modes using real data on stock returns.

ΠΕΡΙΛΗΨΗ

Τα αυτοπαλίνδρομα μοντέλα δεσμευμένης ετεροσκεδαστικότητας (Autoregressive Conditional Heteroscedasticity - ARCH) έχουν εφαρμοστεί με επιτυχία για την πρόβλεψη της διακύμανσης της απόδοσης διαφόρων χρηματοοικονομικών προϊόντων, όπως μετοχές, συναλλαγματικές ισοτιμίες, χρηματιστηριακοί δείκτες, αμοιβαία κεφάλαια κ.ο.κ. Η ακριβής πρόβλεψη της διακύμανσης έχει ιδιαίτερη σημασία στην τιμολόγηση παραγώγων, στην κατασκευή χαρτοφυλακίων, στη μέτρηση και διαχείριση του επενδυτικού κινδύνου.

Η παρούσα διατριβή έχει στόχο να παρουσιάσει μια μέθοδο επιλογής ARCH μοντέλων με βάση την προβλεπτική τους ικανότητα. Συγκεκριμένα, στο κεφάλαιο 2, παρουσιάζονται τόσο τα μονομεταβλητά, όσο και τα πολυμεταβλητά ARCH μοντέλα που υπάρχουν στη βιβλιογραφία, οι μέθοδοι εκτίμησής τους καθώς και τα χαρακτηριστικά των χρηματοοικονομικών σειρών, τα οποία ερμηνεύουν. Μία συστηματική παρουσίαση των ARCH μοντέλων είναι πάρα πολύ χρήσιμη έτσι ώστε να μπορεί ένας ερευνητής να επιλέξει το κατάλληλο μοντέλο για τη συγκεκριμένη εργασία που θέλει.

Στο κεφάλαιο 3, εφαρμόζουμε μία σειρά από Monte Carlo προσομοιώσεις από τις οποίες φαίνεται ότι τα εκτιμώμενα τυποποιημένα κατάλοιπα από διάφορα ARCH μοντέλα κατανέμονται ανεξάρτητα. Μία υπόθεση πολύ σημαντική για τα υπόλοιπα κεφάλαια της διατριβής.

Η πλειοψηφία των μεθόδων επιλογής μοντέλων στην ARCH βιβλιογραφία βασίζεται στην αξιολόγηση της ικανότητας των μοντέλων να περιγράψουν τα δεδομένα. Στο κεφάλαιο 4, εξετάζεται ο έλεγχος υποθέσεων που εισήχθη από τους Xekalaki et al. (2003) για την συγκριτική αξιολόγηση δύο μοντέλων παλινδρόμησης, για την περίπτωση ARCH μοντέλων. Συγκεκριμένα, προτείνεται η σύγκριση ARCH μοντέλων μέσω του ελέγχου της μηδενικής υπόθεσης ότι τα δύο μοντέλα έχουν την ίδια προβλεπτική ικανότητα, έναντι της εναλλακτικής, ότι το μοντέλο με το μικρότερο άθροισμα των τετραγωνικών τυποποιημένων κατάλοιπων έχει την υψηλότερη προβλεπτική ικανότητα. Ο έλεγχος αυτός οδηγεί στην κατασκευή ενός αλγορίθμου επιλογής μοντέλων με κριτήριο βασιζόμενο στα τυποποιημένα σφάλματα πρόβλεψης, τον Standardized Prediction Error Criterion (SPEC) αλγόριθμο επιλογής μοντέλων. Σύμφωνα με τον SPEC αλγόριθμο, από ένα σύνολο ARCH μοντέλων,

αυτό το οποίο έχει το μικρότερο άθροισμα των τετραγωνικών τυποποιημένων κατάλοιπων, επιλέγεται για την πρόβλεψη της δεσμευμένης διακύμανσης της επόμενης χρονικής στιγμής. Η μεθοδολογία που αναπτύσσεται βασίζεται σε ανά δύο συγκρίσεις των υπό εξέταση μοντέλων. Βέβαια, κάθε φορά που ο SPEC αλγόριθμος εφαρμόζεται, το μοντέλο που θα χρησιμοποιηθεί για την πρόβλεψη της διακύμανσης, είναι εν γένει διαφορετικό.

Στα επόμενα κεφάλαια η αξιολόγηση της SPEC μεθοδολογίας επιλογής μοντέλων γίνεται μέσω της χρήσης της σε διάφορες χρηματοοικονομικές εφαρμογές. Συγκεκριμένα, στο κεφάλαιο 5, μία σειρά από στατιστικά κριτήρια αξιολόγησης της προβλεπτικής ικανότητας χρησιμοποιούνται για να δούμε ποιο ARCH μοντέλο έχει τη μεγαλύτερη προβλεπτική ικανότητα για χρονικούς ορίζοντες από μία έως εκατό χρονικές μονάδες μπροστά. Στη συνέχεια, αξιολογούμε την προβλεπτική ικανότητα διαφόρων μεθόδων επιλογής μοντέλων, μεταξύ των οποίων είναι και ο SPEC αλγόριθμος. Τα αποτελέσματα δείχνουν ότι η SPEC μέθοδος έχει πολύ καλή απόδοση στο να επιλέγει το μοντέλο με τη μεγαλύτερη προβλεπτική ικανότητα σε σχέση με άλλες μεθόδους επιλογής μοντέλων περιλαμβανομένων και μεθόδων βασιζόμενων στη χρήση ενός μοναδικού ARCH μοντέλου.

Στο κεφάλαιο 6, αξιολογείται η ικανότητα τόσο του SPEC αλγορίθμου όσο και διαφόρων άλλων μεθόδων επιλογής μοντέλων στο πλαίσιο προβλημάτων τιμολόγησης δικαιωμάτων προαίρεσης στο χρηματιστήριο παραγώγων του Σικάγο. Χρησιμοποιούνται δεδομένα από δικαιώματα προαίρεσης του χρηματιστηριακού δείκτη S&P500. Μέσα από τα αποτελέσματα της σύγκρισης των μεθόδων επιλογής μοντέλων, προκύπτει ότι ο επενδυτής που χρησιμοποιεί τον SPEC αλγόριθμο, έχει την ικανότητα να επιλέγει αυτά τα ARCH μοντέλα τα οποία δίνουν τη μεγαλύτερη δυνατή απόδοση από κάθε άλλη μέθοδο επιλογής μοντέλων. Επειδή, ενίοτε, τα πραγματικά δεδομένα που χρησιμοποιούνται από την αγορά παραγώγων ενδέχεται να μεροληπτούν υπέρ μίας μεθοδολογίας, στο κεφάλαιο 7, προσομοιώνουμε μία αγορά παραγώγων, στην οποία κάθε επενδυτής χρησιμοποιεί μία μέθοδο πρόβλεψης της μελλοντικής τιμής του δικαιώματος προαίρεσης. Και σε αυτή την περίπτωση, οι επενδυτές που βασίζονται στην SPEC μεθοδολογία επιτυγχάνουν τις μέγιστες αποδόσεις.

Τέλος, στο κεφάλαιο 8, προτείνεται μια εναλλακτική μέθοδος επιλογής μοντέλων από n διαθέσιμα μοντέλα $(n \geq 2)$ η οποία οδηγεί στην επιλογή του μοντέλου του οποίου το άθροισμα τετραγώνων των τυποποιημένων σφαλμάτων πρόβλεψης έχει την πιο χαμηλή τιμή. Η μεθοδολογία που χρησιμοποιείται διαφέρει από αυτήν του κεφαλαίου 4, η οποία κάνει χρήση των ανα δύο λόγων των αθροισμάτων τετραγώνων των τυποποιημένων σφαλμάτων πρόβλεψης. Η ακριβής μορφή της κατανομής της ελεγχοσυνάρτησης προσδιορίζεται ως η κατανομή της ελάχιστης συνιστώσας ενός τυχαίου διανύσματος που ακολουθεί την πολυμεταβλητή γάμμα κατανομή. Αυτός ο έλεγχος υποθέσεων μπορεί να εφαρμοστεί για την αξιολόγηση της ικανότητας των μοντέλων να προβλέπουν τόσο την δεσμευμένη μέση τιμή όσο και τη δεσμευμένη διακύμανση. Η διαδικασία ελέγχου εφαρμόζεται για την περίπτωση τριών μοντέλων σε δεδομένα από την Ελληνική κεφαλαιαγορά.

Είναι χρήσιμο να αναφερθεί ότι με βάση τα ευρήματα των παραπάνω κεφαλαίων, η SPEC μέθοδος είναι ένα εργαλείο πολύ χρήσιμο για την επιλογή των κατάλληλων μοντέλων για την πρόβλεψη της διακύμανσης, με εφαρμογές όχι μόνο στην αξιολόγηση χαρτοφυλακίων και στη διαχείριση κινδύνου, αλλά και στη δημιουργία κερδοσκοπικών στρατηγικών στο χρηματιστήριο παραγώγων.

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Chapter 1
Introduction

Autoregressive Conditional Heteroscedasticity (ARCH) models have successfully been employed in order to predict asset return volatility. However, the last two decades numerous formulations of volatility modelling have been proposed and a vast number of studies that evaluate the ability of ARCH models in forecasting volatility have been conducted yielding in several cases contradictory results. This thesis aims at shedding some light in the area of model selection for volatility forecasting. We try to define a unified criterion, for as many classes of ARCH processes as possible, that is based on a rating of the predictability of ARCH models. Subsequently, we evaluate the accuracy of that criterion in suggesting at each point in time the model that will be used in obtaining volatility forecasts.

The number of possible conditional volatility formulations is vast. Therefore, a systematic presentation of the models that have been considered in the ARCH literature can be useful in guiding one's choice of a model for exploiting future volatility, with applications in financial markets. In chapter 2, a number of univariate and multivariate ARCH models, their estimating methods and the characteristics of financial time series, which are captured by volatility models, are presented.

Quite often, the testing procedure requires independence in a sequence of recursive standardized prediction errors, which cannot always be readily deduced particularly in the case of econometric modeling. In chapter 3, on the basis of the results of a series of Monte Carlo simulations, it is conjectured that independence holds and the sum of squared standardized one-step-ahead prediction errors is Chi-square distributed. The results of our simulation are confirmed analytically in chapter 4.

Most of the methods used in the ARCH literature for selecting the appropriate model are based on evaluating the ability of the models to describe the data. In chapter 4, Xekalaki et al.'s (2003) hypothesis test for two regression models is considered in the context of ARCH models. In particular, it is suggested that two ARCH processes can be compared through testing a null hypothesis of equivalence of the models in their predictability. Instead of being based on evaluating the ability of the models to describe the data, the proposed approach is based on evaluating the ability of the models to predict the conditional variance. This comparative evaluation approach leads to what is termed in the sequel the Standardized Prediction Error Criterion (SPEC) model selection algorithm. According to this algorithm, the ARCH model with the lowest sum of squared standardized forecasting errors is selected for predicting future volatility. Each time the model selection method is applied, the model used to predict the conditional variance is revised.

The ability of the SPEC algorithm to predict future volatility is also examined. In the chapter 5, in particular, a number of statistical measures are used to examine the performance of a model to predict future volatility, for forecasting horizons ranging from one day to one hundred days ahead. The results show that the SPEC model selection procedure has a satisfactory performance in selecting that model that generates “better” volatility predictions.

The next two chapters look at the evaluation of the SPEC method not with the use of statistical measures but through assessing the potential added value of the SPEC algorithm in financial applications such as options’ forecasting. So, in chapter 6, we consider assessing model performance through computing option prices based on the volatility forecasts of the underlying asset returns, devising trading rules to trade options on a daily basis and comparing the resulting profits. The comparative evaluation is performed using S&P500 straddle options on the basis of the cumulative profits of traders always using variance forecasts obtained by a single model on the one hand and the cumulative profits of traders using variance forecasts obtained by models suggested by the SPEC algorithm on the other. The results of the study show that traders using this algorithm for deciding which model’s forecasts to use at any given point in time achieve higher cumulative profits than those using only a single model all the time. A comparison of the SPEC algorithm with a set of other model evaluation criteria yields similar findings.

In chapter 7, the evaluation of the presented algorithm is performed by comparing different volatility forecasts in option pricing through the simulation of an options market. Traders employing the SPEC model selection algorithm use the model with the lowest sum of squared standardized one-step-ahead prediction errors for obtaining their volatility forecast. The cumulative profits of the participants in pricing one-day index straddle options always using variance forecasts obtained by GARCH, EGARCH and TARCH models are compared to those made by the participants using variance forecasts obtained by models suggested by the SPEC algorithm. The straddles are priced on the S&P500 index. It is concluded that traders, who base their selection of an ARCH model on the SPEC algorithm, achieve higher profits than those, who use only a single ARCH model. Moreover, the SPEC algorithm is compared with other criteria of model selection that measure the ability of the ARCH models to forecast the realized intra-day volatility. In this case too, the SPEC algorithm users achieve the highest returns. Thus, the SPEC model selection method appears to be a useful tool in selecting the appropriate model for estimating future volatility in pricing derivatives.

In chapter 8, an alternative model selection approach is proposed. It is a multi-model selection procedure, which leads to the selection of the model with the lowest sum of squared standardized one-step-ahead prediction errors. The theoretical framework considered in chapter 8 differs from the one in chapter 4, which is based on pairwise comparisons of the sums of squared standardized one-step-ahead forecasting errors of the candidate models. The form of the exact distribution of the test statistic is explicitly derived as the distribution of the minimum value of n variables that are jointly multivariate gamma distributed. These represent the sums of squared standardized prediction errors of n models. The null hypothesis that the n models are of equivalent predictive ability is therefore tested against the alternative hypothesis that the model with the lowest loss function has the highest predictive ability using this statistic. The suggested testing procedure can be applied in evaluating the accuracy of either the conditional mean or the conditional variance forecasts and is illustrated in the case of three models using real data on index stock returns. Finally, in chapter 9, a brief discussion on topics for future research is provided.

It would be worth mentioning that the SPEC model selection algorithm appears, on the basis of our findings to offer a useful tool in guiding one's choice of the appropriate model for predicting future volatility, with applications in evaluating portfolios, managing financial risk and creating speculative strategies with options.

Chapter 2
Autoregressive Conditional Heteroscedasticity (ARCH) Models: Review of the Literature

2.1. Introduction

Since the first decades of the 20th century, asset returns have been assumed to form an independently and identically distributed (i.i.d) random process with zero mean and constant variance. Bachelier (1900) was the first who contributed the theoretical random walk model for the analysis of speculative prices. For $\{P_t\}$ denoting the discrete time asset price process and $\{y_t\}$ denoting the process of the continuously compounded returns, defined by $y_t = \ln(P_t / P_{t-1})$, the early literature viewed the system that generates the asset price process as a fully unpredictable random walk process:

$$P_t = P_{t-1} + \varepsilon_t$$
$$\varepsilon_t \overset{i.i.d.}{\sim} N(0, \sigma^2),$$

where ε_t is a zero-mean i.i.d. normal process. However, the assumptions of normality, independence and homoscedasticity do not always hold with real data.

Figures 2.1 to 2.3 depict the continuously compounded daily returns of the Chicago Standard and Poor's 500 Composite (S&P500) index, Frankfurt DAX30 stock index and Athens Stock Exchange (ASE) index. The data cover the period from 2nd January 1990 to 27th June 2000. A visual inspection shows clearly, that the mean is constant, but the variance changes over time, so the return series is not a sequence of independently and identically distributed (i.i.d.) random variables. A characteristic of asset returns, which is noticeable from the figures, is the volatility clustering first noted by Mandelbrot (1963): "Large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes". Fama (1970) also observed the alternation between periods of high and low volatility: "Large price changes are followed by large price changes, but of unpredictable sign".

Figure 2.1. S&P500 Continuously Compounded Daily Returns from 2/1/90 to 27/06/00

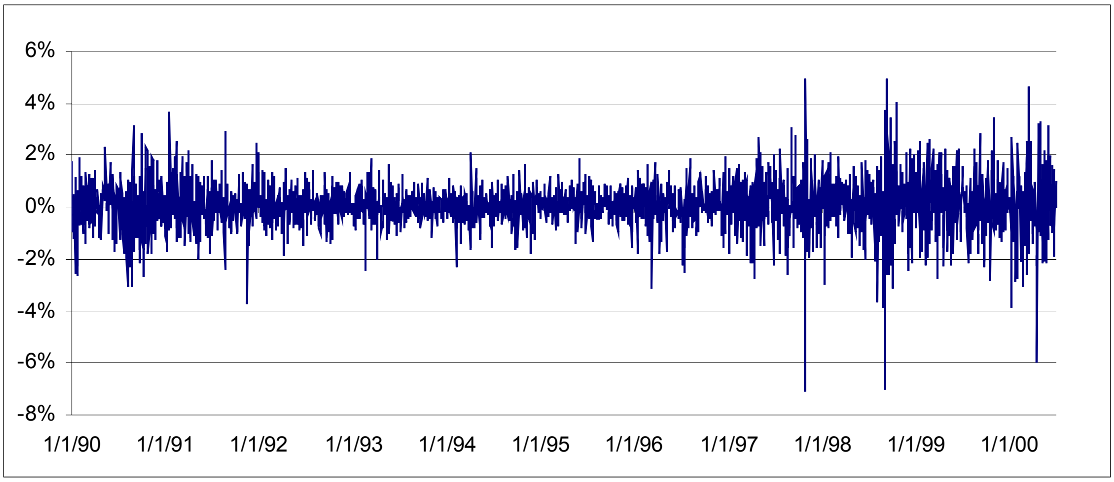


Figure 2.2. DAX 30 Continuously Compounded Daily Returns from 2/1/90 to 27/06/00

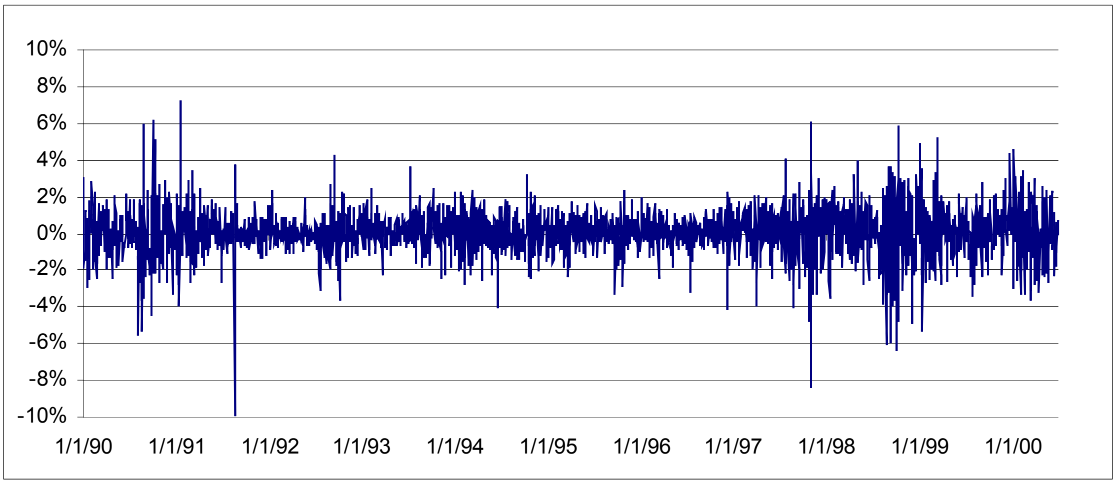
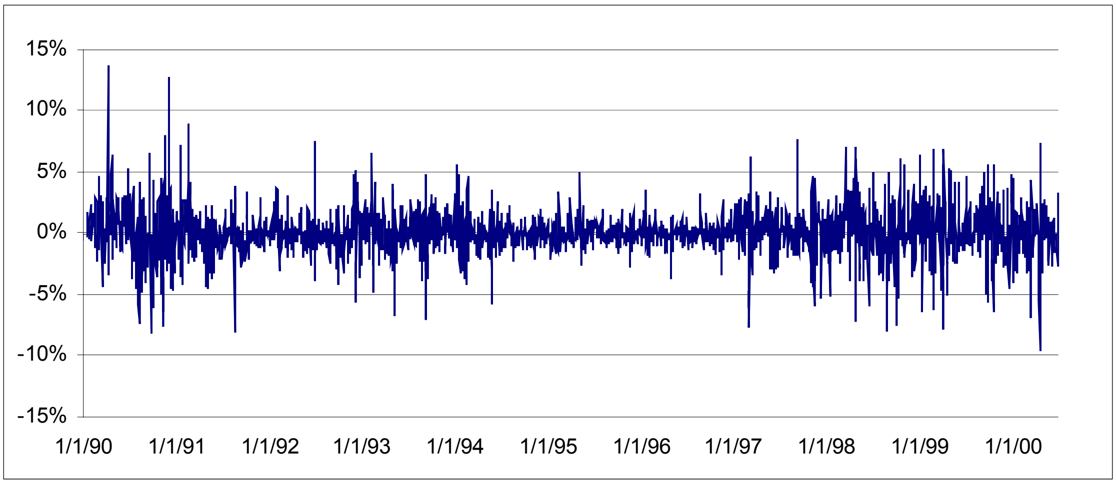
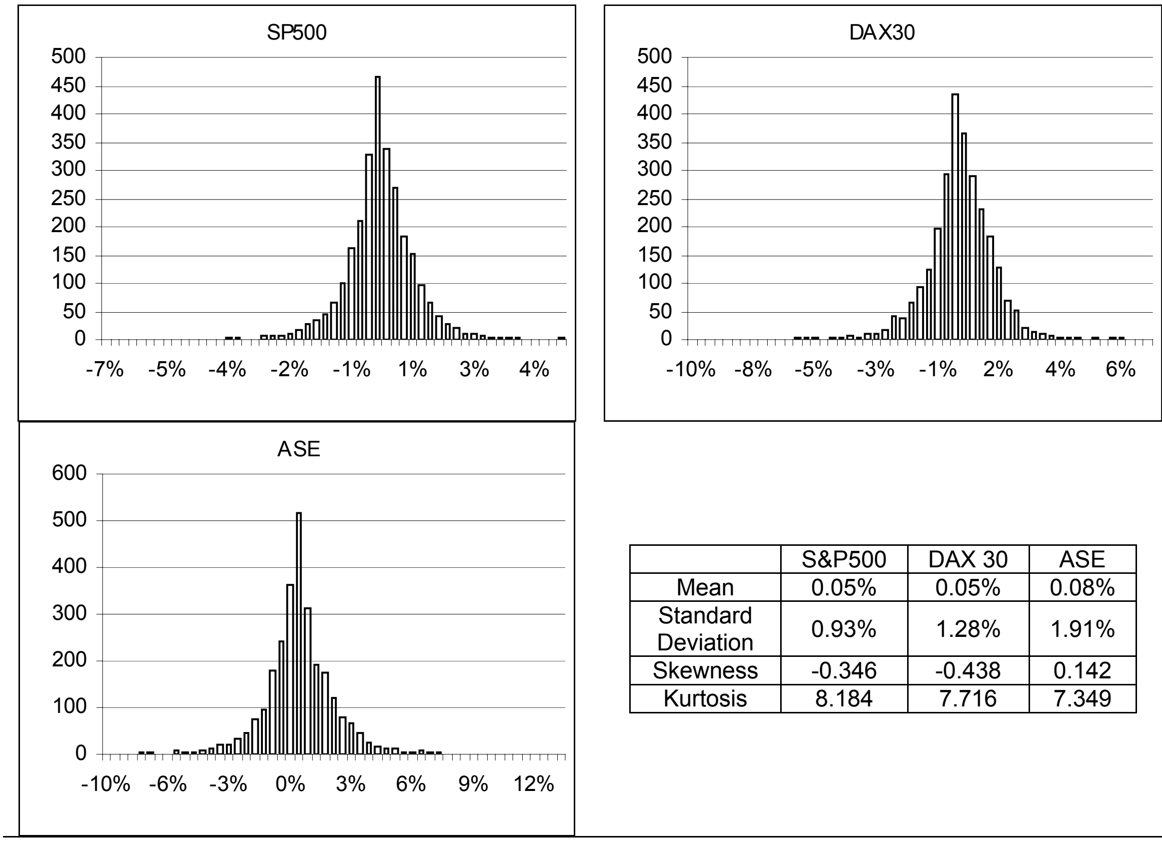


Figure 2.3. ASE Continuously Compounded Daily Returns from 18/1/90 to 27/06/00



A non-constant variance of asset returns should lead to a non-normal distribution. Figure 2.4 represents the histograms and the descriptive statistics of the stock market series plotted in Figures 2.1 to 2.3. Asset returns are highly leptokurtic and slightly asymmetric, a phenomenon correctly observed by Mandelbrot (1963): "The empirical distributions of price changes are usually too "peaked" to be relative to samples from Gaussian populations … the histograms of price changes are indeed unimodal and their central bells remind the Gaussian ogive. But, there are typically so many outliers that ogives fitted to the mean square of price changes are much lower and flatter than the distribution of the data themselves." In the sixties and seventies, the regularity of leptokurtosis led to a literature on modeling asset returns as independently and identically distributed random variables having some thick-tailed distribution (Blattberg and Gonedes (1974), Clark (1973), Hagerman (1978), Mandelbrot (1963,1964), Officer (1972), Praetz (1972)).

Figure 2.4. Histogram and Descriptive Statistics for S&P500, DAX 30 and ASE Stock Market Returns.



	S&P500	DAX 30	ASE
Mean	0.05%	0.05%	0.08%
Standard Deviation	0.93%	1.28%	1.91%
Skewness	-0.346	-0.438	0.142
Kurtosis	8.184	7.716	7.349

These models, although able to capture the leptokurtosis, could not account for the existence of non-linear temporal dependence as the volatility clustering observed from the data. For example, applying an autoregressive model to remove the linear dependence from an asset returns series and testing the residuals for a higher-order dependence using the Brock, Dechert and Scheinkman (BDS) test (Brock et al. (1987), Brock et al. (1991), Brock et al. (1996)), the null hypothesis, that the residuals are i.i.d., is rejected.

In this chapter, a number of univariate and multivariate ARCH models are presented and their estimation is discussed. The main features of what seem to be most widely used ARCH models are described with emphasis on their practical relevance. It is not an attempt to cover the whole of the literature on the technical details of the models, which is very extensive. (A comprehensive survey of the most important theoretical developments in ARCH type modeling covering the period up to 1993 was given by Bollerslev et al. (1994)). The aim is to give the broad framework of the most important models used today in the economic applications. A careful selection of references is provided so that the interested reader can make more detailed examination of particular topics. In particular, an anthology of representations of ARCH models that have been considered in the literature is provided (section 2.2), including representations that have been proposed for accounting for relationships between the conditional mean and the conditional variance (section 2.3) and methods of estimation of their parameters (section 2.4). Generalizations of these models suggested in the literature in multivariate contexts are also discussed (section 2.5). Section 2.6 gives a brief description of other methods of estimating volatility. Finally, section 2.7 is concerned with interpretation and implementation issues of ARCH models in financial applications.

The remaining of the present section looks at the influence that various factors have on a time series and in particular at effects, which as reflected in the data, are known as the “leverage effect”, the “non-trading period effect”, and the “non-synchronous trading effect”.

2.1.1 The Leverage Effect

Black (1976) first noted that often, changes in stock returns display a tendency to be negatively correlated with changes in returns volatility, i.e., volatility tends to rise in response to “bad news” and to fall in response to “good news”. This phenomenon is termed the “leverage effect” and can only be partially interpreted by fixed costs such as financial and operating leverage (see, e.g. Black (1976) and Christie (1982). The asymmetry present in the volatility of stock returns is too large to be fully explained by leverage effect.

Figure 2.5. Daily Log-values and Recursive Standard Deviation of Returns for the S&P500 Stock Market.

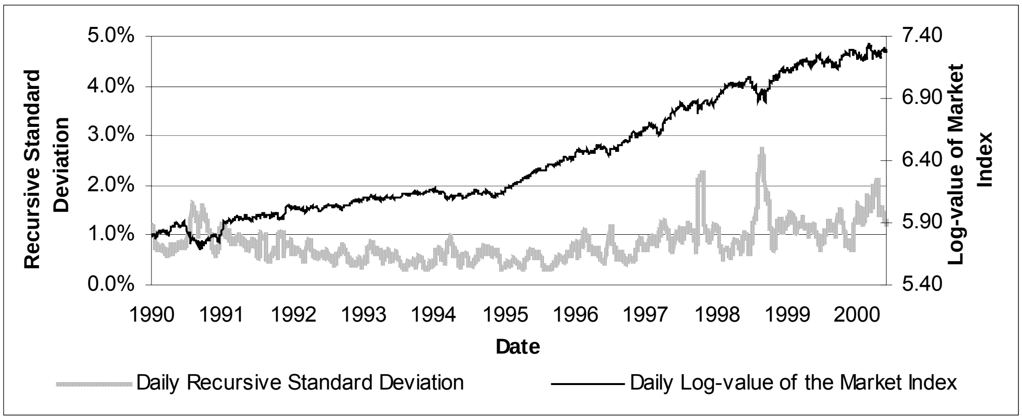


Figure 2.6. Daily Log-values and Recursive Standard Deviation of Returns for the DAX 30 Stock Market.

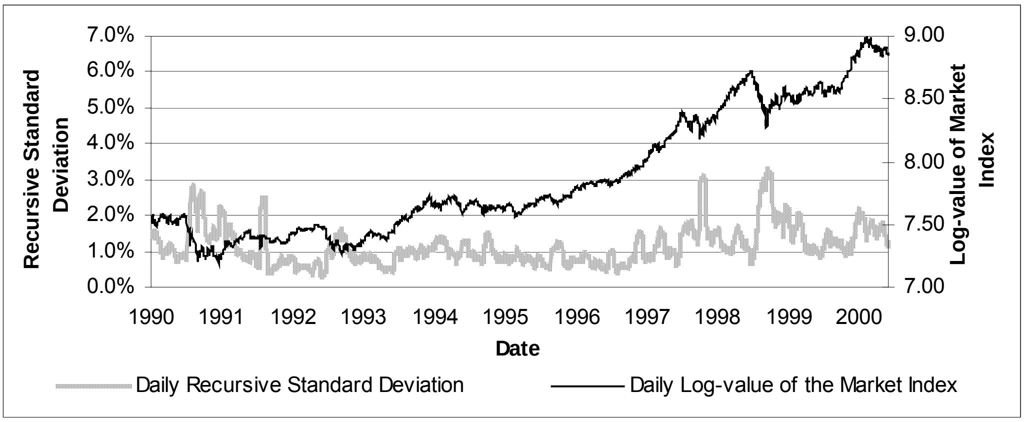


Figure 2.7. Daily Log-values and Recursive Standard Deviation of Returns for the ASE Stock Market.

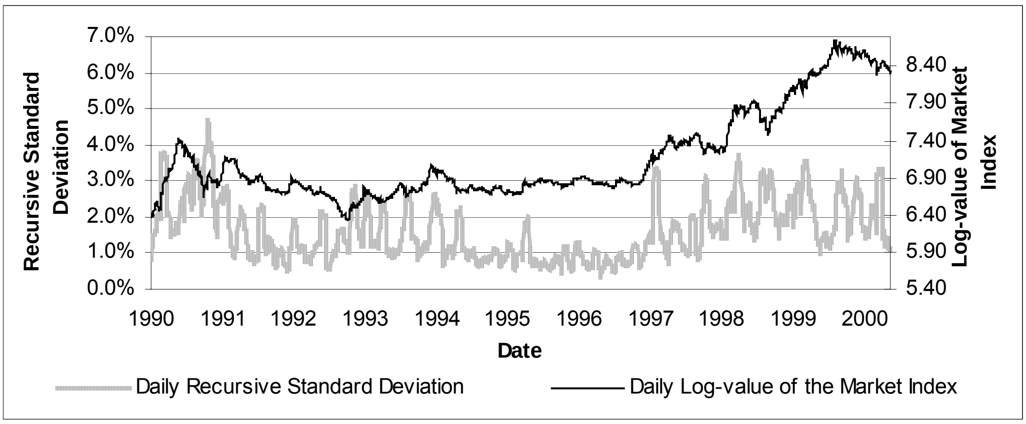


Table 2.1. Mean and Annualized Standard Deviation of the S&P500, DAX 30 and ASE Index Returns.

	Overall	Monday	Tuesday	Wednesday	Thursday	Friday
			S&P500			
Mean	0.05%	0.12%	0.06%	0.07%	-0.01%	0.04%
St. Deviation	14.80%	15.84%	15.43%	12.57%	14.81%	15.22%
N. of observations	2649	505	543	541	532	528
			DAX 30			
Mean	0.05%	0.07%	0.04%	0.09%	0.00%	0.06%
St. Deviation	20.34%	23.91%	19.79%	18.74%	19.49%	19.46%
N. of observations	2625	518	537	530	516	524
			ASE 500			
Mean	0.08%	0.12%	-0.01%	0.06%	-0.01%	0.26%
St. Deviation	30.27%	39.06%	30.60%	25.98%	28.68%	25.16%
N. of observations	2548	494	523	517	519	495

Annualized standard deviation is computed by multiplying the standard deviation of daily returns by $252^{1/2}$, the square root of the number of trading days per year.

We can observe the phenomenon of "leverage effect" by plotting the market prices and their volatility. As a naïve estimate of volatility at day t, the standard deviation of the 22 most recent trading days, $\sigma_t^{(22)} = \sqrt{\sum_{i=t-22}^{t}\left(y_i - \left(\sum_{i=t-22}^{t} y_i \big/ 22\right)\right)^2 \big/ 22}$, is used. Figures 2.5 to 2.7 plot daily log-values of stock market indices and the relevant standard deviations of the continuously compounded returns. The periods of market drops are characterized by a high increase in volatility.

2.1.2 The Non-trading Period Effect

Financial markets appear to be affected by the accumulation of information during non-trading periods as reflected in the prices when the markets reopen following a close. As a result, the variance of returns displays a tendency to increase. This is known as the "non-trading period effect". It is worth noting that the increase in the variance of returns is not nearly proportional to the market close duration as would be anticipated if the information accumulation rate were constant over time. In fact, as Fama (1965) and French and Roll (1986) observed, information accumulates at a lower rate when markets are closed than when they are open. Also, as reflected by the findings of French and Roll (1986) and Baillie and Bollerslev (1989), the returns variance tends to be higher following weekends and holidays than on other days, but not by as much as it would be under a constant news arrival rate. Table 2.1 shows the annualized standard deviations of stock market returns for each day for the indices S&P500, DAX30 and

ASE. The standard deviation on Monday is higher than on other days, mainly for the DAX 30 and ASE indices.

2.1.3 Non-synchronous Trading Effect

The fact that the values of time series are often taken to have been recorded at time intervals of one length when in fact they were recorded at time intervals of other, not necessarily regular, length is an important factor affecting the return series with an effect known as the "non-synchronous trading effect" (see, e.g. Campbell et al. (1997)). For example, the daily prices of securities, usually analyzed, are the closing prices. The closing price of a security is the price at which the last transaction occurred. The last transaction of each security is not implemented at the same time each day. So, it is falsely assumed that the daily prices are equally spaced at 24-hour intervals. The importance of non-synchronous trading was first recognized by Fisher (1966) and further developed by many researchers such as Atchison et al. (1987), Cohen et al. (1978), Cohen et al. (1979, 1983), Dimson (1979), Lo and MacKinlay (1988, 1990a, 1990b), Scholes and Williams (1977).

Non-synchronous trading in the stocks making up an index induces autocorrelation in the return series, primarily when high frequency data are used. To control this, Scholes and Williams (1977) suggested a first order moving average $[MA(1)]$ form for index returns, while Lo and MacKinlay (1988) suggested a first order autoregressive $[AR(1)]$ form. Nelson (1991) wrote "as a practical matter, there is little difference between an $AR(1)$ and an $MA(1)$ when the AR and MA coefficients are small and the autocorrelations at lag one are equal, since the higher-order autocorrelations die out very quickly in the AR model".

2.2. The Autoregressive Conditional Heteroscedasticity (ARCH) Process

Autoregressive Conditional Heteroscedasticity (ARCH) models have been widely used in financial time series analysis and particularly in analyzing the risk of holding an asset, evaluating the price of an option, forecasting time varying confidence intervals and obtaining more efficient estimators under the existence of heteroscedasticity.

Let $\{y_t(\theta)\}$ refer to the univariate discrete time real-valued stochastic process to be predicted (e.g. the rate of return of a particular stock or market portfolio from time $t-1$ to t) where θ is a vector of unknown parameters and $E(y_t(\theta)\,|\,I_{t-1}) \equiv E_{t-1}(y_t(\theta)) \equiv \mu_t(\theta)$ denotes the conditional mean given the information set I_{t-1} (sigma-field) available in time $t-1$. The innovation process for the conditional mean, $\{\varepsilon_t(\theta)\}$, is then given by $\varepsilon_t(\theta) = y_t(\theta) - \mu_t(\theta)$ with corresponding unconditional variance $V(\varepsilon_t(\theta)) = E(\varepsilon_t^2(\theta)) \equiv \sigma^2(\theta)$, zero unconditional mean and $E(\varepsilon_t(\theta)\varepsilon_s(\theta)) = 0$, $\forall t \neq s$. The conditional variance of the process given I_{t-1} is defined by $V(y_t(\theta)\,|\,I_{t-1}) \equiv V_{t-1}(y_t(\theta)) \equiv E_{t-1}(\varepsilon_t^2(\theta)) \equiv \sigma_t^2(\theta)$. Since investors would know the information set I_{t-1} when they make their investment decisions at time $t-1$, the relevant expected return to the investors and volatility are $\mu_t(\theta)$ and $\sigma_t^2(\theta)$, respectively.

An ARCH process, $\{\varepsilon_t(\theta)\}$, can be presented as:

$$\begin{gathered}
\varepsilon_t(\theta) = z_t\sigma_t(\theta) \\
z_t \overset{i.i.d.}{\sim} f\left[E(z_t) = 0, V(z_t) = 1\right] \\
\sigma_t^2(\theta) = g(\sigma_{t-1}(\theta), \sigma_{t-2}(\theta), \ldots; \varepsilon_{t-1}(\theta), \varepsilon_{t-2}(\theta), \ldots; \upsilon_{t-1}, \upsilon_{t-2}, \ldots),
\end{gathered} \tag{2.2.1}$$

where $E(z_t) = 0$, $V(z_t) = 1$, $f(.)$ is the density function of z_t, $\sigma_t(\theta)$ is a time-varying, positive and measurable function of the information set at time $t-1$, υ_t is a vector of predetermined variables included in I_t, and $g(.)$ is a linear or nonlinear functional form. By definition, $\varepsilon_t(\theta)$ is serially uncorrelated with mean zero, but with a time varying conditional variance equal to $\sigma_t^2(\theta)$. The conditional variance is a linear or nonlinear function of lagged values of σ_t and ε_t, and predetermined variables $(\upsilon_{t-1}, \upsilon_{t-2}, \ldots)$ included in I_{t-1}. In the sequel, for notational convenience, no explicit indication of the dependence on the vector of parameters, θ, is given when obvious from the context.

Since very few financial time series have a constant conditional mean of zero, an ARCH model can be presented in a regression form by letting ε_t be the innovation process in a linear regression:

$$\begin{gathered} y_t = x_t'\beta + \varepsilon_t \\ \varepsilon_t \mid I_{t-1} \sim f\left(0, \sigma_t^2\right) \\ \sigma_t^2 = g\left(\sigma_{t-1}(\theta), \sigma_{t-2}(\theta), \ldots; \varepsilon_{t-1}(\theta), \varepsilon_{t-2}(\theta), \ldots; \upsilon_{t-1}, \upsilon_{t-2}, \ldots\right), \end{gathered} \tag{2.2.2}$$

where x_t is a $k \times 1$ vector of endogenous and exogenous explanatory variables included in the information set I_{t-1} and β is a $k \times 1$ vector of unknown parameters.

2.2.1 ARCH Models

In the literature, one can find a large number of specifications of ARCH models that have been considered for the description of the characteristics of financial markets. A wide range of proposed ARCH processes is covered in surveys such as Andersen and Bollerslev (1998c), Bera and Higgins (1993), Bollerslev et al. (1992), Bollerslev et al. (1994), Gouriéroux (1997), Li et al. (2001) and Palm (1996). A good account of the state of the art up to 1995 can be found in Engle (1995).

Engle (1982) introduced the original form of $\sigma_t^2 = g(.)$, in equation (2.2.1), as a linear function of the past q squared innovations:

$$\sigma_t^2 = a_0 + \sum_{i=1}^{q}\left(a_i \varepsilon_{t-i}^2\right). \tag{2.2.3}$$

For the linear ARCH(q) process to be well defined and the conditional variance to be positive, almost surely the parameters must satisfy $a_0 > 0$, $a_i \geq 0$, for $i = 1, \ldots, q$. An equivalent representation of the ARCH(q) process is given by:

$$\sigma_t^2 = a_0 + A(L)\varepsilon_t^2, \tag{2.2.4}$$

where L denotes the lag operator and $A(L) = \left(a_1 L + a_2 L^2 + \ldots + a_q L^q\right)$. Defining $v_t = \varepsilon_t^2 - \sigma_t^2$, the model is rewritten as:

$$\varepsilon_t^2 = a_0 + A(L)\varepsilon_t^2 + v_t. \tag{2.2.5}$$

By its definition, v_t is serially uncorrelated with $E_{t-1}(v_t) = 0$ but neither independently nor identically distributed. The ARCH(q) model is interpreted as an autoregressive process in the squared innovations and is covariance stationary if and only if the roots of $\sum_{i=1}^{q}\left(a_i L^i\right) = 1$ lie outside the unit circle, or, equivalently, the sum of the positive autoregressive parameters is less than one. If the process is covariance stationary, its unconditional variance is equal to $V(\varepsilon_t) \equiv \sigma^2 = a_0\left(1-\sum_{i=1}^{q}(a_i)\right)^{-1}$.

Also, by definition, the innovation process is serially uncorrelated but not independently distributed. On the other hand, the standardized innovations are time invariant distributed. Thus, the unconditional distribution for the innovation process will have fatter tails than the distribution for the standardized innovations. For example, consider the kurtosis for the ARCH(1) process with conditional normally distributed innovations is $E(\varepsilon_t^4)/E(\varepsilon_t^2)^2 = 3(1-\alpha_1^2)/(1-3\alpha_1^2)$ if $3a_1^2 < 1$, and $E(\varepsilon_t^4)/E(\varepsilon_t^2)^2 = \infty$ otherwise, i.e., greater than 3, the kurtosis value of the normal distribution. Generally speaking, an ARCH process always has fatter tails than the normal distribution:

$$E(\varepsilon_t^4)/E(\varepsilon_t^2)^2 = E(\sigma_t^4 z_t^4)/E(\sigma_t^2 z_t^2)^2 = 3E(\sigma_t^4)/E(\sigma_t^2)^2 \geq 3E(\sigma_t^2)^2/E(\sigma_t^2)^2 ,$$

where the first equality comes from the independence of σ_t and z_t, and the inequality is implied by Jensen's inequality.

In empirical applications of the ARCH(q) model, a relatively long lag in the conditional variance equation is often called for, and to avoid problems of negative variance parameter estimates a fixed lag structure is typically imposed (see, for example, Engle (1982, 1983), and Engle and Kraft (1983)). To circumvent this problem, Bollerslev (1986) proposed a generalization of the ARCH(q) process to allow for past conditional variances in the current conditional variance equation, the generalized ARCH, or GARCH(p,q), model:

$$\sigma_t^2 = a_0 + \sum_{i=1}^{q}\left(a_i\varepsilon_{t-i}^2\right) + \sum_{j=1}^{p}\left(b_j\sigma_{t-j}^2\right) = a_0 + A(L)\varepsilon_t^2 + B(L)\sigma_t^2 . \qquad (2.2.6)$$

For $a_0 > 0$, $a_i \geq 0$, $i = 1,...,q$ and $b_j \geq 0$, $j = 1,...,p$, the conditional variance is well defined. Taylor (1986) independently proposed the GARCH model using a different acronym. Nelson and Cao (1992) showed that the non-negativity constraints on the parameters of the process could be substantially weakened, so they should not be imposed in estimation. Provided that the roots of $B(L) = 1$ lie outside the unit circle and the polynomials $1 - B(L)$ and $A(L)$ have no common roots, the positivity constraint is satisfied if all the coefficients in the infinite power series expansion for $B(L)(1-B(L))^{-1}$ are non-negative. In the GARCH(1,2) model, for example, the conditions of non-negativity are that $a_0 \geq 0$, $0 \leq b_1 < 1$, $a_1 \geq 0$ and $b_1 a_1 + a_2 \geq 0$. In the GARCH(2,1) model, the necessary conditions require that $a_0 \geq 0$, $b_1 \geq 0$, $a_1 \geq 0$, $b_1 + b_2 < 1$ and $b_1^2 + 4b_2 \geq 0$. Thus, slightly negative values of parameters, for higher order lags, do not result in negative conditional variance. Rearranging the GARCH(p,q) model, it can be presented as an autoregressive moving average process in the squared innovations of orders $\max(p,q)$ and p, $[ARMA(\max(p,q),p)]$, respectively:

$$\varepsilon_t^2 = a_0 + \sum_{i=1}^{q}\left(a_i \varepsilon_{t-i}^2\right) + \sum_{j=1}^{p}\left(b_j \varepsilon_{t-j}^2\right) - \sum_{j=1}^{p}\left(b_j v_{t-j}\right) + v_t . \qquad (2.2.7)$$

The model is second order stationary if the roots of $A(L) + B(L) = 1$ lie outside the unit circle, or equivalently if $\sum_{i=1}^{q} a_i + \sum_{j=1}^{p} b_j < 1$. Its unconditional variance is equal to $\sigma^2 = a_0 \left(1 - \sum_{i=1}^{q} a_i - \sum_{j=1}^{p} b_j\right)^{-1}$.

Very often, in connection with applications, the estimate for $A(L) + B(L)$ turns out to be very close to unity. This provided an empirical motivation, for the development of the so-called integrated GARCH(p,q) or IGARCH(p,q) model by Engle and Bollerslev (1986):

$$\sigma_t^2 = a_0 + A(L)\varepsilon_t^2 + B(L)\sigma_t^2 \text{, for } A(L) + B(L) = 1 , \qquad (2.2.8)$$

where the polynomial $A(L) + B(L) = 1$ has $d > 0$ unit roots and $\max(p,q) - d$ roots outside the unit circle.

Moreover, Nelson (1990a) showed that the GARCH(1,1) model is strictly stationary even if $a_1 + b_1 > 1$, as long as $E\left(\log\left(b_1 + a_1 z_t^2\right)\right) < 0$. Thus, the conditional variance in IGARCH(1,1) with $a_0 = 0$, collapses to zero almost surely, and in IGARCH(1,1) with $a_0 > 0$ is strictly stationary. Therefore, a process that is integrated in the mean is not stationary in any sense, while an IGARCH process is strictly stationary but covariance non-stationary.

Consider the IGARCH(1,1) model, $\sigma_t^2 = a_0 + a_1 \varepsilon_{t-1}^2 + (1 - a_1)\sigma_{t-1}^2$, where $0 < a_1 < 1$. The conditional variance h-steps in the future takes the form:

$$E_t\left(\sigma_{t+h}^2\right) = \sigma_{t+h|t}^2 = \sigma_t^2 + h a_0 , \qquad (2.2.9)$$

which looks very much like a linear random walk with drift a_0. A linear random walk is strictly non-stationary (no stationary distribution and covariance non-stationary) and it has no unconditional first or second moments. In the case of IGARCH(1,1), the conditional variance is strictly stationary even though its stationary distribution generally lacks unconditional moments. In the case where $a_0 = 0$, equation (2.2.9) reduces to $\sigma^2_{t+h|t} \equiv \sigma^2_t$, a bounded martingale as it cannot take negative values. According to the martingale convergence theorem (Dudley (1989)), a bounded martingale must converge, and, in this case, the only value to which it can converge is zero. Thus, the stationary distributions for σ^2_t and ε_t have moments, but they are all trivially zero. In the case of $a_0 > 0$, Nelson (1990a) showed that there is a non-degenerate stationary distribution for the conditional variance, but with no finite mean or higher moments. The innovation process ε_t then has a stationary distribution with zero mean, but with tails that are so thick that no second or higher order moments exist. Furthermore, if the variable z_t follows the standard normal distribution, Nelson (1990a) showed that:

$$\begin{aligned} E\left(\ln\left(b_1 + a_1 z_t^2\right)\right) &= \ln(2a_1) + \psi(1/2) \\ &\quad + \left(2\pi b_1 a_1^{-1}\right)^{1/2} \Phi\left(0.5;1.5;b_1/2a_1\right) - \left(b_1/a_1\right)_2 F_2\left(1,1;2,1.5;b_1/2a_1\right), \end{aligned} \qquad (2.2.10)$$

where $\psi(.)$ denotes the Euler Psi function, with $\psi(1/2) \approx -1.96351$ (Davis (1965)), $\Phi(.;.;.)$ the confluent hypergeometric function (Lebedev (1972)), and ${}_2F_2(.,.;.,.;.)$ the generalized hypergeometric function (Lebedev (1972)). Bougerol and Picard (1992) extended Nelson's work and showed that the general GARCH(p,q) model is strictly stationary and ergodic. Choudhry (1995), by means of the IGARCH(1,1) model, studied the persistence of stock return volatility in European markets during the 1920's and 1930's and argued that the 1929 stock market crash did not reduce stock market volatility. Using monthly stock returns from 1919 to 1936 in markets of Czechoslovakia, France, Italy, Poland and Spain, Choudhry mentioned that in the GARCH(1,1) model the sum of a_1 and b_1 approaches unity, which implies persistence of a forecast of the conditional variance over all finite horizons.

The GARCH(p,q) model successfully captures several characteristics of financial time series, such as thick tailed returns and volatility clustering. On the other hand, its structure imposes important limitations. The variance only depends on the magnitude and not the sign of ε_t, which is somewhat at odds with the empirical behavior of stock market prices where the "leverage effect" may be present. The models that have been considered so far are symmetric in that only the magnitude and not the positivity or negativity of innovations determines σ_t^2. In order to capture the asymmetry manifested by the data, a new class of models, in which good news and bad news have different predictability for future volatility, was introduced.

The most popular method proposed to capture the asymmetric effects is Nelson's (1991) exponential GARCH, or EGARCH, model. He proposed the following form for the evolution of the conditional variance:

$$\log\left(\sigma_t^2\right) = a_0 + \sum_{i=1}^{\infty} \pi_i g\left(\frac{\varepsilon_{t-i}}{\sigma_{t-i}}\right), \quad \pi_1 \equiv 1, \tag{2.2.11}$$

and accommodated the asymmetric relation between stock returns and volatility changes by making $g(\varepsilon_t/\sigma_t)$ a linear combination of $|\varepsilon_t/\sigma_t|$ and ε_t/σ_t:

$$g(\varepsilon_t/\sigma_t) \equiv \theta\left(|\varepsilon_t/\sigma_t| - E|\varepsilon_t/\sigma_t|\right) + \gamma(\varepsilon_t/\sigma_t), \tag{2.2.12}$$

where θ and γ are constants. By construction, equation (2.2.12) is a zero mean i.i.d. sequence (note that $z_t \equiv \varepsilon_t/\sigma_t$). Over the range $0 < z_t < \infty$, $g(z_t)$ is linear in z_t with slope $\theta + \gamma$ and over the range $-\infty < z_t \leq 0$, $g(z_t)$ is linear with slope $\gamma - \theta$. The first term of (2.2.12), $\theta\left(|z_t| - E|z_t|\right)$, represents the magnitude effect as in the GARCH model, while the second term, $\gamma(z_t)$, represents the leverage effect. To make this tangible, assume that $\theta > 0$ and $\gamma = 0$. The innovation in $\log\left(\sigma_t^2\right)$ is then positive (negative) when the magnitude of z_t is larger (smaller) than its expected value. Assume now that $\theta = 0$ and $\gamma < 0$. In this case the innovation in $\log\left(\sigma_t^2\right)$ is positive (negative) when innovations are negative (positive). Moreover, the conditional variance is positive regardless of whether the π_i coefficients are positive. Thus, in contrast to GARCH models, no inequality constraints need to be imposed for estimation. Nelson (1991) showed that $\log\left(\sigma_t^2\right)$ and ε_t are strictly stationary as long as $\sum_{i=1}^{\infty} \pi_i^2 < \infty$. A natural parameterization is to model the infinite moving average representation of equation (2.2.11) as an autoregressive moving average model:

$$\ln\left(\sigma_t^2\right) = a_0 + \left(1 + \sum_{i=1}^{q} a_i L^i\right)\left(1 - \sum_{j=1}^{p} b_j L^j\right)^{-1} \left(\theta\left(\left|\varepsilon_{t-1}/\sigma_{t-1}\right| - E\left|\varepsilon_{t-1}/\sigma_{t-1}\right|\right) + \gamma\left(\varepsilon_{t-1}/\sigma_{t-1}\right)\right), \quad (2.2.13)$$

or equivalently:

$$\ln\left(\sigma_t^2\right) = a_0 + \left(1 + A(L)\right)\left(1 - B(L)\right)^{-1} g\left(z_{t-1}\right). \quad (2.2.13b)$$

Another popular way to model the asymmetry of positive and negative innovations is the use of indicator functions. Glosten et al. (1993) presented the GJR(p,q) model:

$$\sigma_t^2 = a_0 + \sum_{i=1}^{q}\left(a_i \varepsilon_{t-i}^2\right) + \sum_{i=1}^{q}\left(\gamma_i d\left(\varepsilon_{t-i} < 0\right)\varepsilon_{t-i}^2\right) + \sum_{j=1}^{p}\left(b_j \sigma_{t-j}^2\right), \quad (2.2.14)$$

where γ_i, for $i = 1,...,q$, are parameters that have to be estimated, $d(.)$ denotes the indicator function (i.e. $d\left(\varepsilon_{t-i} < 0\right) = 1$ if $\varepsilon_{t-i} < 0$, and $d\left(\varepsilon_{t-i} < 0\right) = 0$ otherwise). The GJR model allows good news, $\left(\varepsilon_{t-i} > 0\right)$, and bad news, $\left(\varepsilon_{t-i} < 0\right)$, to have differential effects on the conditional variance. Therefore, in the case of the GJR(0,1) model, good news has an impact of a_1, while bad news has an impact of $a_1 + \gamma_1$. For $\gamma_1 > 0$, the "leverage effect" exists.

A similar way to model asymmetric effects on the conditional standard deviation was introduced by Zakoian (1990), and developed further in Rabemananjara and Zakoian (1993), by defining the threshold GARCH, or TGARCH(p,q), model:

$$\sigma_t = a_0 + \sum_{i=1}^{q}\left(a_i \varepsilon_{t-i}^{+}\right) - \sum_{i=1}^{q}\left(\gamma_i \varepsilon_{t-i}^{-}\right) + \sum_{j=1}^{p}\left(b_j \sigma_{t-j}\right), \quad (2.2.15)$$

where $\varepsilon_t^{+} \equiv \varepsilon_t$ if $\varepsilon_t > 0$, $\varepsilon_t^{+} \equiv 0$ otherwise and $\varepsilon_t^{-} \equiv \varepsilon_t - \varepsilon_t^{+}$.

Engle and Ng (1993) recommended the "news impact curve" as a measure of how news is incorporated into volatility estimates by alternative ARCH models. In their recent comparative study of the EGARCH model to the GJR model, Friedmann and Sanddorf-Köhle (2002) proposed a modification of the news impact curve termed the "conditional news impact curve". Engle and Ng argued that the GJR model is better than the EGARCH model because the conditional variance implied by the latter is too high due to its exponential functional form. On the other hand, Friedmann and Sanddorf-Köhle (2002) argued that the EGARCH model does not overstate the predicted volatility.

The number of formulations presented in the financial and econometric literature is vast. In the sequel, the best known variations of ARCH modeling are presented.

Taylor (1986) and Schwert (1989a,b) assumed that the conditional standard deviation is a distributed lag of absolute innovations, and introduced the absolute GARCH, or AGARCH(p,q), model:

$$\sigma_t = a_0 + \sum_{i=1}^{q} a_i \left|\varepsilon_{t-i}\right| + \sum_{j=1}^{p} b_j \sigma_{t-j} \ . \tag{2.2.16}$$

Geweke (1986), Pantula (1986) and Milhøj (1987) suggested a specification in which the log of the conditional variance depends linearly on past logs of squared innovations. Their model is the multiplicative ARCH, or Log-GARCH(p,q), model defined by

$$\ln\left(\sigma_t^2\right) = a_0 + \sum_{i=1}^{q} a_i \ln\left(\varepsilon_{t-i}^2\right) + \sum_{j=1}^{p} b_j \ln\left(\sigma_{t-j}^2\right). \tag{2.2.17}$$

Schwert (1990) built the autoregressive standard deviation, or Stdev-ARCH(q), model:

$$\sigma_t^2 = \left(a_0 + \sum_{i=1}^{q} a_i \left|\varepsilon_{t-i}\right| \right)^2 . \tag{2.2.18}$$

Higgins and Bera (1992) introduced the non-linear ARCH, or NARCH(p,q), model:

$$\sigma_t^\delta = a_0 + \sum_{i=1}^{q} a_i \left|\varepsilon_{t-i}^2\right|^{\delta/2} + \sum_{j=1}^{p} b_j \sigma_{t-j}^\delta \ , \tag{2.2.19}$$

while Engle and Bollerslev (1986) proposed a simpler non-linear ARCH model:

$$\sigma_t^2 = a_0 + a_1 \left|\varepsilon_{t-1}\right|^\delta + b_1 \sigma_{t-1}^2 . \tag{2.2.20}$$

In order to introduce asymmetric effects, Engle (1990), proposed the asymmetric GARCH, or AGARCH(p,q), model:

$$\sigma_t^2 = a_0 + \sum_{i=1}^{q} \left(a_i \varepsilon_{t-i}^2 + \gamma_i \varepsilon_{t-i} \right) + \sum_{j=1}^{p} b_j \sigma_{t-j}^2 \ , \tag{2.2.21}$$

where a negative value of γ_i means that positive returns increase volatility less than negative returns. Moreover, Engle and Ng (1993) presented two more ARCH models that incorporate asymmetry for good and bad news, the non-linear asymmetric GARCH, or NAGARCH(p,q), model:

$$\sigma_t^2 = a_0 + \sum_{i=1}^{q} a_i \left(\varepsilon_{t-i} + \gamma_i \sigma_{t-i} \right)^2 + \sum_{j=1}^{p} b_j \sigma_{t-j}^2 \ , \tag{2.2.22}$$

and the VGARCH(p,q) model:

$$\sigma_t^2 = a_0 + \sum_{i=1}^{q} a_i \left(\varepsilon_{t-i} / \sigma_{t-i} + \gamma_i \right)^2 + \sum_{j=1}^{p} b_j \sigma_{t-j}^2 \ . \tag{2.2.23}$$

Ding et al. (1993) introduced the asymmetric power ARCH, or APARCH(p,q), model, which includes seven ARCH models as special cases (ARCH, GARCH, AGARCH, GJR, TARCH, NARCH and logARCH):

$$\sigma_t^{\delta} = a_0 + \sum_{i=1}^{q} a_i \left(\left|\varepsilon_{t-1}\right| - \gamma_i \varepsilon_{t-i}\right)^{\delta} + \sum_{j=1}^{p} b_j \sigma_{t-j}^{\delta}, \tag{2.2.24}$$

where $a_0 > 0$, $\delta \geq 0$, $b_j \geq 0$, $j = 1,...,p$, $a_i \geq 0$ and $-1 < \gamma_i < 1$, $i = 1,...,q$. The model imposes a Box and Cox (1964) power transformation of the conditional standard deviation process and the asymmetric absolute innovations. The functional form for the conditional standard deviation is familiar to economists as the constant elasticity of substitution (CES) production function. Ling and McAleer (2001) provided sufficient conditions for the stationarity and ergodicity of the APARCH(p,q), model. Brooks et al. (2000) applied the APARCH(1,1) model for 10 series of national stock market index returns. The optimal power transformation was found to be remarkably similar across countries.

Sentana (1995) introduced the quadratic GARCH, or GQARCH(p,q), model of the form:

$$\sigma_t^2 = a_0 + \sum_{i=1}^{q} a_i \varepsilon_{t-i}^2 + \sum_{i=1}^{q} \gamma_i \varepsilon_{t-i} + 2\sum_{i=1}^{q} \sum_{j=i+1}^{q} a_{ij} \varepsilon_{t-i} \varepsilon_{t-j} + \sum_{j=1}^{p} b_j \sigma_{t-j}^2 . \tag{2.2.25}$$

Setting $\gamma_i = 0$, for $i = 1,...,q$, leads to the Augmented ARCH model of Bera and Lee (1990). It does encompass all the ARCH models of quadratic variance functions, but it does not include models in which the variance is quadratic in the absolute value of innovations, as the APARCH model.

Hentschel (1995) gave a complete parametric family of ARCH models. This family nests the most popular symmetric and asymmetric ARCH models, thereby highlighting the relation between the models and their treatment of asymmetry. Hentschel presents the variance equation as:

$$\frac{\sigma_t^{\lambda} - 1}{\lambda} = \omega + a\sigma_{t-1}^{\lambda} f^{v}(\varepsilon_t) + \beta \frac{\sigma_{t-1}^{\lambda} - 1}{\lambda}, \tag{2.2.26}$$

where $f(.)$ denotes the absolute value function of innovations,

$$f(\varepsilon_t) = \left|\varepsilon_t - \beta\right| - \zeta(\varepsilon_t - \beta). \tag{2.2.27}$$

In general, this is a law of the Box-Cox transformation of the conditional standard deviation (as in the case of the APARCH model), and the parameter λ determines the shape of the transformation. For $\lambda > 1$, the transformation of σ_t is convex, while for $\lambda < 1$, it is concave. The parameter v serves to transform the absolute value function. For different restrictions on the parameters in equations (2.2.26) and (2.2.27), almost all the popular symmetric and asymmetric ARCH models are obtained. For example, for $\lambda = 0$, $v = 1$, $\beta = 1$ and free ζ, we obtain Nelson's exponential GARCH model. However, some models, as Sentana's quadratic model, are excluded.

Gouriéroux and Monfort (1992) proposed the qualitative threshold GARCH, or GQTARCH(p,q), model with the following specification:

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \sum_{j=1}^{J} a_{ij} I_j \left(\varepsilon_{t-i}\right) + \sum_{i=1}^{p} b_j \sigma_{t-j}^2 \ . \tag{2.2.28}$$

Assuming constant conditional variance over various observation intervals, Gouriéroux and Monfort (1992) divided the space of ε_t into J intervals and let $I_j\left(\varepsilon_t\right)$ be 1 if ε_t is in the j^{th} interval.

Another important class of models, proposed independently by Cai (1994) and Hamilton and Susmel (1994), is the class of regime switching ARCH models, a natural extension of regime-switching models for the conditional mean, introduced by Hamilton (1989). These models allow the parameters of the ARCH process to come from one of several different regimes, with transitions between regimes governed by an unobserved Markov chain. Let $\tilde{\varepsilon}_t$ be the innovation process and let s_t denote an unobserved random variable that can take on the values $1,2,...,K$. Suppose that s_t can be described by a Markov chain, $P(s_t = j \mid s_{t-1} = i, s_{t-2} = k, ..., \tilde{\varepsilon}_{t-1}, \tilde{\varepsilon}_{t-2}, ...) = p_{ij}$, for $i, j = 1,2,...,K$. The idea is to model the innovation process, $\tilde{\varepsilon}_t$, as $\tilde{\varepsilon}_t \equiv \sqrt{g_{s_t}} \varepsilon_t$, where ε_t is assumed to follow an ARCH process. So, the underlying ARCH variable, ε_t, is multiplied by the constant $\sqrt{g_1}$ when the process is in the regime presented by $s_t = 1$, is multiplied by $\sqrt{g_2}$ when $s_t = 2$, and so on. The factor for the first stage, g_1, is normalized at unity with $g_j \geq 1$ for $j = 2,3,...,K$. The idea is, thus, to model changes in regime as changes in the scale of the process. Dueker (1997) and Hansen (1994) extended the approach to GARCH models.

Fornari and Mele (1995) introduced the volatility-switching ARCH model, or VSARCH(p,q), model:

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} a_i \varepsilon_{t-i}^2 + \gamma S_{t-1} \frac{\varepsilon_{t-1}^2}{\sigma_{t-1}^2} + \sum_{j=1}^{p} b_j \sigma_{t-j}^2 \,, \tag{2.2.29}$$

where S_t is an indicator factor that equals one if $\varepsilon_t > 0$, minus one if $\varepsilon_t < 0$, and $\varepsilon_t^2 / \sigma_t^2$ measures the difference between the forecast of the volatility at time t on the basis of the information set dated at $t-1$, σ_t^2, and the realized value ε_t^2. As Fornari and Mele (1995) mentioned, the volatility-switching model is able to capture a phenomenon that has not been modeled before. It implies that asymmetries can become inverted, with positive innovations inducing more volatility than negative innovations of the same size when the observed value of the conditional variance is lower than expected. Fornari and Mele (1996) built a mixture of the GJR and the VSARCH models, named it asymmetric volatility-switching ARCH, or AVSARCH(p,q), model and estimated it for $p = q = 1$:

$$\sigma_t^2 = a_0 + a_1 \varepsilon_{t-1}^2 + b_1 \sigma_{t-1}^2 + \gamma S_{t-1} \varepsilon_{t-1}^2 + \delta\left(\left(\varepsilon_{t-1}^2 / \sigma_{t-1}^2\right) - k\right) S_{t-1} \,. \tag{2.2.30}$$

The first four terms are the GJR(1,1) model, except that S_t is a dummy that equals one or minus one instead of zero or one, respectively. The last term captures the reversal of asymmetry observed when $\varepsilon_{t-1}^2 / \sigma_{t-1}^2$ reaches k, the threshold value. Note that the AVSARCH model is able to generate kurtosis higher than the GARCH or GJR models.

Hagerud (1996), inspired by the Smooth Transition Autoregressive (STAR) model of Luukkonen et al. (1988), proposed the smooth transition ARCH model. In the STAR model, the conditional mean is a non-linear function of lagged realizations of the series introduced via a transition function. The smooth transition GARCH(p,q) model has the form:

$$\sigma_t^2 = a_0 + \sum_{i=1}^{q} \left(a_i + \gamma_i F(\varepsilon_{t-i})\right) \varepsilon_{t-i}^2 + \sum_{j=1}^{p} b_j \sigma_{t-j}^2 \,, \tag{2.2.31}$$

where $F(.)$ is either the logistic or the exponential transition function, the two most commonly used transition functions for STAR models (for details see Teräsvirta (1994)). The logistic function considered is

$$F(\varepsilon_{t-i}) = \left(1 + \exp(-\theta \varepsilon_{t-i})\right)^{-1} - 0.5 \,, \text{ for } \theta > 0 \,, \tag{2.2.32}$$

and the exponential function is

$$F(\varepsilon_{t-i}) = 1 - \exp\left(-\theta \varepsilon_{t-i}^2\right), \text{ for } \theta > 0 \,. \tag{2.2.33}$$

The two resulting models termed logistic and exponential smooth transition GARCH, or LST-GARCH(p,q) and EST-GARCH(p,q), models, respectively. The smooth transition models allow for the possibility of intermediate positions between different regimes. For $-\infty < \varepsilon_t < \infty$, the logistic transition function takes values in $-0.5 \le F(.) \le 0.5$ and generates data where the dynamics of the conditional variance differ depending on the sign of innovations. On the other hand, the exponential function generates a return process for which the dynamics of the conditional variance depend on the magnitude of the innovations, as for $|\varepsilon_t| \to \infty$ the transition function will be equal to unity, and when $\varepsilon_t = 0$ the transition function is equal to zero. Thus, contrary to the regime switching models, the transition between states is smooth as the conditional variance is a continuous function of innovations. A model similar to the LST-GARCH model was independently proposed by González-Rivera (1996). Recently, Nam et al. (2002) provided an application of a smooth transition ARCH model with a logistic function in the following form

$$\sigma_t^2 = a_0 + a_1\varepsilon_{t-1}^2 + a_2\sigma_{t-1}^2 + \left(b_0 + b_1\varepsilon_{t-1}^2 + b_2\sigma_{t-1}^2\right)F(\varepsilon_{t-1})$$
$$F(\varepsilon_{t-1}) = \left(1 + \exp(-\theta\varepsilon_{t-1})\right)^{-1},$$

which they termed asymmetric nonlinear smooth transition GARCH, or ANST-GARCH model. Nam et al. explored the asymmetric reverting property of short-horizon expected returns and have found that the asymmetric return reversals can be exploited for the contrarian profitability[1]. Note that when $b_0 = b_2 = 0$ the ANST-GARCH model reduces to González-Rivera’s specification. Lubrano (1998) suggested an improvement over these transition functions, introducing an extra parameter, the threshold c, which determines at which magnitude of past innovations the change of regime occurs. The generalized logistic transition function is given by:

$$F(\varepsilon_{t-i}) = \frac{1 - \exp\left(-\theta\varepsilon_{t-i}^2\right)}{1 + \exp\left(-\theta\left(\varepsilon_{t-i}^2 - c^2\right)\right)}. \tag{2.2.34}$$

The exponential transition function can also be generalized in the form:

$$F(\varepsilon_{t-i}) = 1 - \exp\left(-\theta(\varepsilon_{t-i} - c)^2\right). \tag{2.2.35}$$

[1] Contrarian investment strategies are contrary to the general market direction. Interpretation of the contrarian profitability is in a debate between the two competing hypotheses: the time varying rational expectation hypothesis and the stock market overreaction hypothesis. For details see Chan (1988), Chopra et al. (1992), Conrad and Kaul (1993), DeBondt and Thaler (1985, 1987,1989), Lo and MacKinlay (1990b), Veronesi (1999), Zarowin (1990).

Engle and Lee (1993) proposed the component GARCH model in order to investigate the long-run and the short-run movement of volatility. The GARCH(1,1) model can be written as:

$$\sigma_t^2 = \sigma^2 + a_1\left(\varepsilon_{t-1}^2 - \sigma^2\right) + b_1\left(\sigma_{t-1}^2 - \sigma^2\right), \tag{2.2.36}$$

for $\sigma^2 = a_0\left(1 - a_1 - b_1\right)^{-1}$ denoting the unconditional variance. The conditional variance in the GARCH(1,1) model shows mean reversion to the unconditional variance, which is constant for all time. By contrast, the component GARCH, or CGARCH(1,1), model allows mean reversion to a time varying level q_t. The CGARCH(1,1) model is defined as:

$$\begin{aligned} \sigma_t^2 &= q_t + a_1\left(\varepsilon_{t-1}^2 - q_{t-1}\right) + b_1\left(\sigma_{t-1}^2 - q_{t-1}\right) \\ q_t &= a_0 + pq_{t-1} + \phi\left(\varepsilon_{t-1}^2 - \sigma_{t-1}^2\right). \end{aligned} \tag{2.2.37}$$

The difference between the conditional variance and its trend, $\sigma_t^2 - q_t$, is the transitory or short-run component of the conditional variance, while q_t is the time varying long-run volatility. Combining the transitory and permanent equations the model reduces to:

$$\begin{aligned} \sigma_t^2 = &\left(1 - a_1 - b_1\right)\left(1 - p\right)a_0 + \left(a_1 + \phi\right)\varepsilon_{t-1}^2 - \left(a_1 p + \left(a_1 + b_1\right)\phi\right)\varepsilon_{t-2}^2 \\ &+ \left(b_1 - \phi\right)\sigma_{t-1}^2 - \left(b_1 p - \left(a_1 + b_1\right)\phi\right)\sigma_{t-2}^2 , \end{aligned} \tag{2.2.38}$$

which shows that the CGARCH(1,1) is a restricted GARCH(2,2) model. Moreover, because of the existence of the "leverage effect", Engle and Lee (1993) combine the component model with the GJR model to allow shocks to affect the volatility components asymmetrically. The asymmetric component GARCH, or the ACGARCH(1,1), model becomes:

$$\begin{aligned} \sigma_t^2 &= q_t + a_1\left(\varepsilon_{t-1}^2 - q_{t-1}\right) + \gamma_1\left(d\left(\varepsilon_{t-1} < 0\right)\varepsilon_{t-1}^2 - 0.5q_{t-1}\right) + b_1\left(\sigma_{t-1}^2 - q_{t-1}\right) \\ q_t &= a_0 + pq_{t-1} + \phi\left(\varepsilon_{t-1}^2 - \sigma_{t-1}^2\right) + \gamma_2\left(d\left(\varepsilon_{t-1} < 0\right)\varepsilon_{t-1}^2 - 0.5\sigma_{t-1}^2\right), \end{aligned} \tag{2.2.39}$$

where $d(.)$ denotes the indicator function (i.e. $d\left(\varepsilon_{t-i} < 0\right) = 1$ if $\varepsilon_{t-i} < 0$, and $d\left(\varepsilon_{t-i} < 0\right) = 0$ otherwise).

Baillie et al. (1996), motivated by the Fractionally Integrated Autoregressive Moving Average, or ARFIMA, model, presented the Fractionally Integrated Generalized Autoregressive Conditional Heteroscedasticity, or FIGARCH, model. The ARFIMA(k,d,l) model for the discrete time real-valued process $\{y_t\}$, initially developed in Granger (1980) and Granger and Joyeux (1980), is defined as:

$$A(L)(1-L)^d y_t = B(L)\varepsilon_t\,, \qquad (2.2.40)$$

where $A(L)$ and $B(L)$ denote the lag operators of order k and l respectively, and $\{\varepsilon_t\}$ is a mean-zero serially uncorrelated process. The fractional differencing operator, $(1-L)^d$, is usually interpreted in its binomial expansion given by:

$$(1-L)^d = \sum_{j=0}^{\infty} \pi_j L^j \text{ , for } \pi_j = \frac{\Gamma(j-d)}{\Gamma(j+1)\Gamma(-d)} = \prod_{k=0}^{j} \frac{k-1-d}{k}\,, \qquad (2.2.41)$$

where $\Gamma(.)$ denotes the gamma function.

The stationary ARMA process, equation (2.2.40) for $d=0$, is a short memory process, the autocorrelations of which are geometrically bounded:

$$\left|Cor(y_t, y_{t+m})\right| \le cr^m\,,$$

for $m = 1,2,\ldots$, where $c > 0$ and $0 < r < 1$. As $m \to \infty$ the dependence, or memory, between y_t and y_{t+m} decreases rapidly. However, some observed time series appeared to exhibit a substantially larger degree of persistence than allowed for by stationary ARMA processes. For example, Ding et al. (1993) found that the absolute values or powers, particularly squares, of returns on S&P500 index tend to have very slowly decaying autocorrelations. Similar evidence of this feature for other types of financial series is contained in Dacarogna et al. (1993), Mills (1996) and Taylor (1986). Such time series have autocorrelations that seem to satisfy the condition:

$$Cor(y_t, y_{t+m}) \approx cm^{2d-1}\,,$$

as $m \to \infty$, where $c \ne 0$ and $d < 0.5$. Such processes are said to have long memory because the autocorrelations display substantial persistence.

The concept of long memory and fractional Brownian motion was originally developed by Hurst (1951) and extended by Mandelbrot (1963, 1982) and Mandelbrot and Van Ness (1968). However, the ideas became essentially applicable by Granger (1980,1981), Granger and Joyeux (1980) and Hosking (1981). Hurst was a hydrologist who worked on the Nile river dam project. He had studied an 847-years record of the Nile's overflows and observed that larger than average overflows were more likely to be followed by more large overflows. Suddenly, the water flow would change to a lower than average overflow which would be followed by lower than average overflows. Such a process could be examined neither with standard statistical correlation analysis nor by assuming that the water inflow is a random process, so it could be analyzed as a

Brownian motion. Einstein (1905) worked on Brownian motion and found that the distance a random particle covers increases with the square root of time used to measure it, or:

$$d = t^{1/2}, \tag{2.2.42}$$

where d is the distance covered and t is the time index. But this applies only to time series that are in Brownian motion, i.e. mean-zero and unity variance independent processes. Hurst generalized (2.2.42) to account for processes other than Brownian motion in the form:

$$d/s = ct^H . \tag{2.2.43}$$

For any process $\{y_t\}_{t=1}^T$ (e.g. asset returns) with mean $\bar{y}_T = T^{-1}\sum_{t=1}^{T} y_t$, d is given by

$$d = \underset{1 \le k \le T}{Max} \sum_{t=1}^{k} (y_t - \bar{y}_T) - \underset{1 \le k \le T}{Min} \sum_{t=1}^{k} (y_t - \bar{y}_T), \tag{2.2.44}$$

where s is the standard deviation of $\{y_t\}_{t=1}^T$ and c is a constant. The ratio d/s is called *rescaled range* and H is the Hurst exponent. If $\{y_t\}$ is a sequence of independently and identically distributed random variables, then $H = 0.5$. Hurst's investigations for the Nile lead to $H = 0.9$. Thus, the rescaled range was increasing at a faster rate than the square root of time.

The IGARCH(p,q) model in equation (2.2.8) could be rewritten as:

$$\Phi(L)(1-L)\varepsilon_t^2 = a_0 + (1 - B(L))v_t , \tag{2.2.45}$$

where $\Phi(L) \equiv (1 - A(L) - B(L))(1-L)^{-1}$ is of order $[\max(p,q) - 1]$. The FIGARCH model is simply obtained by replacing the first difference operator in equation (2.2.45) with the fractional differencing operator. Rearranging terms in equation (2.2.45) the FIGARCH(p,d,q) model is given as:

$$\sigma_t^2 = a_0 + \left(1 - B(L) - \Phi(L)(1-L)^d\right)\varepsilon_t^2 + B(L)\sigma_t^2 , \tag{2.2.46}$$

which is strictly stationary and ergodic for $0 \le d \le 1$. In contrast to the GARCH and IGARCH models where shocks to the conditional variance either dissipate exponentially or persist indefinitely, for the FIGARCH model the response of the conditional variance to past shocks decays at a slow hyperbolic rate. The sample autocorrelations of the daily absolute returns, or $|y_t|$, as investigated by Ding et al. (1993) and Bollerslev and Mikkelsen (1996) among others, exceed the 95% confidence intervals for no serial dependence for more than 1000 lags. Moreover, the sample autocorrelations for the first difference of absolute returns, $(1-L)|y_t|$, still show statistically significant long-term dependence. On the contrary, the fractional difference of absolute returns, $(1-L)^{0.5}|y_t|$, shows much less long-term dependence. Bollerslev and Mikkelsen (1996) provided evidence that illustrates the importance of using fractional integrated conditional variance models in the context of pricing options with maturity time of one year or longer. Note that the practical importance of the fractional integrated variance models stems from the added flexibility when modeling long run volatility characteristics.

As Mills (1999) stated, the implication of IGARCH models that shocks to the conditional variance persist indefinitely does not reconcile with the persistence observed after large shocks, such as the crash of October 1987, and with the perceived behavior of agents who do not appear to frequently and radically alter the composition of their portfolios. So the widespread observation of the IGARCH behavior may be an artifact of a long memory FIGARCH data generating process. Baillie et al. (1996) provided a simulation experiment that provides considerable support of this line of argument. Beine et al. (2002) applied the FIGARCH(1,d,1) model in order to investigate the effects of official interventions on the volatility of exchange rates. One of their interesting remarks is that measuring the volatility of exchange rates through the FIGARCH model instead of a traditional ARCH model leads to different results. The GARCH and IGARCH models tend to underestimate the effect of the central bank interventions on the volatility of exchange rates. Vilasuso (2002) fitted conditional volatility models to daily spot exchange rates and found that the FIGARCH(1,d,1) model generates superior volatility forecasts compared to those generated by a GARCH(1,1) or IGARCH(1,1) model.

Bollerslev and Mikkelsen (1996) extended the idea of fractional integration to the exponential GARCH model, whereas Tse (1998) built the fractional integration form of the APARCH model. Factorizing the autoregressive polynomial $(1-B(L)) = \Phi(L)(1-L)^d$, where all the roots of $\Phi(z)=0$ lie outside the unit circle, the fractionally integrated exponential GARCH, or FIEGARCH(p,d,q), model is defined as:

$$\ln\left(\sigma_t^2\right) = a_0 + \Phi(L)^{-1}(1-L)^{-d}(1+A(L))g(z_{t-1}). \tag{2.2.47}$$

The fractionally integrated asymmetric power ARCH, or FIAPARCH(p,d,q), model has the following form:

$$\sigma_t^{\delta} = a_0 + \left(1-(1-B(L))^{-1}\Phi(L)(1-L)^{-d}\right)\left(|\varepsilon_t| - \gamma\varepsilon_t\right)^{\delta}. \tag{2.2.48}$$

Finally, Hwang (2001) presented the asymmetric fractionally integrated family GARCH(1,d,1), or ASYMM FIFGARCH(1,d,1), model, which is defined as:

$$\begin{aligned} \sigma_t^{\lambda} &= \frac{k}{1-\delta} + \left(1 - \frac{(1-\varphi L)(1-L)^d}{1-\delta L}\right) f^{v}(\varepsilon_t)\sigma_t^{\lambda} \\ f(\varepsilon_t) &= \left|\frac{\varepsilon_t}{\sigma_t} - b\right| - c\left(\frac{\varepsilon_t}{\sigma_t} - b\right), \end{aligned} \tag{2.2.49}$$

for $|c| \le 1$. Hwang points out that, for different parameter values in (2.2.49), the following fractionally integrated ARCH models are obtained: FIEGARCH, for $\lambda = 0$, $v = 1$, FITGARCH for $\lambda = 1$, $v = 1$, FIGARCH for $\lambda = 2$, $v = 2$, and FINGARCH, for $\lambda = v$ but otherwise unrestricted.

However, Ruiz and Pérez (2003) noted that Hwang's model is poorly specified and does not nest the FIEGARCH model. Thus, they suggested an alternative specification, which is a direct generalization of Hentschel's model in (2.2.26):

$$\begin{aligned} (1-\varphi L)(1-L)^d \frac{\sigma_t^{\lambda} - 1}{\lambda} &= \omega' + a(1+\psi L)\sigma_{t-1}^{\lambda}\left(f^{v}(z_{t-1}) - 1\right) \\ f\left(\frac{\varepsilon_t}{\sigma_t}\right) &= \left|\frac{\varepsilon_t}{\sigma_t} - b\right| - c\left(\frac{\varepsilon_t}{\sigma_t} - b\right) \end{aligned}, \tag{2.2.50}$$

Imposing appropriate restrictions on the parameters of (2.2.50), a number of models are obtained as special cases (e.g. the FIGARCH model in (2.2.46), the FIEGARCH model in (2.2.47), Hentschel's model in (2.2.26)).

Nowicka-Zagrajek and Weron (2001) replaced the constant term in the GARCH(p,q) model with a linear function of i.i.d. stable random variables and defined the randomized GARCH, or R-GARCH(r,p,q), model:

$$\sigma_t^2 = \sum_{i^*=1}^{r}\left(c_{i^*}\eta_{t-i^*}\right) + \sum_{i=1}^{q}\left(a_i\varepsilon_{t-i}^2\right) + \sum_{j=1}^{p}\left(b_j\sigma_{t-j}^2\right), \tag{2.2.51}$$

where $c_{i^*} \ge 0$, $i^* = 1,...,r$, $a_i \ge 0$, $i = 1,...,q$, $b_j \ge 0$, $j = 1,..., p$, the innovations η_t are positive i.i.d. stable random variables expressed by the characteristic function in (2.4.16), and $\{\eta_t\}$ and $\{z_t\}$ are independent.

Müller et al. (1997), based on the hypothesis that participants in a heterogeneous market make volatilities of different time resolutions behave differently, proposed the heterogeneous interval GARCH, or H-GARCH(p,n), model that takes into account the squared price changes over time intervals of different sizes:

$$\sigma_t^2 = a_0 + \sum_{i=1}^{n}\sum_{k=1}^{i} a_{ik}\left(\sum_{i^*=k}^{i}\varepsilon_{t-i^*}^2\right)^2 + \sum_{j=1}^{p}\left(b_j\sigma_{t-j}^2\right), \qquad (2.2.52)$$

where $a_0 > 0$, $a_{ik} \geq 0$, for $i = 1,...,n$, $k = 1,...,i$, $b_j \geq 0$, $j = 1,...,p$.

Many financial markets impose restrictions on the maximum allowable daily change in price. As pointed out by Wei and Chiang (1997), the common practice of ignoring the problem by treating the observed censored observations as if they were actually the equilibrium prices, or dropping the limited prices from the studied sample, leads to the underestimation of conditional volatility. Morgan and Trevor (1997) proposed the Rational Expectation (RE) algorithm (which can be interpreted as an EM algorithm (Dempster et al. (1977)) for censored observations in the presence of heteroscedasticity, which replaces the unobservable components of the likelihood function of the ARCH model by their rational expectations. As an alternative to the RE algorithm, Wei (2002), based on Kodres's (1993) study, proposed a censored-GARCH model and developed a Bayesian estimation procedure for the proposed model. Moreover, on the basis of Kodres's (1988) research, Lee (1999a), Wei (1999) and Calzolari and Fiorentini (1998) developed the class of Tobit-GARCH models.

Brooks et al. (2001) reviewed the most known software packages for estimation of ARCH models, and concluded that the estimation results differ considerably from one another. Table 2.2, in the Appendix, contains the ARCH models that have been presented in this section.

2.3. The Relationship Between Conditional Variance and Conditional Mean

2.3.1 The ARCH in Mean Model

Financial theory suggests that an asset with a higher expected risk would pay a higher return on average. Let y_t denote the rate of return of a particular stock or market portfolio from time t to $t-1$ and rf_t be the return on a riskless asset (i.e. treasury bills). Then, the excess return (asset return minus the return on a riskless asset) can be decomposed into a component anticipated by investors at time $t-1$, μ_t, and a component that was unanticipated, ε_t:

$$y_t - rf_t = \mu_t + \varepsilon_t .$$

The relationship between investors' expected return and risk was presented in an ARCH framework, by Engle et al. (1987). They introduced the ARCH in mean, or ARCH-M, model where the conditional mean is an explicit function of the conditional variance of the process in framework (2.2.1). The estimated coefficient on the expected risk is a measure of the risk-return tradeoff. Thus, the ARCH regression model, in framework (2.2.2), can be presented as:

$$\begin{gathered} y_t = x_t'\beta + \phi\left(\sigma_t^2\right) + \varepsilon_t \\ \varepsilon_t \mid I_{t-1} \sim f\left[0, \sigma_t^2\right] \\ \sigma_t^2 = g\left(\sigma_{t-1}, \sigma_{t-2}, \ldots; \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots; \upsilon_{t-1}, \upsilon_{t-2}, \ldots\right). \end{gathered}$$

where $\phi\left(\sigma_t^2\right)$ represents the risk premium, i.e., the increase in the expected rate of return due to an increase in the variance of the return. Although earlier studies concentrated on detecting a constant risk premium, the ARCH in mean model provided a new approach by which a time varying risk premium can be estimated. The most commonly used specifications of the ARCH-M model are in the form:

$\phi\left(\sigma_t^2\right) = c_0 + c_1\sigma_t^2$, (Nelson (1991), Bollerslev et al. (1994)),

$\phi\left(\sigma_t^2\right) = c_0 + c_1\sigma_t$, (Domowitz and Hakkio (1985), Bollerslev et al. (1988)),

$\phi\left(\sigma_t^2\right) = c_0 + c_1 \log\left(\sigma_t^2\right)$, (Engle et al. (1987)).

A positive as well as a negative risk return tradeoff could be consistent with the financial theory. A positive relationship is expected if we assume a rational risk averse investor who requires a larger risk premium during the times when the payoff of the security is riskier. On the other hand, a negative relationship is expected under the assumption that during relatively riskier periods the investors may want to save more. In applied research work, there is evidence for both positive and negative relationship. French et al. (1987) found positive risk return tradeoff for the excess returns on the S&P500 composite portfolio although not statistically significant in all the examined periods. Nelson (1991) found a negative but insignificant relationship for the excess returns on the Center for Research in Security Prices (CRSP) value weighted market index. Bollerslev et al. (1994) found a positive, not always statistically significant, relationship for the returns on Dow Jones and S&P500 indices. Interesting studies employing the ARCH-M model were conducted by Devaney (2001) and Elyasiani and Mansur (1998). The former examined the tradeoff between conditional variance and excess returns for stocks of the commercial bank sector, while the latter investigated the time varying risk premium for real estate investment trusts.

2.3.2 Volatility and Serial Correlation

LeBaron (1992) found a strong inverse relation between volatility and serial correlation for S&P500, CRSP value weighted market index, Dow Jones and IBM returns. He introduced the exponential autoregressive GARCH, or EXP-GARCH(p,q), model in which the conditional mean is a non-linear function of the conditional variance. Based on LeBaron (1992), the ARCH regression model, in framework (2.2.2), can be presented as:

$$\begin{gathered} y_t = x_t'\beta + \left(c_1 + c_2 \exp\left(-\sigma_t^2 / c_3\right)\right) y_{t-1} + \varepsilon_t \\ \varepsilon_t \mid I_{t-1} \sim f\left[0, \sigma_t^2\right] \\ \sigma_t^2 = g\left(\sigma_{t-1}, \sigma_{t-2}, \ldots; \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots; \upsilon_{t-1}, \upsilon_{t-2}, \ldots\right). \end{gathered} \tag{2.3.1}$$

The model is a mixture of the GARCH model and the exponential AR model of Ozaki (1980). For the data set LeBaron used, c_2 is significantly negative and remarkably robust to the choice of sample period, market index, measurement interval and volatility measure. As LeBaron stated, it is difficult to estimate c_3 in conjunction with c_2 when using a gradient type of algorithm. So, c_3 is set to the sample variance of the series. Generally, the first order autocorrelations are larger for periods of lower volatility and smaller during periods of higher volatility. The accumulation of news[2] and the non-synchronous trading[3] were mentioned as the possible reasons. The stocks do not trade close to the end of the day and information arriving during this period is reflected on the next day's trading, inducing serial correlation. As new information reaches market very slowly, traders optimal action is to do nothing until enough information is accumulated. Because of the non-trading, the trading volume, which is strongly positive related with volatility, lowers. Thus, we have a market with low trade volume and high correlation.

Kim (1989), Sentana and Wadhwani (1991) and Oedegaard (1991) have also investigated the relationship between autocorrelation and volatility and found an inverse relation between volatility and autocorrelation. Moreover, Oedegaard (1991) found that the evidence of autocorrelation, for the S&P500 daily index, decreased over time, possibly because of the introduction of financial derivatives (options and futures) on the index.

2.4. Estimation

2.4.1 Maximum Likelihood Estimation

In ARCH models, the most commonly used method in estimating the vector of unknown parameters, θ, is the method of maximum likelihood (MLE). Under the assumption of independently and identically distributed standardized innovations, $z_t(\theta) \equiv \varepsilon_t(\theta)/\sigma_t(\theta)$, in framework (2.2.2), let us denote their density function as $f(z_t;w)$, where $w \in W \subseteq R^{\breve{w}}$ is the vector of the parameters of f to be estimated. So, for $\psi' = (\theta', w')$ denoting the whole set of the $\breve{\psi} = \breve{\theta} + \breve{w}$ parameters that have to be estimated for the conditional mean, variance and density function, the log-likelihood function for $\{y_t(\theta)\}$ is:

$$l_t(y_t;\psi) = \ln(f(z_t(\theta);w)) - \frac{1}{2}\ln(\sigma_t^2(\theta)). \tag{2.4.1}$$

The full sample log-likelihood function for a sample of T observations is simply:

$$L_T(\{y_t\};\psi) = \sum_{t=1}^{T} l_t(y_t;\psi). \tag{2.4.2}$$

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[2] See section 2.1.2.
[3] See section 2.1.3.

If the conditional density, the mean and the variance functions are differentiable for each possible $\psi \in \Theta \times W \equiv \Psi \subseteq R^{\breve{\psi}}$, the MLE estimator $\hat{\psi}$ for the true parameter vector ψ_0 is found by maximizing equation (2.4.2), or equivalently by solving the equation

$$\sum_{t=1}^{T} \frac{\partial l_t(y_t;\psi)}{\partial \psi} = 0 . \tag{2.4.3}$$

If the density function does not require the estimation of any parameter, as in the case of the normal distribution that is uniquely determined by its first two moments, then $\breve{w} = 0$. In such cases, equation (2.4.3) becomes:

$$\sum_{t=1}^{T} \left(f(z_t(\theta))^{-1} \frac{\partial f(z_t(\theta))}{\partial \theta} \left(-\frac{\partial \mu_t(\theta)}{\partial \theta} \sigma_t^2(\theta)^{-1} - 0.5\sigma_t^2(\theta)^{-3/2} \varepsilon_t(\theta) \frac{\partial \sigma_t^2(\theta)}{\partial \theta} \right) \right) = 0 . \tag{2.4.4}$$

Let us, for example, estimate the parameters of framework (2.2) for normal distributed innovations and the GARCH(p,q) functional form for the conditional variance as given in equation (2.2.6). The density function of the standard normal distribution is:

$$f(z_t) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z_t^2}{2} \right). \tag{2.4.5}$$

For convenience equation (2.2.6) is written as $\sigma_t^2 = \omega' s_t$, where $\omega' = (a_0, a_1, ..., a_q, \beta_1, ..., \beta_p)$ and $s_t = (1, \varepsilon_{t-1}^2, ..., \varepsilon_{t-q}^2, \sigma_{t-1}^2, ..., \sigma_{t-p}^2)$. The vector of parameters that have to be estimated is $\psi' = \theta' = (\beta', \omega')$. For normally distributed standardized innovations, z_t, the log-likelihood function in equation (2.4.1), is:

$$l_t(y_t;\psi) = \ln\left(\frac{1}{\sqrt{2\pi}} \right) - \frac{(y_t - x_t'\beta)^2}{2\sigma_t^2} - \frac{1}{2}\ln(\sigma_t^2),$$

and the full sample log-likelihood function in equation (2.4.2), becomes:

$$L_T(\{y_t\};\theta) = -\frac{1}{2}\left(T\ln(2\pi) + \sum_{t=1}^{T} \frac{(y_t - x_t'\beta)^2}{\sigma_t^2} + \sum_{t=1}^{T} \ln(\sigma_t^2) \right).$$

The first and the second derivatives of the log-likelihood for the t^{th} observation with respect to the variance parameter vector are:

$$\frac{\partial l_t(y_t;\beta,\omega)}{\partial \omega} = \frac{1}{2\sigma_t^2} \frac{\partial \sigma_t^2}{\partial \omega} \left(\frac{\varepsilon_t^2 - \sigma_t^2}{\sigma_t^2} \right),$$

$$\frac{\partial^2 l_t(y_t;\beta,\omega)}{\partial \omega \partial \omega'} = \left(\frac{\varepsilon_t^2 - \sigma_t^2}{\sigma_t^2} \right) \frac{\partial}{\partial \omega'} \left[\frac{1}{2\sigma_t^2} \frac{\partial \sigma_t^2}{\partial \omega} \right] - \frac{1}{2\sigma_t^4} \frac{\partial \sigma_t^2}{\partial \omega} \frac{\partial \sigma_t^2}{\partial \omega'} \frac{\varepsilon_t^2}{\sigma_t^2},$$

where

$$\frac{\partial \sigma_t^2}{\partial \omega} = s_t + \sum_{i=1}^{p} b_i \frac{\partial \sigma_{t-1}^2}{\partial \omega}.$$

The first and second derivatives of the log-likelihood with respect to the mean parameter vector are:

$$\frac{\partial l_t(y_t;\beta,\omega)}{\partial \beta} = \varepsilon_t x_t \sigma_t^{-2} + \frac{1}{2}\sigma_t^2 \frac{\partial \sigma_t^2}{\partial \beta}\left(\frac{\varepsilon_t^2 - \sigma_t^2}{\sigma_t^2}\right),$$

$$\frac{\partial^2 l_t(y_t;\beta,\omega)}{\partial \beta \partial \beta'} =$$

$$= -\sigma_t^{-2} x_t x_t' - \frac{1}{2}\sigma_t^{-4} \frac{\partial \sigma_t^2}{\partial \beta}\frac{\partial \sigma_t^2}{\partial \beta'}\left(\frac{\varepsilon_t^2}{\sigma_t^2}\right) - 2\sigma_t^{-4}\varepsilon_t x_t \frac{\partial \sigma_t^2}{\partial \beta} + \left(\frac{\varepsilon_t^2 - \sigma_t^2}{\sigma_t^2}\right)\frac{\partial}{\partial \beta'}\left[\frac{1}{2}\sigma_t^{-2}\frac{\partial \sigma_t^2}{\partial \beta}\right],$$

where

$$\frac{\partial \sigma_t^2}{\partial \beta} = -2\sum_{i=1}^{q} a_i x_{t-i}\varepsilon_{t-i} + \sum_{j=1}^{p} b_j \frac{\partial \sigma_{t-j}^2}{\partial \beta}.$$

The information matrix corresponding to ω is given as:

$$I_{\omega\omega} = \frac{-1}{T}\sum_{t=1}^{T}\left(E\left(\frac{\partial^2 l_t(y_t;\beta,\omega)}{\partial \omega \partial \omega'}\right)\right) = \frac{1}{2T}\sum_{t=1}^{T}\left(\sigma_t^{-4}\frac{\partial \sigma_t^2}{\partial \omega}\frac{\sigma_t^2}{\partial \omega'}\right).$$

The information matrix corresponding to b is given as:

$$I_{\beta\beta} = \frac{-1}{T}\sum_{t=1}^{T}\left(E\left(\frac{\partial^2 l_t(y_t;\beta,\omega)}{\partial \beta \partial \beta'}\right)\right) = \frac{1}{T}\sum_{t=1}^{T}\left(\sigma_t^{-2} x_t x_t' + 2\sigma_t^{-4}\sum_{i=1}^{q} a_i^2 \varepsilon_{t-i}' x_{t-i} x_{t-i}' \varepsilon_{t-i} + \frac{1}{2}\sum_{j=1}^{p} b_j^2 \left(\frac{\partial \sigma_{t-j}^2}{\partial \beta}\right)^2\right).$$

The elements in the off-diagonal block of the information matrix are zero, i.e.,

$$I_{\omega\beta} = \frac{-1}{T}\sum_{t=1}^{T}\left(E\left(\frac{\partial^2 l_t(y_t;\beta,\omega)}{\partial \omega \partial \beta'}\right)\right) = 0.$$

So, ω can be estimated without loss of asymptotic efficiency based on a consistent estimate of β and vice versa. At this point, it should be noticed that although the block diagonality holds for models as the GARCH, NARCH and Log-GARCH models, it does not hold for asymmetric models, i.e. the EGARCH model, and for the ARCH in mean models. In such cases, the parameters have to be estimated jointly.

Even in the case of the symmetric GARCH(p,q) model with normally distributed innovations, we have to solve a set of $\breve{\theta} = k + p + q + 1$ non-linear equations in (2.4.4). Numerical techniques are used in order to estimate the vector of parameters ψ.

2.4.2 Numerical Estimation Algorithms

The problem faced in non-linear estimation, as in the case of the ARCH models, is that there are no closed form solutions. So, an iterative method has to be applied to obtain a solution. Iterative optimization algorithms work by taking an initial set of values for the parameters, say $\psi^{(0)}$, then performing calculations based on these values to obtain a better set of parameters values $\psi^{(1)}$. This process is repeated until the likelihood function, in equation (2.4.2), no longer improves between iterations. If $\psi^{(0)}$ is a trial value of the estimate, then expanding $L_T\left(\{y_t\};\psi\right)/\partial\psi$ and retaining only the first power of $\psi - \psi^{(0)}$, we obtain

$$\frac{\partial L_T}{\partial \psi} \approx \frac{\partial L_T}{\partial \psi^{(0)}} + \left(\psi - \psi^{(0)}\right)\frac{\partial^2 L_T}{\partial \psi^{(0)} \partial \psi'^{(0)}} .$$

At the maximum, $L_T/\partial\psi$ should equal zero. Rearranging terms, the correction for the initial value, $\psi^{(0)}$, obtained is

$$\left(\psi - \psi^{(0)}\right) = -\frac{\partial L_T}{\partial \psi^{(0)}}\left(\frac{\partial^2 L_T}{\partial \psi^{(0)} \partial \psi'^{(0)}}\right)^{-1} . \tag{2.4.6}$$

Let $\psi^{(i)}$ denote the parameter estimates after the i^{th} iteration. Based on (2.4.6) the Newton-Raphson algorithm computes $\psi^{(i+1)}$ as:

$$\psi^{(i+1)} = \psi^{(i)} - \left(\frac{\partial^2 L_T^{(i)}}{\partial \psi \partial \psi'}\right)^{-1} \frac{\partial L_T^{(i)}}{\partial \psi} . \tag{2.4.7}$$

The scoring algorithm is a method closely related to the Newton-Raphson algorithm and was applied by Engle (1982) to estimate the parameters of the ARCH(p) model. The difference between the Newton-Raphson method and the method of scoring is that the former depends on observed second derivatives, while the latter depends on the expected values of the second derivatives. So, the scoring algorithm computes $\psi^{(i+1)}$ as:

$$\psi^{(i+1)} = \psi^{(i)} + E\left(\frac{\partial^2 L_T^{(i)}}{\partial\psi\partial\psi'}\right)^{-1} \frac{\partial L_T^{(i)}}{\partial\psi}. \tag{2.4.8}$$

An alternative procedure suggested by Berndt et al. (1974), which uses first derivatives only, is the Berndt, Hall, Hall and Hausman (BHHH) algorithm. The BHHH algorithm is similar to the Newton-Raphson algorithm, but, instead of the Hessian (second derivative of the log likelihood function with respect to the vector of unknown parameters), it is based on an approximation formed by the sum of the outer product of the gradient vectors for the contribution of each observation to the objective function. This approximation is asymptotically equivalent to the actual Hessian when evaluated at the parameter values, which maximize the function. The BHHH algorithm computes $\psi^{(i+1)}$ as:

$$\psi^{(i+1)} = \psi^{(i)} + \left(\sum_{t=1}^{T} \frac{\partial l_t^{(i)}}{\partial\psi} \frac{\partial l_t^{(i)}}{\partial\psi'}\right)^{-1} \frac{\partial L_T^{(i)}}{\partial\psi}. \tag{2.4.9}$$

When the outer product is near singular, a ridge correction may be used in order to handle numerical problems and improve the convergence rate. Marquardt (1963) modified the BHHH algorithm by adding a correction matrix to the sum of the outer product of the gradient vectors. The Marquardt updating algorithm is computed as:

$$\psi^{(i+1)} = \psi^{(i)} + \left(\sum_{t=1}^{T} \frac{\partial l_t^{(i)}}{\partial\psi} \frac{\partial l_t^{(i)}}{\partial\psi'} - aI\right)^{-1} \frac{\partial L_T^{(i)}}{\partial\psi}, \tag{2.4.10}$$

where I is the identity matrix and a is a positive number chosen by the algorithm. The effect of this modification is to push the parameter estimates in the direction of the gradient vector. The idea is that when we are far from the maximum, the local quadratic approximation to the function may be a poor guide to its overall shape, so it may be better off simply following the gradient. The correction may provide a better performance at locations far from the optimum, and allows for computation of the direction vector in cases where the Hessian is near singular.

2.4.3 Maximum Likelihood Estimation under Non-Normality

As already mentioned, an attractive feature of the ARCH process is that even though the conditional distribution of the innovations is normal, the unconditional distribution has thicker tails than the normal one. However, the degree of leptokurtosis induced by the ARCH process often does not capture all of the leptokurtosis present in high frequency speculative prices. Thus, there is a fair amount of evidence that the conditional distribution of ε_t is non-normal as well.

To circumvent this problem, Bollerslev (1987) proposed using the standardized t distribution with $\nu > 2$ degrees of freedom:

$$f(z_t;\nu) = \frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)\sqrt{\pi(\nu-2)}}\left(1+\frac{z_t^2}{\nu-2}\right)^{-\frac{\nu+1}{2}}, \quad \nu > 2, \tag{2.4.11}$$

where $\Gamma(.)$ is the gamma function. The degrees of freedom are regarded as parameter to be estimated, $w = (\nu)$. The t distribution is symmetric around zero and for $\nu > 4$ the conditional kurtosis equals $3(\nu-2)(\nu-4)^{-1}$, which exceeds the normal value of three, but for $\nu \to \infty$, (2.4.11) converges to (2.4.5), the standard normal distribution.

Nelson (1991) suggested the use of the generalized error distribution, or GED[4]:

$$f(z_t;\nu) = \frac{\nu \exp\left(-0.5|z_t/\lambda|^\nu\right)}{\lambda 2^{(1+1/\nu)}\Gamma(\nu^{-1})}, \quad \nu > 0, \tag{2.4.12}$$

where ν is the tail-thickness parameter and $\lambda \equiv \sqrt{2^{-2/\nu}\Gamma(\nu^{-1})/\Gamma(3\nu^{-1})}$. (For more details on the GED, see Harvey (1981) and Box and Tiao (1973)). When $\nu = 2$, z_t is standard normally distributed and so (2.4.12) reduces to (2.4.5). For $\nu < 2$, the distribution of z_t has thicker tails than the normal distribution (e.g., for $\nu = 1$, z_t has a double exponential distribution) while for $\nu > 2$, the distribution of z_t has thinner tails than the normal distribution (e.g., for $\nu = \infty$, z_t has a uniform distribution on the interval $(-\sqrt{3},\sqrt{3})$).

The densities presented above account for fat tails but they are symmetric. Lee and Tse (1991) suggested that not only the conditional distribution of innovations may be leptokurtotic, but also asymmetric. Allowing for skewness may be important in modeling interest rates as they are lower bounded by zero and may therefore be skewed. To allow for both skewness and leptokurtosis, they used a Gram Charlier type distribution (see Kendall and Stuart (1969), p.157) with density function given by:

$$f(z_t;\nu,g) = \hat{f}(z_t)\left(1+\frac{\nu}{6}H_3(z_t)+\frac{g}{24}H_4(z_t)\right), \tag{2.4.13}$$

[4] The GED sometimes referred as the exponential power distribution.

where $\hat{f}(.)$ is the standard normal density function, and $H_3(z_t) \equiv z_t^3 - 3z_t$ and $H_4(z_t) \equiv z_t^4 - 6z_t^2 + 3$ are the Hermite polynomials. The quantities ν and g are the measures of skewness and kurtosis, respectively. Jondeau and Rockinger (2001) examined the properties of the Gram Charlier conditional density function and estimated ARCH models with a Gram Charlier density function for a set of exchange rate series.

Bollerslev et al. (1994) applied the generalized t distribution (McDonald and Newey (1988)):

$$f(z_t;\nu,g) = \frac{\nu}{2\sigma_t b g^{1/\nu} B(\nu^{-1},g)\left(1+\left(|\varepsilon_t|^{\nu} \big/ g b^{\nu}\sigma_t^{\nu}\right)\right)^{g+1/\nu}}, \; \nu > 0, \; g > 0 \text{ and } \nu g > 2, \tag{2.4.14}$$

where $B(\nu^{-1},g) \equiv \Gamma(\nu^{-1})\Gamma(g)/\Gamma(\nu^{-1}+g)$ is the beta function and $b \equiv \sqrt{\Gamma(\nu^{-1})\Gamma(g)/\Gamma(3\nu^{-1})\Gamma(g-2\nu^{-1})}$. The generalized t distribution has the advantage that nests both (2.4.11) and (2.4.12). For $\nu = 2$ and $g = 0.5$ times the degrees of freedom, (2.4.14) is set to the t distribution, and for $\nu = \infty$, the GED is obtained. Moreover, the two shape parameters ν and g allow for fitting both the tails and the central part of the conditional distribution.

Lambert and Laurent (2000, 2001) extended the skewed Student t density proposed by Fernandez and Steel (1998) to the ARCH framework, in the following density function:

$$f(z_t;\nu,g) = \frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)\sqrt{\pi(\nu-2)}}\left(\frac{2s}{g+g^{-1}}\right)\left(1+\frac{sz_t+m}{\nu-2}g^{-II_t}\right)^{-\frac{\nu+1}{2}}, \quad \nu > 2, \tag{2.4.15}$$

where g is the asymmetry parameter, ν denotes the number of degrees of freedom of the distribution, $\Gamma(.)$ is the gamma function, $II_t = 1$ if $z_t \geq -ms^{-1}$, and $II_t = -1$ otherwise, $m = \Gamma((\nu-1)/2)\sqrt{(\nu-2)}\left(\Gamma(\nu/2)\sqrt{\pi}\right)^{-1}(g-g^{-1})$ and $s = \sqrt{g^2+g^{-2}-m^2-1}$. Angelidis and Degiannakis (2004), Degiannakis (2004) and Giot Laurent (2003) suggest using ARCH models based on the skewed Student distribution to fully take into account the fat left and right tails of the returns distribution.

De Vries (1991) noted that the unconditional distribution of variaties from an ARCH process can be stable and that under suitable conditions the conditional distribution is stable as well. Stable Paretian conditional distributions have been introduced in ARCH models by Liu and Brorsen (1995), Mittnik et al. (1999), and Panorska et al. (1995). As the stable Paretian distribution does not have an analytical expression for its density function, it is expressed by its characteristic function:

$$\varphi(t, a, \beta, \sigma, \mu) = \exp\left(i\mu t - |\sigma t|^{a} \left(1 - i\beta \frac{t}{|t|} \omega(|t|, a) \right) \right), \tag{2.4.16}$$

where $0 < a \leq 2$ is the characteristic exponent, $-1 \leq \beta \leq 1$ is the skewness parameter, $\sigma > 0$ is the scale parameter, $\mu \in \Re$ is the location parameter, and

$$\omega(|t|, a) = \begin{cases} \tan\frac{\pi a}{2}, & a \neq 1 \\ -\frac{2}{\pi}\log|t|, & a = 1. \end{cases}$$

The standardized innovations, z_t, are assumed as independently, identically stable Pareto distributed random variables with zero location parameter and unit scale parameter. The way that GARCH models are built imposes limits on the heaviness of the tails of their unconditional distribution. Given that a wide range of financial data exhibit remarkable fat tails, this assumption represents a major shortcoming of GARCH models in financial time series analysis. Stable Paretian conditional distributions have been employed in a number of studies, such as Mittnik et al. (1998a, 1998b) and Mittnik and Paolella (2001). Tsionas (1999) established a framework for Monte Carlo posterior inference in models with stable distributed errors by combining a Gibbs sampler with Metropolis independence chains and representing the symmetric stable variates as normal scale mixtures. Mittnik et al. (2002) and Panorska et al. (1995) derived conditions for strict stationarity of GARCH and APARCH models with stable Paretian conditional distributions. De Vries (1991) provided relationships between ARCH and stable processes. Tsionas (2002) compared a stable Paretian model with ARCH errors with a stable Paretian model with stochastic volatility. The Randomized GARCH model with stable Paretian innovations totally skewed to the right and with $0 < a < 1$ was studied by Nowicka-Zagrajek and Weron (2001). They derived the unconditional distributions and analyzed the dependence structure by means of the codifference. It turns out that R-GARCH models with conditional variance dependent on the past can have very heavy tails. The class is very flexible as it includes GARCH models and de Vries process (1991) as special cases.

Hansen (1994) suggested an approach that allows not only the conditional variance to be time varying but also the higher moments of conditional distribution such as skewness and kurtosis. He suggested the autoregressive conditional density, or the ARCD, model, where the density function, $f(z_t; w)$, is presented as:

$$f(z_t; w_t \mid I_{t-1}). \tag{2.4.17}$$

The parameter vector of the conditional density function in (2.4.17) is assumed to be a function of the current information set, I_{t-1}.

Other distributions, that have been employed, include the normal Poisson mixture distribution (Brorsen and Yang (1994), Drost et al. (1998), Jorion (1988), Lin and Yeh (2000), and Vlaar and Palm (1993)), the normal lognormal mixture (Hsieh (1989)), and serially dependent mixture of normally distributed variables (Cai (1994)) or student t distributed variables (Hamilton and Susmel (1994))[5]. Recently, Politis (2003a, 2003b, 2004) developed an implicit ARCH model that gives motivation towards a more *natural* and less ad hoc distribution for the residuals. He proposed to *studentize* the ARCH residuals by dividing with a time-localized measure of standard deviation.

2.4.4 Quasi-Maximum Likelihood Estimation

The assumption of normally distributed standardized innovations is often violated by the data. This has motivated the use of alternative distributional assumptions, presented in the previous section. Alternatively, the MLE based on the normal density may be given a quasi-maximum likelihood interpretation. Bollerslev and Wooldridge (1992), based on Weiss (1986) and Pagan and Sabau (1987), showed that the maximization of the normal log-likelihood function can provide consistent estimates of the parameter vector θ even when the distribution of z_t in non-normal, provided that:

$$E(z_t \mid I_{t-1}) = 0$$
$$E(z_t^2 \mid I_{t-1}) = 1.$$

This estimator is, however, inefficient with the degree of inefficiency increasing with the degree of departure from normality. So, the standard errors of the parameters have to be adjusted. Let $\hat{\theta}$ be the estimate that maximizes the normal log-likelihood function, in

[5] Cai (1994) and Hamilton and Susmel (1994) used the mixtures to estimate the class of regime switching ARCH models, presented in section 2.2.1.

equation (2.4.2), based on the normal density function in (2.4.5), and let θ_0 be the true value. Then, even when z_t is non-normal, under certain regularity conditions:

$$\sqrt{T}\left(\hat{\theta}-\theta_0\right)\xrightarrow{D} N\left(0, A^{-1}BA^{-1}\right), \qquad (2.4.17)$$

where

$$A \equiv \underset{T\to\infty}{p\lim} T^{-1}\sum_{t=1}^{T} E\left(\frac{-\partial^2 l_t(\theta_0)}{\partial\theta\partial\theta'}\right),$$

$$B \equiv \underset{T\to\infty}{p\lim} T^{-1}\sum_{t=1}^{T} E\left(\frac{\partial l_t(\theta_0)}{\partial\theta}\right)\left(\frac{\partial l_t(\theta_0)}{\partial\theta}\right)',$$

for l_t denoting the correctly specified log-likelihood function. The matrices A and B can be consistently estimated by:

$$\hat{A} = -T^{-1}\sum_{t=1}^{T} E\left(\frac{\partial^2 l_t\left(\hat{\theta}\right)}{\partial\theta\partial\theta'} \mid I_{t-1}\right),$$

$$\hat{B} = T^{-1}\sum_{t=1}^{T} E\left(\left(\frac{\partial l_t\left(\hat{\theta}\right)}{\partial\theta}\right)\left(\frac{\partial l_t\left(\hat{\theta}\right)}{\partial\theta'}\right) \mid I_{t-1}\right),$$

where l_t is the incorrectly specified log-likelihood function under the assumption of normal density function. Thus, standard errors for $\hat{\theta}$ that are robust to misspecification of the family of densities can be obtained from the square root of diagonal elements of:

$$T^{-1}\hat{A}^{-1}\hat{B}\hat{A}^{-1}.$$

Recall that if the model is correctly specified and the data are in fact generated by the normal density function, then $A = B$, and, hence, the variance covariance matrix, $T^{-1}\hat{A}^{-1}\hat{B}\hat{A}^{-1}$, reduces to the usual asymptotic variance covariance matrix for maximum likelihood estimation:

$$T^{-1}\hat{A}^{-1}.$$

For symmetric departures from normality, the quasi-maximum likelihood estimation is generally close to the exact MLE. But, for non-symmetric distributions, Engle and González-Rivera (1991), showed that the loss in efficiency may be quite high (Bai and Ng (2001) proposed a procedure for testing conditional symmetry.). In such a case, other methods of estimation should be considered. Lumsdaine (1991, 1996) and Lee and Hansen (1991, 1994) established the consistency and asymptotic normality of the quasi-maximum likelihood estimators of the IGARCH(1,1) model. Lee (1991) extended the asymptotic properties to the IGARCH(1,1) in Mean model, Berkes et al. (2003) and Berkes and Horváth (2003) studied the asymptotic properties of the quasi-maximum likelihood estimators of the GARCH(p,q) model under a set of weaker conditions, and Baille et al. (1996) showed that the quasi-maximum likelihood estimators of the FIGARCH(1,d,0) model are both consistent and asymptotically normally distributed.

2.4.5 Other Estimating Methods

Other estimation methods, except for MLE, have been appeared in the ARCH literature. Harvey et al. (1992) presented the unobserved components structural ARCH, or STARCH, model and proposed an estimation method based on the Kalman filter. These are state space models or factor models in which the innovation is composed of several sources of error where each of the error sources has a heteroscedastic specification of the ARCH form. Since the error components cannot be separately observed given the past observations, the independent variables in the variance equations are not measurable with respect to the available information set, which complicates inference procedures.

Pagan and Hong (1991) applied a nonparametric Kernel estimate of the expected value of squared innovations. Pagan and Schwert (1990) used a collection of nonparametric estimation methods, including Kernels, Fourier series and two-stage least squares regressions. They found that the non-parametric methods did good job in-sample forecasts though the parametric models yielded superior out-of-sample forecasts. Gouriéroux and Monfort (1992) also proposed a nonparametric estimation method in order to estimate the GQTARCH model in equation (2.2.28). Bühlmann and McNeil (2002) proposed a nonparametric estimation iterative algorithm, that requires neither the specification of the conditional variance functional form nor that of the conditional density function, and showed that their algorithm gives more precise estimates of the volatility in the presence of departures from the assumed ARCH specification.

Engle and González-Rivera (1991), Engle and Ng (1993), Gallant and Tauchen (1989), Gallant et al. (1991), Gallant et al. (1993) among others, combined parametric specifications for the conditional variance with a nonparametric estimate of the conditional density function. In a Monte Carlo study, Engle and González-Rivera (1991) found that their semi-parametric method could improve the efficiency of the parameter estimates up to 50 per cent over the QMLE, particularly when the density was highly non-normal and skewed, but it did not seem to capture the total potential gain in efficiency.

Another attractive way to estimate ARCH models without assuming normality is to apply the generalized method of moments (GMM) approach. (For details, see Bates and White (1988), Ferson (1989), Mark (1988), Rich et al. (1991), Simon (1989)). Let us, for example, represent the GARCH(p,q) model as $\sigma_t^2 = \omega' s_t$, where $\omega' = (a_0, a_1, \ldots, a_q, b_1, \ldots, b_p)$ and $s_t = (1, \varepsilon_{t-1}^2, \ldots, \varepsilon_{t-q}^2, \sigma_{t-1}^2, \ldots, \sigma_{t-p}^2)$. Under the assumption of:

$$E((y_t - x_t'\beta)x_t) = 0$$
$$E((\varepsilon_t^2 - \sigma_t^2)s_t) = 0,$$

the parameters could be estimated by GMM by choosing the vector $\theta' = (\beta', \omega')$ so as to minimize:

$$(g(\theta; I_{t-1}))' \hat{S}(g(\theta; I_{t-1})),$$

where

$$g(\theta; I_{t-1}) = \begin{bmatrix} T^{-1}\sum_{t=1}^{T}(y_t - x_t'\beta)x_t \\ T^{-1}\sum_{t=1}^{T}\left((y_t - x_t'\beta)^2 - \omega' s_t\right)s_t \end{bmatrix},$$

and the matrix $\hat{S}$ can be constructed by any of the methods that have been considered in the GMM literature.

Geweke (1988a,b, 1989) argued that a Bayesian approach, rather than the classical one, might be more suitable for estimating ARCH models due to the distinct features of these models. In order to ensure positivity of the conditional variance, some inequality restrictions should be imposed. Although difficult to impose such restrictions in the classical approach, under the Bayesian framework, diffuse priors can incorporate these inequalities. Also, as the main interest in not in the individual parameters but rather in the conditional variance itself, in the Bayesian framework exact posterior distributions of the conditional variance can be obtained.

Giraitis and Robinson (2000) estimated the parameters of the GARCH process using the Whittle estimation technique and demonstrated that the Whittle estimator is strongly consistent and asymptotically normal, provided the GARCH process has finite 8^{th} moment marginal distribution. Whittle (1953) proposed an estimation technique that works in the spectral domain of the process[6]. Moreover, Mikosch and Straumann (2002) showed that the Whittle estimator is consistent as long as the 4^{th} moment is finite and inconsistent when the 4^{th} moment is infinite. Thus, as noted by Mikosch and Straumann, the Whittle estimator for GARCH processes is unreliable as the ARCH models are applied in heavy-tailed data, sometimes without finite 5^{th}, 4^{th}, or even 3^{rd} moments.

Hall and Yao (2003) showed that for heavy tailed innovations, the asymptotic distribution of quasi-maximum likelihood parameter estimators is non-normal and suggested percentile-t subsample bootstrap approximations to estimator distributions.

2.5. Multivariate ARCH Models

All the ARCH models that have been discussed are univariate. However, assets and markets affect each other not only in terms of expected returns but also in terms of volatility. Thus, the accurate estimation of time-varying covariances between asset returns has been crucial for asset pricing and risk management. The generalization of univariate models to a multivariate context leads to a straightforward application of ARCH models to portfolio selection and asset pricing theory.

Let the $(n \times 1)$ vector $\{\mathbf{y}_t\}$ refer to the multivariate discrete time real-valued stochastic process to be predicted, where $E_{t-1}(\mathbf{y}_t) \equiv \boldsymbol{\mu}_t$ denotes the conditional mean. The innovation process for the conditional mean $\boldsymbol{\varepsilon}_t \equiv \mathbf{y}_t - \boldsymbol{\mu}_t$ has an $(n \times n)$ conditional covariance matrix $V_{t-1}(\mathbf{y}_t) \equiv \mathbf{H}_t$. For a system of n regression equations, the natural extension of (2.2) to a multivariate framework could be presented as:

$$\begin{aligned} \mathbf{y}_t &= \mathbf{B}'\mathbf{x}_t + \boldsymbol{\varepsilon}_t \\ \boldsymbol{\varepsilon}_t \mid I_{t-1} &\sim f[0, \mathbf{H}_t] \\ \mathbf{H}_t &= g(\mathbf{H}_{t-1}, \mathbf{H}_{t-2}, \ldots, \boldsymbol{\varepsilon}_{t-1}, \boldsymbol{\varepsilon}_{t-2}, \ldots), \end{aligned} \tag{2.5.1}$$

where $\mathbf{B}$ is a $k \times n$ matrix of unknown parameters, $\mathbf{x}_t$ a $k \times 1$ vector of endogenous and exogenous explanatory variables included in the available information set, I_{t-1}, $f(.)$ the

[6] For further details about the Whittle estimation technique for ARMA processes see Brockwell and Davis (1991).

conditional multivariate density function of innovation process and $g(.)$ a function of the lagged conditional covariance matrices and innovation process.

The natural multivariate extension of the GARCH(p,q) model in equation (2.2.6) is:

$$\mathbf{H}_t = \mathbf{A_0 A_0'} + \sum_{i=1}^{q}\left(\mathbf{A}_i \boldsymbol{\varepsilon}_{t-i} \boldsymbol{\varepsilon}_{t-i}' \mathbf{A}_i'\right) + \sum_{j=1}^{p}\left(\mathbf{B}_j \mathbf{H}_{t-j} \mathbf{B}_j'\right), \tag{2.5.2}$$

where $\mathbf{A_0}$ is a lower triangular matrix with $(n(n+1)/2)$ parameters and, $\mathbf{A}_i$ and $\mathbf{B}_j$ denote $(n \times n)$ matrices with n^2 parameters each. Engle and Kroner (1995), based on an earlier work of Baba et al. (1990), proposed model (2.5.2) to which they referred as the BEKK model. This parameterization guarantees that $\mathbf{H}_t$ is positive definite and requires the estimation of $(n(n+1)/2) + n^2(q+p)$ parameters. For example, for $n = 3$, the multivariate GARCH(1,1) model contains 24 parameters for estimation. Lee (1999b) investigated the output-inflation variability tradeoff using the bivariate BEKK model. Recently, Moschini and Myers (2002), in order to estimate time-varying optimal hedge ratios in commodity markets, modified the BEKK model of (2.5.2) in the form:

$$\mathbf{H}_t = \mathbf{\Gamma}_t'\left(\mathbf{A_0 A_0'} + \sum_{i=1}^{q}\left(\mathbf{A}_i \boldsymbol{\varepsilon}_{t-i} \boldsymbol{\varepsilon}_{t-i}' \mathbf{A}_i'\right) + \sum_{j=1}^{p}\left(\mathbf{B}_j \mathbf{H}_{t-j} \mathbf{B}_j'\right)\right)\mathbf{\Gamma}_t .$$

As Moschini and Myers noted, the covariance matrix is positive definite as long as $\mathbf{\Gamma}_t$ is a positive definite matrix.

A simpler expression of $\mathbf{H}_t$ can be obtained through the use of the $vech(.)$ operator that stacks the lower portion of a $(n \times n)$ matrix as an $(n(n+1)/2) \times 1$ vector. So, the equation (2.5.2) is rewritten as:

$$vech(\mathbf{H}_t) = vech(\mathbf{A_0 A_0'}) + \sum_{i=1}^{q}\left(\tilde{\mathbf{A}}_i vech\left(\boldsymbol{\varepsilon}_{t-i} \boldsymbol{\varepsilon}_{t-i}'\right)\right) + \sum_{j=1}^{p}\left(\tilde{\mathbf{B}}_j vech\left(\mathbf{H}_{t-j}\right)\right), \tag{2.5.3}$$

where $\tilde{\mathbf{A}}_i$ and $\tilde{\mathbf{B}}_j$ are parameter matrices of dimension $(n(n+1)/2 \times n(n+1)/2)$. Engle et al. (1986) published the first paper on multivariate ARCH models applying the multivariate ARCH(2) model. However, in the multivariate expression of the GARCH(p,q) model, serious problems arise: i) the model might not yield a positive definite covariance matrix unless nonlinear inequality restrictions are imposed, and ii) the number of parameters has to be estimated is $(n(n+1)/2)(1 + (n(n+1)/2)(q+p))$, a very large number even for low dimensions of n. For example, for $n = 3$, the multivariate GARCH(1,1) model contains 78 parameters for estimation.

A number of models, considered in the financial literature, have dealt with imposing constraints in multivariate GARCH models in order to reduce the number of parameters that should be estimated. These constraints have to be compatible with a positive definite conditional covariance matrix and must lead to tractable estimation procedures. Bollerslev et al. (1988) proposed the diagonal multivariate GARCH(p,q) model where the $\tilde{\mathbf{A}}_i$ and $\tilde{\mathbf{B}}_j$ matrices are supposed to be diagonal. Thus, the number of parameters is reduced to $(n(n+1)/2)(1+q+p)$. So, for example, for $n = 3$, the diagonal GARCH(1,1) model requires the estimation of 18 parameters. Bollerslev et al. (1988) used this model for analyzing returns on bills, bonds and stocks, while Baillie and Myers (1991), Bera et al. (1991) and Myers (1991) estimated hedge ratios in commodity markets. Ding and Engle (2001) gave sufficient conditions for the diagonal multivariate GARCH(1,1) model to be positive definite and proposed four models, which are nested to the multivariate diagonal multivariate GARCH(1,1) model.

A special case of the BEKK model, for $p = q = 1$, is the factor GARCH model first proposed in Engle (1987). The factor GARCH(1,1) model was constructed to overcome the problem of estimating a vast number of parameters, while retaining the benefits of positive definiteness. The model has the form:

$$\mathbf{H}_t = \mathbf{A_0}\mathbf{A_0'} + \boldsymbol{\lambda}\boldsymbol{\lambda}'\left(\alpha\left(\mathbf{w}'\boldsymbol{\varepsilon}_{\mathbf{t-1}}\right)^2 + \beta\mathbf{w}'\mathbf{H}_{t-1}\mathbf{w}\right), \tag{2.5.4}$$

where α and β are scalars, $\boldsymbol{\lambda}$ and $\mathbf{w}$ are $(n \times 1)$ vectors. The vector $\mathbf{w}$ can be considered as a vector of portfolio weights and it is convenient to restrict in the case $\boldsymbol{\iota}'\mathbf{w} = 1$, where $\boldsymbol{\iota}$ is a vector of ones. This model is a special case of the BEKK model where the matrices $\mathbf{A}_1$ and $\mathbf{B}_1$ have rank 1: $\mathbf{A}_1 = \sqrt{\alpha}\mathbf{w}\boldsymbol{\lambda}'$ and $\mathbf{B}_1 = \sqrt{\beta}\mathbf{w}\boldsymbol{\lambda}'$. The number of parameters is $(n^2 + 5n)/2 + 2$. So for example, for $n = 3$ we have to estimate 14 parameters. The model can be extended to allow for K factors and a higher order GARCH structure. So, the K factor GARCH(p,q) model is represented by

$$\mathbf{H}_t = \mathbf{A_0}\mathbf{A_0'} + \sum_{k=1}^{K}\boldsymbol{\lambda}_k\boldsymbol{\lambda}_k'\left(\sum_{i=1}^{q}\alpha_{k,i}\mathbf{w}_k'\boldsymbol{\varepsilon}_{t-i}\boldsymbol{\varepsilon}_{t-i}'\mathbf{w}_k + \sum_{j=1}^{p}\beta_{k,j}\mathbf{w}_k'\mathbf{H}_{t-j}\mathbf{w}_k\right), \tag{2.5.5}$$

and has $K(2n + p + q) + n(n+1)/2$ free parameters. Engle et al. (1990b) and Ng et al. (1992) applied factor GARCH models on treasury bills and stock returns. Diebold and

Nerlove (1989), Harvey et al. (1992), King et al. (1994) and Alexander (2000) proposed latent factor GARCH models, based on the assumption that only a few factors influence the conditional variances and covariances of asset returns, which are not functions of the information set.

The constant conditional correlation model, introduced by Bollerslev (1990), is a popular method to model multivariate GARCH models, where univariate GARCH models are estimated for each asset and then the correlation matrix is estimated. The time-varying conditional covariances are parameterized to be proportional to the product of the corresponding conditional standard deviations. This assumption greatly simplifies the estimation of the model and reduces the computational cost. Let us assume that the covariance matrix can be decomposed thus $\mathbf{H}_t = \mathbf{\Sigma}_t^{1/2}\mathbf{C}_t\mathbf{\Sigma}_t^{1/2}$, where $\mathbf{\Sigma}_t$ is the diagonal matrix with the conditional variances along the diagonal and $\mathbf{C}_t$ is the matrix of conditional correlations. The constant conditional correlation model assumes that the matrix of conditional correlations is time invariant, so that the temporal variation of $\mathbf{H}_t$ can be determined solely by the conditional variances:

$$\mathbf{H}_t = \mathbf{\Sigma}_t^{1/2}\mathbf{C}\mathbf{\Sigma}_t^{1/2}. \tag{2.5.6}$$

$\mathbf{H}_t$ is positive definite if $\mathbf{C}$ is positive definite and the conditional variances are positive. The number of parameters reduces to $(n(n-1)/2)+n(1+q+p)$. So, for $n=3$ the constant conditional correlation GARCH(1,1) model requires the estimation of 12 parameters. Several authors have considered this representation, e.g. Baillie and Bollerslev (1990), Brown and Ligeralde (1990), Cecchetti et al. (1988), Fornari et al. (2002), Kim (2000), Kroner and Claessens (1991), Kroner and Lastrapes (1991), Kroner and Sultan (1991,1993), Lien and Tse (1998) and Park and Switzer (1995).

However, recent studies have considered test statistics, which reject the constancy of conditional correlation. Bera and Kim (1996), who proposed the Information Matrix test, were led to the rejection of a constant correlation hypothesis for USA, European and Japan stock markets, while Tse (2000), who derived a Lagrange Multiplier test for the conditional correlation stability hypothesis, rejected the hypothesis for Asian stock markets. Tsui and Yu (1999), adopting the Information Matrix test, examined the China stock market and found that the constant conditional correlation hypothesis is not supported. Longin and Solnik (1995) rejected the hypothesis of constant conditional correlation in international equity returns against three alternative sources of variability of the correlation such as a time trend, the presence of threshold and asymmetry and the influence of information variables.

As the hypothesis of constancy of correlation was rejected in a number of papers, Engle (2000) and Engle and Sheppard (2001) introduced a new form of multivariate ARCH model, the Dynamic Conditional Correlation GARCH, or DCC-GARCH(C,M), model. The model is estimated in two steps. The first is a series of univariate GARCH estimates. The second step, using the residuals resulting for the first stage, evaluates the conditional correlation estimator. The success of the DCC-GARCH model depends on the estimability of extremely large time varying covariance matrices. Engle proposed to use the decomposed covariance matrix $\mathbf{H}_t = \mathbf{\Sigma}_t^{1/2}\mathbf{C}_t\mathbf{\Sigma}_t^{1/2}$ and suggested a time varying correlation matrix of the following form:

$$\mathbf{C}_t = \mathbf{Q}_t^{*-1/2}\mathbf{Q}_t\mathbf{Q}_t^{*-1/2}. \qquad (2.5.7)$$

The conditional variances, $\sigma_{k,t}^2$, are estimated as univariate GARCH(p_k, q_k) models, allowing for different lag lengths for each series $k = 1,2,...,n$,

$$\sigma_{k,t}^2 = a_{k,0} + \sum_{i=1}^{q_k}\left(a_{k,i}\varepsilon_{k,t-i}^2\right) + \sum_{j=1}^{p_k}\left(b_{k,j}\sigma_{k,t-j}^2\right). \qquad (2.5.8)$$

The correlation matrix is computed using

$$\mathbf{Q}_t = \left(1 - \sum_{m=1}^{M}a_m - \sum_{c=1}^{C}b_c\right)\overline{\mathbf{Q}} + \sum_{m=1}^{M}a_m\left(\mathbf{z}_{t-m}\mathbf{z}'_{t-m}\right) + \sum_{c=1}^{C}b_c\mathbf{Q}_{t-c}, \qquad (2.5.9)$$

where $\mathbf{z}_t$ are the residuals standardized by their conditional standard deviation, $\overline{\mathbf{Q}}$ is the unconditional covariance of the standardized residuals and $\mathbf{Q}_t^{*-1/2}$ is a diagonal matrix composed of the square roots of the diagonal elements of $\mathbf{Q}_t$. Engle and Sheppard (2001) proved the consistency and asymptotic normality of the two-step estimators as well as the positive definiteness of the covariance matrix. They have also proposed a test of the null hypothesis of constant correlation against an alternative of dynamic conditional correlation. Christodoulakis and Satchell (2002) considered an alternative extension of the constant conditional correlation model of Bollerslev (1990) and developed a bivariate ARCH model with time varying conditional variances and correlations, named Correlated ARCH, or CorrARCH, model.

The multivariate ARCH models, that have been presented, although simplifying the estimation and inference procedures, do not account for empirical regularities such as the asymmetric effects. In to order to capture the "leverage effect" in a multivariate framework, Braun et al. (1995) introduced a bivariate version of the EGARCH model in equation (2.2.13). Sentana (1995), in the presentation of the quadratic GARCH model, applied a multivariate version of his model to U.K. stock returns. Kroner and Ng (1998), following Hentschel's (1995) approach, introduced a general multivariate GARCH model which nests the BEKK, diagonal, factor and constant conditional correlation GARCH models and their natural asymmetric extensions. Their model can be regarded as a multivariate extension of the GJR model in equation (2.2.13). Bekaert and Wu (1997), Ding and Engle (2001) and Tai (2001) have also modified multivariate ARCH models to accommodate asymmetric effects on conditional variances and covariances. Brunetti and Gilbert (1998), based on Bollerslev's (1990) parameterization, proposed the bivariate constant correlation FIGARCH model and Brunetti and Gilbert (2000) applied the model to the crude oil market. Finally, Bayesian analysis of symmetric and asymmetric multivariate ARCH processes was considered in a number of articles such as Aguilar and West (2000), Giakoumatos et al. (2005) and Vrontos et al. (2000, 2001, 2003).

2.6. Other Methods of Volatility Modeling

"*Stochastic volatility*" models (Barndorff-Nielsen et al. (2002), Chib et al. (1998), Giakoumatos (2004), Ghysels et al. (1996), Harvey and Shephard (1993), Jacquier et al. (1994), Shephard (1996), Taylor (1994)), "*implied volatility*" models (Day and Lewis (1988), Latane and Rendleman (1976), Schmalensee and Trippi (1978)), "*historical volatility*" models (Beckers (1983), Garman and Klass (1980), Kunitomo (1992), Parkinson (1980), Rogers and Satchell (1991)) and "*realized volatility*" models are examples from the financial econometric literature of estimating volatility of asset returns.

A typical presentation of a stochastic volatility model can be given by

$$\begin{aligned} \varepsilon_t &= z_{1,t}\sigma e^{0.5\sigma_t} \\ \sigma_t^2 &= a\sigma_{t-1}^2 + z_{2,t} \\ z_{1,t} &\overset{i.i.d.}{\sim} f\left[E(z_t)=0, V(z_t)=1\right] \\ z_{2,t} &\overset{i.i.d.}{\sim} g\left[E(z_t)=0, V(z_t)=\sigma_{z_2}^2\right], \end{aligned} \tag{2.6.1}$$

where σ is a positive scale parameter, $|a| < 1$, and the error terms $z_{1,t}$ and $z_{2,t}$ could be contemporaneously correlated. The additional error term, $z_{2,t}$, in the conditional variance equation makes the stochastic volatility model have no closed form solution. Hence, the estimation of the parameters is a quite difficult task. For this reason, stochastic volatility models are not as popular as the ARCH processes. Jacquier et al. (1994) considered a Markov Chain Monte Carlo (MCMC) framework in order to estimate stochastic volatility models and Jacquier et al. (1999, 2004) extended the MCMC technique to allow for the leverage effect and fat tailed conditional errors. For extensions and applications of MCMC techniques of ARCH models the interested reader may be referred to Brooks et al. (1997), Dellaportas and Roberts (2003), Dellaportas et al. (2002), Giakoumatos et al. (1999), Kaufmann and Fruhwirth-Schnatter (2002) and Nakatsuma (2000). Nelson (1990b) was the first to show that the continuous time limit of an ARCH process, which is a stochastic difference equation, is a diffusion process with stochastic volatility (which is a stochastic differential equation). Duan (1996) extended Nelson's study.

Models based on the daily open, high, low and close asset prices, and exponential smoothing methods, such as the Riskmetrics method by J.P. Morgan, are procedures which are included to the historical volatility models.

Implied volatility is the instantaneous standard deviation of the return on the underlying asset, which would have to be input into a theoretical pricing model in order to yield a theoretical value identical to the price of the option in the marketplace, assuming all other inputs are known. Day and Lewis (1992) examined whether implied volatilities contain incremental information relative to the estimated volatility from ARCH models. Noh et al. (1994) compared the forecasting performance of ARCH and implied volatility models in the context of option pricing. Andersen et al. (2004) reviewed a systematically categorization of various ways of modeling volatility. Recently, Poon and Granger (2001) conducted a comparative review based on the forecasting performance of ARCH, implied volatility, and historical volatility models.

Although the presentation of the above methods of volatility estimation is beyond the scope of this chapter, we briefly refer to the modeling of realized volatility, as it is a recently developed promising area of volatility model building.

2.6.1 Intra-Day Realized Volatility Models

The modeling of realized volatility is based on the idea of using higher frequency data to generate more accurate volatility estimates of lower frequency. Andersen and Bollerslev (1998a) introduced an alternative volatility measure, the "*realized volatility*". For P_t denoting the price of an asset at day t, let the difference of the log-prices,

$$y_{(m),t} = \ln(P_t) - \ln(P_{t-1/m}), \text{ where } t = 1/m, 2/m, \ldots, \tag{2.6.2}$$

denote the discretely observed series of continuously compounded returns with m observations per day. The realized volatility for a horizon of N days ahead is:

$$\tilde{s}^2_{t(N)} = N^{-1} \sum_{i=1}^{N} \sum_{j=1}^{m-1} \left(\ln\left(P_{(j+1/m),t-i}\right) - \ln\left(P_{(j/m),t-i}\right)\right)^2 . \tag{2.6.3}$$

Andersen at al. (2000b, 2001a, 2003) and Andersen at al. (2001b) were the first studies that explored the distributional properties of the realized volatility. The main results are that i) although the distribution of asset returns is non-normal (highly skewed and kurtosed), the distribution of returns scaled by the realized standard deviation is approximately Gaussian and ii) the realized logarithmic standard deviation is also nearly Gaussian. The concept of the realized volatility is based on the "*integrated volatility*", which is central to the stochastic volatility option pricing in Hull and White (1987). Over an interval of length h, the integrated volatility is defined as:

$$y^2_{h,t} = \int_0^h s^2_{t-h+r} dr , \tag{2.6.4}$$

where s_t is the volatility of the instantaneous returns process, generated by the continuous time martingale, $d\ln(P_t) = s_t dW_t$, (W_t is the standard Wiener process). In the case of discrete time with a sample frequency of $h = 1/m$, $y^2_{(1/h),t}$ is an unbiased estimator of $y^2_{h,t}$. As noted by Ebens (1999) and Andersen and Bollerslev (1998a) for daily volatility forecasts, or $(h = 1)$, the discretely sampled daily returns, for $(m = 1)$, constitute a noisy estimator, but the accuracy improves as the sampling frequency is increasing, $(m \to \infty)$. However, the observed tick-by-tick asset prices are available only at discrete points in time and asset returns are characterized by the effect of non-synchronous trading. Thus, the sampling frequency should be as high as the market microstructure features do not induce bias to volatility estimator, i.e. Andersen and Bollerslev (1998a), Andersen et al. (1999), Andersen et al. (2000a), Andersen et al.

(2001a), Areal and Taylor (2000), Kayahan et al. (2002) used a sampling frequency of 5-minites for heavily traded assets. The 5-minites sampling frequency were also used in the majority of the subsequent studies.

Ebens (1999), Giot and Laurent (2001), and Thomakos and Wang (2002) proposed the use of an ARFIMA model, in the form of (2.2.40), in order to fit the logarithmic realized variance. For more information and reference about applications and properties of the realized volatility and the use of intraday data see Andersen (2000), Andersen and Bollerslev (1997), Andersen and Bollerslev (1998b), Andersen et al. (2003), Andersen et al. (2004), Angelidis and Degiannakis (2005a), Barndorff-Nielsen and Shephard (2002, 2005), Bollerslev and Wright (2001), Oomen (2001) and Taylor and Xu (1997).

2.7. Interpretation of ARCH Process

A number of studies have aimed at explaining the prominence of ARCH process in financial applications. Stock (1987, 1988) established the time deformation model, in which economic and calendar time proceed at different speed, and linked the relation between time deformation and ARCH models. Any economic variable evolves on an operational time scale, while in practice it is measured on a calendar time scale. The inappropriate use of calendar time scale leads to volatility clustering since relative to the calendar time, the variable may evolve quicker or slower. The time deformation model for a random variable y_t has the form:

$$\begin{aligned} y_t &= p_t y_{t-1} + \varepsilon_t \\ \varepsilon_t \mid I_{t-1} &\sim N\left(0, \sigma_t^2\right) \\ \sigma_t^2 &= a_0 + a_1 \varepsilon_{t-1}^2. \end{aligned} \tag{2.7.1}$$

According to Stock, when a long segment of operational time elapsed during a unit of calendar time, p_t is small and σ_t^2 is large. In order words, the time varying autoregressive parameter is inversely related to the conditional variance.

Mizrach (1990) developed a model in which the errors, made by the participants of the market on investing, are strongly dependent on all past errors. The highly persistence on the errors forces the volatility of asset returns to have an ARCH like structure.

Gallant et al. (1991), based on some earlier work by Clark (1973), Mandelbrot and Taylor (1967), Tauchen and Pitts (1983), and Westerfield (1977) provided a theoretical interpretation of ARCH effect. Let us assume that the asset returns are defined by a stochastic number of intra-period price revisions so that they can be decomposed to:

$$y_t = \mu_t + \sum_{i=1}^{\omega_t} \zeta_i \ , \tag{2.7.2}$$

where μ_t is the forecastable component, $\zeta_i \overset{i.i.d.}{\sim} N\left(0, s^2\right)$ denotes the incremental changes and ω_t is the number of times new information comes to the market in time t. ω_t is an unobservable random variable and is independent of the incremental changes. In such a case, the asset returns are not normally distributed, as their distribution is a mixture of normal distributions. Rewriting the equation (2.7.2) as: $y_t = \mu_t + s^2 \sqrt{\omega_t} z_t$, with z_t, $t = 1,2,...$ as i.i.d. standard normal variables, the y_t conditional on any information set, ω_t and I_{t-1}, is normally distributed:

$$y_t \mid \left(\omega_t, I_{t-1}\right) \sim N\left(\mu_t, s^2 \omega_t\right). \tag{2.7.3}$$

However, the knowledge of information that flows into the market is an unrealistic assumption. Hence, the y_t conditional on the information set available to the market participants is:

$$y_t \mid I_{t-1} \sim N\left(\mu_t, s^2 E_{t-1}\left(\omega_t\right)\right). \tag{2.7.4}$$

Note that the conditional kurtosis, $3E_{t-1}\left(\omega_t^2\right) / E_{t-1}\left(\omega_t\right)^2$, exceeds 3, as in the ARCH process where the innovation, ε_t, always has fatter tails than its unconditional normal distribution:

$$E\left(\varepsilon_t^4\right) / E\left(\varepsilon_t^2\right)^2 \geq 3. \tag{2.7.5}$$

Lamoureux and Lastrapes (1990) assumed that the number of information arrivals is serially correlated and used the daily trading volume as a proxy variable for the daily information that flows into the stock market. Hence, ω_t can be expressed as an autoregressive process:

$$\omega_t = b_0 + \sum_{i=1}^{\kappa} b_i \omega_{t-i} + z_t$$
$$z_t \overset{iid}{\sim} N(0,1) \qquad (2.7.6)$$

From (2.7.4) we know that $E\left((y_t - \mu_t)^2 \mid I_{t-1}\right) = s^2 \omega_t$, thus (2.7.6) becomes

$$E\left((y_t - \mu_t)^2 \mid I_{t-1}\right) = s^2 b_0 + \sum_{i=1}^{\kappa} b_i E\left((y_{t-i} - \mu_{t-i})^2 \mid I_{t-i-1}\right) + s^2 z_t \,. \qquad (2.7.7)$$

The structure in (2.7.7) expresses the persistence in conditional variance, a characteristic that is captured by the ARCH process. Lamoureux and Lastrapes (1990) used the trading volume as a proxy variable for ω_t. Including the daily trading volume, V_t, as an exogenous variable in the GARCH(1,1) model, they found that its coefficient was highly significant whereas the ARCH coefficients became negligible:

$$\sigma_t^2 = a_0 + a_1 \varepsilon_t^2 + b_1 \sigma_t^2 + \delta V_t \,. \qquad (2.7.8)$$

The heteroscedastic mixture model assumes that $\delta > 0$ and that the persistence of variance as measured by $a_1 + b_1$ should become negligible. Their work provided empirical evidence that the ARCH process is a manifestation of the time dependence on the rate of information arrival to the market.

Brailsford (1996) and Pyun et al. (2000) applied versions of the heteroscedastic mixture model and reported that the degree of persistence reduced as a proxy for information arrival enters into the variance equation. On the other hand, a number of studies (i.e. Abhyankar (1995), Bessembinder and Seguin (1993), Najand and Yung (1991), Locke and Sayers (1993), Sharma et al. (1996)) tested the mixture of distributions hypothesis, for various sets of data, and found that the ARCH coefficients remain statistically significant even after a trading volume is included as an exogenous variable in the model. This contradiction forced Miyakoshi (2002) to reexamine the relation between ARCH effects and rate of information arrival to the market. By using data from the Tokyo Stock Exchange, Miyakoshi showed that for periods with important market announcements, the trading volume affects the return volatility and the ARCH coefficients become negligible, while for periods which lack of "*big news*" the ARCH structure characterizes the conditional variance, adequately. The mixture of distributions hypothesis was also reexamined by Luu and Martens (2002) in the context of "*realized volatility*".

Engle et al. (1990a) evaluated the role of the information arrival process in the determination of volatility in a multivariate framework providing a test of two hypotheses: heat waves and meteor showers. Using meteorological analogies, they supposed that information follows a process like a heat wave so that a hot day in New York is likely to be followed by another hot day in New York but not typically by a hot day in Tokyo. On the other hand, a meteor shower in New York, which rains down on the earth as it turns, will almost surely be followed by one in Tokyo. Thus, the heat wave hypothesis is that the volatility has only country specific autocorrelation, while the meteor shower hypothesis states that volatility in one market spills over to the next. They examined intra daily volatility in the foreign exchange markets, focusing on time periods corresponding to the business hours of different countries. Their research based on the Yen/Dollar exchange rate while the Tokyo, European and New York market are open. They found that the foreign news was more important than the past domestic news. So, the major effect is more like a meteor shower, i.e. Japanese news had a greater impact on the volatility of all markets except the Tokyo market. This is interpreted as evidence that volatility in part arises from trading rather than purely from news. Conrad et al. (1991), Pyun et al. (2000) and Ross (1989) examined the volatility spillover effect across large and small capitalization companies. The main finding is that volatility propagates asymmetrically in sense that the effect of shocks of larger firms on the volatility of smaller companies is more significant than that from smaller firms to larger companies.

Bollerslev and Domowitz (1991) showed how the actual market mechanisms may themselves result in a very different temporal dependence in the volatility of transaction prices, with a particular automated trade execution system inducing a very high degree of persistence in the conditional variance process.

Alternative expositions for theoretical evidence on the sources of ARCH effect have been presented by Attanasio and Wadhwani (1989), Backus et al. (1989), Brock and Kleidon (1990), Diebold and Pauly (1988), Domowitz and Hakkio (1985), Engle and Susmel (1990), Giovannini and Jorion (1989), Hodrick (1989), Hong and Lee (2001), Hsieh (1988), Lai and Pauly (1988), Laux and Ng (1993), Ng (1988), Schwert (1989a), Smith (1987) and Thum (1988). Nelson (1990b) was the first to show how ARCH models can emerge from diffusion processes. The problem of estimation of discretely sampled diffusions, such as ARCH processes, and their relationship with continuous time models has also been considered in the literature (see, e.g., Aït-Sahalia (2001, 2002), and the references therein).

Chapter 3
A Conjecture on the Independence of the Standardized One-Step-Ahead Prediction Errors of the ARCH Model

3.1. *Introduction*

In statistical modeling contexts the use of one-step-ahead prediction errors for testing hypotheses on the forecasting ability of an assumed model has been widely considered. Quite often, the testing procedure requires independence in a sequence of recursive standardized prediction errors, which cannot always be readily deduced particularly in the case of econometric modeling. In this chapter, on the basis of the results of a series of Monte Carlo simulations, it is conjectured that independence holds and the sum of squared standardized one-step-ahead prediction errors is Chi-square distributed. The methodologies used in the remainder of the thesis are based on the assumption that the standardized one-step-ahead prediction errors are a collection of independently and identically distributed variables. Thus, the question of whether the above quantities are indeed independently distributed is crucially important.

3.2. *Monte Carlo Study: Simulating the AR(1)GARCH(1,1) Process*

An ARCH process, ε_t, is presented as:

$$\begin{aligned} \varepsilon_t &= z_t \sigma_t \\ z_t &\overset{i.i.d.}{\sim} N(0,1) \\ \sigma_t^2 &= g(\sigma_{t-1}, \sigma_{t-2}, \ldots, \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots) \end{aligned} \tag{3.2.1}$$

where z_t is a sequence of independently and identically distributed random variables, σ_t is a time-varying, positive measurable function of the information set at time $t-1$ and $g(.)$ could be a linear or nonlinear functional form that has been presented in the ARCH literature.

The squared value of the i.i.d. standard normal process is Chi-square distributed with 1 degree of freedom, or $z_t^2 \sim \chi_1^2$, and the sum of T i.i.d. standard normal processes is Chi-square distributed with T degrees of freedom, or $\sum_{t=1}^{T} z_t^2 \sim \chi_T^2$. The expected value and the variance of the Chi-square distributed process with T degrees of freedom are $E\left(\sum_{t=1}^{T} z_t^2\right) = T$ and $V\left(\sum_{t=1}^{T} z_t^2\right) = 2T$, respectively. Moreover, if z_t is an i.i.d. random sequence then the autocorrelation, $Cor(z_t, z_{t+\tau})$, is approximately $N(0, T^{-1})$ and any transformation of z_t is also an i.i.d. random sequence (see Ding et al. (1993)).

Since very few financial time series have a constant conditional mean of zero, an ARCH model can be presented in a κ^{th} order autoregressive form by letting ε_t be the innovation process in a linear regression:

$$\begin{aligned} y_t &= \sum_{i=1}^{\kappa} (c_i y_{t-i}) + \varepsilon_t \\ \varepsilon_t \mid I_{t-1} &\equiv z_t \sigma_t \sim N(0, \sigma_t^2) \\ z_t &\overset{i.i.d.}{\sim} N(0,1) \\ \sigma_t^2 &= g(\sigma_{t-1}, \sigma_{t-2}, \ldots, \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots) \end{aligned} \tag{3.2.2}$$

The disturbances, ε_t, are normally distributed with time varying conditional variance $\sigma_t^2 = E_{t-1}(\varepsilon_t^2)$. The most commonly used conditional variance function is the GARCH(1,1) model:

$$\sigma_t^2 = a_0 + a_1 \varepsilon_{t-1}^2 + b_1 \sigma_{t-1}^2 .$$

In the sequel, a Monte Carlo simulation is used to provide evidence for the assumption of independently and identically distributed standardized one-step-ahead prediction errors. Our strategy runs as follow:

1) Generate data from the AR(1)GARCH(1,1) process.

- Generate a series of 32.000 values from the standard normal distribution $z_t \overset{i.i.d.}{\sim} N(0,1)$.

- Generate the ARCH process, $\{\varepsilon_t\}_{t=1}^{32000}$, by multiplying the i.i.d. random sequence with a specific conditional variance form, or $\varepsilon_t = z_t\sqrt{\sigma_t^2}$, for $\sigma_t^2 = 0.0001 + 0.12\varepsilon_{t-1}^2 + 0.8\sigma_{t-1}^2$.
- Generate a first order autoregressive processes, $y_t = 0.06y_{t-1} + \varepsilon_t$, for the conditional mean, based on the $\{\varepsilon_t\}_{t=1}^{32000}$ process.

Figure 3.1 plots the simulated processes, Figure 3.2 presents the relevant histograms and descriptive statistics, and Figure 3.3 depicts the histograms of the Chi-square distribution with T degrees of freedom. The figures are presented in the Appendix. The Chi-square distributed process, with T degrees of freedom, is constructed as $\sum_{t=1}^{T} z_t^2$.

According to the literature (e.g. Engle and Mustafa (1992)), the shocks to the variance,

$$E_t\left(\varepsilon_t^2\right) - E_{t-1}\left(\varepsilon_t^2\right) = \varepsilon_t^2 - \sigma_t^2 \equiv v_t,$$

generate a martingale difference sequence (in the sense that it cannot be predicted from its past). These shocks are neither serially independent nor identically distributed. Let us take a glance at the autocorrelations of the variables. z_t has to be serially uncorrelated, the shocks to the variance v_t should be autocorrelated, and the conditional variance σ_t^2 would be highly correlated. As z_t is an i.i.d. random sequence, the transformations of z_t, ($|z_t|^d$, $\forall d > 0$), are uncorrelated in each case. Figure 3.4, in the Appendix, presents the autocorrelation of transformations of the processes $z_t, v_t, \sigma_t, \varepsilon_t$. The half length of the 95% confidence interval for the estimated sample autocorrelation equals $1.96/\sqrt{T} = 0.0113$, if the process is i.i.d. normally distributed. On the other hand, σ_t^2 is autocorrelated at any lag, while both v_t and ε_t are autocorrelated in half of the cases.

Ding and Ganger (1996) and Karanasos (1996) give the autocorrelation function of the squared errors for the GARCH(1,1) process and Karanasos (1999) extends the results to the GARCH(p,q) model. He and Teräsvirta (1999) derive the autocorrelation function of the squared and absolute errors for a family of first order ARCH processes.

The number of estimated autocorrelations that are outside the 95% confidence interval is presented in the Table that follows.

Table 3.1. Percentage of autocorrelations outside the 95% confidence interval ($\tau = 1,2,...,100$).						
$d =$	0.5	1	1.5	2	2.5	3
$Cor\left(\lvert z_t\rvert^d, \lvert z_{t+\tau}\rvert^d\right)$	6%	6%	5%	4%	5%	6%
$Cor\left(\lvert \varepsilon_t\rvert^d, \lvert \varepsilon_{t+\tau}\rvert^d\right)$	36%	44%	45%	47%	47%	44%
$Cor\left(\lvert v_t\rvert^d, \lvert v_{t+\tau}\rvert^d\right)$	56%	54%	44%	30%	20%	15%
$Cor\left(\sigma_t^d, \sigma_{t+\tau}^d\right)$	92%	97%	98%	95%	93%	86%

2) Estimate the parameters of the AR(1)GARCH(1,1) model.

- The AR(1)GARCH(1,1) model is applied, for the data produced from the AR(1)GARCH(1,1) process. Dropping out the first 1000 data, maximum likelihood estimates of the parameters are obtained by numerical maximization of the log-likelihood function, using a rolling sample of constant size equal to 1000[1]. At each of a sequence of points in time, the maximum likelihood parameter vector, $\hat{\theta}_t \equiv \left(\hat{c}_{1,t}, \hat{a}_{0,t}, \hat{a}_{1,t}, \hat{b}_{1,t}\right)$, is being estimated in order to compute the conditional mean and variance:

$$\hat{y}_{t+1|t} = \hat{c}_{1,t} y_t$$

$$\hat{\sigma}^2_{t+1|t} = \hat{a}_{0,t} + \hat{a}_{1,t}\varepsilon^2_{t|t} + \hat{b}_{1,t}\sigma^2_{t|t}.$$

Thus, the model is estimated 30.000 times. Note that $\varepsilon^2_{t|t}$ and $\sigma^2_{t|t}$ belong to the I_t, so are considered as observable.

3) Compute the standardized one-step-ahead prediction errors, $\left\{\hat{z}_{t+1|t}\right\}_{t=1}^{30000}$, $\hat{z}_{t+1|t} = \left(y_{t+1} - \hat{y}_{t+1|t}\right)\hat{\sigma}^{-1}_{t+1|t}$. The SPEC model selection algorithm uses the sum of the squared standardized one-step-ahead prediction errors, or $\sum_{t=1}^{T}\hat{z}^2_{t|t-1}$.

- The one-step-ahead estimated processes are presented in Figure 3.5, Figure 3.6, in the Appendix, presents the relevant histograms and the descriptive statistics, respectively. The one-step-ahead standardized prediction error process, conditional on the information set available at time t, $\hat{z}_{t+1|t} = \left(y_{t+1} - \hat{y}_{t+1|t}\right)\hat{\sigma}^{-1}_{t+1|t}$, is approximately normally distributed, while $\hat{z}^2_{t+1|t}$ is Chi-square distributed with 1 degree of freedom.

[1] Maximum likelihood estimates of the parameters are obtained by numerical maximization of the log-likelihood function using the Marquardt algorithm (Marquardt (1963)). The quasi-maximum likelihood

Moreover, if $\hat{z}^2_{t+1|t}$ is independently Chi-square distributed, $\sum_{t=1}^{T}\hat{z}^2_{t+1|t}$ should be, also, Chi-square distributed with T degrees of freedom, with mean and variance:

$$E\left[\sum_{t=1}^{T}\hat{z}^2_{t+1|t}\right]=T \text{ and } V\left[\sum_{t=1}^{T}\hat{z}^2_{t+1|t}\right]=2T .$$

Figure 3.7, in the Appendix, plots the histograms of $\sum_{t=1}^{T}\hat{z}^2_{t+1|t}$. All the histograms are almost identical to the simulated Chi-squared histograms. Moreover, if $\hat{z}_{t+1|t}$ is an i.i.d. random sequence then the sample autocorrelation, $Cor\left(\hat{z}_{t+1|t},\hat{z}_{t+\tau+1|t+\tau}\right)$, is approximately $N\left(0,T^{-1}\right)$ and the autocorrelation of any transformation of $\hat{z}_{t+1|t}$, $Cor\left(\left|\hat{z}_{t+1|t}\right|^d,\left|\hat{z}_{t+\tau+1|t+\tau}\right|^d\right)$, $\forall d>0$, is also $N\left(0,T^{-1}\right)$. Figure 3.8, in the Appendix, presents the autocorrelation of transformations of the processes $\hat{z}_{t+1|t}$, $\hat{\varepsilon}_{t+1|t}$, $\hat{v}_{t+1|t}$, $\hat{\sigma}_{t+1|t}$.

As the sum of squared standardized one-step-ahead prediction errors is Chi-square distributed, and the transformations of $\hat{z}_{t+1|t}$ are not autocorrelated, the standardized one-step-ahead innovations, $\hat{z}_{t+1|t}$, should be independent.

3.3. *Monte Carlo Study: Simulating the GARCH, EGARCH and TARCH Processes*

In the sequel the assumption that the standardized one-step-ahead prediction errors are independently and identically distributed (or equivalently that the sum of T one-step-ahead prediction errors is Chi-square distributed) is investigated for a higher order of autoregressive process for the conditional mean and the following conditional variance functions:

The GARCH(p,q) model, Bollerslev (1986)

$$\sigma_t^2=a_0+\sum_{i=1}^{q}\left(a_i\varepsilon_{t-i}^2\right)+\sum_{i=1}^{p}\left(b_i\sigma_{t-i}^2\right) \tag{3.3.1}$$

estimator (QMLE) is used, as according to Bollerslev and Wooldridge (1992), it is generally consistent, has a normal limiting distribution and provides asymptotic standard errors that are valid under non-normality.

The EGARCH(p,q) model, Nelson (1991)

$$\ln\left(\sigma_t^2\right) = a_0 + \sum_{i=1}^{q}\left(a_i\left|\frac{\varepsilon_{t-i}}{\sigma_{t-i}}\right| + \gamma_i\left(\frac{\varepsilon_{t-i}}{\sigma_{t-i}}\right)\right) + \sum_{i=1}^{p}\left(b_i \ln\left(\sigma_{t-i}^2\right)\right) \tag{3.3.2}$$

The TARCH(p,q) model, Glosten et al. (1993)

$$\sigma_t^2 = a_0 + \sum_{i=1}^{q}\left(a_i \varepsilon_{t-i}^2\right) + \gamma_1 \varepsilon_{t-1}^2 d_{t-1} + \sum_{i=1}^{p}\left(b_i \sigma_{t-i}^2\right), \tag{3.3.3}$$

where $d_t = 1$ if $\varepsilon_t < 0$, and $d_t = 0$ otherwise.

1. Eight processes have been generated with the coefficients presented in the following Table.

Table 3.2. Coefficients of the simulated processes.

Model	Parameters							
	c_1	c_2	c_3	a_0	a_1	a_2	b_1	γ_1
a) AR(1)GARCH(1,1)	0.05	-	-	0.002	0.05	-	0.91	-
b) AR(1)EGARCH(1,1)	0.05	-	-	0.2	0.05	-	0.2	0.1
c) AR(1)TARCH(1,1)	0.05	-	-	0.002	0.15	-	0.7	-0.08
d) AR(1)GARCH(1,2)	0.05	-	-	0.002	0.05	0.08	0.8	-
e) AR(1)TARCH(1,2)	0.05	-	-	0.002	0.15	0.05	0.7	-0.08
f) AR(3)GARCH(1,1)	0.1	0.03	-0.02	0.002	0.05	-	0.91	-
g) AR(3)EGARCH(1,1)	0.12	0.07	-0.03	0.001	0.05	-	0.2	0.1
h) AR(3)TARCH(1,1)	0.1	0.03	-0.02	0.002	0.15	-	0.7	-0.08

2. Estimate the parameters of the simulated processes.

- At each of a sequence of points in time, the maximum likelihood parameter vector $\hat{\theta}_t \equiv \left(\hat{c}_{1,t}, \hat{c}_{2,t}, \hat{c}_{3,t}, \hat{a}_{0,t}, \hat{a}_{1,t}, \hat{a}_{2,t}, \hat{b}_{1,t}, \hat{\gamma}_{1,t}, \hat{\gamma}_{2,t}\right)$ is being estimated. The models are estimated 30.000 times and the conditional mean and variance are computed in (3.3.4)-(3.3.7):

The κ^{th} order Autoregressive process

$$\hat{y}_{t+1|t} = \sum_{i=1}^{\kappa}\left(\hat{c}_{i,t} y_{t+1-i}\right) \tag{3.3.4}$$

The GARCH(1,q) model

$$\hat{\sigma}_{t+1|t}^2 = \hat{a}_{0,t} + \sum_{i=1}^{q}\left(\hat{a}_{i,t} \varepsilon_{t-i+1|t}^2\right) + \hat{b}_{1,t} \sigma_{t|t}^2 \tag{3.3.5}$$

The EGARCH(1,q) model

$$\hat{\sigma}^2_{t+1|t} = \exp\left(\hat{a}_{0,t} + \hat{a}_{1,t}\left|\frac{\varepsilon_{t|t}}{\sigma_{t|t}}\right| + \hat{\gamma}_{1,t}\left(\frac{\varepsilon_{t|t}}{\sigma_{t|t}}\right) + \hat{b}_{1,t}\ln\left(\sigma^2_{t|t}\right)\right) \tag{3.3.6}$$

The TARCH(1,q) model

$$\hat{\sigma}^2_{t+1|t} = \hat{a}_{0,t} + \sum_{i=1}^{q}\left(\hat{a}_{i,t}\varepsilon^2_{t-i+1|t}\right) + \hat{\gamma}_{1,t}\varepsilon^2_{t|t}d_t + \hat{b}_{1,t}\sigma^2_{t|t}, \tag{3.3.7}$$

where $d_t = 1$ if $\varepsilon_t < 0$, and $d_t = 0$ otherwise.

3. Compute the standardized one-step-ahead prediction errors $\hat{z}_{t+1|t} = \left(y_{t+1} - \hat{y}_{t+1|t}\right)\hat{\sigma}^{-1}_{t+1|t}$ and examine the following properties:

- Histogram, mean and variance of $\left\{\hat{z}^2_{t+1|t}\right\}_{t=1}^{30000}$.
- Histogram, mean and variance of $\left\{\sum_{j=t-T+1}^{t}\hat{z}^2_{j+1|j}\right\}$, for $t = T(T)30000$.[2]
- Sample autocorrelation, $Cor\left(\hat{z}_{t+1|t}, \hat{z}_{t+\tau+1|t+\tau}\right)$, for $\tau = 1(1)100$.
- Sample autocorrelation of transformations of $\hat{z}_{t+1|t}$, $Cor\left(\left|\hat{z}_{t+1|t}\right|^d, \left|\hat{z}_{t+\tau+1|t+\tau}\right|^d\right)$, for $\tau = 1(1)100$ and $d = 0.5(0.5)3$.

Figures 3.9, 3.10 and 3.11, in the Appendix, plot the histograms of $\left\{\hat{z}^2_{t+1|t}\right\}_{t=1}^{30000}$, the histograms of $\left\{\sum_{j=t-T+1}^{t}\hat{z}^2_{j+1|j}\right\}$, for $t = T(T)30000$ and the autocorrelation of the processes $Cor\left(\left|\hat{z}_{t+1|t}\right|^d, \left|\hat{z}_{t+\tau+1|t+\tau}\right|^d\right)$, for $\tau = 1(1)100$ and $d = 0.5(0.5)3$, for each of the eight generated processes. The property of independently and identically distributed standardized one-step-ahead prediction errors holds.

3.4. *Monte Carlo Study: Simulating the GARCH(1,1) Process for Various Coefficient Values*

Simulate one more set of GARCH(1,1) processes in order to investigate if changes in the coefficients change the distribution of squared standardized one-step-ahead prediction errors.

[2] Here, $T = a(b)c$ denotes $T = a, a+b, a+2b, ..., c-b, c$.

- Generate 18 series of 5.000 values from the standard normal distribution $z_t \overset{i.i.d.}{\sim} N(0,1)$.
- Generate 18 GARCH(1,1) processes $\{\varepsilon_t\}_{t=1}^{20000}$ by multiplying the i.i.d. random sequence with σ_t from $\sigma_t^2 = 0.002 + 0.05\varepsilon_{t-1}^2 + b_1^{(k)}\sigma_{t-1}^2$ where $b_1^{(k)} = 0.05 * k$ for $k = 1,2,...,18$.
- Estimate the parameters of the 18 GARCH(1,1) models.
- Compute $\hat{z}_{t+1|t} = \left(y_{t+1} - \hat{y}_{t+1|t}\right)\hat{\sigma}_{t+1|t}^{-1}$.

The histograms of $\left\{\sum_{j=t-T+1}^{t} \hat{z}_{j+1|j}^2\right\}$, for $t = T(T)30000$ and the autocorrelation functions $Cor\left(\left|\hat{z}_{t+1|t}\right|^d, \left|\hat{z}_{t+\tau+1|t+\tau}\right|^d\right)$, $\tau = 1(1)100$ and $d = 0.5(0.5)3$, are similar to these plotted in the previous sections.

3.5. *Conclusion*

The sum of squared standardized one-step-ahead prediction errors is Chi-square distributed, and any transformation of the $\hat{z}_{t+1|t}$ process is not autocorrelated. A property that is robust to the type of conditional variance function, the order of the autoregressive process of the conditional mean and the values of the coefficients, applied. Hence, the simulated evidence provides evidence that the estimated standardized one-step-ahead prediction errors are asymptotically independently standard normally distributed. The results of our simulation are confirmed analytically in the next chapter.

Chapter 4
Predictability and Model Selection in the Context of ARCH Models

4.1. Introduction

The richness of the family of parametric ARCH models certainly complicates the search for the true model, and leaves quite a bit of arbitrariness in the model selection stage. The problem of selecting the model that describes best the movement of the series under study is, therefore, of practical importance. Most of the methods used in the ARCH literature for selecting the appropriate model are based on evaluating the ability of the models to describe the data. An alternative model selection approach is examined based on the evaluation of the predictability of the models in terms of standardized prediction errors.

The aim of this chapter is to develop a model selection method based on the evaluation of the predictability of the ARCH models. Section 4.2 provides a brief description of the methods used in the literature for selecting the appropriate model based on evaluating the ability of the models to describe the data. In section 4.3, Xekalaki et al.'s (2003) model selection method based on a standardized prediction error criterion is examined in the context of ARCH models. In section 4.4 the suggested model selection method is applied using return data for the Athens Stock Exchange (ASE) index over the period August 30th, 1993 to November 4th, 1996, while, in section 4.5, a selection method based on the ability of the models describing the data is investigated. Finally, in section 4.6 a brief discussion of the results is provided.

4.2. Model Selection Methods

Most of the methods used in the literature for selecting the appropriate model are based on evaluating the ability of the models to describe the data. Standard model selection criteria such as the Akaike Information Criterion (AIC) (Akaike (1973)) and the Schwarz Bayesian Criterion (SBC) (Schwarz (1978)) have widely been used in the ARCH literature, despite the fact that their statistical properties in the ARCH context are unknown[1]. These are defined in terms of $l_T(\hat{\theta})$, the maximized value of the log-likelihood function of a model, where $\hat{\theta}$ is the maximum likelihood estimator of θ based on a sample of size T and $\breve{\theta}$ denotes the dimension of θ, thus:

$$AIC = l_T(\hat{\theta}) - \breve{\theta} \tag{4.2.1}$$

$$SBC = l_T(\hat{\theta}) - 2^{-1}\breve{\theta}\ln(T). \tag{4.2.2}$$

In addition, the evaluation of loss functions for alternative models is mainly used in model selection. When we focus on estimation of means, the loss function of choice is typically the mean squared error (MSE):

$$MSE = T^{-1}\sum_{t=1}^{T}\varepsilon_t^2 . \tag{4.2.3}$$

When the same strategy is applied to variance estimation, the choice of the mean squared error is much less clear. Because of high non-linearity in volatility models, a number of researchers constructed heteroscedasticity-adjusted loss functions. Bollerslev et al. (1994) present four types of loss functions:

$$L_1 = \sum_{t=1}^{T}\left(\varepsilon_t^2 - \sigma_t^2\right)^2, \tag{4.2.4}$$

$$L_2 = \sum_{t=1}^{T}\ln\left(\frac{\varepsilon_t^2}{\sigma_t^2}\right)^2, \tag{4.2.5}$$

$$L_3 = \sum_{t=1}^{T}\frac{\left(\varepsilon_t^2 - \sigma_t^2\right)^2}{\sigma_t^4}, \tag{4.2.6}$$

$$L_4 = \sum_{t=1}^{T}\left(\frac{\varepsilon_t^2}{\sigma_t^2} + \ln\left(\sigma_t^2\right)\right). \tag{4.2.7}$$

Pagan and Schwert (1990) used the first two of the loss functions to compare alternative estimators with in-sample and out-of-sample data sets. Andersen et al. (1999), Heynen and Kat (1994), Hol and Koopman (2000), are some examples from the literature that applied loss functions to compare the forecast performance of various volatility models.

Moreover, loss functions have been constructed, based upon the goals of the particular application. West et al. (1993) developed such a criterion based on the portfolio decisions of a risk averse investor. Engle et al. (1993) assumed that the

[1] Kavalieris (1989) provided a thorough discussion for methods of selection of autoregressive models and asymptotic equivalence of the AIC criterion to predictive cross-validation. His work may have some nice

objective was to price options and developed a loss function from the profitability of a particular trading strategy.

4.3. Model Selection Based on a Standardized Prediction Error Criterion (SPEC)

Let $\{y_t(\theta)\}_{t\geq 1}$ refer to the univariate discrete time real-valued stochastic process to be predicted where θ is a vector of unknown parameters. According to section 2.2 of the 2nd chapter, an ARCH process, $\{\varepsilon_t(\theta)\}_{t\geq 1}$, can be presented as:

$$\begin{gathered} y_t(\theta) = x_t'\beta + \varepsilon_t(\theta) \\ \varepsilon_t(\theta) = z_t\sigma_t(\theta) \\ z_t \overset{i.i.d.}{\sim} f[E(z_t)=0, V(z_t)=1] \\ \sigma_t^2(\theta) = g(\sigma_{t-1}(\theta), \sigma_{t-2}(\theta), \ldots; \varepsilon_{t-1}(\theta), \varepsilon_{t-2}(\theta), \ldots; \upsilon_{t-1}, \upsilon_{t-2}, \ldots), \end{gathered} \tag{4.3.1}$$

where x_t is a $k \times 1$ vector of endogenous and exogenous explanatory variables included in the information set I_{t-1}, β is a $k \times 1$ vector of unknown parameters, $f(.)$ is the density function of z_t, $\sigma_t(\theta)$ is a time-varying, positive and measurable function of the information set at time $t-1$, υ_t is a vector of predetermined variables included in I_t, and $g(.)$ is a linear or nonlinear functional form. A wide range of ARCH models is reviewed in section 2.2.1 of chapter 2. In the sequel for notational convenience, no explicit indication of the dependence on the vector of parameters, θ, is given when obvious from the context.

The conditional mean, $\mu_t = E(y_t \mid I_{t-1})$, can be adequately described by a κ^{th} order autoregressive $[AR(\kappa)]$ model:

$$y_t = c_0 + \sum_{i=1}^{\kappa}(c_i y_{t-i}) + \varepsilon_t. \tag{4.3.2}$$

Usually, the conditional mean is either the overall mean or a first order autoregressive process. Theoretically, the $AR(1)$ process allows for the autocorrelation induced by discontinuous (or non-synchronous) trading in the stocks making up an index[2]. Higher

extension to the case of ARCH specification.

[2] For more details on non-synchronous trading see section 2.1.3 of the 2nd chapter.

orders of the autoregressive process are considered in order to investigate if they are adequate to produce more accurate predictions.

Let us assume that a researcher is interested in evaluating the ability of the ARCH models to forecast the conditional variance. Consider the simple case of a regression model: $y_t = x_t'\beta + \varepsilon_t$ where β is a vector of k unknown parameters to be estimated, x_t is a vector of explanatory variables included in the information set at time $t-1$ and $\varepsilon_t \overset{i.i.d.}{\sim} N(0,\sigma^2)$. At time $t-1$, the expected value μ_t of y_t is estimated on the basis of the information available at time $t-1$, i.e. $\hat{y}_{t|t-1} = \hat{\mu}_t = x_t'\hat{\beta}_{t-1}$, where $\hat{\beta}_{t-1} = \left(\mathbf{X}_{t-1}'\mathbf{X}_{t-1}\right)^{-1}\left(\mathbf{X}_{t-1}'\mathbf{Y}_{t-1}\right)$ is the least square estimator of β at time $t-1$, $\mathbf{Y}_t$ is the $(l_t \times 1)$ vector of l_t observations on the dependent variable y_t, and $\mathbf{X}_t$ is the $(l_t \times k)$ matrix whose rows comprise the k-dimensional vectors x_t of the explanatory variables included in the information set, so that $\mathbf{X}_t = \begin{bmatrix} \mathbf{X}_{t-1} \\ x_t' \end{bmatrix}$, $\mathbf{Y}_t = \begin{bmatrix} \mathbf{Y}_{t-1} \\ y_t \end{bmatrix}$. Here $l_0 > k$, $l_{t+1} = l_t + 1$ and $\left|\mathbf{X}_t'\mathbf{X}_t\right| \neq 0$, $t = 0,1,\ldots$. In a manner of speaking, $\hat{y}_{t|t}$ and $\hat{y}_{t|t-1}$ can be considered as in-sample and out-of-sample forecasts, respectively. In other words, $\hat{y}_{t|t}$ is measured on the basis of I_t, the information set available at time t, while $\hat{y}_{t|t-1}$ is measured on the basis of I_{t-1}, the information set available at time $t-1$.

The most commonly used way to model the conditional variance is the GARCH(p,q) process:

$$\sigma_t^2 = a_0 + \sum_{i=1}^{q}\left(a_i\varepsilon_{t-i}^2\right) + \sum_{i=1}^{p}\left(b_i\sigma_{t-i}^2\right), \tag{4.3.3}$$

The GARCH(p,q) process may be rewritten as[3]:

$$\sigma_t^2 = \left(u_t', \eta_t', w_t'\right)\left(v, \zeta, \omega\right),$$

where $u_t' = \left(1, \varepsilon_{t-1}^2, \ldots, \varepsilon_{t-q}^2\right)$, $\eta_t' = 0$, $w_t' = \left(\sigma_{t-1}^2, \ldots, \sigma_{t-p}^2\right)$, $v' = \left(a_0, a_1, \ldots, a_q\right)$, $\zeta' = 0$, $\omega' = \left(b_1, \ldots, b_p\right)$.

[3] The conditional variance is written in the form: $\left(u_t', \eta_t', w_t'\right)\left(v, \zeta, \omega\right)$, which includes the most widely used ARCH models such as the TARCH and the EGARCH processes.

The vector $\theta = (\beta', v', \zeta', \omega')$ denotes the set of parameters to be estimated for both the conditional mean and the conditional variance at time t. In the sequel, the density function $f(.)$, in equation (4.3.1), is assumed to be that of the normal distribution and $\hat{z}_{t|t-1} \equiv \hat{\varepsilon}_{t|t-1}\hat{\sigma}_{t|t-1}^{-1}$ denotes the standardized one-step-ahead prediction errors[4].

The residual $\hat{\varepsilon}_{t|t-1} \equiv y_t - \hat{y}_{t|t-1}$ reflects the difference between the forecast and the observed value of the stochastic process. Xekalaki et al. (2003) suggested measuring the predictive behavior of linear regression models on the basis of the standardized distance between the predicted and the observed value of the dependent random variable. The estimate of the standardized distance was defined by:

$$r_t = \frac{y_t - \hat{y}_{t|t-1}}{\sqrt{V(\hat{y}_{t|t-1})}},$$

where $V(\hat{y}_{t|t-1}) = (\mathbf{Y}_{t-1} - \mathbf{X}_{t-1}\hat{\beta}_{t-1})'(\mathbf{Y}_{t-1} - \mathbf{X}_{t-1}\hat{\beta}_{t-1})(1 + x_t(\mathbf{X}'_{t-1}\mathbf{X}_{t-1})^{-1}x'_t)(l_{t-1} - k)^{-1}$. A scoring rule to rate the performance of the model at time t for a series of T points in time, $(t = 1,...,T)$, was defined by

$$R_T = T^{-1}\sum_{t=1}^{T} r_t^2, \tag{4.3.4}$$

the average of the squared standardized residuals. As an ARCH model estimates simultaneously the conditional mean and the conditional variance, its evaluation is two fold. In the sequel, this approach is adopted using the average of the squared standardized one-step-ahead prediction errors as a scoring rule in order to rate the performance of an ARCH model to forecast both the conditional mean and the conditional variance, in particular,

$$R_T = \frac{\sum_{t=1}^{T} \hat{z}_{t|t-1}^2}{T}. \tag{4.3.5}$$

$\hat{z}_{t|t-1} \equiv \hat{\varepsilon}_{t|t-1}\hat{\sigma}_{t|t-1}^{-1}$ is the estimated standardized distance between the predicted and the observed value of the dependent random variable, when the conditional standard

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[4] Consider the case of the AR(1)GARCH(1,1) model as defined by equations (4.3.2) and (4.3.3), for $\kappa = 1$ and $p = q = 1$, respectively. The estimators of the one-step-ahead prediction error and its variance conditional on the information set available at time $t-1$ are given by $\hat{\varepsilon}_{t|t-1} = y_t - \hat{c}_{0,t-1} - \hat{c}_{1,t-1}y_{t-1}$ and

deviation of the dependent variable given I_{t-1} is defined by an ARCH model, $V(y_t \mid I_{t-1}) \equiv \sigma_t^2$.

Theorem 1: Let (θ_t) denote the vector of unknown parameters to be estimated at time t. Under the assumption of constancy of parameters over time, $(\theta_1) = (\theta_2) = ... = (\theta_T) = (\theta)$, the estimated standardized one-step-ahead prediction errors $\hat{z}_{t|t-1}, \hat{z}_{t+1|t}, ..., \hat{z}_{T|T-1}$ are asymptotically independently standard normally distributed. Symbolically,

$$\hat{z}_{t|t-1} \equiv (y_t - \hat{y}_{t|t-1})\hat{\sigma}_{t|t-1}^{-1} \sim N(0,1),\ t = 1,2,...,T. \tag{4.3.6}$$

Proof: To prove the theorem, we need the following lemmas.

Lemma 1: (Slutsky's theorem) (see, e.g. Greene (1997, p.118)): For a continuous function $g(x_T)$ that is not a function of T, $p\lim g(x_T) = g(p\lim x_T)$.
(Here $p\lim$ denotes the limit in probability as $T \to \infty$.)
The following two Lemmas are implications of Slutsky's theorem.

Lemma 2: (see, e.g. Hamilton, 1994, p. 182): Let $\{X_T\}$ denote a sequence of $(n \times 1)$ random vectors with $p\lim X_T = c$, i.e., $X_T \xrightarrow{p} c$. Let $g(.)$ be a vector-valued function, $g: R^n \to R^m$, which is continuous at c and does not depend on T. Then $g(X_T) \xrightarrow{p} g(c)$.

Lemma 3: (see, e.g. Hamilton (1994, p. 182)): Let $\{X_{1T}\}$ denote a sequence of $(n \times n)$ random matrices with $X_{1T} \xrightarrow{p} C_1$, where C_1 is a non-singular matrix. Let X_{2T} denote a sequence of $(n \times 1)$ random vectors with $X_{2T} \xrightarrow{p} c_2$, where c_2 is a constant. Then, $(X_{1T})^{-1} X_{2T} \xrightarrow{p} (C_1)^{-1} c_2$, or $p\lim(X_{1T})^{-1} X_{2T} = (C_1)^{-1} c_2$.

We now prove the following lemma.

$\hat{\sigma}_{t|t-1}^2 = \hat{a}_{0,t-1} + \hat{a}_{1,t-1}\hat{\varepsilon}_{t-1|t-1}^2 + \hat{b}_{1,t-1}\hat{\sigma}_{t-1|t-1}^2$, respectively. The estimated parameters are indexed by the subscript t to indicate that they may vary with time.

Lemma 4: Let $\{X_{iT}\}$, for $i=1,...,n$, denote a sequence of random vectors with $p\lim X_{iT}=W_i$, where W_i, $i=1,...,n$ are independently and identically distributed with some distribution function $F(.)$. Then $p\lim(X_{1n},X_{2n},...,X_{nT})=(W_1,W_2,...,W_n)$, and $X_{1T},X_{2T},...,X_{nT}$ are asymptotically independently and identically distributed with distribution function $F(.)$.

Proof of Lemma 4: Let $\tilde{g}(.)$ be a vector-valued real function, $\tilde{g}(.): R^n \to R^n$:

$$(x_1,x_2,...,x_n)\to \tilde{g}(x_1,x_2,...,x_n)\equiv (g_1(x_1,x_2,...,x_n), g_2(x_1,x_2,...,x_n),..., g_n(x_1,x_2,...,x_n)).$$

Assume that $\tilde{g}(.)$ is continuous at z_i, $\forall i=1,...,n$, and does not depend on T.

According to Slutsky's theorem (Lemma 1), for a continuous function $g(x_T)$ that is not a function of T, $p\lim g(x_T)=g(p\lim x_T)$. Thus,

$$p\lim \tilde{g}(X_{1T},X_{2T},...,X_{nT})=(g_1(X_1,X_2,...,X_n), g_2(X_1,X_2,...,X_n),..., g_n(X_1,X_2,...,X_n)).$$

By setting $\tilde{g}(x_1,x_2,...,x_n)=(x_1,x_2,...,x_n)$, (i.e. $g_i(x_1,x_2,...,x_n)=x_i, \forall i=1,...,n$), and applying Slutsky's theorem we obtain

$$p\lim \tilde{g}(X_{1T},X_{2T},...,X_{nT})\equiv p\lim(X_{1T},X_{2T},...,X_{nT})=\tilde{g}(W_1,W_2,...,W_n)\equiv(W_1,W_2,...,W_n)$$

Let $F_{(X_{1T},X_{2T},...,X_{nT})}(x_1,x_2,...,x_n)$ denote the joint density distribution of the random variables $X_{1T},X_{2T},...,X_{nT}$. As convergence in probability implies convergence in distribution, we have

$$\lim_{T\to\infty} F_{(X_{1T},X_{2T},...,X_{nT})}(x_1,x_2,...,x_n)=F_{(W_1,W_2,...,W_n)}(x_1,x_2,...,x_n)=$$
$$=F_{W_1}(x_1)\cdot F_{W_2}(x_2)\cdot ...\cdot F_{W_n}(x_n)=\lim_{T\to\infty}F_{X_{1T}}(x_1)\cdot \lim_{T\to\infty}F_{X_{2T}}(x_2)\cdot ...\cdot \lim_{T\to\infty}F_{X_{nT}}(x_n)$$

As the joint density is asymptotically the product of the marginal densities, $X_{1T},X_{2T},...,X_{nT}$ are asymptotically independently distributed, each with distribution function $F(.)$.

Let us now return to the proof of Theorem 1: At time $t-1$, the expected value of y_t is estimated on the basis of the information available at time $t-1$, i.e. $\hat{y}_{t|t-1}=x_t'\hat{\beta}_{t-1}$ and the expected value of the conditional variance is estimated on the basis of the information available at time $t-1$, i.e. $\hat{\sigma}^2_{t|t-1}=(u_t',\eta_t',w_t')(\hat{v}_{t-1},\hat{\zeta}_{t-1},\hat{\omega}_{t-1})$. Note that the elements of the vector (u'_t, η'_t, w'_t) belong to the I_{t-1}, so are considered as known values. The $\hat{z}_{t|t-1}$ can be written as:

$$\hat{z}_{t|t-1} = \frac{(y_t - \hat{y}_{t|t-1})}{\sqrt{\hat{\sigma}^2_{t|t-1}}} =$$

$$= \frac{(x'_t \beta + \varepsilon_t - x'_t \hat{\beta}_{t-1})}{\sqrt{\hat{\sigma}^2_{t|t-1}}} =$$

$$= \frac{\varepsilon_t}{\sqrt{\hat{\sigma}^2_{t|t-1}}} + \frac{(x'_t(\beta - \hat{\beta}_{t-1}))}{\sqrt{\hat{\sigma}^2_{t|t-1}}} =$$

$$= \frac{z_t \sqrt{\sigma^2_t}}{\sqrt{\hat{\sigma}^2_{t|t-1}}} + \frac{(x'_t(\beta - \hat{\beta}_{t-1}))}{\sqrt{\hat{\sigma}^2_{t|t-1}}} =$$

$$= \frac{z_t((u'_t, \eta'_t, w'_t)(v, \zeta, \omega))^{1/2}}{((u'_t, \eta'_t, w'_t)(\hat{v}_{t-1}, \hat{\zeta}_{t-1}, \hat{\omega}_{t-1}))^{1/2}} + \frac{(x'_t(\beta - \hat{\beta}_{t-1}))}{((u'_t, \eta'_t, w'_t)(\hat{v}_{t-1}, \hat{\zeta}_{t-1}, \hat{\omega}_{t-1}))^{1/2}}$$

We assume that a sample of T observations has been used to estimate the vector of unknown parameters. According to Bollerslev (1986), the maximum likelihood estimate $\hat{\theta}_t$ is strongly consistent for θ and asymptotically normal with mean θ. In other words, $p\lim(\hat{\theta}_t) = \theta \Leftrightarrow p\lim(\hat{\beta}'_t, \hat{v}'_t, \hat{\zeta}'_t, \hat{\omega}'_t) = (\beta', v', \zeta', \omega')$, where $p\lim$ denotes limit in probability as the size of the sample, T, goes to infinity. According to Lemma 2:

$$p\lim(\hat{z}_{t|t-1}) =$$

$$= p\lim\left(\frac{z_t((u'_t, \eta'_t, w'_t)(v, \zeta, \omega))^{1/2}}{((u'_t, \eta'_t, w'_t)(\hat{v}_{t-1}, \hat{\zeta}_{t-1}, \hat{\omega}_{t-1}))^{1/2}}\right) + p\lim\left(\frac{(x'_t(\beta - \hat{\beta}_{t-1}))}{((u'_t, \eta'_t, w'_t)(\hat{v}_{t-1}, \hat{\zeta}_{t-1}, \hat{\omega}_{t-1}))^{1/2}}\right) =$$

Then, based on Lemma 3:

$$= \frac{z_t((u'_t, \eta'_t, w'_t)(v, \zeta, \omega))^{1/2}}{(u'_t, \eta'_t, w'_t)(p\lim(\hat{v}_{t-1}, \hat{\zeta}_{t-1}, \hat{\omega}_{t-1}))^{1/2}} + \frac{(x'_t p\lim(\beta - \hat{\beta}_{t-1}))}{(p\lim((u'_t, \eta'_t, w'_t)(\hat{v}_{t-1}, \hat{\zeta}_{t-1}, \hat{\omega}_{t-1})))^{1/2}} =$$

$$= \frac{z_t((u'_t, \eta'_t, w'_t)(v, \zeta, \omega))^{1/2}}{(u'_t, \eta'_t, w'_t)((v, \zeta, \omega))^{1/2}} + \frac{(x'_t p\lim(\beta - \hat{\beta}_{t-1}))}{((u'_t, \eta'_t, w'_t) p\lim(\hat{v}_{t-1}, \hat{\zeta}_{t-1}, \hat{\omega}_{t-1}))^{1/2}} =$$

$$= z_t + \frac{(x'_t)(0)}{((u'_t, \eta'_t, w'_t)(v, \zeta, \omega))^{1/2}} =$$

$$= z_t$$

As convergence in probability implies convergence in distribution, the $\hat{z}_{t|t-1}, \hat{z}_{t+1|t}, ..., \hat{z}_{T|T-1}$ are asymptotically standard normally distributed:

$$\hat{z}_{t|t-1} \xrightarrow{p} z_t \Rightarrow \hat{z}_{t|t-1} \xrightarrow{d} z_t \sim N(0,1)$$

This result, combined with Lemma 4, implies that the $\hat{z}_{t|t-1}, \hat{z}_{t+1|t}, ..., \hat{z}_{T|T-1}$ are asymptotically independently standard normally distributed, i.e.,

$$\hat{z}_{t|t-1} \xrightarrow{d} z_t \overset{i.i.d.}{\sim} N(0,1).$$

Hence, the theorem has been established.

The result of the theorem is valid for all the conditional variance functions with consistent estimators of the parameters.

Remark: As concerns the EGARCH and the TARCH models, the maximum likelihood estimator $\hat{\theta}_t = \left(\hat{\beta}'_t, \hat{v}'_t, \hat{\zeta}'_t, \hat{\omega}'_t\right)$ is consistent and asymptotically normal.

Consider the EGARCH(p,q) model in the following form

$$\ln\left(\sigma_t^2\right) = a_0 + \sum_{i=1}^{q}\left(a_i\left|\frac{\varepsilon_{t-i}}{\sigma_{t-i}}\right| + \gamma_i\left(\frac{\varepsilon_{t-i}}{\sigma_{t-i}}\right)\right) + \sum_{i=1}^{p}\left(b_i \ln\left(\sigma_{t-i}^2\right)\right) \tag{4.3.7}$$

which can be written as:

$$\ln\sigma_t^2 = \left(u'_t, \eta'_t, w'_t\right)\left(v, \zeta, \omega\right)$$

where $u'_t = \left(1, \left|\varepsilon_{t-1}/\sigma_{t-1}\right|, ..., \left|\varepsilon_{t-q}/\sigma_{t-q}\right|\right)$, $\eta'_t = \left(\left[\varepsilon_{t-1}/\sigma_{t-1}\right], ..., \left[\varepsilon_{t-q}/\sigma_{t-q}\right]\right)$,

$w'_t = \left(\ln\sigma_{t-1}^2, ..., \ln\sigma_{t-p}^2\right)$, $v' = \left(a_0, a_1, ..., a_q\right)$, $\zeta' = \left(\gamma_1, ..., \gamma_q\right)$, $\omega' = \left(b_1, ..., b_p\right)$.

According to Nelson (1991), under sufficient regularity conditions, the maximum likelihood estimator $\hat{\theta}_t = \left(\hat{\beta}'_t, \hat{v}'_t, \hat{\zeta}'_t, \hat{\omega}'_t\right)$ is consistent and asymptotically normal. Also, for the TARCH(p,q) process, the conditional variance can take the form:

$$\sigma_t^2 = a_0 + \sum_{i=1}^{q}\left(a_i\varepsilon_{t-i}^2\right) + \gamma\varepsilon_{t-1}^2 d_{t-1} + \sum_{i=1}^{p}\left(b_i\sigma_{t-i}^2\right), \tag{4.3.8}$$

which can be written as:

$$\sigma_t^2 = \left(u'_t, \eta'_t, w'_t\right)\left(v, \zeta, \omega\right)$$

where $u'_t = \left(1, \varepsilon_{t-1}^2, ..., \varepsilon_{t-q}^2\right)$, $\eta'_t = \left(d_{t-1}\varepsilon_{t-1}^2\right)$, $w'_t = \left(\sigma_{t-1}^2, ..., \sigma_{t-p}^2\right)$, $v' = \left(a_0, a_1, ..., a_q\right)$,

$\zeta' = (\gamma)$, $\omega' = (b_1,...,b_p)$, $d_t = 1$ if $\varepsilon_t < 0$, and $d_t = 0$ otherwise.

As pointed out by Glosten et al. (1993), as long as the conditional mean and variance are correctly specified, the maximum likelihood estimates will be consistent and asymptotically normal.

According to Lemma 1, if $p\lim \hat{z}_{t|t-1} = z_t \sim N(0,1)$ and $g(\hat{z}_{t|t-1}) = \sum_{t=1}^{T} (\hat{z}_{t|t-1}^2)$, which is a continuous function, then $p\lim \sum_{t=1}^{T} (\hat{z}_{t|t-1}^2) = \sum_{t=1}^{T} (z_t^2)$. As convergence in probability implies convergence in distribution, $\sum_{t=1}^{T} (\hat{z}_{t|t-1}^2) \xrightarrow{d} \sum_{t=1}^{T} (z_t^2) \sim \chi_T^2$. Hence, as $\hat{z}_{t|t-1}$ are asymptotically standard normal variables, the variable TR_T is asymptotically χ^2 distributed with T degrees of freedom, i.e.,

$$TR_T \xrightarrow{d} \chi_T^2 . \tag{4.3.9}$$

Also, for two processes A and B with T_1 and T_2 observations, respectively, the ratio of the scoring rules $R_{T_1}^{(A)} \equiv T_1^{-1} \sum_{t=1}^{T} \hat{z}_{t|t-1}^{(A)2}$ and $R_{T_2}^{(B)} \equiv T_2^{-1} \sum_{t=1}^{T} \hat{z}_{t|t-1}^{(B)2}$ is F distributed with T_1 and T_2 degrees of freedom, i.e.,

$$R_{T_1T_2} \equiv \frac{R_{T_1}^{(A)}}{R_{T_2}^{(B)}} \sim F_{T_1,T_2} , \tag{4.3.10}$$

if $R_{T_1}^{(A)}$ and $R_{T_2}^{(B)}$ are independently distributed.

According to Kibble (1941), if, for $t = 1,2,...,T$, $\hat{z}_{t|t-1}^{(A)}$ and $\hat{z}_{t|t-1}^{(B)}$ are standard normally distributed variables, following jointly the bivariate standard normal distribution, then the joint distribution of $\left(\frac{T}{2} R_T^{(A)}, \frac{T}{2} R_T^{(B)}\right)$ is the bivariate gamma distribution with probability density function (p.d.f) given by:

$$f_{\frac{T}{2}R_T^{(A)},\frac{T}{2}R_T^{(B)}}(x,y) = \frac{\exp\left(-\frac{x+y}{1-\rho^2}\right)}{\Gamma(T/2)(1-\rho^2)^{T/2}} \sum_{i=0}^{\infty} \left(\frac{\left(\rho/(1-\rho^2)\right)^{2i}}{\Gamma(i+1)\Gamma(i+(T/2))} (xy)^{(T/2)-1-i} \right), x,y > 0 , \tag{4.3.11}$$

where $\Gamma(.)$ is the gamma function and ρ is the correlation coefficient between $\hat{z}_{t|t-1}^{(A)}$ and $\hat{z}_{t|t-1}^{(B)}$, $\rho \equiv Cor\left(\hat{z}_{t|t-1}^{(A)}, \hat{z}_{t|t-1}^{(B)}\right)$. Xekalaki et al. (2003) showed that, when the joint distribution of $\left(\frac{T}{2}R_T^{(A)}, \frac{T}{2}R_T^{(B)}\right)$ is Kibble's bivariate gamma, the distribution of the ratio $Z_T^{(A,B)} \equiv R_T^{(A)} / R_T^{(B)}$ is defined by the following p.d.f.:

$$f_{Z_T^{(A,B)}}(z) = \frac{\left(1-\rho^2\right)^{T/2}}{B\left(T/2, T/2\right)} z^{T/2-1}(1+z)^{-T}\left[1-\left(\frac{2\rho}{z+1}\right)^2 z\right]^{-\frac{T+1}{2}}, z > 0, \tag{4.3.12}$$

where $B\left(\frac{T}{2}, \frac{T}{2}\right) = \Gamma\left(\frac{T}{2}\right)^2 \Big/ \Gamma(T)$. Symbolically,

$$Z_T^{(A,B)} \equiv \sum_{t=1}^{T} \hat{z}_{t|t-1}^{2(B)} \Big/ \sum_{t=1}^{T} \hat{z}_{t|t-1}^{2(A)} \sim CGR(k, \rho), \tag{4.3.13}$$

where $k = T/2$. Xekalaki et al. (2003) referred to the distribution in (4.3.12) as the Correlated gamma ratio (CGR) distribution. A sample of tables of its percentage points and of graphs depicting its probability density function is given in the Appendix.

As pointed out by Xekalaki et al. (2003), $R_T^{(A)}$ and $R_T^{(B)}$ could represent the sum of the squared standardized prediction errors from two regression models (not necessarily nested) but with a common dependent variable. Thus, two regression models can be compared through testing a null hypothesis of equivalence of the models in their predictability against the alternative that model (A) produces “better” predictions. Here, the notion of the equivalence of two models with respect to their predictive ability is considered in Xekalaki et al.’s (2003) sense to be defined implicitly through their mean squared prediction errors. Following Xekalaki et al.’s (2003) rationale, the closest description of the hypothesis to be tested is

H_0: Models A and B have equal mean squared prediction errors

Versus

H_1: Model A has lower mean squared prediction error than model B

using $Z_T^{(A,B)}$ as a test statistic, i.e., using the ratio of the sum of the squared standardized one-step-ahead prediction errors $\hat{z}_{t|t-1}$ of the two competing models. The null hypothesis is rejected if $Z_T^{(A,B)} > CGR(k, \rho, a)$, where $CGR(k, \rho, a)$ is the $100(1-a)$ percentile of the CGR distribution. In the case of independence between $R_T^{(A)}$ and $R_T^{(B)}$, the CGR density function reduces to the form:

$$f_{Z_T^{(A,B)}}\left(Z_T^{(A,B)}\right)=\frac{1}{B\left(T/2,T/2\right)}Z_T^{(A,B)\,T/2-1}\left(1+Z_T^{(A,B)}\right)^{-T}, \tag{4.3.14}$$

which is the p.d.f. of the F distribution with T and T degrees of freedom.

Since very few financial time series have a constant conditional mean of zero, in order to estimate the conditional variance, the conditional mean should have been defined. Thus, both the conditional mean and variance are estimated simultaneously. According to the SPEC model selection algorithm, the models that are considered as having a “better” ability to predict future values of the dependent variable, are those with the lowest sum of squared standardized one-step-ahead prediction errors. It becomes evident, therefore, that these models can potentially be regarded as the most appropriate to use for volatility forecasts too.

4.4. Empirical Results

The suggested model selection procedure is illustrated on data referring to the daily returns of the Athens Stock Exchange (ASE) index. Let $y_t = \ln\left(P_t/P_{t-1}\right)$ denote the continuously compound rate of return from time $t-1$ to t, where P_t is the ASE closing price at time t. The data set covers the period from August 30th, 1993 to November 4th, 1996, a total of 800 trading days. Table 4.1 presents the descriptive statistics. For an estimated kurtosis equal to 7.25 and an estimated skewness equal to 0.08, the distribution of returns is flat (platykurtic) and has a long right tail relative to the normal distribution. The Jarque Bera (JB) statistic (Jarque and Bera (1980)) is used to test whether the series is normally distributed. The test statistic measures the difference of the skewness and kurtosis of the series from those of the normal distribution. The JB statistic is computed as:

$$JB = T\left(S^2+\left((K-3)^2/4\right)\right)/6, \tag{4.4.1}$$

where T is the number of observations, S is the skewness and K is the kurtosis. Under the null hypothesis of a normal distribution, the JB statistic is χ^2 distributed with 2 degrees of freedom. From Table 4.1, the value of the JB statistic obtained is 602.38 with a very low p-value (practically zero). So, the null hypothesis of normality is rejected. In order to determine whether $\{y_t\}$ is a stationary process, the Augmented Dickey Fuller test (ADF) (Dickey and Fuller (1979)) and the nonparametric Phillips Perron (PP) test (Phillips (1987), Phillips and Perron (1988)) are conducted.

The ADF test examines the null hypothesis, $H_0 : \gamma = 0$, versus the alternative, $H_1 : \gamma < 0$, in the following regression:

$$\Delta y_t = c + \gamma y_{t-1} + \sum_{i=1}^{\kappa} \varphi_i \Delta y_{t-i} + \varepsilon_t , \tag{4.4.2}$$

where Δ denotes the difference operator. According to the ADF test, the null hypothesis of non-stationarity is rejected at the 1% level of significance for any lag order up to $\kappa = 12$. The test regression for the PP test is the AR(1) process:

$$\Delta y_t = c + \gamma y_{t-1} + \varepsilon_t . \tag{4.4.3}$$

Table 4.1. Descriptive Statistics of the daily returns of the ASE index

(30th August 1993 to 4th November 1996 (800 observations))

Observations	800
Mean	5.72E-05
Median	-0.00018
Standard Deviation	0.012
Skewness	0.08
Kurtosis	7.25
Jarque Bera (JB)	602.38
probability	<0.000001
Augmented Dickey Fuller (ADF)	-12.67
1% critical value	-3.44
Phillips Perron (PP)	-24.57
1% critical value	-3.44

The skewness of a symmetric distribution, as the normal distribution, is zero. Positive skewness implies that the distribution has a long right tail. Negative skewness implies a long left tail distribution.

The kurtosis of the normal distribution is 3. If the kurtosis exceeds 3, the distribution is peaked (leptokurtic) relative to the normal. If the kurtosis is less than 3, the distribution is flat (platykurtic) relative to the normal.

Under the null hypothesis of a normal distribution, the JB statistic is χ^2 distributed with 2 degrees of freedom. The reported probability is the probability that the JB statistic exceeds, in absolute value, the observed value under the null hypothesis.

ADF: The null hypothesis of non-stationarity is rejected if the ADF value is less than the critical value. (4 lagged differences).

PP: The null hypothesis of non-stationarity is rejected if the PP value is less than the critical value. (4 truncation lags).

While the ADF test corrects for higher order serial correlation by adding lagged differenced terms on the right hand side, the PP test makes a correction to the t statistic of the γ coefficient from the AR(1) regression to account for the serial correlation in ε_t. The correction is nonparametric since an estimate of the spectrum of ε_t at frequency zero, that is robust to heteroscedasticity and autocorrelation of unknown form, is used.

According to the PP test, the null hypothesis is also rejected at the 1% level of significance.

The most commonly used test, for examining the null hypothesis of homoscedasticity against the alternative hypothesis of heteroscedasticity, is Engle's (1982) Lagrange multiplier (LM) test. The ARCH LM test statistic is computed from an auxiliary test regression. To test the null hypothesis of no ARCH effects up to order q in the residuals, the regression model

$$\varepsilon_t^2 = \beta_0 + \sum_{i=1}^{q} \beta_i \varepsilon_{t-i}^2 + u_t \,, \tag{4.4.4}$$

with $\varepsilon_t = y_t - c$ is run. Engle's test statistic is computed as the product of the number of observations times the value of the coefficient of variation R^2 of the auxiliary test regression. From Table 4.2, the values of the LM test statistic for $q = 1,...,8$ are highly significant at any reasonable level.

Table 4.2. Lagrange multiplier (LM) test. Test the null hypothesis of no ARCH effects in the residuals up to order q.

$$\varepsilon_t^2 = \beta_0 + \sum_{i=1}^{q} \beta_i \varepsilon_{t-i}^2 + u_t$$

$$\varepsilon_t = y_t - c$$

Q	LM statistic	p-value
1	108.203	0.00
2	113.315	0.00
3	127.947	0.00
4	128.577	0.00
5	130.691	0.00
6	133.467	0.00
7	131.573	0.00
8	129.496	0.00

The LM statistic is computed as the number of observations times the R^2 from the auxiliary test regression. It converges in distribution to a χ^2_q.

As, according to the results of the above tests, the assumptions of stationarity and ARCH effects seem to be plausible for the process $\{y_t\}$ of daily returns, several ARCH models are considered in the sequel. It is assumed, specifically, that the conditional mean is considered as a κ^{th} order autoregressive process as defined in (4.3.2) and the conditional variance σ_t^2 is assumed to be related to lagged values of ε_t

and σ_t according to a GARCH(p , q) model, an EGARCH(p , q) model or a TARCH(p , q) model as defined by (4.3.3), (4.3.7) and (4.3.8), respectively. Thus, the AR(κ)GARCH(p , q), AR(κ)EGARCH(p , q) and AR(κ)TARCH(p , q) models are applied, for $\kappa = 0,...,4$, $p = 0,1,2$ and $q = 1,2$, yielding a total of 90 cases.

Since, in estimating non-linear models, no closed form expressions are obtainable for the parameter estimators, an iterative method has to be employed. The value of the parameter vector θ that maximizes $l_t(\theta)$, the log likelihood contribution for each observation t , is to be found. Iterative optimization algorithms work by starting with an initial set of values for the parameter vector θ, say $\theta^{(0)}$, and obtaining a set of parameter values $\theta^{(1)}$ which corresponds to a higher value of $l_t(\theta)$. This process is repeated until the objective function $l_t(\theta)$ no longer improves between iterations. In the sequel, the Marquardt algorithm (Marquardt (1963)) is used. This algorithm modifies the Berndt, Hall, Hall and Hausman, or BHHH, algorithm (Berndt et al. (1974)) by adding a correction matrix to the Hessian approximation (i.e., to the sum of the outer product of the gradient vectors for each observation's contribution to the objective function). The Marquardt updating algorithm is computed as:

$$\theta^{(i+1)} = \theta^{(i)} + \left(\sum_{t=1}^{T} \frac{\partial l_t^{(i)}}{\partial \theta} \frac{\partial l_t^{(i)}}{\partial \theta'} - aI \right)^{-1} \sum_{t=1}^{T} \frac{\partial l_t^{(i)}}{\partial \theta}, \tag{4.4.9}$$

where I is the identity matrix and a is a positive number chosen by the algorithm. The effect of this modification is to push the parameter estimates in the direction of the gradient vector. The idea is that when we are far from the maximum, the local quadratic approximation to the function may be a poor guide to its overall shape, so it may be better off to simply follow the gradient. The correction may provide a better performance at locations far from the optimum, and allows for computation of the direction vector in cases where the Hessian is near singular.

The quasi-maximum likelihood estimator (QMLE) is used, as according to Bollerslev and Wooldridge (1992), it is generally consistent, has a limiting normal distribution and provides asymptotic standard errors that are valid under non-normality.

In order to compute the sum of squared standardized one-step-ahead prediction errors, a rolling sample of constant size equal to 500 is used, or $T = 500$, so 300 one-step-ahead daily forecasts are estimated. Combined 90 model specifications and 300 replications for each model, our approach produces a total of 27.000 model estimations. In the chapters follow, ARCH models are estimated for larger number of data windows increasing even more the total number of one-step-ahead estimates. Unfortunately, it is not possible to use updating procedures that help cut down on computing time, expect from fixing, at each point in time, the initial values of the parameters to be estimated to their previously estimated values. On average, the computation of the 90 models requires 12 minutes per trading day[5].

Although, large data sets are often used in the literature for the estimation of ARCH models, we consider here using a not too large sample, which would expectantly incorporate changes in trading behavior more efficiently as the evidence is from various findings in the literature (e.g. Engle et al. 1993, Frey and Michaud 1997 and Angelidis et al. 2004). Moreover, in the 7th chapter samples of 1000 and 2000 observations were considered.

The out-of-sample data set is split into 5 subperiods and the SPEC model selection algorithm is applied in each subperiod separately. Thus, the model selection is revised every 60 trading days and the information set includes daily continuously compound returns of the two most recently years, or 500 trading days. The choice of a 60-day length for each subperiod is arbitrary. The sum of the squared one-step-ahead prediction errors, $\sum_{t=T+1}^{T+s}\left(\hat{z}_{t|t-1}^{2}\right)$, is estimated for each model and presented in Table 4.3, in the end of chapter. The models selected for each subperiod and their sums of the squared standardized one-step-ahead prediction errors are:

Subperiod	Model Selected	$\min\left(\sum_{t=T+1}^{T+s}\left(\hat{z}_{t|t-1}^{2}\right)\right)$
1. 25 August 1995 - 16 November 1995	AR(2) EGARCH(0,1)	21.961
2. 17 November 1995 - 13 February 1996	AR(0) EGARCH(0,1)	76.315
3. 14 February 1996 - 14 May 1996	AR(0) EGARCH(0,1)	42.176
4. 15 May 1996 – 8 August 1996	AR(3) EGARCH(0,1)	27.308
5. 9 August 1996 - 4 November 1996	AR(1) EGARCH(0,1)	43.920

According to the SPEC selection method, the exponential GARCH(0,1) model describes best the conditional variance for the total examined period of 300 trading days. It is selected by the SPEC selection method in each subperiod. Figure 4.1 shows the daily

[5] In the 6th chapter, an options trading strategy that is based on the SPEC algorithm is constructed. The trading game assumes that there is enough time to forecast the next day's option prices. In order to compute option prices for the next trading day, the required computational time per day should be less than 15 minutes.

value of the ASE index and the one-step-ahead conditional standard deviation of its returns.

Figure 4.1. The ASE index and the one step ahead conditional standard deviation of its returns estimated by the EGARCH(0,1)

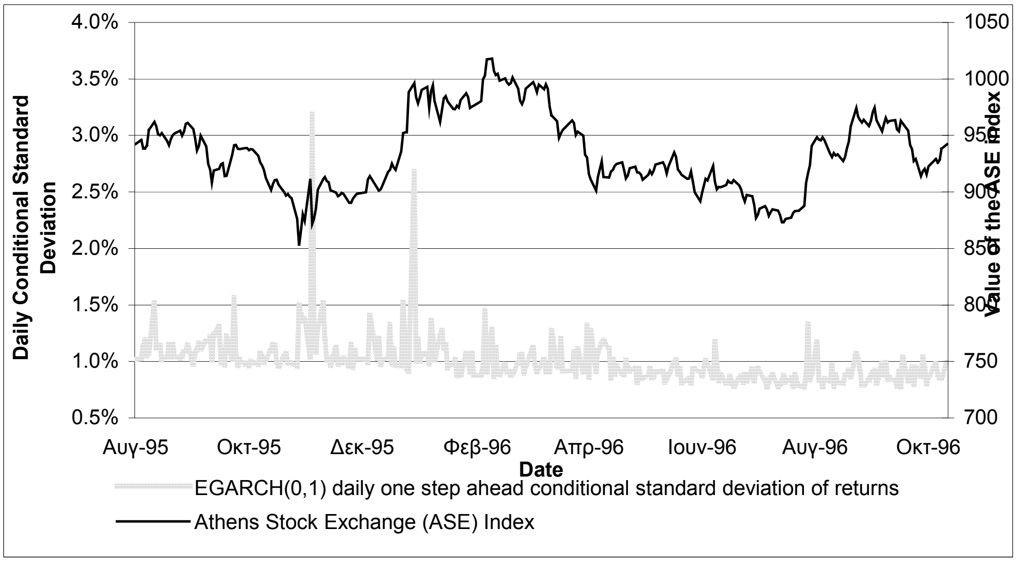


Figure 4.2. The parameters of the estimated EGARCH(0,1) models

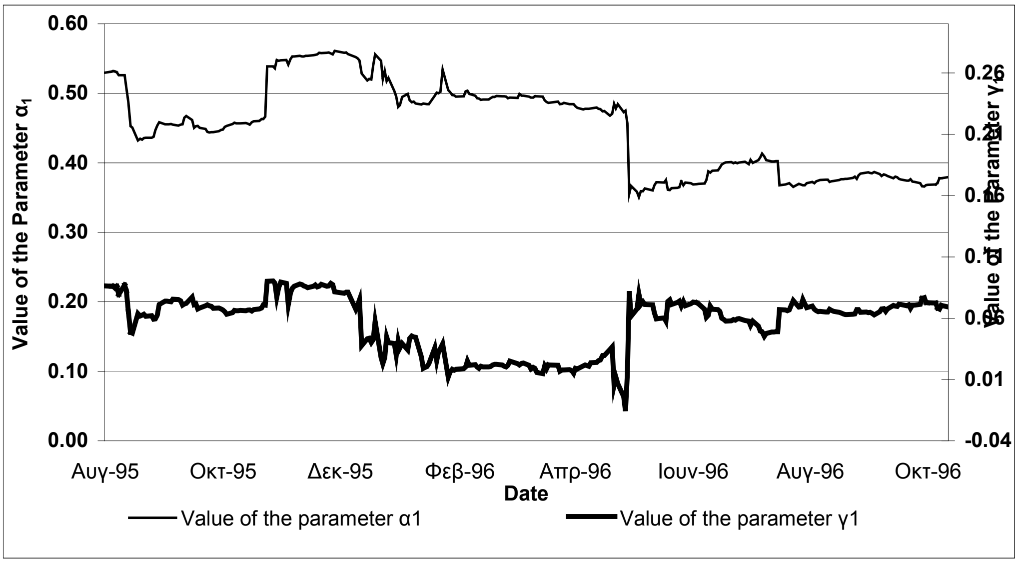
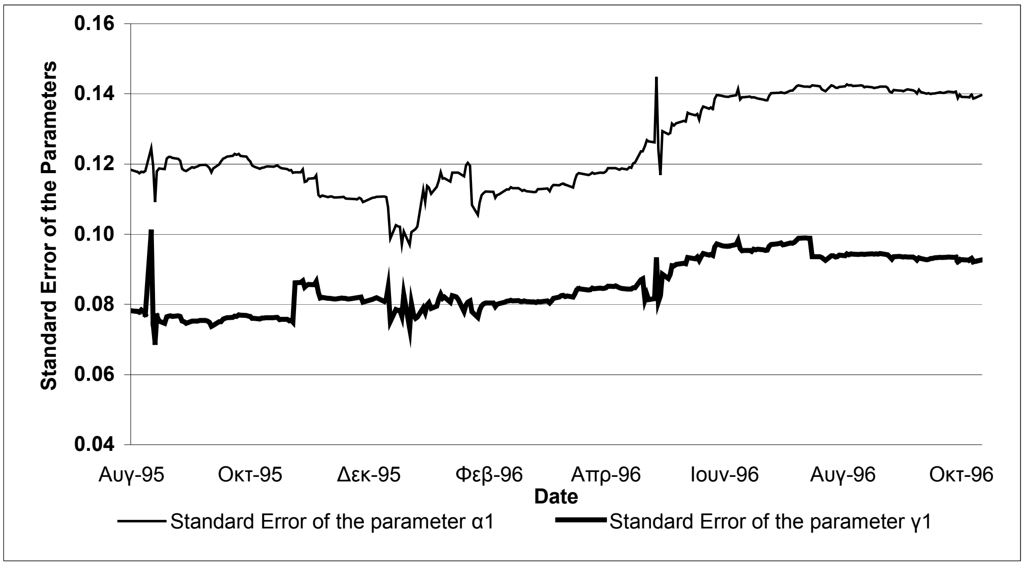


Figure 4.3. The standard error for the parameters of the estimated EGARCH(0,1) models



Despite the fact that an asymmetric model is selected by the SPEC algorithm, there are no asymmetries in the ASE index volatility. According to Figure 4.1, the major episodes of high volatility are not associated with market changes of the same sign. Figure 4.2 presents the values of the parameters a_1 and γ_1 of the 300 estimated EGARCH(0,1) models, while Figure 4.3 depicts the relevant standard errors for the parameters a_1 and γ_1. Obviously, the γ_1 parameter, which allows for the asymmetric effect, is positive but statistically insignificant. Therefore, the asymmetric relation between returns and changes in volatility does not characterize the examined period.

An interesting point is that the higher order of the conditional mean autoregressive process is chosen as adequate to produce more accurate predictions for the first and the fourth subperiods. As concerns the first subperiod, the AR(2)EGARCH(0,1) model

$$y_t = c_0 + c_1 y_{t-1} + c_2 y_{t-2} + \varepsilon_t$$
$$\ln\left(\sigma_t^2\right) = a_0 + a_1\left|\frac{\varepsilon_{t-1}}{\sigma_{t-1}}\right| + \gamma_1\left(\frac{\varepsilon_{t-1}}{\sigma_{t-1}}\right), \tag{4.4.10}$$

is the one with the lowest value of $\sum_{t=501}^{560}\left(\hat{z}_{t|t-1}^2\right)$ equal to 21.961. The hypothesis:

H$_0$: The model AR(2)EGARCH(0,1) has equivalent predictive ability to model X is tested versus

H_1: The model AR(2)EGARCH(0,1) produces "better" predictions than model X , with X denoting any one of the remainder models.

The CGR distribution depends on the correlations among standardized prediction errors from different models. Assuming that replacing the unknown parameters by their estimates should work in large samples, the correlation is estimated from the data. The correlation between the standardized one-step-ahead prediction errors is greater than 0.9 in each case, which is naturally the case for predicted values specially when they are derived from similar model frameworks. If $Z_{60}^{AR(2)EGARCH(0,1),X} \equiv (21.96)^{-1}\sum_{t=501}^{560}\hat{z}_{t|t-1}^{(X)2}$ $> CGR(k = 30, \rho > 0.9, a)$, the null hypothesis of equivalent predictive ability of the models is rejected at $100a\%$ level of significance and the AR(2)EGARCH(0,1) model is regarded as "better" than model X . Table 4.4, in the end of chapter, summarizes the results of the hypothesis tests, for each subperiod.

Figure 4.4, in the end of chapter, depicts the one-step-ahead 95 per cent prediction intervals for the models with the lowest $\sum_{t=T+1}^{T+s}\left(\hat{z}_{t|t-1}^{2}\right)$ in each subperiod. The prediction intervals are constructed as the expected rate of return plus\minus 1.96 times the conditional standard deviation, both measurable to $t-1$ information set: $\hat{\mu}_{t|t-1} \pm 1.96\hat{\sigma}_{t|t-1}$. So, each time next day's prediction interval is plotted, only information available at current day is used. Remark that around November 1995, a volatile period, the prediction interval in Figure 4.4 tracked the movement of the returns quite closely (seven outliers, or 2.33%, were observed).

4.5. An Alternative Approach

In this section an in-sample analysis is performed in order to select the appropriate models describing the data. Then, the selected models are used to estimate the one-step-ahead forecasts. Having assumed that the conditional mean of the returns follows a κ^{th} order autoregressive process, as in (4.5), Richardson and Smith (1994) developed a test for autocorrelation. It is a robust version of the standard Box Pierce (Box and Pierce (1970)) procedure. For p_i denoting the estimated autocorrelation between the returns at time t and $t-i$, the test is formulated as:

$$RS(r) = T\sum_{i=1}^{r}\frac{p_i^2}{1+c_i}, \qquad (4.5.1)$$

where T is the sample size and c_i is the adjustment factor for heteroscedasticity, which is calculated as:

$$c_i = \frac{Cov\left(\bar{y}_t^2, \bar{y}_{t-i}^2\right)}{Var\left(y_t\right)^2}, \qquad (4.5.2)$$

where $\bar{y}_t = y_t - T^{-1}\sum_{t=1}^{T} y_t$. Under the null hypothesis of no autocorrelation, the statistic is asymptotically distributed as χ^2 with r degrees of freedom. If the null hypothesis of no autocorrelation cannot be rejected, then the returns' process is equal to a constant plus the residuals, ε_t. In other words, $\{y_t\}$ follows the AR(0) process. If the null of no autocorrelation is rejected, then $\{y_t\}$ follows the AR(1) process. In order to test for the existence of a higher order autocorrelation, the test is applied on the estimated residuals from the AR(1) model. In this case, the statistic, under the null hypothesis, is asymptotically distributed as χ^2 with $r-1$ degrees of freedom. The test is calculated on 7 autocorrelations $(r=7)$ for 800 observations yielding a value equal to $RS(7) = 14{,}86 > \chi^2_{7,\,0.05}$. As the null hypothesis of no autocorrelation is rejected the test is run on the estimated residuals from the AR(1) model that gives $RS(6) = 12{,}33 < \chi^2_{6,\,0.05}$. Thus, a first order autocorrelation is detected for the returns' process. Note that the AR(1) form allows for the autocorrelation imposed by discontinuous trading.

Having defined the conditional mean equation, the next step is the estimation of the conditional variance function. The AIC and the SBC criteria are used to select the appropriate conditional variance equation. Note that the AIC mainly chooses as best the less parsimonious model. Also, under certain regularity conditions, the SBC is consistent, in the sense that for large samples it leads to the correct model choice, assuming the "true" model does belong to the set of models examined. Thus, the SBC may be preferable to use. As concerns the specific dataset, both the AIC and SBC select the GARCH(1,1) model as the most appropriate function to describe the conditional variance. So, performing an in-sample analysis the AR(1)GARCH(1,1) model is regarded as the most suitable, which is the model applied in most researches. Figure 4.5 presents the in-sample 95 per cent confidence interval for the AR(1)GARCH(1,1) model. There are fourteen observations, or 4.66%, outside the confidence interval.

In order to compare the model selection methods, the choice of the models should be conducted at the same time points. Thus, the Richardson Smith test for autocorrelation detection and the information criteria for model selection are used in each subperiod separately. The models selected for in each subperiod are:

Subperiod	Richardson Smith Model selection	SBC Model Selection	AIC Model Selection
1.	AR(3)	GARCH(1,1)	EGARCH(1,2)
2.	AR(2)	GARCH(2,1)	GARCH(2,1)
3.	AR(0)	GARCH(1,1)	GARCH(1,1)
4.	AR(0)	GARCH(1,1)	GARCH(1,1)
5.	AR(0)	GARCH(1,1)	TARCH(1,1)

Based on Table 4.4, the hypothesis that the model selected by the in-sample analysis is equivalent to the model with minimum value of $\sum_{t=T+1}^{T+s}\left(\hat{z}_{t|t-1}^{2}\right)$ is rejected in the majority of the cases.

Proceeding as in the previous section, the one-step-ahead prediction intervals, for the models selected in each subperiod, are created. As in section 4.5, next day's prediction is based only on information available at current day. Figures 4.6 and 4.7 present the one-step-ahead 95 per cent prediction intervals for the models selected by the SBC and AIC, respectively. There are thirteen observations, or 4.33%, outside the prediction interval for the models selected by the SBC, whereas there are fourteen outliers, or 4.66%, for the models selected by the AIC. Therefore, the importance of selecting a conditional variance model based on its ability to forecast and not on fitting the data gains a lead over. Of course, the construction of the prediction intervals is a naïve way to examine the accuracy of our method's predictability.

4.6. Conclusion

An alternative model selection approach, based on the CGR distribution, was introduced. Instead of being based on evaluating the ability of the models to describe the data (Akaike information and Schwarz Bayesian criteria), the proposed approach is based on evaluating the ability of the models to predict the conditional variance. The method was applied to 800 daily returns of the ASE index, a dataset covers the period from August 30th, 1993 to November 4th, 1996. The first T observations were used to estimate the one-step-ahead prediction of the conditional mean and variance at $T+1$. For $T=500$, a total of 300 one-step-ahead predictions of the conditional mean and variance were obtained. The out-of-sample data set was split to 5 subperiods and the

SPEC model selection algorithm was applied in each subperiod separately. Thus, the model selection was revised every 60 trading days.

The idea of “jumping” from one model to another, as stock market behavior alters, is introduced. The transition from one model to another is done according to the SPEC model selection algorithm. Each time the model selection method is applied, the model is used to predict the conditional variance is revised. Of course, the idea of switching from one regime to another has been already applied to the class of switch regime ARCH models introduced by Cai (1994) and Hamilton and Susmel (1994) and extended by several authors such as Dueker (1997) and Hansen (1994). However, these models allow the parameters of a specific ARCH model to come from one of several different regimes, with transitions between regimes governed by an unobserved Markov chain.

Using an alternative approach, based on evaluating the ability of fitting the data, the conditional mean is first modeled and subsequently, an appropriate form for the conditional variance is chosen. Applying the SPEC model selection algorithm, the null hypothesis, that the model selected by the in-sample analysis is equivalent to the model with minimum value of $\sum_{t=T+1}^{T+s}\left(\hat{z}_{t|t-1}^{2}\right)$, is rejected in the plurality of the cases at less than 5% level of significance. The in-sample model selection methods and the predictability-based method do not coincide in the sifting of the appropriate conditional variance model. Moreover, 2.33% and 4.33% of the data were outside the $\hat{\mu}_{t|t-1} \pm 1.96\hat{\sigma}_{t|t-1}$ prediction interval constructed based on the SPEC and the SBC model selection methods, respectively.

The predictive ability of the SPEC model selection algorithm is further investigated in the next chapters. Among the financial applications where this method could have a potential use are in the fields of portfolio analysis, risk management and trading option derivatives.

Table 4.3. Sum of squared standardized one step ahead prediction errors for each subperiod. The AR(κ)GARCH(p,q), AR(κ)EGARCH(p,q) and AR(κ)TARCH(p,q) models are applied, for κ=0,…,4, p=0,1,2 and q=1,2.

AR(κ) $$y_t = c_0 + \sum_{i=1}^{\kappa}\left(c_i y_{t-i}\right) + \varepsilon_t$$

GARCH(p,q) $$\sigma_t^2 = a_0 + \sum_{i=1}^{q}\left(a_i \varepsilon_{t-i}^2\right) + \sum_{i=1}^{p}\left(b_i \sigma_{t-i}^2\right)$$

EGARCH(p,q) $$\ln\left(\sigma_t^2\right) = a_0 + \sum_{i=1}^{q}\left(a_i\left|\frac{\varepsilon_{t-i}}{\sigma_{t-i}}\right| + \gamma_i\left(\frac{\varepsilon_{t-i}}{\sigma_{t-i}}\right)\right) + \sum_{i=1}^{p}\left(b_i \ln\left(\sigma_{t-i}^2\right)\right)$$

TARCH(p,q) $$\sigma_t^2 = a_0 + \sum_{i=1}^{q}\left(a_i \varepsilon_{t-i}^2\right) + \gamma\varepsilon_{t-1}^2 d_{t-1} + \sum_{i=1}^{p}\left(b_i \sigma_{t-i}^2\right)$$

Table 3.a
25 August 1995 - 16 November 1995 (s=[501,560])

	κ=0*	κ=1	κ=2	κ=3	κ=4
GARCH(p,q)					
p=0, q=1	26.371	25.465	24.843	25.173	26.570
p=0, q=2	30.150	29.493	28.940	29.109	30.835
p=1, q=1	39.076	38.848	38.289	38.496	38.466
p=1, q=2	39.129	38.709	38.159	38.533	38.456
p=2, q=1	39.183	38.304	37.882	37.829	37.889
p=2, q=2	39.511	38.742	38.336	39.223	38.377
TARCH(p,q)					
p=0, q=1	26.795	25.892	25.270	25.683	27.300
p=0, q=2	31.151	30.981	30.442	30.619	32.125
p=1, q=1	39.070	38.624	38.146	38.506	38.550
p=1, q=2	39.016	38.667	38.185	38.660	38.482
p=2, q=1	39.279	37.836	37.422	38.005	38.290
p=2, q=2	40.975	38.732	38.180	38.755	38.398
EGARCH(p,q)					
p=0, q=1	23.770	22.644	21.961	22.047	22.722
p=0, q=2	27.289	27.340	26.731	26.896	28.312
p=1, q=1	44.281	43.555	43.131	43.321	41.934
p=1, q=2	43.754	42.427	41.360	42.235	41.231
p=2, q=1	44.620	43.216	43.138	43.142	42.077
p=2, q=2	43.926	42.915	42.231	42.645	41.138

Table 3.b
17 November 1995 - 13 February 1996 (s=[561,620])

	κ=0*	κ=1	κ=2	κ=3	κ=4
GARCH(p,q)					
p=0, q=1	81.183	79.657	79.913	83.204	89.584
p=0, q=2	88.007	85.947	88.135	89.575	95.825
p=1, q=1	79.571	84.410	85.070	85.671	86.749
p=1, q=2	80.684	85.214	85.554	87.046	89.907
p=2, q=1	79.703	83.700	86.917	84.920	87.420
p=2, q=2	81.230	84.534	85.143	82.863	88.940
TARCH(p,q)					
p=0, q=1	81.505	80.810	81.158	84.704	90.674
p=0, q=2	88.977	88.465	91.004	92.734	98.915
p=1, q=1	81.296	85.321	86.339	87.601	88.412
p=1, q=2	86.517	87.338	88.246	92.729	98.976
p=2, q=1	81.609	86.085	85.458	84.975	90.097
p=2, q=2	89.614	86.608	87.364	91.126	98.289
EGARCH(p,q)					
p=0, q=1	76.315	78.689	78.342	78.551	84.422
p=0, q=2	87.867	91.361	92.862	93.526	101.216
p=1, q=1	88.246	96.778	98.579	99.805	99.650
p=1, q=2	98.798	103.714	105.834	107.774	108.783
p=2, q=1	90.043	98.056	99.570	101.509	101.531
p=2, q=2	93.750	102.953	112.441	105.882	**

Table 3.c
14 February 1996 - 14 May 1996 (s=[621,680])

	κ=0*	κ=1	κ=2	κ=3	κ=4
GARCH(p,q)					
p=0, q=1	45.970	46.740	46.793	47.855	47.882
p=0, q=2	46.138	46.323	46.039	47.496	47.382
p=1, q=1	50.273	50.205	49.959	50.363	49.320
p=1, q=2	50.429	50.097	49.814	50.223	49.330
p=2, q=1	50.650	50.334	49.547	49.917	49.843
p=2, q=2	50.811	50.126	50.051	50.330	48.975
TARCH(p,q)					
p=0, q=1	45.947	46.731	46.749	47.769	47.806
p=0, q=2	46.114	46.311	46.001	47.422	47.263
p=1, q=1	50.461	50.262	50.006	50.396	49.368
p=1, q=2	50.677	50.145	49.830	50.229	49.512
p=2, q=1	50.769	49.491	48.737	50.231	49.613
p=2, q=2	51.664	49.794	50.262	50.548	50.133
EGARCH(p,q)					
p=0, q=1	42.176	42.724	42.688	43.561	43.383
p=0, q=2	43.712	44.279	44.178	45.395	44.838
p=1, q=1	49.382	48.836	48.837	49.369	48.644
p=1, q=2	49.140	48.716	48.592	49.065	48.608
p=2, q=1	49.422	48.384	48.301	48.452	48.380
p=2, q=2	51.970	49.555	**	48.992	**

Table 3.d
15 May 1996 - 8 August 1996 (s=[681,740])

	κ=0*	κ=1	κ=2	κ=3	κ=4
GARCH(p,q)					
p=0, q=1	30.568	30.619	29.473	29.346	29.534
p=0, q=2	31.557	32.105	30.967	30.861	30.813
p=1, q=1	36.016	36.440	35.335	35.175	35.013
p=1, q=2	36.098	36.951	35.846	35.706	35.431
p=2, q=1	35.732	37.374	36.069	36.020	35.628
p=2, q=2	35.859	36.647	36.252	35.446	35.437
TARCH(p,q)					
p=0, q=1	30.747	30.605	29.419	29.352	29.593
p=0, q=2	31.821	31.978	30.804	30.785	30.811
p=1, q=1	36.029	36.326	35.157	35.147	35.075
p=1, q=2	36.117	36.636	35.489	35.482	35.298
p=2, q=1	36.279	37.214	35.789	36.224	35.946
p=2, q=2	35.945	37.646	35.776	36.005	36.030
EGARCH(p,q)					
p=0, q=1	29.252	28.733	27.428	27.308	27.330
p=0, q=2	30.310	30.109	28.772	28.644	28.563
p=1, q=1	35.972	36.142	34.806	34.716	34.754
p=1, q=2	36.251	36.923	35.548	35.477	35.460
p=2, q=1	35.706	37.371	36.176	36.190	36.266
p=2, q=2	35.562	35.109	34.329	34.210	34.777

Table 3.e
9 August 1996 - 4 November 1996 (s=[741,800])

	κ=0*	κ=1	κ=2	κ=3	κ=4
GARCH(p,q)					
p=0, q=1	48.288	47.469	47.437	49.749	50.771
p=0, q=2	50.795	49.575	49.484	51.426	52.236
p=1, q=1	55.915	54.344	54.572	54.967	55.281
p=1, q=2	56.099	54.631	54.872	55.163	55.399
p=2, q=1	55.807	55.420	55.335	56.306	56.075
p=2, q=2	56.102	54.814	55.145	55.137	55.359
TARCH(p,q)					
p=0, q=1	47.179	47.143	47.101	49.494	50.529
p=0, q=2	49.483	49.131	49.030	51.031	51.935
p=1, q=1	53.866	53.341	53.616	53.897	54.272
p=1, q=2	54.065	53.684	53.835	54.075	54.327
p=2, q=1	53.925	54.199	53.999	54.245	56.211
p=2, q=2	54.181	54.482	54.725	55.039	54.846
EGARCH(p,q)					
p=0, q=1	44.260	43.920	44.047	45.908	46.528
p=0, q=2	46.453	45.986	46.035	47.513	47.990
p=1, q=1	52.752	53.271	53.285	53.801	53.944
p=1, q=2	53.233	54.767	54.191	54.450	54.617
p=2, q=1	53.922	55.703	55.410	55.596	55.726
p=2, q=2	52.438	54.052	53.963	**	54.716

*Regress the depedent variable on a constant.
** Model fails to converge at least once.

Table 4.4

Testing the null hypothesis that the model with the lowest sum of the squared standardized one step ahead prediction errors has equivalent predictive ability to model X, with X denoting any of the remainder models.

Table 4.a: 25 August 1995 - 16 November 1995 (1st subperiod)

H_0: The model AR(2)-EGARCH(0,1) is equivalent to model X
versus H_1: The model AR(2)-EGARCH(0,1) is "better" than model X.

Model for Conditional Variance	Model for Conditional Mean: AR(0)	AR(1)	AR(2)	AR(3)	AR(4)
GARCH(0,1)	1.201	1.160	1.131	1.146	1.210
p-value	<0.10	<0.10	<0.25	<0.25	<0.05
GARCH(0,2)	1.373	1.343	1.318	1.326	1.404
p-value	<0.01	<0.01	<0.01	<0.01	<0.01
GARCH(1,1)	1.779	1.769	1.744	1.753	1.752
p-value	<0.01	<0.01	<0.01	<0.01	<0.01
GARCH(1,2)	1.782	1.763	1.738	1.755	1.751
p-value	<0.01	<0.01	<0.01	<0.01	<0.01
GARCH(2,1)	1.784	1.744	1.725	1.723	1.725
p-value	<0.01	<0.01	<0.01	<0.01	<0.01
GARCH(2,2)	1.799	1.764	1.746	1.786	1.748
p-value	<0.01	<0.01	<0.01	<0.01	<0.01
TARCH(0,1)	1.220	1.179	1.151	1.170	1.243
p-value	<0.05	<0.10	<0.25	<0.10	<0.05
TARCH(0,2)	1.418	1.411	1.386	1.394	1.463
p-value	<0.01	<0.01	<0.01	<0.01	<0.01
TARCH(1,1)	1.779	1.759	1.737	1.753	1.755
p-value	<0.01	<0.01	<0.01	<0.01	<0.01
TARCH(1,2)	1.777	1.761	1.739	1.760	1.752
p-value	<0.01	<0.01	<0.01	<0.01	<0.01
TARCH(2,1)	1.789	1.723	1.704	1.731	1.744
p-value	<0.01	<0.01	<0.01	<0.01	<0.01
TARCH(2,2)	1.866	1.764	1.739	1.765	1.748
p-value	<0.01	<0.01	<0.01	<0.01	<0.01
E-GARCH(0,1)	1.082	1.031		1.004	1.035
p-value	<0.25	>0.25		>0.25	>0.25
E-GARCH(0,2)	1.243	1.245	1.217	1.225	1.289
p-value	<0.05	<0.05	<0.05	<0.05	<0.05
E-GARCH(1,1)	2.016	1.983	1.964	1.973	1.909
p-value	<0.01	<0.01	<0.01	<0.01	<0.01
E-GARCH(1,2)	1.992	1.932	1.883	1.923	1.878
p-value	<0.01	<0.01	<0.01	<0.01	<0.01
E-GARCH(2,1)	2.032	1.968	1.964	1.965	1.916
p-value	<0.01	<0.01	<0.01	<0.01	<0.01
E-GARCH(2,2)	2.000	1.954	1.923	1.942	1.873
p-value	<0.01	<0.01	<0.01	<0.01	<0.01

Table 4.b: 17 November 1995 - 13 February 1996 (2nd subperiod)

H_0: The model AR(0)-EGARCH(0,1) is equivalent to model X
versus H_1: The model AR(0)-EGARCH(0,1) is "better" than model X.

Model for Conditional Variance	Model for Conditional Mean: AR(0)	AR(1)	AR(2)	AR(3)	AR(4)
GARCH(0,1)	1.064	1.044	1.047	1.090	1.174
p-value	>0.25	>0.25	>0.25	<0.25	<0.1
GARCH(0,2)	1.153	1.126	1.155	1.174	1.256
p-value	<0.25	<0.25	<0.25	<0.1	<0.05
GARCH(1,1)	1.043	1.106	1.115	1.123	1.137
p-value	>0.25	<0.25	<0.25	<0.25	<0.25
GARCH(1,2)	1.057	1.117	1.121	1.141	1.178
p-value	>0.25	<0.25	<0.25	<0.25	<0.1
GARCH(2,1)	1.044	1.097	1.139	1.113	1.146
p-value	>0.25	<0.25	<0.25	<0.25	<0.25
GARCH(2,2)	1.064	1.108	1.116	1.086	1.165
p-value	>0.25	<0.25	<0.25	<0.25	<0.1
TARCH(0,1)	1.068	1.059	1.063	1.110	1.188
p-value	>0.25	>0.25	>0.25	<0.25	<0.1
TARCH(0,2)	1.166	1.159	1.192	1.215	1.296
p-value	<0.1	<0.1	<0.1	<0.05	<0.05
TARCH(1,1)	1.065	1.118	1.131	1.148	1.159
p-value	>0.25	<0.25	<0.25	<0.25	<0.1
TARCH(1,2)	1.134	1.144	1.156	1.215	1.297
p-value	<0.25	<0.25	<0.25	<0.05	<0.05
TARCH(2,1)	1.069	1.128	1.120	1.113	1.181
p-value	>0.25	<0.25	<0.25	<0.25	<0.1
TARCH(2,2)	1.174	1.135	1.145	1.194	1.288
p-value	<0.1	<0.25	<0.25	<0.1	<0.05
E-GARCH(0,1)		1.031	1.027	1.029	1.106
p-value		>0.25	>0.25	>0.25	<0.25
E-GARCH(0,2)	1.151	1.197	1.217	1.226	1.326
p-value	<0.25	<0.1	<0.05	<0.05	<0.01
E-GARCH(1,1)	1.156	1.268	1.292	1.308	1.306
p-value	<0.25	<0.05	<0.05	<0.05	<0.05
E-GARCH(1,2)	1.295	1.359	1.387	1.412	1.425
p-value	<0.05	<0.01	<0.01	<0.01	<0.01
E-GARCH(2,1)	1.180	1.285	1.305	1.330	1.330
p-value	<0.1	<0.05	<0.05	<0.01	<0.01
E-GARCH(2,2)	1.228	1.349	1.473	1.387	**
p-value	<0.05	<0.01	<0.01	<0.01	

** Model fails to converge at least once.

Table 4.4 (continued)

Testing the null hypothesis that the model with the lowest sum of the squared standardized one step ahead prediction errors has equivalent predictive ability to model X, with X denoting any of the remainder models.

Table 4.c: 14 February 1996 - 14 May 1996 (3rd subperiod)

H_0: The model AR(2)-EGARCH(0,1) is equivalent to model X versus H_1: The model AR(2)-EGARCH(0,1) is "better" than model X.

Model for Conditional Variance	Model for Conditional Mean: AR(0)	AR(1)	AR(2)	AR(3)	AR(4)
GARCH(0,1)	1.090	1.108	1.109	1.135	1.135
p-value	<0.25	<0.25	<0.25	<0.25	<0.25
GARCH(0,2)	1.094	1.098	1.092	1.126	1.123
p-value	<0.25	<0.25	<0.25	<0.25	<0.25
GARCH(1,1)	1.192	1.190	1.185	1.194	1.169
p-value	<0.1	<0.1	<0.1	<0.1	<0.1
GARCH(1,2)	1.196	1.188	1.181	1.191	1.170
p-value	<0.1	<0.1	<0.1	<0.1	<0.1
GARCH(2,1)	1.201	1.193	1.175	1.184	1.182
p-value	<0.1	<0.1	<0.1	<0.1	<0.1
GARCH(2,2)	1.205	1.188	1.187	1.193	1.161
p-value	<0.1	<0.1	<0.1	<0.1	<0.1
TARCH(0,1)	1.089	1.108	1.108	1.133	1.133
p-value	<0.25	<0.25	<0.25	<0.25	<0.25
TARCH(0,2)	1.093	1.098	1.091	1.124	1.121
p-value	<0.25	<0.25	<0.25	<0.25	<0.25
TARCH(1,1)	1.196	1.192	1.186	1.195	1.171
p-value	<0.1	<0.1	<0.1	<0.1	<0.1
TARCH(1,2)	1.202	1.189	1.181	1.191	1.174
p-value	<0.1	<0.1	<0.1	<0.1	<0.1
TARCH(2,1)	1.204	1.173	1.156	1.191	1.176
p-value	<0.1	<0.1	<0.25	<0.1	<0.1
TARCH(2,2)	1.225	1.181	1.192	1.199	1.189
p-value	<0.05	<0.1	<0.1	<0.1	<0.1
E-GARCH(0,1)		1.013	1.012	1.033	1.029
p-value		>0.25	>0.25	>0.25	>0.25
E-GARCH(0,2)	1.036	1.050	1.047	1.076	1.063
p-value	>0.25	>0.25	>0.25	>0.25	>0.25
E-GARCH(1,1)	1.171	1.158	1.158	1.171	1.153
p-value	<0.1	<0.1	<0.1	<0.1	<0.25
E-GARCH(1,2)	1.165	1.155	1.152	1.163	1.153
p-value	<0.1	<0.25	<0.25	<0.1	<0.25
E-GARCH(2,1)	1.172	1.147	1.145	1.149	1.147
p-value	<0.1	<0.25	<0.25	<0.25	<0.25
E-GARCH(2,2)	1.232	1.175	**	1.162	**
p-value	<0.05	<0.1		<0.1	

Table 4.d: 15 May 1996 - 8 August 1996 (4th subperiod)

H_0: The model AR(3)-EGARCH(0,1) is equivalent to model X versus H_1: The model AR(3)-EGARCH(0,1) is "better" than model X.

Model for Conditional Variance	Model for Conditional Mean: AR(0)	AR(1)	AR(2)	AR(3)	AR(4)
GARCH(0,1)	1.119	1.121	1.079	1.075	1.081
p-value	<0.25	<0.25	<0.25	>0.25	<0.25
GARCH(0,2)	1.156	1.176	1.134	1.130	1.128
p-value	<0.25	<0.1	<0.25	<0.25	<0.25
GARCH(1,1)	1.319	1.334	1.294	1.288	1.282
p-value	<0.01	<0.01	<0.05	<0.05	<0.05
GARCH(1,2)	1.322	1.353	1.313	1.308	1.297
p-value	<0.01	<0.01	<0.01	<0.05	<0.05
GARCH(2,1)	1.308	1.369	1.321	1.319	1.305
p-value	<0.01	<0.01	<0.01	<0.01	<0.05
GARCH(2,2)	1.313	1.342	1.328	1.298	1.298
p-value	<0.01	<0.01	<0.01	<0.05	<0.05
TARCH(0,1)	1.126	1.121	1.077	1.075	1.084
p-value	<0.25	<0.25	>0.25	>0.25	<0.25
TARCH(0,2)	1.165	1.171	1.128	1.127	1.128
p-value	<0.1	<0.1	<0.25	<0.25	<0.25
TARCH(1,1)	1.319	1.330	1.287	1.287	1.284
p-value	<0.01	<0.01	<0.05	<0.05	<0.05
TARCH(1,2)	1.323	1.342	1.300	1.299	1.293
p-value	<0.01	<0.01	<0.05	<0.05	<0.05
TARCH(2,1)	1.329	1.363	1.311	1.327	1.316
p-value	<0.01	<0.01	<0.01	<0.01	<0.01
TARCH(2,2)	1.316	1.379	1.310	1.318	1.319
p-value	<0.01	<0.01	<0.01	<0.01	<0.01
E-GARCH(0,1)	1.071	1.052	1.004		1.001
p-value	>0.25	>0.25	>0.25		>0.25
E-GARCH(0,2)	1.110	1.103	1.054	1.049	1.046
p-value	<0.25	<0.25	>0.25	>0.25	>0.25
E-GARCH(1,1)	1.317	1.323	1.275	1.271	1.273
p-value	<0.01	<0.01	<0.05	<0.05	<0.05
E-GARCH(1,2)	1.327	1.352	1.302	1.299	1.299
p-value	<0.01	<0.01	<0.05	<0.05	<0.05
E-GARCH(2,1)	1.308	1.368	1.325	1.325	1.328
p-value	<0.05	<0.01	<0.01	<0.01	<0.01
E-GARCH(2,2)	1.302	1.286	1.257	1.253	1.274
p-value	<0.05	<0.05	<0.05	<0.05	<0.05

** Model fails to converge at least once.

Table 4.4 (continued)

Testing the null hypothesis that the model with the lowest sum of the squared standardized one step ahead prediction errors has equivalent predictive ability to model X, with X denoting any of the remainder models.

Table 4.e: 9 August 1996 - 4 November 1996 (5th subperiod)

H_0: The model AR(1)-EGARCH(0,1) is equivalent to model X
versus H_1: The model AR(1)-EGARCH(0,1) is "better" than model X.

		Model for Conditional Mean				
		AR(0)	AR(1)	AR(2)	AR(3)	AR(4)
Model for Conditional Variance	GARCH(0,1)	1.099	1.081	1.080	1.133	1.156
	p-value	<0.25	<0.25	<0.25	<0.25	<0.25
	GARCH(0,2)	1.157	1.129	1.127	1.171	1.189
	p-value	<0.25	<0.25	<0.25	<0.1	<0.1
	GARCH(1,1)	1.273	1.237	1.243	1.252	1.259
	p-value	<0.05	<0.05	<0.05	<0.05	<0.05
	GARCH(1,2)	1.277	1.244	1.249	1.256	1.261
	p-value	<0.05	<0.05	<0.05	<0.05	<0.05
	GARCH(2,1)	1.271	1.262	1.260	1.282	1.277
	p-value	<0.05	<0.05	<0.05	<0.05	<0.05
	GARCH(2,2)	1.277	1.248	1.256	1.255	1.260
	p-value	<0.05	<0.05	<0.05	<0.05	<0.05
	TARCH(0,1)	1.074	1.073	1.072	1.127	1.150
	p-value	>0.25	>0.25	>0.25	<0.25	<0.25
	TARCH(0,2)	1.127	1.119	1.116	1.162	1.183
	p-value	<0.25	<0.25	<0.25	<0.1	<0.1
	TARCH(1,1)	1.226	1.215	1.221	1.227	1.236
	p-value	<0.05	<0.05	<0.05	<0.05	<0.05
	TARCH(1,2)	1.231	1.222	1.226	1.231	1.237
	p-value	<0.05	<0.05	<0.05	<0.05	<0.05
	TARCH(2,1)	1.228	1.234	1.230	1.235	1.280
	p-value	<0.05	<0.05	<0.05	<0.05	<0.05
	TARCH(2,2)	1.234	1.240	1.246	1.253	1.249
	p-value	<0.05	<0.05	<0.05	<0.05	<0.05
	E-GARCH(0,1)	1.008		1.003	1.045	1.059
	p-value	>0.25		>0.25	>0.25	>0.25
	E-GARCH(0,2)	1.058	1.047	1.048	1.082	1.093
	p-value	>0.25	>0.25	>0.25	<0.25	<0.25
	E-GARCH(1,1)	1.201	1.213	1.213	1.225	1.228
	p-value	<0.1	<0.05	<0.05	<0.05	<0.05
	E-GARCH(1,2)	1.212	1.247	1.234	1.240	1.244
	p-value	<0.05	<0.05	<0.05	<0.05	<0.05
	E-GARCH(2,1)	1.228	1.268	1.262	1.266	1.269
	p-value	<0.05	<0.05	<0.05	<0.05	<0.05
	E-GARCH(2,2)	1.194	1.231	1.229	**	1.246
	p-value	<0.1	<0.05	<0.05		<0.05

** Model fails to converge at least once.

Figure 4.4
One Step Ahead 95% Forecasted Interval for the Models with the Lowest Sum of the Squared Standardized One Step Ahead Prediction Errors

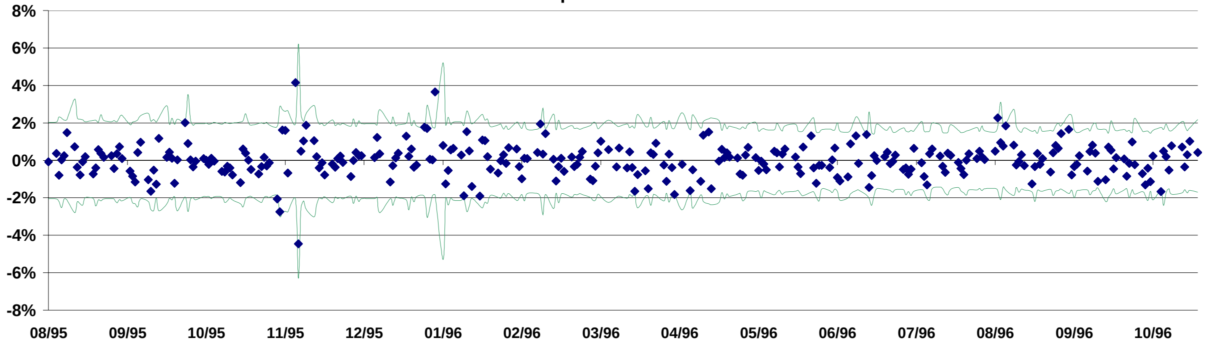


Figure 4.5
In-Sample 95% Confidence Interval for the AR(1) GARCH(1,1) Model

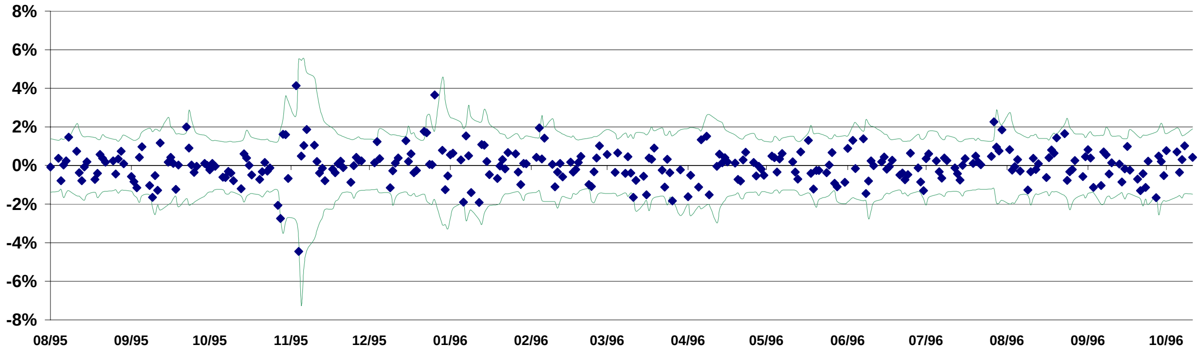


Figure 4.6
One Step Ahead 95% Forecasted Intervals for the Models Selected by the SBC

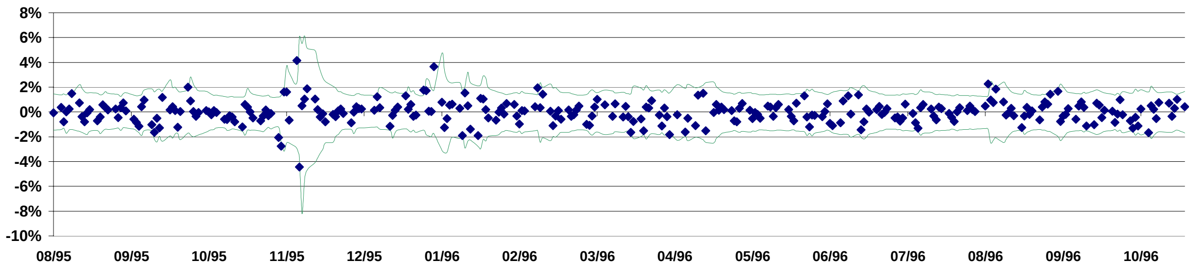
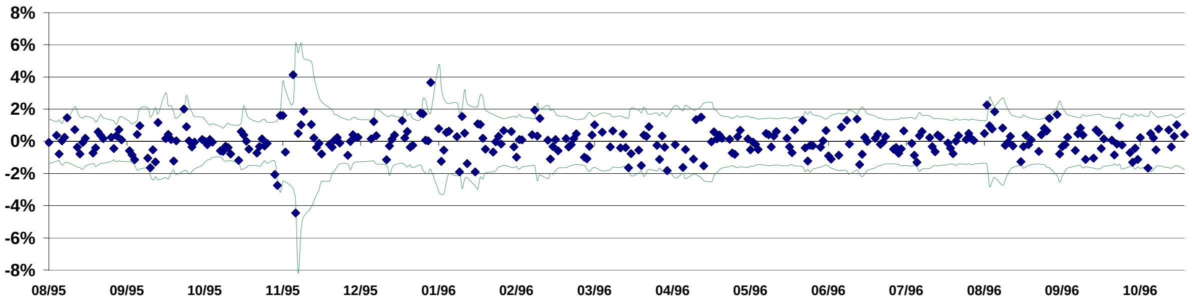


Figure 4.7
One Step Ahead 95% Forecasted Intervals for the Models Selected by the AIC



Chapter 5
Assessing the Performance of the Standardized Prediction Error Criterion Model Selection Algorithm

5.1. *Introduction*

Predicting volatility is of great importance in pricing financial derivatives, selecting portfolios, measuring and managing investment risk more accurately. To evaluate their accuracy, volatility forecasts have to be compared with realized volatility, which cannot be observed. In this chapter, a number of evaluation criteria are used to examine the ability of the SPEC model selection algorithm to indicate the ARCH model that generates "better" volatility predictions, for a forecasting horizon ranging from one day to one hundred days ahead. The results show that the SPEC model selection procedure has a satisfactory performance in selecting that ARCH model that tracks realized volatility closer, for a forecasting horizon ranging from 16 days to 36 days ahead. So, it is possible to use this model selection method in financial applications requiring volatility forecasts for a period longer than one day, i.e. option pricing, risk management. The majority of studies investigate the volatility forecasting accuracy for daily horizons, despite the fact that the practitioners require predictions of lower frequency (the Basle Committee on Banking Supervision (Basle Committee on Banking Supervision, 1998) for the use of Value-at-Risk methods requires the use of 10-days-ahead volatility predictions, whereas fund managers re-balance their portfolios on at least a monthly basis).

In section 5.2 of the present chapter, the forecast recursive relations of the GARCH, TARCH and EGARCH models and the estimation steps comprising the SPEC approach are presented. Section 5.3 provides a brief description of the evaluation criteria and the inter-day realized volatility measures considered. In section 5.4, the ability of the method proposed to select the ARCH model that generates "better" predictions of the volatility, is examined. In section 5.5, the proposed model selection method is compared to other methods of model selection. Finally, in section 5.6, a brief discussion of the results is provided.

5.2. *The forecast recursion relations of ARCH Processes*

For P_t denoting the price of an asset at time t, let $y_t = \ln(P_t / P_{t-1})$ denote the continuously compounded return series of interest. The return series is decomposed into two parts, the predictable and unpredictable component:

$$y_t = E(y_{t|t-1}) + \varepsilon_t \,, \tag{5.2.1}$$

where $E(y_{t|t-1})$ is the conditional mean of return at period t depending upon the information set available at time $t-1$ and ε_t is the prediction error. Usually, the predictable component is either the overall mean or a first order autocorrelated process (imposed by non-synchronous trading[1]). The conditional mean, unfortunately, does not have the ability to give useful predictions. That is why modern financial theory assumes the asset returns are unpredictable. Before the start of the 1980's, the view taken about returns in financial markets was that they behave as random walks and the Brock et al. (1987) (BDS) statistic has widely been used to test the null hypothesis that asset returns are independently and identically distributed. This hypothesis, however, has been rejected in a vast number of applications. A rejection of the null hypothesis is consistent with some types of dependence in the data, which could result in from a linear stochastic system, a nonlinear stochastic system, or a nonlinear deterministic system. Thus, a question arises: "Are the nonlinearities connected with the conditional mean (so, as to be used to predict future returns) or with higher order conditional moments?" Artificial neural networks[2], chaotic dynamical systems[3], nonlinear parametric and nonparametric models[4] are some examples from the literature dealing with conditional mean predictions. ARCH models and Stochastic Volatility models[5] are examples from the literature dealing with conditional variance modeling. However, no nonlinear models that can significantly outperform even the simplest linear model in out-of-sample forecasting seem to exist in the literature (neither in the field of stochastic nonlinear models nor in the field of deterministic chaotic systems). On the other hand, the ARCH processes and

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[1] For more details on non-synchronous trading see section 2.1.3 of the 2nd chapter.
[2] For an overview of the Neural Networks literature see Poggio and Girosi (1990), Hertz et al. (1991), White (1992), Hutchinson et al. (1994).
[3] Brock (1986), Holden (1986), Thompson and Stewart (1986) and Hsieh (1991) review applications of chaotic systems to financial markets.
[4] Priestley (1988), Tong (1990) and Teräsvirta et al. (1994) cover a wide variety of nonlinear models.
[5] See for details Taylor (1994) and Shephard (1996).

Stochastic Volatility models appear to be more appropriate to interpret nonlinearities in financial systems on the basis of the conditional variance. If an ARCH process is the true data generating mechanism, the nonlinearities cannot be exploited to generate improved point predictions relative to a linear model.

In the sequel, the conditional mean is considered as an κ^{th} order autoregressive process defined by

$$E\left(y_{t|t-1}\right) = c_0 + \sum_{i=1}^{\kappa} c_i y_{t-i} \ . \tag{5.2.2}$$

Assuming the unpredictable component in (5.2.1) is an ARCH process, it can be represented as:

$$\begin{gathered} \varepsilon_t = z_t \sigma_t \\ z_t \overset{iid}{\sim} N(0,1) \\ \sigma_t^2 = g\left(\sigma_{t-1}, \sigma_{t-2}, \ldots; \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots; \upsilon_{t-1}, \upsilon_{t-2}, \ldots\right), \end{gathered} \tag{5.2.3}$$

where $\{z_t\}$ is a sequence of independently and identically distributed random variables, σ_t is a time-varying, positive measurable function of the information set at time $t-1$, I_{t-1}, υ_t is a vector of predetermined variables included in I_t and $g(.)$ could be a linear or nonlinear functional form as it is usually assumed in the ARCH literature. A researcher, who is looking for the “best” model, would have in mind a variety of candidate models. The most commonly used conditional variance functions are the GARCH (Bollerslev (1986)), the Exponential GARCH, or EGARCH, (Nelson (1991)) and the Threshold GARCH, or TARCH, (Glosten et al. (1993)) functions. We rewrite these ARCH models from the 4th chapter:

The GARCH(p,q) model

$$\sigma_t^2 = a_0 + \sum_{i=1}^{q}\left(a_i \varepsilon_{t-i}^2\right) + \sum_{i=1}^{p}\left(b_i \sigma_{t-i}^2\right) \tag{5.2.4}$$

The EGARCH(p,q) model

$$\ln\left(\sigma_t^2\right) = a_0 + \sum_{i=1}^{q}\left(a_i \left|\frac{\varepsilon_{t-i}}{\sigma_{t-i}}\right| + \gamma_i\left(\frac{\varepsilon_{t-i}}{\sigma_{t-i}}\right)\right) + \sum_{i=1}^{p}\left(b_i \ln\left(\sigma_{t-i}^2\right)\right) \tag{5.2.5}$$

The TARCH(p,q) model

$$\sigma_t^2 = a_0 + \sum_{i=1}^{q}\left(a_i \varepsilon_{t-i}^2\right) + \gamma \varepsilon_{t-1}^2 d_{t-1} + \sum_{i=1}^{p}\left(b_i \sigma_{t-i}^2\right), \tag{5.2.6}$$

where $d_t = 1$ if $\varepsilon_t < 0$, and $d_t = 0$ otherwise.

Maximum likelihood estimates of the parameters are obtained by numerical maximization of the log-likelihood function using the Marquardt algorithm (Marquardt (1963)). The quasi-maximum likelihood estimator (QMLE) is used, as according to Bollerslev and Wooldridge (1992), it is generally consistent, has a normal limiting distribution and provides asymptotic standard errors that are valid under non-normality.

The majority of practical applications, i.e. option pricing, determination of the value-at-risk, require more than one-day-ahead volatility forecasts. More than one-step-ahead forecasts can be computed by repeated substitution. The forecast recursion relation of the GARCH(p,q) model is:

$$\hat{\sigma}^2_{t+1|t} = a_0^{(t)} + \sum_{i=1}^{q}\left(a_i^{(t)}\varepsilon^2_{t-i+1}\right) + \sum_{i=1}^{p}\left(b_i^{(t)}\sigma^2_{t-i+1}\right) \tag{5.2.7.a}$$

$$\hat{\sigma}^2_{t+s|t} = a_0^{(t)} + \sum_{\substack{i=1 \\ for\ i<s}}^{q}\left(a_i^{(t)}\sigma^2_{t-i+s}\right) + \sum_{\substack{i=s \\ for\ i\geq s}}^{q}\left(a_i^{(t)}\varepsilon^2_{t-i+s}\right) + \sum_{i=1}^{p}\left(b_i^{(t)}\sigma^2_{t-i+s}\right), \tag{5.2.7.b}$$

For $s > t$, the forecast of the predictive error ε_s conditional on information available at time t equals to its zero expected value, $E(\varepsilon_s \mid I_t) = 0$. On the other hand, the estimated value of ε_s^2 measured at time t should be equal to $\sigma^2_{s|t}$ for $s > t$. For $s \leq t$, the predictive error and its square are computed by the model with the available information at time t. The forecast recursion relationship associated with the EGARCH(p,q) model is:

$$\ln\left(\hat{\sigma}^2_{t+1|t}\right) = a_0^{(t)} + \sum_{i=1}^{q}\left(a_i^{(t)}\left|\frac{\varepsilon_{t-i+1}}{\sigma_{t-i+1}}\right| + \gamma_i^{(t)}\left(\frac{\varepsilon_{t-i+1}}{\sigma_{t-i+1}}\right)\right) + \sum_{i=1}^{p}\left(b_i^{(t)}\ln\left(\sigma^2_{t-i+1}\right)\right) \tag{5.2.8.a}$$

$$\ln\left(\hat{\sigma}^2_{t+s|t}\right) = a_0^{(t)} + \sum_{\substack{i=s \\ for\ i\geq s}}^{q}\left(a_i^{(t)}\left|\frac{\varepsilon_{t-i+s}}{\sigma_{t-i+s}}\right| + \gamma_i^{(t)}\left(\frac{\varepsilon_{t-i+s}}{\sigma_{t-i+s}}\right)\right) + \sqrt{\frac{2}{\pi}}\sum_{\substack{i=1 \\ for\ i<s}}^{q}\left(a_i^{(t)}\right) + \sum_{i=1}^{p}\left(b_i^{(t)}\ln\left(\sigma^2_{t-i+s}\right)\right), \tag{5.2.8.b}$$

that associated with the TARCH(p,q) model is:

$$\hat{\sigma}^2_{t+1|t} = a_0^{(t)} + \sum_{i=1}^{q}\left(a_i^{(t)}\varepsilon^2_{t-i+1}\right) + \gamma^{(t)}\varepsilon_t^2 d_t + \sum_{i=1}^{p}\left(b_i^{(t)}\sigma^2_{t-i+1}\right) \tag{5.2.9.a}$$

$$\hat{\sigma}^2_{t+s|t} = a_0^{(t)} + \sum_{\substack{i=1 \\ for\ i<s}}^{q}\left(a_i^{(t)}\sigma^2_{t-i+s}\right) + \sum_{\substack{i=1 \\ for\ i\geq s}}^{q}\left(a_i^{(t)}\varepsilon^2_{t-i+s}\right) + \gamma^{(t)}\sigma^2_{t-1+s}E(d_t) + \sum_{i=1}^{p}\left(b_i^{(t)}\sigma^2_{t-i+s}\right). \tag{5.2.9.b}$$

Here, $E(d_t)$ denotes the percentage of negative innovations out of all innovations. Under the assumption of normally distributed innovations, the expected number of negative shocks is equal to the expected number of positive shocks, or $E(d_t) = 0.5$. The forecast of the conditional variance at time t over a horizon of N days ahead is simply the average of the estimated future variance conditional on information given at time t is given by

$$\sigma^2_{t(N)} = N^{-1} \sum_{i=1}^{N} \hat{\sigma}^2_{t+i|t} \; . \tag{5.2.10}$$

Let us now assume that we are interested in comparing the predictive ability of two ARCH models:

Model A	Model B
$\varepsilon_t^{(A)} = z_{1,t}\sigma_t^{(A)}$	$\varepsilon_t^{(B)} = z_{2,t}\sigma_t^{(B)}$
$z_{1,t} \overset{iid}{\sim} N(0,1)$	$z_{2,t} \overset{iid}{\sim} N(0,1)$
$\sigma_t^{2(A)} = g\left(\sigma_{t-1}^{2(A)},...,\sigma_{t-p}^{2(A)},\varepsilon_{t-1}^{2(A)},...,\varepsilon_{t-q}^{2(A)},\upsilon_{t-1}^{(A)},\upsilon_{t-2}^{(A)},...\right)$	$\sigma_t^{2(B)} = g\left(\sigma_{t-1}^{2(B)},...,\sigma_{t-p}^{2(B)},\varepsilon_{t-1}^{2(B)},...,\varepsilon_{t-q}^{2(B)},\upsilon_{t-1}^{(B)},\upsilon_{t-2}^{(B)},...\right)$

According to the SPEC model selection algorithm, the models that are considered as having a "better" ability to predict future values of the dependent variable are those with the lowest sum of squared standardized one-step-ahead prediction errors. , $\sum_{t=k-T+1}^{k} \hat{z}_t^{2(m)} \equiv \sum_{t=k-T+1}^{k} \hat{\varepsilon}_{t|t-1}^{2(m)} \Big/ \hat{\sigma}_{t|t-1}^{2(m)}$. Here, $\hat{\varepsilon}_{t|t-1}^{(m)} = y_t - x_t'^{(m)}\hat{\beta}_{t-1}^{(m)}$ is the one-step-ahead prediction error of model m, where $\hat{\beta}_{t-1}^{(m)}$ is the estimator of $\beta^{(m)}$ based on the information set that is available at time $t-1$ and $\hat{\sigma}_{t|t-1}^{2(m)}$ is the one-step-ahead conditional variance forecast of model m. It becomes evident, therefore, that these models can potentially be regarded as the most appropriate to use for volatility forecasts too.

Let us assume that M candidate ARCH models are available and that we are looking for the "most suitable" model at each of a sequence of points in time. At time k, selecting a strategy for the most appropriate model to forecast volatility at time $k+1$ ($k = T, T+1,...$) could naturally amount to selecting the model, which, at time k, has the lowest sum of squared standardized one-step-ahead prediction errors, on the basis of the SPEC algorithm. Table 5.1 summarizes the estimation steps comprising this approach. The rows of this table refer to candidate ARCH models, the columns refer to days, while its entries represent the sums of the squares of the T most recent standardized one-step-ahead prediction errors of each of the M models. Each day, the choice of the model to be used to predict the conditional variance for the next day is determined by the entry of the corresponding column of table 5.1 that has the minimum value. In particular, model $m = i$ will be chosen at time $k = T + j$ if it is the one that corresponds to the cell of column $T + j$ that has the minimum value of $\sum_{t=j+1}^{T+j} \hat{z}_t^{2(i)}$.

Table 5.1

The estimation steps required at time k for each model m by the SPEC model selection algorithm. At time k $(k = T, T+1, ...)$, select the model m with the minimum value for the sum of the squares of the T most recent standardized one-step-ahead prediction errors, $\sum_{t=k-T+1}^{k} \hat{z}_t^{2(m)} \equiv \sum_{t=k-T+1}^{k} \hat{\varepsilon}_{t|t-1}^{2(m)} \big/ \hat{\sigma}_{t|t-1}^{2(m)}$.)

Model	Time: $k = T$	$k = T+1$	. . .	$k = T+K$	. . .
$m = 1$	$\sum_{t=1}^{T} \hat{z}_{t\|t-1}^{2(1)}$	$\sum_{t=2}^{T+1} \hat{z}_{t\|t-1}^{2(1)}$	. . .	$\sum_{t=K+1}^{T+K} \hat{z}_{t\|t-1}^{2(1)}$	. . .
$m = 2$	$\sum_{t=1}^{T} \hat{z}_{t\|t-1}^{2(2)}$	$\sum_{t=2}^{T+1} \hat{z}_{t\|t-1}^{2(2)}$	. . .	$\sum_{t=K+1}^{T+K} \hat{z}_{t\|t-1}^{2(2)}$	. . .
.	.	.	.	.	
.	.	.	.	.	
.	.	.	.	.	
$m = M$	$\sum_{t=1}^{T} \hat{z}_{t\|t-1}^{2(M)}$	$\sum_{t=2}^{T+1} \hat{z}_{t\|t-1}^{2(M)}$	. . .	$\sum_{t=K+1}^{T+K} \hat{z}_{t\|t-1}^{2(M)}$	. . .

In the next section, the methodology applied to evaluate the performance of a model in estimating future volatility is presented, while in section 5.4, the ability of the SPEC model selection algorithm to indicate those ARCH models that generate “better” volatility predictions is illustrated on a set of real data on daily returns of the S&P500 stock index.

5.3. *Evaluating the Volatility Forecast Performance*

The main problem in evaluating the predictive performance of a model is the choice of the function one should use to measure the distance between estimations and observations. Evaluating the performance of the variance forecasts requires knowledge of the actual volatility, which is unobservable. Thus, in evaluating the predictive performance of a variance model a question of a dual nature arises: that of determining the realized volatility and of considering the appropriate measure to evaluate the closeness of the forecasts to the corresponding realizations.

5.3.1 Realized Volatility Measures

Practitioners' most popular volatility measures are the average of squared daily returns and the variance of the daily returns. These measures, expressed on a daily basis for a horizon of N days ahead, are:

$$s^2_{t(N)} = N^{-1}\sum_{i=1}^{N} y^2_{t+i} \ , \tag{5.3.1}$$

$$\hat{s}^2_{t(N)} = (N-1)^{-1}\sum_{i=1}^{N}\left(y_{t+i} - \bar{y}_{t(N)}\right)^2 , \tag{5.3.2}$$

respectively, where $\bar{y}_{t(N)} = N^{-1}\sum_{i=1}^{N} y_{t+i}$ is the average return. The inter-day volatity measures are the most popular measures. However, as noted in the literature (e.g. Ebens (1999)), although the squared daily returns are unbiased volatility estimators, they are very noisy. Note that, under the ARCH process, the squared return can be represented by $y_t^2 = z_t^2\sigma_t^2$. It is therefore defined as the product of the true volatility times the square of a normally distributed process. In the present chapter, we decide to use the popular among practitioners inter-day measures while in the 7th chapter an investigation that is based on the intra-day realized volatility is conducted[6].

[6] For details and references about intra-day realized volatility see section 2.7 in chapter 2.

5.3.2 Evaluation Criteria

A large number of forecast evaluation criteria exists in the literature. However, none is generally acceptable. Because of high non-linearity in volatility models and the variety of statistical evaluation criteria, a number of researchers constructed economic criteria based upon the goals of their particular application. West et al. (1993) develop a criterion based on the decisions of a risk averse investor. Engle et al. (1993) assume that the objective is to price options and develop a loss function from the profitability of a particular trading strategy. Gonzalez-Rivera et al. (2004) compared the performance of various volatility models with economic and statistical loss functions and find that there is not a unique model that is the best performer across various loss functions. Brooks and Persand (2003) also found that the forecasting accuracy of the various methods is highly sensitive to the measure used to evaluate them. Hence, different loss functions proposed different models as the most appropriate in volatility forecasting. In the sequel, we focus on statistical criteria to measure the closeness of the forecasts to the realizations, in order to avoid restrictions imposed by economic theory. Moreover, we consider statistical criteria that are robust to non-linearity and heteroscedasticity. Pagan and Schwert (1990) use statistical criteria to compare parametric and non-parametric ARCH models with in-sample and out-of-sample data. Besides, Heynen and Kat (1994) investigate the predictive performance of ARCH and Stochastic Volatility models and Hol and Koopman (2000) compare the predictive ability of Stochastic Volatility and Implied Volatility models. Andersen et al. (1999a) applied heteroscedasticity-adjusted statistics to examine the forecasting performance of intraday returns. Denoting the forecasting variance over an N day period measured at day t by $\sigma^2_{t(N)}$, and the realized variance over the same period by $s^2_{t(N)}$, the following evaluation criteria are considered:

Squared Error (SE): $\left(\sigma^2_{t(N)} - s^2_{t(N)}\right)^2$ (5.3.3)

Absolute Error (AE): $\left|\sigma^2_{t(N)} - s^2_{t(N)}\right|$ (5.3.4)

Heteroscedasticity Adjusted Squared Error (HASE): $\left(1 - s^2_{t(N)} \big/ \sigma^2_{t(N)}\right)^2$ (5.3.5)

Heteroscedasticity Adjusted Absolute Error (HAAE): $\left|1 - s^2_{t(N)} \big/ \sigma^2_{t(N)}\right|$ (5.3.6)

Logarithmic Error (LE): $\ln\left(s^2_{t(N)} \big/ \sigma^2_{t(N)}\right)^2$ (5.3.7)

The first two functions have been widely used in the literature (see, e.g. Brooks and Persand (2003), Heynen and Kat (1994) and West and Cho (1995)). The HASE and

HAAE functions were considered by Andersen et al. (1999b), while the LE function was utilized by Pagan and Schwert (1990).

Usually, the average of the evaluation criteria is computed. However, when simulating an AR(1)GARCH(1,1) process, which is the most commonly used model in financial applications, the distributions of $\left(\sigma_{t(N)}^2 - s_{t(N)}^2\right)$, $\left(1 - s_{t(N)}^2 / \sigma_{t(N)}^2\right)$ and $\ln\left(s_{t(N)}^2 / \sigma_{t(N)}^2\right)$ are asymmetric with extreme outliers. It would therefore be advisable to compute both the mean and the median of the evaluation criteria. Figure 5.1 depicts the histograms of the one-step forecast error distribution from the following simulated process:

$$\begin{aligned} y_t &= 0.001 + 0.1 y_{t-1} + \varepsilon_t \\ \sigma_t^2 &= 0.002 + 0.05\varepsilon_{t-1}^2 + 0.9\sigma_{t-1}^2 \\ \varepsilon_t &= \sigma_t z_t \quad \text{and} \quad z_t \overset{iid}{\sim} N(0,1). \end{aligned} \tag{5.3.8}$$

5.4. *Examining the Performance of the SPEC Model Selection Algorithm*

In this section, the ability of the SPEC model selection algorithm to lead to the ARCH models that track closer future volatility is illustrated on a series of daily log-returns. As follows from section 5.2, the return series can be modeled in the following form:

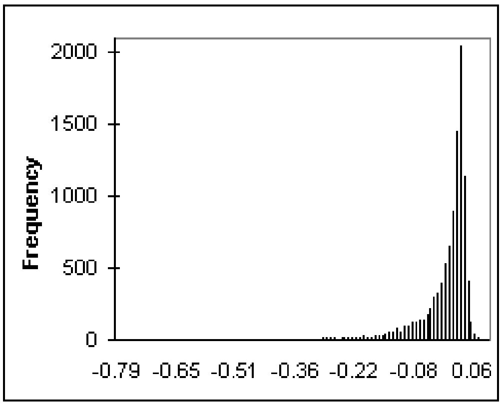
$$\begin{aligned} y_t &= E\left(y_{t|t-1}\right) + \varepsilon_t \\ E\left(y_{t|t-1}\right) &= c_0 + \sum_{i=1}^{\kappa} c_i y_{t-i} \\ \varepsilon_t &= z_t \sigma_t \\ z_t &\overset{iid}{\sim} N(0,1) \\ \sigma_t^2 &= g\left(\sigma_{t-1}(\theta), \sigma_{t-2}(\theta), \ldots; \varepsilon_{t-1}(\theta), \varepsilon_{t-2}(\theta), \ldots; \upsilon_{t-1}, \upsilon_{t-2}, \ldots\right) \end{aligned} \tag{5.4.1}$$

In the sequel, the above form is considered in connection with the ARCH models defined by (5.2.4), (5.2.5) and (5.2.6), for $\kappa = 0,1,2,3,4$, $p = 0,1,2$ and $q = 1,2$, thus yielding a total of 85 cases[7].

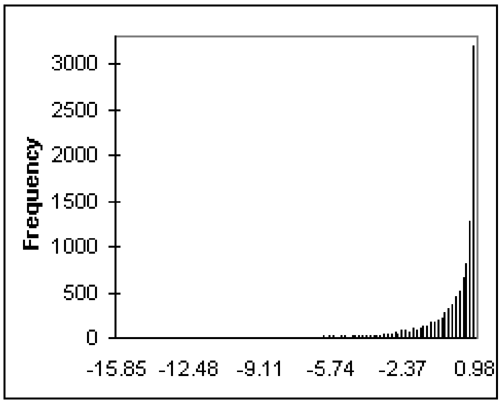
[7] Numerical maximization of the log-likelihood function, for the EGARCH(2,2) model, frequently failed to converge. So the five EGARCH models for $p = q = 2$ were excluded.

Figure 5.1.

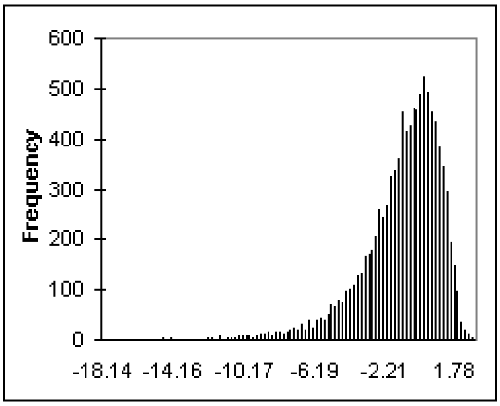
Histogram of $\sigma^2_{t(1)} - y^2_t$ from an AR(1)GARCH(1,1) simulated process



Histogram of $1 - y^2_t \big/ \sigma^2_{t(1)}$ from an AR(1)GARCH(1,1) simulated process



Histogram of $\ln\left(y^2_t \big/ \sigma^2_{t(1)}\right)$ from an AR(1)GARCH(1,1) simulated process



The data set consists of 1661 S&P500 stock index daily returns in the period from November 24th, 1993 to June 26th, 2000. The ARCH processes are estimated using a rolling sample of constant size equal to 500[8]. Thus, the first one-step-ahead volatility prediction, $\hat{\sigma}^2_{t+1|t}$, is available at time $t = 500$. Applying the SPEC model selection algorithm, the sum of squared standardized one-step-ahead prediction errors, $\sum_{t=1}^{T} \hat{z}^2_{t|t-1}$, was estimated considering various values for T, and, in particular, $T = 5(5)80$. This is

[8] Section 6.5 provides motivation for the choice of a 500-observations window.

an indirect way to examine the performance of the SPEC model selection algorithm for various values of T. Thus, the evaluation criteria were applied on the one-step-ahead forecasts using $1661-500-80=1081$ data points, on the two-step-ahead forecasts using $1661-500-81=1080$ data points, …, and on the k^{th}-step-ahead forecasts using $1081-k+1$ data points.

Adopting Brooks and Persand's (2003) approach we consider evaluating multi-step-ahead forecasts based on overlapping time periods. In particular, most of the studies in the literature evaluate the multi-step forecasts using non-overlapping time periods in order to infer about the statistical significance of the ranking.

Our main purpose is to examine the application potential of the SPEC algorithm of selection of models on the basis of their forecasting ability in terms of volatility. So, the mean and the median value of each of the 5 evaluation criteria, in equations (5.3.3)-(5.3.7), were computed, yielding a total of 10 evaluation criteria for each forecasting horizon from one day to one hundred days ahead. However, volatility is expressed either as the variance or as the standard deviation. Thus, in order to examine possible differences between forecasting the variance and its square root, the evaluation criteria were, also, applied on the standard deviation. Therefore, $\sigma^2_{t(N)}$ and $s^2_{t(N)}$, in equations (5.3.3)-(5.3.7), were replaced by $\sigma_{t(N)}$ and $s_{t(N)}$, respectively and 10 more evaluation criteria were computed. In total, 20 evaluation criteria were computed for a horizon ranging from one trading day to five trading months. In section 5.3.1, two realized volatility measures were mentioned. As, qualitatively, they are of the same nature, in the sequel, we base the analysis on the realized volatility as defined by $s^2_{t(N)}$.

It was examined whether the ARCH models selected by the SPEC algorithm achieve the lowest value of the evaluation criteria. The main focus was on the median values of the criteria and mainly on the heteroscedasticity adjusted criteria since they are more robust to asymmetry. The comparative evaluation is performed by computing the loss functions for variance forecasts always obtained by a single model on the one hand, and for variance forecasts obtained by models picked by the SPEC algorithm on the other. Table 5.2, in the Appendix, presents the minimum and maximum values of the evaluation criteria that were achieved by each of the 85 ARCH models and the ARCH models suggested by the SPEC model selection algorithm. The SPEC algorithm is applied for 16 values for T, and, in particular, $T=5(5)80$. The minSPEC (maxSPEC) value refers to the minimum (maximum) of the 16 values of the evaluation criteria achieved by the models selected by the SPEC algorithm. Moreover, for each of the 85 estimated ARCH models the evaluation criteria have been computed. The minARCH (maxARCH) value refers to the minimum (maximum) of the 85 values of the evaluation criteria achieved by the ARCH models.

Figure 5.2, in the Appendix, shows, for each evaluation criterion and each forecasting horizon, whether ARCH models selected by the SPEC algorithm achieve the lowest value of the evaluation criteria. In the first part of Figure 5.2, the performance of the models, which are selected by the SPEC algorithm, on the basis of the conditional variance is depicted, while, the second part refers to their performance on forecasting standard deviation. The general conclusion is that the SPEC algorithm lead to the selection of the ARCH processes which track closer the realized volatility in the majority of the cases. Specifically, for the forecasting horizon ranging from 11 to 52 days, the models selected by the SPEC algorithm achieve the lowest criteria values, irrespectively of the evaluation criteria. The percentage of cases, that the models selected by the SPEC algorithm achieve the lowest value of the evaluation criteria, is higher around the forecasting horizon ranging from 16 to 36 days ahead, or 4 to 7 trading weeks ahead. Table 5.3, in the Appendix, presents the percentage of cases the models selected by the SPEC algorithm perform “better” as judged by the evaluation criteria, for 3 different horizon ranges. Note that, in terms of the MSE and MAE criteria, none of the models chosen by the SPEC algorithm appears to perform better in any of the forecasting horizons considered. But, in terms of the median values of the criteria and the heteroscedasticity adjusted criteria, which are robust to asymmetry, the models selected by the SPEC algorithm appear to have a better performance in all the forecasting horizons considered.

It is interesting to note that, via the evaluation criteria, the suggested sample size, T, for the SPEC model selection algorithm can be determined. The SPEC model selection algorithm has been applied for $T = 5(5)80$. In the sequel, the value of T for which the SPEC selection method achieves the best performance according to the evaluation criteria used, is examined. Figure 5.3 shows a plot of the average T, suggested by the evaluation criteria, across the forecasting horizons. The bar charts of Figure 5.3 are a graphical representation of the number of evaluation criteria by which the performance of the models selected by the SPEC algorithm were judged “better” than the performance of any other single model (measured on the right hand side vertical axis).

For a 16 to 36 day ahead forecasting horizon, the appropriate T, as concerns the specific data, ranges around 20 days with a standard deviation of 3.6 days. Table 5.4, in the Appendix, provides more details for the sample size of the SPEC selection method suggested by the evaluation criteria and its standard deviation for both the entire 16 to 36 day ahead forecasting horizon and for each day individually. The SPEC model selection algorithm shows a better performance for a sample size of about 20 days.

Figure 5.3. Sample size of the SPEC model selection algorithm, suggested by the Evaluation Criteria.

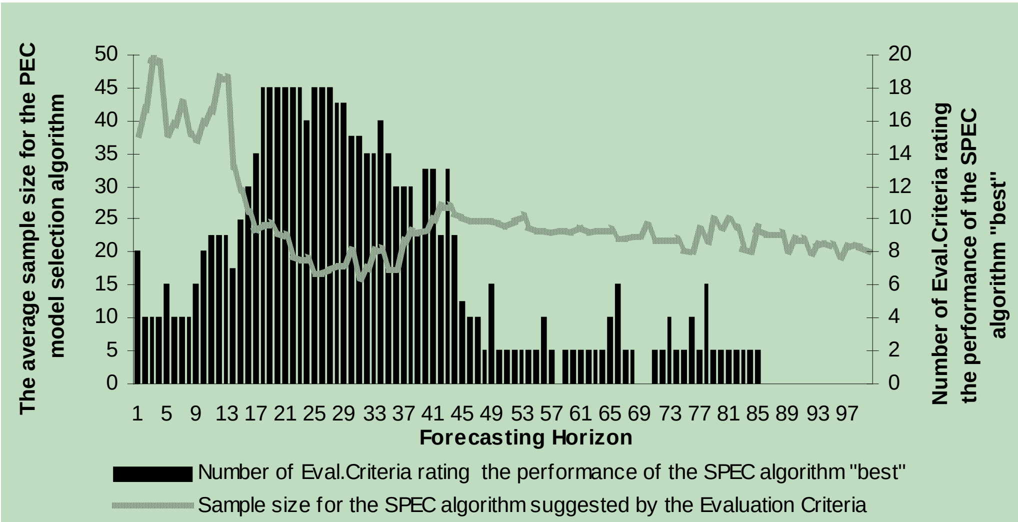
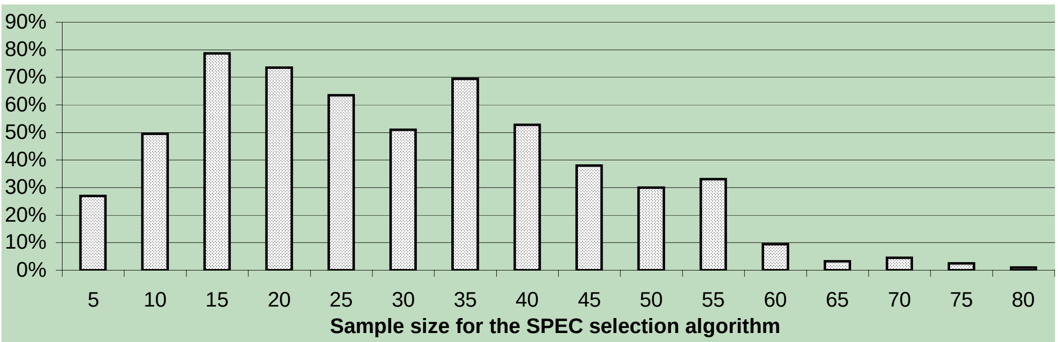
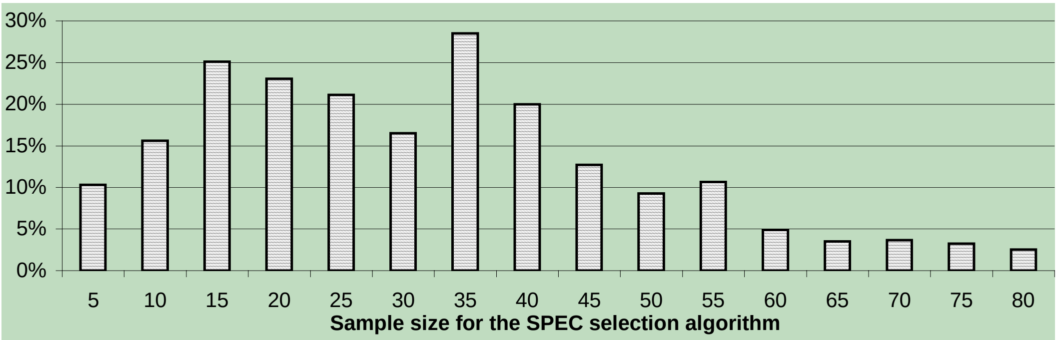


Figure 5.4.a. The percentage of evaluation criteria rating the performance of the SPEC algorithm 'best'. Forecasting horizon ranging from 16 to 36 days.



In order to test the importance of selecting the appropriate T, for the model selection method suggested, the evaluation criteria were run for $T = 5(5)80$. The results are indeed in support of a sample size of around 20 days for the SPEC algorithm to manifest a better performance. Figure 5.4 presents the percentage of the evaluation criteria by which the SPEC algorithm, with specific T, selects those ARCH models that generate "better" volatility predictions. For T ranging from 15 to 35, the SPEC selection method appears to have the highest performance.

Figure 5.4.b. The percentage of evaluation criteria rating the performance of the SPEC algorithm 'best'. Forecasting horizon ranging from 1 to 100 days.



5.5. *Comparison of the SPEC Criterion to Other Methods of Model Selection*

Most of the methods used in the time series literature for selecting the appropriate model are based on evaluating the ability of the models to describe the data. Standard model selection criteria such as the Akaike Information Criterion [AIC] and the Schwarz Bayesian Criterion [SBC] have widely been used in the ARCH literature, despite the fact that their statistical properties in the ARCH context are unknown. These are defined in terms of $l_n(\hat{\theta})$, the maximized value of the log-likelihood function of a model, where $\hat{\theta}$ is the maximum likelihood estimator of the parameter vector θ based on a sample of size n and $\breve{\theta}$ denotes the dimension of θ, thus:

$$AIC = l_n(\hat{\theta}) - \breve{\theta}$$

$$SBC = l_n(\hat{\theta}) - 2^{-1}\breve{\theta}\ln(n).$$

In addition, model selection is mainly based on the evaluation of some loss function for each of the competing models. In this section, the statistical criteria, which were considered in section 5.3 as measures in evaluating the predictive performance of a variance model, are considered as criteria for the selection of ARCH models. In particular, the model selection methods presented in Table 5.5, are considered and their ability to predict future volatility is investigated.

Applying the SPEC model selection algorithm, the sum of squared standardized one-step-ahead prediction errors, $\sum_{t=1}^{T}\hat{\varepsilon}_{t|t-1}^{2}\Big/\hat{\sigma}_{t|t-1}^{2}$, was estimated considering various values for T. Therefore, each of the model selection criteria, in Table 5.5**,** was computed considering various values for T, and, in particular, $T = 10(10)80$. The AIC and SBC criteria were computed based on the rolling sample of constant size equal to 500, or $n = 500$, that is used at each time to estimate the parameters of the models. Based on Table 5.1, selecting a strategy for each method of model selection naturally amounts to selecting the model, which, at time k, has the lowest value of the formula is indicated in Table 5.5.

Tables 5.6.1 to 5.6.11, in the 5th Appendix, presents the percentage of cases the models selected by each model selection method perform "better" as judged by the evaluation criteria, for 3 different horizon ranges. As concerns the AIC and SBC selection methods, they do not achieve the lowest value of the evaluation criteria in almost all the cases, which is indicative of the inability of the in-sample model selection methods to suggest the models with superior volatility forecasting performance. The general conclusion is that the loss functions presented in Table 5.5 do not led to the selection of the ARCH processes which track closer the realized volatility. The HASEVar, HAAEVar and HASEDev criteria show a better performance, as they select the ARCH models with the lowest value of the evaluation criteria, around the forecasting horizon ranging from 16 to 36 days ahead. So, they might be used in selecting that model that generates "better" volatility predictions. In order to investigate whether the suggested model selection method or the loss functions indicate the ARCH models that track closer the realized volatility, the predictive ability of these loss functions must be compared to the volatility forecasting ability of the SPEC criterion, and mainly for a forecasting horizon ranging from 16 days to 36 days ahead.

Of main interest is whether the ARCH models selected by the SPEC algorithm yield values for the evaluation criteria that are lower than those corresponding to the

ARCH models selected by the model selection methods summarized in Table 5.5. Tables 5.7.1 to 5.7.11, in the Appendix, presents the percentage of times the ARCH models selected by the SPEC algorithm achieve lower values for the corresponding evaluation criteria and the specific forecasting horizons than the models selected by the other model selection methods. As concerns forecasting horizons of 4 to 7 trading weeks ahead the performance of the SPEC algorithm is by far the best.

Table 5.5. Methods of selection of ARCH models. $\sigma^2_{t(N)}$ denotes the forecasting variance over an N day period measured at day t and $s^2_{t(N)}$ denotes the realized variance over the same period.

1. Square Error of Conditional Variance (SEVar):
$$\sum_{t=1}^{T}\left(\left(\sigma^2_{t(N)} - s^2_{t(N)}\right)^2\right) \tag{5.5.1}$$
2. Absolute Error of Conditional Variance (AEVar):
$$\sum_{t=1}^{T}\left(\left|\sigma^2_{t(N)} - s^2_{t(N)}\right|\right) \tag{5.5.2}$$
3. Square Error of Conditional Standard Deviation (SEDev):
$$\sum_{t=1}^{T}\left(\left(\sigma_{t(N)} - s_{t(N)}\right)^2\right) \tag{5.5.3}$$
4. Absolute Error of Conditional Standard Deviation (AEDev):
$$\sum_{t=1}^{T}\left(\left|\sigma_{t(N)} - s_{t(N)}\right|\right) \tag{5.5.4}$$
5. Heteroscedasticity Adjusted Squared Error of Cond. Variance (HASEVar):
$$\sum_{t=1}^{T}\left(\left(1 - s^2_{t(N)} \big/ \sigma^2_{t(N)}\right)^2\right) \tag{5.5.5}$$
6. Heteroscedasticity Adjusted Absolute Error of Cond. Variance (HAAEVar):
$$\sum_{t=1}^{T}\left(\left|1 - s^2_{t(N)} \big/ \sigma^2_{t(N)}\right|\right) \tag{5.5.6}$$
7. Heteroscedasticity Adjusted Squared Error of Cond. St. Deviation (HASEDev):
$$\sum_{t=1}^{T}\left(\left(1 - s_{t(N)} \big/ \sigma_{t(N)}\right)^2\right) \tag{5.5.7}$$
8. Heteroscedasticity Adjusted Absolute Error of Cond. St. Deviation (HAAEDev):
$$\sum_{t=1}^{T}\left(\left|1 - s_{t(N)} \big/ \sigma_{t(N)}\right|\right) \tag{5.5.8}$$
9. Logarithmic Error of Conditional Variance (LEVar):
$$\sum_{t=1}^{T}\left(\ln\left(s^2_{t(N)} \big/ \sigma^2_{t(N)}\right)^2\right) \tag{5.5.9}$$
10. Akaike Information Criterion (AIC):

$$AIC = l_n\left(\hat{\theta}\right) - \breve{\theta} \tag{5.5.10}$$

11. Schwarz Bayesian Criterion (SBC):

$$SBC = l_n\left(\hat{\theta}\right) - 2^{-1}\breve{\theta}\ln(n) \tag{5.5.11}$$

The SPEC model selection algorithm performs “better” than the other methods of model selection in about 90% of the cases. This percentage is lower when the SPEC algorithm is compared to the HASEVar, HAAEVar and HASEDev methods. Nevertheless, even in such cases, the opponent methods select the ARCH models that track closer future volatility much less frequently than the SPEC algorithm. The percentage of times, an opponent to the SPEC algorithm selects the most appropriate models in forecasting future volatility, is highest in the case of the HAAEVar method. However, only in the 23% of cases, the ARCH models selected by the HAAEVar method perform "better" than the models selected by the SPEC criterion, for any of the 3 horizon ranges.

5.6. *Conclusions*

The SPEC method, for selecting an ARCH model among several competing models was suggested, amounts to choosing the model with the lowest sum of squared standardized forecasting errors. A number of evaluation criteria, for forecasting horizons ranging from one day to one hundred days ahead, were applied and it was found that the ARCH models, selected by the SPEC model selection algorithm, generate “better” predictions of the volatility. Here, Brooks and Persand’s (2003) evaluation approach was adopted and multi-step-ahead forecasts were evaluated based on overlapping time periods. Alternatively, one might like to consider non-overlapping time periods and apply other evaluation schemes, such as those proposed by Diebold and Mariano (1995), Hansen and Lund (2003) or Hansen et al. (2003). Thus, the SPEC selection method appears to be a useful tool in guiding one’s choice of the appropriate model for estimating future volatility, with applications in evaluating portfolios, derivatives and financial risk.

Granger and Pesaran (2000a, 2000b) addressed the problem of forecast evaluation in the context of a realistic decision problem. They noted that “each forecast is linked with a value or cost function, as making a forecast error will cause a cost to some decision maker”. In the next chapter, we consider evaluating the SPEC method along such lines. Specifically, the performance of the SPEC algorithm is examined through the use of economic loss functions and in particular, the cumulative returns from trading volatility forecasts.

Chapter 6
Using the Standardized Prediction Error Criterion for ARCH Model Selection in Forecasting Option Prices

6.1. *Introduction*

The common way to measure the performance of volatility forecasting models is through assessing their ability to predict future volatility. However, as volatility is unobservable, there is no natural metric for measuring the accuracy of any particular model. Noh et al. (1994) considered assessing model performance through computing option prices based on the volatility forecasts of the underlying asset returns, devising trading rules to trade options on a daily basis and comparing the resulting profits.

Within this framework, the present chapter examines the performance of a number of ARCH-model based methods of predicting volatility in pricing options. The focus is on a method that allows the trader flexibility as to the choice of the model to use for prediction at each of a sequence of points in time based on the SPEC algorithm. The comparative evaluation is performed using options data on the basis of the cumulative profits of traders always using variance forecasts obtained by a single model on the one hand and the cumulative profits of traders using variance forecasts obtained by models suggested by the SPEC algorithm on the other. The results of the study show that traders using this algorithm for deciding which model's forecasts to use at any given point in time achieve higher cumulative profits than those using only a single model all the time. A comparison of the SPEC algorithm with a set of other model evaluation criteria yields similar findings.

Noh et al. (1994) considered the problem of assessing the performance of two model based methods for volatility-forecasting, the ARCH modeling based method and the implied volatility regression method, by trading options. The ARCH models provide one common conditional volatility estimate for both call and put option prices, while the implied-volatility forecasting method provides different volatility estimates for call and put option prices. Over the April 1986 to December 1991 period, for S&P500 index options, the ARCH model based forecasting method led to a greater profit than the rule based on the implied volatility regression model. In particular, by the trading strategy based on the ARCH model a daily profit of 0.89% was earned, while by the implied volatility method a daily loss of 1.26% was made.

A comparative evaluation is performed through comparing their volatility forecasts in terms of the profits of traders pricing derivatives in a real market based on these forecasts. The focus is on the model forecasting ability rating based selection algorithm considered in 4th chapter. According to the SPEC algorithm, each trading day, the ARCH model with the lowest sum of squared standardized one-step-ahead prediction errors is selected for estimating future volatility of underlying asset returns. The advantage of this method over other single model based methods lies in the fact that the trader is flexible as to the choice of the model at each of a sequence of points in time. Forecasts of option prices used in the comparative evaluation are calculated using the Black and Scholes (BS) pricing formula (Black and Scholes 1973). The obtained results indicate that the SPEC has a satisfactory performance in selecting the ARCH models that yield *better* volatility predictions for option pricing. It is demonstrated in particular, that over the period from March 1998 to June 2000, taking into consideration a transaction cost of $2, an agent who would consider using this model selection algorithm could make a daily profit of 1.46% from trading S&P500 index options. Section 6.2 provides a brief description of the BS pricing formula and introduces the reader to the notion of trading options and computing the relative cash flows. Section 6.3 presents the trading rules considered by Noh et al. (1994) for the performance of volatility-forecasting methods. In sections 6.4 to 6.7, the cash flows, from trading options based on i) a set of ARCH processes, ii) the SPEC model selection algorithm, and iii) a number of other methods of model selection, are computed. Finally, in section 6.8, a brief discussion of the results is provided.

6.2. *Options*

An option is a security that gives its owner the right, not the obligation, to buy or sell an asset at a fixed price (exercise price) within a specified period of time, subject to certain conditions. There are two main types of options: calls and puts. A call option is the right to buy a number of shares, of the underlying asset, at a fixed price on or before the maturity day. A put option is a right to sell a number of shares, of the underlying asset, at a fixed price on or before the maturity day. A straddle option is the purchase (or sale) of both a call and a put option, of the underlying asset, with the same expiration day. The maturity day is the latest date that the option can be exercised. If the option can be exercised only on the maturity day, it is termed a European option, whereas an American option can be exercised on or before the expiration day.

The purchaser of a call (put) option acquires the right to buy (sell) a share of a stock for a given price on or before time T and pays for the right at the time of purchase. On the other hand, the writer of this call (put) collects both the option price today and the obligation to deliver (buy) one share of stock in the future for the exercise price, if the purchaser of the call (put) demands.

6.2.1 Stock and Exercise Price Relationship

The exercise price of the "at the money" option is equal to the price of the underlying asset. The exercise price of the "near the money" option is approximately the same as the price of the underlying asset. A call (put) option is said to be "in the money" if its exercise price is less (greater) than the current price of the underlying asset. A call (put) option is said to be "out of the money" if its exercise price is greater (less) than the price of the underlying asset.

6.2.2 Black and Scholes Option Pricing Formula

The pricing of options is a cornerstone of financial literature. The BS option pricing model is a very important and useful model in estimating the fair value of an option. Based on the law of one price or no arbitrage condition, the option pricing models of Black and Scholes (1973) and Merton (1973) gained an almost immediate acceptance among academics and investments professionals[1]. Their approach can be used to price any security whose payoffs depend on the prices of other securities. The main idea is to create a costless self-financing portfolio strategy, whereby long positions are completely financed by short positions, which can replicate the payoff of the derivative. Under the no-arbitrage condition, the dynamic strategy reduces to a partial differential equation subject to a set of boundary conditions that are determined by the specific terms of the derivative security.

[1] The 1997 Nobel Prize in Economics was awarded to Robert C. Merton and Myron S. Scholes for their work, along with Fischer Black, in developing the Fischer-Black options pricing model. Black, who died in 1995, would undoubtedly have shared in the prize had he still been alive. (American Mathematical Society; www.ams.org/new-in-math/nobel1997econ.html).
"An early version of Black and Scholes (1973) was submitted in the summer of 1970, but both the Journal of Political Economy and the Review of Economics and Statistics rejected the paper – perhaps because the ideas were so new and/or because Black was not an academic. After revising the approach and receiving encouragement from the University of Chicago professors Merton Miller and Eugene Fama, an article testing the model empirically was published in 1972 in the Journal of Finance, (Black and Scholes 1972). The proof of the model was published in 1973 in the Journal of Political Economy, published by the University of Chicago". (Daigler 1994, page 128).

The pricing of index options is based on the Black & Scholes option pricing formula (Black and Scholes (1973)). In particular, the forecast price of a call and a put option at time $t+1$ given the information available at time t, with τ days to maturity, denoted, respectively, by $C_{t+1|t}^{(\tau)}$ and $P_{t+1|t}^{(\tau)}$ are given by

$$C_{t+1|t}^{(\tau)} = S_t N(d_1) - Ke^{-rf_t \tau} N(d_2),$$

$$P_{t+1|t}^{(\tau)} = -S_t N(-d_1) + Ke^{-rf_t \tau} N(-d_2),$$

where (6.2.1)

$$d_1 = \frac{\ln\left(S_t/K\right) + \left(rf_t + 1/2\left(\sigma_{t+1|t}^{(\tau)}\right)^2\right)\tau}{\sigma_{t+1|t}^{(\tau)}\sqrt{\tau}} \text{ and } d_2 = d_1 - \sigma_{t+1|t}^{(\tau)}\sqrt{\tau}.$$

Here, S_t is the market closing price of the stock (or portfolio) at time t (used as a forecast for S_{t+1}), rf_t is the daily continuously compounded risk free interest rate and K is the exercise (or strike) price at maturity day, while, $N(.)$ and $\sigma_{t+1|t}^{(\tau)}$ denote, respectively, the cumulative normal distribution function and the standard deviation of the rate of return during the life of the option, from $t+1$ until the maturity day, given the information available at time t.

6.2.3 An Example in Computing Theoretical Option Prices

Consider a trader who wants to evaluate the BS theoretical price of a European call and put option with three months to expiry. The stock price is \$60, the strike price is \$65, the risk free rate is 8% per annum (the return of three month treasury bills), the dividend yield is 5% per annum and the volatility is 30% per annum. Thus, $S_t = 60$, $K = 65$, $\tau = 0.25$, $rf_t = 0.08$, $\gamma_t = 0.05$ and $\sigma_t = 0.3$. Computing:

$$d_1 = -0.409, \quad N(d_1) = 0.341, \quad N(-d_1) = 0.659,$$
$$d_2 = -0.559, \quad N(d_2) = 0.288, \quad N(-d_2) = 0.712,$$

the price of the call option is: $C_t = 60e^{-0.05*0.25}0.341 - 65e^{-0.08*0,25}0.288 = \$1.8674$ and the price of the put option is: $P_t = -60e^{-0.05*0.25}0.659 + 65e^{-0.08*0.25}0.712 = \$6.3256$.

6.2.4 Option Strategies and Cash Flows

Suppose the price of stock at time t is S_t and the price of a call and put option, with expiration day T and exercise price K, are C_t and P_t, respectively. In terms of cash flows, the purchaser of an option (a long option position) always has an initial negative cash flow, the price of the option, and a future cash flow that is at worst zero. The writer of the option (a sort option position) has an initial positive cash flow followed by a terminal cash flow that is at best zero. At expiration day, T, the call option is exercised only if $S_T > K$. Thus, the cash flow, at time T, of the call purchaser is[2]:

$$\max(0, S_T - K) - e^{rf_t(T-t)}C_t = \begin{cases} -e^{rf_t(T-t)}C_t & if\ S_T \le K \\ S_T - K - e^{rf_t(T-t)}C_t & if\ S_T > K. \end{cases} \quad (6.2.2)$$

The cash flow of the call writer is opposite to that of the call purchaser:

$$e^{rf_t(T-t)}C_t - \max(0, S_T - K) = \begin{cases} e^{rf_t(T-t)}C_t & if\ S_T \le K \\ e^{rf_t(T-t)}C_t + K - S_T & if\ S_T > K. \end{cases} \quad (6.2.3)$$

Moreover, the put is exercised only if $S_T < K$. Thus, at maturity day, the cash flow of the put purchaser is:

$$\max(0, K - S_T) - e^{rf_t(T-t)}P_t = \begin{cases} -e^{rf_t(T-t)}P_t & if\ S_T \ge K \\ K - S_T - e^{rf_t(T-t)}P_t & if\ S_T < K \end{cases} \quad (6.2.4)$$

and the cash flow of the put writer is:

$$e^{rf_t(T-t)}P_t - \max(0, K - S_T) = \begin{cases} e^{rf_t(T-t)}P_t & if\ S_T \ge K \\ e^{rf_t(T-t)}P_t + S_T - K & if\ S_T < K. \end{cases} \quad (6.2.5)$$

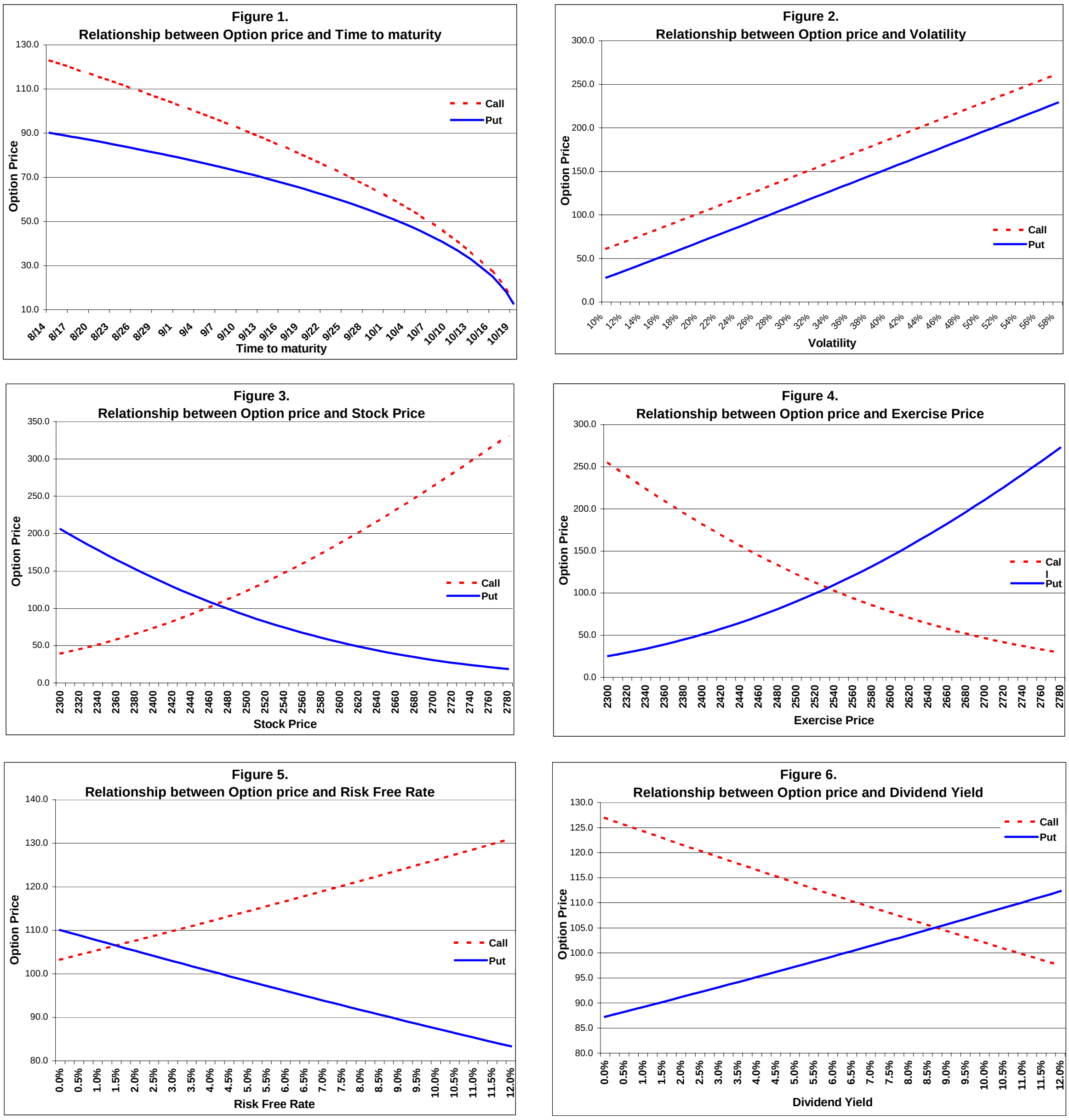
Figure 6.7 presents the profit and loss performance of buying and writing options. A long straddle position is an option strategy in which a call and a put of the same exercise price, maturity and underlying terms are purchased. This position is called a straddle since it will profit from a substantial change in the stock price in either direction. Traders purchase a straddle under one of two circumstances. The first circumstance exists when a large change in the stock price is expected, but the direction of the change is unknown. Examples include an upcoming announcement of earning, uncertain takeover or merger speculation, a court case for damages, a new product announcement, or an uncertain economic announcement such as inflation figures or a change in the prime interest rate. A straddle seems a risk free trading strategy when a large change in the price of a stock is expected.

However, in the real world, this is not necessarily the case. If the general view of the market is that there will be a big jump in the stock price soon, the option prices should reflect the increase in the potential volatility of the stock. A trader will find options on the stock to be significantly more expensive than options on a similar stock for which no jump is expected. For a straddle to be an effective strategy, the

[2] It is assumed that investors, at time t, are borrowing and lending at the same risk free rate, rf_t. Thus, the money during the period from t to T is investing with a daily return of $(1 + rf_t) \approx \exp(rf_t)$.

trader must believe that big movements in the stock price are likely and this belief must be different from that of most of the other market participants.

Figures 6.1-6.6. Relationship between option prices and variables involved in the BS formula.



The second circumstance in which straddles are purchased occurs when the trader estimates that the true future volatility of the stock will be greater that the volatility that is currently impounded in the option prices. Note that although the long straddle has theoretically unlimited potential profit and limited risk, it should not be viewed as a low risk strategy. Options can lose their value very quickly, and in the case of a straddle, there is twice the amount of erosion of time value as compared to the purchase of a call or put. The opposite of a long straddle strategy is a short straddle position. This strategy has unlimited risk and limited profit potential, and is therefore only appropriate for experienced investors with a high tolerance for risk. The short straddle will profit from limited stock movement and will suffer losses if the underlying asset moves substantially in either direction. Figure 6.8 presents the payoffs of taking long and sort straddle positions. At expiration day, T , the cash flows of taking a long and a sort straddle position are:

$$\begin{aligned} &|S_T - K| - e^{rf_t(T-t)}(C_t + P_t) \\ &e^{rf_t(T-t)}(C_t + P_t) - |S_T - K|, \end{aligned} \tag{6.2.6}$$

respectively.

Figure 6.7. The cash flows of taking long and sort positions in call and put options.

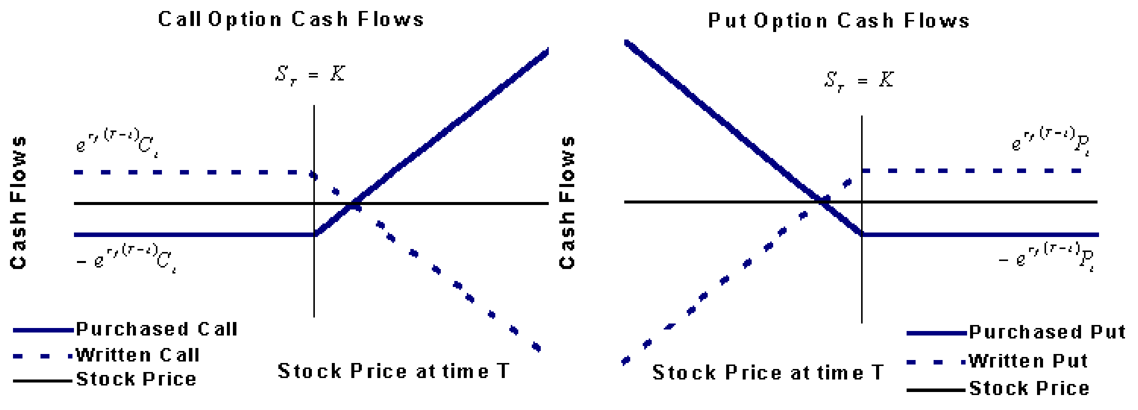
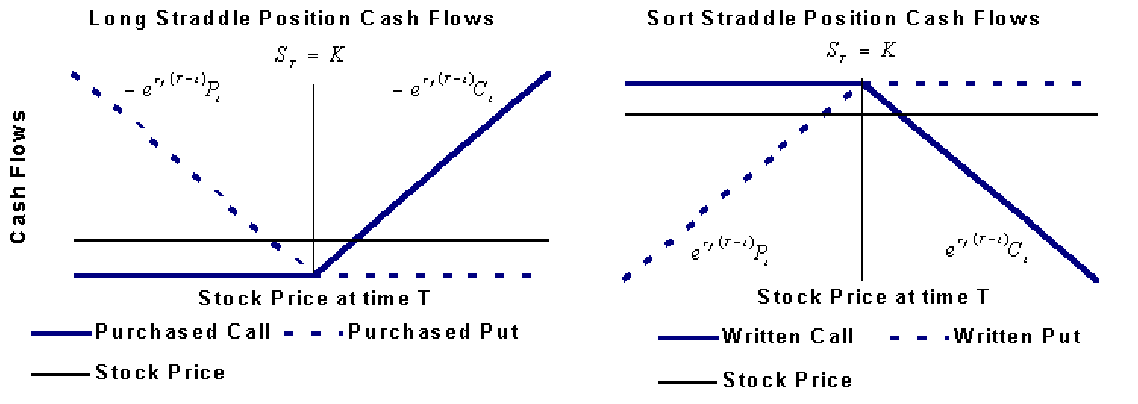


Figure 6.8. The cash flows of taking long and sort straddle positions.



6.2.5 An Example of Straddle Trading

Consider a trader who feels that the price of a certain stock, currently valued at $54 by the market, will move significantly in the next three months. The trader could create a straddle by buying both a put and a call with a strike price of $55 and an expiration date in three months. Suppose that the call and the put costs are \$5 and \$4, respectively. The most that can be lost is the amount paid, or \$9, if the stock price moves to \$55. If the stock price moves above \$63 or below \$45, the long position earns a profit. In the case of taking a short straddle position, the maximum profit is the premium received, or \$9. The maximum loss is unlimited, and the sort position will lose if the stock price moves above \$63 or below \$45.

6.3. *Assessing the Performance of Volatility Forecasting Methods*

Noh et al. (1994) devised rules to trade "near the money" straddles. If the straddle price forecast is greater than the market straddle price, the straddle is bought. If the straddle price forecast is less than the market straddle price, the straddle is sold, i.e.

If $C_{t+1|t}^{(\tau)} + P_{t+1|t}^{(\tau)} > P_t^{(\tau)} + C_t^{(\tau)} \Rightarrow$ The straddle is bought at time t. (6.3.1)

If $C_{t+1|t}^{(\tau)} + P_{t+1|t}^{(\tau)} < P_t^{(\tau)} + C_t^{(\tau)} \Rightarrow$ The straddle is sold at time t. (6.3.2)

The strategy can be understood with the help of the following example: On Monday, after the stock market closes[3], Tuesday's price of an option that expires on Friday, is estimated. The remaining life of the option is 3 days, from Tuesday to Friday. If option's prediction price on Tuesday is higher than the observed option price on Monday, the option is bought in order to be sold on Tuesday. If the predicted option price on Tuesday is lower than the observed option price on Monday, the option is sort-sold in order to be bought on Tuesday.

Monday	t
Tuesday	$t+1$
Wednesday	$t+2$
Thursday	$t+3$
Friday	$t+4$

The rate of return from trading an option is:

$$RT_t = \frac{C_t + P_t - C_{t-1} - P_{t-1}}{C_{t-1} + P_{t-1}}, \text{ on buying a straddle,} \quad (6.3.3)$$

$$RT_t = \frac{-C_t - P_t + C_{t-1} + P_{t-1}}{C_{t-1} + P_{t-1}}, \text{ on sort-selling a straddle.} \quad (6.3.4)$$

[3] The trading strategy assumes that there is enough time to forecast the option prices given all the information at time t (closing prices of stocks) so as the trader to be able to decide the trading of an option at time t (before the option market closes). For example, the Chicago Stock Market closes at 3:00 pm local time and the Chicago Board of Option Exchange closes as 3:15 pm local time.

Note that the transaction costs, X, should be taken into account. If this is the case, the net rate of return from trading an option is given as:

$$NRT_t = RT_t - \frac{X}{C_{t-1} + P_{t-1}}. \tag{6.3.5}$$

Moreover, a filter can be applied in the trading strategy, so as to trade an option only when the difference between forecast and today's option price exceeds the amount of the filter.

Noh et al. (1994) applied the AR(1)-GARCH(1,1) model in order to forecast the future volatility. Forecasts of option prices, on the next trading day, are calculated using the BS option pricing formula and conditional volatility forecasts. The volatility during the life of the option is computed as the square root of the average forecast conditional variance:

$$\sigma_{t+1|t}^{(\tau)} = \left(\tau^{-1} \sum_{i=2}^{\tau+1} \hat{\sigma}_{t+i|t}^{2} \right)^{1/2}, \tag{6.3.6}$$

where $\hat{\sigma}_{t+i|t}^{2}$ denotes the prediction of the conditional variance at time $t+i$ given the information set available at time t.

Noh et al. (1994) assessed the performance of the AR(1)-GARCH(1,1) model for straddles written on the S&P500 index over the period from April 1986 to December 1991 and found that the model earns a profit of \$0.885 per straddle in excess of a \$0.25 transaction cost and applying a \$0.5 filter. Gonzalez-Rivera et al. (2004) had also evaluated the ability of various volatility models in predicting one-period ahead call options on the S&P500 index, with expiration dates ranging from January 2000 through November 2000, and found that simple models like the EWMA of Riskmetrics^{TM} (1995) performed as well as sophisticated ARCH specifications.

6.4. *Option Pricing Using a Set of ARCH Processes and Model Selection Algorithms*

The GARCH(1,1) is the most commonly used model in financial applications. The question that arises at this point is: "*Why should one use the simple GARCH(1,1) model instead of using a higher order of GARCH(p,q) model, an asymmetric ARCH model, or even a more complicated form of an ARCH process?*". There is a vast number of ARCH models. Which one should be preferred? The volatility prediction model, which gives the highest rate of return in trading options, should be the preferable one. Moreover, under the assumption that the BS formula describes perfectly the dynamics of the market that affects the price of the option, the model gives the most precise prediction of conditional volatility should be the model that gives the highest rate of return. Unfortunately, an important limitation still remains. Even if one could find the model, which predicts the volatility precisely, it is well known that the BS formula does not describe the dynamics pricing the options perfectly.[4] Moreover, the validity of the variance forecasts depends on which option pricing formula is used. Engle et al. (1997) used Hull and White's (1987) modification to the BS formula for pricing straddles on a simulated options market. A series of studies such as Barone-Adesi et al. (2004), Duan (1995), Duan et al. (1999), Heston and Nandi (2000), Ritchken and Trevor (1999) and Sabbatini and Linton (1998), derived ARCH-based option pricing models assuming that a specific ARCH process generates the variance of the asset. . However, despite its limitations, the BS pricing formula has had a wide acceptance by floor traders on option exchanges.

In this chapter, since the ARCH-based option pricing models considered in the literature for the various models being compared are different or no ARCH based pricing formula exists for some of them, the BS option pricing model is adopted. In the sequel, a variety of volatility prediction models are estimated using S&P500 stock index daily returns and the rate of return from trading straddles, based on the volatility predictions, is calculated. The SPEC model selection algorithm is subsequently applied in order to choose for each particular day the *appropriate* ARCH model for estimating the price of an option. The day-by-day rates of return are reflective of the corresponding predictive performances of the models. Comparing the results, provides an indirect comparative assessment of a trading strategy based on option prices forecasts provided by any one of these models to the trading strategy of deciding each day on the basis of the option price forecast by the model selected by the SPEC algorithm as the *most appropriate* for that particular day.

6.5. *Trading Straddles Based on a Set Of ARCH Processes*

For $y_t = \ln(S_t / S_{t-1})$ denoting the continuously compound rate of return from time $t-1$ to t, where S_t is the asset price at time t, a set of ARCH models are estimated. The conditional mean is considered as a κ^{th} order autoregressive process ($AR(\kappa)$):

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[4] There is a large number of articles that examine the biasedness of BS formula. For details see Daigler (1994) and Natenberg (1994).

$$
\begin{aligned}
y_t &= \mu_t + z_t \sigma_t \\
\mu_t &= c_0 + \sum_{i=1}^{\kappa} \left(c_i y_{t-i} \right) \\
z_t &\overset{i.i.d.}{\sim} N(0,1),
\end{aligned}
\tag{6.5.1}
$$

and the conditional variance is regarded as a GARCH(p,q), an EGARCH(p,q) or a TARCH(p,q) function of the forms (5.2.4) – (5.2.6) considered in chapter 5. Thus, the AR(κ)GARCH(p,q), AR(κ)EGARCH(p,q) and AR(κ)TARCH(p,q) models are applied, for $\kappa = 0,...,4$, $p = 0,1,2$ and $q = 1,2$, yielding a total of 85 cases[5]. The conditional variance for the GARCH(p,q) process may be rewritten as:

$\sigma_t^2 = \left(u_t', \eta_t', w_t' \right)\left(v, \zeta, \omega \right)$,

where $u_t' = \left(1, \varepsilon_{t-1}^2, ..., \varepsilon_{t-q}^2\right)$, $\eta_t' = 0$, $w_t' = \left(\sigma_{t-1}^2, ..., \sigma_{t-p}^2\right)$, $v' = \left(a_0, a_1, ..., a_q\right)$, $\zeta' = 0$, (6.5.2)

$\omega' = \left(b_1, ..., b_p\right)$.

For the EGARCH(p,q) process, the conditional variance can be expressed as:

$\ln \sigma_t^2 = \left(u_t', \eta_t', w_t' \right)\left(v, \zeta, \omega \right)$,

where $u_t' = \left(1, \left|\varepsilon_{t-1}/\sigma_{t-1}\right|, ..., \left|\varepsilon_{t-q}/\sigma_{t-q}\right|\right)$, $\eta_t' = \left(\varepsilon_{t-1}/\sigma_{t-1}, ..., \varepsilon_{t-q}/\sigma_{t-q}\right)$, (6.5.3)

$w_t' = \left(\ln \sigma_{t-1}^2, ..., \ln \sigma_{t-p}^2\right)$, $v' = \left(a_0, a_1, ..., a_q\right)$, $\zeta' = \left(\gamma_1, ..., \gamma_q\right)$, $\omega' = \left(b_1, ..., b_p\right)$.

Also, for the TARCH(p,q) process, the conditional variance can take the form:

$\sigma_t^2 = \left(u_t', \eta_t', w_t' \right)\left(v, \zeta, \omega \right)$,

where $u_t' = \left(1, \varepsilon_{t-1}^2, ..., \varepsilon_{t-q}^2\right)$, $\eta_t' = \left(d_{t-1}\varepsilon_{t-1}^2\right)$, $w_t' = \left(\sigma_{t-1}^2, ..., \sigma_{t-p}^2\right)$, $v' = \left(a_0, a_1, ..., a_q\right)$, (6.5.4)

$\zeta' = (\gamma)$, $\omega' = \left(b_1, ..., b_p\right)$, $d_t = 1$ if $\varepsilon_t < 0$, and $d_t = 0$ otherwise.

In general, the conditional variance forecast recursion relations (5.2.7) – (5.2.9), in the 5th chapter, could be presented as:

$$\hat{\sigma}_{t+s|t}^2 \equiv E\left(\sigma_{t+s}^2 \mid I_t\right) = E\left(u_{t+s}', \eta_{t+s}', w_{t+s}' \mid I_t\right)\left(v^{(t)}, \zeta^{(t)}, \omega^{(t)}\right) = \left(u_{t+s|t}', \eta_{t+s|t}', w_{t+s|t}'\right)\left(v^{(t)}, \zeta^{(t)}, \omega^{(t)}\right). \tag{6.5.5}$$

In the 4th chapter we have considered a comparative evaluation of two ARCH models (i.e. model A and model B), on the basis of their ability to predict the future values of the dependent variable and its volatility forecasts. For $\hat{\varepsilon}_{t+1|t} \equiv y_{t+1} - \hat{y}_{t+1|t}$ and $\hat{\sigma}_{t+1|t}^2$ denoting the one-step-ahead prediction error and the prediction of the conditional variance at time $t+1$ given the information available at time t , the predictive abilities of models A and B can be compared through testing a null

[5] Numerical maximization of the log-likelihood function, for the E-GARCH(2,2) model, frequently failed to converge. So the five E-GARCH models for $p = q = 2$ were excluded.

hypothesis that the two models are of equivalent predictive ability. Let us assume that a set of candidate ARCH models is available and the *most suitable* model is sought for predicting conditional volatility. The ARCH model, with the lowest value of the sum of the T most recently estimated squared standardized one-step-ahead prediction errors, $\sum_{t=1}^{T} \hat{\varepsilon}_{t+1|t}^{2} \big/ \hat{\sigma}_{t+1|t}^{2}$, can be considered for obtaining one-step-ahead forecast of the conditional volatility. Consider a set of M competing ARCH processes, which have been estimated T times, using a rolling sample of s observations. The SPEC algorithm for selecting the *most suitable* of M candidate models at each of a series of points in time is comprised of the following steps.

- For model m, ($m = 1,2,...,M$) and for each point in time t, $(t = s, s+1,...,T+s-1)$, the vector of coefficients is estimated using a rolling sample of s observations[6]:

$$\hat{\theta}^{(m)(t)} \equiv \left(\hat{\beta}^{(m)(t)}, \hat{v}^{(m)(t)}, \hat{\zeta}^{(m)(t)}, \hat{\omega}^{(m)(t)}\right).$$

- Using the vector of coefficients $\hat{\theta}^{(m)(t)}$, estimate the vector:

$$\left(\hat{y}_{t+1|t}^{(m)}, \hat{\sigma}_{t+1|t}^{2(m)}\right).$$

- Compute:

$$\hat{z}_{t+1|t}^{2(m)} \equiv \frac{\left(y_{t+1} - \hat{y}_{t+1|t}^{(m)}\right)^2}{\hat{\sigma}_{t+1|t}^{2(m)}}.$$

- Compute:

$$R_{T+s}^{(m)} \equiv \sum_{t=s}^{T+s-1} \hat{z}_{t+1|t}^{2(m)}.$$

The *most suitable* model to forecast volatility at time $T+s$ is the model m with the minimum value of $R_{T+s}^{(m)}$. The algorithm is repeated for each of a sequence of points in time for the selection of the *most appropriate* model to be used for obtaining a volatility forecast for the next point in time.

In a theoretical framework, the SPEC model selection method would be able to select the model with the better prediction of the conditional variance of the dependent variable. The question of whether a trader using models for volatility forecasts picked by the SPEC algorithm makes profits from option pricing is investigated in the sequel. Its advantage in predicting realized volatility, for forecast horizons ranging from one day ahead to one hundred days ahead, has also been

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6 $\hat{\beta}^{(m)(t)} = \left(\hat{c}_o^{(m)(t)}, \hat{c}_1^{(m)(t)},....,\hat{c}_k^{(m)(t)}\right)'$

examined in the previous chapter but next sections look into the added value from using the algorithm in the case of real world options data. The profit from trading options will be used for measuring the performance of variance forecasts. Next section describes the strategy an agent follows to trade straddles and the method of measuring the total return of the strategy.

The data set consists of 1064 S&P500 stock index daily returns in the period from March 14th, 1996 to June 2nd, 2000. Larger data sets are often used for the estimation of ARCH models. However, as already noted in the 4th chapter, the use of a restricted sample size incorporates changes in trading behavior more efficiently. Among others, Angelidis, Benos and Degiannakis (2004), Engle et al. (1993) and Frey and Michaud (1997) supported the use of restricted samples and provided empirical evidence that they better capture changes in market activity. Also, Hoppe (1998) investigating the issue of the sample size in the context of Value-at-Risk, argued that a smaller sample could lead to more accurate estimates than a larger one. On the other hand, in the next chapter we consider samples of 500, 1000 and 2000 observations and demonstrate that the results of our simulation study are not appreciably affected by the sample size.

In the sequel, a rolling sample of constant size equal to 500 is considered. Hence, the first one-step-ahead volatility prediction, $\hat{\sigma}^2_{t+1|t}$, is available at time $t = 500$, or on March 11th, 1998. Maximum likelihood estimates of the parameters are obtained by numerical maximization of the log-likelihood function using the Marquardt algorithm (Marquardt 1963), a modification of the BHHH algorithm (Berndt et al. 1974). The quasi-maximum likelihood estimator is used, as according to Bollerslev and Wooldridge (1992), it is generally consistent, has a normal limiting distribution and provides asymptotic standard errors that are valid under non-normality.

The S&P500 index options[7] data were obtained from the Datastream for the period from March 11th, 1998 through June 2nd, 2000, totally 564 trading days. Unfortunately, the data record is not adequate for all the trading days. Proper data are available for 456 trading days and contains information for the closing price of the call and put options, exercise price, expiration date, and the number of contracts traded. In total, 49500 call and put prices were collected. However, in order to minimize the biasedness of the BS formula, only the straddle options with exercise prices closest to the index level, maturity longer than 10 trading days and trading volume greater than 100 were considered from the entire dataset for each trading day. The choice of these data points was based on considerations for the optimal performance of the option pricing model. Sabbatini and Linton (1998) employing Duan's (1995) option pricing model in estimating the volatility of the Swiss market index, for example, used under such considerations a two-year period of daily closing prices of "near the money" options and with a period to maturity of at least 15 days. In our case, "near the money" trading is considered since practice has shown that the BS pricing model tends to misprice "deep out of the money" and "deep in the money" options (see, e.g. Black (1975), Merton (1976) and MacBeth and Merville (1979, 1980)), while it works better for "near the money" options (see, e.g. Daigler (1994, p. 153)).

Table 6.1. Mean and standard deviation of the S&P500 option prices and their trading volumes for the trading days collected in the data record (11 March 1998 – 2 June 2000)

Type of Option	Trading Days	Option Prices		Trading Volume	
		Mean	Standard Deviation	Mean	Standard Deviation
Call	456	33,6	10,09	1418	1861
Put	456	28,1	11,91	1680	2185
Straddle	456	61,7	18,44	1549	2032

Also, a maturity period of length no shorter than 10 trading days is considered to avoid mispricings attributable to causes of practical as well as of theoretical nature. In particular, experience has shown that traders pay less and less attention to the values generated by the pricing model as expiration approaches (e.g. Natenberg (1994, p. 398)). From the theoretical point of view, there is often a departure from the BS model's assumption that stock prices are realizations of a continuous diffusion process, as in most markets the underlying contracts conform to a combination of both a diffusion process and a jump process[8]. According to Dumas et al. (1998) the volatility estimation of close to expiration options is extremely sensitive to possible measurement errors. Most of the time, asset prices change smoothly and continuously with no gaps. However, every now and then a gap will occur, instantaneously sending the price to a new level. These prices will again be followed

[7] S&P500 index options are traded on the Chicago Board Options Exchange (CBOE).

[8] A variation of the BS model, which assumes that the underlying contract follows a jump diffusion process, has been developed. See for example Merton (1976) and Beckers (1981). Unfortunately, the model is considerably more complex mathematically than the traditional BS model. Moreover, in addition to the five customary inputs, the model also requires two new inputs: the average size of a jump in the underlying market and the frequency with which such jumps are likely to occur. Unless the trader can adequately estimate these new inputs, the values generated by a jump diffusion model may be no better, and might be worse, than those generated by the traditional model. Most traders take the view that whatever weakness are encountered in a traditional model can be best offset through intelligent decision making based on actual trading experience, rather than through the use of a more complex jump diffusion model.

Table 6.2. Mean and standard deviation of the S&P500 option prices based on the ARCH volatility forecasts for the trading days collected in the data record (11 March 1998 – 2 June 2000).

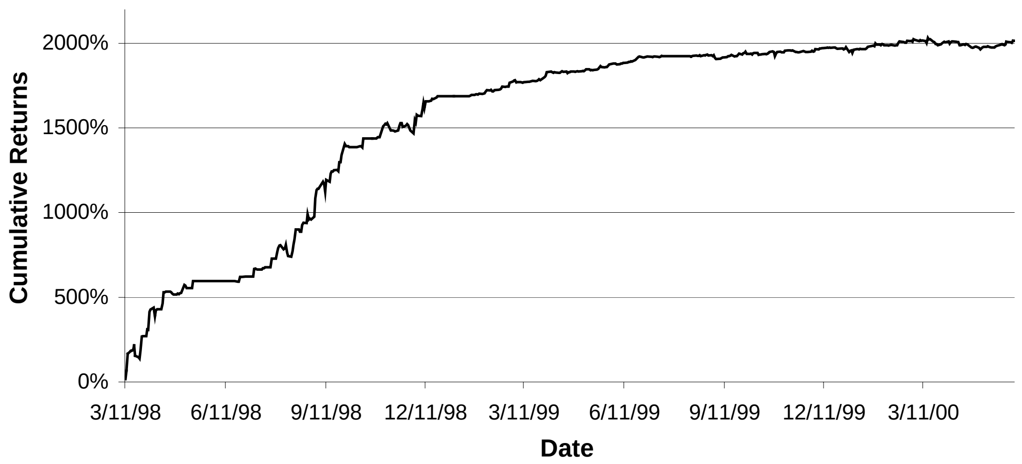
ARCH Model	Call Option		Put Option		Straddle Option		ARCH Model	Call Option		Put Option		Straddle Option	
	Mean	Stand. Dev.	Mean	Stand. Dev.	Mean	Stand. Dev.		Mean	Stand. Dev.	Mean	Stand. Dev.	Mean	Stand. Dev.
AR(0)GARCH(0,1)	29.2	9.0	25.2	7.3	54.5	14.9	AR(3)TARCH(1,1)	30.3	9.9	26.3	9.1	56.5	17.8
AR(1)GARCH(0,1)	29.1	9.1	25.2	7.4	54.3	15.1	AR(4)TARCH(1,1)	30.4	9.9	26.4	9.1	56.7	17.9
AR(2)GARCH(0,1)	29.1	9.1	25.2	7.4	54.3	15.1	AR(0)TARCH(1,2)	29.5	9.9	25.5	9.0	54.9	17.6
AR(3)GARCH(0,1)	29.1	9.1	25.1	7.4	54.2	15.1	AR(1)TARCH(1,2)	30.2	9.8	26.2	9.0	56.5	17.6
AR(4)GARCH(0,1)	29.1	9.1	25.1	7.4	54.2	15.1	AR(2)TARCH(1,2)	30.3	9.8	26.3	9.1	56.5	17.6
AR(0)GARCH(0,2)	29.4	8.9	25.4	7.3	54.8	14.8	AR(3)TARCH(1,2)	30.1	9.8	26.1	9.0	56.2	17.5
AR(1)GARCH(0,2)	29.4	8.9	25.5	7.3	54.9	14.8	AR(4)TARCH(1,2)	30.1	9.7	26.1	8.9	56.2	17.3
AR(2)GARCH(0,2)	29.5	8.9	25.5	7.3	55.0	14.9	AR(0)TARCH(2,1)	29.4	9.8	25.4	8.9	54.7	17.4
AR(3)GARCH(0,2)	29.7	9.0	25.7	7.4	55.5	15.0	AR(1)TARCH(2,1)	30.1	9.9	26.1	9.0	56.3	17.6
AR(4)GARCH(0,2)	29.8	9.0	25.9	7.5	55.7	15.1	AR(2)TARCH(2,1)	30.4	10.1	26.4	9.3	56.8	18.1
AR(0)GARCH(1,1)	31.1	10.1	27.1	9.4	58.2	18.4	AR(3)TARCH(2,1)	30.3	10.0	26.3	9.2	56.6	17.9
AR(1)GARCH(1,1)	30.9	10.1	27.0	9.3	57.9	18.2	AR(4)TARCH(2,1)	30.4	10.0	26.4	9.1	56.8	17.9
AR(2)GARCH(1,1)	30.9	10.2	26.9	9.4	57.8	18.3	AR(0)TARCH(2,2)	29.4	10.0	25.4	9.2	54.8	18.0
AR(3)GARCH(1,1)	30.8	10.2	26.9	9.4	57.7	18.5	AR(1)TARCH(2,2)	30.1	9.9	26.1	9.2	56.2	17.9
AR(4)GARCH(1,1)	30.8	10.2	26.8	9.5	57.6	18.5	AR(2)TARCH(2,2)	30.3	10.1	26.3	9.3	56.5	18.1
AR(0)GARCH(1,2)	31.0	10.1	27.1	9.4	58.1	18.2	AR(3)TARCH(2,2)	30.1	10.1	26.1	9.3	56.2	18.1
AR(1)GARCH(1,2)	31.0	10.2	27.1	9.4	58.1	18.3	AR(4)TARCH(2,2)	30.1	10.0	26.1	9.2	56.2	17.9
AR(2)GARCH(1,2)	31.0	10.2	27.0	9.4	57.9	18.4	AR(0)EGARCH(0,1)	28.6	9.1	24.7	7.3	53.3	15.0
AR(3)GARCH(1,2)	30.9	10.2	26.9	9.4	57.8	18.5	AR(1)EGARCH(0,1)	28.9	9.1	24.9	7.3	53.8	15.1
AR(4)GARCH(1,2)	30.8	10.2	26.9	9.5	57.7	18.5	AR(2)EGARCH(0,1)	28.9	9.1	24.9	7.3	53.7	15.1
AR(0)GARCH(2,1)	31.0	10.1	27.0	9.3	58.0	18.2	AR(3)EGARCH(0,1)	28.8	9.1	24.9	7.3	53.7	15.0
AR(1)GARCH(2,1)	30.8	10.1	26.9	9.3	57.7	18.1	AR(4)EGARCH(0,1)	28.8	9.1	24.9	7.3	53.7	15.0
AR(2)GARCH(2,1)	30.8	10.2	26.8	9.3	57.6	18.3	AR(0)EGARCH(0,2)	27.7	8.8	23.7	7.0	51.5	14.5
AR(3)GARCH(2,1)	30.8	10.1	26.8	9.3	57.5	18.2	AR(1)EGARCH(0,2)	28.1	8.9	24.2	7.1	52.3	14.7
AR(4)GARCH(2,1)	30.7	10.1	26.7	9.4	57.4	18.3	AR(2)EGARCH(0,2)	28.1	8.9	24.1	7.1	52.3	14.6
AR(0)GARCH(2,2)	30.9	10.0	26.9	9.2	57.8	18.0	AR(3)EGARCH(0,2)	28.1	8.8	24.1	7.1	52.1	14.5
AR(1)GARCH(2,2)	30.9	10.1	27.0	9.2	57.9	18.1	AR(4)EGARCH(0,2)	28.0	8.8	24.1	7.1	52.1	14.5
AR(2)GARCH(2,2)	30.9	10.1	26.9	9.2	57.9	18.1	AR(0)EGARCH(1,1)	28.2	9.0	24.3	7.5	52.5	15.1
AR(3)GARCH(2,2)	30.9	10.2	26.9	9.3	57.8	18.3	AR(1)EGARCH(1,1)	29.0	9.1	25.0	7.6	53.9	15.3
AR(4)GARCH(2,2)	30.8	10.2	26.8	9.4	57.7	18.4	AR(2)EGARCH(1,1)	28.8	9.1	24.8	7.6	53.6	15.4
AR(0)TARCH(0,1)	29.8	9.1	25.8	7.6	55.5	15.3	AR(3)EGARCH(1,1)	28.8	9.1	24.8	7.6	53.6	15.3
AR(1)TARCH(0,1)	30.0	9.1	26.0	7.6	56.1	15.3	AR(4)EGARCH(1,1)	28.8	9.2	24.8	7.7	53.7	15.5
AR(2)TARCH(0,1)	30.1	9.0	26.1	7.5	56.2	15.1	AR(0)EGARCH(1,2)	27.8	9.1	23.8	7.5	51.6	15.3
AR(3)TARCH(0,1)	30.0	9.0	26.1	7.5	56.1	15.1	AR(1)EGARCH(1,2)	28.8	9.4	24.8	7.7	53.5	15.8
AR(4)TARCH(0,1)	30.1	9.0	26.1	7.5	56.3	15.1	AR(2)EGARCH(1,2)	28.7	9.3	24.7	7.7	53.4	15.6
AR(0)TARCH(0,2)	29.0	9.1	24.9	7.5	53.9	15.2	AR(3)EGARCH(1,2)	28.6	9.3	24.6	7.6	53.3	15.6
AR(1)TARCH(0,2)	29.3	8.9	25.2	7.3	54.5	14.9	AR(4)EGARCH(1,2)	28.7	9.4	24.7	7.7	53.3	15.7
AR(2)TARCH(0,2)	29.3	9.0	25.3	7.4	54.6	15.0	AR(0)EGARCH(2,1)	28.2	9.0	24.3	7.5	52.5	15.0
AR(3)TARCH(0,2)	29.5	9.0	25.4	7.4	54.9	15.1	AR(1)EGARCH(2,1)	29.1	9.0	25.1	7.6	54.1	15.3
AR(4)TARCH(0,2)	29.6	9.0	25.6	7.5	55.2	15.2	AR(2)EGARCH(2,1)	29.1	9.0	25.1	7.6	54.1	15.3
AR(0)TARCH(1,1)	29.4	9.8	25.3	8.9	54.7	17.4	AR(3)EGARCH(2,1)	29.1	9.1	25.1	7.7	54.1	15.4
AR(1)TARCH(1,1)	30.3	9.9	26.3	9.1	56.5	17.8	AR(4)EGARCH(2,1)	29.1	9.1	25.1	7.6	54.1	15.4
AR(2)TARCH(1,1)	30.4	10.0	26.4	9.2	56.8	18.0							

by a smooth diffusion process until another gap will occur. Further, as Natenberg (1994, p.397) commented: "since a gap in the market will have its greatest effect on "at the money" options close to expiration[9], it is these options that are likely to be mispriced by the traditional BS pricing model with its continuous diffusion process."

As mentioned before, we have on the one hand traders who always choose to use one and the same ARCH model for their forecasts and traders who at each point in time choose to use the ARCH model suggested by the SPEC algorithm on the other. This leads us to comparing 86 forecasting methods: 85 single-model methods, one for each of 85 ARCH models, each amounting to the utilization of the forecasts of one and the same model at any point in time and the SPEC model selection algorithm.

The average and the standard deviation of the collected S&P500 option prices are presented in Table 6.1. On each trading day, for each of the 85 ARCH models, the call and put option prices are forecasted. Table 6.2 presents the mean and the standard deviation of the predicted option prices, indicatively, for 12 of the 85 ARCH models. The ARCH forecasts for both call and put options are lower than the actual option prices, which is in accordance to Noh's et al. (1994) research.

Figure 6.9. Cumulative rate of return of the AR(3)EGARCH(1,1) agent from trading straddles on the S&P500 index (11 March 1998 – 2 June 2000).



[9] A gap in the market has its greatest effect on a high Gamma option and "at the money" options close to expiration have the highest Gamma. Delta, Lambda, Gamma, Theta, Vega and Rho comprise the pricing sensitivities and represent the key relationships between the individual characteristics of the option and the option price. For more details on options sensitivities see appendix 6.2 of the 6th chapter.

Let us assume that there are 85 traders and each trader employs an ARCH model to forecast future volatility and straddle prices. Each trading day, if the straddle price forecast is greater than the market straddle price, the straddle is bought, otherwise the straddle is sold. For each trader, the daily rate of return from trading straddles for 456 days is computed as in equations (6.3.3) to (6.3.4) and is presented in the second column of Table 6.3[10], in the Appendix 6.1. According to the t-ratios, computed as ratios of the mean to the standard deviation divided by the square root of the trading days, all the traders achieve profits significantly greater than zero. However, the trader who employs the AR(3)EGARCH(1,1) model achieves the highest profits. The AR(3)EGARCH(1,1) agent makes 4.42 per cent per day trading for 456 days, with a t-ratio of 5.32. Figure 6.9 depicts the cumulative returns of the AR(3)EGARCH(1,1) agent from trading straddles on a daily basis. However, each time an agent trades a contract has to pay a transaction cost. Taking into consideration a transaction cost of $2, which reflects the bid – ask spread[11], the rate of return would naturally be lower. Table 6.3 also presents for each trader the net rate of return after a trading cost of $2, as computed in (6.3.5).

Table 6.4. ARCH models that yield the highest rate of return from trading straddles on the S&P500 index (11 March 1998 – 2 June 2000).

Trans. Cost - Filter	Model	Mean	St.Dev	t-ratio	p-value	Trading Days	Total Returns
$0.00 - $0.00	AR(3)EGARCH(1,1)	4.42%	17.75%	5.32	0.00	456	2015%
$2.00 - $0.00	AR(3)EGARCH(1,1)	0.77%	17.21%	0.95	0.34	456	349%
$2.00 - $1.25	AR(2)EGARCH(1,1)	0.90%	17.34%	1.06	0.29	421	378%
$2.00 - $1.75	AR(0)GARCH(2,2)	1.06%	18.46%	1.13	0.26	385	408%
$2.00 - $2.00	AR(0)GARCH(1,2)	1.10%	18.53%	1.16	0.25	381	420%
$2.00 - $2.25	AR(0)GARCH(1,2)	1.35%	18.50%	1.39	0.17	362	490%
$2.00 - $2.75	AR(4)GARCH(0,2)	1.60%	17.60%	1.69	0.09	346	553%
$2.00 - $3.50	AR(3)GARCH(0,2)	1.89%	18.11%	1.87	0.06	322	607%

However, a rational trader will trade straddles only when profits are predicted to exceed transaction costs. So, straddles are traded only when the absolute

[10] Because of the large amount of data, Table 6.3, in the Appendix, is decomposed into four parts.
[11] Bid price is the price that a trader is offering to pay for the option. Ask price is the price that a trader is offering to sell the option. The ask price is higher that the bid price and the amount by which the ask exceeds the bid is referred to as the bid – ask spread. The exchange sets the upper limits for the bid – ask spread. For example, according to the CBOE rules, the width is supposed to be a dollar wide for contracts above $20. However, Exchange rules allow for doubling and even tripling the width depending upon the market conditions. For a retail investor, cost is higher and varies significantly from broker to broker. The actual amount charged is usually calculated as a fixed cost plus a proportion of the dollar amount of the trade, i.e. from a discount broker the purchase of contracts of $10.000 would cost $145 in commissions. Retail commissions from full service brokers are higher.

difference between forecast and today's option price exceeds the amount of the filter, F_{il}, yielding a net rate of return of:

$$NRT_t = \begin{cases} RT_t - \dfrac{X}{C_{t-1}+P_{t-1}} & \text{, if } \left|C_{t|t-1}^{(\tau)} + P_{t|t-1}^{(\tau)} - C_{t-1} - P_{t-1}\right| > F_{il} \\ 0 & \text{, otherwise.} \end{cases} \tag{6.5.6}$$

Various values for the filter are applied (i.e. \$1.25, \$1.75, \$2.00, \$2.25, \$2.75, \$3.50). Notice that although the "before transaction cost" profits are significantly greater than zero, applying a \$2 transaction cost, the profits are not significantly greater than zero for any of the agents. According to Table 6.4, the models that achieve the highest rate of return are not the same for each filter strategy.

6.6. *Trading Straddles Based on the SPEC Model Selection Algorithm*

The main purpose is to examine the application of the SPEC algorithm of selection of volatility models on the basis of forecasting option prices and creating trading strategies that yield abnormal returns. The term "abnormal returns" refers to profits that are uncorrelated with the market rate of return as the "at the money" straddle trading is a delta neutral[12] trading strategy. According to the SPEC model selection algorithm, the *most appropriate* model, among a set of candidate ARCH models, to forecast one-day-ahead volatility is the model with the lowest sum of the most recently estimated squared standardized one-step-ahead prediction errors. To price an option, we need a forecast of the average daily variance over the lifetime of the option, as given in (6.3.6), whereas the SPEC algorithm looks at the sum of the one-step-ahead forecasts over some horizon T. Indeed, the SPEC method of model selection does not pick the model that would have had the best performance in estimating the volatility for option pricing but indicates the model that would have had the best performance in forecasting the one-day-ahead volatility. In the 4th chapter we have shown that $\hat{z}_{t+1|t}^{(m)} \equiv \dfrac{\left(y_{t+1} - \hat{y}_{t+1|t}^{(m)}\right)}{\hat{\sigma}_{t+1|t}^{(m)}}$ is asymptotically standard normally distributed for all the considered volatility specifications. On the other hand, there is not a uniform method to compare the models based on average variance over the lifetime of the option because the distribution of $\hat{z}_{t+s|t}^{(m)}$, for $s > 1$, is not common for all the

[12] Delta is the change in the option price for a given change in the stock price. An option is termed delta neutral when the sum total of all the positive and negative deltas adds up to approximately zero. The rate of return of a delta neutral trading strategy is indifferent to any change in the underlying stock price. For more details on Delta see appendix 6.2 of the 6th chapter.

ARCH models. However, our findings indicate that the models picked by the SPEC algorithm appear to be at the same time the models that have had the best performance in estimating volatility for option pricing. Therefore, we investigate the gains from the use of the SPEC algorithm, a criterion that evaluates the one-step-ahead volatility predictions, in pricing options with lifetime greater than one day.

Table 6.5. Daily rates of return from trading straddles on the S&P500 index based on the ARCH models selected by the SPEC model selection algorithm. Applying the SPEC model selection algorithm, $\sum_{t=1}^{T}\hat{z}_{t|t-1}^{2}$ was estimated considering various values for T, and, in particular, $T=5(5)80$. I.e. SPEC(5) corresponds to the SPEC model selection algorithm for $T=5$.

	Without transaction cost				$2 transaction cost											
	Without filter				Without filter				$1.25 filter				$1.75 filter			
Model Selection Method	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days
SPEC(5)	4.06%	17.84%	4.86	456	0.41%	17.42%	0.50	456	0.73%	17.99%	0.83	416	0.80%	18.38%	0.86	395
SPEC(10)	4.05%	17.84%	4.84	456	0.40%	17.47%	0.48	456	0.74%	18.03%	0.84	414	0.83%	18.20%	0.92	401
SPEC(15)	4.04%	17.84%	4.83	456	0.38%	17.49%	0.47	456	0.86%	18.00%	0.97	414	0.96%	18.24%	1.06	398
SPEC(20)	3.92%	17.87%	4.68	456	0.27%	17.51%	0.33	456	0.83%	18.07%	0.93	412	0.92%	18.22%	1.01	400
SPEC(25)	4.04%	17.84%	4.84	456	0.39%	17.47%	0.48	456	0.96%	17.96%	1.08	414	1.10%	18.17%	1.21	399
SPEC(30)	4.06%	17.84%	4.86	456	0.41%	17.47%	0.50	456	0.88%	18.07%	0.99	413	0.91%	18.27%	0.99	396
SPEC(35)	3.87%	17.88%	4.62	456	0.22%	17.52%	0.27	456	0.82%	18.00%	0.93	412	0.94%	18.29%	1.02	394
SPEC(40)	3.96%	17.86%	4.73	456	0.31%	17.50%	0.37	456	0.90%	18.08%	1.01	407	0.99%	18.23%	1.08	395
SPEC(45)	3.70%	17.91%	4.41	456	0.04%	17.56%	0.05	456	0.88%	18.13%	0.98	406	0.97%	18.32%	1.05	392
SPEC(50)	3.69%	17.92%	4.40	456	0.04%	17.56%	0.05	456	0.67%	18.09%	0.75	411	0.78%	18.23%	0.85	399
SPEC(55)	4.17%	17.81%	5.00	456	0.52%	17.26%	0.64	456	0.73%	17.91%	0.82	406	0.94%	18.12%	1.03	392
SPEC(60)	3.84%	17.88%	4.59	456	0.19%	17.35%	0.24	456	0.50%	17.95%	0.56	407	0.69%	18.09%	0.76	396
SPEC(65)	4.11%	17.82%	4.92	456	0.46%	17.29%	0.57	456	0.65%	17.63%	0.76	414	0.79%	17.84%	0.89	401
SPEC(70)	4.12%	17.82%	4.94	456	0.47%	17.28%	0.58	456	0.55%	17.73%	0.63	411	0.67%	17.88%	0.75	401
SPEC(75)	4.08%	17.83%	4.89	456	0.43%	17.31%	0.53	456	0.80%	17.76%	0.91	404	0.89%	18.02%	0.98	391
SPEC(80)	3.81%	17.89%	4.55	456	0.16%	17.47%	0.20	456	0.54%	18.06%	0.60	404	0.64%	18.33%	0.69	391

	$2 transaction cost															
	$2.00 filter				$2.25 filter				$2.75 filter				$3.50 filter			
Model Selection Method	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days
SPEC(5)	0.78%	18.06%	0.85	384	1.04%	18.31%	1.09	370	1.23%	18.70%	1.23	351	2.00%	18.64%	1.95	329
SPEC(10)	0.92%	18.06%	1.01	394	1.01%	18.32%	1.07	382	1.24%	18.24%	1.29	363	1.53%	18.72%	1.50	334
SPEC(15)	1.15%	18.20%	1.24	387	1.20%	18.43%	1.26	377	1.04%	18.47%	1.07	362	1.44%	18.49%	1.42	331
SPEC(20)	1.07%	18.05%	1.17	395	1.16%	18.31%	1.24	383	1.02%	18.38%	1.06	366	1.31%	18.35%	1.32	338
SPEC(25)	1.30%	18.11%	1.42	389	1.36%	18.26%	1.46	382	1.35%	18.19%	1.42	367	1.57%	18.12%	1.60	342
SPEC(30)	1.10%	18.21%	1.19	386	1.20%	18.43%	1.27	376	1.30%	18.45%	1.33	358	1.52%	18.13%	1.55	342
SPEC(35)	1.12%	18.23%	1.21	384	1.21%	18.46%	1.26	374	1.21%	18.50%	1.23	356	1.59%	18.36%	1.59	338
SPEC(40)	1.17%	18.24%	1.25	382	1.31%	18.58%	1.35	367	1.32%	18.58%	1.33	351	1.78%	18.52%	1.75	330
SPEC(45)	1.12%	18.24%	1.20	383	1.23%	18.51%	1.28	371	1.20%	18.48%	1.22	356	1.60%	18.46%	1.58	334
SPEC(50)	0.77%	18.45%	0.83	389	1.02%	18.42%	1.08	378	1.09%	18.64%	1.10	356	1.42%	18.57%	1.41	337
SPEC(55)	0.98%	18.30%	1.05	383	1.05%	18.55%	1.09	372	1.20%	18.66%	1.21	356	1.50%	18.55%	1.49	338
SPEC(60)	0.68%	18.30%	0.73	386	0.71%	18.46%	0.75	379	1.11%	18.33%	1.15	358	1.23%	18.54%	1.23	341
SPEC(65)	0.83%	18.06%	0.91	390	0.92%	18.33%	0.97	377	1.62%	18.32%	1.65	352	1.55%	18.38%	1.56	339
SPEC(70)	0.70%	18.15%	0.76	388	0.79%	18.47%	0.83	373	1.17%	18.96%	1.15	347	1.47%	18.49%	1.45	333
SPEC(75)	0.92%	18.28%	0.97	379	0.98%	18.48%	1.02	370	1.27%	18.88%	1.26	349	1.44%	18.60%	1.40	327
SPEC(80)	0.86%	18.33%	0.91	378	0.97%	18.48%	1.01	371	1.06%	18.81%	1.07	356	0.92%	18.86%	0.90	334

Applying the SPEC model selection algorithm, the sum of squared standardized one-step-ahead prediction errors, $\sum_{t=1}^{T} \hat{z}_{t|t-1}^{2}$, was estimated considering various values for T, and, in particular, $T = 5(5)80$. Thus, it is assumed that there are 16 traders each of which uses on each trading day, the ARCH model picked by the SPEC algorithm to forecast volatility and straddle prices for the next trading day. Table 6.5 presents, for each trader following the SPEC model selection strategy, the net rate of return from trading straddles on a daily basis. With transaction costs of \$2 and a filter of \$3.5, the trader utilizing the SPEC algorithm with $T = 5$ achieves the highest rate of return. The agent based on the SPEC(5) forecast algorithm makes 2.00% per day trading for 329 days, with a t-ratio of 1.95.

Table 6.6. Number of ARCH models selected by the SPEC(5) algorithm for trading straddles on the S&P500 index with transaction costs of \$2.00 and a \$3.5 filter (11 March 1998 – 2 June 2000), classified by the types of models considered for their conditional means and variances.

		Type of Conditional Mean Model					
		AR(0)	AR(1)	AR(2)	AR(3)	AR(4)	Total
Type of Conditional Variance Model	GARCH(0,1)	27	6	4	4	4	45
	GARCH(0,2)	1		4	4	6	15
	GARCH(1,1)	7	2	1	2	9	21
	GARCH(1,2)	9	2	3	2	4	20
	GARCH(2,1)	1	1			1	3
	GARCH(2,2)	5	3			2	10
	TARCH(0,1)	1		2		3	6
	TARCH(0,2)					2	2
	TARCH(1,1)	5	9	1	3	1	19
	TARCH(1,2)	12	2		3	2	19
	TARCH(2,1)	2	2	6	9	5	24
	TARCH(2,2)	3	5	2	3	6	19
	EGARCH(0,1)	16	12	2	10	1	41
	EGARCH(0,2)	1	6	5	16	8	36
	EGARCH(1,1)	1				1	2
	EGARCH(1,2)	9	5	5	7	6	32
	EGARCH(2,1)	8	1	1	3	2	15
	Total	108	56	36	66	63	329

The models picked by the SPEC(5) algorithm are presented in Table 6.6. So, for example, the model with AR(0) conditional mean and GARCH(0,1) conditional variance was picked on 27 trading days. The selection algorithm chooses higher orders of the conditional mean autoregressive process for half the number of trading days. As concerns the conditional variance function, the GARCH, E-GARCH and

TARCH models are suggested as the *most suitable* in the 35%, 27%, and 38% of the cases, respectively. Consequently, the SPEC algorithm does not appear to be noticeably biased towards selecting a specific type of model.

In order to compare the strategy performances over the entire sample, agents are assumed to invest at the risk free rate when they do not trade. Thus, the net rate of return is now computed as:

$$NRT_t = \begin{cases} \dfrac{C_t + P_t - C_{t-1} - P_{t-1} - X}{C_{t-1} + P_{t-1}} & \text{,if } C_{t|t-1}^{(\tau)} + P_{t|t-1}^{(\tau)} - C_{t-1} - P_{t-1} > F_{il} \\ \dfrac{C_{t-1} + P_{t-1} - C_t - P_t - X}{C_{t-1} + P_{t-1}} + r_{f_t} & \text{,if } C_{t-1} + P_{t-1} - C_{t|t-1}^{(\tau)} - P_{t|t-1}^{(\tau)} > F_{il} \\ rf_t & \text{, otherwise.} \end{cases} \quad (6.6.1)$$

For $X = \$2.00$ and $F_{il} = \$3.50$, the trader using the AR(3)GARCH(0,2) forecasts makes a daily profit of 1.35% with a corresponding standard deviation of 15.24% and a t-ratio of 1.89 (or p-value 0.06). On the other hand, the agent that follows the SPEC(5) model selection algorithm achieves a profit of 1.46% per day with a corresponding standard deviation of 15.85% and a t-ratio of 1.97 (or p-value 0.05). Even marginally, the SPEC(5) model selection algorithm achieves higher cumulative returns than those of any other trader who is based only on a single ARCH model. Moreover, a t-ratio of 1.97 indicates that profits from the SPEC(5) algorithm are significantly different from zero. Thus, the SPEC model selection algorithm has a satisfactory performance in selecting those models that generate *better* volatility predictions.

6.7. *Trading Straddles Based on Other Methods Of Model Selection*

As we already mentioned in the 5th chapter, most of the methods used in the time series literature for selecting the appropriate model are based on evaluating the ability of the models to describe the data. Standard model selection criteria such as the AIC and the SBC information criteria have widely been used in the ARCH literature. In addition, the evaluation of loss functions for alternative models is mainly used in model selection. When the focus is mainly on estimation of means, the loss function of choice is typically the mean squared error. However, when the same strategy is applied to variance estimation, the choice of the mean squared error is much less clear. Because of high non-linearity in volatility models a number of researchers constructed heteroscedasticity adjusted loss functions. Denoting the forecasting variance over an N day period measured at day t by $\sigma^2_{t(N)} = N^{-1}\sum_{i=1}^{N}\hat{\sigma}^2_{t+i|t}$, and the realized variance over the same period by $s^2_{t(N)} = N^{-1}\sum_{i=1}^{N} y^2_{t+i}$, a set of statistical criteria to measure the closeness of the forecasts to the realizations were presented in Table 5.5 of the 5th chapter.

Applying the SPEC model selection algorithm, the sum of squared standardized one-step-ahead prediction errors, $\sum_{t=1}^{T}\hat{\varepsilon}^2_{t|t-1}\big/\hat{\sigma}^2_{t|t-1}$, was estimated considering various values for T. Therefore, each of the model selection criteria, in Table 5.5 of the 5th chapter**,** was computed considering various values for T, and, in particular, $T = 10(10)80$. The AIC and SBC criteria were computed based on the rolling sample of constant size equal to 500 that is used at each time to estimate the parameters of the models. Selecting a strategy based on any of several competing methods of model selection naturally amounts to selecting the ARCH model that, at each of a sequence of points in time, has the lowest value of the evaluation function.

Table 6.7. Daily rate of return from trading straddles on the S&P500 index based on the SPEC model selection algorithm and the ARCH model selection algorithms with transaction costs of $2.00 and a $3.5 filter. The column "sample size" refers to the sample size, T, for which the corresponding model selection algorithm leads to the highest rate of return. Agents are assumed to invest at the risk free rate when they do not trade. The net rate of return is computed as in equation (6.6.1).

Trans. Cost – Filter	Model Selection Method	Sample size	Mean	Stand. Dev.	t-ratio	Days
$2.00 -$3.50	SPEC	T=5	1.46%	15.85%	1.97	456
$2.00 -$3.50	AIC	-	0.90%	15.57%	1.23	456
$2.00 -$3.50	SBC	-	1.06%	15.93%	1.42	456
$2.00 -$3.50	SEVar	T = 40	0.61%	16.34%	0.80	456
$2.00 -$3.50	AEVar	T = 60	0.76%	15.82%	1.03	456
$2.00 -$3.50	SEDev	T = 60	0.74%	16.29%	0.97	456
$2.00 -$3.50	AEDev	T = 60	0.81%	15.96%	1.08	456
$2.00 -$3.50	HASEVar	T = 10	1.10%	15.98%	1.47	456
$2.00 -$3.50	HAAEVar	T = 40	1.24%	16.12%	1.65	456
$2.00 -$3.50	HASEDev	T = 20	0.90%	16.32%	1.18	456
$2.00 -$3.50	HAAEDev	T = 30	1.12%	16.47%	1.45	456
$2.00 -$3.50	LEVar	T = 80	0.75%	15.92%	1.00	456

Assuming that the agents invest at the risk free rate when they do not trade, Table 6.7 presents the daily rate of returns from trading straddles on the S&P500 index based on the ARCH models selected by the 11 model selection methods presented in this section. Detailed tables for the daily rate of return from trading straddles based on the ARCH models selected by the 11 model selection methods are presented in Tables 6.8 to 6.17, in the Appendix 6.1. After transaction costs of $2, the agents based on the HASEVar, HAAEVar, HASEDev and HAAEDev criteria achieve the higher returns. Moreover, a trader who selects the volatility forecasts models according to the standard model selection criteria, SBC and AIC, makes a cumulative profit higher than in the case he/she would select ARCH models based on the heteroscedasticity unadjusted and logarithmic error functions. However, in none of the cases, the daily returns came out to be significantly different from zero (according to the t-ratios of Table 6.7) or higher than the returns achieved by the SPEC algorithm.

The net rate of return is computed according to equation (6.6.1). The SPEC model selection algorithm, for $T = 5$, leads to the highest profit of 1.46% per day and a t-ratio of 1.97. Of the remaining model selection criteria considered in this section, the HAAEVar selection algorithm, for $T = 40$, yielded the highest daily profit (1.24%) with a corresponding standard deviation of 16.12% and a t-ratio of 1.65. Thus, none of the model selection algorithms considered appears to lead to daily returns that are higher than the returns attained using the SPEC algorithm. Even the AR(3)GARCH(0,2) model, which yields a daily profit of 1.35%, achieves a higher rate of return than that of the models picked by the other model selection criteria. This is an indication of the superiority of the SPEC algorithm over the other methods in the ability to select the models that would produce accurate volatility estimations for option pricing predictions.

6.8. *Discussion*

Selecting a model that can produce accurate volatility predictions for pricing next day's options is an intriguing problem. In this chapter, a number of single ARCH model-based methods of predicting volatility were compared to poly-model SPEC algorithm method in terms of profits from trading real options of the S&P500 index returns. Over the March 1998 to June 2000 period, forecasts of option prices were calculated by feeding the volatility estimated by the ARCH models into the BS option pricing model, which is commonly used in option exchanges worldwide despite the fact that it assumes a constant variance for the rate of return. Actually, as in the case of Noh et al.'s (1994) study, our results imply that option prices can be predicted even with the use of a misspecified model if asset volatilities can be predicted. The results of our study showed that the SPEC algorithm outperformed all of the single ARCH model-based methods as well as a set of other methods of model selection.

Moreover, in the 7th chapter we make a comparative study among a set of ARCH model selection algorithms in order to examine which method yields the highest profits by trading straddles, in a simulated options market, based on variance forecast option prices. The simulated option market was considered to avoid the bias induced by the use of actual option prices. The results also showed that the SPEC algorithm for $T=5$ achieved the highest rate of return. One may therefore infer that the evidence is rather in support of the assumption that the increase in profits is due to improved volatility prediction and that the SPEC model selection algorithm offers a potential tool in picking the model that would yield the *best* volatility prediction. If the increase in profits were random, the SPEC algorithm would not achieve the highest profits in both the simulated market and in the present study that is based on real world options data.

Chapter 7
Evaluating Volatility Forecasts in Option Pricing in the Context of a Simulated Options Market

7.1. Introduction

The evaluation of the SPEC algorithm is performed by comparing different volatility forecasts in option pricing through the simulation of an options market. Traders employing the SPEC model selection algorithm use the model with the lowest sum of squared standardized one-step-ahead prediction errors for obtaining their volatility forecast. The cumulative profits of the participants in pricing one-day index straddle options always using variance forecasts obtained by GARCH, EGARCH and TARCH models are compared to those made by the participants using variance forecasts obtained by models suggested by the SPEC algorithm. The straddles are priced on the S&P500 index. It is concluded that traders, who base their selection of an ARCH model on the SPEC algorithm, achieve higher profits than those, who use only a single ARCH model. Moreover, the SPEC algorithm is compared with other criteria of model selection that measure the ability of the ARCH models to forecast the realized intra-day volatility. In this case too, the SPEC algorithm users achieve the highest returns. Thus, the SPEC model selection method appears to be a useful tool in selecting the appropriate model for estimating future volatility in pricing derivatives.

In this chapter, inspired by Engle et al.'s (1993) approach to assess incremental profits for a set of competing forecasts of the variance for a given portfolio, we examine the usage of the SPEC model selection algorithm, in pricing contingent claims. The goal of the present chapter is to evaluate the SPEC algorithm for volatility model selection through the simulation of an options market. In particular, section 7.2 presents Engle et al.'s (1993) data generated set-up of evaluating volatility forecasts. In sections 7.3 and 7.4, based on Engle et al.'s (1993) technique, the suggested model selection method is evaluated using daily return data for the S&P500 stock index over the period from June 26th, 1991 to October 18th, 2002. The use of a model selection method is a tedious procedure as it presupposes the estimation of a set of models. In order to examine whether there is any added value in using the suggested model selection algorithm instead of any other method of using only a single ARCH model in the study, the performance of the SPEC algorithm in investigated against a set of such methods for a range of ARCH models. The results of section 7.3 provide evidence that this is indeed the case since they indicate that the SPEC model selection algorithm offers a useful tool in providing information related to the appropriate model. In section 7.4, the algorithm is compared with other methods of model selection. In particular, model selection criteria that measure the accuracy of the models to predict the realized volatility are constructed. The SPEC method is then compared with those model selection methods. Clearly, the SPEC algorithm outperforms all of the other methods of model selection considered. Samples of 500 and 2000 observations were also considered, in the 7.5 section, leading to similar findings, thus demonstrating that the results of the simulation study are not appreciably affected by the sample size. Finally, in section 7.6 a brief discussion of the results is provided.

7.2. Evaluation of Variance Forecasts with Simulated Option Prices

As Engle et al. (1997 p.120) noted, “a natural criterion for choosing between any pair of competing methods to forecast the variance of the rate of return on an asset would be the expected incremental profit from replacing the lesser forecast with the better one’’. Engle et al. (1993) considered evaluating variance forecasts of the NYSE index using generated index option prices instead of actual ones, thus avoiding the perennial problems inherent in observed option prices. The wildcard delivery option on cash-settled options (the right of an option buyer to exercise up an option at the closing price for a period of time after the close of stock market), the existence of bid-ask spread and transaction costs, the non-synchronous coexistence of option and stock prices, are some of the difficulties that are induced in empirical studies by the use of the actual index-option prices. In particular, Engle et al. (1993) used a set of competing methods to generate alternative daily forecasts for the variance of the returns on the NYSE index and applied these forecasts to price one-day options on $1 shares of the NYSE index. The moving average variance, the ordinary least squares, the ARMA(1,1) in the squared residuals and the GARCH(1,1) models were applied for three sample lengths of i) 300 days, ii) 1000 days, and iii) 5000 days. The four models and the three sample lengths produce 12 variance forecasts predicting methods. To these, Engle et al. (1993) added 3 more predicting methods by considering the average of all daily forecasts, the daily minimum and the daily maximum forecasts. As reported by Kane and Marks (1987), the average of conditionally independent forecasts converges rapidly to a perfect forecast, so that any failure of the average forecast might be indicative of departures from quality and conditional independence of the individual forecasts. As a check for the presence of bias, Engle et al. (1993) added the minimum and maximum of the daily forecasts. So, for example, in case of a significant downward bias, the maximum forecast will beat the minimum forecast and all of the individual forecasts that are more severe biased.

Each agent applies a variance-forecast method and trades one-day options on a \$1 share of the NYSE portfolio. The exercise price is taken to be $\exp(rf_t)$. Thus, for $S_t = 1$, $\tau = 1$, $K = \exp(rf_t)$, $\sigma^{(1)}_{t+1|t} \equiv \hat{\sigma}_{t+1|t}$, $C^{(1)}_{t+1|t} \equiv C_{t+1|t}$ and $P^{(1)}_{t+1|t} \equiv P_{t+1|t}$, the Black & Scholes option pricing formula (equation 6.2.1 in chapter 6) reduces to:

$$C_{t+1|t} = P_{t+1|t} = 2N\left(0.5\hat{\sigma}_{t+1|t}\right) - 1. \tag{7.2.1}$$

The way in which the simulated options market operates is the following: The daily differences in the variance forecasts of the various methods considered lead to different reservation prices for one-day options on the index considered. These, in turn, trigger option trading among fictitious agents, each using one of the forecast methods considered. A trader with a higher (or lower) variance forecast and, hence, with a higher (or lower) reservation price for the option would buy (or sell) a straddle on a \$1 share of index considered from any of the remaining traders with lower (or higher) reservation prices for the option. A straddle option is the purchase (or sale) of both a call and a put option, of the underlying asset, with the same maturity day. The straddle trading is used because a straddle, that has its stock price equal to the exercise price, is Delta neutral. Delta[1] is the change in the option price for a given change in the stock price:

$$\Delta_{CALL} = \frac{\partial C}{\partial S} = e^{-\gamma\tau} N\left(d_1\right) > 0,$$

and

$$\Delta_{PUT} = \frac{\partial P}{\partial S} = e^{-\gamma\tau}\left(N\left(d_1\right) - 1\right) < 0.$$

The day t payoff to agent i from holding the straddle is:

$$\pi_t = \max\left(\exp\left(y_t\right) - \exp\left(r_t\right), \exp\left(r_t\right) - \exp\left(y_t\right)\right), \tag{7.2.2}$$

which is identical for each agent. A trade between two agents, i and i^*, is executed at the average of the reservation prices of the two agents, that is, at the bid/ask

[1] Delta, Lambda, Gamma, Theta, Vega and Rho comprise the option sensitivities and represent the key relationships between the individual characteristics of the option and the option price. For more details on options sensitivities see appendix 6.2 of the 6th chapter.

prices. The transaction that is executed at the average of the bid and ask price, yields to agent i a profit given by

$$\pi_{t+1}^{(i,i^*)} = \begin{cases} \pi_{t+1} - \left(C_{t+1|t,(i)} + C_{t+1|t,(i^*)}\right), & \text{for } C_{t+1|t,(i)} > C_{t+1|t,(i^*)} \\ \left(C_{t+1|t,(i)} + C_{t+1|t,(i^*)}\right) - \pi_{t+1}, & \text{for } C_{t+1|t,(i)} < C_{t+1|t,(i^*)} \end{cases}. \tag{7.2.3}$$

In Engle et al. (1993), the GARCH(1,1) forecast method achieves the highest cumulative profits for the three sample lengths. Moreover, the GARCH(1,1) method for a rolling sample of 1000 observations yields the highest profit, dominating even the average of all variance forecast methods.

7.3. Evaluating the SPEC Model Selection Algorithm on Simulated Options

In the 5th chapter, a number of statistical evaluation criteria were applied in order to examine the ability of the SPEC model selection algorithm to select the ARCH model that best predicts future volatility, for forecast horizons ranging from one day ahead to one hundred days ahead. The results showed that the SPEC model selection procedure has a satisfactory performance in selecting that model that generates “better” volatility predictions. Moreover, in the 6th chapter we made a comparative study among a set of ARCH model selection algorithms in order to examine which method yields the highest profits in straddle trading based on volatility forecasts using actual option price data. The results showed that the SPEC algorithm for $T=5$ achieved the highest rate of return.

In the sequel, the performance of the SPEC algorithm as an ARCH model selection criterion is evaluated in the context of a simulated options market in order to avoid biases induced by the use of actual index-option prices. In particular, following Engle et al.’s (1993) approach, an economic criterion to evaluate the SPEC model selection algorithm is adopted: the profit from variance forecasts in pricing one-day index straddle options. A simulated market of option trading among 104 fictitious agents is created, whereby traders use variance forecasts obtained by the models of their choice to price a straddle on the S&P500 index for the next day. The performance of the SPEC algorithm is evaluated through comparing the different volatility forecasts. The comparison is performed on the basis of the cumulative profits of traders each of which always uses volatility forecasts obtained by the same GARCH, EGARCH or TARCH model on the one hand and cumulative profits by traders using volatility forecasts obtained by models suggested by the SPEC criterion on the other. So, traders can be thought of a having different “methods” or "strategies" for obtaining variance forecasts (amounting to the utilization of the forecasts of a model at each point in time) and can be classified into two categories: Those who choose to always use one and the same ARCH model and those who at each point in time choose to use the ARCH model suggested by the SPEC algorithm. The variance forecast methods that are compared are: 85 selection "methods" (strategies), one for each of 85 ARCH models, each amounting to the utilization of the forecasts of the same model at any point in time, the SPEC model selection algorithm for 16 different sample sizes, the average, the minimum and the maximum of all daily forecasts methods.

The data set consists of S&P500 stock index daily returns in the period from June 26th, 1991 to October 18th, 2002, totally 2853 trading days.

The conditional mean is considered as a κ^{th} order autoregressive process:

$$y_t = \mu_t + z_t \sigma_t ,$$
$$\mu_t = c_0 + \sum_{i=1}^{\kappa} \left(c_i y_{t-i} \right), \tag{7.3.1}$$
$$z_t \overset{i.i.d.}{\sim} N(0,1) .$$

Usually, the conditional mean is either the overall mean or a first order autoregressive process. Theoretically, the $AR(1)$ process allows for the autocorrelation induced by discontinuous (or non-synchronous) trading in the stocks making up an index. For more details on non-synchronous trading see section 2.1.3 of the 2nd chapter.

The conditional variance is regarded as a GARCH(p,q), an EGARCH(p,q) and a TARCH(p,q) function in the forms of (5.2.4), (5.2.5) and (5.2.6) of the 5th chapter, respectively. Thus, the AR(κ)GARCH(p,q), AR(κ)EGARCH(p,q) and AR(κ)TARCH(p,q) models are applied, for $\kappa = 0,...,4$, $p = 0,1,2$ and $q = 1,2$, yielding a total of 85 cases. Numerical maximization of the log-likelihood function, for the E-GARCH(2,2) model, frequently failed to converge. So the five E-GARCH models for $p = q = 2$ were excluded. Maximum likelihood estimates of the parameters are obtained by numerical maximization of the log-likelihood function using the Marquardt algorithm (Marquardt (1963)). The quasi-maximum likelihood estimator (QMLE) is used, as according to Bollerslev and Wooldridge (1992), it is generally consistent, has a limiting normal distribution and provides asymptotic standard errors that are valid under non-normality. The one step-ahead volatility forecasts of the models are:

One-step-ahead forecast of the GARCH(p,q) model

$$\hat{\sigma}^2_{t+1|t} = a_0^{(t)} + \sum_{i=1}^{q}\left(a_i^{(t)}\varepsilon^2_{t-i+1}\right) + \sum_{i=1}^{p}\left(b_i^{(t)}\sigma^2_{t-i+1}\right). \tag{7.3.2}$$

One-step-ahead forecast of the EGARCH(p,q) model

$$\hat{\sigma}^2_{t+1|t} = \exp\left(a_0^{(t)} + \sum_{i=1}^{q}\left(a_i^{(t)}\left|\frac{\varepsilon_{t-i+1}}{\sigma_{t-i+1}}\right| + \gamma_i^{(t)}\left(\frac{\varepsilon_{t-i+1}}{\sigma_{t-i+1}}\right)\right) + \sum_{i=1}^{p}\left(b_i^{(t)}\ln\left(\sigma^2_{t-i+1}\right)\right)\right). \tag{7.3.3}$$

One-step-ahead forecast of the TARCH(p,q) model

$$\hat{\sigma}^2_{t+1|t} = a_0^{(t)} + \sum_{i=1}^{q}\left(a_i^{(t)}\varepsilon^2_{t-i+1}\right) + \gamma^{(t)}\varepsilon_t^2 d_t + \sum_{i=1}^{p}\left(b_i^{(t)}\sigma^2_{t-i+1}\right), \tag{7.3.4}$$

where $d_t = 1$ if $\varepsilon_t < 0$, and $d_t = 0$ otherwise. The ARCH processes are estimated using a rolling sample of constant size equal to 1000. Thus, the first one-step-ahead volatility prediction, $\hat{\sigma}^2_{t+1|t}$, is available at time $t = 1000$.

The SPEC model selection algorithm is applied for various values of T, and, in particular, for $T = 5(5)80$. Let us consider the set of M candidate ARCH models of the form,

$$y_t = x_t'^{(m)}\beta^{(m)} + \varepsilon_t^{(m)},$$

where

$$\varepsilon_t^{(m)} = z_{1,t}\sigma_t^{(m)},$$

$$z_{1,t} \overset{iid}{\sim} N(0,1),$$

and

$$\sigma_t^{2(m)} = g\left(\sigma_{t-1}^{2(m)},\ldots,\sigma_{t-p}^{2(m)},\varepsilon_{t-1}^{2(m)},\ldots,\varepsilon_{t-q}^{2(m)},\upsilon_{t-1}^{(m)},\upsilon_{t-2}^{(m)},\ldots\right),$$

where the superscript m refers to model m, m=1, 2, …, M. Assume that, at each of a series of points in time, we are interested in looking for the most suitable of the M competing models for obtaining a volatility forecast. According to the SPEC model selection algorithm, the model with the lowest sum of squared standardized one-step-ahead prediction errors is considered as having a better ability to predict the conditional variance of the dependent variable. Thus, at time k, selecting a strategy for the most appropriate model to forecast volatility at time $k+1$ ($k = T, T+1,\ldots$) could naturally amount to selecting the model, which, at time k, has the lowest value of standardized one-step-ahead prediction errors, $\sum_{t=k-T+1}^{k}\hat{z}_t^{2(m)} \equiv \sum_{t=k-T+1}^{k}\hat{\varepsilon}^{2(m)}_{t|t-1}\Big/\hat{\sigma}^{2(m)}_{t|t-1}$. The estimation steps comprising the SPEC model selection algorithm are summarized in

Table 5.1 in chapter 5. Thus, based on the SPEC model selection algorithm, sixteen agents are assumed to take part in the simulated options market.

Table 7.1

The annualised daily profits per competitor per straddle for trades that are at the average of the bid/ask prices.

Rank	Algorithm	Profit	T-Ratio	Rank	Algorithm	Profit	T-Ratio	Rank	Algorithm	Profit	T-Ratio
1	SPEC(T=5)	22.34%	6.76	36	AR(2)EGARCH(2,1)	8.71%	4.12	71	AR(2)GARCH(1,2)	-4.69%	-1.77
2	SPEC(T=10)	20.09%	6.64	37	AR(0)EGARCH(2,1)	8.64%	3.98	72	AR(4)GARCH(1,2)	-5.04%	-1.85
3	SPEC(T=15)	17.94%	6.53	38	AR(4)EGARCH(1,1)	8.44%	4.36	73	AR(0)EGARCH(0,2)	-6.45%	-1.89
4	SPEC(T=25)	16.58%	6.70	39	AR(4)TARCH(1,2)	8.11%	3.36	74	AR(1)EGARCH(0,2)	-6.90%	-2.02
5	SPEC(T=20)	16.56%	6.49	40	AR(0)TARCH(1,1)	7.80%	3.58	75	AR(2)EGARCH(0,2)	-7.20%	-2.11
6	AR(0)EGARCH(1,2)	14.44%	5.49	41	AR(1)EGARCH(2,1)	7.62%	3.69	76	AR(3)EGARCH(0,2)	-7.60%	-2.24
7	SPEC(T=50)	14.42%	6.35	42	AR(3)EGARCH(2,1)	7.53%	3.71	77	AR(4)EGARCH(0,2)	-7.83%	-2.31
8	SPEC(T=40)	14.29%	5.91	43	AR(0)TARCH(2,1)	7.26%	3.33	78	MAXIMUM	-10.38%	-2.27
9	SPEC(T=30)	13.93%	5.78	44	AR(4)EGARCH(2,1)	6.87%	3.44	79	AR(0)TARCH(0,2)	-11.66%	-4.01
10	SPEC(T=45)	13.85%	5.73	45	AR(2)TARCH(1,1)	6.72%	3.01	80	AR(2)TARCH(0,2)	-12.18%	-4.19
11	SPEC(T=35)	13.80%	5.69	46	AR(1)TARCH(1,1)	6.50%	2.82	81	AR(3)TARCH(0,2)	-12.57%	-4.26
12	SPEC(T=80)	13.49%	5.75	47	AR(2)TARCH(2,1)	6.14%	2.87	82	AR(1)TARCH(0,2)	-12.69%	-4.35
13	SPEC(T=55)	13.10%	5.56	48	AR(1)TARCH(2,1)	6.02%	2.75	83	AR(4)TARCH(0,2)	-13.00%	-4.30
14	SPEC(T=70)	13.04%	5.48	49	AR(3)TARCH(2,1)	5.91%	2.82	84	AR(0)GARCH(0,2)	-13.35%	-4.48
15	SPEC(T=60)	12.84%	5.43	50	AR(3)TARCH(1,1)	5.61%	2.60	85	AR(1)GARCH(0,2)	-13.80%	-4.66
16	SPEC(T=65)	12.70%	5.39	51	AR(4)TARCH(1,1)	5.54%	2.56	86	AR(2)GARCH(0,2)	-13.84%	-4.66
17	AR(0)TARCH(2,2)	12.61%	5.65	52	AR(4)TARCH(2,1)	4.65%	2.25	87	AR(3)GARCH(0,2)	-14.29%	-4.72
18	AR(1)EGARCH(1,2)	12.54%	4.88	53	AR(0)GARCH(2,1)	-0.49%	-0.19	88	AR(4)GARCH(0,2)	-14.33%	-4.64
19	AR(2)EGARCH(1,2)	12.44%	4.96	54	AR(0)GARCH(1,2)	-0.66%	-0.27	89	AR(0)EGARCH(0,1)	-16.93%	-5.43
20	AR(3)EGARCH(1,2)	12.12%	4.88	55	AR(0)GARCH(2,2)	-0.68%	-0.27	90	AR(1)EGARCH(0,1)	-17.79%	-5.61
21	SPEC(T=75)	12.02%	5.11	56	AR(0)GARCH(1,1)	-1.50%	-0.59	91	AR(2)EGARCH(0,1)	-17.91%	-5.59
22	AR(4)EGARCH(1,2)	12.01%	4.82	57	AR(1)GARCH(2,1)	-1.59%	-0.60	92	AR(3)EGARCH(0,1)	-18.22%	-5.68
23	AR(0)EGARCH(1,1)	11.32%	5.41	58	AR(3)GARCH(2,2)	-1.89%	-0.71	93	AR(4)EGARCH(0,1)	-18.27%	-5.68
24	AR(1)TARCH(2,2)	11.04%	4.95	59	AR(1)GARCH(2,2)	-1.94%	-0.74	94	AR(0)GARCH(0,1)	-20.26%	-6.39
25	AR(0)TARCH(1,2)	10.88%	4.52	60	AR(2)GARCH(2,1)	-1.99%	-0.75	95	AR(1)GARCH(0,1)	-20.49%	-6.35
26	AR(2)TARCH(2,2)	10.74%	4.88	61	AR(3)GARCH(2,1)	-2.00%	-0.75	96	AR(2)GARCH(0,1)	-20.89%	-6.45
27	AR(2)EGARCH(1,1)	10.69%	5.31	62	AR(2)GARCH(2,2)	-2.62%	-1.00	97	AR(3)GARCH(0,1)	-21.10%	-6.45
28	AR(2)TARCH(1,2)	10.31%	4.17	63	AR(1)GARCH(1,2)	-2.70%	-1.03	98	AR(0)TARCH(0,1)	-21.29%	-6.84
29	AR(1)EGARCH(1,1)	10.24%	4.89	64	AR(4)GARCH(2,1)	-2.72%	-1.02	99	AR(4)GARCH(0,1)	-21.64%	-6.54
30	AR(3)TARCH(2,2)	10.05%	4.68	65	AR(1)GARCH(1,1)	-3.22%	-1.26	100	AR(1)TARCH(0,1)	-21.90%	-6.94
31	AR(4)TARCH(2,2)	9.41%	4.31	66	AR(3)GARCH(1,1)	-3.29%	-1.24	101	AR(2)TARCH(0,1)	-22.00%	-6.95
32	AVERAGE	9.28%	9.33	67	AR(2)GARCH(1,1)	-3.63%	-1.40	102	AR(3)TARCH(0,1)	-22.24%	-7.08
33	AR(3)EGARCH(1,1)	9.23%	4.72	68	AR(4)GARCH(1,1)	-3.64%	-1.37	103	AR(4)TARCH(0,1)	-22.25%	-7.02
34	AR(1)TARCH(1,2)	8.94%	3.53	69	AR(4)GARCH(2,2)	-3.65%	-1.37	104	MINIMUM	-37.99%	-8.20
35	AR(3)TARCH(1,2)	8.89%	3.77	70	AR(3)GARCH(1,2)	-4.28%	-1.62				

Each agent, who follows the SPEC algorithm, selects the ARCH model with the lowest sum of T squared standardized one-step-ahead prediction errors, $\sum_{t=1}^{T} \hat{z}_{t|t-1}^{2}$, in order to forecast next day's variance. As in Engle et al. (1993), three more daily forecasts are added: the average of all daily forecasts, the daily minimum and daily maximum forecasts. In the sequel, the resulting forecast methods will be referred to as the AVERAGE, the MINIMUM and the MAXIMUM method, respectively.

Thus, the simulated options market that has been created is comprised by 104 competitors. Each trader applies a trading strategy for the period ranging from October 4th 1995 to October 18th, 2002 on the S&P500 index, totally 1773 trading days. For 1773 trading days and 104 agents, the ith agent's daily profit per straddle is computed as:

$$\pi^{(i)} = \sum_{t=1}^{1773} \left(\left(\sum_{i^*=1}^{103} \pi_t^{(i,i^*)} \middle/ 103 \right) \middle/ 1773 \right). \qquad (7.3.5)$$

Any method that yields superior profits relative to the AVERAGE method appears more suitable in predicting volatility for pricing contingent claims. Table 7.1 presents the profits per competitor per straddle and the corresponding t-ratios (ratio of average daily profit to its standard deviation divided by the square root of the trading days). The agents based on the SPEC model selection algorithm clearly outperform the others. All the SPEC model selection based algorithms achieve returns higher than the AVERAGE method. The highest annualized daily returns are achieved by the SPEC(5) model selection algorithm, which is in accordance to previous chapter's results.

Moreover, the agents that employ the SPEC model selection algorithm rank at the sixteenth of the twenty-two top positions. The MINIMUM forecast takes the last positions and the MAXIMUM forecast achieves negative and statistically significant returns, an indication that neither a downward nor an upward forecast bias, that could affect profits significantly, is present. It is interesting to note that the EGARCH(1,2) and the TARCH(2,2) model selection algorithms perform distinctly better that the remaining ARCH models. The more flexible models, which account for the leverage effect and have a higher order of p,q, outperform the parsimonious models (i.e. GARCH(0,1), TARCH(0,1) and EGARCH(0,1)). Degiannakis (2004), Giot and Laurent (2003), Hansen and Lunde (2003) and Vilasuso (2002), among others, have found that more flexible models beat the forecasting ability of the parsimonious ones. Of course, as the number of candidate models increases, the probability of finding models with superior predictive ability will increase as well. Note that in our simulation study, we include 3 conditional variance specifications and in the 2nd chapter we have presented 31 conditional variance specifications in the context of the ARCH framework. However, the investigation of the SPEC algorithm performance with a set of more flexible ARCH models, which account for recent developments in the area of asset returns volatility, is suggested for further research.

Table 7.2

Ranks of the methods based on the SPEC model selection algorithm and of the AVERAGE method by dropping out the least profitable agent at a time.

Number of traders	Algorithm																
	SPEC (T=5)	SPEC (T=10)	SPEC (T=15)	SPEC (T=20)	SPEC (T=25)	SPEC (T=30)	SPEC (T=35)	SPEC (T=40)	SPEC (T=45)	SPEC (T=50)	SPEC (T=55)	SPEC (T=60)	SPEC (T=65)	SPEC (T=70)	SPEC (T=75)	SPEC (T=80)	AVERAGE
104	1	2	3	5	4	9	11	8	10	6	12	15	17	14	22	13	31
103	1	2	3	5	4	9	10	8	11	6	12	15	17	14	22	13	32
102	1	2	3	5	4	9	10	7	11	6	12	15	17	14	21	13	32
101	1	2	3	5	4	9	10	7	11	6	12	15	17	14	21	13	32
100	1	2	3	5	4	9	10	7	11	6	12	14	17	15	21	13	32
95	1	2	3	5	4	9	10	6	11	7	12	14	17	15	21	13	35
90	1	2	3	5	4	9	10	7	11	8	12	14	17	16	21	13	36
85	1	2	3	4	5	9	10	7	11	8	12	15	18	16	21	13	37
80	1	2	3	4	5	9	10	7	11	8	12	15	18	16	22	14	40
75	1	2	3	4	5	8	10	7	11	9	13	15	18	16	23	14	41
70	1	2	3	5	4	8	10	7	11	9	13	14	17	16	22	15	41
65	1	2	3	5	4	8	9	7	11	10	13	14	18	16	23	15	42
60	1	2	3	5	4	8	9	7	11	10	13	14	17	16	23	15	43
55	1	2	3	5	4	8	9	7	11	10	13	14	15	16	24	17	43
50	1	2	3	5	4	7	9	8	12	10	14	13	15	17	24	18	43
45	1	2	3	5	4	8	9	7	11	10	13	14	15	17	24	18	42
40	1	2	3	5	4	8	9	7	11	10	12	13	15	16	21	17	39
35	1	2	3	5	4	8	9	7	11	10	12	13	15	17	22	19	
30	1	2	4	5	3	9	8	7	11	10	13	12	14	17	21	19	
25	1	2	4	5	3	9	7	8	10	11	13	12	15	18	22	19	
20	1	2	3	5	4	10	7	8	9	11	13	12	17	19		20	
15	1	2	3	5	4	9	7	8	10	12	13	11					
14	1	2	3	5	4	10	7	8	9	12	13	11					
13	1	2	3	5	4	10	7	9	8	12	13	11					
12	1	2	3	5	4	11	7	9	10	12		8					
11	1	2	3	5	4	11	7	9	10			8					
10	1	2	3	5	4		7	9	10			8					
9	1	2	3	5	4		8	7				9					
8	1	2	3	5	4		8	7									
7	1	2	3	5	4			7									
6	1	2	3	5	4												
5	1	2	3	5	4												
4	1	2	3		4												
3	1	2	3														
2	1	2															

7.3.1 Ranking of Methods Dropping Out the Least Profitable Agent

An interesting question to investigate is whether the performance of the SPEC algorithm is unaffected by the models that are included in the simulation. This is examined in the sequel by repeatedly running the simulation, each time having dropped out the trader using the least profitable method and calculating the cumulative profits of the remaining participating agents. If the performance of the algorithm is not affected by the models considered, the profits of participants who trade options using the SPEC algorithm should occupy the top places of the ranking. The resulting ranks of the SPEC algorithm based methods and the AVERAGE method are summarized in Table 7.2. The first column shows the number of participants in each group and the rows present the ranking of the SPEC model selection methods and the AVERAGE method within each group. As there are 104 traders, 103 groups are created. Although there are some slight changes in the rank, the traders based on the SPEC model selection algorithm keep the first places in the ranking. The SPEC(5) model selection algorithm achieves the highest returns in all the cases, thus indicating that the forecasting ability is not sensitive to the models that are used. On the other hand, the AVERAGE method deteriorates as the group becomes smaller. An expected feature as the sample becomes smaller by dropping out the least accurate forecasts.

7.3.2 Exercise Price and Relative Profits

Following Engle et al.'s (1993) approach, the sensitivity of agents' profits to exercise price is examined. Table 7.3 shows the ranking and cumulative profits of the competitors trading one-day straddles with exercise prices equal to e^{5rf_t} and e^{-3rf_t}. The call and put option prices are calculated as:

$$C_{t+1|t} = N\left(\frac{2(1-K)rf_t+\sigma^2_{t+1|t}}{2\sigma_{t+1|t}}\right) - e^{(K-1)rf_t} N\left(\frac{2(1-K)rf_t-\sigma^2_{t+1|t}}{2\sigma_{t+1|t}}\right), \tag{7.3.6}$$

and

$$P_{t+1|t} = C_{t+1|t} + e^{(K-1)rf_t} - 1,$$

for $K = 5, -3$. The rank of the traders does not change significantly. So, the cumulative profits in the simulated market are not sensitive to the exercise price that is used.

7.4. Comparing Methods of Model Selection on Simulated Options

The selection of the appropriate model is one of the most challenging areas in statistical modeling. Usually, a researcher has to choose among a set of candidate models. Methods of model selection examine the ability of the models either to describe or to forecast the variable under investigation.

Table 7.3

The rank and annualized daily profits of the competitors trading one-day straddles with different exercise prices.

Forecasts	e^{5rf_t}		e^{-3rf_t}		Forecasts	e^{5rf_t}		e^{-3rf_t}		Forecasts	e^{5rf_t}		e^{-3rf_t}	
	Profit	Rank	Profit	Rank		Profit	Rank	Profit	Rank		Profit	Rank	Profit	Rank
SPEC(T=5)	22.46%	1	22.52%	1	AR(1)GARCH(1,2)	-2.48%	64	-2.47%	64	AR(1)TARCH(2,1)	5.95%	48	5.94%	48
SPEC(T=10)	20.29%	2	20.34%	2	AR(2)GARCH(1,2)	-4.45%	71	-4.44%	71	AR(2)TARCH(2,1)	6.12%	47	6.12%	47
SPEC(T=15)	17.84%	3	17.89%	3	AR(3)GARCH(1,2)	-4.03%	70	-4.02%	70	AR(3)TARCH(2,1)	5.92%	49	5.91%	49
SPEC(T=20)	16.42%	5	16.46%	5	AR(4)GARCH(1,2)	-4.77%	72	-4.77%	72	AR(4)TARCH(2,1)	4.63%	52	4.63%	52
SPEC(T=25)	16.50%	4	16.53%	4	AR(0)GARCH(2,1)	-0.23%	53	-0.22%	53	AR(0)TARCH(2,2)	12.69%	16	12.70%	16
SPEC(T=30)	13.81%	10	13.84%	9	AR(1)GARCH(2,1)	-1.46%	57	-1.45%	57	AR(1)TARCH(2,2)	11.03%	24	11.03%	24
SPEC(T=35)	13.79%	11	13.82%	11	AR(2)GARCH(2,1)	-1.76%	60	-1.76%	60	AR(2)TARCH(2,2)	10.79%	26	10.79%	26
SPEC(T=40)	14.21%	8	14.24%	8	AR(3)GARCH(2,1)	-1.78%	61	-1.77%	61	AR(3)TARCH(2,2)	10.14%	29	10.14%	29
SPEC(T=45)	13.81%	9	13.83%	10	AR(4)GARCH(2,1)	-2.44%	63	-2.44%	63	AR(4)TARCH(2,2)	9.50%	31	9.50%	31
SPEC(T=50)	14.40%	6	14.41%	6	AR(0)GARCH(2,2)	-0.36%	54	-0.35%	54	AR(0)EGARCH(0,1)	-17.14%	89	-17.17%	89
SPEC(T=55)	13.41%	12	13.43%	12	AR(1)GARCH(2,2)	-1.67%	59	-1.66%	59	AR(1)EGARCH(0,1)	-17.99%	90	-18.01%	90
SPEC(T=60)	12.83%	15	12.85%	15	AR(2)GARCH(2,2)	-2.33%	62	-2.32%	62	AR(2)EGARCH(0,1)	-18.15%	91	-18.17%	91
SPEC(T=65)	12.55%	17	12.56%	17	AR(3)GARCH(2,2)	-1.58%	58	-1.57%	58	AR(3)EGARCH(0,1)	-18.44%	92	-18.47%	92
SPEC(T=70)	12.89%	14	12.90%	14	AR(4)GARCH(2,2)	-3.33%	67	-3.33%	67	AR(4)EGARCH(0,1)	-18.52%	93	-18.55%	93
SPEC(T=75)	11.88%	22	11.89%	22	AR(0)TARCH(0,1)	-21.45%	98	-21.47%	98	AR(0)EGARCH(0,2)	-7.03%	73	-7.06%	73
SPEC(T=80)	13.34%	13	13.35%	13	AR(1)TARCH(0,1)	-22.06%	100	-22.08%	100	AR(1)EGARCH(0,2)	-7.48%	74	-7.50%	74
MINIMUM	-38.14%	104	-38.24%	104	AR(2)TARCH(0,1)	-22.13%	101	-22.15%	101	AR(2)EGARCH(0,2)	-7.78%	75	-7.81%	75
MAXIMUM	-10.43%	78	-10.30%	78	AR(3)TARCH(0,1)	-22.38%	102	-22.40%	102	AR(3)EGARCH(0,2)	-8.20%	76	-8.22%	76
AVERAGE	9.46%	32	9.46%	32	AR(4)TARCH(0,1)	-22.39%	103	-22.42%	103	AR(4)EGARCH(0,2)	-8.47%	77	-8.50%	77
AR(0)GARCH(0,1)	-20.47%	94	-20.49%	94	AR(0)TARCH(0,2)	-11.33%	79	-11.35%	79	AR(0)EGARCH(1,1)	11.15%	23	11.16%	23
AR(1)GARCH(0,1)	-20.70%	95	-20.72%	95	AR(1)TARCH(0,2)	-12.38%	82	-12.41%	82	AR(1)EGARCH(1,1)	10.07%	30	10.07%	30
AR(2)GARCH(0,1)	-21.05%	96	-21.07%	96	AR(2)TARCH(0,2)	-11.92%	80	-11.95%	80	AR(2)EGARCH(1,1)	10.52%	27	10.52%	27
AR(3)GARCH(0,1)	-21.26%	97	-21.29%	97	AR(3)TARCH(0,2)	-12.26%	81	-12.28%	81	AR(3)EGARCH(1,1)	9.12%	33	9.12%	33
AR(4)GARCH(0,1)	-21.82%	99	-21.85%	99	AR(4)TARCH(0,2)	-12.68%	83	-12.70%	83	AR(4)EGARCH(1,1)	8.34%	38	8.34%	38
AR(0)GARCH(0,2)	-12.95%	84	-12.97%	84	AR(0)TARCH(1,1)	7.68%	40	7.68%	40	AR(0)EGARCH(1,2)	14.32%	7	14.33%	7
AR(1)GARCH(0,2)	-13.43%	85	-13.45%	85	AR(1)TARCH(1,1)	6.39%	46	6.39%	46	AR(1)EGARCH(1,2)	12.38%	18	12.39%	18
AR(2)GARCH(0,2)	-13.51%	86	-13.54%	86	AR(2)TARCH(1,1)	6.70%	45	6.70%	45	AR(2)EGARCH(1,2)	12.32%	19	12.33%	19
AR(3)GARCH(0,2)	-13.91%	87	-13.94%	87	AR(3)TARCH(1,1)	5.56%	50	5.56%	50	AR(3)EGARCH(1,2)	12.03%	20	12.04%	20
AR(4)GARCH(0,2)	-13.96%	88	-13.98%	88	AR(4)TARCH(1,1)	5.39%	51	5.38%	51	AR(4)EGARCH(1,2)	11.94%	21	11.95%	21
AR(0)GARCH(1,1)	-1.32%	56	-1.31%	56	AR(0)TARCH(1,2)	10.80%	25	10.81%	25	AR(0)EGARCH(2,1)	8.51%	37	8.52%	37
AR(1)GARCH(1,1)	-3.03%	65	-3.02%	65	AR(1)TARCH(1,2)	8.92%	35	8.93%	35	AR(1)EGARCH(2,1)	7.51%	41	7.50%	41
AR(2)GARCH(1,1)	-3.44%	68	-3.43%	68	AR(2)TARCH(1,2)	10.37%	28	10.37%	28	AR(2)EGARCH(2,1)	8.64%	36	8.64%	36
AR(3)GARCH(1,1)	-3.10%	66	-3.09%	66	AR(3)TARCH(1,2)	8.99%	34	8.98%	34	AR(3)EGARCH(2,1)	7.42%	42	7.42%	42
AR(4)GARCH(1,1)	-3.44%	69	-3.44%	69	AR(4)TARCH(1,2)	8.17%	39	8.17%	39	AR(4)EGARCH(2,1)	6.81%	44	6.81%	44
AR(0)GARCH(1,2)	-0.40%	55	-0.40%	55	AR(0)TARCH(2,1)	7.12%	43	7.12%	43					

The Akaike information criterion (Akaike (1973)) and the Schwarz Bayesian criterion (Schwarz (1978)) are model selection methods that are based on the maximized value of the log-likelihood function and evaluate the ability of the models to describe the data. In the case we are interesting in using a model for forecasting, the evaluation of the models would naturally be based on their ability to produce valuable forecasts. Loss functions, which measure either the distance between actual and predicted values or the benefit from the use of these forecasts, are used to evaluate the forecasting ability of the models. Poon and Granger (2003) reviewed a detailed record of volatility forecasting loss functions and relative references.

In the sequel, the SPEC model selection algorithm is compared with other criteria of selection that measure the ability of the models to forecast volatility again on the basis of the profits of the participants in a simulated options market. Denoting the realized at time $t+1$ by s_{t+1}^2, the following loss functions were considered:

1. Mean Square Error of Variance (MSEV):

$$T^{-1}\sum_{t=1}^{T}\left(\hat{\sigma}_{t+1|t}^{2}-s_{t+1}^{2}\right)^{2}. \tag{7.4.1}$$

2. Mean Absolute Error of Variance (MAEV):

$$T^{-1}\sum_{t=1}^{T}\left|\hat{\sigma}_{t+1|t}^{2}-s_{t+1}^{2}\right|. \tag{7.4.2}$$

3. Mean Square Error of Deviation (MSED):

$$T^{-1}\sum_{t=1}^{T}\left(\hat{\sigma}_{t+1|t}-s_{t+1}\right)^{2}. \tag{7.4.3}$$

4. Mean Absolute Error of Deviation (MAED):

$$T^{-1}\sum_{t=1}^{T}\left|\hat{\sigma}_{t+1|t}-s_{t+1}\right|. \tag{7.4.4}$$

5. Heteroscedasticity Adjusted Mean Squared Error of Variance (HAMSEV):

$$T^{-1}\sum_{t=1}^{T}\left(1-s_{t+1}^{2}\big/\hat{\sigma}_{t+1|t}^{2}\right)^{2}. \tag{7.4.5}$$

6. Heteroscedasticity Adjusted Mean Absolute Error of Variance (HAMAEV):

$$T^{-1}\sum_{t=1}^{T}\left|1-s_{t+1}^{2}\big/\hat{\sigma}_{t+1|t}^{2}\right|. \tag{7.4.6}$$

7. Heteroscedasticity Adjusted Mean Squared Error of Deviation (HAMSED):

$$T^{-1}\sum_{t=1}^{T}\left(1-s_{t+1}\big/\hat{\sigma}_{t+1|t}\right)^{2}. \tag{7.4.7}$$

8. Heteroscedasticity Adjusted Mean Absolute Error of Deviation (HAMAED):

$$T^{-1}\sum_{t=1}^{T}\left|1-s_{t+1}\big/\hat{\sigma}_{t+1|t}\right|. \tag{7.4.8}$$

9. Mean Logarithmic Error of Variance (MLEV):

$$T^{-1}\sum_{t=1}^{T}\ln\left(s_{t+1}^{2}/\hat{\sigma}_{t+1|t}^{2}\right)^{2} . \tag{7.4.9}$$

10. Gaussian Maximum Likelihood Error of Variance (GMLEV):

$$T^{-1}\sum_{t=1}^{T}\left(\ln\left(\hat{\sigma}_{t+1|t}^{2}\right)+\left(\frac{s_{t+1}^{2}}{\hat{\sigma}_{t+1|t}^{2}}\right)\right). \tag{7.4.10}$$

11. Gaussian Maximum Likelihood Error of Deviation (GMLED):

$$T^{-1}\sum_{t=1}^{T}\left(\ln\left(\hat{\sigma}_{t+1|t}\right)+\left(\frac{s_{t+1}}{\hat{\sigma}_{t+1|t}}\right)\right), \tag{7.4.11}$$

where T is the number of the one-step-ahead volatility forecasts. The first four loss functions have been widely used in applied studies. The heteroscedasticity adjusted functions were introduced by Andersen et al. (1999) and Bollerslev and Ghysels (1996), while mean logarithmic error function was utilized by Pagan and Schwert (1990). The GMLE function, which was presented in Bollerslev et al. (1994), measures the forecast error according to the likelihood function that is used in estimating the models.

As the actual volatility is unobservable, the common way to determine the daily realized volatility is the squared daily returns, which is an unbiased but noisy volatility estimator. Andersen and Bollerslev (1998a) introduced the use of the sum squared finely sampled high frequency data as an alternative volatility measure. For a detailed description of the realized intra-day volatility, the interested reader is referred to section 2.6.1 in chapter 2 and references therein. Based on Andersen et al. (1999), Andersen et al. (2001b) and Kayahan et al. (2002), we compute the realized intra-day volatility of day t as:

$$s_{t}^{2}=\sum_{j=1}^{m-1}\left(\ln\left(P_{(j+1/m),t}\right)-\ln\left(P_{(j/m),t}\right)\right)^{2} , \tag{7.4.12}$$

where $P_{(m),t}$ denotes five-minute linearly interpolated prices of S&P500 at day t with m observations per day. The intra-day quotation data are available from April 28th 1997 to October 18th 2002 and were provided by Olsen and Associates.

Each loss function is computed for $T=10(10)80$. In order to compare the SPEC algorithm with the 11 loss functions, a simulated options market is created. Each agent selects the ARCH model with the lowest value of its the loss function in order to forecast next day's variance. The simulated market is consisting of 99 traders: the 12 model selection algorithms for 8 different sample sizes (including the SPEC algorithm), the average, the minimum and the maximum of all daily forecasts methods. The comparison is carried out on the basis of the annualized daily profits of the participants.

Table 7.4

The annualised daily profits per competitor per straddle for trades that are at the average of the bid/ask prices.

Rank	Model Selection Algorithm	Profit	T-Ratio	Rank	Model Selection Algorithm	Profit	T-Ratio	Rank	Model Selection Algorithm	Profit	T-Ratio
1	SPEC(T=10)	16.42%	3.51	34	MAEV(T=10)	-7.72%	-1.82	67	GMLEV(T=60)	-10.97%	-2.88
2	SPEC(T=20)	13.73%	3.09	35	HAMSED(T=20)	-7.79%	-1.96	68	GMLEV(T=30)	-11.02%	-2.95
3	SPEC(T=50)	13.59%	3.17	36	MSEV(T=70)	-8.06%	-2.17	69	MAED(T=80)	-11.07%	-2.90
4	AVERAGE	13.30%	4.67	37	MSEV(T=60)	-8.08%	-2.19	70	GMLEV(T=50)	-11.08%	-2.91
5	SPEC(T=30)	11.87%	2.74	38	GMLEV(T=20)	-8.35%	-2.06	71	HAMAED(T=40)	-11.20%	-2.93
6	SPEC(T=40)	11.64%	2.68	39	HAMAED(T=20)	-8.38%	-2.10	72	GMLEV(T=70)	-11.25%	-2.96
7	SPEC(T=60)	11.50%	2.65	40	MSEV(T=50)	-8.46%	-2.19	73	MSED(T=80)	-11.57%	-3.09
8	SPEC(T=70)	10.63%	2.48	41	MSEV(T=80)	-8.58%	-2.35	74	GMLED(T=70)	-11.75%	-3.02
9	SPEC(T=80)	8.62%	2.03	42	GMLED(T=50)	-8.60%	-2.34	75	MLEV(T=70)	-11.80%	-3.04
10	HAMSEV(T=10)	0.87%	0.21	43	HAMAEV(T=40)	-8.70%	-2.40	76	HAMAED(T=80)	-11.83%	-3.05
11	HAMAEV(T=10)	0.53%	0.13	44	HAMSED(T=40)	-8.94%	-2.41	77	GMLED(T=60)	-12.11%	-3.13
12	HAMSEV(T=20)	0.38%	0.09	45	HAMSED(T=60)	-9.30%	-2.54	78	MSED(T=60)	-12.42%	-3.27
13	HAMSEV(T=30)	0.04%	0.01	46	GMLED(T=20)	-9.61%	-2.41	79	HAMAED(T=50)	-12.44%	-3.29
14	HAMSED(T=10)	-0.29%	-0.07	47	MSED(T=70)	-9.90%	-2.58	80	MSED(T=30)	-12.45%	-3.12
15	HAMSEV(T=60)	-0.96%	-0.26	48	MSEV(T=20)	-9.92%	-2.40	81	MAEV(T=50)	-12.46%	-3.21
16	HAMSEV(T=80)	-1.05%	-0.28	49	HAMAEV(T=70)	-9.95%	-2.70	82	MLEV(T=30)	-12.65%	-3.22
17	MAX	-1.18%	-0.21	50	HAMAEV(T=50)	-10.04%	-2.81	83	MAED(T=50)	-12.68%	-3.24
18	HAMSEV(T=70)	-1.22%	-0.34	51	GMLED(T=30)	-10.11%	-2.66	84	HAMAED(T=70)	-12.87%	-3.23
19	GMLEV(T=10)	-1.66%	-0.40	52	HAMAEV(T=60)	-10.14%	-2.79	85	MAED(T=70)	-13.39%	-3.44
20	GMLED(T=10)	-1.93%	-0.47	53	MLEV(T=60)	-10.26%	-2.70	86	MSEV(T=30)	-13.44%	-3.34
21	HAMSEV(T=50)	-2.95%	-0.78	54	MLEV(T=20)	-10.32%	-2.58	87	MAEV(T=60)	-13.46%	-3.45
22	MLEV(T=10)	-3.20%	-0.77	55	HAMSED(T=30)	-10.38%	-2.72	88	HAMAED(T=60)	-13.70%	-3.44
23	HAMSEV(T=40)	-3.33%	-0.87	56	GMLEV(T=80)	-10.46%	-2.79	89	MAEV(T=20)	-13.74%	-3.30
24	HAMAED(T=10)	-3.81%	-0.94	57	MLEV(T=50)	-10.46%	-2.79	90	MAED(T=20)	-14.23%	-3.46
25	MSED(T=10)	-4.01%	-0.95	58	GMLED(T=40)	-10.51%	-2.76	91	MAEV(T=70)	-14.25%	-3.60
26	MSEV(T=10)	-4.28%	-1.01	59	HAMAEV(T=30)	-10.54%	-2.83	92	MAED(T=40)	-14.28%	-3.62
27	MAED(T=10)	-5.19%	-1.23	60	MLEV(T=40)	-10.55%	-2.82	93	MAEV(T=30)	-14.30%	-3.55
28	GMLEV(T=40)	-5.84%	-1.57	61	GMLED(T=80)	-10.61%	-2.76	94	MAEV(T=80)	-14.33%	-3.71
29	HAMAEV(T=80)	-6.44%	-1.77	62	MSED(T=20)	-10.67%	-2.64	95	MAED(T=30)	-14.53%	-3.60
30	HAMSED(T=80)	-6.66%	-1.80	63	MLEV(T=80)	-10.72%	-2.86	96	MAED(T=60)	-14.60%	-3.82
31	HAMSED(T=70)	-7.18%	-1.91	64	MSED(T=40)	-10.73%	-2.79	97	MAEV(T=40)	-16.09%	-4.06
32	HAMAEV(T=20)	-7.52%	-1.89	65	MSED(T=50)	-10.78%	-2.89	98	HAMAED(T=30)	-16.14%	-4.06
33	MSEV(T=40)	-7.71%	-1.97	66	HAMSED(T=50)	-10.82%	-2.86	99	MIN	-33.42%	-6.12

The resulting ranking of the criteria is summarized in Table 7.4. For each model selection criterion, the highest annualized daily profits are given along with the values of the corresponding t-ratios defined as in Table 7.1 and the sample sizes (values of T) at which the maximum returns are attained (in parentheses).

The results in the table indicate that traders who are based on the SPEC algorithm achieve the highest returns, despite the use of the realized intra-day volatility by the loss functions. Moreover, the SPEC method appears more suitable in predicting volatility for pricing contingent claims, as it is the only model selection method that produces returns higher that the AVERAGE algorithm does. An interesting point is that, with the exception of HAMSEV, all the algorithms achieve their highest returns for $T = 10$.

Table 7.5

The annualised daily profits per competitor per straddle for trades that are at the average of the bid/ask prices, using rolling samples of 500 observations.

Rank	Algorithm	Profit	T-Ratio	Rank	Algorithm	Profit	T-Ratio	Rank	Algorithm	Profit	T-Ratio
1	SPEC(T=5)	21.79%	5.4	36	AR(3)TARCH(2,1)	5.74%	1.6	71	AR(0)GARCH(0,2)	-8.80%	-2.5
2	SPEC(T=30)	19.92%	6.3	37	AR(2)EGARCH(1,2)	5.72%	1.8	72	AR(4)TARCH(0,2)	-8.97%	-2.5
3	SPEC(T=10)	19.59%	5.3	38	AR(2)EGARCH(2,1)	5.60%	2.0	73	AR(1)EGARCH(0,2)	-9.02%	-2.9
4	SPEC(T=25)	19.03%	6.0	39	AR(1)EGARCH(1,2)	4.48%	1.4	74	AR(2)TARCH(0,2)	-9.04%	-2.5
5	SPEC(T=35)	18.55%	6.4	40	AR(0)TARCH(2,1)	4.46%	1.2	75	AR(1)TARCH(0,2)	-9.06%	-2.5
6	SPEC(T=20)	18.41%	5.5	41	AR(3)TARCH(0,1)	4.06%	1.2	76	AR(3)GARCH(2,1)	-9.14%	-2.8
7	SPEC(T=15)	18.22%	5.3	42	AR(2)TARCH(0,1)	4.01%	1.3	77	AR(2)GARCH(2,1)	-9.22%	-3.0
8	AR(4)TARCH(1,2)	15.87%	6.1	43	AR(3)EGARCH(1,2)	3.85%	1.2	78	AR(4)GARCH(1,1)	-9.26%	-2.9
9	SPEC(T=45)	15.73%	5.7	44	AR(4)EGARCH(1,2)	3.73%	1.1	79	AR(1)GARCH(1,1)	-9.26%	-3.1
10	AR(0)TARCH(1,2)	15.06%	5.9	45	AR(4)TARCH(0,1)	3.25%	1.0	80	AR(3)TARCH(0,2)	-9.29%	-2.5
11	AR(2)TARCH(1,2)	14.94%	5.8	46	AR(2)EGARCH(1,1)	2.61%	0.9	81	AR(4)EGARCH(0,2)	-9.32%	-2.8
12	AR(1)TARCH(1,2)	14.85%	5.7	47	AR(3)EGARCH(2,1)	2.30%	0.8	82	AR(3)EGARCH(0,2)	-9.37%	-2.9
13	AR(3)TARCH(1,2)	14.81%	5.8	48	AR(4)EGARCH(2,1)	1.76%	0.6	83	AR(2)EGARCH(0,2)	-9.38%	-3.0
14	SPEC(T=40)	14.47%	5.2	49	AR(1)EGARCH(2,1)	1.72%	0.6	84	AR(1)GARCH(2,1)	-9.60%	-3.0
15	SPEC(T=60)	13.99%	5.0	50	AR(1)EGARCH(1,1)	1.60%	0.5	85	AR(4)GARCH(1,2)	-9.97%	-3.0
16	AVERAGE	13.65%	12.3	51	AR(4)EGARCH(1,1)	1.25%	0.4	86	AR(3)GARCH(1,2)	-10.35%	-3.1
17	SPEC(T=80)	13.60%	4.5	52	AR(3)EGARCH(1,1)	1.07%	0.4	87	AR(1)GARCH(1,2)	-10.52%	-3.3
18	SPEC(T=50)	13.59%	5.0	53	AR(4)TARCH(1,1)	-1.56%	-0.4	88	AR(1)GARCH(0,2)	-11.07%	-3.2
19	SPEC(T=55)	13.22%	4.7	54	AR(3)TARCH(1,1)	-1.68%	-0.4	89	AR(2)GARCH(1,2)	-11.19%	-3.5
20	SPEC(T=70)	13.15%	4.7	55	AR(1)TARCH(1,1)	-1.79%	-0.4	90	AR(0)GARCH(0,1)	-11.39%	-3.2
21	SPEC(T=65)	13.13%	4.7	56	AR(0)TARCH(1,1)	-1.81%	-0.4	91	AR(2)GARCH(0,2)	-11.51%	-3.3
22	AR(1)TARCH(2,2)	12.82%	4.7	57	AR(2)TARCH(1,1)	-2.41%	-0.6	92	AR(1)GARCH(0,1)	-11.62%	-3.1
23	SPEC(T=75)	12.62%	4.3	58	AR(0)GARCH(1,1)	-2.87%	-1.2	93	AR(2)GARCH(0,1)	-12.20%	-3.3
24	AR(2)TARCH(2,2)	10.19%	3.8	59	AR(0)GARCH(2,2)	-3.58%	-1.2	94	AR(3)GARCH(0,2)	-12.37%	-3.5
25	AR(3)TARCH(2,2)	9.69%	3.6	60	AR(0)GARCH(2,1)	-3.83%	-1.5	95	AR(3)GARCH(0,1)	-12.56%	-3.3
26	AR(0)TARCH(2,2)	9.43%	3.2	61	AR(0)GARCH(1,2)	-4.12%	-1.4	96	AR(4)GARCH(0,2)	-12.68%	-3.4
27	AR(4)TARCH(2,2)	9.35%	3.4	62	AR(0)EGARCH(0,2)	-6.81%	-2.1	97	AR(4)GARCH(0,1)	-12.92%	-3.4
28	AR(2)TARCH(2,1)	8.56%	2.4	63	AR(1)GARCH(2,2)	-7.36%	-2.3	98	AR(1)EGARCH(0,1)	-16.17%	-4.4
29	AR(0)TARCH(0,1)	8.35%	2.9	64	AR(3)GARCH(1,1)	-7.62%	-2.5	99	AR(2)EGARCH(0,1)	-16.94%	-4.5
30	AR(1)TARCH(2,1)	7.84%	2.3	65	AR(4)GARCH(2,2)	-7.69%	-2.3	100	AR(0)EGARCH(0,1)	-17.14%	-4.6
31	AR(4)TARCH(2,1)	7.03%	1.9	66	AR(0)TARCH(0,2)	-7.70%	-2.2	101	AR(3)EGARCH(0,1)	-17.52%	-4.7
32	AR(0)EGARCH(1,2)	6.87%	2.3	67	AR(4)GARCH(2,1)	-7.81%	-2.4	102	AR(4)EGARCH(0,1)	-17.92%	-4.7
33	AR(0)EGARCH(2,1)	6.42%	2.4	68	AR(2)GARCH(2,2)	-7.88%	-2.4	103	MAXIMUM	-18.60%	-3.3
34	AR(0)EGARCH(1,1)	6.07%	2.2	69	AR(3)GARCH(2,2)	-8.13%	-2.4	104	MINIMUM	-33.35%	-5.9
35	AR(1)TARCH(0,1)	5.82%	1.8	70	AR(2)GARCH(1,1)	-8.16%	-2.7				

7.5 Investigate the performance of the SPEC algorithm using larger sample sizes for the estimation of the ARCH models

As has been noted in the literature, although the use of the entire set of available data is common practice in forecasting volatility, at least for some cases, a restricted sample size could generate more accurate one-step-ahead forecasts, since it incorporates changes in trading behaviour more efficiently. For example, Hoppe (1998) examined the issue of the sample size, in the context of value-at-risk, and argued that a smaller sample could lead to more accurate estimates than a larger one. Frey and Michaud (1997) supported the use of small sample sizes in order to capture the structural changes over time due to changes in trading behaviour. Angelidis, Benos and Degiannakis (2004) noted similar findings.

In order to investigate whether the use of a rolling sample size of 1000 observations induces a bias on the results of the simulation, we re-run the simulation study with larger datasets. We used rolling samples of 500 and 2000 observations and we found out that the results in the previous sections are not appreciably different when using sample sizes of 500, 1000 or 2000 observations.

Tables 7.5 and 7.6 present the profits per competitor per straddle and the corresponding t-ratios when we use rolling samples of 500 and 2000 observations, respectively. There is no qualitative difference among the used sample sizes. The SPEC algorithm performs best for low values of T, (T=5, 10), in the new simulation studies, which is in complete agreement with the originally obtained results on the basis of a 1000-observation rolling sample. The MINIMUM forecast takes the last positions and the MAXIMUM forecast achieves negative and statistically significant returns, an indication that neither a downward nor an upward forecast bias, that could affect profits significantly, is present.

As there is no qualitative difference between the use of sample sizes of 500 and 2000 observations, we present the results based on the sample size of 500. Dropping out the trader with the least profitable method at a time, the cumulative profits of the participants in the simulated market are calculated. The SPEC(5) model selection algorithm achieves the highest returns in all the cases, thus indicating that the forecasting ability is not sensitive to the models that are used. As concerns the sample size of 500 observations, Table 7.7 presents the transitivity of the profitability of competitors, who employ the SPEC model selection algorithm and the AVERAGE method. Table 7.8 shows the ranking and cumulative profits of the competitors trading straddles with exercise prices equal to e^{5rf_t}, e^{rf_t} and e^{-3rf_t}. The rank of the traders does not change significantly. So, the cumulative profits in the simulated market are not sensitive to the exercise price is used. The results of Table 7.7 and 7.8 are almost identical to those presented in the previous section for a sample size of 1000 observations.

Table 7.6

The annualised daily profits per competitor per straddle for trades that are at the average of the bid/ask prices, using rolling samples of 2000 observations.

Rank	Algorithm	Profit	T-Ratio	Rank	Algorithm	Profit	T-Ratio	Rank	Algorithm	Profit	T-Ratio
1	SPEC(T=5)	19.08%	5.63	36	AR(4)EGARCH(2,1)	9.82%	5.27	71	AR(1)GARCH(2,1)	-5.20%	-2.16
2	SPEC(T=10)	17.29%	5.69	37	AR(0)EGARCH(2,1)	9.82%	5.12	72	AR(3)GARCH(2,1)	-5.22%	-2.18
3	SPEC(T=40)	16.52%	7.09	38	AR(3)EGARCH(2,1)	9.51%	5.14	73	MAXIMUM	-9.42%	-2.04
4	SPEC(T=55)	16.24%	7.10	39	AR(3)TARCH(2,2)	9.42%	4.25	74	AR(0)EGARCH(0,2)	-10.19%	-2.82
5	SPEC(T=50)	15.89%	7.04	40	AVERAGE	9.37%	8.13	75	AR(4)EGARCH(0,2)	-10.25%	-2.83
6	SPEC(T=25)	15.43%	6.26	41	AR(2)EGARCH(2,1)	8.98%	4.56	76	AR(3)EGARCH(0,2)	-10.29%	-2.86
7	SPEC(T=65)	15.41%	6.88	42	AR(0)TARCH(1,1)	8.93%	4.07	77	AR(2)EGARCH(0,2)	-10.58%	-2.95
8	SPEC(T=45)	15.38%	6.65	43	AR(0)TARCH(2,1)	8.88%	4.20	78	AR(1)EGARCH(0,2)	-10.63%	-2.95
9	SPEC(T=35)	15.26%	6.63	44	AR(1)EGARCH(2,1)	8.73%	4.30	79	AR(0)TARCH(0,2)	-11.50%	-3.70
10	SPEC(T=15)	15.19%	5.47	45	AR(1)TARCH(2,1)	7.60%	3.56	80	AR(0)GARCH(0,2)	-12.52%	-4.03
11	SPEC(T=20)	14.85%	5.73	46	AR(2)TARCH(1,1)	7.16%	3.30	81	AR(1)TARCH(0,2)	-13.32%	-4.30
12	SPEC(T=60)	14.82%	6.44	47	AR(1)TARCH(1,1)	7.02%	3.22	82	AR(4)TARCH(0,2)	-13.58%	-4.28
13	SPEC(T=70)	14.61%	6.37	48	AR(2)TARCH(2,1)	6.66%	3.19	83	AR(2)TARCH(0,2)	-13.60%	-4.35
14	SPEC(T=30)	14.51%	6.26	49	AR(4)TARCH(1,1)	6.53%	3.25	84	AR(3)TARCH(0,2)	-13.65%	-4.35
15	AR(0)EGARCH(1,2)	14.22%	5.43	50	AR(4)TARCH(2,1)	6.50%	3.16	85	AR(1)GARCH(0,2)	-14.02%	-4.49
16	AR(1)EGARCH(1,1)	13.57%	7.18	51	AR(3)TARCH(1,1)	5.60%	2.66	86	AR(4)GARCH(0,2)	-14.29%	-4.50
17	AR(2)EGARCH(1,1)	13.26%	7.09	52	AR(3)TARCH(2,1)	4.64%	2.25	87	AR(2)GARCH(0,2)	-14.50%	-4.60
18	AR(0)TARCH(1,2)	12.92%	5.02	53	AR(0)GARCH(1,2)	-0.12%	-0.05	88	AR(3)GARCH(0,2)	-14.50%	-4.58
19	AR(1)EGARCH(1,2)	12.85%	5.05	54	AR(0)GARCH(2,2)	-0.29%	-0.12	89	AR(0)EGARCH(0,1)	-16.91%	-5.10
20	AR(4)EGARCH(1,1)	12.74%	6.85	55	AR(1)GARCH(2,2)	-1.64%	-0.66	90	AR(1)EGARCH(0,1)	-17.02%	-5.08
21	AR(0)EGARCH(1,1)	12.67%	6.68	56	AR(1)GARCH(1,2)	-1.74%	-0.73	91	AR(2)EGARCH(0,1)	-17.42%	-5.20
22	AR(2)EGARCH(1,2)	12.50%	5.00	57	AR(4)GARCH(2,2)	-1.82%	-0.74	92	AR(3)EGARCH(0,1)	-17.50%	-5.20
23	AR(0)TARCH(2,2)	12.31%	5.47	58	AR(3)GARCH(2,2)	-2.96%	-1.17	93	AR(4)EGARCH(0,1)	-17.87%	-5.26
24	AR(4)EGARCH(1,2)	12.21%	4.81	59	AR(3)GARCH(1,2)	-3.11%	-1.23	94	AR(3)GARCH(0,1)	-20.29%	-6.19
25	AR(3)EGARCH(1,1)	12.16%	6.75	60	AR(0)GARCH(1,1)	-3.12%	-1.27	95	AR(2)GARCH(0,1)	-20.59%	-6.30
26	AR(1)TARCH(2,2)	12.01%	5.18	61	AR(2)GARCH(1,2)	-3.36%	-1.35	96	AR(0)GARCH(0,1)	-20.75%	-6.43
27	AR(2)TARCH(1,2)	12.00%	4.79	62	AR(2)GARCH(2,2)	-3.41%	-1.36	97	AR(1)GARCH(0,1)	-20.87%	-6.33
28	SPEC(T=75)	12.00%	5.49	63	AR(4)GARCH(1,2)	-3.70%	-1.47	98	AR(0)TARCH(0,1)	-21.15%	-6.22
29	AR(3)EGARCH(1,2)	11.15%	4.49	64	AR(3)GARCH(1,1)	-3.74%	-1.46	99	AR(4)GARCH(0,1)	-21.17%	-6.36
30	AR(1)TARCH(1,2)	10.99%	4.24	65	AR(0)GARCH(2,1)	-4.00%	-1.67	100	AR(1)TARCH(0,1)	-21.34%	-6.26
31	AR(3)TARCH(1,2)	10.49%	4.16	66	AR(1)GARCH(1,1)	-4.15%	-1.68	101	AR(2)TARCH(0,1)	-21.59%	-6.36
32	AR(2)TARCH(2,2)	10.22%	4.56	67	AR(4)GARCH(1,1)	-4.23%	-1.66	102	AR(3)TARCH(0,1)	-21.78%	-6.37
33	SPEC(T=80)	10.21%	4.62	68	AR(4)GARCH(2,1)	-4.34%	-1.72	103	AR(4)TARCH(0,1)	-22.14%	-6.39
34	AR(4)TARCH(1,2)	9.84%	3.84	69	AR(2)GARCH(2,1)	-4.77%	-1.98	104	MINIMUM	-43.47%	-9.24
35	AR(4)TARCH(2,2)	9.84%	4.54	70	AR(2)GARCH(1,1)	-5.08%	-2.03				

7.6. Discussion

Adopting Engle et al.'s (1993) approach to comparing several variance forecast methods using an economic value criterion, the performance of the SPEC model selection algorithm was examined. Simulating an options market, in order to avoid problems related to observed actual option prices, 104 traders were assumed to trade one-day straddles on $1 shares of the S&P500 index, for the period from October 4th 1995 to October 18th, 2002 (1773 trading days). Traders were also assumed to use variance forecast methods of their choice. The variance forecast methods considered were: 85 selection "methods" (strategies), one for each of 85 ARCH models, each amounting to the utilization of the forecasts of the same model at any point in time, the SPEC model selection algorithm for 16 different sample sizes, the average, the minimum and the maximum of all daily forecasts methods. Traders using SPEC algorithm based methods appear to achieve higher profits than traders using any of the 85 single ARCH model based methods considered in the simulation. Moreover, traders, who apply the SPEC model selection algorithm for sample sizes $T = 5(5)25$, appear to achieve the highest profits, a conclusion which is in agreement to chapter's 6 findings in the case of real index-option prices. The ability of the SPEC model selection algorithm was also compared with loss functions that measure the ability of the models to forecast volatility. Even though, the other criteria (loss functions) used the realized intra-day volatility, the SPEC algorithm, for $T = 10$, led to the highest profits. It appears, therefore, that the results support the conclusion that the increase in profits cannot be attributed to chance but to improved volatility prediction. Hence, the SPEC selection method offers a useful model selection tool in estimating future volatility, with applications in pricing derivatives.

Table 7.7

Rank of the methods based on the SPEC model selection algorithm by dropping out the least profitable agent at a time, using rolling samples of 500 observations.

	Algorithm																
Ranks in Groups by Size	SPEC (T=5)	SPEC (T=10)	SPEC (T=15)	SPEC (T=20)	SPEC (T=25)	SPEC (T=30)	SPEC (T=35)	SPEC (T=40)	SPEC (T=45)	SPEC (T=50)	SPEC (T=55)	SPEC (T=60)	SPEC (T=65)	SPEC (T=70)	SPEC (T=75)	SPEC (T=80)	AVERAGE
104	1	3	7	6	4	2	5	14	9	18	19	15	21	20	23	17	16
103	1	3	7	6	4	2	5	14	9	17	19	15	21	20	23	18	16
102	1	3	7	6	4	2	5	14	9	17	19	15	21	20	23	18	16
101	1	3	7	6	4	2	5	14	9	16	19	15	21	20	23	18	17
100	1	3	7	6	4	2	5	14	9	16	19	15	20	21	23	18	17
95	1	3	6	7	4	2	5	14	8	16	19	15	20	21	23	17	18
90	1	3	6	7	4	2	5	13	8	16	18	15	19	20	23	17	21
85	1	2	6	7	4	3	5	13	8	16	19	15	18	20	23	17	22
80	1	2	5	7	4	3	6	12	8	16	19	15	18	20	22	17	23
75	1	2	5	7	4	3	6	10	8	16	19	15	18	20	22	17	23
70	1	2	4	7	5	3	6	10	8	14	19	16	18	20	21	17	23
65	1	2	4	7	5	3	6	10	8	12	20	13	18	19	21	17	23
60	1	2	4	6	5	3	7	10	8	11	21	13	18	19	20	17	25
55	1	2	4	6	5	3	7	10	8	11	21	12	16	19	20	15	27
50	1	2	4	6	5	3	7	10	8	11	21	12	15	19	20	16	28
45	1	2	4	6	5	3	7	10	8	11	21	12	14	18	20	15	29
40	1	2	4	6	5	3	7	11	9	13	21	16	18	19	20	14	31
35	1	2	4	6	5	3	7	11	8	13	20	15	16	19	21	14	31
30	1	2	4	6	5	3	7	10	8	14	20	15	16	19	22	13	
25	1	2	4	6	5	3	7	9	8	14	20	16	15	19	21	13	
20	1	2	4	6	5	3	7	9	8	14	19	17	16	18		10	
15	1	2	5	6	4	3	7	11	8	13						15	
14	1	2	5	6	4	3	7	12	8	14							
13	1	2	5	6	4	3	7	12	8								
12	1	2	5	6	4	3	7	10	8								
11	1	2	5	6	4	3	7	10	8								
10	1	2	5	6	4	3	7	10	8								
9	1	2	5	6	4	3	7		8								
8	1	2	6	5	4	3	7		8								
7	1	2	6	5	4	3	7										
6	1	2	6	5	3	4											
5	1	2		5	4	3											
4	1	2			4	3											
3	1	2				3											
2	1	2															

Table 7.8

The rank and annualized daily profits of the competitors trading one-day straddles with different exercise prices, using rolling samples of 500 observations.

Forecasts	e^{5rf_t}		e^{-3rf_t}		Forecasts	e^{5rf_t}		e^{-3rf_t}		Forecasts	e^{5rf_t}		e^{-3rf_t}	
	Profit	Rank	Profit	Rank		Profit	Rank	Profit	Rank		Profit	Rank	Profit	Rank
SPEC(T=5)	21.60%	1	21.66%	1	AR(1)GARCH(1,2)	-10.53%	87	-10.52%	87	AR(1)EGARCH(2,1)	1.78%	49	1.78%	49
SPEC(T=10)	19.43%	3	19.48%	3	AR(2)GARCH(1,2)	-11.24%	89	-11.23%	89	AR(2)EGARCH(2,1)	5.65%	38	5.65%	38
SPEC(T=15)	18.09%	7	18.12%	7	AR(3)GARCH(1,2)	-10.41%	86	-10.40%	86	AR(3)EGARCH(2,1)	2.34%	47	2.34%	47
SPEC(T=20)	18.30%	6	18.33%	6	AR(4)GARCH(1,2)	-10.03%	85	-10.02%	85	AR(4)EGARCH(2,1)	1.80%	48	1.80%	48
SPEC(T=25)	18.93%	4	18.95%	4	AR(0)GARCH(2,1)	-3.87%	60	-3.87%	60	AR(0)TARCH(0,1)	8.39%	29	8.39%	29
SPEC(T=30)	19.83%	2	19.85%	2	AR(1)GARCH(2,1)	-9.58%	84	-9.58%	84	AR(1)TARCH(0,1)	5.86%	35	5.86%	35
SPEC(T=35)	18.47%	5	18.48%	5	AR(2)GARCH(2,1)	-9.26%	79	-9.25%	77	AR(2)TARCH(0,1)	4.04%	42	4.04%	42
SPEC(T=40)	14.40%	14	14.42%	14	AR(3)GARCH(2,1)	-9.18%	76	-9.17%	76	AR(3)TARCH(0,1)	4.09%	41	4.09%	41
SPEC(T=45)	15.66%	9	15.67%	9	AR(4)GARCH(2,1)	-7.87%	67	-7.86%	67	AR(4)TARCH(0,1)	3.28%	45	3.28%	45
SPEC(T=50)	13.52%	17	13.54%	17	AR(0)GARCH(2,2)	-3.63%	59	-3.62%	59	AR(0)TARCH(0,2)	-7.66%	64	-7.68%	65
SPEC(T=55)	13.17%	19	13.18%	19	AR(1)GARCH(2,2)	-7.36%	63	-7.35%	63	AR(1)TARCH(0,2)	-9.04%	75	-9.05%	75
SPEC(T=60)	13.93%	15	13.94%	15	AR(2)GARCH(2,2)	-7.93%	68	-7.92%	68	AR(2)TARCH(0,2)	-9.01%	74	-9.02%	74
SPEC(T=65)	13.05%	21	13.06%	21	AR(3)GARCH(2,2)	-8.18%	69	-8.17%	69	AR(3)TARCH(0,2)	-9.26%	78	-9.27%	78
SPEC(T=70)	13.08%	20	13.09%	20	AR(4)GARCH(2,2)	-7.74%	66	-7.73%	66	AR(4)TARCH(0,2)	-8.94%	72	-8.95%	72
SPEC(T=75)	12.54%	23	12.55%	23	AR(0)EGARCH(0,1)	-17.09%	100	-17.11%	100	AR(0)TARCH(1,1)	-1.73%	56	-1.75%	56
SPEC(T=80)	13.52%	18	13.53%	18	AR(1)EGARCH(0,1)	-16.15%	98	-16.16%	98	AR(1)TARCH(1,1)	-1.71%	55	-1.73%	55
MINIMUM	-32.93%	104	-33.02%	104	AR(2)EGARCH(0,1)	-16.91%	99	-16.92%	99	AR(2)TARCH(1,1)	-2.33%	57	-2.35%	57
MAXIMUM	-18.96%	103	-18.83%	103	AR(3)EGARCH(0,1)	-17.48%	101	-17.49%	101	AR(3)TARCH(1,1)	-1.59%	54	-1.61%	54
AVERAGE	13.63%	16	13.62%	16	AR(4)EGARCH(0,1)	-17.87%	102	-17.88%	102	AR(4)TARCH(1,1)	-1.47%	53	-1.49%	53
AR(0)GARCH(0,1)	-11.35%	90	-11.37%	90	AR(0)EGARCH(0,2)	-6.74%	62	-6.76%	62	AR(0)TARCH(1,2)	15.07%	10	15.06%	10
AR(1)GARCH(0,1)	-11.60%	92	-11.61%	92	AR(1)EGARCH(0,2)	-8.97%	73	-8.99%	73	AR(1)TARCH(1,2)	14.87%	12	14.86%	12
AR(2)GARCH(0,1)	-12.17%	93	-12.19%	93	AR(2)EGARCH(0,2)	-9.32%	83	-9.34%	83	AR(2)TARCH(1,2)	14.96%	11	14.95%	11
AR(3)GARCH(0,1)	-12.52%	95	-12.54%	95	AR(3)EGARCH(0,2)	-9.31%	80	-9.33%	82	AR(3)TARCH(1,2)	14.83%	13	14.83%	13
AR(4)GARCH(0,1)	-12.89%	97	-12.90%	97	AR(4)EGARCH(0,2)	-9.25%	77	-9.27%	79	AR(4)TARCH(1,2)	15.90%	8	15.89%	8
AR(0)GARCH(0,2)	-8.73%	71	-8.75%	71	AR(0)EGARCH(1,1)	6.11%	34	6.11%	34	AR(0)TARCH(2,1)	4.52%	39	4.51%	39
AR(1)GARCH(0,2)	-11.03%	88	-11.04%	88	AR(1)EGARCH(1,1)	1.60%	50	1.61%	50	AR(1)TARCH(2,1)	7.89%	30	7.88%	30
AR(2)GARCH(0,2)	-11.46%	91	-11.47%	91	AR(2)EGARCH(1,1)	2.61%	46	2.61%	46	AR(2)TARCH(2,1)	8.60%	28	8.59%	28
AR(3)GARCH(0,2)	-12.30%	94	-12.32%	94	AR(3)EGARCH(1,1)	1.10%	52	1.10%	52	AR(3)TARCH(2,1)	5.79%	36	5.78%	36
AR(4)GARCH(0,2)	-12.61%	96	-12.63%	96	AR(4)EGARCH(1,1)	1.30%	51	1.30%	51	AR(4)TARCH(2,1)	7.08%	31	7.07%	31
AR(0)GARCH(1,1)	-2.90%	58	-2.90%	58	AR(0)EGARCH(1,2)	6.91%	32	6.91%	32	AR(0)TARCH(2,2)	9.48%	26	9.47%	26
AR(1)GARCH(1,1)	-9.32%	81	-9.31%	81	AR(1)EGARCH(1,2)	4.45%	40	4.46%	40	AR(1)TARCH(2,2)	12.84%	22	12.84%	22
AR(2)GARCH(1,1)	-8.23%	70	-8.21%	70	AR(2)EGARCH(1,2)	5.73%	37	5.74%	37	AR(2)TARCH(2,2)	10.21%	24	10.21%	24
AR(3)GARCH(1,1)	-7.68%	65	-7.67%	64	AR(3)EGARCH(1,2)	3.89%	43	3.89%	43	AR(3)TARCH(2,2)	9.72%	25	9.71%	25
AR(4)GARCH(1,1)	-9.32%	82	-9.31%	80	AR(4)EGARCH(1,2)	3.77%	44	3.77%	44	AR(4)TARCH(2,2)	9.38%	27	9.38%	27
AR(0)GARCH(1,2)	-4.16%	61	-4.15%	61	AR(0)EGARCH(2,1)	6.46%	33	6.46%	33					

Chapter 8
The Distribution of the Minimum Component of a Vector Having a Multivariate Gamma Function

8.1. Introduction

Numerous methods of model evaluation have been derived in the statistical literature. Most of them are based on measuring the ability of the models to fit in the data (i.e. Akaike 1973 and Schwarz 1978). In the case where we are interested in evaluating a model's forecasting ability, a loss function, which takes into consideration the characteristics of the predicting variable as well as the utility of the forecasts, is mainly constructed. For example, loss functions that are robust to heteroscedasticity are used by Andersen et al. (1999), Heynen and Kat (1994) and Pagan and Schwert (1990) for evaluating the predictive ability of volatility forecasting models because of high non-linearity of the variable under investigation. Engle et al. (1993), Granger (2001), Granger and Pesaran (2000) and West et al. (1993), among others, defined loss functions that evaluated the models according to their predictions' utility. As Hendry and Clements (2001) noted "it seems natural that a stock broker measures the value of forecasts by their monetary return, not their mean squared error". Although loss functions are measures of accuracy, which are constructed based upon the goals of their particular application, in the majority of the cases, their statistical properties are unknown. The superiority of a loss function against others cannot be judged by a statistical-theoretical ground but just from their empirical motivations.

Even though we cannot investigate the statistical properties of a loss function, we are capable to use it for measuring whether two forecasts have statistically equal forecasting accuracy. Diebold and Mariano (1995) derived a test of the null hypothesis of no difference in the accuracy of two competing forecasts. In particular, for $\left\{\hat{y}_t^{(m_1)}\right\}_{t=1}^{T}$ and $\left\{\hat{y}_t^{(m_2)}\right\}_{t=1}^{T}$ denoting two forecasts of the variable under investigation $\left\{y_t\right\}_{t=1}^{T}$, Diebold and Mariano considered the time-t loss associated with forecast m_i, for $i=1,2$, to be an arbitrary function of realization and prediction, $f\left(y_t, \hat{y}_t^{(m_i)}\right)$. The null hypothesis of equal forecast accuracy is $E\left(f\left(y_t, \hat{y}_t^{(m_1)}\right)\right) = E\left(f\left(y_t, \hat{y}_t^{(m_2)}\right)\right)$.

On the other hand, Xekalaki et al. (2003), based on a loss function, derived a two-model hypothesis testing procedure to test whether the models have equal ability in predicting the dependent variable of a regression model. In the 4th chapter, the hypothesis test was extended in comparing the ability of two models to forecast the conditional variance of ARCH models. Their hypothesis test is based on the sum of squared standardized one-step-ahead prediction errors, $X_{m_i} = 2^{-1}\sum_{t=1}^{T}\frac{\left(y_t - \hat{y}_{t|t-1}^{(m_i)}\right)^2}{V\left(\hat{y}_{t|t-1}^{(m_i)}\right)}$. The loss function, X_{m_i}, is asymptotically chi-square distributed, whereas the ratio X_{m_2}/X_{m_1} follows the CGR distribution. The null hypothesis, that models m_1 and m_2 have equal predictability against the alternative that model m_1 has a better predictive ability, is rejected at the $100p\%$ level of significance if X_{m_2}/X_{m_1} is greater than the $100(1-p)$ percentile of the CGR distribution. Moreover, the SPEC algorithm of ARCH model selection was considered based on the former hypothesis test. According to the SPEC model selection algorithm, the model, which, among a set of n models, m_i, $i = 1,2,...,n$, has the lowest sum of squared standardized one-step-ahead prediction errors, is considered as having a superior ability to predict the conditional variance of the dependent variable. In the previous chapter, the performance of the SPEC algorithm was evaluated by comparing different volatility forecasts in option pricing through the simulation of an options market and concluded that traders, who base their selection of an ARCH model on the SPEC algorithm, achieve higher profits than those, who use other methods of model selection.

In the present chapter, the exact from of the distribution of the loss function for the model with the lowest value, $X_{(1)} \equiv \min\left(X_{m_1}, X_{m_2}, ..., X_{m_n}\right)$, is determined in order to derive the statistical properties of $X_{(1)}$ on which the SPEC model selection algorithm is based. In section 8.2, the cumulative distribution function of $X_{(1)}$, named minimum multivariate gamma (MMG) distribution, is derived while section 8.3 provides the percentage points of the tri-variate version. Based on the MMG distribution function, in the 8.4 section, a testing procedure is constructed where the null hypothesis that n available models are of equivalent predictive ability is tested against the alternative hypothesis that the model $m_{(1)}$ with the lowest value of the loss function X_{m_i} has the highest predictive ability. According to authors' knowledge, the hypothesis tests, which exist in the forecasting literature, compare the ability of two models in producing accurate predictions. The advantage of the MMG hypothesis test is that it takes into consideration the forecasting ability of n $(n \geq 2)$ candidate models in order to infer whether the model $m_{(1)}$ has the highest predictive ability. In section 8.5 the suggested hypothesis test is applied using return data for the Athens Stock Exchange (ASE) index over the period August 30th, 1993 to November 4th and a short discussion is provided in section 8.6.

8.2. The Distribution of the Minimum Component of a Vector $\mathbf{X} = (X_1, X_2, ..., X_n)$ Having a Multivariate Gamma Distribution

In the sequel, two theorems are provided, which are subsequently used for the derivation of the cumulative function of the MMG distribution. Theorem 1 defines the cumulative distribution function of $X_{(1)}$ under the assumption that $X_1, X_2, ..., X_n$ are identically but not independently distributed. Theorem 2 derives the cumulative distribution function of n random variables having Krishnamoorthy and Parthasarathy's (1951) multivariate gamma distribution. Finally, Lemma 3 combines the two theorems and develops the cumulative function of $X_{(1)}$ when $X_1, X_2, ..., X_n$ are multivariate gamma distributed.

8.2.1 Determining the Cumulative Function of the Minimum Component, $X_{(1)}$

Theorem 1:

Let $X_1, X_2, ..., X_n$ be non-negative, identically distributed random variables with distribution function $F_{X_i}(.) = F_{X_1}(.)$, for $i = 1,2,...,n$. Denote by $X_{(1)}, X_{(2)}, ..., X_{(n)}$ the same variables arranged in an ascending order. Let the joint distribution function of $(X_1, X_2, ..., X_n)$ be $F_{X_1, X_2, ..., X_n}(x_1, x_2, ..., x_n) = P(X_1 \leq x_1, X_2 \leq x_2, ..., X_n \leq x_n)$ and denote their joint probability density function by $f_{X_1,X_2,\dots,X_n}(x_1,x_2,\dots,x_n)$. Then, the cumulative distribution function of $X_{(1)} \equiv \min(X_1, X_2,\dots,X_n)$ is

$$F_{X_{(1)}}(x) = \sum_{j=1}^{n}(-1)^{j-1}\sum_{n\ j} F_{X_{i_1},X_{i_2},\dots,X_{i_j},}(x,x,\dots,x), \tag{8.2.1}$$

where $\sum_{n\ j}$ denotes summation over the set $\{i_k = i_{k-1}+1, i_{k-1}+2,\dots,(n\wedge n-j+k)\}$, for $k = 1,2,\dots,j$, ($i_0 = 0$ [1]), i.e., $\sum_{n\ j} x_{i_1 i_2 \dots i_j} = \sum_{i_1=1}^{n\wedge n-j+1}\ \sum_{i_2=i_1+1}^{n\wedge n-j+2} \dots \sum_{i_k=i_{k-1}+1}^{n\wedge n-j+k} \dots \sum_{i_{j-1}=i_{j-2}+1}^{n\wedge n-1}\ \sum_{i_j=i_{j-1}+1}^{n} x_{i_1 i_2 \dots i_j}$.

Proof:

We have by the definition of the cumulative distribution function of a random variable that:

$$F_{X_{(1)}}(x) = P(X_{(1)} \le x) = 1 - P(X_{(1)} > x) = 1 - P(X_1 > x, X_2 > x,\dots,X_n > x) =$$

$$= 1 - \int_x^\infty \dots \int_x^\infty \left(\int_x^\infty \left(\int_x^\infty f_{X_1,X_2,X_3,\dots,X_n}(x_1,x_2,x_3,\dots,x_n)\, dx_1 \right) dx_2 \right) dx_3 \dots dx_n =$$

$$= 1 - \int_x^\infty \dots \int_x^\infty \left(\int_x^\infty \left(\int_{-\infty}^\infty f_{X_1,X_2,X_3,\dots,X_n}(x_1,x_2,x_3,\dots,x_n)\, dx_1 - \int_{-\infty}^x f_{X_1,X_2,X_3,\dots,X_n}(x_1,x_2,x_3,\dots,x_n)\, dx_1 \right) dx_2 \right) dx_3 \dots dx_n =$$

$$= 1 - \int_x^\infty \dots \int_x^\infty \left(\left(\int_{-\infty}^\infty \int_{-\infty}^\infty f_{X_1,X_2,X_3,\dots,X_n}(x_1,x_2,x_3,\dots,x_n)\, dx_1 dx_2 - \int_{-\infty}^\infty \int_{-\infty}^x f_{X_1,X_2,X_3,\dots,X_n}(x_1,x_2,x_3,\dots,x_n)\, dx_1 dx_2 \right.\right.$$

$$\left.\left. - \int_{-\infty}^x \int_{-\infty}^\infty f_{X_1,X_2,X_3,\dots,X_n}(x_1,x_2,x_3,\dots,x_n)\, dx_1 dx_2 + \int_{-\infty}^x \int_{-\infty}^x f_{X_1,X_2,X_3,\dots,X_n}(x_1,x_2,x_3,\dots,x_n)\, dx_1 dx_2 \right)\right) dx_3 \dots dx_n = \dots$$

$$\dots = -(-1)\sum_{i=1}^n F_{X_i}(x) - (-1)^2 \sum_{n\ 2} F_{X_{i_1}X_{i_2}}(x,x) - (-1)^3 \sum_{n\ 3} F_{X_{i_1}X_{i_2}X_{i_3}}(x,x,x) - (-1)^4 \sum_{n\ 4} F_{X_{i_1}X_{i_2}X_{i_3}X_{i_4}}(x,x,x,x)$$

$$- \dots - (-1)^{n-1} \sum_{n\ n-1} F_{X_{i_1}X_{i_2}X_{i_3}X_{i_4}\dots X_{i_{n-1}}}(x,x,x,x,\dots,x) - (-1)^n F_{X_1X_2X_3X_4\dots X_n}(x,x,x,x,\dots,x),$$

where $\sum_{n\ j}$ denotes summation over the set $\{i_k = i_{k-1}+1, i_{k-1}+2,\dots,(n\wedge n-j+k)\}$, for $k = 1,2,\dots,j$, ($i_0 = 0$), i.e., $\sum_{n\ j} x_{i_1 i_2 \dots i_j} = \sum_{i_1=1}^{n\wedge n-j+1}\ \sum_{i_2=i_1+1}^{n\wedge n-j+2} \dots \sum_{i_k=i_{k-1}+1}^{n\wedge n-j+k} \dots \sum_{i_{j-1}=i_{j-2}+1}^{n\wedge n-1}\ \sum_{i_j=i_{j-1}+1}^{n} x_{i_1 i_2 \dots i_j}$.

Noting that $\sum_{i=1}^n F_{X_i}(x) = \sum_{n\ 1} F_{X_i}(x)$ and $F_{X_1X_2\dots X_n}(x,x,\dots,x) = \sum_{n\ n} F_{X_1X_2\dots X_n}(x,x,\dots,x)$, the above relationship leads to (8.2.1) and, hence, to the result.

[1] $(n\wedge n-j+k) \equiv \min(n, n-j+k)$.

8.2.2 Determining the Cumulative Function of the Multivariate Gamma Distribution, $F_{X_1,\ldots,X_n}(x,\ldots,x)$

Theorem 2:

Suppose that $(X_1, X_2, \ldots, X_n)$ are n random variables having Krishnamoorthy and Parthasarathy's (1951) multivariate gamma distribution with joint probability density function given by

$$f(x_1, x_2, \ldots, x_n) = \begin{cases} x_1^{a-1}e^{-x_1} \big/ \Gamma(a) & , n = 1 \\ \prod_{i=1}^{n} f(x_i) \sum_{r=0}^{\infty} \frac{a_{(r)}}{r!} \left\{ \sum_{j=2}^{n} \sum_{j} C_{i_1 i_2 \ldots i_j} \prod_{k=1}^{j} \frac{L(x_{i_k}, a)}{a} \right\}^r & , n \geq 2, \end{cases} \tag{8.2.2}$$

where $f(x_i)$ denotes the marginal density of X_i, $i = 1, \ldots, n$, for $x_i \geq 0$, $a > 0$,

$$C_{12\ldots n} = -(-1)^n \begin{vmatrix} 0 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{12} & 0 & \cdots & \rho_{2n} \\ \cdots & \cdots & & \cdots \\ \rho_{1n} & \rho_{2n} & \cdots & 0 \end{vmatrix}$$

and

$$\left\{ \frac{L(x_k, a)}{a} \right\}^r = \frac{L_r(x_k, a)}{a_{(r)}} = \frac{(-1)^r \frac{d^r}{dx_k^r}\left(x_k^{r+a-1}e^{-x_k}\right)}{a_{(r)} x_k^{a-1} e^{-x_k}}. \tag{8.2.3}$$

Let us denote the joint distribution function of $(X_1, X_2, \ldots, X_n)$ by $F_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n)$. Then,

$$F_{X_1,\ldots,X_n}(x,\ldots,x) = \int_0^x \ldots \int_0^x f(x_1,\ldots,x_n)\,dx_1 \ldots dx_n =$$

$$= \sum_{0 \leq r_1, r_2, \ldots, r_{12\ldots n} < \infty} \left(a_{\left(\sum_{j=1}^{n} \sum_{n} \sum_{j} r_{i_1 i_2 \ldots i_j}\right)} \prod_{j=1}^{n} \prod_{n} \prod_{j} \frac{C_{i_1 i_2 \ldots i_j}^{r_{i_1 i_2 \ldots i_j}}}{r_{i_1 i_2 \ldots i_j}!} \prod_{k=1}^{n} \frac{I_{\sum_{j=1}^{n} \sum_k r_{i_1 i_2 \ldots i_j}}(x, a)}{a_{\left(\sum_{j=1}^{n} \sum_k r_{i_1 i_2 \ldots i_j}\right)}} \right),$$

where $I_r(x, a) = \int_0^x L_r(x_k, a) f(x_k)\,dx_k$ and can be evaluated for $r = 0,1,2,3,4$ by relationships (8.2.13) to (8.2.17) and for $r > 5$ by relationship (8.2.18) given below.

Proof:

Defining $C_{i_1} = 0$, we can write (8.2.2) as

$$f(x_1, x_2, ..., x_n) = \prod_{i=1}^{n} f(x_i) \sum_{r=0}^{\infty} \frac{a_{(r)}}{r!} \left\{ \sum_{j=1}^{n} \sum_{n \quad j} C_{i_1 i_2 ... i_j} \prod_{k=1}^{j} \frac{L(x_{i_k}, a)}{a} \right\}^r, n \geq 1.$$

Making use of the expansion $(1-x)^{-a} = \sum_{r=0}^{\infty} \frac{a_{(r)}}{r!} x^r$, we obtain

$$f(x_1, x_2, ..., x_n) = \prod_{i=1}^{n} f(x_i) \left(1 - \sum_{j=1}^{n} \sum_{n \quad j} C_{i_1 i_2 ... i_j} \prod_{k=1}^{j} \frac{L(x_{i_k}, a)}{a} \right)^{-a}.$$

Noting that, for any $\left(1 - \sum_{i=1}^{n} x_i\right)^{-a} \equiv \sum_{r_1 r_2 ... r_n} a_{\left(\sum_{i=1}^{n} r_i\right)} \prod_{i=1}^{n} \frac{x_i^{r_i}}{r_i!}$ the above expression reduces to

$$f(x_1, x_2, ..., x_n) = \prod_{i=1}^{n} f(x_i) \sum_{0 \leq r_1, r_2, ..., r_{12...n} < \infty} \left(a_{\left(\sum_{j=1}^{n} \sum_{n \quad j} r_{i_1 i_2 ... i_j} \right)} \prod_{j=1}^{n} \prod_{n \quad j} \frac{1}{r_{i_1 i_2 ... i_j}!} \left\{ C_{i_1 i_2 ... i_j} \prod_{k=1}^{j} \frac{L(x_{i_k}, a)}{a} \right\}^{r_{i_1 i_2 ... i_j}} \right), \tag{8.2.4}$$

where $\prod_{n \quad j}$ denotes the product $\prod_{n \quad j} x_{i_1 i_2 ... i_j} = \prod_{i_1 = 1}^{n \wedge n - j + 1} \prod_{i_2 = i_1 + 1}^{n \wedge n - j + 2} ... \prod_{i_m = i_{m-1} + 1}^{n \wedge n - j + m} ... \prod_{i_j = i_{j-1} + 1}^{n} x_{i_1 i_2 ... i_j}$ and by defining $r_{i_1} = 0$. As the $\sum_{n \quad j} r_{i_1 i_2 ... i_j}$ has $\binom{n}{j}$ terms, then the $\sum_{j=1}^{n} \sum_{n \quad j} r_{i_1 i_2 ... i_j}$ has $\sum_{j=1}^{n} \binom{n}{j} = 2^n - 1$ terms[2]. Rewriting $\prod_{i=1}^{n} f(x_i)$ as $\prod_{j=1}^{n} f(x_j)$ and using Lemma 1, we have

$$f(x_1, x_2, ..., x_n) = \sum_{0 \leq r_1, r_2, ..., r_{12...n} < \infty} \left(a_{\left(\sum_{j=1}^{n} \sum_{n \quad j} r_{i_1 i_2 ... i_j} \right)} \left(\prod_{j=1}^{n} \prod_{n \quad j} \frac{C_{i_1 i_2 ... i_j}^{r_{i_1 i_2 ... i_j}}}{r_{i_1 i_2 ... i_j}!} \right) \left(\prod_{k=1}^{n} f(x_k) \left\{ \frac{L(x_k, a)}{a} \right\}^{\sum_{j=1}^{n} \Sigma_k r_{i_1 i_2 ... i_j}} \right) \right),$$

where Σ_k denotes summation over the set:

$\{i_1 = 1, 2, ..., k; i_m = i_{m-1} + 1, i_{m-1} + 2, ..., n - j + m, m = 2, 3, ..., j\}$.

Finally using relationship (8.2.3), we have

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[2] There are $2^n - 1$ terms in the summation $\sum_{0 \leq r_1, r_2, ..., r_{12...n} < \infty}$, i.e., for $n = 3$, the terms correspond to the indices $r_1, r_2, r_3, r_{12}, r_{13}, r_{23}, r_{123}$.

$$f(x_1, x_2, ..., x_n) = \sum_{0 \le r_1, r_2, ..., r_{12...n} < \infty} \left(a_{\left(\sum_{j=1}^{n}\sum_{n}\sum_{j} r_{i_1 i_2 ... i_j}\right)} \left(\prod_{j=1}^{n} \prod_{n} \prod_{j} \frac{C_{i_1 i_2 ... i_j}^{r_{i_1 i_2 ... i_j}}}{r_{i_1 i_2 ... i_j}!} \right) \left(\prod_{k=1}^{n} f(x_k) \frac{L_{\sum_{j=1}^{n}\sum_k r_{i_1 i_2 ... i_j}}(x_k, a)}{a_{\left(\sum_{j=1}^{n}\sum_k r_{i_1 i_2 ... i_j}\right)}} \right) \right). \quad (8.2.5)$$

Thus,

$$F_{X_1, ..., X_n}(x_1, x_2, ..., x_n) = \int_0^{x_1} ... \int_0^{x_n} f(t_1, ..., t_n)\, dt_1 .... dt_n$$

$$= \int_0^{x_1} ... \int_0^{x_n} \sum_{0 \le r_1, r_2, ..., r_{12...n} < \infty} \left(a_{\left(\sum_{j=1}^{n}\sum_{n}\sum_{j} r_{i_1 i_2 ... i_j}\right)} \prod_{j=1}^{n} \prod_{n} \prod_{j} \frac{C_{i_1 i_2 ... i_j}^{r_{i_1 i_2 ... i_j}}}{r_{i_1 i_2 ... i_j}!} \prod_{k=1}^{n} f(t_k) \frac{L_{\sum_{j=1}^{n}\sum_k r_{i_1 i_2 ... i_j}}(t_k, a)}{a_{\left(\sum_{j=1}^{n}\sum_k r_{i_1 i_2 ... i_j}\right)}} \right) dt_1 .... dt_n =$$

$$= \sum_{0 \le r_1, r_2, ..., r_{12...n} < \infty} \left(a_{\left(\sum_{j=1}^{n}\sum_{n}\sum_{j} r_{i_1 i_2 ... i_j}\right)} \prod_{j=1}^{n} \prod_{n} \prod_{j} \frac{C_{i_1 i_2 ... i_j}^{r_{i_1 i_2 ... i_j}}}{r_{i_1 i_2 ... i_j}!} \prod_{k=1}^{n} \frac{\int_0^{x_n} f(t_k) L_{\sum_{j=1}^{n}\sum_k r_{i_1 i_2 ... i_j}}(t_k, a) dt_k}{a_{\left(\sum_{j=1}^{n}\sum_k r_{i_1 i_2 ... i_j}\right)}} \right) =$$

$$= \sum_{0 \le r_1, r_2, ..., r_{12...n} < \infty} \left(a_{\left(\sum_{j=1}^{n}\sum_{n}\sum_{j} r_{i_1 i_2 ... i_j}\right)} \left(\prod_{j=1}^{n} \prod_{n} \prod_{j} \frac{C_{i_1 i_2 ... i_j}^{r_{i_1 i_2 ... i_j}}}{r_{i_1 i_2 ... i_j}!} \right) \left(\prod_{k=1}^{n} \frac{I_{\sum_{j=1}^{n}\sum_k r_{i_1 i_2 ... i_j}}(x_k, a)}{a_{\left(\sum_{j=1}^{n}\sum_k r_{i_1 i_2 ... i_j}\right)}} \right) \right), \quad (8.2.6)$$

where $I_r(x_k, a) = \int_0^{x_k} f(t_k) L_r(t_k, a)\, dt_k$ with $\Gamma(a) = \int_0^{\infty} e^{-t} t^{a-1} dt$ and $\Gamma_x(a)$ denoting the incomplete gamma function defined by $\Gamma_x(a) = \int_x^{\infty} e^{-t} t^{a-1} dt$, $x > 0$.

For $r = 0,1,2,...$, the Laguerre (1879) Polynomials, $L_r(x_k, a) = \dfrac{(-1)^r \dfrac{d^r}{dx_k^r}\left(x_k^{r+a-1} e^{-x_k}\right)}{x_k^{a-1} e^{-x_k}}$, are computed as

$$L_0(x_k, a) \equiv L_0^{(a-1)}(x_k) = 1 \quad (8.2.7)$$

$$L_1(x_k, a) \equiv (-1) L_1^{(a-1)}(x_k) = x_k - a \quad (8.2.8)$$

$$L_2(x_k, a) \equiv (-1)^2 2 L_2^{(a-1)}(x_k) = a + a^2 - 2x_k - 2a x_k + x_k^2 \quad (8.2.9)$$

$$L_3(x_k, a) \equiv (-1)^3 3! L_3^{(a-1)}(x_k) = -2a - 3a^2 - a^3 + 6x_k + 9a x_k + 3a^2 x_k - 6x_k^2 - 3a x_k^2 + x_k^3 \quad (8.2.10)$$

$$\begin{aligned} L_4(x_k, a) &\equiv (-1)^4 4! L_4^{(a-1)}(x_k) = 6a + 11a^2 + 6a^3 + a^4 - 24x_k - 44a x_k - 24a^2 x_k \\ &- 4a^3 x_k + 36x_k^2 + 30a x_k^2 + 6a^2 x_k^2 - 12x_k^3 - 4a x_k^3 + x_k^4 \end{aligned} \quad (8.2.11)$$

In order to compute Laguerre Polynomials of higher order we can use the following recursive formula,

$$L_r(x_k,a)=(-1)^r\left((a+2r-2-x_k)(-1)^{r-1}L_{r-1}(x_k,a)-(r-1)(a+r-2)(-1)^{r-2}L_{r-2}(x_k,a)\right). \tag{8.2.12}$$

As concerns the integral $I_r(x_k,a)=\int_0^{x_k} f(t)L_r(t,a)\,dt$, for $r=0,1,2,\ldots$, is computed as

$$I_0(x_k,a)=1-\frac{\Gamma_{x_k}(a)}{\Gamma(a)} \tag{8.2.13}$$

$$I_1(x_k,a)=\frac{a\Gamma_{x_k}(a)-\Gamma_{x_k}(a+1)}{\Gamma(a)} \tag{8.2.14}$$

$$I_2(x_k,a)=-\frac{(1+a)\left(a\Gamma_{x_k}(a)-2\Gamma_{x_k}(a+1)\right)+\Gamma_{x_k}(a+2)}{\Gamma(a)} \tag{8.2.15}$$

$$I_3(x_k,a)=\frac{-(2+a)\left((1+a)\left(a\Gamma_{x_k}(a)-3\Gamma_{x_k}(a+1)\right)+3\Gamma_{x_k}(a+2)\right)+\Gamma_{x_k}(a+3)}{\Gamma(a)} \tag{8.2.16}$$

$$I_4(x_k,a)=\frac{(3+a)\left(-(2+a)\left((1+a)\left(a\Gamma_{x_k}(a)-4\Gamma_{x_k}(a+1)\right)+6\Gamma_{x_k}(a+2)\right)+4\Gamma_{x_k}(a+3)\right)-\Gamma_{x_k}(a+4)}{\Gamma(a)} \tag{8.2.17}$$

According to Lemma 2, a generalized form of $I_r(x_k,a)$ is the following

$$I_r(x_k,a)=\int_0^{x_k}\frac{t^{a-1}e^{-t}L_r(t,a)}{\Gamma(a)}dt=\frac{(-1)^r a_{(r)}}{\Gamma(a)}\sum_{l=0}^{r}\frac{(-r)_{(l)}}{a_{(l)}l!}\left(\Gamma(a+l)-\Gamma_{x_k}(a+l)\right). \tag{8.2.18}$$

Thus, the form of $F_{X_1,\ldots,X_n}(x,\ldots,x)$ is

$$\sum_{0\le r_1,r_2,\ldots,r_{12\ldots n}<\infty}\left(a_{\left(\sum_{j=1}^{n}\sum_{n}\sum_{j} r_{i_1i_2\ldots i_j}\right)}\left(\prod_{j=1}^{n}\prod_{n}\prod_{j}\frac{C_{i_1i_2\ldots i_j}^{r_{i_1i_2\ldots i_j}}}{r_{i_1i_2\ldots i_j}!}\right)\left(\prod_{k=1}^{n}\frac{I_{\sum_{j=1}^{n}\Sigma_k r_{i_1i_2\ldots i_j}}(x,a)}{a_{\left(\sum_{j=1}^{n}\Sigma_k r_{i_1i_2\ldots i_j}\right)}}\right)\right), \tag{8.2.19}$$

with $I_r(x,a)$ as given by relationships (8.2.13) to (8.2.18).

<u>Lemma 1:</u>

$$\prod_{j=1}^{n} f(x_j)\prod_{n}\prod_{j}\frac{1}{r_{i_1i_2\ldots i_j}!}\left\{C_{i_1i_2\ldots i_j}\prod_{k=1}^{j}\frac{L(x_{i_k},a)}{a}\right\}^{r_{i_1i_2\ldots i_j}}=\left(\prod_{j=1}^{n}\prod_{n}\prod_{j}\frac{C_{i_1i_2\ldots i_j}^{r_{i_1i_2\ldots i_j}}}{r_{i_1i_2\ldots i_j}!}\right)\left(\prod_{k=1}^{n} f(x_k)\left\{\frac{L(x_k,a)}{a}\right\}^{\sum_{j=1}^{n}\Sigma_k r_{i_1i_2\ldots i_j}}\right)$$

,where $\Sigma_k r_{i_1i_2\ldots i_j}=\sum_{i_1=1}^{k}\sum_{i_2=i_1+1}^{n-j+2}\cdots\sum_{i_h=i_{h-1}+1}^{n-j+h}\cdots\sum_{i_j=i_{j-1}+1}^{n} r_{i_1i_2\ldots i_j}$.

Proof:

For simplicity, let us denote $\dfrac{C_{i_1 i_2 \ldots i_j}^{r_{i_1 i_2 \ldots i_j}}}{r_{i_1 i_2 \ldots i_j}!}\left\{\prod_{k=1}^{j}\dfrac{L\left(x_{i_k},a\right)}{a}\right\}^{r_{i_1 i_2 \ldots i_j}} \equiv H_{r_{i_1 i_2 \ldots i_j}}\left(x_{i_1},x_{i_2},\ldots,x_{i_j};a\right)$. So

$$\prod_{j=1}^{n} f\left(x_j\right)\prod_{n\ j}\frac{1}{r_{i_1 i_2 \ldots i_j}!}\left\{C_{i_1 i_2 \ldots i_j}\prod_{k=1}^{j}\frac{L\left(x_{i_k},a\right)}{a}\right\}^{r_{i_1 i_2 \ldots i_j}} = \prod_{j=1}^{n} f\left(x_j\right)\prod_{n\ j}\frac{C_{i_1 i_2 \ldots i_j}^{r_{i_1 i_2 \ldots i_j}}}{r_{i_1 i_2 \ldots i_j}!}\left\{\prod_{k=1}^{j}\frac{L\left(x_{i_k},a\right)}{a}\right\}^{r_{i_1 i_2 \ldots i_j}}$$

$$= \prod_{j=1}^{n} f\left(x_j\right)\prod_{n\ j} H_{r_{i_1 i_2 \ldots i_j}}\left(x_{i_1},x_{i_2},\ldots,x_{i_j};a\right) =$$

$$= f\left(x_1\right)\prod_{n\ 1} H_{r_{i_1}}\left(x_{i_1};a\right) \cdot f\left(x_2\right)\prod_{n\ 2} H_{r_{i_1 i_2}}\left(x_{i_1},x_{i_2};a\right)\cdot\ldots\cdot f\left(x_n\right)\prod_{n\ n} H_{r_{i_1 i_2 \ldots i_n}}\left(x_{i_1},x_{i_2},\ldots,x_{i_n};a\right)$$

$$= f\left(x_1\right)\prod_{i_1=1}^{n\wedge n-1+1} H_{r_{i_1}}\left(x_{i_1};a\right) \cdot f\left(x_2\right)\prod_{i_1=1}^{n\wedge n-1+1}\ \prod_{i_2=i_1+1}^{n\wedge n-2+2} H_{r_{i_1 i_2}}\left(x_{i_1},x_{i_2};a\right)\cdot\ldots$$

$$\cdot f\left(x_n\right)\prod_{i_1=1}^{n\wedge n-n+1}\ \prod_{i_2=i_1+11}^{n\wedge n-n+2}\cdots\prod_{i_n=i_{n-1}+1}^{n} H_{r_{i_1 i_2 \ldots i_n}}\left(x_{i_1},x_{i_2},\ldots,x_{i_n};a\right) =$$

$$= \begin{pmatrix}
f\left(x_1\right) & H_{r_1}\left(x_1;a\right) & H_{r_2}\left(x_2;a\right) & H_{r_3}\left(x_3;a\right) & H_{r_4}\left(x_4;a\right) & . \ . \ . & H_{r_n}\left(x_n;a\right) \\
f\left(x_2\right) & & H_{r_{12}}\left(x_1,x_2;a\right) & H_{r_{13}}\left(x_1,x_3;a\right) & H_{r_{14}}\left(x_1,x_4;a\right) & . \ . \ . & H_{r_{1n}}\left(x_1,x_n;a\right) \\
 & & & H_{r_{23}}\left(x_2,x_3;a\right) & H_{r_{24}}\left(x_2,x_4;a\right) & . \ . \ . & H_{r_{2n}}\left(x_2,x_n;a\right) \\
 & & & & H_{r_{34}}\left(x_3,x_4;a\right) & . \ . \ . & H_{r_{3n}}\left(x_3,x_n;a\right) \\
 & & & & & . \ . \ . & H_{r_{n-1n}}\left(x_{n-1},x_n;a\right) \\
 & & & & & . \ . \ . & . \\
 & & & & & . \ . \ . & . \\
 & & & & & . \ . \ . & . \\
f\left(x_n\right) & & & & & & H_{r_{12\ldots n}}\left(x_1,x_2,\ldots,x_n;a\right)
\end{pmatrix} =$$

$$= f\left(x_1\right)\left[\frac{C_1^{r_1}}{r_1!}\cdots\frac{C_n^{r_n}}{r_n!}\left\{\frac{L\left(x_1,a\right)}{a}\right\}^{r_1}\cdots\left\{\frac{L\left(x_n,a\right)}{a}\right\}^{r_n}\right].$$

$$f\left(x_2\right)\left[\begin{array}{l}
\dfrac{C_{12}^{r_{12}}}{r_{12}!}\cdots\dfrac{C_{1n}^{r_{1n}}}{r_{1n}!}\left\{\dfrac{L\left(x_1,a\right)}{a}\dfrac{L\left(x_2,a\right)}{a}\right\}^{r_{12}}\cdots\left\{\dfrac{L\left(x_1,a\right)}{a}\dfrac{L\left(x_n,a\right)}{a}\right\}^{r_{1n}}\cdot\ldots \\
\ldots\cdot\dfrac{C_{23}^{r_{23}}}{r_{23}!}\cdots\dfrac{C_{2n}^{r_{2n}}}{r_{2n}!}\left\{\dfrac{L\left(x_2,a\right)}{a}\dfrac{L\left(x_3,a\right)}{a}\right\}^{r_{23}}\cdots\left\{\dfrac{L\left(x_2,a\right)}{a}\dfrac{L\left(x_n,a\right)}{a}\right\}^{r_{2n}}\cdot\ldots \\
\ldots\cdot\dfrac{C_{n-1\ n}^{r_{n-1\ n}}}{r_{n-1\ n}!}\left\{\dfrac{L\left(x_{n-1},a\right)}{a}\dfrac{L\left(x_n,a\right)}{a}\right\}^{r_{n-1\ n}}
\end{array}\right]$$

$$
\cdots\cdot
f(x_n)\left[\frac{C_{12\ldots n}^{r_{12\ldots n}}}{r_{12\ldots n}!}\left\{\frac{L(x_1,a)}{a}\frac{L(x_2,a)}{a}\cdots\frac{L(x_n,a)}{a}\right\}^{r_{12\ldots n}}\right]
$$

$$
=\left(\prod_{j=1}^{n}\prod_{n}\prod_{j}\frac{C_{i_1 i_2\ldots i_j}^{r_{i_1 i_2\ldots i_j}}}{r_{i_1 i_2\ldots i_j}!}\right)\left(\prod_{k=1}^{n} f(x_k)\left\{\frac{L(x_k,a)}{a}\right\}^{\sum_{j=1}^{n}\sum_k r_{i_1 i_2\ldots i_j}}\right).
$$

Lemma 2:

The generalized form of $I_r(x_k,a)=\int_0^{x_k} f(t)L_r(t,a)\,dt$ is computed as:

$$
I_r(x_k,a)=\frac{(-1)^r a_{(r)}}{\Gamma(a)}\sum_{l=0}^{r}\frac{(-r)_{(l)}}{a_{(r)}l!}\left(\Gamma(a+l)-\Gamma_{x_k}(a+l)\right).
$$

Proof:

$I_r(x_k,a)=\int_0^{x_k}\frac{t^{a-1}e^{-t}}{\Gamma(a)}L_r(t,a)\,dt$, where $L_r(t,a)=(-1)^r\frac{d^r}{dt^r}\left(t^{r+a-1}e^{-t}\right)\Big/t^{a-1}e^{-t}$.

But, $\frac{d^r}{dt^r}\left(t^{r+a-1}e^{-t}\right)=\sum_{l=0}^{r}\binom{r}{l}\frac{d^{r-l}}{dt^{r-l}}t^{r+a-1}\frac{d^l}{dt^l}e^{-t}=\sum_{l=0}^{r}\frac{r^{(l)}}{l!}(r+a-1)^{(r-l)}t^{r+a-1+r+l}(-1)^l e^{-t}$,

where $A^{(B)}=A(A-1)\ldots(A-B+1)=(-1)^B(-A)(-A+1)\ldots(-A+B-1)=(-1)^B(-A)_{(B)}$.

Hence, $\frac{d^r}{dt^r}\left(t^{r+a-1}e^{-t}\right)=\sum_{l=0}^{r}\frac{(-1)^l r^{(l)}}{l!}(r+a-1)^{(r-l)}t^{a+l-1}e^{-t}$

and $L_r(t,a)=(-1)^r\sum_{l=0}^{r}\frac{(-1)^l r^{(l)}}{l!}(r+a-1)^{(r-l)}t^l=\sum_{l=0}^{r}\frac{(-r)_{(l)}(-t)^l}{l!}(-r-a+1)_{(r-l)}$.

As, $(A-B)_{(B+C)}=(A-B)_{(B)}(A)_{(C)}$, we find that:

$$
L_r(t,a)=\sum_{l=0}^{r}\frac{(-r)_{(l)}(-t)^l}{l!}(-r-a+1)_{(r)}(-a+1)_{(-l)}.
$$

Using i) $(-r-a+1)_{(r)}=(-1)^r a_{(r)}$ and ii) $(-a+1)_{(-l)}=(-1)^l\big/a_{(l)}$, we arrive at:

$$
L_r(t,a)=\sum_{l=0}^{r}\frac{(-r)_{(l)}(-1)^r a_{(r)}(-1)^l(-t)^l}{a_{(l)}l!}=(-1)^r a_{(r)}\sum_{l=0}^{r}\frac{(-r)_{(l)}t^l}{a_{(l)}l!}=(-1)^r a_{(r)}\,{}_1F_1(-r;a;t),
$$

where ${}_1F_1(-a;b;z)=\sum_{i=0}^{a}\frac{(-a)_{(i)}z^i}{b_{(i)}i!}$ is the Kummer (1836) confluent hypergeometric function.

Thus,

$$I_r(x_k,a)=\frac{(-1)^r a_{(r)}}{\Gamma(a)}\sum_{l=0}^{r}\left(\frac{(-r)_{(l)}}{a_{(l)}l!}\int_0^{x_k} t^{l+a-1}e^{-t}\,dt\right)=\frac{(-1)^r a_{(r)}}{\Gamma(a)}\sum_{l=0}^{r}\left(\frac{(-r)_{(l)}}{a_{(l)}l!}\left(\Gamma(a+l)-\Gamma_{x_k}(a+l)\right)\right).$$

8.2.3 Determining the Distribution of the Minimum Component $X_{(1)}$ of a Vector $(X_1,X_2,...,X_n)$ Having a Multivariate Gamma Distribution

__*Lemma 3:*__

Suppose that $(X_1,X_2,...,X_n)$ are n random variables having Krishnamoorthy and Parthasarathy's (1951) multivariate gamma distribution with cumulative distribution function given by (8.2.19) and parameters a and $C_{12...n}$. Then, the cumulative distribution function of $X_{(1)}\equiv\min(X_1,X_2,...,X_n)$ is computed in (8.2.20).

*Proof:*

Combining (8.2.1) with (8.2.19), the form of $F_{X_{(1)}}(x;a,C_{12...n})$, the MMG cumulative distribution function is,

$$F_{X_{(1)}}(x;a,C_{12...n})=\sum_{j=1}^{n}(-1)^{j-1}\sum_{n\ j}F_{X_{i_1},X_{i_2},...,X_{i_j},}(x,x,...,x)=$$

$$=\sum_{j=1}^{n}(-1)^{j-1}\sum_{n\ j}\ \sum_{0\le r_{i_1},r_{i_1i_2},...,r_{i_1i_2...i_j}<\infty}\left(a_{\left(\sum_{j=1}^{n}\sum_{n\ j}r_{i_1i_2...i_j}\right)}\left(\prod_{j=1}^{n}\prod_{n\ j}\frac{C_{i_1i_2...i_j}^{r_{i_1i_2...i_j}}}{r_{i_1i_2...i_j}!}\right)\prod_{k=1}^{n}\frac{I_{\sum_{j=1}^{n}\sum_k r_{i_1i_2...i_j}}(x,a)}{a_{\left(\sum_{j=1}^{n}\sum_k r_{i_1i_2...i_j}\right)}}\right).\qquad(8.2.20)$$

8.3. Tabulating the Distribution Function of $X_{(1)}$

In the sequel, we compute selected values of $F_{X_{(1)}}(\omega_{1-p};a,C_{12...n})$, for various values of $\omega_{1-p}\ge 0$, p, α and the non-diagonal elements of $C_{12...n}$ using the tri-variate version of Krishnamoorthy and Parthasarathy's distribution.

The form of $F_{X_1,\dots,X_n}(x,\dots,x)$, for $n = 1,2,3$ is computed as:

For $n = 1$, $F_{X_1}(x) = \sum_{0 \le r_1 < \infty} \left(\frac{C_1^{r_1}}{r_1!} I_{r_1}(x,a) \right)$. As we have defined $C_{i_1} = 0$ and $r_{i_1} = 0$, we take that $F_{X_1}(x) = I_0(x,a) = 1 - \frac{\Gamma_x(a)}{\Gamma(a)}$. For $n = 2$, $F_{X_1,X_2}(x,x) = \sum_{0 \le r_{12} < \infty} \left(a_{(r_{12})} \frac{C_{12}^{r_{12}}}{r_{12}!} \left(\frac{I_{r_{12}}(x,a)}{a_{(r_{12})}} \right)^2 \right)$

and for $n = 3$,

$$F_{X_1,X_2,X_3}(x,x,x) =$$

$$\sum_{0 \le r_{12}, r_{13}, r_{23}, r_{123} < \infty} \left(a_{(r_{12}+r_{13}+r_{23}+r_{123})} \frac{C_{12}^{r_{12}}}{r_{12}!} \frac{C_{13}^{r_{13}}}{r_{13}!} \frac{C_{23}^{r_{23}}}{r_{23}!} \frac{C_{123}^{r_{123}}}{r_{123}!} \frac{I_{r_{12}+r_{13}+r_{123}}(x,a)}{a_{(r_{12}+r_{13}+r_{123})}} \frac{I_{r_{12}+r_{23}+r_{123}}(x,a)}{a_{(r_{12}+r_{23}+r_{123})}} \frac{I_{r_{13}+r_{23}+r_{123}}(x,a)}{a_{(r_{13}+r_{23}+r_{123})}} \right).$$

From (8.2.20) and as $F_{X_i}(x) = F_{X_1}(x)$, $\forall i = 1,\dots,n$, we find that

$$F_{X_{(1)}}(x; a, C_{123}) = 3F_{X_1}(x) - \sum_{i_1=1}^{2} \sum_{i_2=2}^{3} F_{X_{i_1},X_{i_2}}(x,x) + F_{X_1,X_2,X_3}(x,x,x).$$

$F_{X_{(1)}}(\omega_{1-p}; a, C_{123})$ is computed for various values of $\omega_{1-p} \ge 0$, $0 < p < 1$, $a > 0$ and $0 \le \rho_{i,i+1} < 1$, for $i = 1,2$, the non-diagonal elements of C_{123}. Tables of selected values of $F_{X_{(1)}}(\omega_{1-p}; a, C_{123})$ as well as graphs depicting the cumulative density function of the tri-variate MMG are presented in the Appendix 8.

8.4. Hypothesis Testing for the True Value of the Minimum Component $X_{(1)}$

The distribution function of $X_{(1)} \equiv \min(X_{m_1}, X_{m_2}, \dots, X_{m_n})$, when $X_{m_1}, X_{m_2}, \dots, X_{m_n}$ are identically distributed random variables with Krishnamoorthy and Parthasarathy's multivariate gamma distribution function, can be used to compare the predictability of a set of models.

Let us assume that we are interested in examining the ability of m_i, for $i = 1,2,\dots,n$, ARCH models in predicting the one-step-ahead conditional variance of the dependent variable. Consider the ARCH process, $\{\varepsilon_t(\theta^{(m_i)})\}$, as innovations in a linear regression

$$
\begin{gathered}
y_t = x'^{(m_i)}_{t-1}\beta^{(m_i)} + \varepsilon_t\left(\theta^{(m_i)}\right) \\
\varepsilon_t\left(\theta^{(m_i)}\right) = z_t\sigma_t\left(\theta^{(m_i)}\right) \\
z_t \overset{i.i.d.}{\sim} N(0,1) \\
\sigma_t^2\left(\theta^{(m_i)}\right) = g\left(\left\{\varepsilon_{t-i}\left(\theta^{(m_i)}\right)\right\};\left\{\sigma_{t-j}\left(\theta^{(m_i)}\right)\right\};\left\{\upsilon^{(m_i)}_{t-k}\right\}\forall i \geq 1, \forall j \geq 1, \forall k \geq 1\right),
\end{gathered}
\tag{8.4.1}
$$

where $x_t^{(m_i)}$ is a vector of endogenous and exogenous explanatory variables included in the information set $I_t^{(m_i)}$, $\theta^{(m_i)}$ is a vector of unknown parameters, $\beta^{(m_i)}$ belongs to $\theta^{(m_i)}$, $\sigma_t\left(\theta^{(m_i)}\right)$ is a measurable function of the information set at time $t-1$ that represents the conditional variance of $\varepsilon_t\left(\theta^{(m_i)}\right)$, $\upsilon_t^{(m_i)}$ is a vector of predetermined variables included in $I_t^{(m_i)}$, and $g(.)$ is a linear or non-linear functional form. In the 4th chapter it was shown that under the assumption of constancy of parameters over time, $\left(\theta_1^{(m_i)}\right)=\left(\theta_2^{(m_i)}\right)=...=\left(\theta_T^{(m_i)}\right)=\left(\theta^{(m_i)}\right)$, the estimated standardized one-step-ahead prediction errors $\hat{z}^{(m_i)}_{t|t-1}, \hat{z}^{(m_i)}_{t+1|t},..., \hat{z}^{(m_i)}_{T|T-1}$ are asymptotically independently standard normally distributed,

$$
\hat{z}^{(m_i)}_{t|t-1} \equiv \left(y_t - \hat{y}^{(m_i)}_{t|t-1}\right)\hat{\sigma}^{-1(m_i)}_{t|t-1} \overset{i.i.d.}{\sim} N(0,1),
\tag{8.4.2}
$$

where $\hat{y}^{(m_i)}_{t|t-1} = x'^{(m_i)}_{t-1}\hat{\beta}^{(m_i)}_{t-1}$ and $\hat{\sigma}^{(m_i)}_{t|t-1}$ is the one-step-ahead conditional standard deviation whose computation depends on the functional form of the m_i ARCH process. Kibble (1941) showed that if two variables follow jointly the standard normal distribution, then the joint distribution of $\left(X_{m_1} \equiv 2^{-1}\sum_{t=1}^{T}\hat{z}^{2(m_1)}_{t|t-1}, X_{m_2} \equiv 2^{-1}\sum_{t=1}^{T}\hat{z}^{2(m_2)}_{t|t-1}\right)$ is the bivariate gamma. Krishnamoorthy and Parthasarathy extended Kibble's distribution to n variables. The null hypothesis of equivalent predictive ability of models m_i, for $i = 1,2,...,n$, can be tested against the alternative hypothesis that model $m_{(1)}$ (the model with the lowest half-sum of squared standardized one-step-ahead prediction errors) is superior in forecasting the one-step-ahead conditional variance:

H$_0$: Models m_i are of equivalent predictive ability,

versus,

H$_1$: Model $m_{(1)}$ has the highest predictive ability.

The null hypothesis is rejected if the test statistic $X_{(1)} \equiv \min_{m_i}\left(2^{-1}\sum_{t=1}^{T}\hat{z}^{2(m_i)}_{t|t-1}\right)$ exceeds the $100(1-p)$ percentile of the MMG cumulative distribution function, for $a = T/2$, $C_{12...n} = \begin{bmatrix} 1 & \rho_{12} & ... & \rho_{1n} \\ \rho_{1,2} & 1 & ... & \rho_{2n} \\ ... & ... & & ... \\ \rho_{1n} & \rho_{2n} & ... & 1 \end{bmatrix}$ and $\rho_{ij} \equiv Cor\left(\hat{z}^{(i)}_{t|t-1}, \hat{z}^{(j)}_{t|t-1}\right)$.

Obviously, this hypothesis test can be applied for comparing the ability of models in predicting the conditional mean. Consider the case that the m_i, for $i = 1,2,...,n$, models are in the form of a linear regression

$$\begin{aligned} y_t &= x_t'^{(m_i)}\beta^{(m_i)} + \varepsilon_t^{(m_i)} \\ \varepsilon_t^{(m_i)} &\overset{i.i.d.}{\sim} N\left(0, \sigma^{2(m_i)}\right), \end{aligned} \quad (8.4.3.)$$

where $\beta^{(m_i)}$ is a vector of $k^{(m_i)}$ unknown parameters to be estimated and $x_t^{(m_i)}$ is a vector of explanatory variables included in I_{t-1}. In such a case, the quantity X_{m_i} is computed as $X_{m_i} = 2^{-1}\sum_{t=1}^{T}\frac{\left(y_t - x_t'^{(m_i)}\hat{\beta}^{(m_i)}_{t-1}\right)^2}{V\left(\hat{y}^{(m_i)}_{t|t-1}\right)}$, where $\hat{\beta}^{(m_i)}_{t-1} = \left(\mathbf{X}'^{(m_i)}_{t-1}\mathbf{X}^{(m_i)}_{t-1}\right)^{-1}\left(\mathbf{X}'^{(m_i)}_{t-1}\mathbf{Y}_{t-1}\right)$ is the least square estimator of $\beta^{(m_i)}$ at time $t-1$,

$$V\left(\hat{y}^{(m_i)}_{t|t-1}\right) = \left(\mathbf{Y}_{t-1} - \mathbf{X}^{(m_i)}_{t-1}\hat{\beta}^{(m_i)}_{t-1}\right)'\left(\mathbf{Y}_{t-1} - \mathbf{X}^{(m_i)}_{t-1}\hat{\beta}^{(m_i)}_{t-1}\right)\left(1 + x_t^{(m_i)}\left(\mathbf{X}'^{(m_i)}_{t-1}\mathbf{X}^{(m_i)}_{t-1}\right)^{-1}x_t'^{(m_i)}\right)\left(l_{t-1} - k^{(m_i)}\right)^{-1},$$

$\mathbf{Y}_t$ is the $(l_t \times 1)$ vector of l_t observations on the dependent variable y_t, and $\mathbf{X}_t^{(m_i)}$ is the $\left(l_t \times k^{(m_i)}\right)$ matrix of $x_t^{(m_i)}$ explanatory variables, so that $\mathbf{X}_t^{(m_i)} = \begin{bmatrix} \mathbf{X}^{(m_i)}_{t-1} \\ x_t'^{(m_i)} \end{bmatrix}$, $\mathbf{Y}_t = \begin{bmatrix} \mathbf{Y}_{t-1} \\ y_t \end{bmatrix}$, $l_1 > k^{(m_i)}$, $l_{t+1} = l_t + 1$ and $\left|\mathbf{X}_t'^{(m_i)}\mathbf{X}_t^{(m_i)}\right| \neq 0$, $t = 1,2,...,T$.

8.5. An Empirical Application

A thorough investigation of the predictive ability of ARCH model with the lowest sum of standardized one-step-ahead prediction errors was conducted in chapters 5 to 7. In the present section we are not trying to evaluate the usage of selecting the model with the minimum value of the test statistic, $X_{(1)}$, but we illustrate the application of the MMG hypothesis test. The reader who is interested in the predictability of the ARCH models with $X_{(1)}$ is referred to the relevant chapters mentioned above.

For $y_t = \ln(P_t / P_{t-1})$ denoting the daily log-returns, where P_t is the ASE closing price at day t, we estimate three ARCH processes. More specifically, framework (8.4.1) is considered as a first order autoregressive process, AR(1), and the conditional variance is modelled as Glosten's et al. (1993) Threshold ARCH, or TARCH(p,q), process:

$$\begin{gathered} y_t = c_0 + c_1 y_{t-1} + \varepsilon_t \\ \varepsilon_t = z_t \sigma_t \\ z_t \overset{i.i.d.}{\sim} N(0,1) \\ \sigma_t^2 = a_0 + \sum_{i=1}^{q}\left(a_i \varepsilon_{t-i}^2\right) + \gamma \varepsilon_{t-1}^2 d\left(\varepsilon_{t-1} \le 0\right) + \sum_{i=1}^{p}\left(b_i \sigma_{t-i}^2\right), \end{gathered} \tag{8.5.1}$$

where $d(\varepsilon_t \le 0) = 1$ if $\varepsilon_t \le 0$, and $d(\varepsilon_t \le 0) = 0$ otherwise. The TARCH(p,q) model allows a response of conditional variance to news with different coefficients for good $(\varepsilon_{t-1} > 0)$ and bad $(\varepsilon_{t-1} \le 0)$ news. Therefore, good news has an impact of $\sum_{i=1}^{q} a_i$, while bad news has an impact of $\sum_{i=1}^{q}(a_i) + \gamma$. For $\gamma = 0$ the TARCH model reduces to Bollerslev's (1986) representation of the GARCH(p,q) model. We arbitrarily choose to estimate the GARCH(0,1), the GARCH(1,1) and the TARCH(2,2) models using the same data set of the 4th chapter, on the Athens Stock Exchange (ASE) index which cover the period from 30th of August 1993 to 4th November of 1996. A rolling sample of constant size equal to 500 is used and 300 one-day-ahead conditional mean and variance forecasts are computed, which are divided into 5 sub-groups of 60 trading days each. The half sum of squared standardized prediction errors of the three models, $X_{m_i} \equiv 2^{-1} \sum_{t=1}^{60} \hat{z}_{t|t-1}^{2(m_i)}$, for $i = 1,2,3$, is computed separately for each sub-group and they are presented in Table 8.1. The standardized one-day-ahead prediction error of the TARCH(p,q) model is computed as

$$\hat{z}_{t|t-1} = \frac{y_t - \hat{c}_{0,t-1} - \hat{c}_{1,t-1} y_{t-1}}{\sqrt{\hat{a}_{0,t-1} + \sum_{i=1}^{q}\left(\hat{a}_{i,t-1} \hat{\varepsilon}_{t-i|t-1}^2\right) + \hat{\gamma}_{t-1} \hat{\varepsilon}_{t-1|t-1}^2 d\left(\hat{\varepsilon}_{t-1|t-1} \le 0\right) + \sum_{i=1}^{p}\left(\hat{b}_{i,t-1} \hat{\sigma}_{t-i|t-1}^2\right)}},$$

where $\left(\hat{c}_{o,t-1}, \hat{c}_{1,t-1}, \hat{a}_{i,t-1}, i = 0,..., q, \hat{\gamma}_{t-1}, \hat{b}_{i,t-1}, i = 1,..., p\right)$ is the estimating vector of unknown parameters θ given the available information at time $t-1$.

Table 8.1. The half-sum of squared standardized one-day-ahead prediction errors of the three estimated ARCH models, $X_{m_i} \equiv 2^{-1}\sum_{t=1}^{60} \hat{z}_{t|t-1}^{2(m_i)}$, for $i = 1,2,3$.

Sub-period	AR(1)-GARCH(0,1)	AR(1)-GARCH(1,1)	AR(1)-TARCH(2,2)
1.	12.73	19.42	19.37
2.	39.83	42.21	43.30
3.	23.37	25.10	24.90
4.	15.31	18.22	18.82
5.	23.73	27.17	27.24

The AR(1)-GARCH(0,1) model appears to achieve the lowest value of the test statistic in all the sub-periods. The null hypothesis

H_0: All the three models are of equivalent predictive ability

would, therefore, be interesting to be tested versus the alternative

H_1: The model AR(1)-GARCH(0,1) has the highest predictive ability.

For any level of significance greater that $1 - F_{X_{(1)}}\left(\omega_{1-p}; a = 30, C_{123}\right)$ the null hypothesis is rejected at $100p\%$ level of significance. Hence the evidence is in support of the hypothesis that the AR(1)-GARCH(0,1) model has the highest predictive ability. Using Table 8.2, one can test the above hypotheses for each sub-period. Note that $\rho_{ij} > 95\%$, for each model in every sub-group. The null hypothesis is rejected at any level of significance greater than or equal to $1 - F_{X_{(1)}}\left(\omega_{1-p}; a = 30, C_{123}\right)$.

Table 8.2. Selected values of the cumulative density function, $F_{X_{(1)}}\left(\omega_{1-p}; a = 30, C_{123}\right)$.

Sub-period	$F_{X_{(1)}}\left(\omega_{1-p}; a = 30, C_{123}\right)$
1.	0.00008
2.	0.98347
3.	0.16725
4.	0.00178
5.	0.18515

We find that the null hypothesis is rejected at any reasonable level of significance only in the second sub-period. Despite the fact that the AR(1)-GARCH(0,1) model has the lowest value of the test statistic in all the periods, it is not selected by the MMG test among the three candidate models as the most accurate in forecasting the one-day-ahead ASE index volatility.

8.6. Conclusion

The present chapter investigates the selection of a model from a set of available models making simultaneous use of the information that is available from the candidate forecasting models. The approach to compare statistically the predictive accuracy of a set of forecasting models is commonly through pair wise comparisons. However, the hypothesis testing procedure considered in this chapter, although complicated, provides the researchers with a tool that allows the study of the joint fluctuations of the prediction errors of the models. The presented multivariate test can be applied in the selection of models forecasting either the conditional mean or the conditional variance. In future work, we plan to study the gains in the forecasting accuracy that the use of the MMG test would achieve compared to methods based on the use of classical two-model comparisons in empirical applications. Morever, it should be pointed out that the practical applicability of the MMG test could be extended to a comparison of a group of models of arbitrary size. Instead of relying on tabulated values for the distribution of the minimum of a multivariate Gamma distribution, one might approximate the quantiles of the minimum by a Monte Carlo computer simulation. Approached such that of Hansen (2001) and Hansen and Lunde (2003) where the p-values of the test statistic are obtained by using the bootstrap method of Politis and Romano (1994) could be very instructive.

Chapter 9
Scope for Further Research

The present study provided an evaluation of the ability of a model selection criterion in selecting the appropriate model to predict the conditional variance. According to that criterion, named SPEC algorithm, the ARCH model with the lowest sum of squared standardized one-step-ahead forecasting errors is selected for predicting one-step-ahead future volatility. Two different theoretical frameworks have been considered. One based on pairwise comparisons of the sums of squared standardized one-step-ahead forecasting errors of the candidate models (chapter 4) and one utilizing their overall minimum (chapter 8). In chapters 5 to 7, we considered various approaches to explore whether the models picked by the SPEC method achieve the highest predictive ability compared to those picked by other methods of model selection, including single-model methods.

In the sequel, we refer to a number of topics worth future exploration.

- An important issue is the theoretical motivation of the SPEC algorithm application in ARCH models with non-normally distributed conditional innovations. According to the SPEC method of model selection, either in the case of a non-normal conditional distribution for the residuals, the ARCH model with the lowest sum of squared standardized one-step-ahead forecasting errors should be the most appropriate in forecasting one-step-ahead volatility. However, the theoretical background in the case of other distributions such as the student-t, the generalized error distribution and the skew student-t distribution has to be further explored. Politis (2003b, 2004) considered transforming the innovations to empirical ratios that are normally distributed by dividing the ARCH process with a time-localized measure of standard deviation. Such approached may add power in the applicability of the SPEC method.
- In the previous chapters, we included 3 conditional variance specifications, the GARCH, the EGARCH and the TARCH models, but in the 2nd chapter we have presented 31 conditional variance specifications in the context of the ARCH framework. Hence, investigating the performance of the SPEC algorithm over a set of more flexible ARCH models, which account for recent developments in the area of asset returns volatility, would be an interesting problem.

- Recent developments in financial forecasting have provided evidence on statistically significant predictability of asset returns. As already mentioned, in most of the cases, the predictable component is either the overall mean or a first order autocorrelated process. In the present thesis, the mean specification was considered as an autoregressive process. However, artificial neural networks (Poggio and Girosi (1990), Hertz et al. (1991), White (1992), Hutchinson et al. (1994)), chaotic dynamical systems (Brock (1986), Holden (1986), Thompson and Stewart (1986) and Hsieh (1991)), nonlinear parametric and nonparametric models (Tong (1990) and Teräsvirta et al. (1994)) are some examples from the literature dealing with conditional mean predictions. It would be interesting to investigate whether there is added value in applying the SPEC model selection method for such models for the conditional mean specification.
- We have investigated the added value of the SPEC model selection method in forecasting volatility for options pricing. Value-at-Risk (VaR) at a given probability level p, is the predicted amount of financial loss of a portfolio over a given time horizon. The forecasting of the VaR number is another area of applied financial statistics that the added value of the SPEC method should be explored. Angelidis and Degiannakis (2005b), Billio and Pelizzon (2000), Brooks and Persand (2003) and Giot and Laurent (2003a, 2003b) are examples of recent studies that investigate the forecasting ability of ARCH models in predicting the VaR number.
- In section 2.6.1 we have noted the use of intra-day data as an alternative volatility measure that introduced by Andersen and Bollerslev (1998a). In the 7th chapter, the ability of the SPEC model selection algorithm was compared with loss functions that used the realized intra-day volatility and the SPEC algorithm led to the highest profits. However, the SPEC method can be compared to models that are based on intra-day datasets, like the ARFIMA methodology described in equation 2.2.40 of the 2nd chapter. As concerns the question whether an ARFIMA model, which uses intra-day data, delivers more accurate volatility forecasts than an ARCH model, which is based on daily returns, the answer may not always positive. For example, Giot and Laurent (2004) concluded that an adequately specified ARCH model has equivalent predictive ability with an ARFIMA specification in predicting the one-day-ahead VaR. So, a future application of the SPEC model selection method on inter-day and intra-day models would be interested.
- The SPEC algorithm is interesting to be applied in more data sets such as stocks, stock indices, bonds, commodities and exchange rates.

- At each point in time at which the SPEC method is applied, a specific model is picked as the most appropriate for predicting future volatility. In Table 6.6 of the 6th chapter, we have seen that the SPEC algorithm does not appear to be noticeably biased towards selecting a specific type of model. However, there are studies in the literature such as Christoffersen and Jacobs (2003), Ferreira and Lopez (2003) and Lopez and Walter (2001), which support the assumption that the simplest model specifications are chosen a disproportionately large percentage of the time. On the other hand, Angelidis, Benos and Degiannakis (2004), Degiannakis (2004), Giot and Laurent (2003a, 2004) among others concluded that the more flexible an ARCH model is, the more adequate it is in volatility forecasting, compared to parsimonious models. For further research, it may be interesting to investigate whether the selection of specific models is related to certain economic factors.
- The MMG hypothesis testing is a multi-model selection procedure, which leads to the selection of the model with the lowest sum of squared standardized one-step-ahead prediction errors. The form of the exact distribution of the test statistic is explicitly derived as the distribution of the minimum value of n variables that are multivariate gamma distributed. The derived exact distribution of the test statistic should not be considered only as the theoretical justification of the SPEC algorithm. Studying the gains in forecasting accuracy in empirical applications from the use of the proposed test procedure is worth exploration.

Appendices of Chapters 2, 3, 4, 5, 6 and 8

Appendix 2

- ***The ARCH models that have been presented***

Table 2.2. The ARCH models that have been presented in Section 2.1. The reader who is interested in more information for an ARCH model should recur to the equation referred in the last column.

ARCH(q)	$\sigma_t^2 = a_0 + \sum_{i=1}^{q} \left(a_i \varepsilon_{t-i}^2\right)$	Engle (1982)	(2.2.3)
GARCH(p,q)	$\sigma_t^2 = a_0 + \sum_{i=1}^{q} \left(a_i \varepsilon_{t-i}^2\right) + \sum_{j=1}^{p} \left(b_j \sigma_{t-j}^2\right)$	Bollerslev (1986)	(2.2.6)
IGARCH(p,q)	$\sigma_t^2 = a_0 + \sum_{i=1}^{q} \left(a_i \varepsilon_{t-i}^2\right) + \sum_{j=1}^{p} \left(b_j \sigma_{t-j}^2\right)$, for $\sum_{i=1}^{q} a_i + \sum_{j=1}^{p} b_j = 1$	Engle and Bollerslev (1986)	(2.2.8)
EGARCH(p,q)	$\ln\left(\sigma_t^2\right) = a_0 + \left(1 + \sum_{i=1}^{q} a_i L^i\right)\left(1 - \sum_{j=1}^{p} b_j L^j\right)^{-1}$ $\left(\theta\left(\left\lvert\varepsilon_{t-1}/\sigma_{t-1}\right\rvert - E\left\lvert\varepsilon_{t-1}/\sigma_{t-1}\right\rvert\right) + \gamma\left(\varepsilon_{t-1}/\sigma_{t-1}\right)\right)$	Nelson (1991)	(2.2.13)
GJR(p,q)	$\sigma_t^2 = a_0 + \sum_{i=1}^{q} \left(a_i \varepsilon_{t-i}^2\right) + \sum_{i=1}^{q} \left(\gamma_i d\left(\varepsilon_{t-i} < 0\right)\varepsilon_{t-i}^2\right) + \sum_{j=1}^{p} \left(b_j \sigma_{t-j}^2\right)$	Glosten et al. (1993)	(2.2.14)
TGARCH(p,q)	$\sigma_t = a_0 + \sum_{i=1}^{q} \left(a_i \varepsilon_{t-i}^+\right) - \sum_{i=1}^{q} \left(\gamma_i \varepsilon_{t-i}^-\right) + \sum_{j=1}^{p} \left(b_j \sigma_{t-j}\right)$	Zakoian (1990)	(2.2.15)
AGARCH(p,q)	$\sigma_t = a_0 + \sum_{i=1}^{q} a_i \left\lvert\varepsilon_{t-i}\right\rvert + \sum_{j=1}^{p} b_j \sigma_{t-j}$	Taylor (1986) Schwert (1989a,b)	(2.2.16)
Ln-GARCH(p,q)	$\ln\left(\sigma_t^2\right) = a_0 + \sum_{i=1}^{q} a_i \ln\left(\varepsilon_{t-i}^2\right) + \sum_{j=1}^{p} b_j \ln\left(\sigma_{t-j}^2\right).$	Geweke (1986) Pantula (1986)	(2.2.17)
Stdev-ARCH(q)	$\sigma_t^2 = \left(a_0 + \sum_{i=1}^{q} a_i \left\lvert\varepsilon_{t-i}\right\rvert\right)^2$	Schwert (1990)	(2.2.18)
NARCH(p,q)	$\sigma_t^\delta = a_0 + \sum_{i=1}^{q} a_i \left\lvert\varepsilon_{t-i}^2\right\rvert^{\delta/2} + \sum_{j=1}^{p} b_j \sigma_{t-j}^\delta$	Higgins and Bera (1992)	(2.2.19)
AGARCH(p,q)	$\sigma_t^2 = a_0 + \sum_{i=1}^{q} \left(a_i \varepsilon_{t-i}^2 + \gamma_i \varepsilon_{t-i}\right) + \sum_{j=1}^{p} b_j \sigma_{t-j}^2$	Engle (1990)	(2.2.21)
NAGARCH(p,q)	$\sigma_t^2 = a_0 + \sum_{i=1}^{q} a_i \left(\varepsilon_{t-i} + \gamma_i \sigma_{t-i}\right)^2 + \sum_{j=1}^{p} b_j \sigma_{t-j}^2$	Engle and Ng (1993)	(2.2.22)
VGARCH(p,q)	$\sigma_t^2 = a_0 + \sum_{i=1}^{q} a_i \left(\varepsilon_{t-i}/\sigma_{t-i} + \gamma_i\right)^2 + \sum_{j=1}^{p} b_j \sigma_{t-j}^2$	Engle and Ng (1993)	(2.2.23)
APARCH(p,q)	$\sigma_t^\delta = a_0 + \sum_{i=1}^{q} a_i \left(\left\lvert\varepsilon_{t-1}\right\rvert - \gamma_i \varepsilon_{t-i}\right)^\delta + \sum_{j=1}^{p} b_j \sigma_{t-j}^\delta$	Ding et al. (1993)	(2.2.24)
GQARCH(p,q)	$\sigma_t^2 = a_0 + \sum_{i=1}^{q} a_i \varepsilon_{t-i}^2 + \sum_{i=1}^{q} \gamma_i \varepsilon_{t-i} + 2\sum_{i=1}^{q} \sum_{j=i+1}^{q} a_{ij} \varepsilon_{t-i} \varepsilon_{t-j} + \sum_{j=1}^{p} b_j \sigma_{t-j}^2$	Sentana (1995)	(2.2.25)

GQTARCH(p,q)	$\sigma_t^2 = \omega + \sum_{i=1}^{q}\sum_{j=1}^{J} a_{ij} I_j(\varepsilon_{t-i}) + \sum_{i=1}^{p} b_j \sigma_{t-j}^2$	Gouriéroux and Monfort (1992)	(2.2.28)
VSARCH(p,q):	$\sigma_t^2 = \omega + \sum_{i=1}^{q} a_i \varepsilon_{t-i}^2 + \gamma S_{t-1} \frac{\varepsilon_{t-1}^2}{\sigma_{t-1}^2} + \sum_{j=1}^{p} b_j \sigma_{t-j}^2$	Fornari Mele (1995)	(2.2.29)
AVSARCH(p,q)	$\sigma_t^2 = a_0 + \sum_{i=1}^{q} a_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} b_j \sigma_{t-j}^2 + \gamma S_{t-1}\varepsilon_{t-1}^2 + \delta\left(\left(\varepsilon_{t-1}^2 / \sigma_{t-1}^2\right) - k\right) S_{t-}$	Fornari and Mele (1995)	(2.2.30)
LST-GARCH(p,q)	$\sigma_t^2 = a_0 + \sum_{i=1}^{q} (a_i + \gamma_i F(\varepsilon_{t-i}))\varepsilon_{t-i}^2 + \sum_{j=1}^{p} b_j \sigma_{t-j}^2$ $F(\varepsilon_{t-i}) = (1 + \exp(-\theta\varepsilon_{t-i}))^{-1} - 0.5$	Hagerud (1996)	(2.2.32)
EST-GARCH(p,q)	$\sigma_t^2 = a_0 + \sum_{i=1}^{q} (a_i + \gamma_i F(\varepsilon_{t-i}))\varepsilon_{t-i}^2 + \sum_{j=1}^{p} b_j \sigma_{t-j}^2$ $F(\varepsilon_{t-i}) = 1 - \exp(-\theta\varepsilon_{t-i}^2)$	Hagerud (1996)	(2.2.33)
GLST-GARCH(p,q)	$\sigma_t^2 = a_0 + \sum_{i=1}^{q} (a_i + \gamma_i F(\varepsilon_{t-i}))\varepsilon_{t-i}^2 + \sum_{j=1}^{p} b_j \sigma_{t-j}^2$ $F(\varepsilon_{t-i}) = \frac{1 - \exp(-\theta\varepsilon_{t-i}^2)}{1 + \exp(-\theta(\varepsilon_{t-i}^2 - c^2))}$	Lubrano (1998)	(2.2.34)
GEST-GARCH(p,q)	$\sigma_t^2 = a_0 + \sum_{i=1}^{q} (a_i + \gamma_i F(\varepsilon_{t-i}))\varepsilon_{t-i}^2 + \sum_{j=1}^{p} b_j \sigma_{t-j}^2$ $F(\varepsilon_{t-i}) = 1 - \exp(-\theta(\varepsilon_{t-i} - c)^2)$	Lubrano (1998)	(2.2.35)
CGARCH(1,1)	$\sigma_t^2 = q_t + a_1(\varepsilon_{t-1}^2 - q_{t-1}) + b_1(\sigma_{t-1}^2 - q_{t-1})$ $q_t = a_0 + pq_{t-1} + \phi(\varepsilon_{t-1}^2 - \sigma_{t-1}^2)$	Engle and Lee (1993)	(2.2.37)
ACGARCH(1,1)	$\sigma_t^2 = q_t + a_1(\varepsilon_{t-1}^2 - q_{t-1}) + \gamma_1(d(\varepsilon_{t-1} < 0)\varepsilon_{t-1}^2 - 0.5q_{t-1}) + b_1(\sigma_{t-1}^2 - q_{t-}$ $q_t = a_0 + pq_{t-1} + \phi(\varepsilon_{t-1}^2 - \sigma_{t-1}^2) + \gamma_2(d(\varepsilon_{t-1} < 0)\varepsilon_{t-1}^2 - 0.5\sigma_{t-1}^2)$	Engle and Lee (1993)	(2.2.39)
FIGARCH(p,d,q)	$\sigma_t^2 = a_0 + (1 - B(L) - \Phi(L)(1-L)^d)\varepsilon_t^2 + B(L)\sigma_t^2$	Baillie et al. (1996)	(2.2.46)
FIEGARCH(p,d,q)	$\ln(\sigma_t^2) = a_0 + \Phi(L)^{-1}(1-L)^{-d}(1 + A(L))g(z_{t-1})$	Bollerslev and Mikkelsen (1996)	(2.2.47)
FIAPARCH(p,d,q)	$\sigma_t^\delta = a_0 + (1 - (1 - B(L))^{-1}\Phi(L)(1-L)^{-d})(\lvert\varepsilon_t\rvert - \gamma\varepsilon_t)^\delta$	Tse (1998)	(2.2.48)
ASYMM FIFGARCH(1,d,1)	$\sigma_t^\lambda = \frac{k}{1-\delta} + \left(1 - \frac{(1-\varphi L)(1-L)^d}{1-\delta L}\right) f^v(\varepsilon_t)\sigma_t^\lambda$ $f(\varepsilon_t) = \left\lvert \frac{\varepsilon_t}{\sigma_t} - b \right\rvert - c\left(\frac{\varepsilon_t}{\sigma_t} - b\right),$	Hwang (2001)	(2.2.49)
ASYMM FIFGARCH(1,d,1) modified	$(1-\varphi L)(1-L)^d \frac{\sigma_t^\lambda - 1}{\lambda} = \omega' + a(1 + \psi L)\sigma_{t-1}^\lambda\left(f^v(z_{t-1}) - 1\right)$ $f\left(\frac{\varepsilon_t}{\sigma_t}\right) = \left\lvert \frac{\varepsilon_t}{\sigma_t} - b \right\rvert - c\left(\frac{\varepsilon_t}{\sigma_t} - b\right)$	Ruiz and Perez (2003)	(2.2.50)

R-GARCH(r,p,q)	$\sigma_t^2 = \sum_{i^*=1}^{r}\left(c_{i^*}\eta_{t-i^*}\right) + \sum_{i=1}^{q}\left(a_i\varepsilon_{t-i}^2\right) + \sum_{j=1}^{p}\left(b_j\sigma_{t-j}^2\right)$	Nowicka-Zagrajek and Weron (2001)	(2.2.51)
H-GARCH(p,n)	$\sigma_t^2 = a_0 + \sum_{i=1}^{n}\sum_{k=1}^{i} a_{ik}\left(\sum_{i^*=k}^{i}\varepsilon_{t-i^*}^2\right)^2 + \sum_{j=1}^{p}\left(b_j\sigma_{t-j}^2\right)$	Müller et al. (1997)	(2.2.52)

Appendix 3

- ***Figure 3.1. The simulated processes***
- ***Figure 3.2. Histograms and descriptive statistics of the simulated processes***
- ***Figure 3.3. Histograms of simulated Chi-square distributed process with T degrees of freedom***
- ***Figure 3.4. Autocorrelation of transformations of the processes*** z_t, ε_t, v_t, σ_t
- ***Figure 3.5. The one-step-ahead estimated processes***
- ***Figure 3.6. Histograms and descriptive statistics of the one-step-ahead estimated processes***
- ***Figure 3.7. Histograms and descriptive statistics of*** $\left\{\sum_{j=t-T+1}^{t} \hat{z}_{j+1|j}^2\right\}$, ***for*** $t = T(T)30000$
- ***Figure 3.8. Autocorrelation of transformations of the processes*** $\hat{z}_{t+1|t}$, $\hat{\varepsilon}_{t+1|t}$, $\hat{v}_{t+1|t}$, $\hat{\sigma}_{t+1|t}$
- ***Figure 3.9. Histograms and descriptive statistics of the squared standardized one-step-ahead prediction errors***
- ***Figure 3.10. Histograms and descriptive statistics of*** $\left\{\sum_{j=t-T+1}^{t} \hat{z}_{j+1|j}^2\right\}$, ***for*** $t = T(T)30000$
- ***Figure 3.11. Autocorrelation of transformations of the processes*** $Cor\left(\left|\hat{z}_{t+1|t}\right|^d, \left|\hat{z}_{t+\tau+1|t+\tau}\right|^d\right)$, $d = 0.5(0.5)3$, $\tau = 1(1)100$

Figure 3.1. The simulated processes.

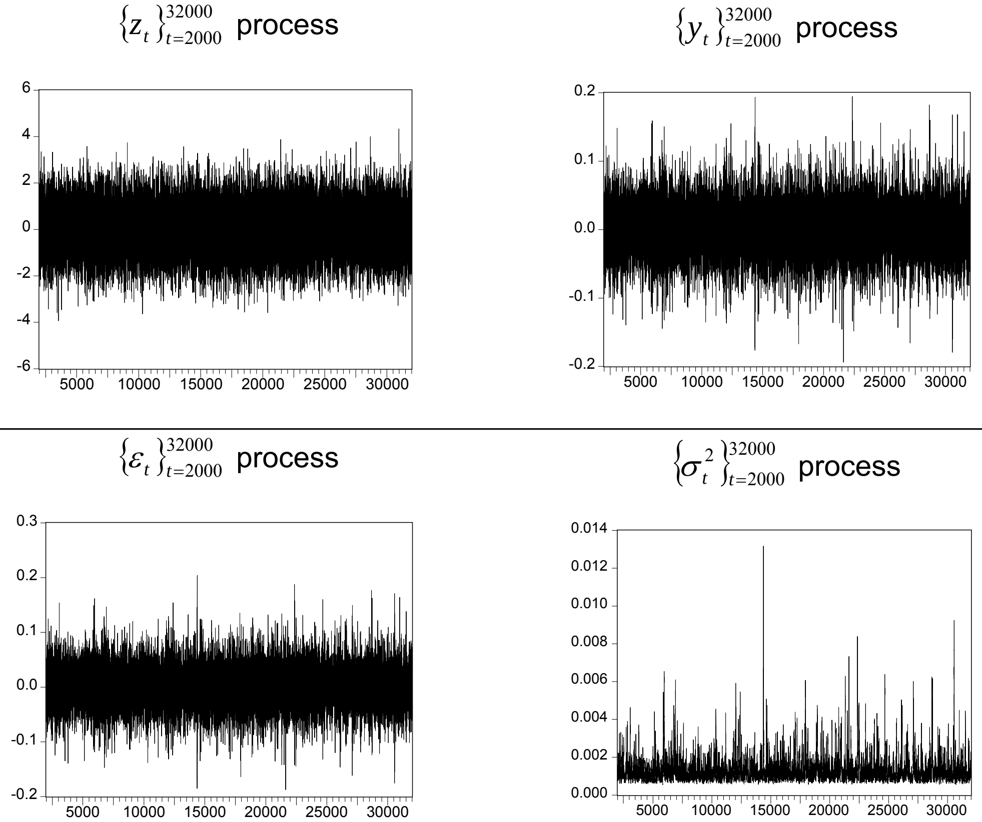
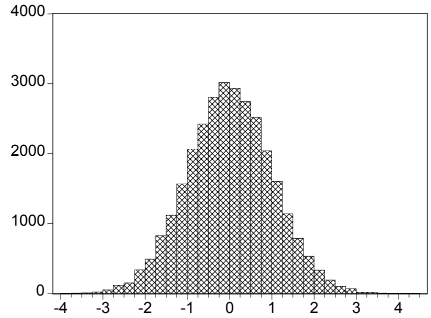


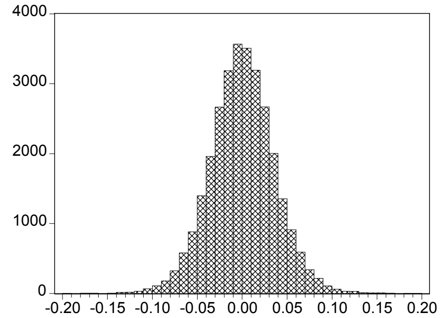
Figure 3.2. Histograms and descriptive statistics of the simulated processes.

$\{z_t\}_{t=2000}^{32000}$ process



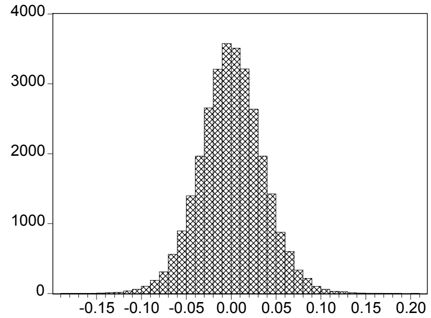
Mean	0.006139
Median	0.001398
Maximum	4.327886
Minimum	-3.948533
Std. Dev.	0.999047
Skewness	0.015896
Kurtosis	3.004320

$\{y_t\}_{t=2000}^{32000}$ process



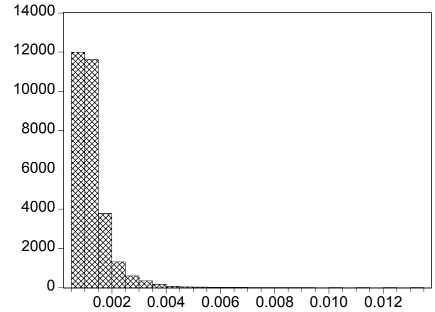
Mean	0.000232
Median	6.16E-05
Maximum	0.194647
Minimum	-0.194074
Std. Dev.	0.035562
Skewness	0.023013
Kurtosis	3.712639

$\{\varepsilon_t\}_{t=2000}^{32000}$ process



Mean	0.000218
Median	3.81E-05
Maximum	0.204209
Minimum	-0.187684
Std. Dev.	0.035500
Skewness	0.017497
Kurtosis	3.703915

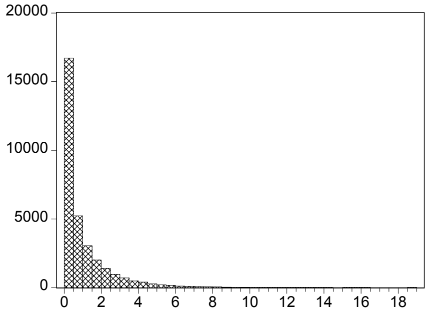
$\{\sigma_t^2\}_{t=2000}^{32000}$ process



Mean	0.001256
Median	0.001089
Maximum	0.013171
Minimum	0.000536
Std. Dev.	0.000620
Skewness	3.575072
Kurtosis	30.14328

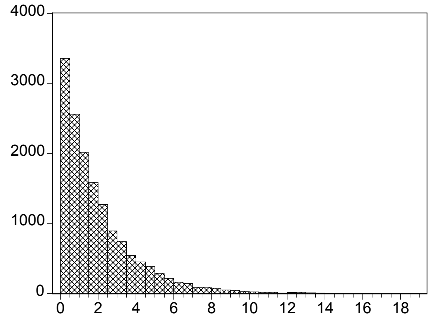
Figure 3.3. Histograms of simulated Chi-square distributed process with T degrees of freedom.

$T = 1$



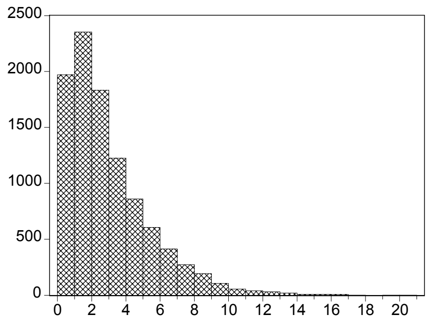
Mean	0.996122
Variance	1.995353
Observations	32000.00

$T = 2$



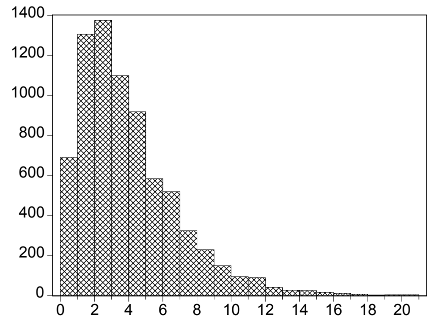
Mean	1.996369
Variance	3.997227
Observations	15000.00

$T = 3$



Mean	2.994554
Variance	5.951681
Observations	10000.00

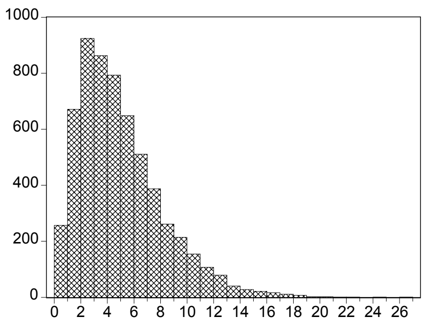
$T = 4$



Mean	3.992739
Variance	8.121678
Observations	7500.000

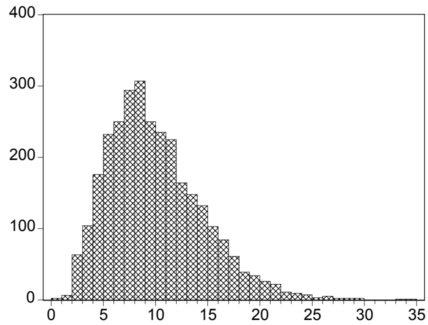
Figure 3.3. Histograms of simulated Chi-square distributed process with T degrees of freedom.

$T = 5$



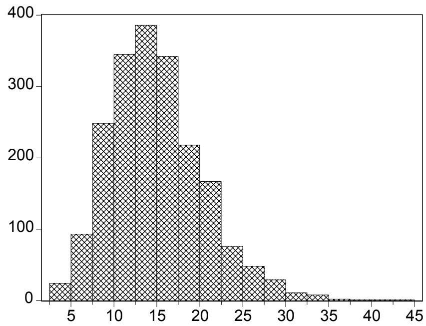
Mean	4.990924
Variance	10.11472
Observations	6000.000

$T = 10$



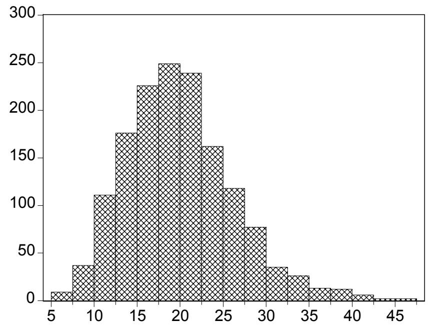
Mean	9.981847
Variance	20.56190
Observations	3000.000

$T = 15$



Mean	14.97277
Variance	30.02824
Observations	2000.000

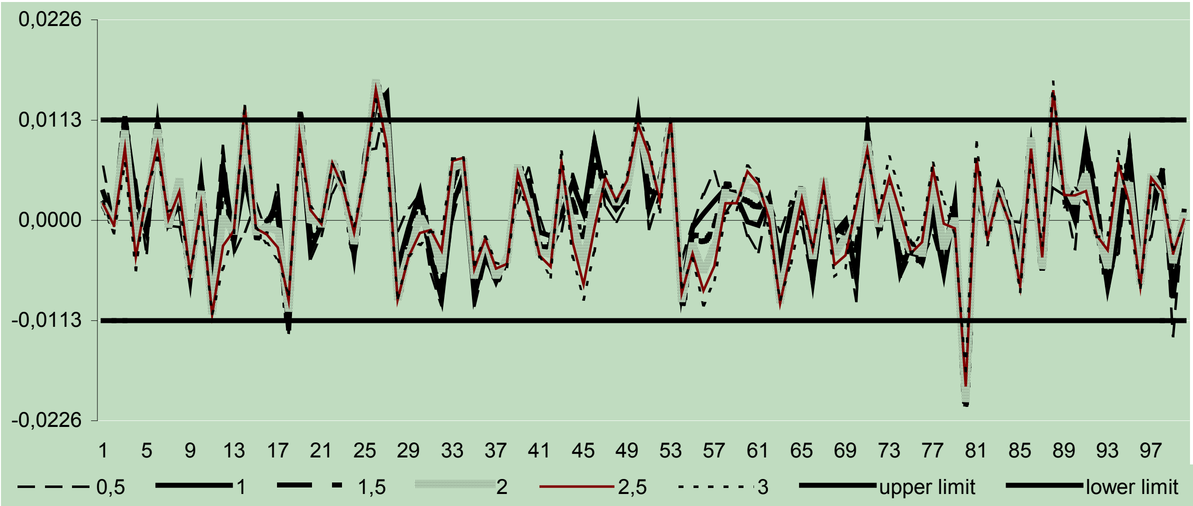
$T = 20$



Mean	19.96369
Variance	39.83026
Observations	1500.000

Figure 3.4. Autocorrelation of transformations of the processes $z_t, \varepsilon_t, v_t, \sigma_t$.

$$Cor\left(\left|z_t\right|^d, \left|z_{t+\tau}\right|^d\right),\ d = 0.5(0.5)3,\ \tau = 1(1)100$$



$$Cor\left(\left|\varepsilon_t\right|^d, \left|\varepsilon_{t+\tau}\right|^d\right),\ d = 0.5(0.5)3,\ \tau = 1(1)100$$

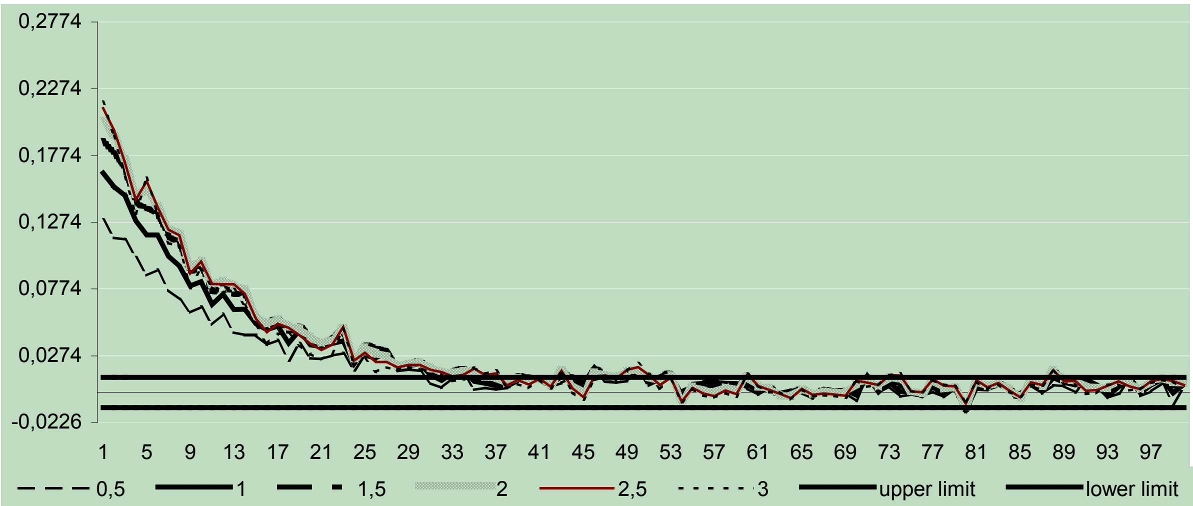
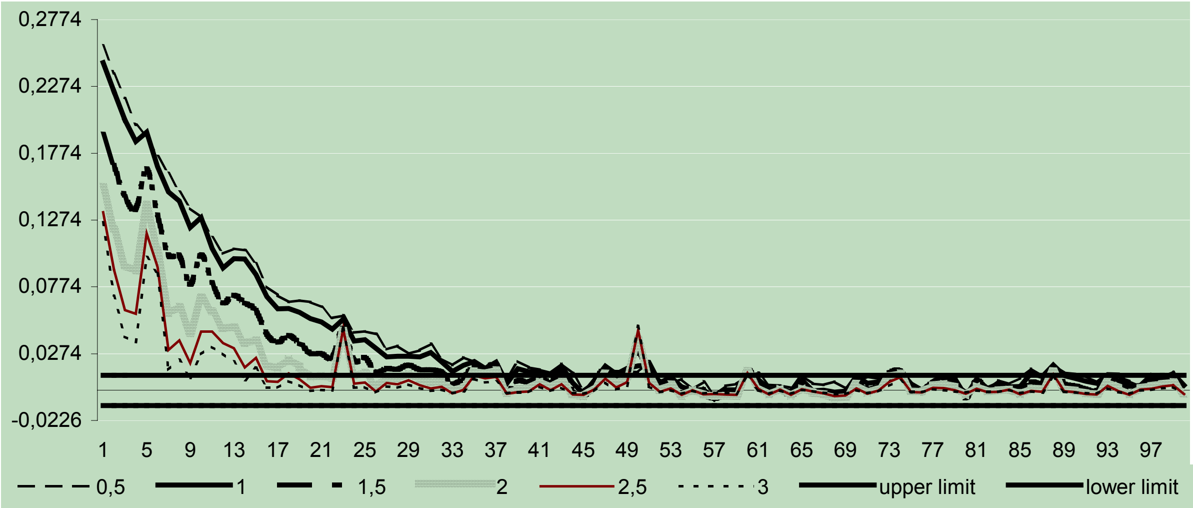


Figure 3.4. Autocorrelation of transformations of the processes $z_t, \varepsilon_t, v_t, \sigma_t$.

$$Cor\left(\left|v_t\right|^d, \left|v_{t+\tau}\right|^d\right),\ d = 0.5(0.5)3,\ \tau = 1(1)100$$



$$Cor\left(\sigma_t^d, \sigma_{t+\tau}^d\right),\ d = 0.5(0.5)3,\ \tau = 1(1)100$$

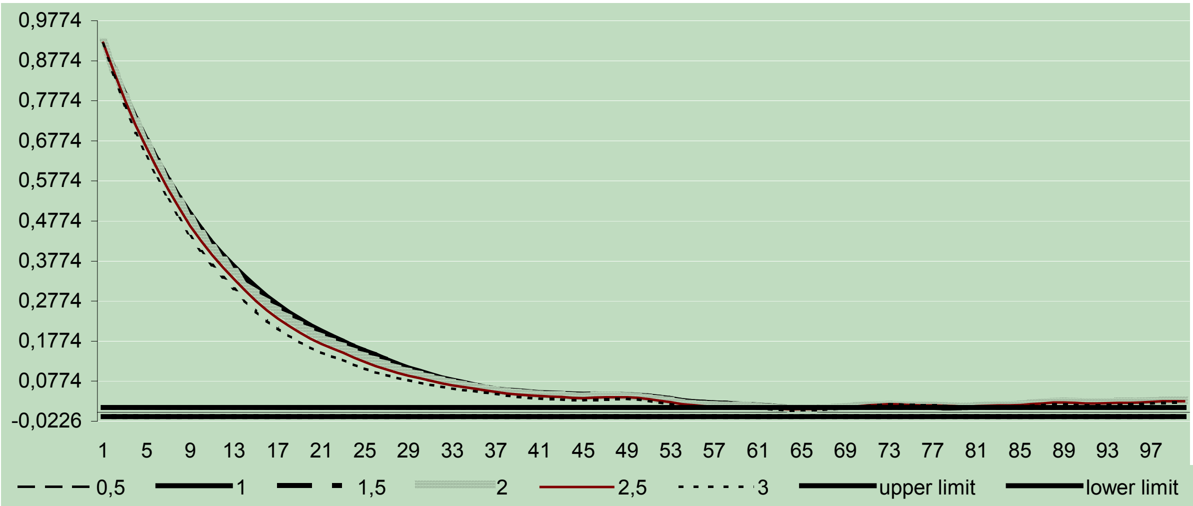


Figure 3.5. The one-step-ahead estimated processes.

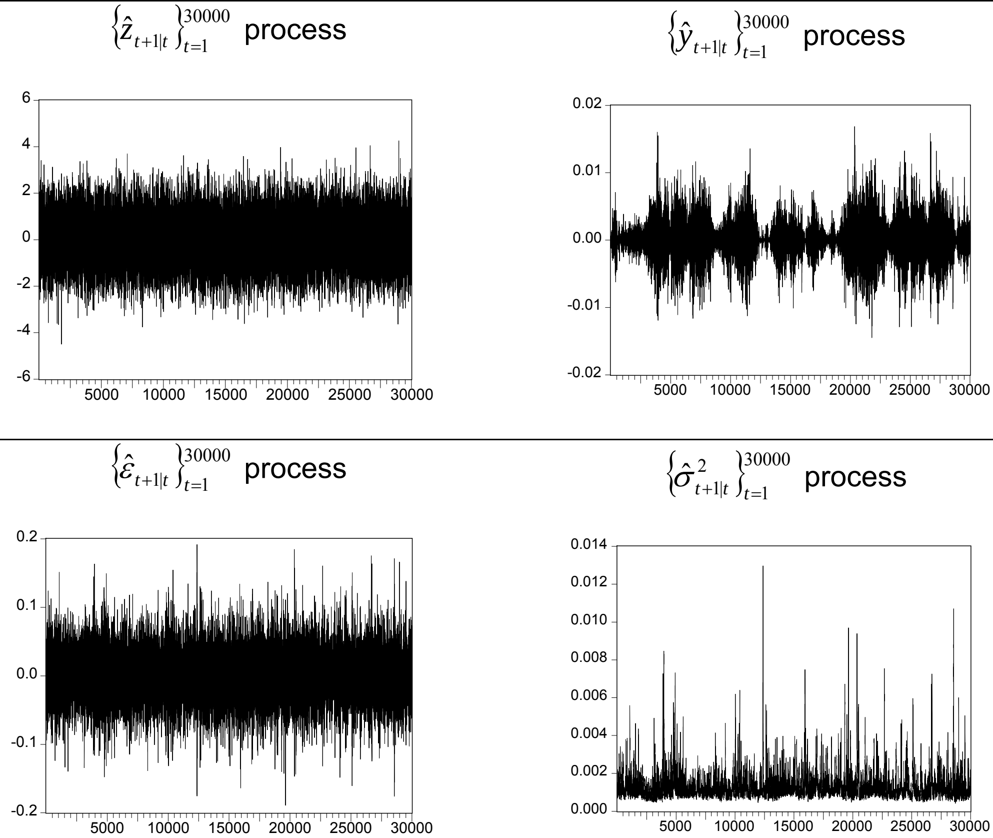
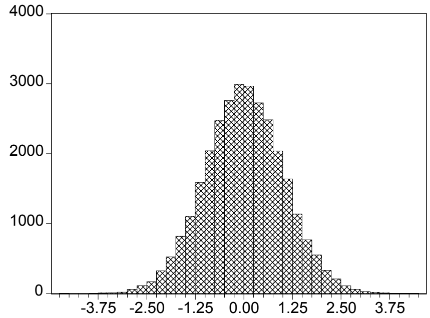


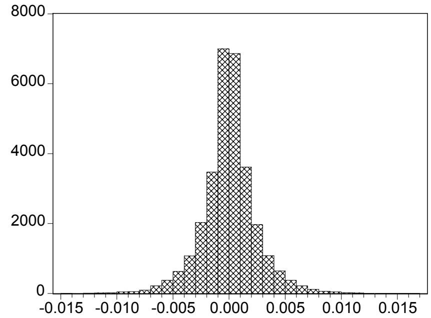
Figure 3.6. Histograms and descriptive statistics of the one-step-ahead estimated processes.

$\left\{\hat{z}_{t+1|t}\right\}_{t=1}^{30000}$ process



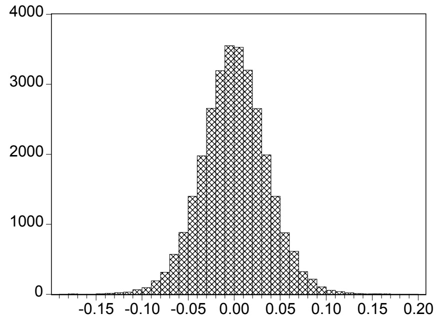
Mean	0.006218
Median	0.003156
Maximum	4.256031
Minimum	-4.508652
Std. Dev.	1.004477
Skewness	0.016695
Kurtosis	3.027268

$\left\{\hat{y}_{t+1|t}\right\}_{t=1}^{30000}$ process



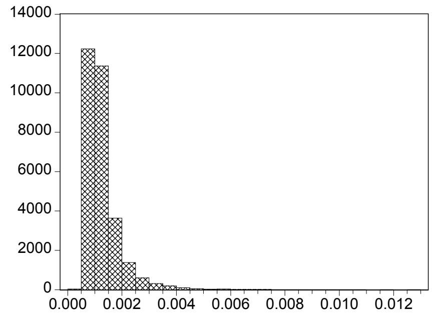
Mean	1.73E-05
Median	4.14E-07
Maximum	0.016871
Minimum	-0.014472
Std. Dev.	0.002390
Skewness	0.086885
Kurtosis	5.895584

$\left\{\hat{\varepsilon}_{t+1|t}\right\}_{t=1}^{30000}$ process



Mean	0.000214
Median	9.65E-05
Maximum	0.191628
Minimum	-0.189207
Std. Dev.	0.035514
Skewness	0.014996
Kurtosis	3.699045

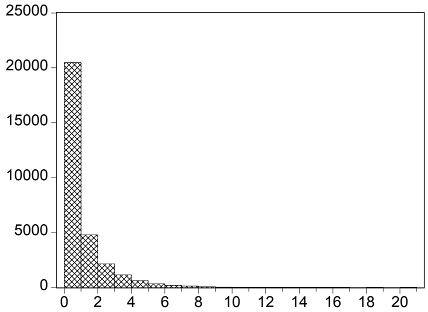
$\left\{\hat{\sigma}^2_{t+1|t}\right\}_{t=1}^{30000}$ process



Mean	0.001259
Median	0.001081
Maximum	0.012977
Minimum	0.000408
Std. Dev.	0.000670
Skewness	3.879618
Kurtosis	32.67293

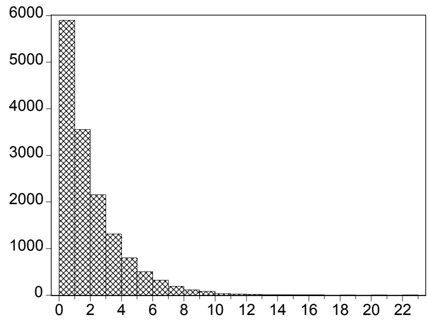
Figure 3.7. Histograms and descriptive statistics of $\left\{\sum_{j=t-T+1}^{t} \hat{z}_{j+1|j}^{2}\right\}, t = T(T)30000$.

$T = 1$



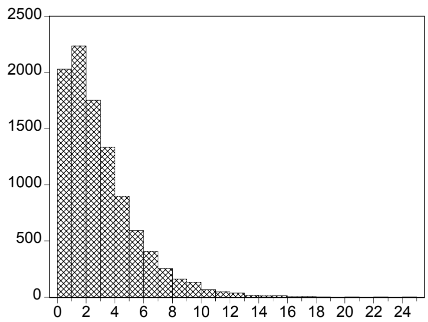
Mean	1.008980
Variance	2.064259
Observations	30000.00

$T = 2$



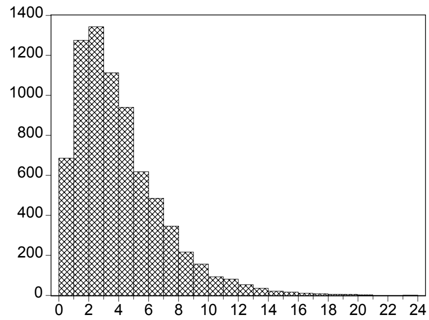
Mean	2.017960
Variance	4.128895
Observations	15000.00

$T = 3$



Mean	3.026939
Variance	6.229548
Observations	10000.00

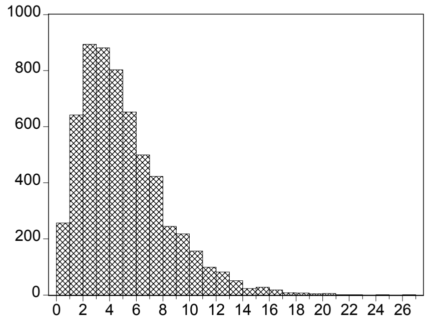
$T = 4$



Mean	4.035919
Variance	8.334519
Observations	7500.000

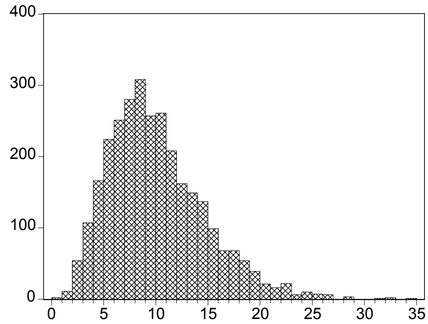
Figure 3.7. Histograms and descriptive statistics of $\left\{\sum_{j=t-T+1}^{t}\hat{z}^{2}_{j+1|j}\right\}, t = T(T)30000$.

$T = 5$



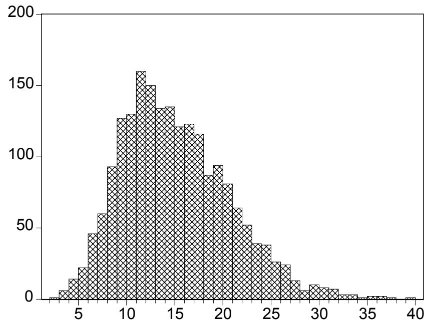
Mean	5.044899
Variance	10.39188
Observations	6000.000

$T = 10$



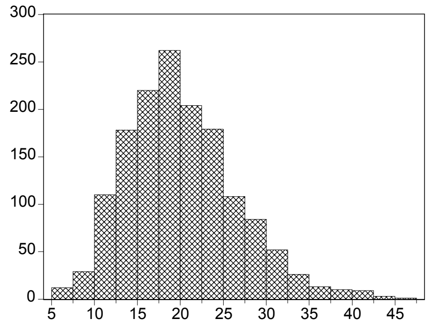
Mean	10.08980
Variance	21.34030
Observations	3000.000

$T = 15$



Mean	15.13470
Variance	31.63887
Observations	2000.000

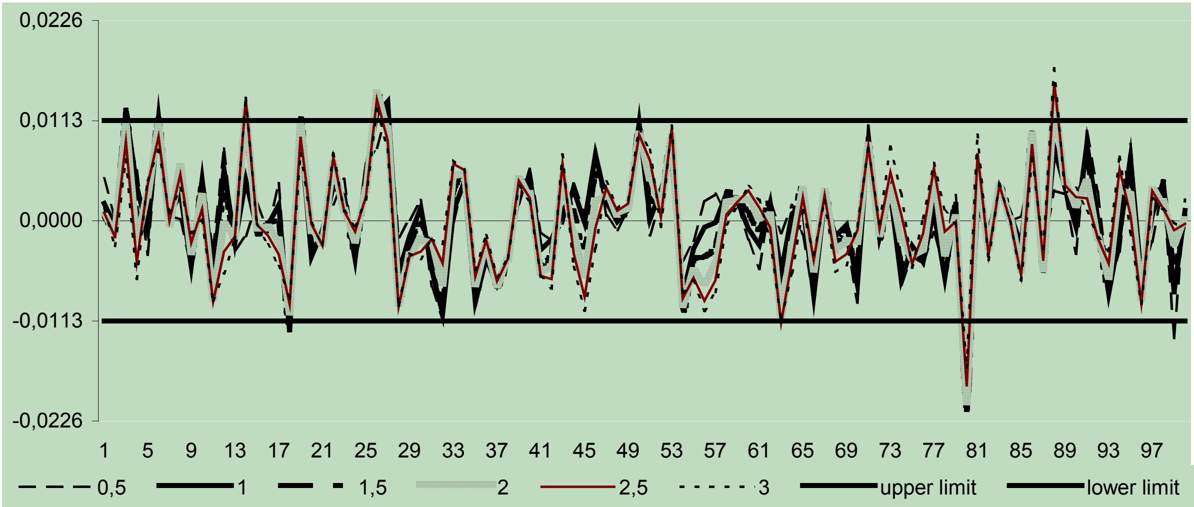
$T = 20$



Mean	20.17960
Variance	41.31167
Observations	1500.000

Figure 3.8. Autocorrelation of transformations of the processes $\hat{z}_{t+1|t}$, $\hat{\varepsilon}_{t+1|t}$, $\hat{v}_{t+1|t}$, $\hat{\sigma}_{t+1|t}$.

$$Cor\left(\left|\hat{z}_{t+1|t}\right|^{d}, \left|\hat{z}_{t+\tau+1|t+\tau}\right|^{d}\right),\ d = 0.5(0.5)3,\ \tau = 1(1)100$$



$$Cor\left(\left|\hat{\varepsilon}_{t+1|t}\right|^{d}, \left|\hat{\varepsilon}_{t+\tau+1|t+\tau}\right|^{d}\right),\ d = 0.5(0.5)3,\ \tau = 1(1)100$$

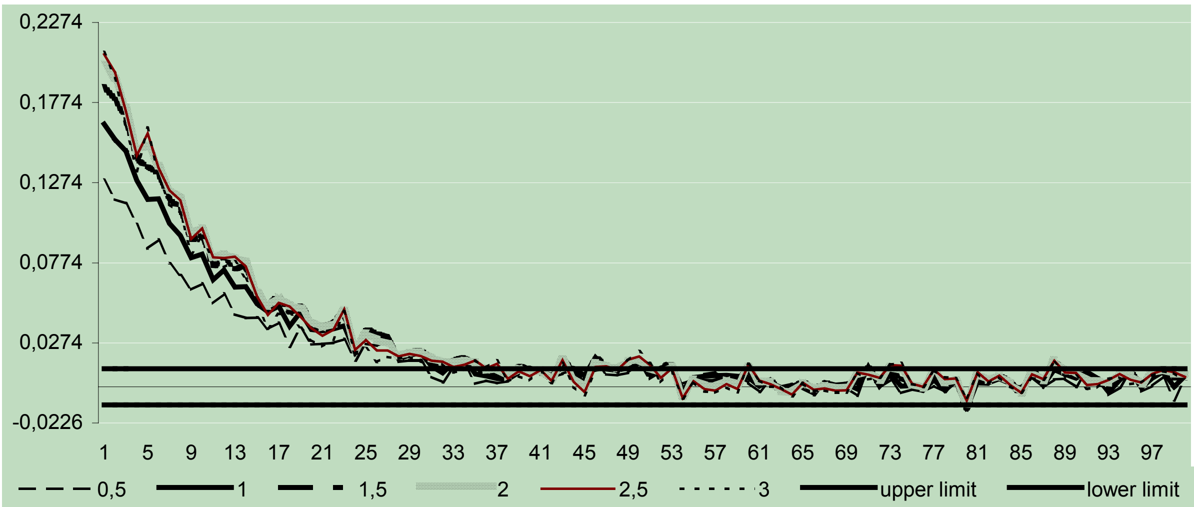
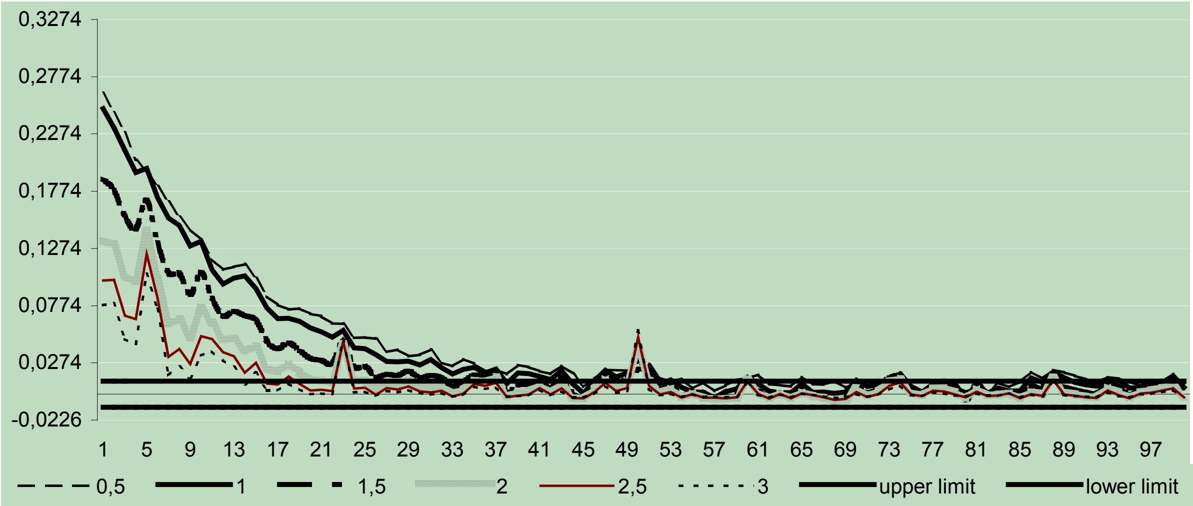


Figure 3.8. Autocorrelation of transformations of the processes $\hat{z}_{t+1|t}$, $\hat{\varepsilon}_{t+1|t}$, $\hat{v}_{t+1|t}$, $\hat{\sigma}_{t+1|t}$.

$$Cor\left(\left|\hat{v}_{t+1|t}\right|^{d},\left|\hat{v}_{t+\tau+1|t+\tau}\right|^{d}\right),\ d=0.5(0.5)3,\ \tau=1(1)100$$



$$Cor\left(\hat{\sigma}^{d}_{t+1|t},\hat{\sigma}^{d}_{t+\tau+1|t+\tau}\right),\ d=0.5(0.5)3,\ \tau=1(1)100$$

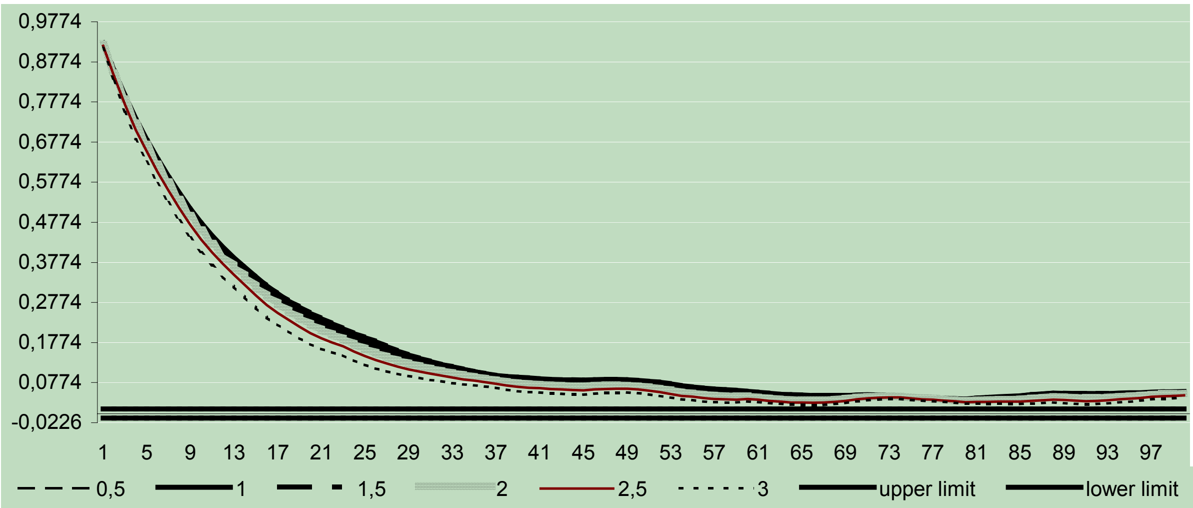
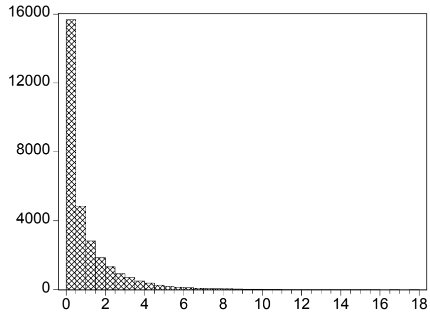


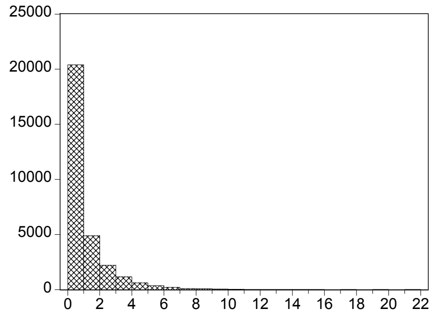
Figure 3.9. Histograms and descriptive statistics of the squared standardized one-step-ahead prediction errors.

a) AR(1)GARCH(1,1)



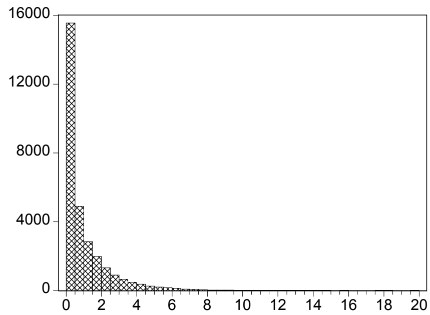
Mean	1.009548
Variance	2.086989
Observations	30000.00

b) AR(1)EGARCH(1,1)



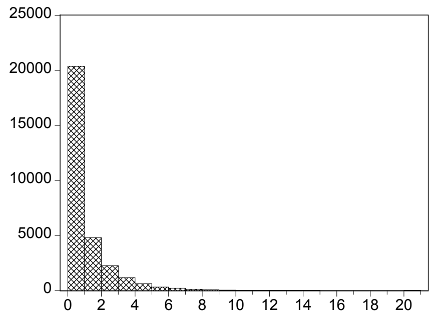
Mean	1.013434
Variance	2.088944
Observations	30000.00

c) AR(1)TARCH(1,1)



Mean	1.009935
Variance	2.062381
Observations	30000.00

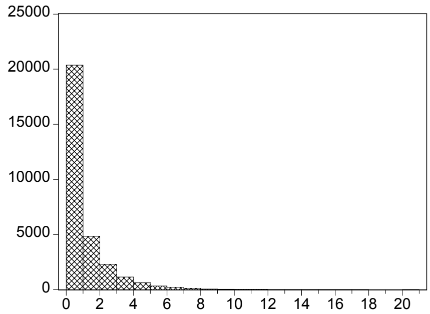
d) AR(1)GARCH(1,2)



Mean	1.013076
Variance	2.068613
Observations	30000.00

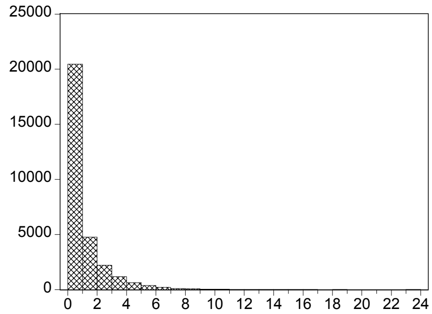
Figure 3.9. Histograms and descriptive statistics of the squared standardized one-step-ahead prediction errors.

e) AR(1)TARCH(1,2)



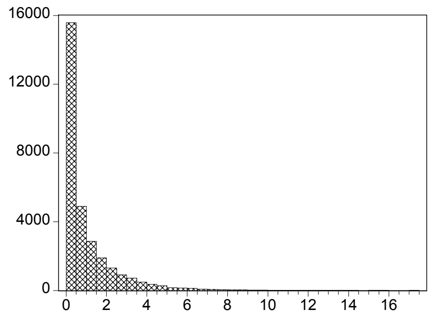
Mean	1.014403
Variance	2.126908
Observations	30000.00

f) AR(3)GARCH(1,1)



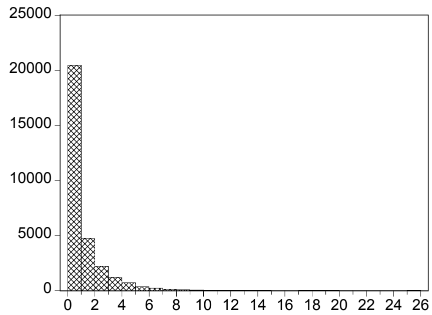
Mean	1.013256
Variance	2.119161
Observations	30000.00

g) AR(3)EGARCH(1,1)



Mean	1.011744
Variance	2.072332
Observations	30000.00

h) AR(3)TARCH(1,1)

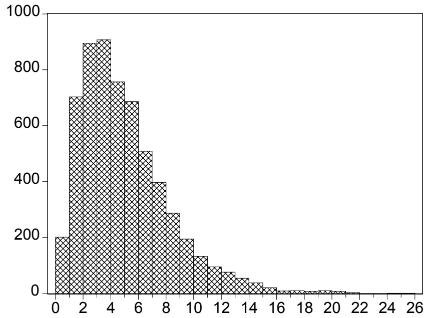


Mean	1.014928
Variance	2.067768
Observations	30000.00

Figure 3.10. Histograms and descriptive statistics of $\left\{\sum_{j=t-T+1}^{t} \hat{z}^2_{j+1|j}\right\}, t = T(T)30000$.

$T=5$

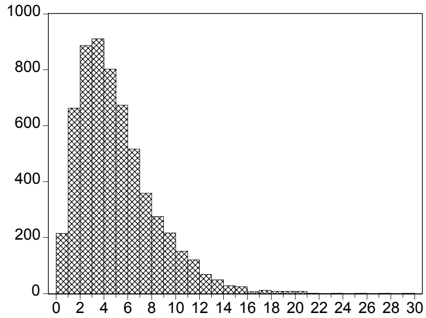
a) AR(1)GARCH(1,1)



Mean	5.047740
Variance	10.42947
Observations	6000.000

$T=5$

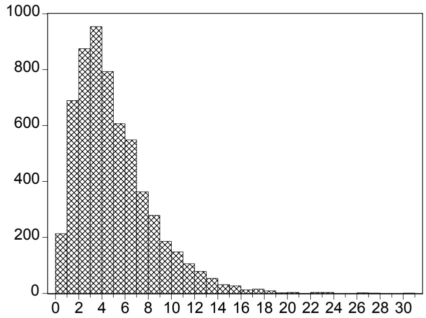
c) AR(1)EGARCH(1,1)



Mean	5.067171
Variance	10.48590
Observations	6000.000

$T=5$

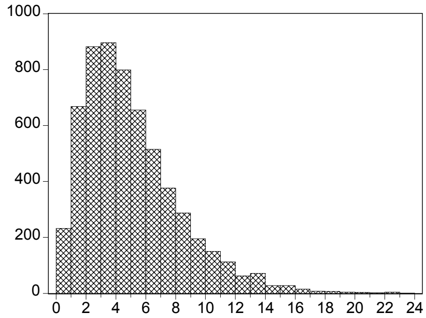
d) AR(1)TARCH(1,1)



Mean	5.049674
Variance	10.75704
Observations	6000.000

$T=5$

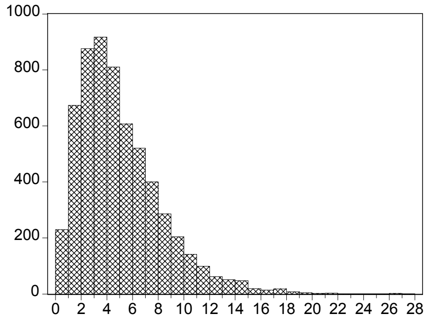
e) AR(1)GARCH(1,2)



Mean	5.065382
Variance	10.44275
Observations	6000.000

$T=5$

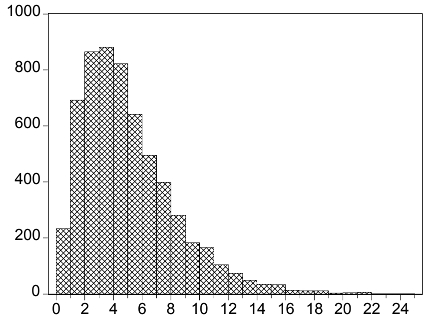
f) AR(1)TARCH(1,2)



Mean	5.072014
Variance	10.74956
Observations	6000.000

$T=5$

g) AR(3)GARCH(1,1)

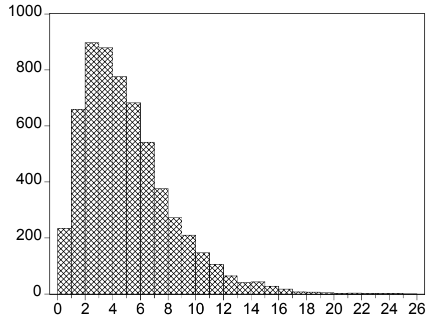


Mean	5.066279
Variance	10.69325
Observations	6000.000

Figure 3.10. Histograms and descriptive statistics of $\left\{\sum_{j=t-T+1}^{t} \hat{z}^2_{j+1|j}\right\}, t = T(T)30000$.

$T = 5$

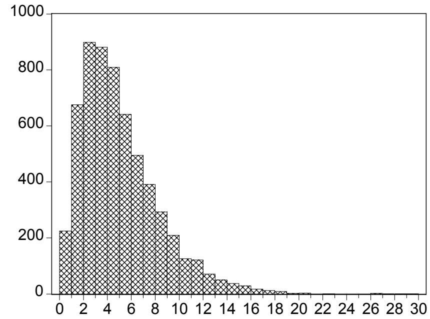
h) AR(3)EGARCH(1,1)



Mean	5.058722
Variance	10.50795
Observations	6000.000

$T = 5$

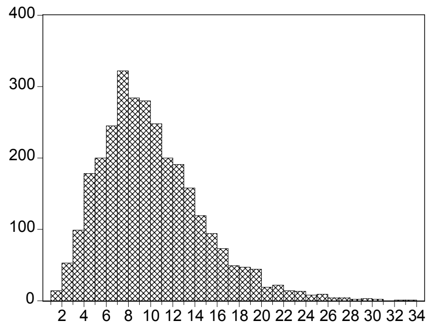
i) AR(3)TARCH(1,1)



Mean	5.074639
Variance	10.71557
Observations	6000.000

$T = 10$

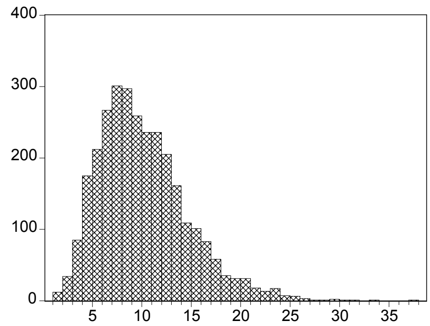
a) AR(1)GARCH(1,1)



Mean	10.09548
Variance	21.49013
Observations	3000.000

$T = 10$

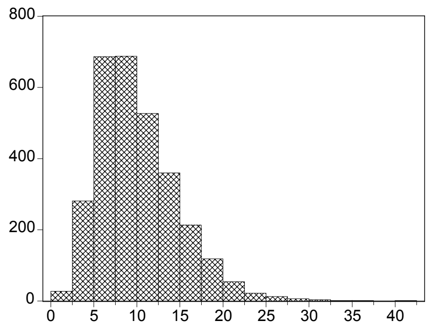
b) AR(1)EGARCH(1,1)



Mean	10.13434
Variance	20.45036
Observations	3000.000

$T = 10$

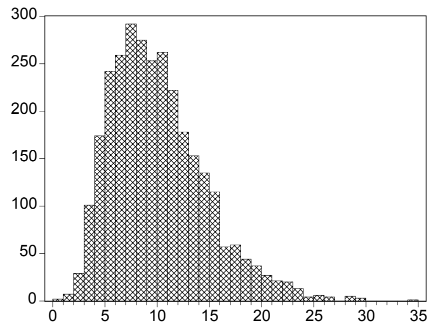
c) AR(1)TARCH(1,1)



Mean	10.09935
Variance	21.81810
Observations	3000.000

$T = 10$

d) AR(1)GARCH(1,2)

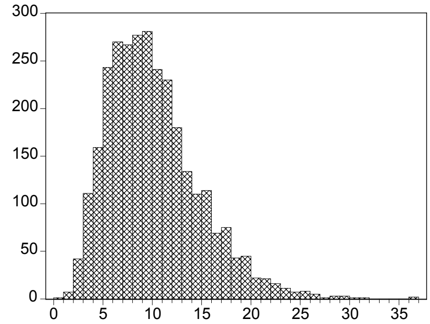


Mean	10.13076
Variance	20.71675
Observations	3000.000

Figure 3.10. Histograms and descriptive statistics of $\left\{\sum_{j=t-T+1}^{t} \hat{z}_{j+1|j}^{2}\right\}, t = T(T)30000$.

$T = 10$

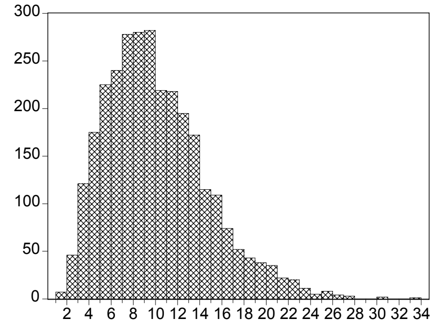
e) AR(1)TARCH(1,2)



Mean	10.14403
Variance	21.69734
Observations	3000.000

$T = 10$

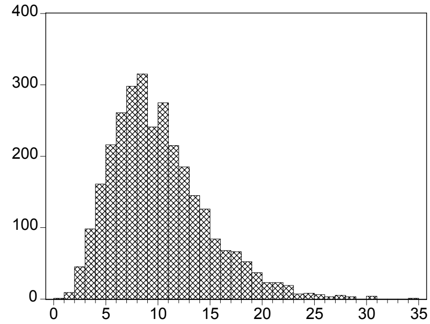
f) AR(3)GARCH(1,1)



Mean	10.13256
Variance	21.12526
Observations	3000.000

$T = 10$

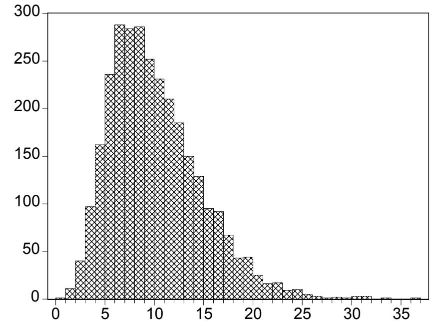
g) AR(3)EGARCH(1,1)



Mean	10.11744
Variance	21.04838
Observations	3000.000

$T = 10$

h) AR(3)TARCH(1,1)

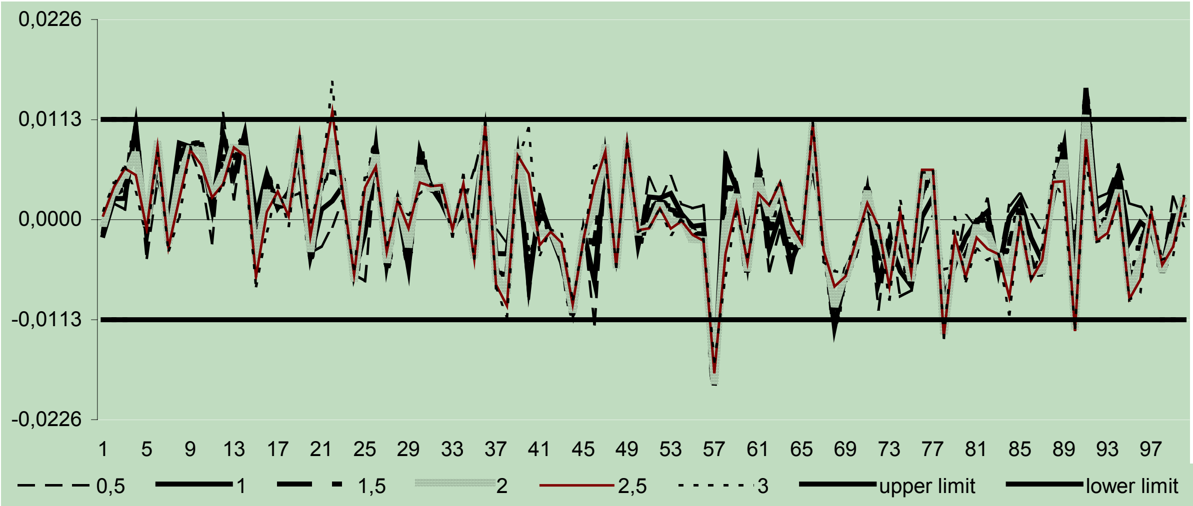


Mean	10.14928
Variance	21.53689
Observations	3000.000

Figure 3.11. Autocorrelation of transformations of the processes

$$Cor\left(\left|\hat{z}_{t+1|t}\right|^{d},\left|\hat{z}_{t+\tau+1|t+\tau}\right|^{d}\right),\ d=0.5(0.5)3,\ \tau=1(1)100$$

a) AR(1)GARCH(1,1)



b) AR(1)EGARCH(1,1)

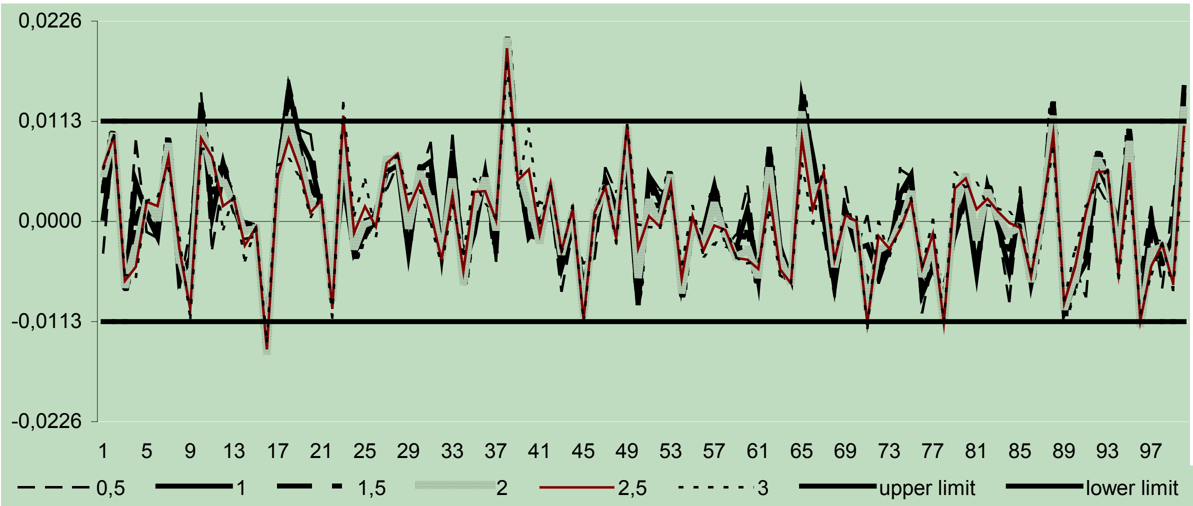
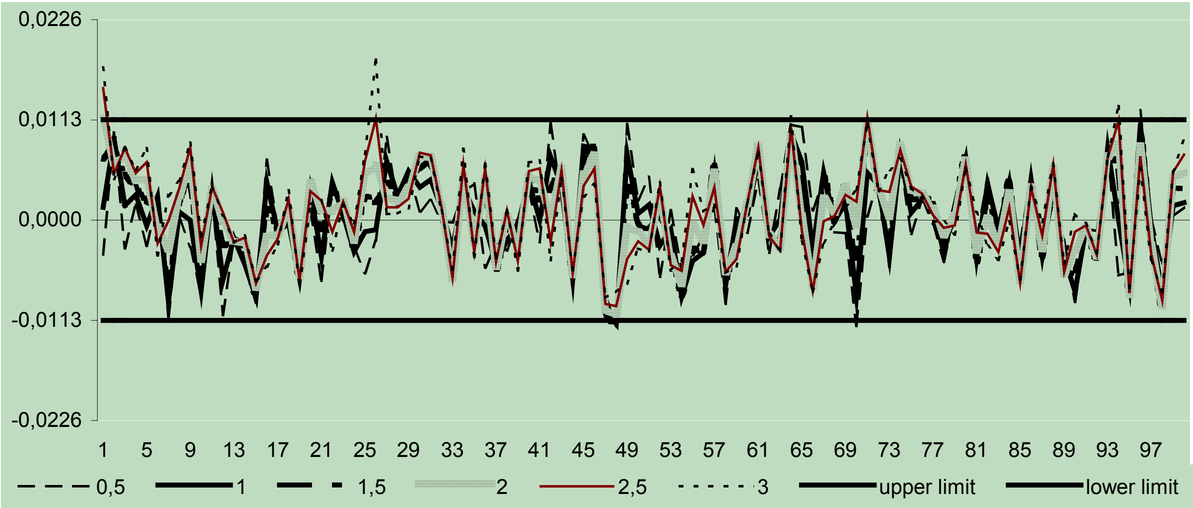


Figure 3.11. Autocorrelation of transformations of the processes

$$Cor\left(\left|\hat{z}_{t+1|t}\right|^{d}, \left|\hat{z}_{t+\tau+1|t+\tau}\right|^{d}\right),\ d = 0.5(0.5)3\ ,\ \tau = 1(1)100$$

c) AR(1)TARCH(1,1)



d) AR(1)GARCH(1,2)

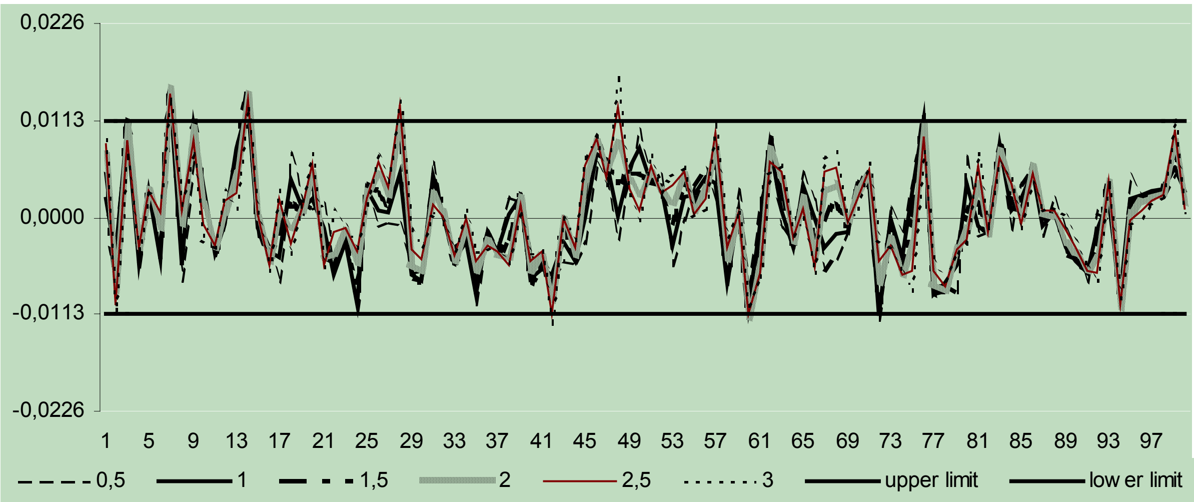
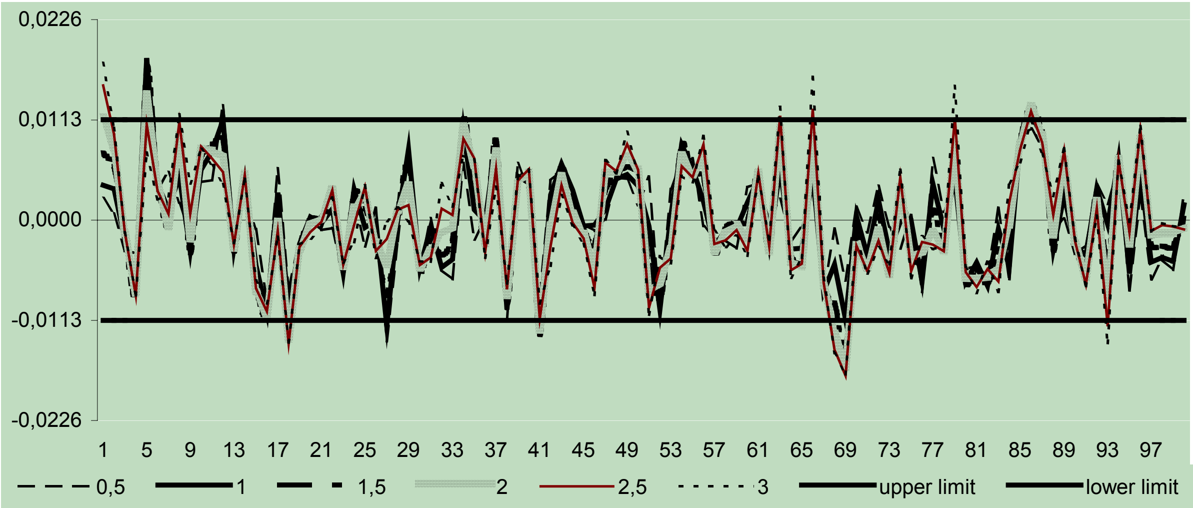


Figure 3.11. Autocorrelation of transformations of the processes

$$Cor\left(\left|\hat{z}_{t+1|t}\right|^{d}, \left|\hat{z}_{t+\tau+1|t+\tau}\right|^{d}\right),\ d = 0.5(0.5)3,\ \tau = 1(1)100$$

e) AR(1)TARCH(1,2)



f) AR(3)GARCH(1,1)

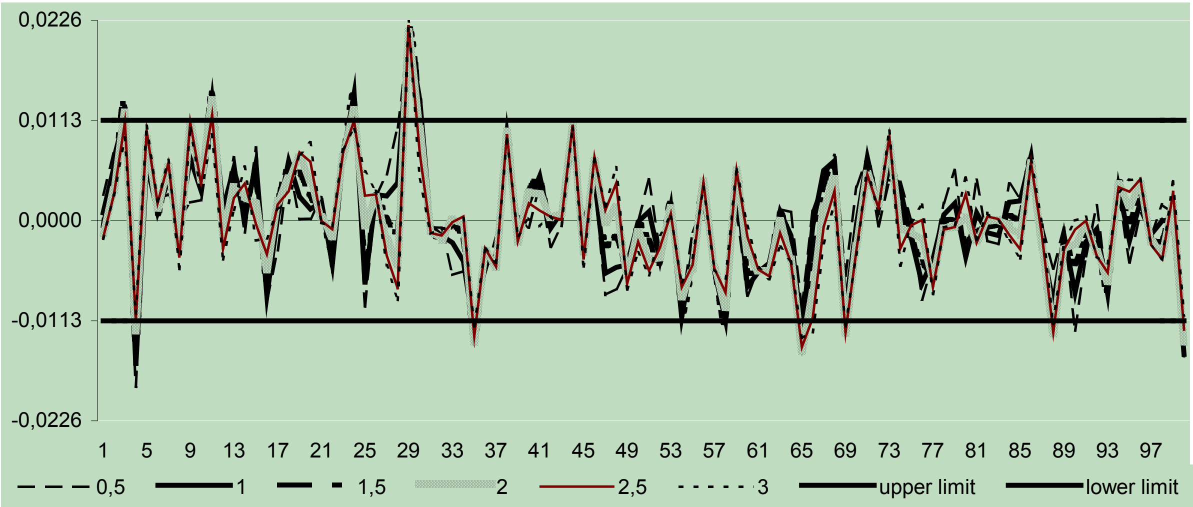
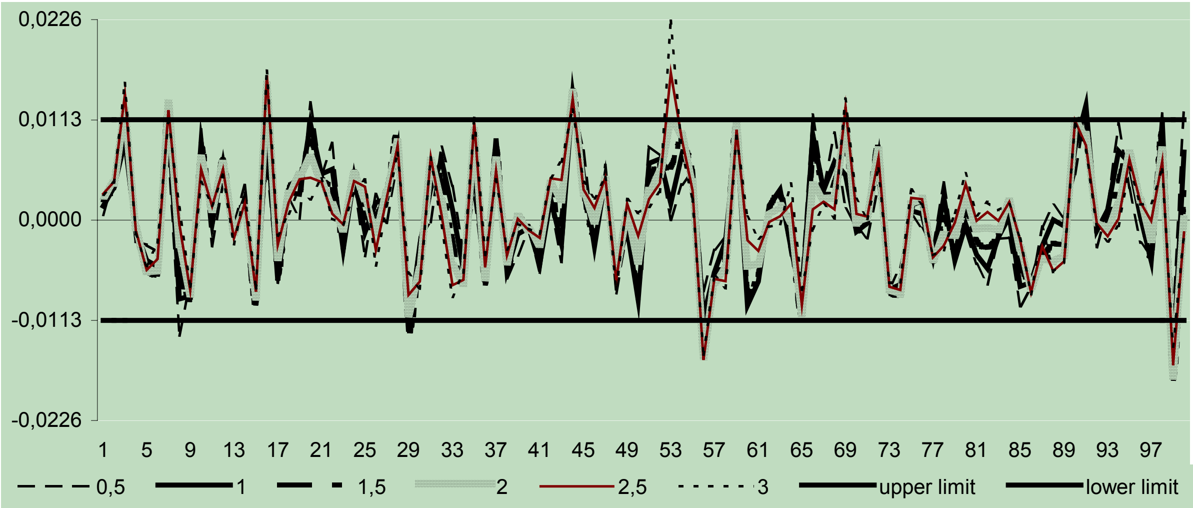


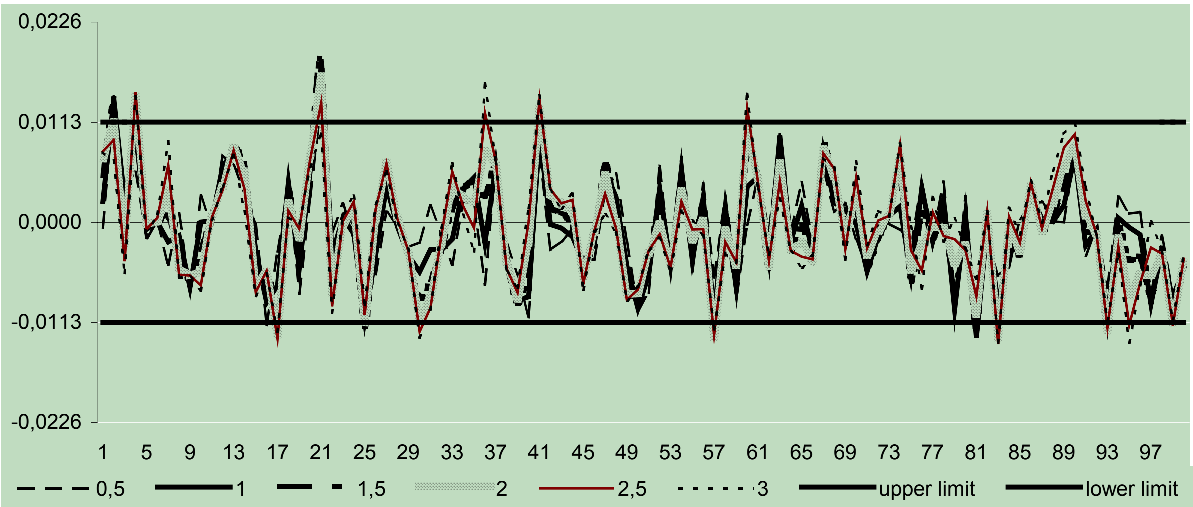
Figure 3.11. Autocorrelation of transformations of the processes

$$Cor\left(\left|\hat{z}_{t+1|t}\right|^{d}, \left|\hat{z}_{t+\tau+1|t+\tau}\right|^{d}\right),\ d = 0.5(0.5)3\,,\ \tau = 1(1)100$$

g) AR(3)EGARCH(1,1)



h) AR(3)TARCH(1,1)



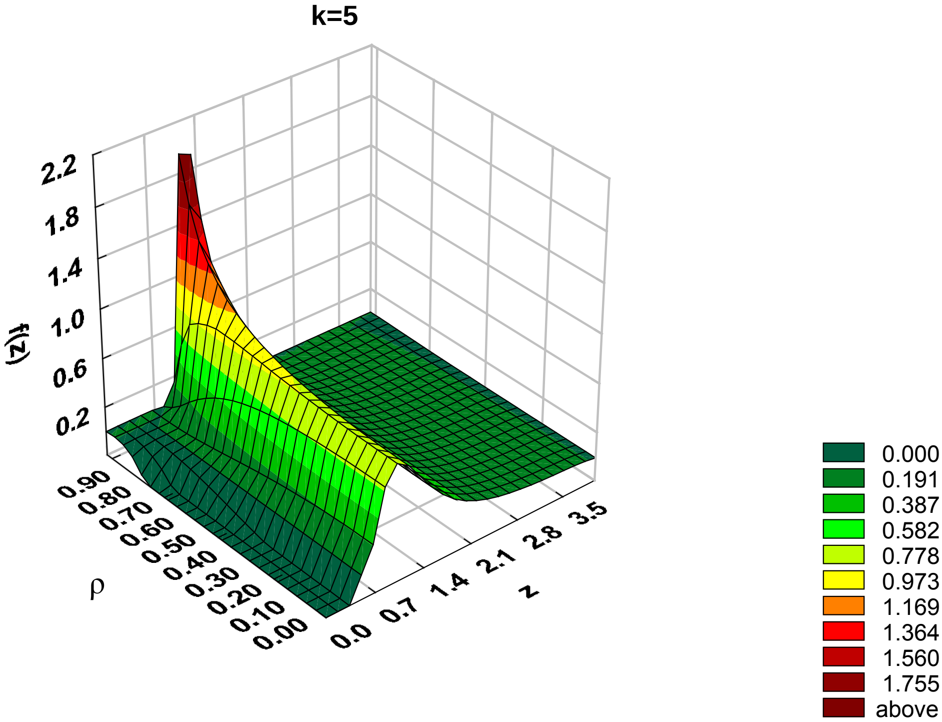
Appendix 4

- ***Figures 4.8-4.14. The probability density function of the Correlated Gamma Ratio Distribution***

- ***Table 4.5. Percentage points of the Correlated Gamma Ratio Distribution***

Figure 4.8. The probability density function of the Correlated Gamma Ratio Distribution

$$f(z)=\frac{(1-\rho^2)^k}{B(k,k)}z^{k-1}(1+z)^{-2k}\left[1-\left(\frac{2\rho}{z+1}\right)^2 z\right]^{-\frac{2k+1}{2}}$$, for $z \geq 0$, $0 \leq \rho < 1$, $k=5$



The probability density function of the Correlated Gamma Ratio Distribution for $k=5$ and $\rho = 0.1, 0.5, 0.7, 0.9$

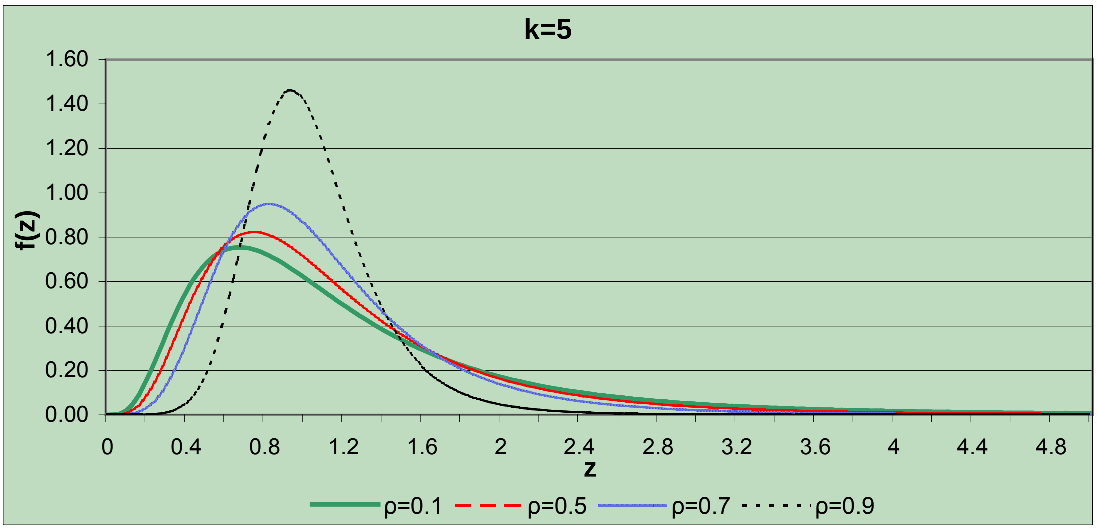
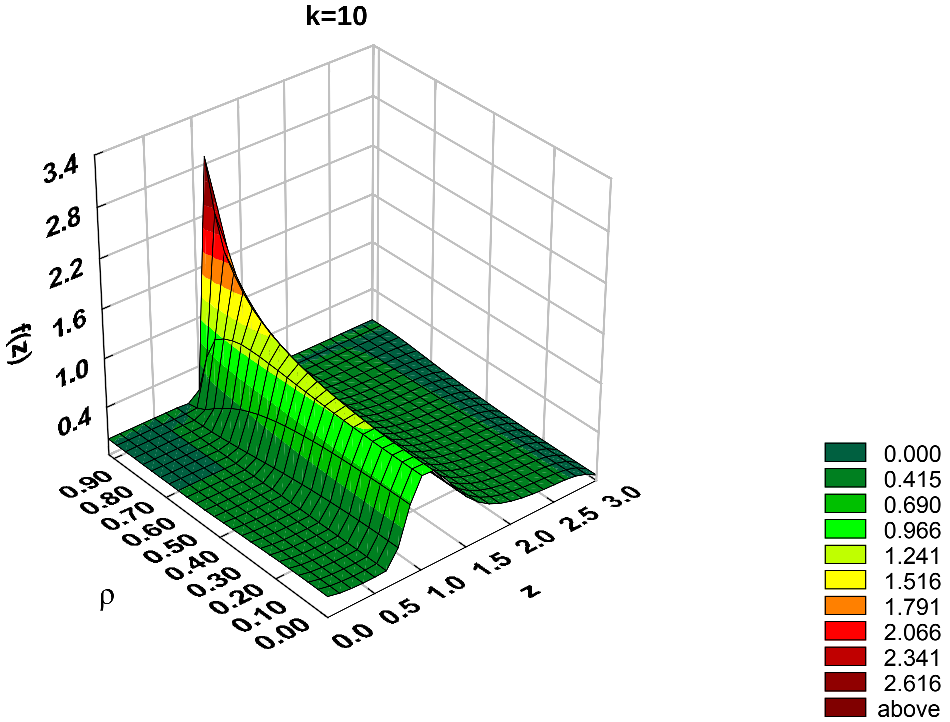


Figure 4.9. The probability density function of the Correlated Gamma Ratio Distribution

$$f(z)=\frac{\left(1-\rho^2\right)^k}{B(k,k)}z^{k-1}(1+z)^{-2k}\left[1-\left(\frac{2\rho}{z+1}\right)^2 z\right]^{-\frac{2k+1}{2}}$$, for $z\geq 0$, $0\leq\rho<1$, $k=10$



The probability density function of the Correlated Gamma Ratio Distribution for $k=10$ and $\rho=0.1,0.5,0.7,0.9$

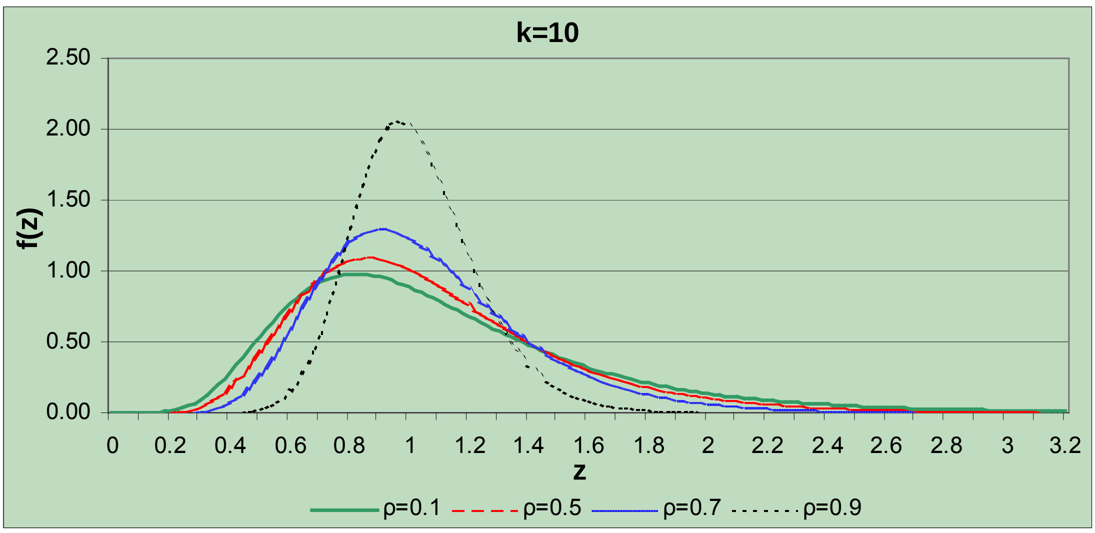
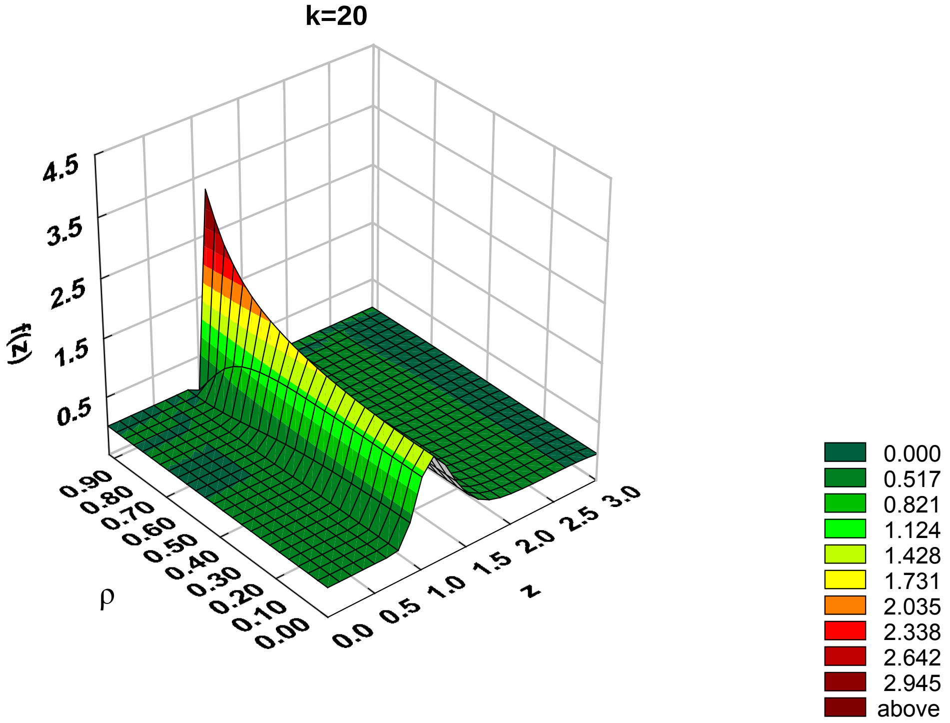


Figure 4.10. The probability density function of the Correlated Gamma Ratio Distribution

$$f(z) = \frac{(1-\rho^2)^k}{B(k,k)} z^{k-1}(1+z)^{-2k}\left[1-\left(\frac{2\rho}{z+1}\right)^2 z\right]^{-\frac{2k+1}{2}}$$, for $z \geq 0$, $0 \leq \rho < 1$, $k = 20$



The probability density function of the Correlated Gamma Ratio Distribution for $k = 20$ and $\rho = 0.1, 0.5, 0.7, 0.9$

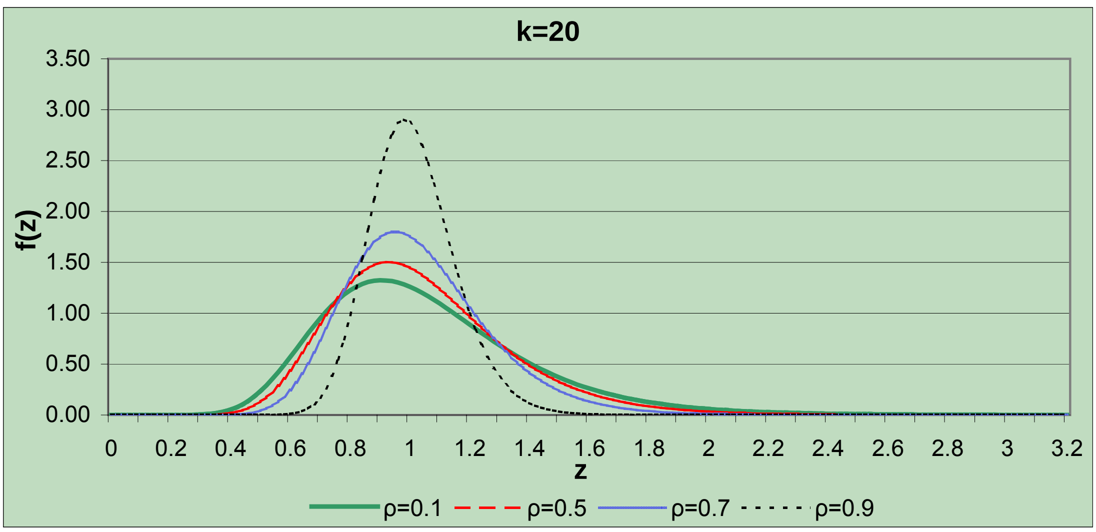
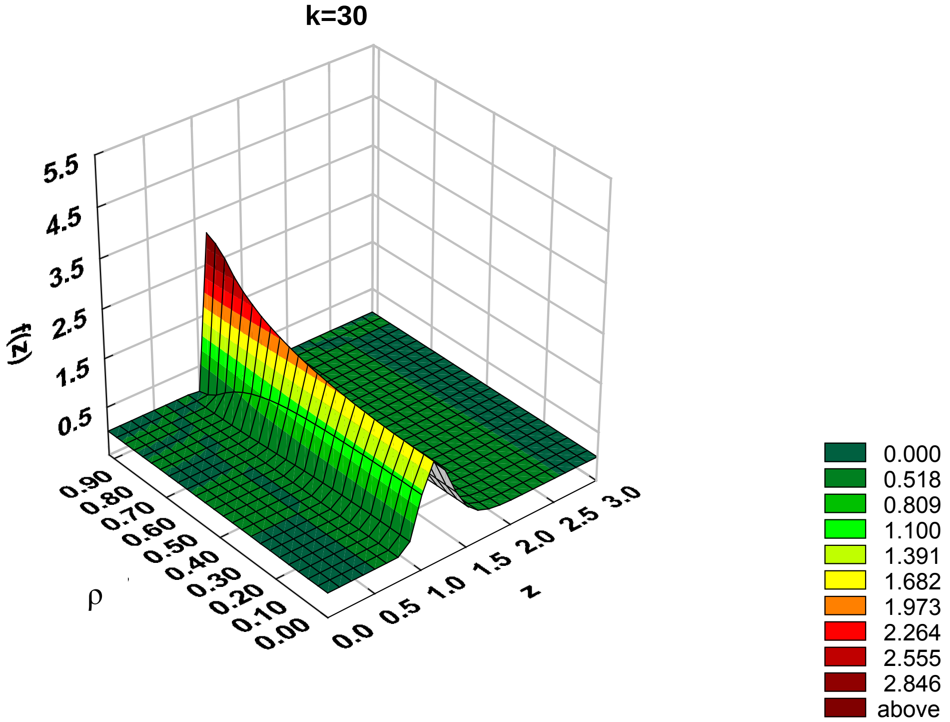


Figure 4.11. The probability density function of the Correlated Gamma Ratio Distribution

$$f(z)=\frac{\left(1-\rho^2\right)^k}{B(k,k)}z^{k-1}(1+z)^{-2k}\left[1-\left(\frac{2\rho}{z+1}\right)^2 z\right]^{-\frac{2k+1}{2}}, \text{ for } z\geq 0,\ 0\leq\rho<1,\ k=30$$



The probability density function of the Correlated Gamma Ratio Distribution for $k=30$ and $\rho=0.1,0.5,0.7,0.9$

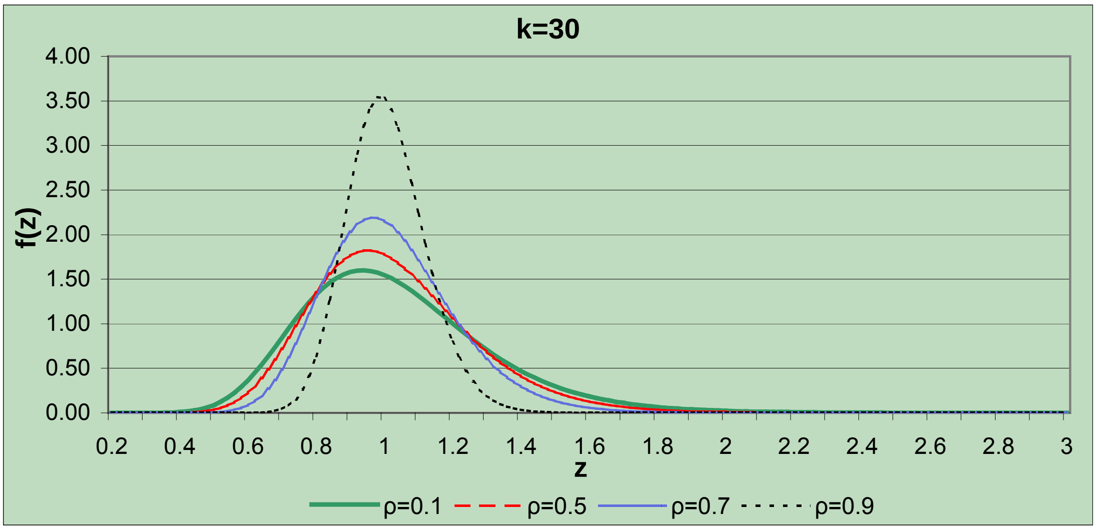
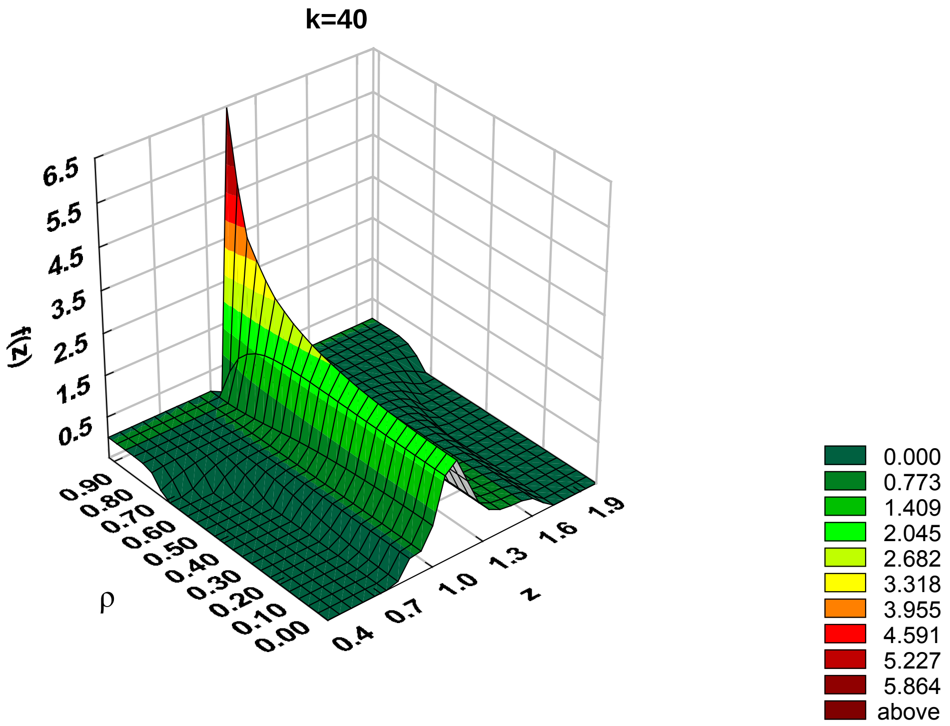


Figure 4.12. The probability density function of the Correlated Gamma Ratio Distribution

$$f(z)=\frac{\left(1-\rho^2\right)^k}{B(k,k)}z^{k-1}(1+z)^{-2k}\left[1-\left(\frac{2\rho}{z+1}\right)^2 z\right]^{-\frac{2k+1}{2}}$$, for $z\ge 0$, $0\le\rho<1$, $k=40$



The probability density function of the Correlated Gamma Ratio Distribution for $k=40$ and $\rho=0.1,0.5,0.7,0.9$

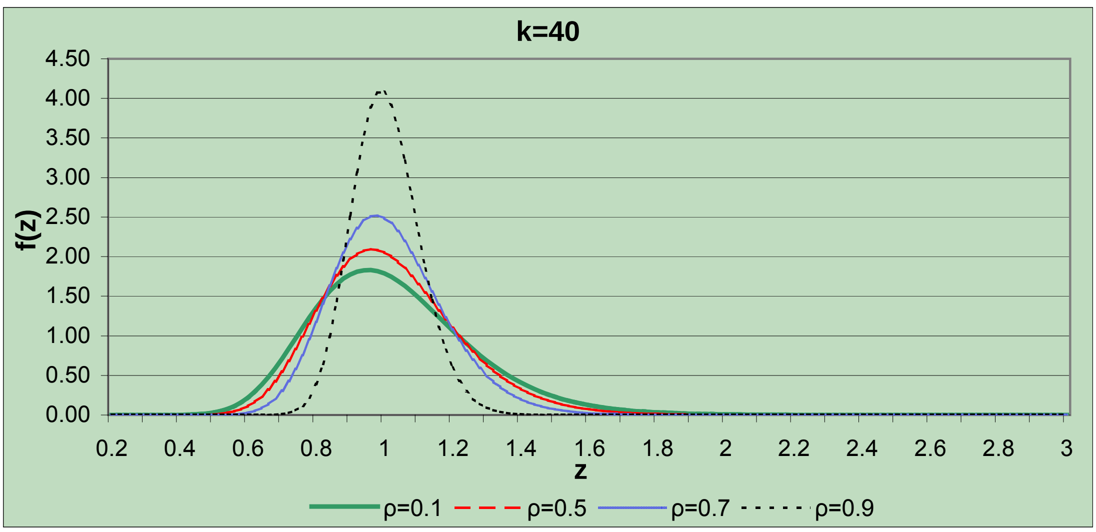
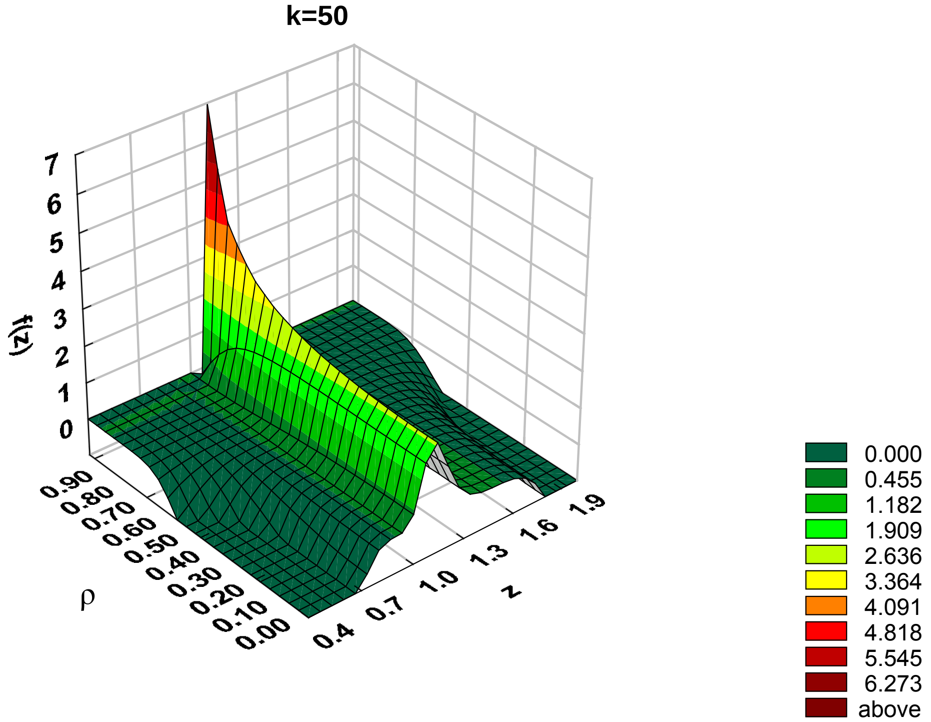


Figure 4.13. The probability density function of the Correlated Gamma Ratio Distribution

$$f(z)=\frac{\left(1-\rho^2\right)^k}{B(k,k)}z^{k-1}(1+z)^{-2k}\left[1-\left(\frac{2\rho}{z+1}\right)^2 z\right]^{-\frac{2k+1}{2}}$$, for $z\geq 0$, $0\leq\rho<1$, $k=50$



The probability density function of the Correlated Gamma Ratio Distribution for $k=50$ and $\rho=0.1,0.5,0.7,0.9$

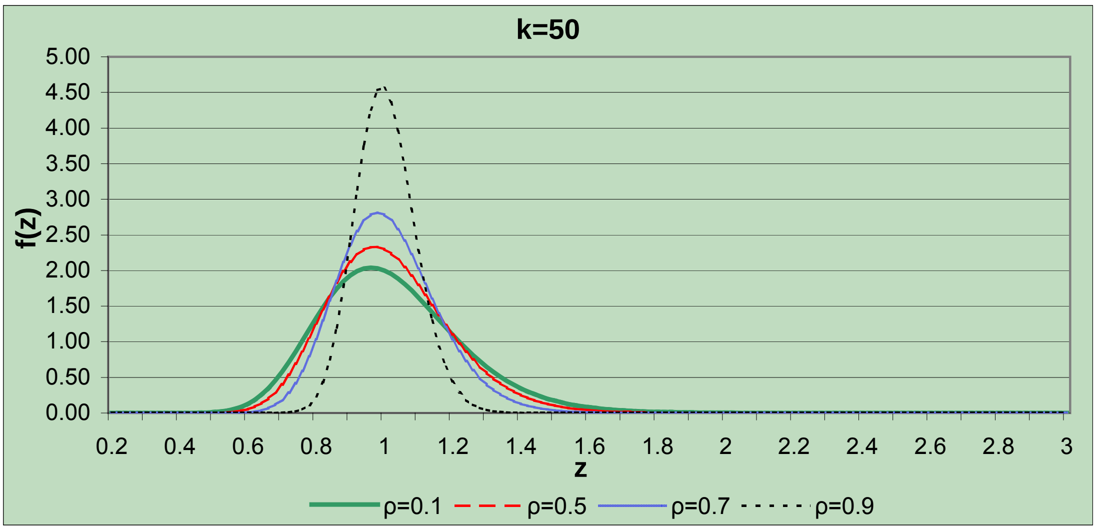
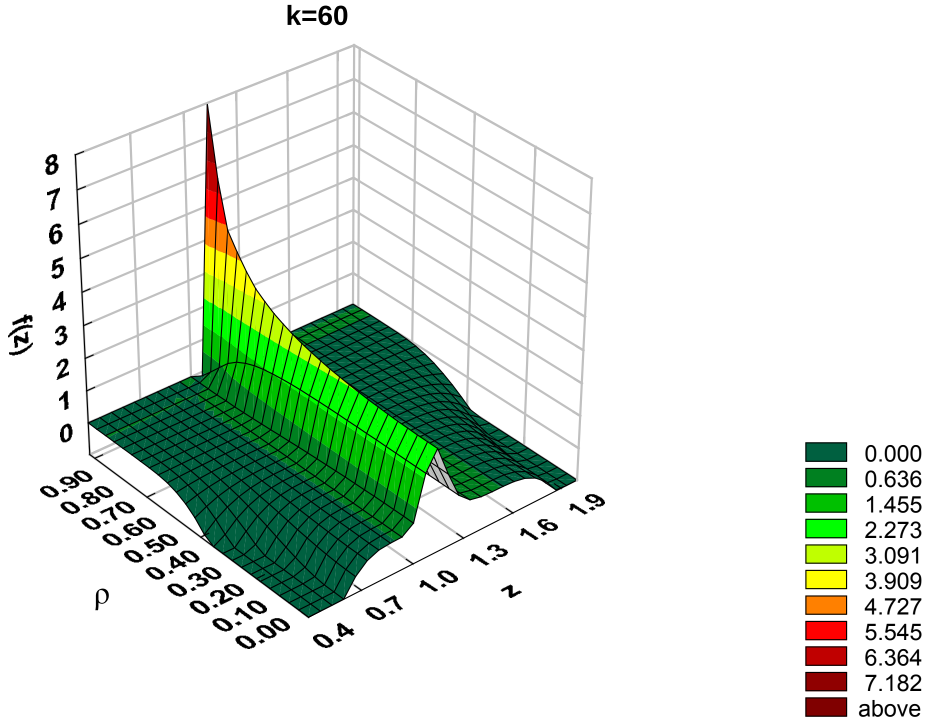


Figure 4.14. The probability density function of the Correlated Gamma Ratio Distribution

$$f(z)=\frac{\left(1-\rho^2\right)^k}{B(k,k)}z^{k-1}(1+z)^{-2k}\left[1-\left(\frac{2\rho}{z+1}\right)^2 z\right]^{-\frac{2k+1}{2}}$$, for $z \geq 0$, $0 \leq \rho < 1$, $k = 60$



The probability density function of the Correlated Gamma Ratio Distribution for $k = 60$ and $\rho = 0.1, 0.5, 0.7, 0.9$

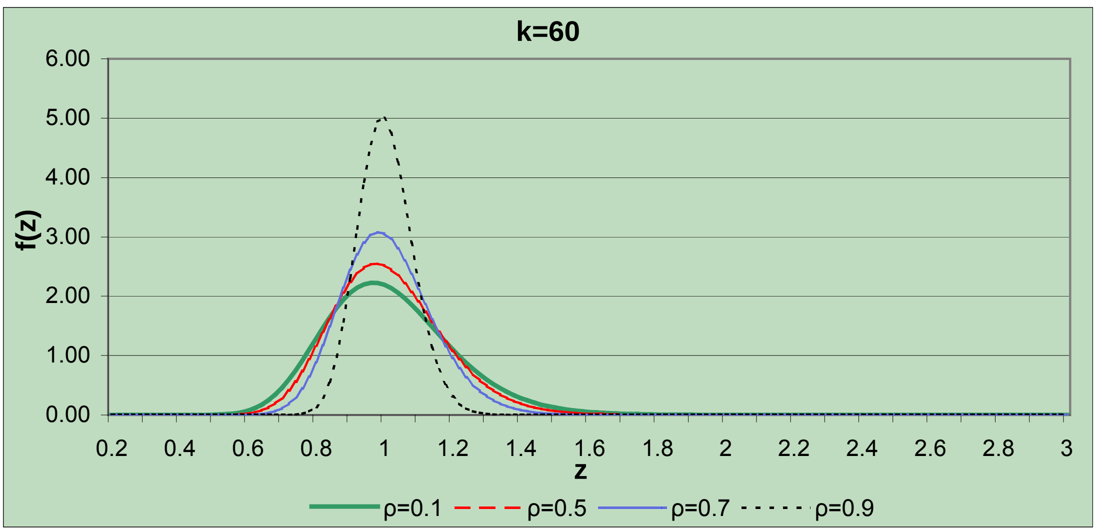
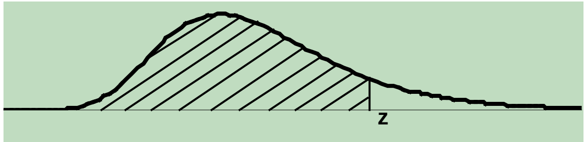


Table 4.5. Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.25$

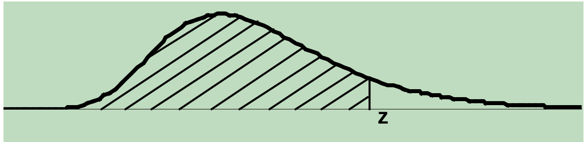
$$\Phi(z) = \int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
1	3.008	3.004	2.993	2.974	2.947	2.913	2.871	2.821	2.763	2.697
2	2.064	2.062	2.057	2.048	2.035	2.019	1.999	1.975	1.947	1.915
3	1.782	1.781	1.777	1.771	1.762	1.751	1.736	1.72	1.7	1.678
4	1.64	1.639	1.636	1.631	1.624	1.615	1.603	1.59	1.574	1.557
5	1.551	1.55	1.548	1.544	1.538	1.53	1.521	1.509	1.496	1.481
6	1.49	1.489	1.487	1.484	1.478	1.472	1.463	1.453	1.442	1.429
7	1.445	1.444	1.442	1.439	1.434	1.428	1.421	1.412	1.402	1.39
8	1.41	1.409	1.407	1.404	1.4	1.395	1.388	1.38	1.37	1.359
9	1.381	1.381	1.379	1.376	1.372	1.367	1.361	1.354	1.345	1.335
10	1.358	1.357	1.356	1.353	1.35	1.345	1.339	1.332	1.324	1.315
11	1.338	1.338	1.336	1.334	1.33	1.326	1.321	1.314	1.306	1.297
12	1.321	1.321	1.32	1.317	1.314	1.31	1.305	1.298	1.291	1.283
13	1.307	1.306	1.305	1.303	1.3	1.296	1.291	1.285	1.278	1.27
14	1.294	1.293	1.292	1.29	1.287	1.283	1.279	1.273	1.266	1.259
15	1.282	1.282	1.281	1.279	1.276	1.272	1.268	1.262	1.256	1.249
16	1.272	1.272	1.271	1.269	1.266	1.262	1.258	1.253	1.247	1.24
17	1.263	1.262	1.261	1.259	1.257	1.254	1.249	1.244	1.239	1.232
18	1.254	1.254	1.253	1.251	1.249	1.245	1.241	1.237	1.231	1.224
19	1.247	1.246	1.245	1.244	1.241	1.238	1.234	1.229	1.224	1.218
20	1.24	1.239	1.238	1.237	1.234	1.231	1.227	1.223	1.218	1.212
21	1.233	1.233	1.232	1.23	1.228	1.225	1.221	1.217	1.212	1.206
22	1.227	1.227	1.226	1.224	1.222	1.219	1.216	1.211	1.206	1.201
23	1.222	1.221	1.22	1.219	1.217	1.214	1.21	1.206	1.201	1.196
24	1.216	1.216	1.215	1.214	1.212	1.209	1.205	1.201	1.197	1.191
25	1.212	1.211	1.21	1.209	1.207	1.204	1.201	1.197	1.192	1.187
26	1.207	1.207	1.206	1.204	1.202	1.2	1.197	1.193	1.188	1.183
27	1.203	1.202	1.202	1.2	1.198	1.196	1.193	1.189	1.184	1.179
28	1.199	1.198	1.198	1.196	1.194	1.192	1.189	1.185	1.181	1.176
29	1.195	1.195	1.194	1.192	1.191	1.188	1.185	1.181	1.177	1.172
30	1.191	1.191	1.19	1.189	1.187	1.185	1.182	1.178	1.174	1.169

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.25$

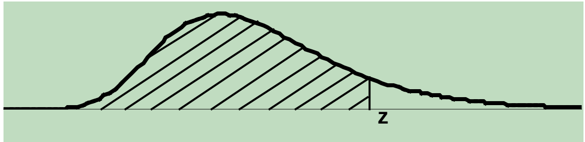
$$\Phi(z) = \int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
31	1.188	1.188	1.187	1.186	1.184	1.181	1.178	1.175	1.171	1.166
32	1.185	1.184	1.184	1.182	1.181	1.178	1.175	1.172	1.168	1.163
33	1.181	1.181	1.181	1.179	1.178	1.175	1.172	1.169	1.165	1.161
34	1.179	1.178	1.178	1.176	1.175	1.172	1.17	1.166	1.163	1.158
35	1.176	1.176	1.175	1.174	1.172	1.17	1.167	1.164	1.16	1.156
36	1.173	1.173	1.172	1.171	1.169	1.167	1.164	1.161	1.158	1.153
37	1.17	1.17	1.17	1.168	1.167	1.165	1.162	1.159	1.155	1.151
38	1.168	1.168	1.167	1.166	1.164	1.162	1.16	1.157	1.153	1.149
39	1.166	1.165	1.165	1.164	1.162	1.16	1.157	1.154	1.151	1.147
40	1.163	1.163	1.163	1.161	1.16	1.158	1.155	1.152	1.149	1.145
41	1.161	1.161	1.16	1.159	1.158	1.156	1.153	1.15	1.147	1.143
42	1.159	1.159	1.158	1.157	1.156	1.154	1.151	1.148	1.145	1.141
43	1.157	1.157	1.156	1.155	1.154	1.152	1.149	1.147	1.143	1.139
44	1.155	1.155	1.154	1.153	1.152	1.15	1.148	1.145	1.141	1.138
45	1.153	1.153	1.153	1.151	1.15	1.148	1.146	1.143	1.14	1.136
46	1.152	1.151	1.151	1.15	1.148	1.146	1.144	1.141	1.138	1.134
47	1.15	1.15	1.149	1.148	1.147	1.145	1.142	1.14	1.136	1.133
48	1.148	1.148	1.147	1.146	1.145	1.143	1.141	1.138	1.135	1.131
49	1.146	1.146	1.146	1.145	1.143	1.141	1.139	1.137	1.133	1.13
50	1.145	1.145	1.144	1.143	1.142	1.14	1.138	1.135	1.132	1.128
51	1.143	1.143	1.143	1.142	1.14	1.138	1.136	1.134	1.131	1.127
52	1.142	1.142	1.141	1.14	1.139	1.137	1.135	1.132	1.129	1.126
53	1.14	1.14	1.14	1.139	1.137	1.136	1.134	1.131	1.128	1.125
54	1.139	1.139	1.138	1.137	1.136	1.134	1.132	1.13	1.127	1.123
55	1.138	1.137	1.137	1.136	1.135	1.133	1.131	1.128	1.125	1.122
56	1.136	1.136	1.136	1.135	1.133	1.132	1.13	1.127	1.124	1.121
57	1.135	1.135	1.134	1.133	1.132	1.13	1.128	1.126	1.123	1.12
58	1.134	1.134	1.133	1.132	1.131	1.129	1.127	1.125	1.122	1.119
59	1.133	1.132	1.132	1.131	1.13	1.128	1.126	1.124	1.121	1.118
60	1.131	1.131	1.131	1.13	1.129	1.127	1.125	1.123	1.12	1.117

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.25$

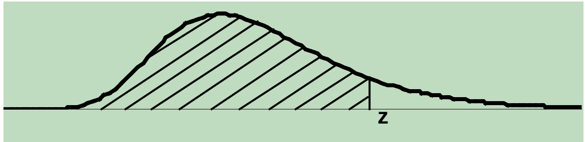
$$\Phi(z) = \int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
1	2.623	2.54	2.448	2.346	2.234	2.111	1.974	1.822	1.646	1.431
2	1.879	1.839	1.794	1.743	1.687	1.624	1.554	1.474	1.379	1.26
3	1.652	1.623	1.591	1.555	1.515	1.47	1.419	1.36	1.29	1.2
4	1.536	1.513	1.487	1.458	1.426	1.389	1.348	1.3	1.243	1.169
5	1.464	1.444	1.422	1.398	1.37	1.339	1.303	1.262	1.212	1.148
6	1.413	1.396	1.377	1.355	1.331	1.303	1.272	1.235	1.191	1.133
7	1.376	1.361	1.343	1.324	1.302	1.277	1.248	1.215	1.175	1.122
8	1.347	1.333	1.317	1.299	1.279	1.256	1.23	1.199	1.162	1.114
9	1.323	1.31	1.295	1.279	1.26	1.239	1.215	1.186	1.152	1.107
10	1.304	1.292	1.278	1.262	1.245	1.225	1.202	1.175	1.143	1.101
11	1.287	1.276	1.263	1.248	1.232	1.213	1.192	1.166	1.136	1.096
12	1.273	1.262	1.25	1.236	1.221	1.203	1.182	1.159	1.13	1.091
13	1.261	1.251	1.239	1.226	1.211	1.194	1.174	1.152	1.124	1.087
14	1.25	1.24	1.229	1.216	1.202	1.186	1.167	1.146	1.119	1.084
15	1.24	1.231	1.22	1.208	1.195	1.179	1.161	1.14	1.115	1.081
16	1.232	1.223	1.212	1.201	1.188	1.173	1.156	1.135	1.111	1.078
17	1.224	1.215	1.205	1.194	1.182	1.167	1.15	1.131	1.107	1.076
18	1.217	1.209	1.199	1.188	1.176	1.162	1.146	1.127	1.104	1.073
19	1.211	1.202	1.193	1.183	1.171	1.157	1.142	1.123	1.101	1.071
20	1.205	1.197	1.188	1.177	1.166	1.153	1.138	1.12	1.098	1.07
21	1.199	1.191	1.183	1.173	1.162	1.149	1.134	1.117	1.096	1.068
22	1.194	1.187	1.178	1.168	1.158	1.145	1.131	1.114	1.093	1.066
23	1.189	1.182	1.174	1.164	1.154	1.142	1.128	1.111	1.091	1.065
24	1.185	1.178	1.17	1.161	1.15	1.138	1.125	1.109	1.089	1.063
25	1.181	1.174	1.166	1.157	1.147	1.135	1.122	1.106	1.087	1.062
26	1.177	1.17	1.162	1.154	1.144	1.133	1.12	1.104	1.086	1.061
27	1.173	1.167	1.159	1.151	1.141	1.13	1.117	1.102	1.084	1.059
28	1.17	1.163	1.156	1.148	1.138	1.127	1.115	1.1	1.082	1.058
29	1.167	1.16	1.153	1.145	1.136	1.125	1.113	1.098	1.081	1.057
30	1.164	1.157	1.15	1.142	1.133	1.123	1.111	1.097	1.079	1.056

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.25$

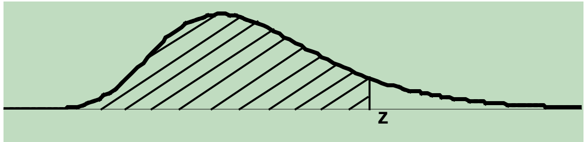
$$\Phi(z) = \int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
31	1.161	1.155	1.148	1.14	1.131	1.121	1.109	1.095	1.078	1.055
32	1.158	1.152	1.145	1.137	1.129	1.119	1.107	1.093	1.077	1.054
33	1.155	1.15	1.143	1.135	1.127	1.117	1.105	1.092	1.075	1.054
34	1.153	1.147	1.141	1.133	1.125	1.115	1.104	1.09	1.074	1.053
35	1.151	1.145	1.138	1.131	1.123	1.113	1.102	1.089	1.073	1.052
36	1.148	1.143	1.136	1.129	1.121	1.111	1.101	1.088	1.072	1.051
37	1.146	1.141	1.134	1.127	1.119	1.11	1.099	1.087	1.071	1.05
38	1.144	1.139	1.132	1.125	1.117	1.108	1.098	1.085	1.07	1.05
39	1.142	1.137	1.131	1.124	1.116	1.107	1.096	1.084	1.069	1.049
40	1.14	1.135	1.129	1.122	1.114	1.105	1.095	1.083	1.068	1.048
41	1.138	1.133	1.127	1.12	1.113	1.104	1.094	1.082	1.067	1.048
42	1.136	1.131	1.125	1.119	1.111	1.103	1.093	1.081	1.067	1.047
43	1.135	1.13	1.124	1.117	1.11	1.101	1.092	1.08	1.066	1.047
44	1.133	1.128	1.122	1.116	1.109	1.1	1.09	1.079	1.065	1.046
45	1.132	1.127	1.121	1.115	1.107	1.099	1.089	1.078	1.064	1.046
46	1.13	1.125	1.12	1.113	1.106	1.098	1.088	1.077	1.063	1.045
47	1.129	1.124	1.118	1.112	1.105	1.097	1.087	1.076	1.063	1.045
48	1.127	1.122	1.117	1.111	1.104	1.096	1.086	1.076	1.062	1.044
49	1.126	1.121	1.116	1.109	1.103	1.095	1.086	1.075	1.061	1.044
50	1.124	1.12	1.114	1.108	1.101	1.094	1.085	1.074	1.061	1.043
51	1.123	1.118	1.113	1.107	1.1	1.093	1.084	1.073	1.06	1.043
52	1.122	1.117	1.112	1.106	1.099	1.092	1.083	1.072	1.06	1.042
53	1.121	1.116	1.111	1.105	1.098	1.091	1.082	1.072	1.059	1.042
54	1.119	1.115	1.11	1.104	1.097	1.09	1.081	1.071	1.058	1.042
55	1.118	1.114	1.109	1.103	1.097	1.089	1.08	1.07	1.058	1.041
56	1.117	1.113	1.108	1.102	1.096	1.088	1.08	1.07	1.057	1.041
57	1.116	1.112	1.107	1.101	1.095	1.087	1.079	1.069	1.057	1.04
58	1.115	1.111	1.106	1.1	1.094	1.087	1.078	1.068	1.056	1.04
59	1.114	1.11	1.105	1.099	1.093	1.086	1.078	1.068	1.056	1.04
60	1.113	1.109	1.104	1.098	1.092	1.085	1.077	1.067	1.055	1.039

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.20$

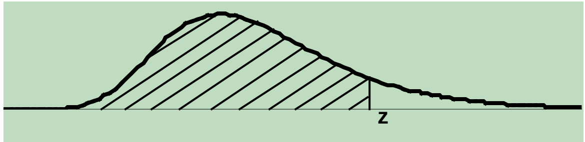
$$\Phi(z) = \int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
1	4.013	4.006	3.988	3.958	3.915	3.861	3.794	3.714	3.622	3.518
2	2.483	2.48	2.472	2.459	2.44	2.416	2.387	2.352	2.311	2.265
3	2.062	2.06	2.055	2.046	2.033	2.017	1.997	1.973	1.945	1.914
4	1.856	1.855	1.851	1.844	1.834	1.821	1.806	1.787	1.765	1.741
5	1.732	1.73	1.727	1.721	1.713	1.702	1.689	1.674	1.656	1.635
6	1.646	1.645	1.642	1.637	1.63	1.621	1.61	1.596	1.58	1.562
7	1.584	1.583	1.58	1.576	1.57	1.561	1.551	1.539	1.525	1.509
8	1.536	1.535	1.533	1.528	1.523	1.515	1.506	1.495	1.482	1.468
9	1.497	1.497	1.494	1.491	1.485	1.478	1.47	1.46	1.448	1.435
10	1.466	1.465	1.463	1.459	1.454	1.448	1.44	1.431	1.42	1.407
11	1.439	1.438	1.436	1.433	1.429	1.423	1.415	1.407	1.396	1.385
12	1.416	1.416	1.414	1.411	1.406	1.401	1.394	1.386	1.376	1.365
13	1.397	1.396	1.394	1.391	1.387	1.382	1.375	1.368	1.359	1.348
14	1.379	1.379	1.377	1.374	1.371	1.365	1.359	1.352	1.343	1.333
15	1.364	1.364	1.362	1.359	1.356	1.351	1.345	1.338	1.329	1.32
16	1.35	1.35	1.348	1.346	1.342	1.338	1.332	1.325	1.317	1.308
17	1.338	1.338	1.336	1.334	1.33	1.326	1.32	1.314	1.306	1.297
18	1.327	1.327	1.325	1.323	1.32	1.315	1.31	1.304	1.296	1.288
19	1.317	1.316	1.315	1.313	1.31	1.306	1.3	1.294	1.287	1.279
20	1.308	1.307	1.306	1.304	1.301	1.297	1.292	1.286	1.279	1.271
21	1.299	1.299	1.297	1.295	1.292	1.288	1.284	1.278	1.271	1.263
22	1.291	1.291	1.29	1.287	1.285	1.281	1.276	1.271	1.264	1.257
23	1.284	1.283	1.282	1.28	1.277	1.274	1.269	1.264	1.257	1.25
24	1.277	1.277	1.275	1.274	1.271	1.267	1.263	1.258	1.251	1.244
25	1.271	1.27	1.269	1.267	1.265	1.261	1.257	1.252	1.246	1.239
26	1.265	1.264	1.263	1.261	1.259	1.255	1.251	1.246	1.24	1.233
27	1.259	1.259	1.258	1.256	1.253	1.25	1.246	1.241	1.235	1.229
28	1.254	1.253	1.252	1.251	1.248	1.245	1.241	1.236	1.23	1.224
29	1.249	1.248	1.247	1.246	1.243	1.24	1.236	1.231	1.226	1.22
30	1.244	1.244	1.243	1.241	1.239	1.236	1.232	1.227	1.222	1.215

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.20$

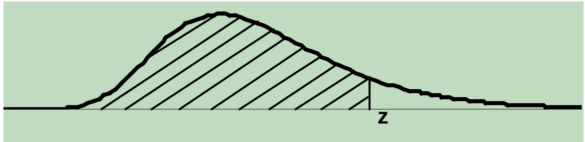
$$\Phi(z) = \int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
31	1.24	1.239	1.238	1.237	1.234	1.231	1.228	1.223	1.218	1.212
32	1.235	1.235	1.234	1.232	1.23	1.227	1.224	1.219	1.214	1.208
33	1.231	1.231	1.23	1.229	1.226	1.223	1.22	1.215	1.21	1.204
34	1.228	1.227	1.226	1.225	1.223	1.22	1.216	1.212	1.207	1.201
35	1.224	1.224	1.223	1.221	1.219	1.216	1.213	1.208	1.204	1.198
36	1.22	1.22	1.219	1.218	1.216	1.213	1.209	1.205	1.2	1.195
37	1.217	1.217	1.216	1.214	1.212	1.21	1.206	1.202	1.197	1.192
38	1.214	1.214	1.213	1.211	1.209	1.207	1.203	1.199	1.194	1.189
39	1.211	1.211	1.21	1.208	1.206	1.204	1.2	1.196	1.192	1.186
40	1.208	1.208	1.207	1.205	1.203	1.201	1.198	1.194	1.189	1.184
41	1.205	1.205	1.204	1.203	1.201	1.198	1.195	1.191	1.187	1.181
42	1.202	1.202	1.201	1.2	1.198	1.195	1.192	1.189	1.184	1.179
43	1.2	1.2	1.199	1.197	1.195	1.193	1.19	1.186	1.182	1.177
44	1.197	1.197	1.196	1.195	1.193	1.19	1.187	1.184	1.179	1.175
45	1.195	1.195	1.194	1.192	1.191	1.188	1.185	1.182	1.177	1.172
46	1.193	1.192	1.191	1.19	1.188	1.186	1.183	1.179	1.175	1.17
47	1.19	1.19	1.189	1.188	1.186	1.184	1.181	1.177	1.173	1.168
48	1.188	1.188	1.187	1.186	1.184	1.182	1.179	1.175	1.171	1.166
49	1.186	1.186	1.185	1.184	1.182	1.18	1.177	1.173	1.169	1.165
50	1.184	1.184	1.183	1.182	1.18	1.178	1.175	1.171	1.167	1.163
51	1.182	1.182	1.181	1.18	1.178	1.176	1.173	1.17	1.166	1.161
52	1.18	1.18	1.179	1.178	1.176	1.174	1.171	1.168	1.164	1.159
53	1.178	1.178	1.177	1.176	1.174	1.172	1.169	1.166	1.162	1.158
54	1.176	1.176	1.175	1.174	1.173	1.17	1.168	1.164	1.161	1.156
55	1.175	1.174	1.174	1.172	1.171	1.169	1.166	1.163	1.159	1.155
56	1.173	1.173	1.172	1.171	1.169	1.167	1.164	1.161	1.157	1.153
57	1.171	1.171	1.17	1.169	1.168	1.165	1.163	1.16	1.156	1.152
58	1.17	1.169	1.169	1.168	1.166	1.164	1.161	1.158	1.154	1.15
59	1.168	1.168	1.167	1.166	1.164	1.162	1.16	1.157	1.153	1.149
60	1.167	1.166	1.166	1.165	1.163	1.161	1.158	1.155	1.152	1.148

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.20$

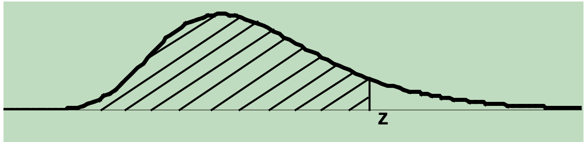
$$\Phi(z) = \int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
1	3.4	3.269	3.125	2.966	2.792	2.602	2.394	2.163	1.902	1.591
2	2.212	2.153	2.088	2.015	1.935	1.846	1.746	1.634	1.503	1.34
3	1.878	1.837	1.792	1.742	1.686	1.623	1.553	1.473	1.378	1.259
4	1.712	1.68	1.645	1.605	1.561	1.511	1.455	1.39	1.314	1.216
5	1.611	1.584	1.555	1.521	1.484	1.442	1.394	1.339	1.273	1.189
6	1.542	1.518	1.492	1.463	1.43	1.393	1.351	1.303	1.245	1.17
7	1.491	1.47	1.446	1.42	1.391	1.358	1.32	1.276	1.224	1.156
8	1.451	1.432	1.411	1.387	1.36	1.33	1.295	1.255	1.207	1.144
9	1.419	1.402	1.382	1.36	1.335	1.307	1.275	1.238	1.194	1.135
10	1.393	1.377	1.359	1.338	1.315	1.289	1.259	1.224	1.182	1.128
11	1.371	1.356	1.339	1.319	1.298	1.273	1.245	1.212	1.173	1.121
12	1.352	1.338	1.322	1.304	1.283	1.26	1.233	1.202	1.165	1.115
13	1.336	1.322	1.307	1.29	1.27	1.248	1.223	1.193	1.157	1.11
14	1.322	1.309	1.294	1.277	1.259	1.238	1.214	1.185	1.151	1.106
15	1.309	1.296	1.282	1.267	1.249	1.229	1.205	1.178	1.145	1.102
16	1.298	1.286	1.272	1.257	1.24	1.22	1.198	1.172	1.14	1.099
17	1.287	1.276	1.263	1.248	1.232	1.213	1.191	1.166	1.136	1.096
18	1.278	1.267	1.254	1.24	1.224	1.206	1.185	1.161	1.132	1.093
19	1.269	1.259	1.247	1.233	1.218	1.2	1.18	1.156	1.128	1.09
20	1.262	1.251	1.24	1.226	1.211	1.194	1.175	1.152	1.124	1.088
21	1.255	1.244	1.233	1.22	1.206	1.189	1.17	1.148	1.121	1.085
22	1.248	1.238	1.227	1.215	1.201	1.184	1.166	1.144	1.118	1.083
23	1.242	1.232	1.221	1.209	1.196	1.18	1.162	1.141	1.115	1.081
24	1.236	1.227	1.216	1.204	1.191	1.176	1.158	1.138	1.113	1.08
25	1.231	1.222	1.211	1.2	1.187	1.172	1.155	1.135	1.11	1.078
26	1.226	1.217	1.207	1.196	1.183	1.168	1.152	1.132	1.108	1.076
27	1.221	1.212	1.203	1.192	1.179	1.165	1.148	1.129	1.106	1.075
28	1.217	1.208	1.199	1.188	1.175	1.162	1.146	1.127	1.104	1.073
29	1.212	1.204	1.195	1.184	1.172	1.159	1.143	1.124	1.102	1.072
30	1.208	1.2	1.191	1.181	1.169	1.156	1.14	1.122	1.1	1.071

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.20$

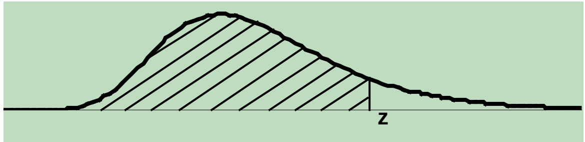
$$\Phi(z) = \int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
31	1.205	1.197	1.188	1.178	1.166	1.153	1.138	1.12	1.098	1.07
32	1.201	1.193	1.184	1.174	1.163	1.15	1.135	1.118	1.097	1.068
33	1.198	1.19	1.181	1.172	1.16	1.148	1.133	1.116	1.095	1.067
34	1.194	1.187	1.178	1.169	1.158	1.145	1.131	1.114	1.094	1.066
35	1.191	1.184	1.176	1.166	1.155	1.143	1.129	1.112	1.092	1.065
36	1.188	1.181	1.173	1.164	1.153	1.141	1.127	1.111	1.091	1.064
37	1.186	1.178	1.17	1.161	1.151	1.139	1.125	1.109	1.09	1.063
38	1.183	1.176	1.168	1.159	1.149	1.137	1.123	1.108	1.088	1.062
39	1.18	1.173	1.166	1.157	1.147	1.135	1.122	1.106	1.087	1.062
40	1.178	1.171	1.163	1.154	1.145	1.133	1.12	1.105	1.086	1.061
41	1.175	1.169	1.161	1.152	1.143	1.131	1.119	1.103	1.085	1.06
42	1.173	1.167	1.159	1.15	1.141	1.13	1.117	1.102	1.084	1.059
43	1.171	1.164	1.157	1.149	1.139	1.128	1.116	1.101	1.083	1.059
44	1.169	1.162	1.155	1.147	1.137	1.127	1.114	1.1	1.082	1.058
45	1.167	1.16	1.153	1.145	1.136	1.125	1.113	1.098	1.081	1.057
46	1.165	1.158	1.151	1.143	1.134	1.124	1.112	1.097	1.08	1.057
47	1.163	1.157	1.15	1.142	1.133	1.122	1.11	1.096	1.079	1.056
48	1.161	1.155	1.148	1.14	1.131	1.121	1.109	1.095	1.078	1.055
49	1.159	1.153	1.146	1.138	1.13	1.12	1.108	1.094	1.077	1.055
50	1.158	1.151	1.145	1.137	1.128	1.118	1.107	1.093	1.076	1.054
51	1.156	1.15	1.143	1.136	1.127	1.117	1.106	1.092	1.076	1.054
52	1.154	1.148	1.142	1.134	1.126	1.116	1.105	1.091	1.075	1.053
53	1.153	1.147	1.14	1.133	1.124	1.115	1.103	1.09	1.074	1.053
54	1.151	1.145	1.139	1.131	1.123	1.113	1.102	1.089	1.073	1.052
55	1.15	1.144	1.137	1.13	1.122	1.112	1.101	1.089	1.073	1.052
56	1.148	1.143	1.136	1.129	1.121	1.111	1.1	1.088	1.072	1.051
57	1.147	1.141	1.135	1.128	1.12	1.11	1.1	1.087	1.071	1.051
58	1.145	1.14	1.134	1.127	1.118	1.109	1.099	1.086	1.071	1.05
59	1.144	1.139	1.132	1.125	1.117	1.108	1.098	1.085	1.07	1.05
60	1.143	1.137	1.131	1.124	1.116	1.107	1.097	1.085	1.07	1.049

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.15$

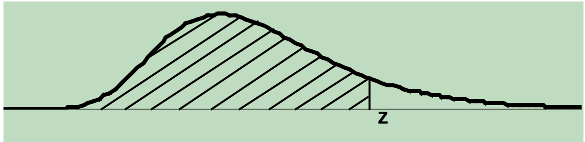
$$\Phi(z) = \int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
1	5.689	5.679	5.649	5.599	5.528	5.438	5.327	5.197	5.045	4.873
2	3.092	3.088	3.076	3.056	3.028	2.992	2.949	2.896	2.836	2.767
3	2.449	2.447	2.439	2.426	2.408	2.385	2.356	2.322	2.282	2.237
4	2.149	2.147	2.141	2.131	2.117	2.099	2.077	2.051	2.021	1.986
5	1.97	1.969	1.964	1.956	1.945	1.93	1.912	1.89	1.865	1.837
6	1.851	1.849	1.845	1.838	1.829	1.816	1.801	1.782	1.761	1.736
7	1.764	1.763	1.759	1.753	1.744	1.733	1.72	1.703	1.684	1.662
8	1.698	1.697	1.693	1.688	1.68	1.67	1.658	1.643	1.626	1.606
9	1.645	1.644	1.641	1.636	1.629	1.62	1.609	1.595	1.579	1.561
10	1.602	1.601	1.599	1.594	1.587	1.579	1.568	1.556	1.541	1.525
11	1.566	1.566	1.563	1.559	1.553	1.545	1.535	1.523	1.51	1.494
12	1.536	1.535	1.533	1.529	1.523	1.515	1.506	1.495	1.483	1.468
13	1.51	1.509	1.507	1.503	1.497	1.49	1.482	1.471	1.459	1.445
14	1.487	1.486	1.484	1.48	1.475	1.468	1.46	1.45	1.439	1.426
15	1.466	1.466	1.464	1.46	1.455	1.449	1.441	1.432	1.421	1.408
16	1.448	1.448	1.446	1.442	1.438	1.431	1.424	1.415	1.405	1.393
17	1.432	1.431	1.429	1.426	1.422	1.416	1.409	1.4	1.39	1.379
18	1.417	1.417	1.415	1.412	1.407	1.402	1.395	1.387	1.377	1.366
19	1.404	1.403	1.402	1.399	1.394	1.389	1.382	1.374	1.365	1.354
20	1.392	1.391	1.389	1.387	1.383	1.377	1.371	1.363	1.354	1.344
21	1.381	1.38	1.378	1.376	1.372	1.367	1.36	1.353	1.344	1.334
22	1.37	1.37	1.368	1.365	1.362	1.357	1.351	1.343	1.335	1.325
23	1.361	1.36	1.358	1.356	1.352	1.347	1.342	1.335	1.326	1.317
24	1.352	1.351	1.35	1.347	1.343	1.339	1.333	1.326	1.318	1.309
25	1.343	1.343	1.341	1.339	1.335	1.331	1.325	1.319	1.311	1.302
26	1.336	1.335	1.334	1.331	1.328	1.323	1.318	1.311	1.304	1.295
27	1.328	1.328	1.326	1.324	1.321	1.316	1.311	1.305	1.297	1.289
28	1.321	1.321	1.32	1.317	1.314	1.31	1.305	1.298	1.291	1.283
29	1.315	1.314	1.313	1.311	1.308	1.304	1.299	1.292	1.285	1.277
30	1.309	1.308	1.307	1.305	1.302	1.298	1.293	1.287	1.28	1.272

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.15$

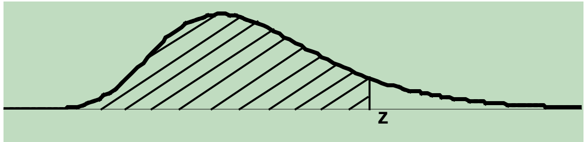
$$\Phi(z)=\int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
31	1.303	1.303	1.301	1.299	1.296	1.292	1.287	1.282	1.275	1.267
32	1.298	1.297	1.296	1.294	1.291	1.287	1.282	1.276	1.27	1.262
33	1.292	1.292	1.291	1.289	1.286	1.282	1.277	1.272	1.265	1.258
34	1.287	1.287	1.286	1.284	1.281	1.277	1.273	1.267	1.261	1.253
35	1.283	1.282	1.281	1.279	1.276	1.273	1.268	1.263	1.256	1.249
36	1.278	1.278	1.277	1.275	1.272	1.268	1.264	1.259	1.252	1.245
37	1.274	1.274	1.272	1.27	1.268	1.264	1.26	1.255	1.249	1.241
38	1.27	1.269	1.268	1.266	1.264	1.26	1.256	1.251	1.245	1.238
39	1.266	1.265	1.264	1.262	1.26	1.256	1.252	1.247	1.241	1.234
40	1.262	1.262	1.261	1.259	1.256	1.253	1.249	1.244	1.238	1.231
41	1.258	1.258	1.257	1.255	1.253	1.249	1.245	1.24	1.235	1.228
42	1.255	1.255	1.254	1.252	1.249	1.246	1.242	1.237	1.231	1.225
43	1.252	1.251	1.25	1.248	1.246	1.243	1.239	1.234	1.228	1.222
44	1.248	1.248	1.247	1.245	1.243	1.24	1.236	1.231	1.226	1.219
45	1.245	1.245	1.244	1.242	1.24	1.237	1.233	1.228	1.223	1.216
46	1.242	1.242	1.241	1.239	1.237	1.234	1.23	1.225	1.22	1.214
47	1.239	1.239	1.238	1.236	1.234	1.231	1.227	1.223	1.217	1.211
48	1.237	1.236	1.235	1.234	1.231	1.228	1.225	1.22	1.215	1.209
49	1.234	1.234	1.233	1.231	1.229	1.226	1.222	1.218	1.212	1.206
50	1.231	1.231	1.23	1.228	1.226	1.223	1.22	1.215	1.21	1.204
51	1.229	1.228	1.227	1.226	1.224	1.221	1.217	1.213	1.208	1.202
52	1.226	1.226	1.225	1.223	1.221	1.218	1.215	1.211	1.206	1.2
53	1.224	1.224	1.223	1.221	1.219	1.216	1.213	1.208	1.203	1.198
54	1.222	1.221	1.22	1.219	1.217	1.214	1.21	1.206	1.201	1.196
55	1.219	1.219	1.218	1.217	1.214	1.212	1.208	1.204	1.199	1.194
56	1.217	1.217	1.216	1.214	1.212	1.21	1.206	1.202	1.197	1.192
57	1.215	1.215	1.214	1.212	1.21	1.208	1.204	1.2	1.195	1.19
58	1.213	1.213	1.212	1.21	1.208	1.206	1.202	1.198	1.194	1.188
59	1.211	1.211	1.21	1.208	1.206	1.204	1.2	1.196	1.192	1.186
60	1.209	1.209	1.208	1.206	1.204	1.202	1.198	1.195	1.19	1.185

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.15$

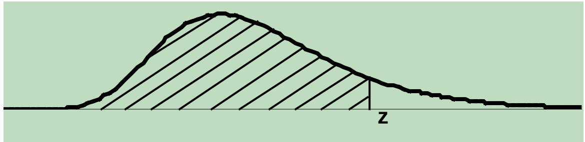
$$\Phi(z) = \int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
1	4.681	4.467	4.232	3.975	3.695	3.391	3.059	2.697	2.295	1.828
2	2.689	2.603	2.507	2.401	2.284	2.155	2.013	1.853	1.67	1.447
3	2.186	2.128	2.064	1.994	1.915	1.828	1.731	1.621	1.493	1.334
4	1.947	1.903	1.853	1.799	1.738	1.67	1.593	1.507	1.405	1.276
5	1.804	1.768	1.727	1.681	1.63	1.574	1.51	1.436	1.35	1.24
6	1.708	1.676	1.641	1.601	1.557	1.508	1.452	1.388	1.312	1.215
7	1.638	1.61	1.578	1.543	1.504	1.46	1.41	1.352	1.284	1.196
8	1.584	1.558	1.53	1.498	1.463	1.423	1.377	1.325	1.262	1.182
9	1.541	1.518	1.492	1.462	1.43	1.393	1.351	1.302	1.245	1.17
10	1.506	1.484	1.46	1.433	1.402	1.368	1.329	1.284	1.23	1.16
11	1.476	1.456	1.434	1.408	1.38	1.348	1.311	1.269	1.218	1.152
12	1.451	1.432	1.411	1.387	1.36	1.33	1.295	1.255	1.207	1.144
13	1.429	1.412	1.391	1.369	1.343	1.315	1.282	1.244	1.198	1.138
14	1.411	1.393	1.374	1.353	1.329	1.301	1.27	1.234	1.19	1.133
15	1.394	1.377	1.359	1.339	1.315	1.289	1.259	1.224	1.183	1.128
16	1.379	1.363	1.346	1.326	1.304	1.279	1.25	1.216	1.176	1.123
17	1.365	1.35	1.333	1.315	1.293	1.269	1.241	1.209	1.17	1.119
18	1.353	1.339	1.322	1.304	1.284	1.26	1.234	1.202	1.165	1.116
19	1.342	1.328	1.312	1.295	1.275	1.252	1.227	1.196	1.16	1.112
20	1.332	1.318	1.303	1.286	1.267	1.245	1.22	1.191	1.156	1.109
21	1.323	1.31	1.295	1.278	1.26	1.238	1.214	1.186	1.151	1.106
22	1.314	1.301	1.287	1.271	1.253	1.232	1.209	1.181	1.148	1.104
23	1.306	1.294	1.28	1.264	1.246	1.226	1.203	1.177	1.144	1.101
24	1.298	1.287	1.273	1.258	1.241	1.221	1.199	1.173	1.141	1.099
25	1.292	1.28	1.267	1.252	1.235	1.216	1.194	1.169	1.138	1.097
26	1.285	1.274	1.261	1.246	1.23	1.211	1.19	1.165	1.135	1.095
27	1.279	1.268	1.255	1.241	1.225	1.207	1.186	1.162	1.132	1.093
28	1.273	1.262	1.25	1.236	1.221	1.203	1.182	1.159	1.13	1.091
29	1.268	1.257	1.245	1.232	1.216	1.199	1.179	1.156	1.127	1.09
30	1.263	1.252	1.241	1.227	1.212	1.195	1.176	1.153	1.125	1.088

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.15$

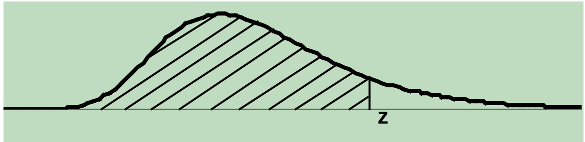
$$\Phi(z) = \int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
31	1.258	1.248	1.236	1.223	1.208	1.192	1.172	1.15	1.123	1.086
32	1.253	1.243	1.232	1.219	1.205	1.188	1.169	1.147	1.121	1.085
33	1.249	1.239	1.228	1.215	1.201	1.185	1.167	1.145	1.119	1.084
34	1.245	1.235	1.224	1.212	1.198	1.182	1.164	1.143	1.117	1.082
35	1.241	1.231	1.221	1.209	1.195	1.179	1.161	1.14	1.115	1.081
36	1.237	1.228	1.217	1.205	1.192	1.177	1.159	1.138	1.113	1.08
37	1.233	1.224	1.214	1.202	1.189	1.174	1.157	1.136	1.112	1.079
38	1.23	1.221	1.211	1.199	1.186	1.171	1.154	1.134	1.11	1.078
39	1.227	1.218	1.208	1.196	1.184	1.169	1.152	1.132	1.108	1.077
40	1.223	1.215	1.205	1.194	1.181	1.167	1.15	1.131	1.107	1.076
41	1.22	1.212	1.202	1.191	1.179	1.164	1.148	1.129	1.106	1.075
42	1.217	1.209	1.199	1.189	1.176	1.162	1.146	1.127	1.104	1.074
43	1.215	1.206	1.197	1.186	1.174	1.16	1.144	1.126	1.103	1.073
44	1.212	1.204	1.194	1.184	1.172	1.158	1.143	1.124	1.102	1.072
45	1.209	1.201	1.192	1.182	1.17	1.156	1.141	1.123	1.1	1.071
46	1.207	1.199	1.19	1.179	1.168	1.154	1.139	1.121	1.099	1.07
47	1.204	1.196	1.187	1.177	1.166	1.153	1.138	1.12	1.098	1.069
48	1.202	1.194	1.185	1.175	1.164	1.151	1.136	1.119	1.097	1.069
49	1.2	1.192	1.183	1.173	1.162	1.149	1.135	1.117	1.096	1.068
50	1.197	1.19	1.181	1.171	1.16	1.148	1.133	1.116	1.095	1.067
51	1.195	1.188	1.179	1.17	1.159	1.146	1.132	1.115	1.094	1.067
52	1.193	1.186	1.177	1.168	1.157	1.145	1.13	1.114	1.093	1.066
53	1.191	1.184	1.175	1.166	1.155	1.143	1.129	1.112	1.092	1.065
54	1.189	1.182	1.174	1.164	1.154	1.142	1.128	1.111	1.091	1.065
55	1.187	1.18	1.172	1.163	1.152	1.14	1.126	1.11	1.09	1.064
56	1.186	1.178	1.17	1.161	1.151	1.139	1.125	1.109	1.09	1.063
57	1.184	1.177	1.169	1.16	1.149	1.138	1.124	1.108	1.089	1.063
58	1.182	1.175	1.167	1.158	1.148	1.136	1.123	1.107	1.088	1.062
59	1.18	1.173	1.166	1.157	1.147	1.135	1.122	1.106	1.087	1.062
60	1.179	1.172	1.164	1.155	1.145	1.134	1.121	1.105	1.086	1.061

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.10$

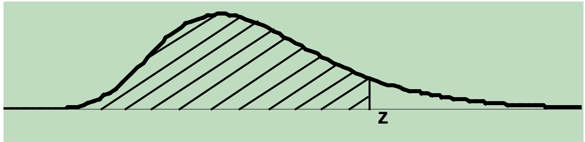
$$\Phi(z) = \int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
1	9.05	9.032	8.977	8.886	8.758	8.594	8.393	8.156	7.881	7.571
2	4.107	4.101	4.082	4.051	4.007	3.951	3.882	3.8	3.705	3.597
3	3.055	3.051	3.039	3.02	2.992	2.957	2.914	2.863	2.804	2.737
4	2.589	2.586	2.578	2.564	2.543	2.517	2.485	2.447	2.403	2.353
5	2.323	2.32	2.313	2.302	2.285	2.264	2.239	2.208	2.172	2.131
6	2.147	2.145	2.14	2.13	2.116	2.098	2.076	2.05	2.02	1.985
7	2.022	2.021	2.016	2.007	1.995	1.979	1.96	1.937	1.911	1.88
8	1.928	1.927	1.922	1.914	1.904	1.89	1.872	1.852	1.828	1.801
9	1.854	1.853	1.848	1.841	1.832	1.819	1.803	1.785	1.763	1.738
10	1.794	1.793	1.789	1.782	1.773	1.762	1.747	1.73	1.71	1.688
11	1.744	1.743	1.739	1.733	1.725	1.714	1.701	1.685	1.667	1.645
12	1.702	1.701	1.697	1.692	1.684	1.674	1.662	1.647	1.629	1.61
13	1.666	1.665	1.661	1.656	1.649	1.639	1.628	1.614	1.597	1.579
14	1.634	1.633	1.63	1.625	1.618	1.609	1.598	1.585	1.57	1.552
15	1.606	1.606	1.603	1.598	1.591	1.583	1.572	1.56	1.545	1.528
16	1.582	1.581	1.578	1.574	1.568	1.559	1.549	1.537	1.523	1.507
17	1.56	1.559	1.556	1.552	1.546	1.538	1.529	1.517	1.504	1.488
18	1.54	1.539	1.537	1.533	1.527	1.519	1.51	1.499	1.486	1.471
19	1.522	1.521	1.519	1.515	1.509	1.502	1.493	1.483	1.47	1.456
20	1.506	1.505	1.503	1.499	1.493	1.486	1.478	1.468	1.456	1.442
21	1.491	1.49	1.488	1.484	1.479	1.472	1.464	1.454	1.442	1.429
22	1.477	1.476	1.474	1.47	1.465	1.459	1.451	1.441	1.43	1.417
23	1.464	1.463	1.461	1.458	1.453	1.446	1.439	1.429	1.419	1.406
24	1.452	1.451	1.449	1.446	1.441	1.435	1.428	1.419	1.408	1.396
25	1.441	1.44	1.438	1.435	1.43	1.424	1.417	1.408	1.398	1.386
26	1.431	1.43	1.428	1.425	1.42	1.415	1.407	1.399	1.389	1.377
27	1.421	1.42	1.418	1.415	1.411	1.405	1.398	1.39	1.38	1.369
28	1.412	1.411	1.409	1.406	1.402	1.397	1.39	1.382	1.372	1.361
29	1.403	1.403	1.401	1.398	1.394	1.388	1.382	1.374	1.364	1.354
30	1.395	1.395	1.393	1.39	1.386	1.381	1.374	1.366	1.357	1.347

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.10$

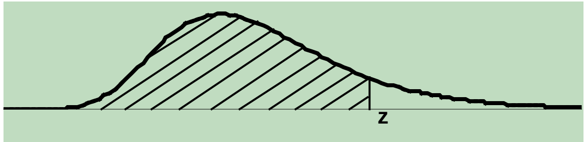
$$\Phi(z) = \int_0^z \frac{(1-\rho^2)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
31	1.388	1.387	1.385	1.382	1.378	1.373	1.367	1.359	1.35	1.34
32	1.38	1.38	1.378	1.375	1.371	1.366	1.36	1.353	1.344	1.334
33	1.373	1.373	1.371	1.369	1.365	1.36	1.354	1.346	1.338	1.328
34	1.367	1.366	1.365	1.362	1.358	1.354	1.348	1.34	1.332	1.322
35	1.361	1.36	1.359	1.356	1.352	1.348	1.342	1.335	1.326	1.317
36	1.355	1.354	1.353	1.35	1.347	1.342	1.336	1.329	1.321	1.312
37	1.349	1.349	1.347	1.345	1.341	1.337	1.331	1.324	1.316	1.307
38	1.344	1.343	1.342	1.339	1.336	1.331	1.326	1.319	1.311	1.302
39	1.339	1.338	1.337	1.334	1.331	1.326	1.321	1.314	1.307	1.298
40	1.334	1.333	1.332	1.329	1.326	1.322	1.316	1.31	1.302	1.293
41	1.329	1.328	1.327	1.325	1.321	1.317	1.312	1.305	1.298	1.289
42	1.324	1.324	1.323	1.32	1.317	1.313	1.307	1.301	1.294	1.285
43	1.32	1.32	1.318	1.316	1.313	1.308	1.303	1.297	1.29	1.282
44	1.316	1.315	1.314	1.312	1.309	1.304	1.299	1.293	1.286	1.278
45	1.312	1.311	1.31	1.308	1.305	1.301	1.296	1.29	1.282	1.274
46	1.308	1.307	1.306	1.304	1.301	1.297	1.292	1.286	1.279	1.271
47	1.304	1.304	1.302	1.3	1.297	1.293	1.288	1.282	1.276	1.268
48	1.3	1.3	1.299	1.297	1.294	1.29	1.285	1.279	1.272	1.265
49	1.297	1.296	1.295	1.293	1.29	1.286	1.281	1.276	1.269	1.261
50	1.293	1.293	1.292	1.29	1.287	1.283	1.278	1.273	1.266	1.258
51	1.29	1.29	1.289	1.286	1.284	1.28	1.275	1.27	1.263	1.256
52	1.287	1.287	1.285	1.283	1.28	1.277	1.272	1.267	1.26	1.253
53	1.284	1.283	1.282	1.28	1.277	1.274	1.269	1.264	1.257	1.25
54	1.281	1.28	1.279	1.277	1.274	1.271	1.266	1.261	1.255	1.248
55	1.278	1.278	1.276	1.274	1.272	1.268	1.264	1.258	1.252	1.245
56	1.275	1.275	1.274	1.272	1.269	1.265	1.261	1.256	1.25	1.243
57	1.272	1.272	1.271	1.269	1.266	1.263	1.258	1.253	1.247	1.24
58	1.27	1.269	1.268	1.266	1.264	1.26	1.256	1.251	1.245	1.238
59	1.267	1.267	1.266	1.264	1.261	1.258	1.253	1.248	1.242	1.236
60	1.265	1.264	1.263	1.261	1.259	1.255	1.251	1.246	1.24	1.233

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.10$

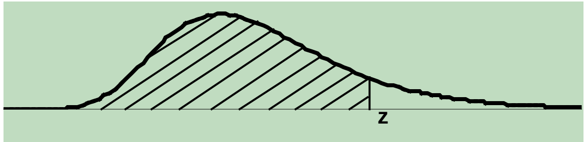
$$\Phi(z) = \int_0^z \frac{(1-\rho^2)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
1	7.223	6.838	6.416	5.955	5.456	4.917	4.336	3.707	3.022	2.249
2	3.475	3.34	3.191	3.027	2.848	2.651	2.436	2.197	1.928	1.607
3	2.661	2.576	2.481	2.377	2.263	2.137	1.997	1.84	1.66	1.441
4	2.296	2.233	2.162	2.083	1.997	1.901	1.794	1.673	1.533	1.36
5	2.085	2.034	1.976	1.912	1.841	1.762	1.674	1.573	1.457	1.31
6	1.946	1.902	1.853	1.798	1.737	1.669	1.593	1.506	1.404	1.276
7	1.846	1.807	1.764	1.716	1.662	1.602	1.534	1.457	1.366	1.251
8	1.77	1.735	1.696	1.653	1.605	1.55	1.489	1.419	1.337	1.232
9	1.71	1.679	1.643	1.604	1.559	1.51	1.454	1.389	1.313	1.216
10	1.662	1.632	1.6	1.563	1.522	1.476	1.424	1.365	1.294	1.203
11	1.621	1.594	1.564	1.529	1.491	1.448	1.4	1.344	1.278	1.192
12	1.587	1.561	1.533	1.501	1.465	1.425	1.379	1.326	1.264	1.183
13	1.557	1.533	1.506	1.476	1.442	1.404	1.361	1.311	1.252	1.175
14	1.532	1.509	1.483	1.455	1.423	1.386	1.345	1.298	1.241	1.167
15	1.509	1.487	1.463	1.436	1.405	1.371	1.331	1.286	1.231	1.161
16	1.489	1.468	1.445	1.419	1.389	1.356	1.319	1.275	1.223	1.155
17	1.471	1.451	1.429	1.404	1.375	1.344	1.308	1.266	1.215	1.15
18	1.455	1.435	1.414	1.39	1.363	1.332	1.297	1.257	1.209	1.145
19	1.44	1.421	1.401	1.377	1.351	1.322	1.288	1.249	1.202	1.141
20	1.426	1.408	1.388	1.366	1.341	1.312	1.28	1.242	1.196	1.137
21	1.414	1.397	1.377	1.356	1.331	1.303	1.272	1.235	1.191	1.134
22	1.402	1.386	1.367	1.346	1.322	1.295	1.265	1.229	1.186	1.13
23	1.392	1.376	1.357	1.337	1.314	1.288	1.258	1.223	1.182	1.127
24	1.382	1.366	1.348	1.329	1.306	1.281	1.252	1.218	1.177	1.124
25	1.373	1.357	1.34	1.321	1.299	1.274	1.246	1.213	1.173	1.121
26	1.364	1.349	1.332	1.314	1.292	1.268	1.241	1.208	1.17	1.119
27	1.356	1.342	1.325	1.307	1.286	1.262	1.235	1.204	1.166	1.117
28	1.349	1.334	1.318	1.3	1.28	1.257	1.231	1.2	1.163	1.114
29	1.341	1.328	1.312	1.294	1.274	1.252	1.226	1.196	1.16	1.112
30	1.335	1.321	1.306	1.289	1.269	1.247	1.222	1.192	1.157	1.11

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.10$

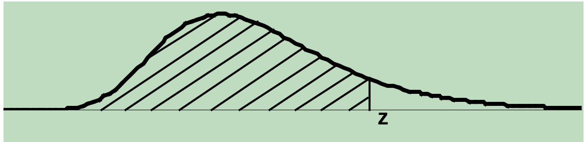
$$\Phi(z) = \int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
31	1.328	1.315	1.3	1.283	1.264	1.243	1.218	1.189	1.154	1.108
32	1.322	1.309	1.295	1.278	1.259	1.238	1.214	1.186	1.151	1.106
33	1.317	1.304	1.29	1.273	1.255	1.234	1.21	1.183	1.149	1.105
34	1.311	1.299	1.285	1.269	1.251	1.23	1.207	1.18	1.146	1.103
35	1.306	1.294	1.28	1.264	1.247	1.227	1.204	1.177	1.144	1.101
36	1.301	1.289	1.275	1.26	1.243	1.223	1.2	1.174	1.142	1.1
37	1.296	1.285	1.271	1.256	1.239	1.22	1.197	1.171	1.14	1.098
38	1.292	1.28	1.267	1.252	1.235	1.216	1.194	1.169	1.138	1.097
39	1.288	1.276	1.263	1.249	1.232	1.213	1.192	1.167	1.136	1.096
40	1.283	1.272	1.259	1.245	1.229	1.21	1.189	1.164	1.134	1.094
41	1.28	1.268	1.256	1.242	1.226	1.207	1.186	1.162	1.132	1.093
42	1.276	1.265	1.252	1.238	1.223	1.205	1.184	1.16	1.131	1.092
43	1.272	1.261	1.249	1.235	1.22	1.202	1.182	1.158	1.129	1.091
44	1.269	1.258	1.246	1.232	1.217	1.199	1.179	1.156	1.127	1.09
45	1.265	1.255	1.243	1.229	1.214	1.197	1.177	1.154	1.126	1.089
46	1.262	1.251	1.24	1.227	1.212	1.195	1.175	1.152	1.124	1.088
47	1.259	1.248	1.237	1.224	1.209	1.192	1.173	1.15	1.123	1.087
48	1.256	1.246	1.234	1.221	1.207	1.19	1.171	1.149	1.122	1.086
49	1.253	1.243	1.231	1.219	1.204	1.188	1.169	1.147	1.12	1.085
50	1.25	1.24	1.229	1.216	1.202	1.186	1.167	1.145	1.119	1.084
51	1.247	1.237	1.226	1.214	1.2	1.184	1.165	1.144	1.118	1.083
52	1.244	1.235	1.224	1.212	1.198	1.182	1.164	1.142	1.116	1.082
53	1.242	1.232	1.221	1.209	1.196	1.18	1.162	1.141	1.115	1.081
54	1.239	1.23	1.219	1.207	1.194	1.178	1.16	1.14	1.114	1.081
55	1.237	1.227	1.217	1.205	1.192	1.176	1.159	1.138	1.113	1.08
56	1.234	1.225	1.215	1.203	1.19	1.175	1.157	1.137	1.112	1.079
57	1.232	1.223	1.213	1.201	1.188	1.173	1.156	1.136	1.111	1.078
58	1.23	1.221	1.211	1.199	1.186	1.171	1.154	1.134	1.11	1.078
59	1.228	1.219	1.209	1.197	1.184	1.17	1.153	1.133	1.109	1.077
60	1.226	1.217	1.207	1.195	1.183	1.168	1.151	1.132	1.108	1.076

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.05$

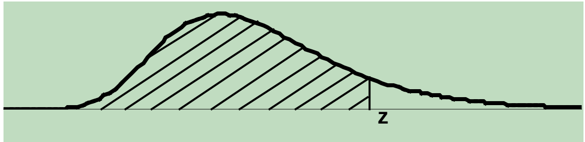
$$\Phi(z) = \int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
1	19.202	19.158	19.02	18.808	18.50	18.109	17.62	17.06	16.40	15.663
2	6.388	6.377	6.342	6.283	6.202	6.097	5.968	5.816	5.64	5.441
3	4.284	4.277	4.257	4.224	4.177	4.117	4.043	3.956	3.855	3.74
4	3.438	3.433	3.419	3.396	3.362	3.32	3.267	3.205	3.133	3.051
5	2.978	2.975	2.963	2.945	2.919	2.885	2.844	2.795	2.739	2.674
6	2.687	2.684	2.674	2.659	2.637	2.609	2.575	2.535	2.487	2.434
7	2.484	2.481	2.473	2.46	2.441	2.417	2.388	2.353	2.312	2.265
8	2.333	2.331	2.324	2.312	2.296	2.275	2.249	2.218	2.182	2.141
9	2.217	2.215	2.209	2.198	2.184	2.164	2.141	2.113	2.081	2.044
10	2.124	2.122	2.117	2.107	2.093	2.076	2.055	2.029	2	1.966
11	2.048	2.046	2.041	2.032	2.019	2.003	1.984	1.96	1.933	1.902
12	1.984	1.982	1.977	1.969	1.957	1.943	1.924	1.902	1.877	1.848
13	1.929	1.928	1.923	1.915	1.905	1.891	1.874	1.853	1.829	1.802
14	1.882	1.881	1.876	1.869	1.859	1.846	1.83	1.81	1.788	1.762
15	1.841	1.84	1.835	1.829	1.819	1.807	1.791	1.773	1.752	1.727
16	1.804	1.803	1.799	1.793	1.784	1.772	1.757	1.74	1.72	1.697
17	1.772	1.771	1.767	1.761	1.752	1.741	1.727	1.711	1.691	1.669
18	1.743	1.742	1.738	1.732	1.724	1.713	1.7	1.684	1.666	1.644
19	1.717	1.716	1.712	1.706	1.698	1.688	1.675	1.66	1.643	1.622
20	1.693	1.692	1.688	1.683	1.675	1.665	1.653	1.638	1.621	1.602
21	1.671	1.67	1.667	1.661	1.654	1.644	1.633	1.619	1.602	1.583
22	1.651	1.65	1.647	1.642	1.635	1.625	1.614	1.6	1.584	1.566
23	1.632	1.631	1.629	1.624	1.617	1.608	1.597	1.584	1.568	1.55
24	1.615	1.614	1.612	1.607	1.6	1.591	1.581	1.568	1.553	1.536
25	1.599	1.599	1.596	1.591	1.585	1.576	1.566	1.553	1.539	1.522
26	1.585	1.584	1.581	1.577	1.57	1.562	1.552	1.54	1.526	1.51
27	1.571	1.57	1.567	1.563	1.557	1.549	1.539	1.527	1.514	1.498
28	1.558	1.557	1.555	1.55	1.544	1.536	1.527	1.515	1.502	1.487
29	1.546	1.545	1.542	1.538	1.532	1.525	1.516	1.504	1.491	1.476
30	1.534	1.534	1.531	1.527	1.521	1.514	1.505	1.494	1.481	1.466

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.05$

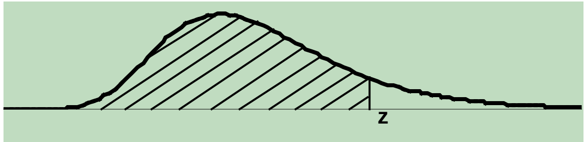
$$\Phi(z) = \int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
31	1.524	1.523	1.52	1.516	1.511	1.504	1.495	1.484	1.472	1.457
32	1.513	1.513	1.51	1.506	1.501	1.494	1.485	1.475	1.462	1.448
33	1.504	1.503	1.501	1.497	1.491	1.485	1.476	1.466	1.454	1.44
34	1.494	1.494	1.491	1.488	1.482	1.476	1.467	1.457	1.446	1.432
35	1.486	1.485	1.483	1.479	1.474	1.467	1.459	1.449	1.438	1.425
36	1.477	1.477	1.475	1.471	1.466	1.459	1.451	1.442	1.431	1.418
37	1.469	1.469	1.467	1.463	1.458	1.452	1.444	1.434	1.423	1.411
38	1.462	1.461	1.459	1.456	1.451	1.445	1.437	1.428	1.417	1.404
39	1.455	1.454	1.452	1.449	1.444	1.438	1.43	1.421	1.41	1.398
40	1.448	1.447	1.445	1.442	1.437	1.431	1.424	1.415	1.404	1.392
41	1.441	1.44	1.438	1.435	1.431	1.425	1.417	1.408	1.398	1.386
42	1.435	1.434	1.432	1.429	1.424	1.419	1.411	1.403	1.393	1.381
43	1.429	1.428	1.426	1.423	1.418	1.413	1.406	1.397	1.387	1.376
44	1.423	1.422	1.42	1.417	1.413	1.407	1.4	1.392	1.382	1.371
45	1.417	1.416	1.415	1.412	1.407	1.402	1.395	1.386	1.377	1.366
46	1.412	1.411	1.409	1.406	1.402	1.396	1.39	1.381	1.372	1.361
47	1.406	1.406	1.404	1.401	1.397	1.391	1.385	1.377	1.367	1.356
48	1.401	1.401	1.399	1.396	1.392	1.387	1.38	1.372	1.363	1.352
49	1.396	1.396	1.394	1.391	1.387	1.382	1.375	1.367	1.358	1.348
50	1.392	1.391	1.389	1.387	1.383	1.377	1.371	1.363	1.354	1.344
51	1.387	1.387	1.385	1.382	1.378	1.373	1.367	1.359	1.35	1.34
52	1.383	1.382	1.381	1.378	1.374	1.369	1.362	1.355	1.346	1.336
53	1.378	1.378	1.376	1.373	1.37	1.365	1.358	1.351	1.342	1.332
54	1.374	1.374	1.372	1.369	1.366	1.361	1.354	1.347	1.339	1.329
55	1.37	1.37	1.368	1.365	1.362	1.357	1.351	1.343	1.335	1.325
56	1.366	1.366	1.364	1.362	1.358	1.353	1.347	1.34	1.331	1.322
57	1.363	1.362	1.361	1.358	1.354	1.349	1.343	1.336	1.328	1.319
58	1.359	1.358	1.357	1.354	1.351	1.346	1.34	1.333	1.325	1.315
59	1.355	1.355	1.353	1.351	1.347	1.342	1.337	1.33	1.322	1.312
60	1.352	1.351	1.35	1.347	1.344	1.339	1.333	1.327	1.319	1.309

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.05$

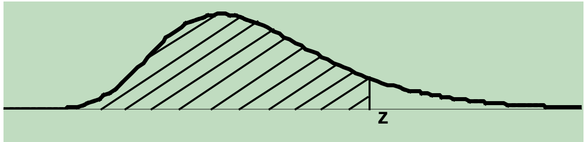
$$\Phi(z) = \int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
1	14.835	13.92	12.91	11.83	10.65	9.392	8.041	6.596	5.049	3.368
2	5.217	4.969	4.696	4.397	4.072	3.719	3.336	2.919	2.456	1.923
3	3.611	3.467	3.309	3.135	2.944	2.736	2.507	2.255	1.971	1.633
4	2.959	2.856	2.742	2.616	2.478	2.327	2.159	1.973	1.76	1.503
5	2.601	2.52	2.429	2.33	2.22	2.098	1.964	1.813	1.64	1.428
6	2.373	2.305	2.229	2.145	2.053	1.951	1.837	1.709	1.56	1.377
7	2.213	2.154	2.088	2.016	1.935	1.846	1.747	1.634	1.503	1.34
8	2.094	2.042	1.984	1.919	1.847	1.768	1.679	1.578	1.46	1.312
9	2.002	1.954	1.902	1.843	1.779	1.706	1.625	1.533	1.425	1.29
10	1.927	1.884	1.836	1.783	1.723	1.657	1.582	1.497	1.397	1.272
11	1.866	1.826	1.782	1.732	1.677	1.616	1.546	1.467	1.374	1.256
12	1.815	1.778	1.736	1.69	1.638	1.581	1.516	1.442	1.354	1.243
13	1.771	1.736	1.697	1.654	1.605	1.551	1.49	1.42	1.337	1.232
14	1.733	1.7	1.663	1.622	1.577	1.525	1.467	1.401	1.322	1.222
15	1.7	1.669	1.634	1.595	1.551	1.502	1.447	1.384	1.309	1.213
16	1.67	1.641	1.607	1.57	1.529	1.482	1.43	1.369	1.297	1.205
17	1.644	1.616	1.584	1.548	1.509	1.464	1.414	1.356	1.287	1.198
18	1.62	1.593	1.563	1.529	1.491	1.448	1.399	1.344	1.277	1.192
19	1.599	1.573	1.544	1.511	1.474	1.433	1.386	1.333	1.269	1.186
20	1.58	1.554	1.526	1.495	1.459	1.42	1.375	1.323	1.261	1.181
21	1.562	1.538	1.51	1.48	1.446	1.407	1.364	1.313	1.253	1.176
22	1.545	1.522	1.496	1.466	1.433	1.396	1.354	1.305	1.247	1.171
23	1.53	1.508	1.482	1.454	1.421	1.385	1.344	1.297	1.24	1.167
24	1.516	1.494	1.47	1.442	1.411	1.376	1.336	1.29	1.234	1.163
25	1.503	1.482	1.458	1.431	1.401	1.367	1.328	1.283	1.229	1.159
26	1.491	1.47	1.447	1.421	1.391	1.358	1.32	1.276	1.224	1.156
27	1.48	1.46	1.437	1.411	1.382	1.35	1.313	1.27	1.219	1.153
28	1.469	1.449	1.427	1.402	1.374	1.343	1.307	1.265	1.215	1.15
29	1.459	1.44	1.418	1.394	1.366	1.336	1.3	1.26	1.211	1.147
30	1.45	1.431	1.41	1.386	1.359	1.329	1.294	1.255	1.207	1.144

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.05$

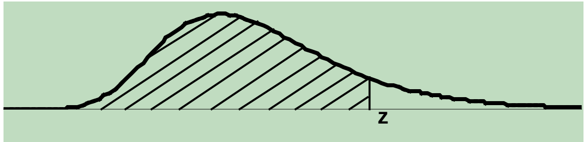
$$\Phi(z) = \int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
31	1.441	1.422	1.402	1.378	1.352	1.323	1.289	1.25	1.203	1.141
32	1.432	1.414	1.394	1.371	1.346	1.317	1.284	1.245	1.199	1.139
33	1.425	1.407	1.387	1.365	1.339	1.311	1.279	1.241	1.196	1.137
34	1.417	1.4	1.38	1.358	1.334	1.306	1.274	1.237	1.193	1.135
35	1.41	1.393	1.374	1.352	1.328	1.301	1.269	1.233	1.189	1.132
36	1.403	1.386	1.367	1.346	1.323	1.296	1.265	1.229	1.186	1.13
37	1.396	1.38	1.362	1.341	1.317	1.291	1.261	1.226	1.184	1.128
38	1.39	1.374	1.356	1.335	1.313	1.287	1.257	1.223	1.181	1.127
39	1.384	1.368	1.35	1.33	1.308	1.282	1.253	1.219	1.178	1.125
40	1.378	1.363	1.345	1.326	1.303	1.278	1.25	1.216	1.176	1.123
41	1.373	1.358	1.34	1.321	1.299	1.274	1.246	1.213	1.174	1.122
42	1.368	1.353	1.336	1.316	1.295	1.271	1.243	1.21	1.171	1.12
43	1.363	1.348	1.331	1.312	1.291	1.267	1.24	1.208	1.169	1.118
44	1.358	1.343	1.327	1.308	1.287	1.264	1.236	1.205	1.167	1.117
45	1.353	1.339	1.322	1.304	1.283	1.26	1.233	1.202	1.165	1.116
46	1.348	1.334	1.318	1.3	1.28	1.257	1.231	1.2	1.163	1.114
47	1.344	1.33	1.314	1.297	1.276	1.254	1.228	1.198	1.161	1.113
48	1.34	1.326	1.31	1.293	1.273	1.251	1.225	1.195	1.159	1.112
49	1.336	1.322	1.307	1.29	1.27	1.248	1.223	1.193	1.157	1.11
50	1.332	1.318	1.303	1.286	1.267	1.245	1.22	1.191	1.156	1.109
51	1.328	1.315	1.3	1.283	1.264	1.242	1.218	1.189	1.154	1.108
52	1.324	1.311	1.296	1.28	1.261	1.24	1.215	1.187	1.152	1.107
53	1.321	1.308	1.293	1.277	1.258	1.237	1.213	1.185	1.151	1.106
54	1.317	1.305	1.29	1.274	1.255	1.235	1.211	1.183	1.149	1.105
55	1.314	1.301	1.287	1.271	1.253	1.232	1.209	1.181	1.148	1.104
56	1.311	1.298	1.284	1.268	1.25	1.23	1.207	1.179	1.146	1.103
57	1.308	1.295	1.281	1.266	1.248	1.228	1.205	1.178	1.145	1.102
58	1.305	1.292	1.279	1.263	1.245	1.225	1.203	1.176	1.144	1.101
59	1.302	1.289	1.276	1.26	1.243	1.223	1.201	1.174	1.142	1.1
60	1.299	1.287	1.273	1.258	1.241	1.221	1.199	1.173	1.141	1.099

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.01$

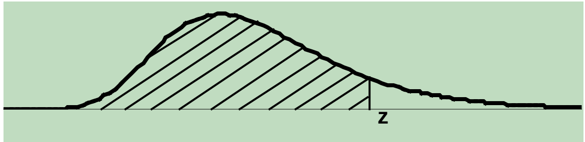
$$\Phi(z) = \int_0^z \frac{(1-\rho^2)^k}{B(k,k)} x^{k-1} (1+x)^{-2k} \left[1 - \left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1 - a$$



	ρ									
k	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
1	100	100	100	100	99.95	97.546	94.60	91.141	87.15	82.665
2	15.977	15.942	15.83	15.66	15.41	15.096	14.70	14.25	13.72	13.121
3	8.466	8.449	8.399	8.316	8.199	8.048	7.865	7.647	7.396	7.11
4	6.029	6.018	5.986	5.932	5.856	5.759	5.64	5.499	5.336	5.151
5	4.849	4.841	4.817	4.777	4.721	4.649	4.561	4.457	4.336	4.198
6	4.155	4.149	4.13	4.098	4.053	3.996	3.926	3.842	3.746	3.636
7	3.698	3.692	3.676	3.65	3.612	3.564	3.506	3.436	3.355	3.263
8	3.372	3.367	3.354	3.331	3.299	3.257	3.207	3.146	3.077	2.997
9	3.128	3.124	3.112	3.092	3.063	3.027	2.982	2.929	2.867	2.797
10	2.938	2.934	2.923	2.905	2.88	2.847	2.807	2.759	2.704	2.641
11	2.785	2.782	2.772	2.755	2.732	2.702	2.666	2.622	2.572	2.515
12	2.659	2.656	2.647	2.632	2.611	2.583	2.55	2.51	2.464	2.411
13	2.554	2.551	2.542	2.528	2.509	2.483	2.452	2.415	2.372	2.323
14	2.464	2.461	2.453	2.44	2.422	2.398	2.369	2.335	2.295	2.249
15	2.386	2.384	2.376	2.364	2.347	2.325	2.297	2.265	2.227	2.184
16	2.318	2.316	2.309	2.297	2.281	2.26	2.235	2.204	2.168	2.128
17	2.258	2.256	2.25	2.239	2.223	2.203	2.179	2.15	2.116	2.078
18	2.205	2.203	2.197	2.186	2.172	2.153	2.13	2.102	2.07	2.033
19	2.157	2.155	2.149	2.139	2.126	2.108	2.085	2.059	2.029	1.993
20	2.114	2.112	2.107	2.097	2.084	2.067	2.045	2.02	1.991	1.957
21	2.075	2.073	2.068	2.059	2.046	2.029	2.009	1.985	1.957	1.925
22	2.04	2.038	2.033	2.024	2.011	1.996	1.976	1.953	1.926	1.895
23	2.007	2.005	2	1.992	1.98	1.965	1.946	1.923	1.897	1.867
24	1.977	1.975	1.97	1.962	1.951	1.936	1.918	1.896	1.871	1.842
25	1.949	1.947	1.943	1.935	1.924	1.909	1.892	1.871	1.847	1.819
26	1.923	1.922	1.917	1.909	1.899	1.885	1.868	1.848	1.824	1.797
27	1.899	1.898	1.893	1.886	1.876	1.862	1.846	1.826	1.803	1.777
28	1.877	1.875	1.871	1.864	1.854	1.841	1.825	1.806	1.783	1.758
29	1.856	1.855	1.85	1.843	1.834	1.821	1.805	1.787	1.765	1.74
30	1.836	1.835	1.831	1.824	1.814	1.802	1.787	1.769	1.748	1.724

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.01$

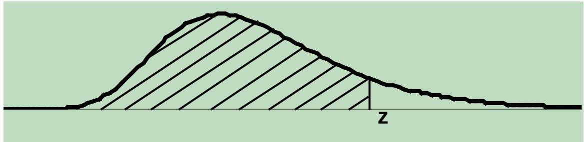
$$\Phi(z) = \int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
31	1.818	1.816	1.813	1.806	1.797	1.785	1.77	1.752	1.732	1.708
32	1.8	1.799	1.795	1.789	1.78	1.768	1.754	1.736	1.716	1.693
33	1.784	1.783	1.779	1.773	1.764	1.752	1.738	1.721	1.702	1.679
34	1.768	1.767	1.763	1.757	1.749	1.737	1.724	1.707	1.688	1.666
35	1.754	1.752	1.749	1.743	1.734	1.723	1.71	1.694	1.675	1.654
36	1.74	1.738	1.735	1.729	1.721	1.71	1.697	1.681	1.663	1.642
37	1.726	1.725	1.722	1.716	1.708	1.697	1.684	1.669	1.651	1.63
38	1.714	1.713	1.709	1.704	1.696	1.685	1.673	1.658	1.64	1.62
39	1.702	1.701	1.697	1.692	1.684	1.674	1.661	1.647	1.629	1.609
40	1.69	1.689	1.686	1.68	1.673	1.663	1.651	1.636	1.619	1.6
41	1.679	1.678	1.675	1.669	1.662	1.652	1.64	1.626	1.609	1.59
42	1.668	1.667	1.664	1.659	1.652	1.642	1.63	1.616	1.6	1.581
43	1.658	1.657	1.654	1.649	1.642	1.632	1.621	1.607	1.591	1.573
44	1.649	1.648	1.645	1.64	1.632	1.623	1.612	1.598	1.582	1.564
45	1.639	1.638	1.635	1.63	1.623	1.614	1.603	1.59	1.574	1.556
46	1.63	1.629	1.626	1.622	1.615	1.606	1.595	1.582	1.566	1.549
47	1.622	1.621	1.618	1.613	1.606	1.597	1.587	1.574	1.559	1.541
48	1.613	1.612	1.61	1.605	1.598	1.59	1.579	1.566	1.551	1.534
49	1.605	1.604	1.602	1.597	1.59	1.582	1.571	1.559	1.544	1.527
50	1.598	1.597	1.594	1.589	1.583	1.574	1.564	1.552	1.537	1.521
51	1.59	1.589	1.587	1.582	1.576	1.567	1.557	1.545	1.531	1.514
52	1.583	1.582	1.579	1.575	1.569	1.56	1.55	1.538	1.524	1.508
53	1.576	1.575	1.573	1.568	1.562	1.554	1.544	1.532	1.518	1.502
54	1.569	1.568	1.566	1.561	1.555	1.547	1.538	1.526	1.512	1.496
55	1.563	1.562	1.559	1.555	1.549	1.541	1.531	1.52	1.506	1.491
56	1.556	1.556	1.553	1.549	1.543	1.535	1.526	1.514	1.501	1.485
57	1.55	1.549	1.547	1.543	1.537	1.529	1.52	1.509	1.495	1.48
58	1.544	1.544	1.541	1.537	1.531	1.524	1.514	1.503	1.49	1.475
59	1.539	1.538	1.535	1.531	1.525	1.518	1.509	1.498	1.485	1.47
60	1.533	1.532	1.53	1.526	1.52	1.513	1.504	1.493	1.48	1.465

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.01$

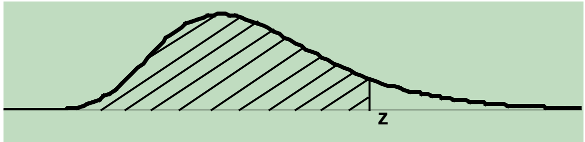
$$\Phi(z) = \int_0^z \frac{(1-\rho^2)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
1	77.666	72.171	66.18	59.724	52.79	45.398	37.55	29.277	20.566	11.419
2	12.45	11.707	10.894	10.008	9.05	8.018	6.91	5.721	4.442	3.04
3	6.791	6.437	6.049	5.625	5.164	4.666	4.128	3.545	2.907	2.184
4	4.944	4.714	4.46	4.183	3.882	3.554	3.197	2.808	2.376	1.876
5	4.044	3.873	3.684	3.477	3.251	3.004	2.734	2.438	2.106	1.715
6	3.512	3.375	3.223	3.056	2.874	2.674	2.455	2.213	1.94	1.614
7	3.159	3.044	2.917	2.776	2.622	2.453	2.267	2.061	1.826	1.544
8	2.908	2.808	2.697	2.575	2.441	2.294	2.132	1.95	1.743	1.493
9	2.718	2.629	2.532	2.423	2.304	2.173	2.028	1.866	1.68	1.453
10	2.569	2.49	2.402	2.304	2.197	2.078	1.946	1.799	1.629	1.421
11	2.45	2.377	2.297	2.208	2.109	2.001	1.88	1.744	1.587	1.394
12	2.351	2.284	2.21	2.128	2.037	1.936	1.825	1.698	1.553	1.372
13	2.268	2.206	2.137	2.061	1.976	1.882	1.778	1.66	1.523	1.353
14	2.197	2.139	2.074	2.003	1.924	1.836	1.737	1.626	1.497	1.337
15	2.136	2.081	2.02	1.953	1.878	1.795	1.702	1.597	1.475	1.322
16	2.082	2.03	1.973	1.909	1.838	1.76	1.672	1.572	1.455	1.309
17	2.034	1.985	1.931	1.87	1.803	1.728	1.644	1.549	1.438	1.298
18	1.992	1.945	1.893	1.835	1.771	1.7	1.62	1.529	1.422	1.288
19	1.954	1.909	1.86	1.804	1.743	1.674	1.598	1.51	1.407	1.278
20	1.919	1.877	1.829	1.776	1.717	1.651	1.577	1.493	1.394	1.27
21	1.888	1.847	1.801	1.75	1.694	1.63	1.559	1.478	1.383	1.262
22	1.86	1.82	1.776	1.727	1.672	1.611	1.542	1.464	1.372	1.255
23	1.833	1.795	1.753	1.705	1.652	1.593	1.527	1.451	1.361	1.248
24	1.809	1.772	1.731	1.685	1.634	1.577	1.513	1.439	1.352	1.242
25	1.787	1.751	1.711	1.667	1.617	1.562	1.499	1.428	1.343	1.236
26	1.766	1.732	1.693	1.65	1.602	1.548	1.487	1.418	1.335	1.231
27	1.747	1.713	1.676	1.634	1.587	1.535	1.476	1.408	1.328	1.226
28	1.729	1.696	1.66	1.619	1.573	1.522	1.465	1.399	1.321	1.221
29	1.712	1.68	1.645	1.605	1.56	1.511	1.455	1.39	1.314	1.216
30	1.696	1.665	1.63	1.592	1.548	1.5	1.445	1.382	1.308	1.212

Percentage Points of the Correlated Gamma Ratio Distribution for $a = 0.01$

$$\Phi(z) = \int_0^z \frac{\left(1-\rho^2\right)^k}{B(k,k)} x^{k-1}(1+x)^{-2k}\left[1-\left(\frac{2\rho}{x+1}\right)^2 x\right]^{-\frac{2k+1}{2}} dx = 1-a$$



	ρ									
k	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
31	1.681	1.651	1.617	1.579	1.537	1.49	1.436	1.375	1.302	1.208
32	1.667	1.637	1.604	1.568	1.526	1.48	1.428	1.367	1.296	1.204
33	1.654	1.625	1.593	1.556	1.516	1.471	1.42	1.361	1.291	1.201
34	1.641	1.613	1.581	1.546	1.506	1.462	1.412	1.354	1.286	1.197
35	1.629	1.602	1.571	1.536	1.497	1.454	1.405	1.348	1.281	1.194
36	1.618	1.591	1.56	1.527	1.489	1.446	1.398	1.342	1.276	1.191
37	1.607	1.58	1.551	1.518	1.48	1.439	1.391	1.337	1.272	1.188
38	1.597	1.571	1.542	1.509	1.472	1.431	1.385	1.331	1.268	1.185
39	1.587	1.561	1.533	1.501	1.465	1.425	1.379	1.326	1.264	1.183
40	1.577	1.552	1.524	1.493	1.458	1.418	1.373	1.321	1.26	1.18
41	1.568	1.544	1.516	1.485	1.451	1.412	1.368	1.317	1.256	1.178
42	1.56	1.536	1.508	1.478	1.444	1.406	1.362	1.312	1.252	1.175
43	1.551	1.528	1.501	1.471	1.438	1.4	1.357	1.308	1.249	1.173
44	1.544	1.52	1.494	1.465	1.432	1.395	1.353	1.304	1.246	1.171
45	1.536	1.513	1.487	1.458	1.426	1.389	1.348	1.3	1.243	1.169
46	1.529	1.506	1.481	1.452	1.42	1.384	1.343	1.296	1.24	1.166
47	1.522	1.499	1.474	1.446	1.415	1.379	1.339	1.292	1.237	1.165
48	1.515	1.493	1.468	1.44	1.409	1.374	1.335	1.289	1.234	1.163
49	1.508	1.487	1.462	1.435	1.404	1.37	1.331	1.285	1.231	1.161
50	1.502	1.481	1.457	1.43	1.399	1.365	1.327	1.282	1.228	1.159
51	1.496	1.475	1.451	1.425	1.395	1.361	1.323	1.279	1.226	1.157
52	1.49	1.469	1.446	1.42	1.39	1.357	1.319	1.276	1.223	1.156
53	1.484	1.464	1.441	1.415	1.386	1.353	1.316	1.273	1.221	1.154
54	1.479	1.458	1.436	1.41	1.381	1.349	1.312	1.27	1.219	1.152
55	1.473	1.453	1.431	1.406	1.377	1.345	1.309	1.267	1.216	1.151
56	1.468	1.448	1.426	1.401	1.373	1.342	1.306	1.264	1.214	1.149
57	1.463	1.443	1.422	1.397	1.369	1.338	1.303	1.262	1.212	1.148
58	1.458	1.439	1.417	1.393	1.366	1.335	1.3	1.259	1.21	1.146
59	1.453	1.434	1.413	1.389	1.362	1.331	1.297	1.256	1.208	1.145
60	1.449	1.43	1.409	1.385	1.358	1.328	1.294	1.254	1.206	1.144

Appendix 5

- ***Table 5.2. The table presents the minimum and maximum values of the evaluation criteria that were achieved by each of the ARCH models and the ARCH models suggested by the SPEC model selection algorithm, respectively, for a subset of the forecasting horizon which ranges from 2 to 100 days***
- ***Figure 5.2. The plots indicate whether the ARCH models selected by the SPEC algorithm achieve the lowest value of the evaluation criterion, for a forecasting horizon ranging from one day to one hundred days ahead***
- ***Table 5.3. The percentage of times the ARCH models selected by the SPEC algorithm perform "better" as judged by the evaluation criteria***
- ***Table 5.4. Average sample size for the SPEC model selection algorithm suggested by the Evaluation Criteria for both the entire 16 to 36 day ahead forecasting horizon and for each day individually***
- ***Tables 5.6.1 to 5.6.11 presents the percentage of cases the models selected by each model selection method perform "better" as judged by the evaluation criteria***
- ***Tables 5.7.1 to 5.7.11 present the percentage of times the ARCH models selected by the SPEC algorithm perform "better" than the ARCH models selected by the other model selection methods***

Table 5.2

The table presents the minimum and maximum values of the evaluation criteria that were achieved by each of the ARCH models and the ARCH models suggested by the SPEC model selection algorithm, respectively, for a subset of the forecasting horizon which ranges from 5 to 100 days. The first panel refers to the variance, (k=2), whereas the second panel accounts for the standard deviation, (k=1). The evaluation criteria are the annualized mean and median values of the following loss functions:

Squared Error (SE) : $\left(\sigma_{t(T)}^{k} - s_{t(T)}^{k}\right)^{2}$

Absolute Error (AE) : $\left|\sigma_{t(T)}^{k} - s_{t(T)}^{k}\right|$

Heteroscedasticity Adjusted Squared Error (HASE) : $\left(1 - s_{t(T)}^{k} / \sigma_{t(T)}^{k}\right)^{2}$

Heteroscedasticity Adjusted Absolute Error (HAAE): $\left|1 - s_{t(T)}^{k} / \sigma_{t(T)}^{k}\right|$

Logarithmic Error (LE): $\ln\left(s_{t(T)}^{k} / \sigma_{t(T)}^{k}\right)^{2}$

Evaluation Criteria for the Conditional Variance, k=2

Forecast Horizon in days

	5	10	15	16	18	20	22	24	26	28	30	32	34	35	40	45	50	60	70	80	90	100
MeanSE																						
minSPEC	**0.187**	**0.126**	**0.097**	**0.092**	**0.083**	**0.077**	**0.072**	**0.069**	**0.067**	**0.065**	**0.064**	**0.063**	**0.062**	**0.061**	**0.059**	**0.056**	**0.053**	**0.049**	**0.045**	**0.041**	**0.038**	**0.036**
maxSPEC	0.208	0.140	0.107	0.102	0.094	0.089	0.084	0.082	0.079	0.077	0.075	0.073	0.072	0.072	0.068	0.065	0.061	0.056	0.050	0.046	0.043	0.041
minARCH	**0.177**	**0.116**	**0.087**	**0.083**	**0.076**	**0.070**	**0.066**	**0.063**	**0.061**	**0.058**	**0.057**	**0.055**	**0.054**	**0.054**	**0.050**	**0.047**	**0.044**	**0.040**	**0.037**	**0.034**	**0.031**	**0.030**
maxARCH	0.220	0.160	0.129	0.123	0.114	0.107	0.101	0.098	0.096	0.094	0.094	0.094	0.094	0.095	0.095	0.094	0.093	0.092	0.091	0.092	0.091	0.091
MeanAE																						
minSPEC	**1.964**	**1.683**	**1.541**	**1.513**	**1.458**	**1.423**	**1.402**	**1.395**	**1.386**	**1.378**	**1.366**	**1.360**	**1.361**	**1.363**	**1.356**	**1.353**	**1.351**	**1.340**	**1.329**	**1.317**	**1.305**	**1.291**
maxSPEC	2.176	1.869	1.708	1.674	1.605	1.550	1.532	1.527	1.521	1.509	1.498	1.490	1.493	1.492	1.472	1.465	1.464	1.448	1.431	1.415	1.388	1.371
minARCH	**1.844**	**1.588**	**1.492**	**1.471**	**1.432**	**1.409**	**1.395**	**1.386**	**1.374**	**1.359**	**1.350**	**1.337**	**1.335**	**1.331**	**1.314**	**1.292**	**1.283**	**1.263**	**1.241**	**1.233**	**1.214**	**1.201**
maxARCH	2.217	1.963	1.840	1.812	1.761	1.722	1.701	1.692	1.677	1.667	1.660	1.656	1.666	1.670	1.671	1.677	1.681	1.692	1.702	1.701	1.692	1.687
MedSE																						
minSPEC	**0.0119**	**0.0077**	**0.0066**	**0.0064**	**0.0055**	**0.0052**	**0.0051**	**0.0050**	**0.0050**	**0.0048**	**0.0047**	**0.0047**	**0.0045**	**0.0043**	**0.0047**	**0.0044**	**0.0046**	**0.0041**	**0.0045**	**0.0050**	**0.0053**	**0.0056**
maxSPEC	0.0162	0.0115	0.0090	0.0090	0.0074	0.0068	0.0066	0.0063	0.0060	0.0058	0.0055	0.0054	0.0053	0.0055	0.0053	0.0056	0.0058	0.0061	0.0059	0.0064	0.0070	0.0074
minARCH	**0.0094**	**0.0068**	**0.0061**	**0.0056**	**0.0059**	**0.0057**	**0.0055**	**0.0056**	**0.0052**	**0.0049**	**0.0048**	**0.0048**	**0.0045**	**0.0047**	**0.0044**	**0.0043**	**0.0047**	**0.0043**	**0.0042**	**0.0048**	**0.0048**	**0.0045**
maxARCH	0.0151	0.0119	0.0098	0.0096	0.0084	0.0082	0.0083	0.0085	0.0085	0.0082	0.0079	0.0079	0.0075	0.0078	0.0082	0.0082	0.0092	0.0106	0.0113	0.0118	0.0121	0.0132
MedAE																						
minSPEC	**1.089**	**0.877**	**0.815**	**0.799**	**0.742**	**0.723**	**0.717**	**0.708**	**0.708**	**0.690**	**0.684**	**0.683**	**0.668**	**0.658**	**0.683**	**0.661**	**0.681**	**0.642**	**0.672**	**0.706**	**0.729**	**0.749**
maxSPEC	1.274	1.073	0.947	0.951	0.859	0.827	0.810	0.797	0.776	0.758	0.740	0.738	0.728	0.745	0.726	0.751	0.759	0.779	0.766	0.801	0.835	0.860
minARCH	**0.970**	**0.824**	**0.784**	**0.751**	**0.770**	**0.755**	**0.740**	**0.748**	**0.722**	**0.703**	**0.696**	**0.690**	**0.674**	**0.682**	**0.666**	**0.653**	**0.689**	**0.657**	**0.650**	**0.694**	**0.690**	**0.671**
maxARCH	1.228	1.089	0.988	0.981	0.915	0.906	0.909	0.920	0.923	0.907	0.890	0.890	0.868	0.881	0.906	0.906	0.958	1.028	1.063	1.088	1.100	1.150
MeanHASE																						
minSPEC	**208.9**	**146.0**	**111.0**	**104.7**	**93.2**	**85.1**	**78.9**	**75.2**	**73.3**	**71.3**	**70.3**	**69.7**	**68.5**	**68.2**	**65.9**	**63.4**	**61.7**	**62.4**	**61.3**	**59.1**	**57.3**	**57.5**
maxSPEC	250.3	170.1	131.0	128.1	121.0	114.6	113.9	111.1	109.0	106.1	103.2	101.0	98.6	97.7	93.3	89.5	85.8	81.9	76.3	71.1	67.2	66.2
minARCH	**236.8**	**158.5**	**117.6**	**111.5**	**99.5**	**91.9**	**86.5**	**82.1**	**78.9**	**76.6**	**75.0**	**73.7**	**72.5**	**72.3**	**68.1**	**64.6**	**60.5**	**57.2**	**54.5**	**51.5**	**48.1**	**47.9**
maxARCH	414.9	279.8	232.7	226.3	214.7	203.8	196.3	189.0	183.6	178.7	174.5	170.9	167.5	166.1	159.1	153.1	147.9	141.3	136.3	132.2	128.7	126.8
MeanHAAE																						
minSPEC	**64.52**	**56.37**	**52.23**	**51.22**	**49.53**	**48.15**	**47.53**	**47.37**	**47.38**	**47.33**	**47.15**	**47.32**	**47.51**	**47.68**	**47.99**	**48.49**	**49.09**	**50.71**	**51.74**	**52.64**	**53.31**	**54.09**
maxSPEC	68.19	60.02	56.63	56.30	55.61	55.29	55.49	55.52	55.48	55.14	54.76	54.63	54.79	54.78	54.32	54.61	55.13	55.98	56.32	56.65	56.86	57.32
minARCH	**70.92**	**61.82**	**57.56**	**56.56**	**54.71**	**53.32**	**52.69**	**52.21**	**51.80**	**51.33**	**50.95**	**50.90**	**51.11**	**51.18**	**51.17**	**50.88**	**50.26**	**50.48**	**50.53**	**50.73**	**50.18**	**50.25**
maxARCH	91.12	82.42	79.46	79.09	78.44	77.80	77.13	76.70	76.40	76.39	76.43	76.62	76.87	76.99	77.49	77.90	78.38	78.82	79.31	80.12	80.66	81.69

Table 5.2 (continued)

	Forecast Horizon in days																					
	5	10	15	16	18	20	22	24	26	28	30	32	34	35	40	45	50	60	70	80	90	100
MedHASE																						
minSPEC	**19.52**	**12.27**	**10.81**	**10.01**	**9.10**	**8.42**	**8.45**	**8.14**	**7.95**	**7.56**	**7.77**	**7.54**	**7.44**	**7.43**	**7.80**	**7.38**	**8.08**	**8.41**	**8.22**	**9.39**	**8.33**	**8.74**
maxSPEC	21.91	15.01	11.88	11.16	11.31	10.54	10.39	10.35	10.25	10.01	10.66	10.39	10.27	10.22	10.77	10.96	10.95	11.10	11.97	11.28	11.29	11.94
minARCH	**19.53**	**12.50**	**10.08**	**9.82**	**9.69**	**9.02**	**9.24**	**9.43**	**9.22**	**8.51**	**8.85**	**8.42**	**8.43**	**8.86**	**8.45**	**8.40**	**8.72**	**8.36**	**8.57**	**8.38**	**8.80**	**8.33**
maxARCH	28.21	20.02	17.90	17.26	15.91	16.05	15.27	15.08	15.30	14.95	14.90	15.31	15.77	15.53	15.46	15.09	15.92	16.42	19.19	22.97	27.93	36.67
MedHAAE																						
minSPEC	**44.19**	**35.03**	**32.88**	**31.64**	**30.17**	**29.02**	**29.07**	**28.54**	**28.19**	**27.49**	**27.88**	**27.46**	**27.27**	**27.25**	**27.93**	**27.17**	**28.43**	**29.01**	**28.67**	**30.65**	**28.87**	**29.57**
maxSPEC	46.80	38.75	34.47	33.41	33.63	32.47	32.23	32.17	32.02	31.63	32.65	32.24	32.05	31.98	32.82	33.10	33.10	33.31	34.60	33.58	33.61	34.55
minARCH	**44.20**	**35.35**	**31.75**	**31.33**	**31.12**	**30.04**	**30.39**	**30.70**	**30.37**	**29.17**	**29.74**	**29.02**	**29.03**	**29.77**	**29.07**	**28.97**	**29.52**	**28.91**	**29.28**	**28.95**	**29.67**	**28.86**
maxARCH	53.11	44.75	42.31	41.55	39.89	40.07	39.07	38.83	39.11	38.67	38.60	39.13	39.71	39.41	39.32	38.85	39.90	40.52	43.81	47.93	52.85	60.55
MeanLE																						
minSPEC	**68.24**	**39.93**	**31.75**	**30.99**	**29.04**	**27.71**	**27.07**	**26.43**	**26.07**	**25.85**	**25.66**	**25.40**	**25.24**	**25.16**	**24.94**	**24.85**	**24.84**	**25.09**	**25.16**	**25.22**	**25.22**	**25.55**
maxSPEC	77.35	46.44	37.00	35.93	33.76	32.67	32.14	31.50	31.09	30.71	30.34	30.02	29.86	29.72	29.16	28.93	28.89	28.89	28.37	28.03	27.70	27.74
minARCH	**63.02**	**40.18**	**33.82**	**33.10**	**31.36**	**29.90**	**29.03**	**28.31**	**27.70**	**27.19**	**26.84**	**26.42**	**26.21**	**26.06**	**25.40**	**24.77**	**24.41**	**23.99**	**23.40**	**23.01**	**22.44**	**22.27**
maxARCH	79.54	52.04	45.60	44.76	43.33	42.18	41.65	41.27	40.98	40.91	40.83	40.71	40.81	40.81	40.78	40.94	41.10	42.78	44.33	45.35	45.72	46.73
MedLE																						
minSPEC	**22.80**	**14.22**	**10.94**	**10.65**	**10.04**	**8.65**	**9.27**	**8.49**	**8.02**	**8.19**	**7.59**	**7.72**	**8.17**	**8.33**	**8.36**	**8.20**	**9.10**	**9.06**	**9.62**	**9.73**	**9.52**	**9.14**
maxSPEC	26.63	17.41	13.00	12.72	12.03	11.30	11.22	11.42	11.29	10.87	10.79	10.49	10.38	10.65	10.77	10.81	11.51	12.07	12.94	12.35	12.36	13.03
minARCH	**21.60**	**13.53**	**11.27**	**11.10**	**10.41**	**9.46**	**9.86**	**10.03**	**9.81**	**9.32**	**9.14**	**8.65**	**8.92**	**9.11**	**9.07**	**8.68**	**9.19**	**8.61**	**9.34**	**9.46**	**9.22**	**8.83**
maxARCH	32.39	22.40	18.06	17.53	16.90	16.38	15.43	15.85	15.30	14.79	14.31	13.37	13.68	14.18	14.83	15.71	17.07	18.56	21.65	21.97	23.58	26.19

Evaluation Criteria for the Conditional Standard Deviation, k=1

	Forecast Horizon in days																					
	5	10	15	16	18	20	22	24	26	28	30	32	34	35	40	45	50	60	70	80	90	100
MeanSE																						
minSPEC	**0.535**	**0.398**	**0.335**	**0.323**	**0.301**	**0.288**	**0.277**	**0.271**	**0.267**	**0.265**	**0.263**	**0.260**	**0.259**	**0.259**	**0.255**	**0.250**	**0.245**	**0.239**	**0.234**	**0.227**	**0.220**	**0.216**
maxSPEC	0.617	0.459	0.383	0.371	0.349	0.338	0.332	0.327	0.322	0.318	0.314	0.310	0.309	0.308	0.300	0.294	0.288	0.279	0.266	0.254	0.243	0.237
minARCH	**0.507**	**0.396**	**0.343**	**0.333**	**0.316**	**0.301**	**0.292**	**0.283**	**0.277**	**0.271**	**0.268**	**0.265**	**0.262**	**0.261**	**0.252**	**0.242**	**0.235**	**0.224**	**0.214**	**0.206**	**0.196**	**0.191**
maxARCH	0.656	0.513	0.448	0.435	0.414	0.400	0.392	0.386	0.380	0.375	0.371	0.367	0.364	0.363	0.358	0.353	0.347	0.345	0.343	0.341	0.334	0.333
MeanAE																						
minSPEC	**4.944**	**4.111**	**3.759**	**3.698**	**3.583**	**3.504**	**3.466**	**3.448**	**3.431**	**3.416**	**3.388**	**3.381**	**3.386**	**3.393**	**3.394**	**3.410**	**3.433**	**3.464**	**3.466**	**3.484**	**3.493**	**3.491**
maxSPEC	5.396	4.519	4.142	4.074	3.913	3.822	3.798	3.792	3.783	3.757	3.728	3.713	3.725	3.725	3.687	3.696	3.721	3.733	3.734	3.737	3.706	3.692
minARCH	**4.697**	**4.001**	**3.766**	**3.718**	**3.632**	**3.576**	**3.543**	**3.535**	**3.515**	**3.487**	**3.465**	**3.432**	**3.430**	**3.423**	**3.401**	**3.360**	**3.356**	**3.338**	**3.315**	**3.327**	**3.308**	**3.284**
maxARCH	5.460	4.762	4.483	4.428	4.337	4.261	4.235	4.228	4.203	4.184	4.166	4.156	4.173	4.180	4.194	4.229	4.268	4.356	4.443	4.497	4.507	4.549
MedSE																						
minSPEC	**0.1322**	**0.0823**	**0.0739**	**0.0699**	**0.0597**	**0.0551**	**0.0560**	**0.0549**	**0.0538**	**0.0521**	**0.0499**	**0.0480**	**0.0474**	**0.0478**	**0.0498**	**0.0470**	**0.0490**	**0.0515**	**0.0531**	**0.0538**	**0.0542**	**0.0536**
maxSPEC	0.1768	0.1218	0.0840	0.0806	0.0742	0.0696	0.0677	0.0665	0.0646	0.0605	0.0602	0.0613	0.0596	0.0628	0.0617	0.0647	0.0700	0.0689	0.0745	0.0717	0.0654	0.0685
minARCH	**0.1211**	**0.0783**	**0.0653**	**0.0632**	**0.0624**	**0.0596**	**0.0573**	**0.0575**	**0.0565**	**0.0536**	**0.0548**	**0.0499**	**0.0507**	**0.0523**	**0.0500**	**0.0516**	**0.0518**	**0.0500**	**0.0506**	**0.0531**	**0.0516**	**0.0486**
maxARCH	0.1693	0.1278	0.1077	0.1046	0.0967	0.0920	0.0929	0.0827	0.0860	0.0785	0.0819	0.0805	0.0859	0.0896	0.0864	0.0936	0.1001	0.1131	0.1401	0.1410	0.1492	0.1542
MedAE																						
minSPEC	**3.636**	**2.868**	**2.719**	**2.645**	**2.444**	**2.347**	**2.367**	**2.344**	**2.319**	**2.282**	**2.234**	**2.190**	**2.177**	**2.186**	**2.232**	**2.168**	**2.213**	**2.269**	**2.304**	**2.319**	**2.327**	**2.315**
maxSPEC	4.205	3.491	2.898	2.838	2.723	2.638	2.601	2.578	2.543	2.459	2.453	2.476	2.442	2.506	2.484	2.544	2.646	2.625	2.729	2.678	2.557	2.617
minARCH	**3.479**	**2.798**	**2.555**	**2.514**	**2.499**	**2.441**	**2.394**	**2.398**	**2.377**	**2.316**	**2.340**	**2.234**	**2.252**	**2.286**	**2.236**	**2.272**	**2.275**	**2.235**	**2.250**	**2.305**	**2.271**	**2.204**
maxARCH	4.115	3.576	3.282	3.234	3.110	3.034	3.047	2.876	2.932	2.802	2.861	2.837	2.931	2.993	2.939	3.059	3.163	3.362	3.743	3.755	3.863	3.927
MeanHASE																						
minSPEC	**18.50**	**14.04**	**11.92**	**11.59**	**10.81**	**10.32**	**9.91**	**9.68**	**9.57**	**9.47**	**9.41**	**9.37**	**9.31**	**9.30**	**9.18**	**9.07**	**8.99**	**9.19**	**9.23**	**9.16**	**9.08**	**9.22**
maxSPEC	19.98	15.57	13.59	13.38	13.00	12.68	12.77	12.68	12.62	12.48	12.31	12.19	12.06	11.99	11.65	11.45	11.26	11.21	10.85	10.53	10.26	10.25
minARCH	**20.34**	**15.77**	**13.49**	**13.06**	**12.25**	**11.68**	**11.34**	**11.04**	**10.82**	**10.64**	**10.52**	**10.45**	**10.32**	**10.25**	**9.91**	**9.57**	**9.30**	**9.05**	**8.74**	**8.45**	**8.11**	**8.03**
maxARCH	28.78	23.16	21.45	21.21	20.76	20.28	19.97	19.68	19.47	19.28	19.13	19.01	18.90	18.85	18.63	18.44	18.30	18.11	18.01	17.93	17.83	17.81

Table 5.2 (continued)

	Forecast Horizon in days																					
	5	10	15	16	18	20	22	24	26	28	30	32	34	35	40	45	50	60	70	80	90	100
MeanHAAE																						
minSPEC	**29.45**	**24.54**	**22.61**	**22.24**	**21.57**	**21.05**	**20.78**	**20.68**	**20.64**	**20.60**	**20.47**	**20.50**	**20.57**	**20.63**	**20.75**	**20.97**	**21.25**	**21.82**	**22.17**	**22.54**	**22.79**	**23.00**
maxSPEC	30.68	25.95	24.17	24.00	23.64	23.45	23.41	23.41	23.39	23.24	23.07	23.01	23.08	23.08	22.90	23.06	23.31	23.62	23.82	23.97	24.09	24.24
minARCH	**30.46**	**26.09**	**24.45**	**24.12**	**23.53**	**23.09**	**22.82**	**22.72**	**22.58**	**22.41**	**22.24**	**22.05**	**21.98**	**21.95**	**21.83**	**21.66**	**21.66**	**21.64**	**21.59**	**21.71**	**21.58**	**21.52**
maxARCH	36.16	31.96	30.45	30.28	30.01	29.78	29.49	29.33	29.23	29.26	29.31	29.42	29.57	29.63	29.93	30.17	30.44	30.66	31.45	32.19	32.62	33.28
MedHASE																						
minSPEC	**5.560**	**3.343**	**2.845**	**2.602**	**2.444**	**2.179**	**2.252**	**2.146**	**1.972**	**2.001**	**2.003**	**1.887**	**1.991**	**2.012**	**2.147**	**1.944**	**2.176**	**2.243**	**2.259**	**2.348**	**2.260**	**2.271**
maxSPEC	6.158	4.088	3.167	2.888	3.001	2.757	2.702	2.759	2.645	2.724	2.678	2.676	2.552	2.618	2.773	2.718	2.767	2.940	3.195	2.951	2.913	3.147
minARCH	**5.366**	**3.329**	**2.523**	**2.652**	**2.564**	**2.372**	**2.441**	**2.485**	**2.444**	**2.270**	**2.247**	**2.173**	**2.269**	**2.273**	**2.198**	**2.124**	**2.264**	**2.144**	**2.242**	**2.356**	**2.185**	**2.124**
maxARCH	7.855	5.423	4.480	4.480	4.143	4.020	3.904	3.863	3.960	3.813	3.723	3.698	3.733	3.664	3.711	3.899	3.961	4.259	5.022	5.826	6.754	7.586
MedHAAE																						
minSPEC	**23.58**	**18.29**	**16.87**	**16.13**	**15.63**	**14.76**	**15.01**	**14.65**	**14.04**	**14.15**	**14.15**	**13.74**	**14.11**	**14.19**	**14.65**	**13.94**	**14.75**	**14.98**	**15.03**	**15.32**	**15.03**	**15.07**
maxSPEC	24.82	20.22	17.80	16.99	17.32	16.60	16.44	16.61	16.26	16.51	16.36	16.36	15.98	16.18	16.65	16.49	16.63	17.15	17.88	17.18	17.07	17.74
minARCH	**23.17**	**18.25**	**15.89**	**16.29**	**16.01**	**15.40**	**15.62**	**15.76**	**15.63**	**15.07**	**14.99**	**14.74**	**15.06**	**15.08**	**14.83**	**14.57**	**15.05**	**14.64**	**14.97**	**15.35**	**14.78**	**14.57**
maxARCH	28.03	23.29	21.16	21.17	20.36	20.05	19.76	19.66	19.90	19.53	19.29	19.23	19.32	19.14	19.26	19.75	19.90	20.64	22.41	24.14	25.99	27.54
MeanLE																						
minSPEC	**17.06**	**9.98**	**7.94**	**7.75**	**7.26**	**6.93**	**6.77**	**6.61**	**6.52**	**6.46**	**6.41**	**6.35**	**6.31**	**6.29**	**6.24**	**6.21**	**6.21**	**6.27**	**6.29**	**6.30**	**6.30**	**6.39**
maxSPEC	19.34	11.61	9.25	8.98	8.44	8.17	8.03	7.88	7.77	7.68	7.59	7.51	7.47	7.43	7.29	7.23	7.22	7.22	7.09	7.01	6.92	6.94
minARCH	**15.75**	**10.04**	**8.46**	**8.27**	**7.84**	**7.47**	**7.26**	**7.08**	**6.92**	**6.80**	**6.71**	**6.61**	**6.55**	**6.52**	**6.35**	**6.19**	**6.10**	**6.00**	**5.85**	**5.75**	**5.61**	**5.57**
maxARCH	19.89	13.01	11.40	11.19	10.83	10.55	10.41	10.32	10.24	10.23	10.21	10.18	10.20	10.20	10.20	10.23	10.27	10.70	11.08	11.34	11.43	11.68
MedLE																						
minSPEC	**5.701**	**3.555**	**2.734**	**2.661**	**2.510**	**2.162**	**2.319**	**2.123**	**2.006**	**2.049**	**1.896**	**1.930**	**2.043**	**2.083**	**2.089**	**2.050**	**2.276**	**2.264**	**2.404**	**2.432**	**2.380**	**2.285**
maxSPEC	6.657	4.352	3.249	3.179	3.007	2.825	2.806	2.854	2.821	2.718	2.698	2.623	2.595	2.662	2.693	2.703	2.877	3.018	3.235	3.089	3.090	3.257
minARCH	**5.400**	**3.383**	**2.818**	**2.774**	**2.603**	**2.364**	**2.465**	**2.507**	**2.452**	**2.329**	**2.285**	**2.163**	**2.229**	**2.278**	**2.268**	**2.169**	**2.297**	**2.152**	**2.335**	**2.366**	**2.306**	**2.207**
maxARCH	8.099	5.599	4.514	4.382	4.224	4.096	3.857	3.963	3.825	3.698	3.577	3.344	3.420	3.546	3.708	3.928	4.267	4.639	5.412	5.491	5.894	6.548

Figure 5.2a

The plots indicate whether the ARCH models selected by the SPEC algorithm achieve the lowest value of the evaluation criterion, for a forecasting horizon ranging from one day to one hundred days ahead. The value 2 indicates that the ARCH model selected by the SPEC algorithm achieves the lowest value for the corresponding criterion and the specific forecasting horizon. The value 1 indicates the opposite. The realized volatilty measure is expressed as in (4.1). The evaluation criteria are the mean and the median values of the functions defined by (4.3), (4.4), (4.5), (4.6) and (4.7).

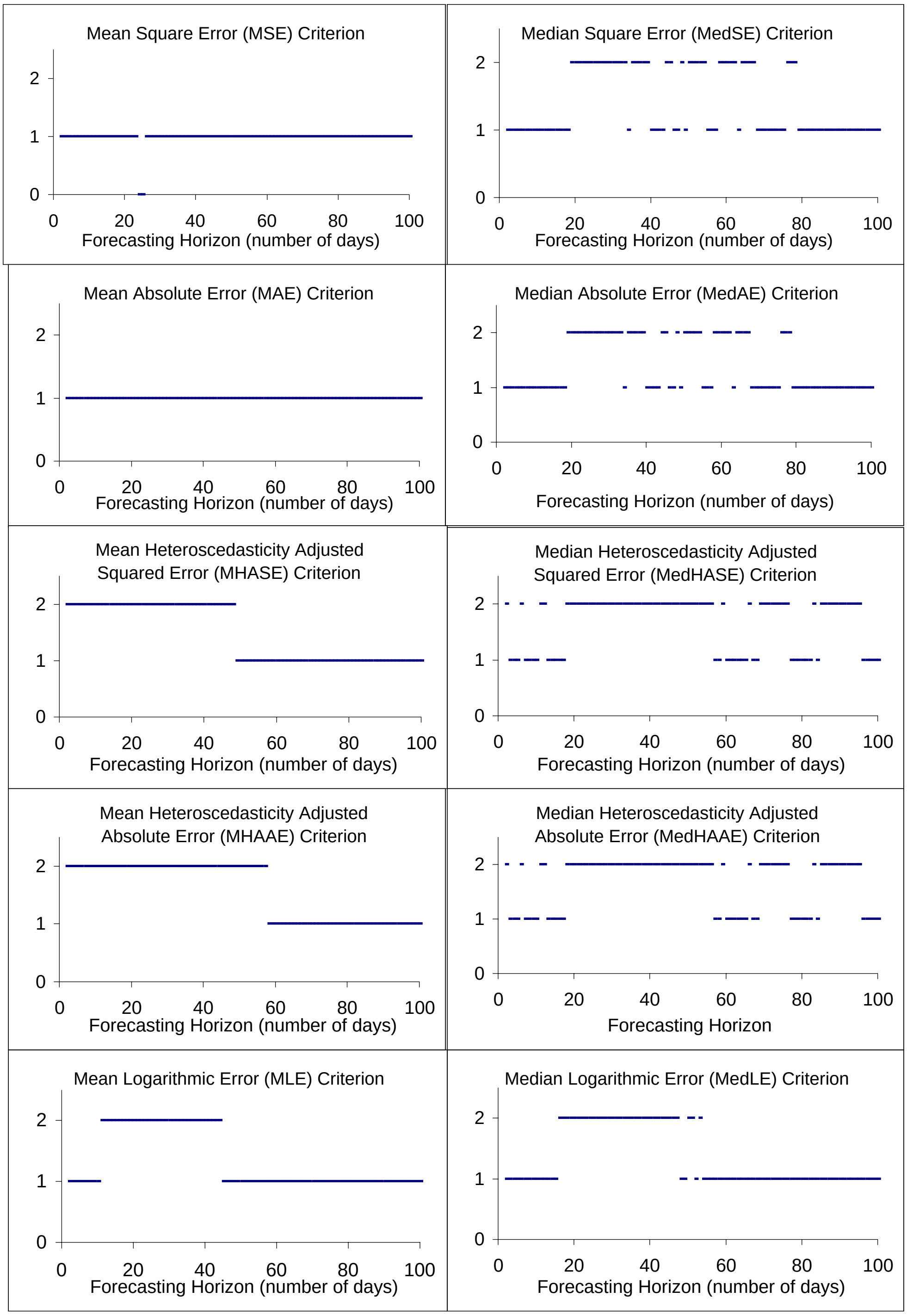


Figure 5.2b

The plots indicate whether the ARCH models selected by the SPEC algorithm achieve the lowest value of the evaluation criterion, for a forecasting horizon ranging from one day to one hundred days ahead. The value 2 indicates that the ARCH model selected by the SPEC algorithm achieves the lowest value for the corresponding criterion and the specific forecasting horizon. The value 1 indicates the opposite. The realized volatilty measure is expressed as the square root of (4.1). The evaluation criteria are the mean and the median values of the functions defined by (4.3), (4.4), (4.5), (4.6) and (4.7).

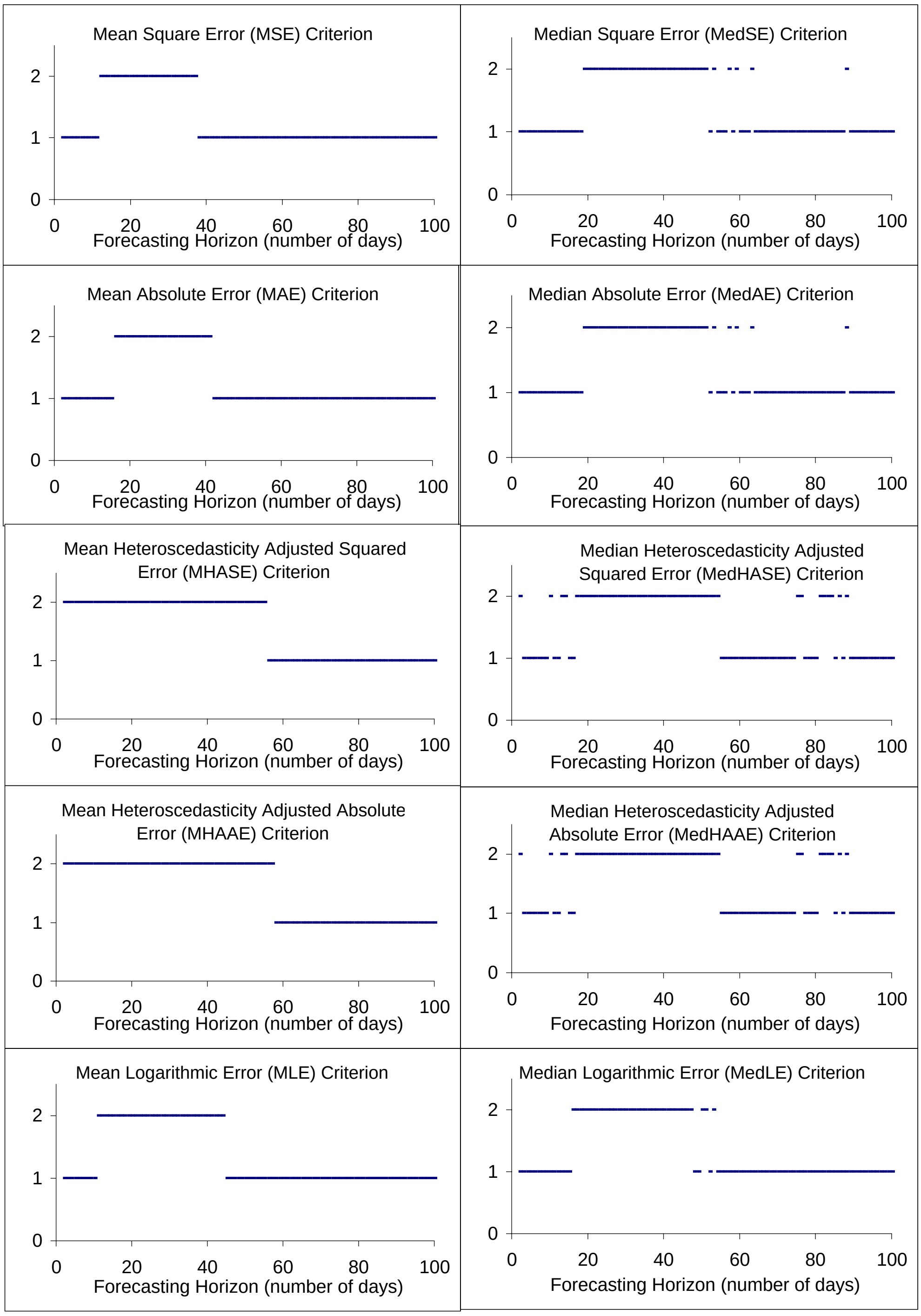


Table 5.3

The percentage of times the ARCH models selected by the SPEC algorithm perform "better" as judged by the evaluation criteria. The first and the second panel correspond to the mean and the median of the evaluation criteria, respectively. The left and the right part of the panels correspond to the volatility expressed as the varian and the standard deviation of the returns, respectively.

Days ahead forecasting horizon	Mean									
	Variance					Standard Deviation				
	MSE	MAE	MHASE	MHAAE	MLE	MSE	MAE	MHASE	MHAAE	MLE
1-100	0%	0%	47%	56%	34%	26%	26%	54%	56%	34%
11-52	0%	0%	88%	100%	79%	62%	62%	100%	100%	79%
16-36	0%	0%	100%	100%	100%	100%	100%	100%	100%	100%
Days ahead forecasting horizon	**Median**									
	Variance					Standard Deviation				
	MedSE	MedAE	MedHASE	MedHAAE	MedLE	MedSE	MedAE	MedHASE	MedHAAE	MedLE
1-100	40%	40%	65%	65%	35%	38%	38%	50%	50%	35%
11-52	64%	64%	88%	88%	83%	81%	81%	93%	93%	83%
16-36	86%	86%	95%	95%	100%	90%	90%	100%	100%	100%

MSE: Mean Square Error

MAE: Mean Absolute Error

MHASE: Mean Heteroscedasticity Adjusted Squared Error

MHAAE: Mean Heteroscedasticity Adjusted Absolute Error

MLE: Mean Logarithmic Error

MedSE: Median Square Error

MedAE: Median Absolute Error

minSPEC

maxSPEC

MedLE: Median Logarithmic Error

Table 5.4

Average sample size for the SPEC model selection algorithm suggested by the Evaluation Criteria for both the entire 16 to 36 day ahead forecasting horizon and for each day individually.

	Average sample size suggested by the Evaluation Criteria rating the performance of the SPEC selection algorithm "best".			Average sample size suggested by all the Evaluation Criteria considered.		
Forecasting Horizon (in number of days ahead)	Number of Criteria	Average sample size	Standard Deviation	Number of Criteria	Average sample size	Standard Deviation
16-36	366	19.7	3.6	420	19.9	3.7
16	12	23.8	1.7	20	26.0	2.5
17	14	20.7	1.5	20	23.5	2.9
18	18	24.7	2.8	20	24.3	2.7
19	18	25.0	3.3	20	24.3	3.3
20	18	23.6	3.3	20	23.0	3.3
21	18	23.3	3.4	20	22.5	3.4
22	18	20.0	3.8	20	19.5	3.6
23	18	19.4	3.8	20	19.0	3.7
24	18	19.4	3.8	20	19.0	3.7
25	18	17.2	2.9	20	17.0	2.8
26	18	17.2	2.9	20	17.0	2.8
27	18	17.8	3.6	20	17.5	3.4
28	18	18.3	3.1	20	18.0	2.9
29	18	18.3	3.1	20	18.0	2.9
30	18	20.6	6.5	20	20.0	6.2
31	18	16.1	1.4	20	16.0	1.4
32	18	17.8	3.6	20	17.5	3.4
33	16	15.6	0.8	20	20.0	6.2
34	18	21.1	4.7	20	20.5	4.5
35	18	17.8	3.6	20	17.5	3.4
36	18	17.8	2.9	20	17.5	2.8

Table 5.6.1

The percentage of times the ARCH models selected by the SEVar method perform "better" as judged by the evaluation criteria. The first and the second panel correspond to the mean and the median of the evaluation criteria, respectively. The left and the right part of the panels correspond to the volatility expressed as the variance and the standard deviation of the returns, respectively.

Days ahead forecasting horizon	Mean									
	Variance					Standard Deviation				
	MSE	MAE	MHASE	MHAAE	MLE	MSE	MAE	MHASE	MHAAE	MLE
1-100	2%	2%	1%	1%	3%	2%	2%	1%	1%	3%
11-52	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
16-36	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%

Days ahead forecasting horizon	Median									
	Variance					Standard Deviation				
	MedSE	MedAE	MedHASE	MedHAAE	MedLE	MedSE	MedAE	MedHASE	MedHAAE	MedLE
1-100	4%	4%	2%	2%	3%	2%	2%	1%	1%	3%
11-52	2%	2%	0%	0%	0%	0%	0%	0%	0%	0%
16-36	5%	5%	0%	0%	0%	0%	0%	0%	0%	0%

MSE: Mean Square Error

MAE: Mean Absolute Error

MHASE: Mean Heteroscedasticity Adjusted Squared Error

MHAAE: Mean Heteroscedasticity Adjusted Absolute Error

MLE: Mean Logarithmic Error

MedSE: Median Square Error

MedAE: Median Absolute Error

MedHASE: Median Heteroscedasticity Adjusted Squared Error

MedHAAE: Median Heteroscedasticity Adjusted Absolute Error

MedLE: Median Logarithmic Error

Table 5.6.2

The percentage of times the ARCH models selected by the AEVar method perform "better" as judged by the evaluation criteria. The first and the second panel correspond to the mean and the median of the evaluation criteria, respectively. The left and the right part of the panels correspond to the volatility expressed as the variance and the standard deviation of the returns, respectively.

Days ahead forecasting horizon	Mean									
	Variance					Standard Deviation				
	MSE	MAE	MHASE	MHAAE	MLE	MSE	MAE	MHASE	MHAAE	MLE
1-100	1%	2%	0%	0%	3%	2%	3%	0%	1%	3%
11-52	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
16-36	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%

Days ahead forecasting horizon	Median									
	Variance					Standard Deviation				
	MedSE	MedAE	MedHASE	MedHAAE	MedLE	MedSE	MedAE	MedHASE	MedHAAE	MedLE
1-100	9%	9%	1%	1%	2%	6%	6%	1%	1%	2%
11-52	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
16-36	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%

Table 5.6.3

The percentage of times the ARCH models selected by the SEDev method perform "better" as judged by the evaluation criteria. The first and the second panel correspond to the mean and the median of the evaluation criteria, respectively. The left and the right part of the panels correspond to the volatility expressed as the variance and the standard deviation of the returns, respectively.

Days ahead forecasting horizon	Mean									
	Variance					Standard Deviation				
	MSE	MAE	MHASE	MHAAE	MLE	MSE	MAE	MHASE	MHAAE	MLE
1-100	1%	3%	1%	0%	3%	2%	3%	1%	1%	3%
11-52	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
16-36	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%

Days ahead forecasting horizon	Median									
	Variance					Standard Deviation				
	MedSE	MedAE	MedHASE	MedHAAE	MedLE	MedSE	MedAE	MedHASE	MedHAAE	MedLE
1-100	8%	8%	1%	1%	2%	5%	5%	1%	1%	2%
11-52	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
16-36	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%

Table 5.6.4

The percentage of times the ARCH models selected by the AEDev method perform "better" as judged by the evaluation criteria. The first and the second panel correspond to the mean and the median of the evaluation criteria, respectively. The left and the right part of the panels correspond to the volatility expressed as the variance and the standard deviation of the returns, respectively.

Days ahead forecasting horizon	Mean									
	Variance					Standard Deviation				
	MSE	MAE	MHASE	MHAAE	MLE	MSE	MAE	MHASE	MHAAE	MLE
1-100	1%	3%	0%	0%	3%	2%	3%	0%	1%	3%
11-52	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
16-36	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%

Days ahead forecasting horizon	Median									
	Variance					Standard Deviation				
	MedSE	MedAE	MedHASE	MedHAAE	MedLE	MedSE	MedAE	MedHASE	MedHAAE	MedLE
1-100	9%	9%	1%	1%	2%	6%	6%	1%	1%	2%
11-52	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
16-36	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%

MSE: Mean Square Error

MAE: Mean Absolute Error

MHASE: Mean Heteroscedasticity Adjusted Squared Error

MHAAE: Mean Heteroscedasticity Adjusted Absolute Error

MLE: Mean Logarithmic Error

MedSE: Median Square Error

MedAE: Median Absolute Error

MedHASE: Median Heteroscedasticity Adjusted Squared Error

MedHAAE: Median Heteroscedasticity Adjusted Absolute Error

MedLE: Median Logarithmic Error

Table 5.6.5

The percentage of times the ARCH models selected by the HASEVar method perform "better" as judged by the evaluation criteria. The first and the second panel correspond to the mean and the median of the evaluation criteria, respectively. The left and the right part of the panels correspond to the volatility expressed as the variance and the standard deviation of the returns, respectively.

Days ahead forecasting horizon	Mean									
	Variance					Standard Deviation				
	MSE	MAE	MHASE	MHAAE	MLE	MSE	MAE	MHASE	MHAAE	MLE
1-100	0%	0%	8%	16%	0%	0%	11%	12%	34%	0%
11-52	0%	0%	0%	14%	0%	0%	26%	5%	57%	0%
16-36	0%	0%	0%	5%	0%	0%	52%	0%	90%	0%

Days ahead forecasting horizon	Median									
	Variance					Standard Deviation				
	MedSE	MedAE	MedHASE	MedHAAE	MedLE	MedSE	MedAE	MedHASE	MedHAAE	MedLE
1-100	40%	40%	21%	21%	22%	26%	26%	20%	20%	22%
11-52	67%	67%	45%	45%	48%	57%	57%	45%	45%	48%
16-36	95%	95%	81%	81%	86%	86%	86%	81%	81%	86%

Table 5.6.6

The percentage of times the ARCH models selected by the HAAEVar method perform "better" as judged by the evaluation criteria. The first and the second panel correspond to the mean and the median of the evaluation criteria, respectively. The left and the right part of the panels correspond to the volatility expressed as the variance and the standard deviation of the returns, respectively.

Days ahead forecasting horizon	Mean									
	Variance					Standard Deviation				
	MSE	MAE	MHASE	MHAAE	MLE	MSE	MAE	MHASE	MHAAE	MLE
1-100	2%	1%	4%	16%	0%	1%	12%	9%	34%	0%
11-52	0%	0%	0%	14%	0%	0%	29%	0%	57%	0%
16-36	0%	0%	0%	5%	0%	0%	57%	0%	90%	0%

Days ahead forecasting horizon	Median									
	Variance					Standard Deviation				
	MedSE	MedAE	MedHASE	MedHAAE	MedLE	MedSE	MedAE	MedHASE	MedHAAE	MedLE
1-100	36%	36%	26%	26%	24%	26%	26%	24%	24%	24%
11-52	64%	64%	52%	52%	52%	60%	60%	50%	50%	52%
16-36	90%	90%	100%	100%	100%	86%	86%	95%	95%	100%

Table 5.6.7

The percentage of times the ARCH models selected by the HASEDev method perform "better" as judged by the evaluation criteria. The first and the second panel correspond to the mean and the median of the evaluation criteria, respectively. The left and the right part of the panels correspond to the volatility expressed as the variance and the standard deviation of the returns, respectively.

Days ahead forecasting horizon	Mean									
	Variance					Standard Deviation				
	MSE	MAE	MHASE	MHAAE	MLE	MSE	MAE	MHASE	MHAAE	MLE
1-100	1%	2%	3%	6%	0%	2%	2%	5%	8%	0%
11-52	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
16-36	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
Days ahead forecasting horizon	Median									
	Variance					Standard Deviation				
	MedSE	MedAE	MedHASE	MedHAAE	MedLE	MedSE	MedAE	MedHASE	MedHAAE	MedLE
1-100	25%	25%	25%	25%	25%	21%	21%	27%	27%	25%
11-52	50%	50%	43%	43%	45%	48%	48%	48%	48%	45%
16-36	86%	86%	81%	81%	86%	76%	76%	90%	90%	86%

MSE: Mean Square Error

MAE: Mean Absolute Error

MHASE: Mean Heteroscedasticity Adjusted Squared Error

MHAAE: Mean Heteroscedasticity Adjusted Absolute Error

MLE: Mean Logarithmic Error

MedSE: Median Square Error

MedAE: Median Absolute Error

MedHASE: Median Heteroscedasticity Adjusted Squared Error

MedHAAE: Median Heteroscedasticity Adjusted Absolute Error

MedLE: Median Logarithmic Error

Table 5.6.8

The percentage of times the ARCH models selected by the HAAEDev method perform "better" as judged by the evaluation criteria. The first and the second panel correspond to the mean and the median of the evaluation criteria, respectively. The left and the right part of the panels correspond to the volatility expressed as the variance and the standard deviation of the returns, respectively.

Days ahead forecasting horizon	Mean									
	Variance					Standard Deviation				
	MSE	MAE	MHASE	MHAAE	MLE	MSE	MAE	MHASE	MHAAE	MLE
1-100	3%	2%	2%	2%	2%	2%	2%	2%	3%	2%
11-52	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
16-36	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
Days ahead forecasting horizon	Median									
	Variance					Standard Deviation				
	MedSE	MedAE	MedHASE	MedHAAE	MedLE	MedSE	MedAE	MedHASE	MedHAAE	MedLE
1-100	3%	3%	8%	8%	6%	5%	5%	7%	7%	6%
11-52	5%	5%	5%	5%	5%	7%	7%	10%	10%	5%
16-36	10%	10%	10%	10%	10%	10%	10%	19%	19%	10%

Table 5.6.9

The percentage of times the ARCH models selected by the LEVar method perform "better" as judged by the evaluation criteria. The first and the second panel correspond to the mean and the median of the evaluation criteria, respectively. The left and the right part of the panels correspond to the volatility expressed as the variance and the standard deviation of the returns, respectively.

Days ahead forecasting horizon	Mean									
	Variance					Standard Deviation				
	MSE	MAE	MHASE	MHAAE	MLE	MSE	MAE	MHASE	MHAAE	MLE
1-100	1%	2%	0%	0%	3%	2%	2%	0%	0%	3%
11-52	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
16-36	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
Days ahead forecasting horizon	Median									
	Variance					Standard Deviation				
	MedSE	MedAE	MedHASE	MedHAAE	MedLE	MedSE	MedAE	MedHASE	MedHAAE	MedLE
1-100	8%	8%	1%	1%	2%	5%	5%	1%	1%	2%
11-52	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
16-36	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%

Table 5.6.10

The percentage of times the ARCH models selected by the AIC method perform "better" as judged by the evaluation criteria. The first and the second panel correspond to the mean and the median of the evaluation criteria, respectively. The left and the right part of the panels correspond to the volatility expressed as the variance and the standard deviation of the returns, respectively.

Days ahead forecasting horizon	Mean									
	Variance					Standard Deviation				
	MSE	MAE	MHASE	MHAAE	MLE	MSE	MAE	MHASE	MHAAE	MLE
1-100	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
11-52	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
16-36	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%

Days ahead forecasting horizon	Median									
	Variance					Standard Deviation				
	MedSE	MedAE	MedHASE	MedHAAE	MedLE	MedSE	MedAE	MedHASE	MedHAAE	MedLE
1-100	1%	1%	0%	0%	0%	0%	0%	0%	0%	0%
11-52	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
16-36	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%

Table 5.6.11

The percentage of times the ARCH models selected by the SBC method perform "better" as judged by the evaluation criteria. The first and the second panel correspond to the mean and the median of the evaluation criteria, respectively. The left and the right part of the panels correspond to the volatility expressed as the variance and the standard deviation of the returns, respectively.

Days ahead forecasting horizon	Mean									
	Variance					Standard Deviation				
	MSE	MAE	MHASE	MHAAE	MLE	MSE	MAE	MHASE	MHAAE	MLE
1-100	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
11-52	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
16-36	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%

Days ahead forecasting horizon	Median									
	Variance					Standard Deviation				
	MedSE	MedAE	MedHASE	MedHAAE	MedLE	MedSE	MedAE	MedHASE	MedHAAE	MedLE
1-100	0%	0%	0%	0%	0%	1%	1%	0%	0%	0%
11-52	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
16-36	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%

Table 5.7.1

The percentage of times the ARCH models selected by the SPEC method perform "better" than the ARCH models selected by the SEVar criterion as judged by the evaluation criteria. The first and the second panel correspond to the mean and the median of the evaluation criteria, respectively. The left and the right part of the panels correspond to the volatility expressed as the variance and the standard deviation of the returns, respectively.

Days ahead forecasting horizon	Mean									
	Variance					Standard Deviation				
	MSE	MAE	MHASE	MHAAE	MLE	MSE	MAE	MHASE	MHAAE	MLE
1-100	96%	92%	100%	100%	95%	97%	94%	100%	99%	95%
11-52	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
16-36	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
Days ahead forecasting horizon	Median									
	Variance					Standard Deviation				
	MedSE	MedAE	MedHASE	MedHAAE	MedLE	MedSE	MedAE	MedHASE	MedHAAE	MedLE
1-100	84%	84%	97%	97%	92%	85%	85%	95%	95%	92%
11-52	86%	86%	100%	100%	100%	88%	88%	100%	100%	100%
16-36	90%	90%	100%	100%	100%	95%	95%	100%	100%	100%

MSE: Mean Square Error

MAE: Mean Absolute Error

MHASE: Mean Heteroscedasticity Adjusted Squared Error

MHAAE: Mean Heteroscedasticity Adjusted Absolute Error

MLE: Mean Logarithmic Error

MedSE: Median Square Error

MedAE: Median Absolute Error

MedHASE: Median Heteroscedasticity Adjusted Squared Error

MedHAAE: Median Heteroscedasticity Adjusted Absolute Error

MedLE: Median Logarithmic Error

Table 5.7.2

The percentage of times the ARCH models selected by the SPEC method perform "better" than the ARCH models selected by the AEVar criterion as judged by the evaluation criteria. The first and the second panel correspond to the mean and the median of the evaluation criteria, respectively. The left and the right part of the panels correspond to the volatility expressed as the variance and the standard deviation of the returns, respectively.

Days ahead forecasting horizon	Mean									
	Variance					Standard Deviation				
	MSE	MAE	MHASE	MHAAE	MLE	MSE	MAE	MHASE	MHAAE	MLE
1-100	97%	94%	100%	100%	95%	97%	95%	100%	100%	95%
11-52	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
16-36	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
Days ahead forecasting horizon	Median									
	Variance					Standard Deviation				
	MedSE	MedAE	MedHASE	MedHAAE	MedLE	MedSE	MedAE	MedHASE	MedHAAE	MedLE
1-100	88%	88%	99%	99%	96%	91%	91%	98%	98%	96%
11-52	95%	95%	100%	100%	100%	100%	100%	100%	100%	100%
16-36	95%	95%	100%	100%	100%	100%	100%	100%	100%	100%

Table 5.7.3

The percentage of times the ARCH models selected by the SPEC method perform "better" than the ARCH models selected by the SEDev criterion as judged by the evaluation criteria. The first and the second panel correspond to the mean and the median of the evaluation criteria, respectively. The left and the right part of the panels correspond to the volatility expressed as the variance and the standard deviation of the returns, respectively.

Days ahead forecasting horizon	Mean									
	Variance					Standard Deviation				
	MSE	MAE	MHASE	MHAAE	MLE	MSE	MAE	MHASE	MHAAE	MLE
1-100	96%	93%	100%	100%	95%	97%	95%	100%	100%	95%
11-52	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
16-36	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
Days ahead forecasting horizon	Median									
	Variance					Standard Deviation				
	MedSE	MedAE	MedHASE	MedHAAE	MedLE	MedSE	MedAE	MedHASE	MedHAAE	MedLE
1-100	88%	88%	99%	99%	96%	91%	91%	98%	98%	96%
11-52	95%	95%	100%	100%	100%	100%	100%	100%	100%	100%
16-36	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%

Table 5.7.4

The percentage of times the ARCH models selected by the SPEC method perform "better" than the ARCH models selected by the AEDev criterion as judged by the evaluation criteria. The first and the second panel correspond to the mean and the median of the evaluation criteria, respectively. The left and the right part of the panels correspond to the volatility expressed as the variance and the standard deviation of the returns, respectively.

Days ahead forecasting horizon	Mean									
	Variance					Standard Deviation				
	MSE	MAE	MHASE	MHAAE	MLE	MSE	MAE	MHASE	MHAAE	MLE
1-100	97%	93%	100%	100%	95%	97%	95%	100%	100%	95%
11-52	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
16-36	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
Days ahead forecasting horizon	**Median**									
	Variance					Standard Deviation				
	MedSE	MedAE	MedHASE	MedHAAE	MedLE	MedSE	MedAE	MedHASE	MedHAAE	MedLE
1-100	87%	87%	98%	98%	96%	91%	91%	98%	98%	96%
11-52	93%	93%	100%	100%	100%	100%	100%	100%	100%	100%
16-36	95%	95%	100%	100%	100%	100%	100%	100%	100%	100%

MSE: Mean Square Error

MAE: Mean Absolute Error

MHASE: Mean Heteroscedasticity Adjusted Squared Error

MHAAE: Mean Heteroscedasticity Adjusted Absolute Error

MLE: Mean Logarithmic Error

MedSE: Median Square Error

MedAE: Median Absolute Error

MedHASE: Median Heteroscedasticity Adjusted Squared Error

MedHAAE: Median Heteroscedasticity Adjusted Absolute Error

MedLE: Median Logarithmic Error

Table 5.7.5

The percentage of times the ARCH models selected by the SPEC method perform "better" than the ARCH models selected by the HASEVar criterion as judged by the evaluation criteria. The first and the second panel correspond to the mean and the median of the evaluation criteria, respectively. The left and the right part of the panels correspond to the volatility expressed as the variance and the standard deviation of the returns, respectively.

Days ahead forecasting horizon	Mean									
	Variance					Standard Deviation				
	MSE	MAE	MHASE	MHAAE	MLE	MSE	MAE	MHASE	MHAAE	MLE
1-100	93%	89%	100%	100%	94%	94%	91%	99%	98%	94%
11-52	100%	98%	100%	100%	100%	100%	100%	100%	100%	100%
16-36	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
Days ahead forecasting horizon	**Median**									
	Variance					Standard Deviation				
	MedSE	MedAE	MedHASE	MedHAAE	MedLE	MedSE	MedAE	MedHASE	MedHAAE	MedLE
1-100	36%	36%	92%	92%	84%	48%	48%	90%	90%	84%
11-52	33%	33%	90%	90%	81%	55%	55%	90%	90%	81%
16-36	38%	38%	90%	90%	81%	52%	52%	90%	90%	81%

Table 5.7.6

The percentage of times the ARCH models selected by the SPEC method perform "better" than the ARCH models selected by the HAAEVar criterion as judged by the evaluation criteria. The first and the second panel correspond to the mean and the median of the evaluation criteria, respectively. The left and the right part of the panels correspond to the volatility expressed as the variance and the standard deviation of the returns, respectively.

Days ahead forecasting horizon	Mean									
	Variance					Standard Deviation				
	MSE	MAE	MHASE	MHAAE	MLE	MSE	MAE	MHASE	MHAAE	MLE
1-100	0%	60%	99%	99%	93%	94%	89%	98%	96%	93%
11-52	0%	95%	100%	100%	100%	100%	98%	100%	100%	100%
16-36	0%	100%	100%	100%	100%	100%	100%	100%	100%	100%
Days ahead forecasting horizon	**Median**									
	Variance					Standard Deviation				
	MedSE	MedAE	MedHASE	MedHAAE	MedLE	MedSE	MedAE	MedHASE	MedHAAE	MedLE
1-100	36%	36%	92%	92%	83%	59%	57%	88%	88%	83%
11-52	26%	26%	93%	93%	79%	52%	52%	88%	88%	79%
16-36	19%	19%	90%	90%	76%	43%	43%	86%	86%	76%

Table 5.7.7

The percentage of times the ARCH models selected by the SPEC method perform "better" than the ARCH models selected by the HASEDev criterion as judged by the evaluation criteria. The first and the second panel correspond to the mean and the median of the evaluation criteria, respectively. The left and the right part of the panels correspond to the volatility expressed as the variance and the standard deviation of the returns, respectively.

Days ahead forecasting horizon	Mean									
	Variance					Standard Deviation				
	MSE	MAE	MHASE	MHAAE	MLE	MSE	MAE	MHASE	MHAAE	MLE
1-100	82%	88%	100%	100%	93%	94%	91%	99%	98%	93%
11-52	100%	95%	100%	100%	100%	100%	100%	100%	100%	100%
16-36	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
Days ahead forecasting horizon	Median									
	Variance					Standard Deviation				
	MedSE	MedAE	MedHASE	MedHAAE	MedLE	MedSE	MedAE	MedHASE	MedHAAE	MedLE
1-100	45%	45%	88%	88%	88%	60%	60%	87%	87%	88%
11-52	45%	45%	93%	93%	93%	60%	60%	93%	93%	93%
16-36	33%	33%	95%	95%	100%	43%	43%	100%	100%	100%

MSE: Mean Square Error

MAE: Mean Absolute Error

MHASE: Mean Heteroscedasticity Adjusted Squared Error

MHAAE: Mean Heteroscedasticity Adjusted Absolute Error

MLE: Mean Logarithmic Error

MedSE: Median Square Error

MedAE: Median Absolute Error

MedHASE: Median Heteroscedasticity Adjusted Squared Error

MedHAAE: Median Heteroscedasticity Adjusted Absolute Error

MedLE: Median Logarithmic Error

Table 5.7.8

The percentage of times the ARCH models selected by the SPEC method perform "better" than the ARCH models selected by the HAAEDev criterion as judged by the evaluation criteria. The first and the second panel correspond to the mean and the median of the evaluation criteria, respectively. The left and the right part of the panels correspond to the volatility expressed as the variance and the standard deviation of the returns, respectively.

Days ahead forecasting horizon	Mean									
	Variance					Standard Deviation				
	MSE	MAE	MHASE	MHAAE	MLE	MSE	MAE	MHASE	MHAAE	MLE
1-100	30%	86%	100%	99%	95%	96%	92%	99%	98%	95%
11-52	71%	90%	100%	100%	100%	100%	100%	100%	100%	100%
16-36	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
Days ahead forecasting horizon	Median									
	Variance					Standard Deviation				
	MedSE	MedAE	MedHASE	MedHAAE	MedLE	MedSE	MedAE	MedHASE	MedHAAE	MedLE
1-100	71%	71%	94%	94%	92%	80%	81%	93%	93%	92%
11-52	86%	86%	100%	100%	95%	86%	86%	100%	100%	95%
16-36	90%	90%	100%	100%	100%	95%	95%	100%	100%	100%

Table 5.7.9

The percentage of times the ARCH models selected by the SPEC method perform "better" than the ARCH models selected by the LEVar criterion as judged by the evaluation criteria. The first and the second panel correspond to the mean and the median of the evaluation criteria, respectively. The left and the right part of the panels correspond to the volatility expressed as the variance and the standard deviation of the returns, respectively.

Days ahead forecasting horizon	Mean									
	Variance					Standard Deviation				
	MSE	MAE	MHASE	MHAAE	MLE	MSE	MAE	MHASE	MHAAE	MLE
1-100	97%	94%	100%	100%	96%	97%	95%	100%	100%	96%
11-52	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
16-36	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
Days ahead forecasting horizon	Median									
	Variance					Standard Deviation				
	MedSE	MedAE	MedHASE	MedHAAE	MedLE	MedSE	MedAE	MedHASE	MedHAAE	MedLE
1-100	90%	90%	100%	100%	97%	92%	92%	99%	99%	97%
11-52	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
16-36	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%

Table 5.7.10

The percentage of times the ARCH models selected by the SPEC method perform "better" than the ARCH models selected by the AIC criterion as judged by the evaluation criteria. The first and the second panel correspond to the mean and the median of the evaluation criteria, respectively. The left and the right part of the panels correspond to the volatility expressed as the variance and the standard deviation of the returns, respectively.

Days ahead forecasting horizon	Mean									
	Variance					Standard Deviation				
	MSE	MAE	MHASE	MHAAE	MLE	MSE	MAE	MHASE	MHAAE	MLE
1-100	99%	97%	100%	100%	96%	100%	96%	100%	100%	96%
11-52	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
16-36	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
Days ahead forecasting horizon	Median									
	Variance					Standard Deviation				
	MedSE	MedAE	MedHASE	MedHAAE	MedLE	MedSE	MedAE	MedHASE	MedHAAE	MedLE
1-100	89%	89%	100%	100%	100%	93%	93%	99%	99%	100%
11-52	95%	95%	100%	100%	100%	100%	100%	100%	100%	100%
16-36	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%

Table 5.7.11

The percentage of times the ARCH models selected by the SPEC method perform "better" than the ARCH models selected by the SBC criterion as judged by the evaluation criteria. The first and the second panel correspond to the mean and the median of the evaluation criteria, respectively. The left and the right part of the panels correspond to the volatility expressed as the variance and the standard deviation of the returns, respectively.

Days ahead forecasting horizon	Mean									
	Variance					Standard Deviation				
	MSE	MAE	MHASE	MHAAE	MLE	MSE	MAE	MHASE	MHAAE	MLE
1-100	98%	97%	100%	100%	96%	99%	96%	100%	100%	96%
11-52	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
16-36	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
Days ahead forecasting horizon	Median									
	Variance					Standard Deviation				
	MedSE	MedAE	MedHASE	MedHAAE	MedLE	MedSE	MedAE	MedHASE	MedHAAE	MedLE
1-100	91%	91%	99%	99%	97%	92%	92%	99%	99%	97%
11-52	98%	98%	100%	100%	100%	100%	100%	100%	100%	100%
16-36	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%

Appendix 6

- ***Table 6.3. Daily rate of return from trading straddles on the S&P500 index based on the 85 ARCH volatility models.***
- ***Table 6.8. Daily rate of return from trading straddles on the S&P500 index based on the ARCH models selected by the SEVar model selection method.***
- ***Table 6.9. Daily rate of return from trading straddles on the S&P500 index based on the ARCH models selected by the AEVar model selection method.***
- ***Table 6.10. Daily rate of return from trading straddles on the S&P500 index based on the ARCH models selected by the SEDev model selection method.***
- ***Table 6.11. Daily rate of return from trading straddles on the S&P500 index based on the ARCH models selected by the AEDev model selection method.***
- ***Table 6.12. Daily rate of return from trading straddles on the S&P500 index based on the ARCH models selected by the HASEVar model selection method.***
- ***Table 6.13. Daily rate of return from trading straddles on the S&P500 index based on the ARCH models selected by the HAAEVar model selection method.***
- ***Table 6.14. Daily rate of return from trading straddles on the S&P500 index based on the ARCH models selected by the HASEDev model selection method.***
- ***Table 6.15. Daily rate of return from trading straddles on the S&P500 index based on the ARCH models selected by the HAAEDev model selection method.***
- ***Table 6.16. Daily rate of return from trading straddles on the S&P500 index based on the ARCH models selected by the LEVar model selection method.***
- ***Table 6.17. Daily rate of return from trading straddles on the S&P500 index based on the ARCH models selected by the AIC and SBC model selection methods.***

- ***Construction of the Black and Scholes Option Pricing Formula.***
- ***Options Sensitivities.***

Table 6.3.a. Daily rate of return from trading straddles on the S&P500 index based on the 85 ARCH volatility forecasts (11 March 1998 – 2 June 2000).

ARCH Model	Without transaction cost				$2 transaction cost											
					Without filter				$1.25 filter				$1.75 filter			
	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days
AR(0)GARCH(0,1)	3.45%	17.96%	4.10	456	-0.21%	17.60%	-0.25	456	0.43%	17.94%	0.48	409	0.59%	18.43%	0.63	382
AR(1)GARCH(0,1)	3.45%	17.96%	4.10	456	-0.20%	17.60%	-0.25	456	0.47%	17.96%	0.53	408	0.58%	18.36%	0.62	385
AR(2)GARCH(0,1)	3.45%	17.96%	4.10	456	-0.20%	17.60%	-0.25	456	0.45%	17.92%	0.51	410	0.59%	18.34%	0.63	386
AR(3)GARCH(0,1)	3.42%	17.97%	4.06	456	-0.24%	17.60%	-0.29	456	0.47%	17.94%	0.53	409	0.57%	18.29%	0.62	388
AR(4)GARCH(0,1)	3.42%	17.97%	4.06	456	-0.24%	17.60%	-0.29	456	0.47%	17.94%	0.53	409	0.56%	18.27%	0.61	389
AR(0)GARCH(0,2)	3.75%	17.90%	4.48	456	0.10%	17.46%	0.12	456	0.82%	17.19%	0.96	402	0.85%	17.46%	0.96	383
AR(1)GARCH(0,2)	3.63%	17.93%	4.32	456	-0.02%	17.50%	-0.03	456	0.48%	18.10%	0.53	404	0.70%	18.49%	0.74	381
AR(2)GARCH(0,2)	3.64%	17.93%	4.34	456	-0.01%	17.50%	-0.01	456	0.47%	18.12%	0.52	403	0.66%	18.51%	0.70	380
AR(3)GARCH(0,2)	3.64%	17.92%	4.34	456	-0.01%	17.50%	-0.01	456	0.48%	18.20%	0.53	397	0.65%	18.52%	0.69	380
AR(4)GARCH(0,2)	3.81%	17.89%	4.55	456	0.16%	17.45%	0.19	456	0.55%	18.23%	0.60	393	0.67%	18.68%	0.69	373
AR(0)GARCH(1,1)	3.73%	17.91%	4.44	456	0.07%	17.52%	0.09	456	0.58%	18.21%	0.65	405	0.74%	18.62%	0.78	383
AR(1)GARCH(1,1)	3.89%	17.87%	4.65	456	0.24%	17.46%	0.29	456	0.56%	18.29%	0.61	403	0.58%	18.65%	0.61	386
AR(2)GARCH(1,1)	3.85%	17.88%	4.60	456	0.20%	17.47%	0.25	456	0.68%	18.11%	0.76	402	0.81%	18.47%	0.86	384
AR(3)GARCH(1,1)	3.91%	17.87%	4.67	456	0.25%	17.46%	0.31	456	0.65%	18.18%	0.72	405	0.75%	18.59%	0.79	384
AR(4)GARCH(1,1)	4.02%	17.84%	4.81	456	0.37%	17.44%	0.45	456	0.50%	18.24%	0.55	405	0.60%	18.58%	0.64	383
AR(0)GARCH(1,2)	4.09%	17.83%	4.90	456	0.44%	17.41%	0.54	456	0.70%	18.13%	0.78	407	1.03%	18.45%	1.09	386
AR(1)GARCH(1,2)	4.16%	17.81%	4.99	456	0.51%	17.40%	0.62	456	0.74%	18.21%	0.81	405	0.95%	18.55%	1.01	386
AR(2)GARCH(1,2)	4.21%	17.80%	5.05	456	0.56%	17.39%	0.68	456	0.67%	18.04%	0.76	412	0.92%	18.52%	0.98	387
AR(3)GARCH(1,2)	4.09%	17.83%	4.90	456	0.44%	17.42%	0.54	456	0.71%	18.06%	0.80	410	0.93%	18.43%	1.00	391
AR(4)GARCH(1,2)	4.20%	17.80%	5.04	456	0.55%	17.39%	0.68	456	0.75%	18.23%	0.83	401	0.86%	18.48%	0.92	389
AR(0)GARCH(2,1)	3.81%	17.89%	4.54	456	0.15%	17.48%	0.19	456	0.54%	18.22%	0.60	406	0.61%	18.39%	0.65	386
AR(1)GARCH(2,1)	3.95%	17.86%	4.72	456	0.30%	17.42%	0.36	456	0.72%	18.01%	0.81	404	0.88%	18.49%	0.93	380
AR(2)GARCH(2,1)	3.86%	17.88%	4.61	456	0.21%	17.47%	0.25	456	0.68%	18.13%	0.75	400	0.78%	18.45%	0.83	380
AR(3)GARCH(2,1)	3.88%	17.88%	4.63	456	0.22%	17.47%	0.27	456	0.66%	17.87%	0.75	412	0.77%	18.25%	0.83	392
AR(4)GARCH(2,1)	3.98%	17.85%	4.76	456	0.33%	17.44%	0.40	456	0.57%	18.05%	0.63	405	0.62%	18.13%	0.68	391
AR(0)GARCH(2,2)	4.04%	17.84%	4.84	456	0.39%	17.43%	0.48	456	0.71%	18.13%	0.79	407	**1.06%**	**18.46%**	**1.13**	**385**
AR(1)GARCH(2,2)	4.10%	17.83%	4.91	456	0.45%	17.41%	0.55	456	0.69%	18.05%	0.77	412	0.61%	18.20%	0.67	392
AR(2)GARCH(2,2)	4.22%	17.80%	5.06	456	0.57%	17.39%	0.70	456	0.69%	18.03%	0.78	413	0.77%	18.45%	0.82	387
AR(3)GARCH(2,2)	4.18%	17.81%	5.01	456	0.52%	17.40%	0.64	456	0.62%	17.90%	0.70	414	0.67%	18.15%	0.73	393
AR(4)GARCH(2,2)	4.21%	17.80%	5.05	456	0.56%	17.39%	0.68	456	0.69%	18.07%	0.77	405	0.64%	18.21%	0.70	390
AR(0)TARCH(0,1)	3.43%	17.97%	4.08	456	-0.22%	17.61%	-0.27	456	0.38%	18.18%	0.41	397	0.43%	18.49%	0.45	381
AR(1)TARCH(0,1)	3.56%	17.94%	4.24	456	-0.09%	17.58%	-0.11	456	0.43%	18.19%	0.47	399	0.57%	18.68%	0.59	372
AR(2)TARCH(0,1)	3.69%	17.92%	4.40	456	0.04%	17.53%	0.05	456	0.38%	18.20%	0.42	399	0.70%	18.80%	0.72	370
AR(3)TARCH(0,1)	3.47%	17.96%	4.13	456	-0.18%	17.60%	-0.22	456	0.47%	18.25%	0.51	396	0.61%	18.69%	0.63	375
AR(4)TARCH(0,1)	3.27%	18.00%	3.88	456	-0.38%	17.65%	-0.46	456	0.45%	18.30%	0.49	394	0.61%	18.69%	0.63	375
AR(0)TARCH(0,2)	3.41%	17.97%	4.05	455	-0.25%	17.55%	-0.30	455	0.42%	17.75%	0.48	412	0.54%	18.15%	0.60	393
AR(1)TARCH(0,2)	3.54%	17.95%	4.20	455	-0.12%	17.54%	-0.15	455	0.49%	17.96%	0.55	403	0.53%	18.18%	0.58	390
AR(2)TARCH(0,2)	3.53%	17.95%	4.20	455	-0.12%	17.54%	-0.15	455	0.49%	17.93%	0.55	404	0.73%	17.96%	0.80	387
AR(3)TARCH(0,2)	3.54%	17.95%	4.21	455	-0.11%	17.53%	-0.14	455	0.51%	18.04%	0.57	402	0.83%	18.15%	0.89	379
AR(4)TARCH(0,2)	3.51%	17.95%	4.17	455	-0.15%	17.54%	-0.18	455	0.59%	18.10%	0.66	398	0.81%	18.23%	0.86	376
AR(0)TARCH(1,1)	4.13%	17.84%	4.94	455	0.48%	17.28%	0.59	455	0.41%	17.60%	0.47	423	0.30%	17.68%	0.34	408
AR(1)TARCH(1,1)	3.83%	17.91%	4.57	455	0.18%	17.38%	0.22	455	0.64%	17.67%	0.74	417	0.79%	17.95%	0.88	400
AR(2)TARCH(1,1)	4.19%	17.82%	5.02	455	0.54%	17.27%	0.67	455	0.70%	17.75%	0.80	413	0.79%	18.01%	0.87	399

Table 6.3.b. Daily rate of return from trading straddles on the S&P500 index based on the 85 ARCH volatility forecasts (11 March 1998 – 2 June 2000).

ARCH Model	Without transaction cost				$2 transaction cost											
					Without filter				$1.25 filter				$1.75 filter			
	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days
AR(3)TARCH(1,1)	4.26%	17.79%	5.11	456	0.61%	17.24%	0.75	456	0.61%	17.73%	0.70	415	0.74%	17.97%	0.83	402
AR(4)TARCH(1,1)	4.12%	17.82%	4.94	456	0.47%	17.27%	0.58	456	0.61%	17.71%	0.70	417	0.74%	17.90%	0.83	405
AR(0)TARCH(1,2)	4.39%	17.78%	5.27	455	0.74%	17.24%	0.91	455	0.67%	17.56%	0.79	424	0.56%	17.57%	0.65	409
AR(1)TARCH(1,2)	3.64%	17.96%	4.31	454	-0.02%	17.54%	-0.02	454	0.63%	17.52%	0.73	421	0.81%	17.74%	0.92	407
AR(2)TARCH(1,2)	3.43%	18.00%	4.06	454	-0.22%	17.59%	-0.27	454	0.52%	17.46%	0.61	417	0.69%	17.60%	0.79	406
AR(3)TARCH(1,2)	3.77%	17.92%	4.49	455	0.11%	17.39%	0.14	455	0.39%	17.66%	0.45	419	0.59%	17.84%	0.66	405
AR(4)TARCH(1,2)	3.76%	17.94%	4.47	454	0.11%	17.41%	0.13	454	0.45%	17.59%	0.52	418	0.50%	17.84%	0.57	405
AR(0)TARCH(2,1)	3.69%	17.93%	4.39	455	0.04%	17.40%	0.05	455	0.54%	17.45%	0.64	420	0.46%	17.49%	0.52	403
AR(1)TARCH(2,1)	3.65%	17.94%	4.34	455	0.00%	17.37%	0.00	455	0.11%	17.70%	0.13	418	0.40%	17.88%	0.44	403
AR(2)TARCH(2,1)	3.91%	17.89%	4.66	455	0.25%	17.35%	0.31	455	0.45%	17.83%	0.51	412	0.58%	17.98%	0.64	400
AR(3)TARCH(2,1)	3.78%	17.90%	4.51	456	0.13%	17.36%	0.16	456	0.28%	17.68%	0.32	415	0.40%	17.90%	0.45	402
AR(4)TARCH(2,1)	3.80%	17.90%	4.53	455	0.14%	17.36%	0.18	455	0.19%	17.58%	0.21	412	0.32%	17.67%	0.37	399
AR(0)TARCH(2,2)	3.43%	18.00%	4.06	454	-0.22%	17.48%	-0.27	454	0.11%	17.75%	0.13	422	0.23%	17.69%	0.27	405
AR(1)TARCH(2,2)	3.42%	17.99%	4.06	455	-0.23%	17.57%	-0.28	455	0.45%	17.71%	0.52	416	0.58%	17.84%	0.66	405
AR(2)TARCH(2,2)	3.67%	17.94%	4.36	455	0.02%	17.52%	0.02	455	0.58%	17.62%	0.68	418	0.68%	17.80%	0.78	408
AR(3)TARCH(2,2)	3.41%	18.01%	4.03	454	-0.25%	17.59%	-0.30	454	0.48%	17.52%	0.56	424	0.55%	17.68%	0.63	414
AR(4)TARCH(2,2)	3.45%	18.00%	4.08	454	-0.21%	17.59%	-0.26	454	0.42%	17.69%	0.48	416	0.49%	17.77%	0.55	410
AR(0)EGARCH(0,1)	3.38%	17.98%	4.01	456	-0.27%	17.61%	-0.33	456	0.23%	17.86%	0.27	416	0.19%	17.80%	0.21	393
AR(1)EGARCH(0,1)	3.37%	17.98%	4.00	456	-0.28%	17.62%	-0.34	456	0.38%	18.00%	0.43	408	0.51%	18.14%	0.56	394
AR(2)EGARCH(0,1)	3.35%	17.98%	3.98	456	-0.30%	17.62%	-0.37	456	0.38%	18.00%	0.43	408	0.37%	17.93%	0.41	392
AR(3)EGARCH(0,1)	3.37%	17.98%	4.00	456	-0.28%	17.62%	-0.34	456	0.38%	17.96%	0.43	410	0.38%	17.96%	0.41	391
AR(4)EGARCH(0,1)	3.37%	17.98%	4.00	456	-0.28%	17.62%	-0.34	456	0.38%	17.96%	0.43	410	0.38%	17.96%	0.41	391
AR(0)EGARCH(0,2)	2.53%	18.12%	2.98	456	-1.12%	17.79%	-1.35	456	-0.46%	17.34%	-0.55	431	-0.37%	17.21%	-0.44	420
AR(1)EGARCH(0,2)	2.90%	18.06%	3.43	456	-0.75%	17.71%	-0.91	456	-0.44%	17.60%	-0.52	422	-0.06%	17.52%	-0.07	408
AR(2)EGARCH(0,2)	2.89%	18.06%	3.41	456	-0.77%	17.72%	-0.92	456	-0.44%	17.58%	-0.52	423	-0.08%	17.52%	-0.09	408
AR(3)EGARCH(0,2)	2.86%	18.07%	3.38	456	-0.79%	17.72%	-0.95	456	-0.45%	17.54%	-0.53	425	-0.16%	17.46%	-0.19	412
AR(4)EGARCH(0,2)	2.86%	18.07%	3.38	456	-0.79%	17.72%	-0.95	456	-0.47%	17.52%	-0.55	426	-0.19%	17.45%	-0.22	413
AR(0)EGARCH(1,1)	4.14%	17.82%	4.96	456	0.49%	17.28%	0.60	456	0.32%	17.30%	0.39	427	0.50%	17.33%	0.58	415
AR(1)EGARCH(1,1)	4.33%	17.77%	5.20	456	0.67%	17.24%	0.84	456	0.81%	17.52%	0.95	423	0.84%	17.39%	0.97	406
AR(2)EGARCH(1,1)	4.40%	17.75%	5.29	456	0.75%	17.22%	0.93	456	**0.90%**	**17.34%**	**1.06**	**421**	0.85%	17.35%	0.99	407
AR(3)EGARCH(1,1)	**4.42%**	**17.75%**	**5.32**	**456**	**0.77%**	**17.21%**	**0.95**	**456**	0.83%	17.56%	0.97	419	0.84%	17.25%	0.98	406
AR(4)EGARCH(1,1)	4.39%	17.76%	5.28	456	0.74%	17.23%	0.92	456	0.68%	17.43%	0.80	425	0.83%	17.19%	0.97	407
AR(0)EGARCH(1,2)	3.16%	18.02%	3.74	456	-0.49%	17.51%	-0.60	456	-0.14%	17.75%	-0.17	426	-0.14%	17.42%	-0.17	414
AR(1)EGARCH(1,2)	3.63%	17.93%	4.33	456	-0.02%	17.41%	-0.02	456	0.19%	17.82%	0.22	416	0.25%	18.10%	0.28	402
AR(2)EGARCH(1,2)	3.27%	18.00%	3.88	456	-0.38%	17.58%	-0.46	456	0.11%	17.85%	0.12	417	0.19%	18.11%	0.20	401
AR(3)EGARCH(1,2)	3.41%	17.97%	4.06	456	-0.24%	17.45%	-0.29	456	0.04%	17.93%	0.04	415	0.04%	18.19%	0.04	401
AR(4)EGARCH(1,2)	3.53%	17.95%	4.20	456	-0.12%	17.43%	-0.15	456	0.16%	17.91%	0.18	412	0.26%	18.18%	0.28	396
AR(0)EGARCH(2,1)	4.07%	17.83%	4.87	456	0.42%	17.30%	0.52	456	0.29%	17.32%	0.34	427	0.43%	17.34%	0.51	416
AR(1)EGARCH(2,1)	4.31%	17.78%	5.18	456	0.66%	17.25%	0.81	456	0.76%	17.63%	0.88	418	0.86%	17.46%	0.98	403
AR(2)EGARCH(2,1)	4.40%	17.76%	5.29	456	0.74%	17.22%	0.92	456	0.80%	17.29%	0.95	427	0.81%	17.39%	0.94	407
AR(3)EGARCH(2,1)	4.32%	17.77%	5.19	456	0.67%	17.24%	0.83	456	0.79%	17.77%	0.91	420	0.82%	17.33%	0.95	402
AR(4)EGARCH(2,1)	4.33%	17.77%	5.21	456	0.68%	17.24%	0.84	456	0.78%	17.48%	0.92	427	0.89%	17.40%	1.03	407

Table 6.3.c. Daily rate of return from trading straddles on the S&P500 index based on the 85 ARCH volatility forecasts (11 March 1998 – 2 June 2000).

ARCH Model	$2 transaction cost															
	$2.00 filter				$2.25 filter				$2.75 filter				$3.50 filter			
	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days
AR(0)GARCH(0,1)	0.61%	18.60%	0.63	374	0.95%	18.55%	0.98	363	1.29%	18.51%	1.30	348	1.36%	18.88%	1.31	328
AR(1)GARCH(0,1)	0.61%	18.59%	0.63	374	0.88%	18.53%	0.90	363	1.07%	18.30%	1.09	347	1.19%	18.64%	1.15	327
AR(2)GARCH(0,1)	0.57%	18.58%	0.59	374	0.91%	18.54%	0.93	362	1.07%	18.30%	1.09	347	1.19%	18.64%	1.15	327
AR(3)GARCH(0,1)	0.60%	18.56%	0.63	376	0.82%	18.53%	0.85	364	1.24%	18.54%	1.25	348	1.15%	18.59%	1.12	329
AR(4)GARCH(0,1)	0.55%	18.55%	0.57	375	0.82%	18.53%	0.85	364	1.24%	18.54%	1.25	348	1.15%	18.59%	1.12	329
AR(0)GARCH(0,2)	1.04%	17.36%	1.16	374	1.14%	17.60%	1.23	362	1.44%	17.50%	1.52	344	1.48%	17.62%	1.51	326
AR(1)GARCH(0,2)	0.92%	18.44%	0.96	367	1.04%	18.27%	1.08	359	1.26%	18.56%	1.26	346	1.51%	17.66%	1.54	324
AR(2)GARCH(0,2)	0.89%	18.44%	0.92	367	1.03%	18.22%	1.07	361	1.17%	18.46%	1.18	350	1.77%	18.01%	1.76	321
AR(3)GARCH(0,2)	0.90%	18.53%	0.93	365	1.06%	18.31%	1.10	359	1.21%	18.55%	1.22	347	**1.89%**	**18.11%**	**1.87**	**322**
AR(4)GARCH(0,2)	0.81%	18.54%	0.83	364	1.09%	18.32%	1.12	358	**1.60%**	**17.60%**	**1.69**	**346**	1.88%	18.12%	1.86	321
AR(0)GARCH(1,1)	0.99%	18.60%	1.03	370	1.06%	18.74%	1.08	363	1.19%	19.13%	1.15	341	1.75%	19.40%	1.61	317
AR(1)GARCH(1,1)	0.64%	18.82%	0.66	377	0.98%	19.04%	0.98	362	1.25%	19.17%	1.21	344	1.44%	19.16%	1.33	313
AR(2)GARCH(1,1)	0.82%	18.55%	0.86	380	1.08%	18.82%	1.10	364	1.16%	18.61%	1.16	350	1.44%	18.97%	1.36	321
AR(3)GARCH(1,1)	0.79%	18.71%	0.82	379	0.97%	18.56%	1.01	368	0.91%	18.80%	0.91	353	1.64%	18.72%	1.57	320
AR(4)GARCH(1,1)	0.69%	18.64%	0.72	378	0.77%	18.73%	0.79	372	0.96%	18.66%	0.97	357	1.25%	18.65%	1.22	328
AR(0)GARCH(1,2)	**1.10%**	**18.53%**	**1.16**	**381**	**1.35%**	**18.50%**	**1.39**	**362**	1.23%	18.61%	1.23	345	1.39%	19.24%	1.28	317
AR(1)GARCH(1,2)	0.95%	18.75%	0.98	376	0.85%	18.74%	0.87	365	0.99%	19.21%	0.96	344	1.60%	19.20%	1.50	322
AR(2)GARCH(1,2)	0.96%	18.71%	1.00	378	0.91%	18.76%	0.93	369	0.89%	18.92%	0.89	356	1.46%	18.91%	1.41	334
AR(3)GARCH(1,2)	0.72%	18.42%	0.76	380	0.75%	18.74%	0.76	365	0.91%	19.01%	0.90	353	1.42%	18.97%	1.36	332
AR(4)GARCH(1,2)	0.82%	18.61%	0.86	379	0.69%	18.68%	0.71	368	1.08%	19.07%	1.06	348	1.28%	19.18%	1.20	325
AR(0)GARCH(2,1)	0.98%	18.08%	1.05	375	0.93%	18.16%	0.98	367	1.24%	18.68%	1.23	343	1.46%	19.15%	1.37	321
AR(1)GARCH(2,1)	0.95%	18.74%	0.98	368	1.00%	18.86%	1.01	363	1.07%	19.06%	1.04	341	1.49%	19.13%	1.38	315
AR(2)GARCH(2,1)	0.80%	18.65%	0.83	370	0.86%	18.77%	0.87	365	0.82%	18.78%	0.82	347	1.28%	19.12%	1.19	317
AR(3)GARCH(2,1)	0.87%	18.36%	0.93	384	0.83%	18.47%	0.87	375	0.89%	18.86%	0.88	350	1.58%	18.71%	1.51	321
AR(4)GARCH(2,1)	0.69%	18.34%	0.74	381	0.80%	18.53%	0.83	372	0.85%	18.85%	0.84	351	1.26%	18.94%	1.20	325
AR(0)GARCH(2,2)	1.04%	18.55%	1.09	376	1.03%	18.24%	1.08	363	1.25%	18.60%	1.24	345	1.28%	18.98%	1.21	324
AR(1)GARCH(2,2)	0.73%	18.40%	0.77	379	0.87%	18.69%	0.89	366	1.00%	18.98%	0.98	346	1.31%	19.17%	1.23	324
AR(2)GARCH(2,2)	0.74%	18.54%	0.77	375	0.83%	18.70%	0.85	367	1.04%	18.94%	1.03	352	1.37%	18.99%	1.30	328
AR(3)GARCH(2,2)	0.75%	18.41%	0.80	380	0.78%	18.68%	0.81	369	0.98%	18.97%	0.97	353	1.37%	18.99%	1.30	328
AR(4)GARCH(2,2)	0.63%	18.29%	0.68	386	0.73%	18.59%	0.76	373	1.21%	18.65%	1.22	353	1.41%	19.15%	1.33	324
AR(0)TARCH(0,1)	0.75%	18.41%	0.78	371	0.91%	18.64%	0.92	360	1.05%	19.07%	1.02	342	1.53%	19.16%	1.42	317
AR(1)TARCH(0,1)	0.59%	18.75%	0.60	369	0.83%	18.75%	0.84	357	1.21%	18.85%	1.18	338	1.76%	19.46%	1.60	311
AR(2)TARCH(0,1)	0.70%	18.90%	0.71	366	0.69%	19.00%	0.68	358	1.27%	18.89%	1.23	336	1.81%	19.44%	1.64	311
AR(3)TARCH(0,1)	0.72%	18.90%	0.73	366	0.94%	18.78%	0.95	357	1.22%	18.73%	1.21	342	1.79%	19.39%	1.64	313
AR(4)TARCH(0,1)	0.86%	18.52%	0.89	368	0.96%	18.76%	0.97	358	1.45%	18.99%	1.40	334	1.76%	19.43%	1.60	312
AR(0)TARCH(0,2)	0.54%	18.18%	0.58	385	0.59%	18.36%	0.62	377	0.69%	18.38%	0.71	357	1.01%	18.61%	0.98	330
AR(1)TARCH(0,2)	0.78%	18.23%	0.83	375	0.88%	18.48%	0.91	364	1.16%	18.47%	1.17	344	1.25%	18.64%	1.21	325
AR(2)TARCH(0,2)	0.81%	18.34%	0.85	370	0.82%	18.46%	0.85	365	0.95%	18.75%	0.94	347	1.20%	18.65%	1.16	325
AR(3)TARCH(0,2)	0.88%	18.36%	0.92	370	0.93%	18.50%	0.96	364	1.02%	18.84%	1.01	348	1.29%	18.74%	1.23	321
AR(4)TARCH(0,2)	0.89%	18.41%	0.93	368	0.89%	18.50%	0.92	364	1.02%	18.85%	1.01	349	1.54%	19.08%	1.46	324
AR(0)TARCH(1,1)	0.39%	17.87%	0.44	397	0.62%	17.72%	0.69	392	0.62%	17.62%	0.69	383	0.60%	17.49%	0.65	369
AR(1)TARCH(1,1)	0.85%	18.04%	0.94	395	0.86%	18.14%	0.94	390	0.84%	18.03%	0.90	377	1.23%	18.29%	1.26	351
AR(2)TARCH(1,1)	0.87%	18.11%	0.95	392	0.98%	18.11%	1.06	389	0.96%	18.10%	1.03	379	1.26%	18.32%	1.30	356

Table 6.3.d. Daily rate of return from trading straddles on the S&P500 index based on the 85 ARCH volatility forecasts (11 March 1998 – 2 June 2000).

ARCH Model	$2 transaction cost															
	$2.00 filter				$2.25 filter				$2.75 filter				$3.50 filter			
	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days
AR(3)TARCH(1,1)	0.77%	18.07%	0.85	397	0.83%	18.21%	0.90	388	0.98%	18.13%	1.05	378	1.12%	18.30%	1.15	357
AR(4)TARCH(1,1)	0.75%	18.00%	0.83	400	0.88%	18.16%	0.95	391	1.00%	18.19%	1.06	375	1.05%	18.41%	1.07	351
AR(0)TARCH(1,2)	0.67%	17.63%	0.76	404	0.74%	17.76%	0.83	396	0.73%	17.52%	0.82	386	0.49%	17.19%	0.55	370
AR(1)TARCH(1,2)	0.71%	17.70%	0.81	401	0.75%	17.81%	0.84	395	0.83%	17.84%	0.91	383	1.05%	18.27%	1.09	357
AR(2)TARCH(1,2)	0.69%	17.67%	0.78	403	0.73%	17.77%	0.82	398	0.84%	17.81%	0.93	384	1.07%	18.34%	1.10	354
AR(3)TARCH(1,2)	0.56%	17.92%	0.63	401	0.61%	18.02%	0.67	396	0.92%	18.05%	1.00	383	0.93%	18.36%	0.96	360
AR(4)TARCH(1,2)	0.74%	17.81%	0.82	394	0.76%	17.83%	0.85	393	0.88%	17.94%	0.96	386	1.06%	18.33%	1.09	358
AR(0)TARCH(2,1)	0.44%	17.59%	0.49	397	0.50%	17.68%	0.55	391	0.48%	17.30%	0.54	380	0.32%	17.18%	0.35	367
AR(1)TARCH(2,1)	0.42%	18.04%	0.46	395	0.33%	18.01%	0.36	391	0.26%	18.12%	0.28	378	0.48%	18.52%	0.49	360
AR(2)TARCH(2,1)	0.68%	18.10%	0.75	393	0.73%	18.18%	0.79	389	0.64%	18.29%	0.67	375	0.84%	18.58%	0.85	354
AR(3)TARCH(2,1)	0.52%	18.02%	0.57	393	0.51%	18.17%	0.55	386	0.62%	18.12%	0.67	376	0.72%	18.50%	0.73	356
AR(4)TARCH(2,1)	0.35%	17.84%	0.39	391	0.44%	17.97%	0.48	383	0.52%	18.05%	0.56	374	0.69%	18.57%	0.69	344
AR(0)TARCH(2,2)	0.45%	17.58%	0.51	400	0.53%	17.63%	0.59	396	0.63%	17.80%	0.69	384	0.46%	17.43%	0.50	370
AR(1)TARCH(2,2)	0.50%	17.78%	0.56	400	0.59%	17.80%	0.66	397	0.69%	18.06%	0.75	385	0.83%	18.46%	0.85	358
AR(2)TARCH(2,2)	0.63%	17.75%	0.71	400	0.60%	17.86%	0.67	395	0.76%	18.02%	0.83	386	1.03%	18.11%	1.08	365
AR(3)TARCH(2,2)	0.61%	17.85%	0.69	405	0.62%	17.94%	0.69	401	0.66%	17.98%	0.72	388	0.86%	18.47%	0.89	363
AR(4)TARCH(2,2)	0.62%	17.83%	0.70	404	0.71%	17.96%	0.79	397	0.69%	18.01%	0.75	386	1.12%	18.40%	1.15	356
AR(0)EGARCH(0,1)	0.38%	17.75%	0.42	383	0.49%	17.86%	0.53	377	0.61%	18.16%	0.64	357	0.94%	18.16%	0.94	334
AR(1)EGARCH(0,1)	0.31%	18.19%	0.33	375	0.51%	18.12%	0.54	366	0.73%	18.51%	0.73	348	1.12%	18.40%	1.11	330
AR(2)EGARCH(0,1)	0.33%	18.12%	0.35	378	0.52%	18.14%	0.55	365	0.73%	18.49%	0.74	349	1.12%	18.40%	1.11	330
AR(3)EGARCH(0,1)	0.31%	18.07%	0.34	380	0.52%	18.14%	0.54	365	0.71%	18.44%	0.73	351	1.11%	18.32%	1.11	333
AR(4)EGARCH(0,1)	0.31%	18.07%	0.34	380	0.52%	18.17%	0.55	364	0.71%	18.44%	0.73	351	1.11%	18.32%	1.11	333
AR(0)EGARCH(0,2)	-0.34%	17.31%	-0.40	415	-0.08%	17.45%	-0.10	402	0.18%	17.51%	0.21	388	0.51%	17.68%	0.55	362
AR(1)EGARCH(0,2)	0.00%	17.59%	0.00	398	0.26%	17.48%	0.29	390	0.30%	17.58%	0.34	376	0.77%	18.00%	0.81	351
AR(2)EGARCH(0,2)	0.07%	17.37%	0.08	400	0.26%	17.48%	0.29	390	0.33%	17.59%	0.36	375	0.78%	17.97%	0.82	352
AR(3)EGARCH(0,2)	-0.06%	17.24%	-0.07	409	0.05%	17.36%	0.05	401	0.19%	17.55%	0.21	382	0.72%	17.91%	0.76	355
AR(4)EGARCH(0,2)	-0.06%	17.24%	-0.07	409	0.06%	17.36%	0.07	401	0.19%	17.55%	0.21	382	0.72%	17.88%	0.76	356
AR(0)EGARCH(1,1)	0.40%	17.02%	0.47	410	0.34%	17.08%	0.40	406	0.46%	17.05%	0.53	389	0.72%	17.41%	0.79	369
AR(1)EGARCH(1,1)	0.97%	17.52%	1.09	390	1.01%	17.67%	1.11	382	1.18%	17.94%	1.26	366	1.50%	18.05%	1.54	344
AR(2)EGARCH(1,1)	0.85%	17.39%	0.98	397	0.95%	17.43%	1.08	392	1.10%	17.72%	1.20	375	1.31%	17.82%	1.39	354
AR(3)EGARCH(1,1)	0.91%	17.41%	1.04	397	1.02%	17.52%	1.15	388	1.13%	17.81%	1.22	370	1.61%	17.96%	1.68	351
AR(4)EGARCH(1,1)	0.86%	17.30%	1.00	401	0.90%	17.45%	1.02	393	1.20%	17.74%	1.30	371	1.69%	17.95%	1.76	349
AR(0)EGARCH(1,2)	-0.09%	17.48%	-0.11	408	-0.11%	17.60%	-0.12	401	0.14%	17.84%	0.15	385	0.32%	17.82%	0.34	367
AR(1)EGARCH(1,2)	0.23%	18.21%	0.25	396	0.51%	18.07%	0.56	388	0.73%	18.47%	0.76	368	0.87%	18.71%	0.87	349
AR(2)EGARCH(1,2)	0.49%	17.75%	0.55	395	0.54%	17.84%	0.59	390	0.68%	18.01%	0.73	378	0.84%	18.34%	0.85	350
AR(3)EGARCH(1,2)	0.12%	18.23%	0.13	398	0.41%	17.86%	0.45	393	0.60%	18.06%	0.65	380	0.92%	18.46%	0.94	353
AR(4)EGARCH(1,2)	0.32%	18.27%	0.35	392	0.66%	17.91%	0.73	386	0.86%	18.07%	0.92	375	1.01%	18.41%	1.03	353
AR(0)EGARCH(2,1)	0.20%	17.01%	0.24	413	0.24%	17.18%	0.28	403	0.45%	17.13%	0.52	393	0.70%	17.44%	0.77	374
AR(1)EGARCH(2,1)	0.88%	17.48%	1.00	393	0.94%	17.53%	1.06	390	0.93%	17.91%	1.00	371	1.11%	18.27%	1.14	354
AR(2)EGARCH(2,1)	0.85%	17.36%	0.98	399	0.94%	17.46%	1.07	393	1.09%	17.63%	1.20	380	1.26%	18.13%	1.31	354
AR(3)EGARCH(2,1)	0.92%	17.43%	1.04	393	0.99%	17.53%	1.12	388	0.96%	17.77%	1.05	376	1.17%	18.12%	1.21	354
AR(4)EGARCH(2,1)	0.95%	17.48%	1.08	399	0.99%	17.55%	1.12	395	1.10%	17.66%	1.22	378	1.35%	18.08%	1.41	356

Table 6.8. Daily rate of return from trading straddles on the S&P500 index based on the ARCH models selected by the SEVar model selection method (11 March 1998 – 2 June 2000).

	Without transaction cost				$2 transaction cost											
					Without filter				$1.25 filter				$1.75 filter			
Sample size	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days
T = 10	3.09%	18.03%	3.66	456	-0.56%	17.68%	-0.68	456	-0.13%	17.81%	-0.14	419	-0.04%	17.57%	-0.04	397
T = 20	3.13%	18.02%	3.71	456	-0.52%	17.69%	-0.63	456	-0.42%	17.82%	-0.48	423	-0.13%	17.79%	-0.15	409
T = 30	3.28%	18.00%	3.89	456	-0.37%	17.48%	-0.46	456	-0.14%	17.69%	-0.17	416	-0.16%	17.97%	-0.18	401
T = 40	3.03%	18.04%	3.58	456	-0.63%	17.66%	-0.76	456	-0.32%	18.20%	-0.36	411	0.05%	18.10%	0.05	393
T = 50	3.04%	18.04%	3.60	456	-0.61%	17.55%	-0.74	456	-0.35%	17.99%	-0.39	416	-0.20%	18.07%	-0.22	404
T = 60	3.18%	18.01%	3.77	456	-0.47%	17.52%	-0.57	456	-0.25%	17.97%	-0.28	416	0.04%	17.84%	0.04	403
T = 70	3.28%	18.00%	3.89	456	-0.37%	17.49%	-0.45	456	-0.08%	17.96%	-0.09	415	0.23%	17.86%	0.25	400
T = 80	3.32%	17.99%	3.94	456	-0.34%	17.49%	-0.41	456	-0.06%	17.98%	-0.07	415	0.18%	18.26%	0.20	392

	$2 transaction cost															
	$2.00 filter				$2.25 filter				$2.75 filter				$3.50 filter			
Sample size	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days
T = 10	0.14%	17.35%	0.16	392	0.22%	17.42%	0.25	387	0.11%	17.33%	0.12	379	0.56%	17.59%	0.59	351
T = 20	-0.02%	17.62%	-0.02	404	0.10%	17.71%	0.11	397	0.12%	17.88%	0.13	389	0.45%	18.23%	0.48	368
T = 30	0.06%	17.86%	0.07	394	0.20%	17.99%	0.21	386	0.16%	18.10%	0.17	374	0.49%	18.32%	0.51	351
T = 40	0.24%	18.00%	0.26	384	0.34%	18.16%	0.36	376	0.41%	18.34%	0.43	368	0.78%	18.69%	0.78	349
T = 50	-0.04%	18.00%	-0.04	395	0.21%	17.89%	0.23	389	0.41%	18.08%	0.44	377	0.74%	18.50%	0.75	355
T = 60	0.16%	17.65%	0.18	400	0.24%	17.83%	0.27	391	0.38%	18.10%	0.41	378	0.65%	18.51%	0.66	356
T = 70	0.19%	17.35%	0.22	394	0.26%	17.52%	0.30	385	0.32%	17.72%	0.35	376	0.56%	17.98%	0.59	356
T = 80	0.11%	17.79%	0.12	384	0.16%	17.97%	0.17	375	0.29%	18.16%	0.30	366	0.52%	18.58%	0.51	341

Table 6.9. Daily rate of return from trading straddles on the S&P500 index based on the ARCH models selected by the AEVar model selection method (11 March 1998 – 2 June 2000).

	Without transaction cost				$2 transaction cost											
					Without filter				$1.25 filter				$1.75 filter			
Sample size	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days
T = 10	4.06%	17.83%	4.86	456	0.41%	17.30%	0.51	456	0.44%	17.23%	0.53	422	0.28%	17.10%	0.33	410
T = 20	3.21%	18.01%	3.81	456	-0.44%	17.50%	-0.54	456	-0.36%	17.79%	-0.42	425	-0.21%	18.00%	-0.23	407
T = 30	2.89%	18.06%	3.42	456	-0.76%	17.57%	-0.92	456	-0.39%	17.88%	-0.45	419	-0.12%	18.01%	-0.13	402
T = 40	3.32%	17.99%	3.95	456	-0.33%	17.47%	-0.40	456	-0.19%	17.69%	-0.22	413	-0.24%	17.80%	-0.28	402
T = 50	3.14%	18.02%	3.72	456	-0.51%	17.50%	-0.62	456	-0.05%	17.93%	-0.05	411	0.11%	18.08%	0.12	395
T = 60	3.20%	18.01%	3.79	456	-0.45%	17.63%	-0.55	456	0.03%	17.76%	0.04	419	0.25%	17.94%	0.28	401
T = 70	3.35%	17.98%	3.98	456	-0.30%	17.58%	-0.36	456	0.34%	17.91%	0.38	409	0.33%	18.08%	0.36	395
T = 80	3.23%	18.00%	3.83	456	-0.42%	17.65%	-0.51	456	0.16%	17.66%	0.18	411	0.26%	18.07%	0.28	391

	$2 transaction cost															
	$2.00 filter				$2.25 filter				$2.75 filter				$3.50 filter			
Sample size	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days
T = 10	0.48%	16.91%	0.57	406	0.48%	16.95%	0.57	404	0.46%	16.97%	0.54	389	0.85%	17.33%	0.94	364
T = 20	0.10%	17.88%	0.11	398	0.16%	17.96%	0.18	393	0.33%	18.06%	0.36	383	0.63%	18.41%	0.65	363
T = 30	0.09%	17.90%	0.10	395	0.30%	17.86%	0.33	386	0.59%	18.12%	0.63	369	0.76%	18.39%	0.77	349
T = 40	-0.04%	17.81%	-0.05	389	0.20%	17.77%	0.22	380	0.43%	17.79%	0.46	361	0.64%	18.08%	0.65	345
T = 50	0.10%	18.21%	0.11	388	0.27%	18.12%	0.29	382	0.48%	18.16%	0.50	363	0.70%	18.35%	0.70	339
T = 60	0.32%	18.01%	0.36	397	0.57%	18.01%	0.62	386	0.66%	17.94%	0.71	373	0.99%	18.17%	1.01	346
T = 70	0.53%	17.95%	0.58	389	0.80%	17.92%	0.87	379	0.72%	17.77%	0.77	364	0.90%	18.21%	0.92	345
T = 80	0.35%	17.82%	0.39	385	0.59%	17.76%	0.65	376	0.68%	17.95%	0.72	365	0.96%	18.29%	0.97	346

Table 6.10. Daily rate of return from trading straddles on the S&P500 index based on the ARCH models selected by the SEDev model selection method (11 March 1998 – 2 June 2000).

	Without transaction cost				$2 transaction cost											
					Without filter				$1.25 filter				$1.75 filter			
Sample size	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days
T = 10	3.68%	17.92%	4.39	456	0.03%	17.38%	0.04	456	0.04%	17.35%	0.05	421	0.07%	17.61%	0.08	408
T = 20	3.31%	17.99%	3.93	456	-0.34%	17.47%	-0.42	456	-0.39%	17.68%	-0.46	429	-0.24%	17.91%	-0.27	414
T = 30	3.03%	18.04%	3.59	456	-0.62%	17.54%	-0.76	456	-0.55%	17.64%	-0.64	421	-0.32%	17.82%	-0.36	402
T = 40	3.10%	18.03%	3.67	456	-0.56%	17.51%	-0.68	456	-0.40%	17.61%	-0.47	421	-0.20%	17.65%	-0.23	408
T = 50	2.81%	18.08%	3.32	456	-0.84%	17.58%	-1.03	456	-0.43%	17.96%	-0.49	415	-0.13%	17.85%	-0.15	402
T = 60	3.18%	18.01%	3.77	456	-0.47%	17.54%	-0.58	456	-0.15%	17.96%	-0.17	416	0.15%	17.88%	0.17	401
T = 70	3.24%	18.00%	3.85	456	-0.41%	17.50%	-0.50	456	-0.08%	17.87%	-0.09	418	0.21%	17.84%	0.24	401
T = 80	3.29%	17.99%	3.90	456	-0.36%	17.50%	-0.44	456	-0.12%	18.01%	-0.13	413	0.24%	18.04%	0.26	393

	$2 transaction cost															
	$2.00 filter				$2.25 filter				$2.75 filter				$3.50 filter			
Sample size	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days
T = 10	0.00%	16.94%	0.00	403	0.09%	17.04%	0.11	396	-0.03%	16.91%	-0.03	389	0.40%	17.39%	0.44	359
T = 20	-0.05%	17.83%	-0.06	406	0.04%	17.90%	0.04	401	0.21%	18.08%	0.23	388	0.49%	18.45%	0.51	367
T = 30	-0.14%	17.71%	-0.15	395	-0.02%	17.82%	-0.02	388	0.15%	18.03%	0.16	376	0.41%	18.28%	0.42	353
T = 40	-0.06%	17.62%	-0.07	397	0.04%	17.85%	0.04	386	0.49%	18.03%	0.52	368	0.56%	18.30%	0.58	351
T = 50	0.00%	17.76%	0.00	394	0.06%	17.89%	0.07	388	0.47%	18.06%	0.50	372	0.65%	18.50%	0.66	353
T = 60	0.29%	17.72%	0.33	396	0.42%	17.90%	0.46	387	0.80%	17.98%	0.86	375	0.93%	18.41%	0.95	357
T = 70	0.37%	17.68%	0.41	396	0.48%	17.88%	0.53	386	0.70%	17.97%	0.75	376	0.79%	18.40%	0.81	354
T = 80	0.33%	17.88%	0.36	388	0.39%	18.03%	0.42	381	0.58%	18.24%	0.61	370	0.79%	18.58%	0.80	348

Table 6.11. Daily rate of return from trading straddles on the S&P500 index based on the ARCH models selected by the AEDev model selection method (11 March 1998 – 2 June 2000).

	Without transaction cost				$2 transaction cost											
					Without filter				$1.25 filter				$1.75 filter			
Sample size	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days
T = 10	3.52%	17.95%	4.18	456	-0.14%	17.42%	-0.17	456	-0.18%	17.55%	-0.21	423	-0.37%	17.42%	-0.43	411
T = 20	3.66%	17.92%	4.36	456	0.00%	17.38%	0.00	456	-0.04%	17.61%	-0.04	421	-0.02%	17.90%	-0.02	406
T = 30	2.97%	18.05%	3.52	456	-0.68%	17.55%	-0.83	456	-0.40%	17.80%	-0.46	418	-0.24%	17.93%	-0.27	403
T = 40	3.21%	18.01%	3.81	456	-0.44%	17.50%	-0.54	456	-0.28%	17.70%	-0.32	415	-0.36%	17.86%	-0.40	402
T = 50	2.94%	18.05%	3.48	456	-0.71%	17.56%	-0.86	456	-0.34%	17.79%	-0.39	418	-0.10%	17.95%	-0.11	399
T = 60	3.18%	18.01%	3.76	456	-0.48%	17.64%	-0.58	456	0.08%	17.85%	0.09	415	0.30%	18.00%	0.34	399
T = 70	2.91%	18.06%	3.44	456	-0.75%	17.69%	-0.90	456	0.04%	17.77%	0.05	414	0.10%	18.00%	0.11	396
T = 80	3.29%	17.99%	3.90	456	-0.36%	17.52%	-0.44	456	0.08%	17.63%	0.10	414	0.15%	17.92%	0.17	393

	$2 transaction cost															
	$2.00 filter				$2.25 filter				$2.75 filter				$3.50 filter			
Sample size	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days
T = 10	-0.19%	17.23%	-0.22	408	-0.15%	17.24%	-0.17	406	-0.16%	17.13%	-0.18	395	0.10%	17.62%	0.11	368
T = 20	0.21%	17.78%	0.23	399	0.27%	17.84%	0.30	395	0.46%	17.93%	0.51	385	0.76%	18.35%	0.79	362
T = 30	-0.17%	18.08%	-0.19	396	0.08%	18.04%	0.09	386	0.10%	17.98%	0.10	372	0.49%	18.01%	0.51	355
T = 40	-0.28%	18.08%	-0.30	391	-0.05%	18.02%	-0.06	383	0.13%	18.04%	0.13	365	0.57%	18.09%	0.59	347
T = 50	-0.15%	18.04%	-0.16	393	0.04%	17.97%	0.05	386	0.34%	17.95%	0.36	367	0.66%	18.07%	0.68	350
T = 60	0.33%	18.10%	0.36	394	0.60%	18.06%	0.65	385	0.68%	17.96%	0.74	373	1.02%	18.17%	1.06	352
T = 70	0.13%	18.09%	0.15	391	0.41%	18.09%	0.44	380	0.41%	18.02%	0.44	366	0.76%	18.08%	0.78	351
T = 80	0.19%	17.97%	0.21	390	0.50%	17.88%	0.54	380	0.70%	17.96%	0.75	371	0.99%	18.00%	1.04	357

Table 6.12. Daily rate of return from trading straddles on the S&P500 index based on the ARCH models selected by the HASEVar model selection method (11 March 1998 – 2 June 2000).

	Without transaction cost				$2 transaction cost											
					Without filter				$1.25 filter				$1.75 filter			
Sample size	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days
T = 10	4.04%	17.84%	4.84	456	0.39%	17.47%	0.47	456	0.68%	17.96%	0.78	419	1.07%	18.00%	1.19	400
T = 20	3.95%	17.86%	4.72	456	0.29%	17.48%	0.36	456	0.64%	18.02%	0.72	416	0.94%	18.05%	1.04	399
T = 30	3.67%	17.92%	4.37	456	0.01%	17.59%	0.02	456	0.51%	18.06%	0.58	417	0.66%	18.34%	0.72	400
T = 40	3.29%	17.99%	3.90	456	-0.37%	17.67%	-0.44	456	0.17%	18.15%	0.19	416	0.28%	18.42%	0.31	398
T = 50	3.37%	17.98%	4.01	456	-0.28%	17.51%	-0.34	456	0.14%	17.96%	0.15	412	0.34%	18.13%	0.37	397
T = 60	3.53%	17.95%	4.20	456	-0.12%	17.44%	-0.15	456	0.28%	17.96%	0.32	412	0.34%	17.92%	0.38	402
T = 70	3.38%	17.98%	4.02	456	-0.27%	17.48%	-0.33	456	0.04%	18.05%	0.05	410	0.26%	18.17%	0.29	398
T = 80	3.35%	17.98%	3.98	456	-0.30%	17.49%	-0.37	456	0.07%	18.09%	0.08	408	0.22%	18.20%	0.25	398

	$2 transaction cost															
	$2.00 filter				$2.25 filter				$2.75 filter				$3.50 filter			
Sample size	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days
T = 10	1.04%	18.13%	1.13	392	1.08%	18.33%	1.15	383	1.15%	18.49%	1.19	367	1.45%	18.47%	1.45	341
T = 20	0.86%	18.16%	0.94	392	0.87%	18.30%	0.94	386	0.97%	18.35%	1.02	372	1.22%	18.72%	1.21	347
T = 30	0.79%	18.26%	0.85	390	0.82%	18.47%	0.87	381	1.12%	18.60%	1.16	370	1.41%	18.65%	1.41	349
T = 40	0.27%	18.54%	0.29	392	0.49%	18.48%	0.52	383	0.73%	18.59%	0.76	372	1.22%	18.25%	1.27	360
T = 50	0.34%	18.32%	0.37	388	0.42%	18.49%	0.44	379	0.58%	18.54%	0.60	372	1.00%	18.17%	1.04	354
T = 60	0.12%	17.80%	0.13	390	0.16%	17.92%	0.18	383	0.37%	18.02%	0.40	373	0.74%	18.08%	0.78	357
T = 70	0.05%	18.04%	0.06	386	0.04%	18.23%	0.05	378	0.32%	18.45%	0.33	363	0.52%	18.40%	0.53	347
T = 80	0.03%	18.08%	0.03	386	0.02%	18.27%	0.02	378	0.24%	18.44%	0.25	365	0.45%	18.37%	0.46	350

Table 6.13. Daily rate of return from trading straddles on the S&P500 index based on the ARCH models selected by the HAAEVar model selection method (11 March 1998 – 2 June 2000).

	Without transaction cost				$2 transaction cost											
					Without filter				$1.25 filter				$1.75 filter			
Sample size	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days
T = 10	3.48%	17.96%	4.14	456	-0.17%	17.63%	-0.21	456	0.23%	17.95%	0.26	422	0.43%	18.17%	0.47	407
T = 20	4.14%	17.82%	4.97	456	0.49%	17.45%	0.60	456	0.81%	17.93%	0.92	415	1.02%	18.21%	1.12	398
T = 30	3.62%	17.93%	4.31	456	-0.03%	17.60%	-0.04	456	0.34%	18.11%	0.38	413	0.45%	18.36%	0.49	398
T = 40	3.56%	17.94%	4.23	456	-0.09%	17.58%	-0.11	456	0.39%	18.20%	0.43	410	0.47%	18.50%	0.50	392
T = 50	3.80%	17.89%	4.54	456	0.15%	17.40%	0.18	456	0.60%	17.86%	0.68	412	0.59%	18.16%	0.65	396
T = 60	3.80%	17.89%	4.54	456	0.15%	17.39%	0.18	456	0.36%	17.93%	0.41	413	0.42%	18.09%	0.46	396
T = 70	3.70%	17.91%	4.41	456	0.04%	17.42%	0.05	456	0.48%	17.95%	0.54	411	0.55%	18.19%	0.61	398
T = 80	3.39%	17.97%	4.03	456	-0.26%	17.49%	-0.32	456	0.21%	18.07%	0.23	408	0.29%	18.37%	0.32	394

	$2 transaction cost															
	$2.00 filter				$2.25 filter				$2.75 filter				$3.50 filter			
Sample size	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days
T = 10	0.36%	18.21%	0.39	404	0.50%	18.11%	0.55	397	0.44%	18.22%	0.47	378	0.90%	18.45%	0.91	346
T = 20	0.98%	18.31%	1.06	393	1.17%	18.24%	1.25	384	1.23%	18.18%	1.30	370	1.59%	18.02%	1.64	345
T = 30	0.60%	18.32%	0.65	388	0.85%	18.34%	0.90	375	1.08%	18.54%	1.11	363	1.50%	18.44%	1.51	343
T = 40	0.80%	18.23%	0.85	380	0.84%	18.42%	0.88	372	1.05%	18.60%	1.07	360	1.66%	18.74%	1.63	337
T = 50	0.63%	18.47%	0.66	382	0.69%	18.61%	0.72	375	0.85%	18.82%	0.85	354	1.19%	18.70%	1.17	337
T = 60	0.46%	18.28%	0.49	386	0.57%	18.46%	0.60	377	0.91%	18.84%	0.91	355	1.19%	18.78%	1.16	339
T = 70	0.65%	18.47%	0.69	384	0.72%	18.63%	0.75	377	1.04%	19.10%	1.02	352	1.28%	19.06%	1.23	337
T = 80	0.38%	18.61%	0.40	382	0.54%	18.57%	0.56	374	0.79%	18.96%	0.78	353	1.02%	18.94%	0.99	337

Table 6.14. Daily rate of return from trading straddles on the S&P500 index based on the ARCH models selected by the HASEDev model selection method (11 March 1998 – 2 June 2000).

	Without transaction cost				$2 transaction cost											
					Without filter				$1.25 filter				$1.75 filter			
Sample size	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days
T = 10	3.34%	17.98%	3.97	456	-0.31%	17.71%	-0.37	456	0.10%	18.00%	0.12	417	0.18%	17.65%	0.20	401
T = 20	3.76%	17.90%	4.49	456	0.11%	17.56%	0.13	456	0.35%	18.15%	0.39	414	0.66%	18.18%	0.73	397
T = 30	3.92%	17.87%	4.69	456	0.27%	17.35%	0.33	456	0.58%	17.85%	0.66	415	0.67%	18.01%	0.74	404
T = 40	3.55%	17.94%	4.22	456	-0.10%	17.44%	-0.13	456	0.14%	18.03%	0.16	410	0.26%	18.29%	0.28	395
T = 50	3.39%	17.97%	4.03	456	-0.26%	17.48%	-0.32	456	0.03%	17.96%	0.03	415	0.19%	18.14%	0.20	400
T = 60	2.83%	18.07%	3.34	456	-0.83%	17.58%	-1.00	456	-0.55%	18.07%	-0.63	416	-0.33%	18.23%	-0.36	401
T = 70	2.78%	18.08%	3.29	456	-0.87%	17.59%	-1.05	456	-0.52%	18.01%	-0.59	417	-0.41%	18.17%	-0.45	406
T = 80	2.92%	18.06%	3.46	456	-0.73%	17.56%	-0.88	456	-0.53%	18.06%	-0.60	414	-0.24%	18.05%	-0.27	401
	$2 transaction cost															
	$2.00 filter				$2.25 filter				$2.75 filter				$3.50 filter			
Sample size	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days
T = 10	0.19%	17.48%	0.21	394	0.23%	17.56%	0.26	389	0.32%	17.76%	0.35	378	0.59%	18.11%	0.62	353
T = 20	0.56%	18.27%	0.60	391	0.61%	18.41%	0.65	384	0.88%	18.53%	0.92	373	1.17%	18.68%	1.17	348
T = 30	0.66%	18.13%	0.73	398	0.68%	18.27%	0.74	392	0.85%	18.40%	0.90	375	1.13%	18.41%	1.15	353
T = 40	0.22%	18.41%	0.23	389	0.43%	18.39%	0.45	380	0.68%	18.52%	0.70	370	1.12%	18.60%	1.13	353
T = 50	0.19%	18.31%	0.20	392	0.37%	18.20%	0.40	385	0.59%	18.40%	0.62	372	1.06%	18.56%	1.07	352
T = 60	-0.33%	18.45%	-0.36	390	-0.09%	18.40%	-0.09	381	0.13%	18.68%	0.13	364	0.48%	18.68%	0.48	352
T = 70	-0.42%	18.42%	-0.46	393	-0.40%	18.62%	-0.42	384	-0.11%	19.04%	-0.11	361	0.10%	18.96%	0.10	347
T = 80	-0.18%	18.25%	-0.20	390	-0.18%	18.36%	-0.19	385	0.08%	18.58%	0.08	369	0.29%	18.52%	0.30	354

Table 6.15. Daily rate of return from trading straddles on the S&P500 index based on the ARCH models selected by the HAAEDev model selection method (11 March 1998 – 2 June 2000).

	Without transaction cost				$2 transaction cost											
					Without filter				$1.25 filter				$1.75 filter			
Sample size	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days
T = 10	2.69%	18.09%	3.18	456	-0.96%	17.80%	-1.15	456	-0.71%	18.18%	-0.80	421	-0.67%	18.44%	-0.73	406
T = 20	4.02%	17.84%	4.81	456	0.37%	17.36%	0.45	456	0.32%	17.74%	0.37	415	0.33%	17.92%	0.38	406
T = 30	4.01%	17.85%	4.79	456	0.35%	17.32%	0.44	456	0.62%	17.78%	0.70	415	0.60%	17.92%	0.67	408
T = 40	3.47%	17.96%	4.13	456	-0.18%	17.45%	-0.22	456	-0.03%	17.89%	-0.03	411	0.07%	18.15%	0.07	398
T = 50	2.84%	18.07%	3.36	456	-0.81%	17.57%	-0.99	456	-0.60%	17.88%	-0.68	418	-0.60%	18.13%	-0.67	406
T = 60	2.98%	18.05%	3.52	456	-0.67%	17.54%	-0.82	456	-0.53%	17.94%	-0.60	415	-0.50%	18.20%	-0.55	402
T = 70	2.84%	18.07%	3.36	456	-0.81%	17.57%	-0.98	456	-0.59%	18.04%	-0.67	417	-0.57%	18.28%	-0.63	405
T = 80	2.88%	18.06%	3.41	456	-0.77%	17.56%	-0.94	456	-0.48%	18.10%	-0.54	414	-0.49%	18.34%	-0.54	402
	$2 transaction cost															
	$2.00 filter				$2.25 filter				$2.75 filter				$3.50 filter			
Sample size	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days
T = 10	-0.57%	18.33%	-0.62	398	-0.35%	18.16%	-0.38	393	-0.36%	18.32%	-0.38	380	-0.04%	18.87%	-0.04	348
T = 20	0.49%	17.81%	0.55	399	0.52%	17.96%	0.57	391	0.74%	18.02%	0.80	383	1.16%	18.11%	1.22	363
T = 30	0.81%	17.86%	0.91	399	0.88%	17.94%	0.97	394	1.14%	18.08%	1.23	382	1.38%	18.40%	1.43	365
T = 40	0.23%	18.16%	0.25	386	0.40%	18.14%	0.43	377	0.70%	18.26%	0.73	366	0.84%	18.63%	0.85	350
T = 50	-0.58%	18.26%	-0.63	400	-0.43%	18.20%	-0.46	393	-0.14%	18.54%	-0.15	372	0.31%	18.84%	0.31	347
T = 60	-0.46%	18.33%	-0.50	396	-0.27%	18.27%	-0.29	389	0.11%	18.61%	0.11	367	0.43%	18.82%	0.43	347
T = 70	-0.52%	18.40%	-0.56	399	-0.36%	18.37%	-0.38	391	0.08%	18.76%	0.09	367	0.27%	18.87%	0.27	346
T = 80	-0.51%	18.47%	-0.55	395	-0.35%	18.37%	-0.38	390	0.05%	18.72%	0.05	368	0.21%	18.78%	0.21	349

Table 6.16. Daily rate of return from trading straddles on the S&P500 index based on the ARCH models selected by the LEVar model selection method (11 March 1998 – 2 June 2000).

	Without transaction cost				$2 transaction cost											
					Without filter				$1.25 filter				$1.75 filter			
Sample size	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days
T = 10	3.52%	17.95%	4.18	456	-0.14%	17.41%	-0.17	456	-0.42%	17.62%	-0.49	421	-0.40%	17.87%	-0.45	407
T = 20	3.32%	17.99%	3.94	456	-0.33%	17.47%	-0.41	456	-0.34%	17.78%	-0.39	419	-0.35%	18.00%	-0.39	408
T = 30	2.81%	18.07%	3.32	456	-0.84%	17.57%	-1.02	456	-0.69%	17.84%	-0.79	423	-0.35%	17.80%	-0.39	406
T = 40	3.31%	17.99%	3.93	456	-0.34%	17.48%	-0.42	456	-0.06%	17.87%	-0.07	418	0.06%	17.81%	0.07	404
T = 50	3.01%	18.04%	3.56	456	-0.64%	17.53%	-0.78	456	-0.28%	17.98%	-0.32	412	0.08%	17.99%	0.09	392
T = 60	3.29%	17.99%	3.90	456	-0.37%	17.51%	-0.45	456	-0.24%	17.80%	-0.27	419	0.18%	17.76%	0.20	401
T = 70	3.18%	18.01%	3.78	456	-0.47%	17.55%	-0.57	456	-0.23%	17.72%	-0.27	415	0.17%	17.61%	0.19	397
T = 80	3.07%	18.03%	3.63	456	-0.58%	17.55%	-0.71	456	-0.18%	17.73%	-0.20	413	0.23%	17.69%	0.26	391

	$2 transaction cost															
	$2.00 filter				$2.25 filter				$2.75 filter				$3.50 filter			
Sample size	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days	Mean	Stand. Dev.	t ratio	Days
T = 10	-0.44%	17.34%	-0.51	403	-0.34%	17.36%	-0.39	397	-0.29%	17.17%	-0.33	388	-0.05%	17.71%	-0.06	361
T = 20	-0.31%	18.09%	-0.34	403	-0.19%	18.24%	-0.20	394	-0.26%	18.19%	-0.28	380	0.15%	18.33%	0.15	358
T = 30	-0.32%	17.94%	-0.35	399	-0.24%	18.01%	-0.26	394	-0.04%	17.81%	-0.04	378	0.38%	17.93%	0.40	357
T = 40	0.12%	18.02%	0.13	393	0.28%	18.20%	0.30	382	0.46%	18.22%	0.49	363	0.93%	18.35%	0.94	342
T = 50	0.14%	18.08%	0.15	386	0.17%	18.22%	0.18	380	0.44%	18.18%	0.46	363	0.78%	18.35%	0.79	344
T = 60	0.23%	17.86%	0.26	394	0.28%	18.01%	0.30	387	0.40%	17.91%	0.43	372	0.79%	18.14%	0.82	350
T = 70	0.21%	17.62%	0.24	394	0.30%	17.82%	0.32	384	0.55%	18.02%	0.58	369	0.85%	18.20%	0.87	345
T = 80	0.23%	17.71%	0.25	390	0.31%	17.94%	0.33	379	0.60%	18.12%	0.63	366	0.96%	18.25%	0.98	347

Table 6.17. Daily rate of return from trading straddles on the S&P500 index based on the ARCH models selected by the AIC and SBC model selection methods (11 March 1998 – 2 June 2000).

	Without transaction cost				$2 transaction cost											
					Without filter				$1.25 filter				$1.75 filter			
Sample size	Mean	Stand. Dev.	T ratio	Days	Mean	Stand. Dev.	T ratio	Days	Mean	Stand. Dev.	T ratio	Days	Mean	Stand. Dev.	t ratio	Days
AIC	4.05%	17.84%	4.85	456	0.40%	17.31%	0.49	456	0.66%	17.69%	0.76	414	0.78%	17.75%	0.88	401
SBC	3.74%	17.90%	4.47	456	0.09%	17.37%	0.11	456	0.52%	17.82%	0.59	412	0.65%	18.15%	0.71	394

	$2 transaction cost															
	$2.00 filter				$2.25 filter				$2.75 filter				$3.50 filter			
Sample size	Mean	Stand. Dev.	T ratio	Days	Mean	Stand. Dev.	T ratio	Days	Mean	Stand. Dev.	T ratio	Days	Mean	Stand. Dev.	t ratio	Days
AIC	0.90%	17.86%	1.00	394	0.96%	18.00%	1.05	386	1.03%	17.95%	1.10	366	1.21%	18.22%	1.21	333
SBC	0.88%	18.23%	0.94	385	0.95%	18.37%	1.01	378	0.98%	18.25%	1.03	364	1.41%	18.49%	1.40	338

Appendix 6.1.
Construction of the Black and Scholes Option Pricing Formula

The lines following present the Black and Scholes approach in constructing their option pricing formula. Suppose we have an option whose value, $C(S,t)$, depends only on the stock price, S, and time, t. It is not necessary at this stage to determine whether C is a call or put. Let's create a riskless hedge portfolio, consisting of a long position in the stock (buy the stock) and short position in the option (sell the option) under the assumption investors have full access to information, are borrowing and lending at the continuously compounded risk free interest rate and are trading continuously in a frictionless capital market with no transaction costs, no taxes, no short sales constraints. Moreover, we assume that the stock price follows a geometric Brownian motion:

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t),$$

where μ is the expected instantaneous rate of return on the underlying asset, σ is the instantaneous variance of the rate of return and $B(t)$ is a standard Brownian motion. If we write as Q_S the number of stocks and Q_C the number of options then the value V_H of that riskless hedge portfolio and its changes dV_H in short intervals will be determined as:

$$\begin{aligned} V_H &= Q_S S + Q_C C \\ d\,V_H &= Q_S dS + Q_C dC \end{aligned}$$

If we assume that the short position (writing call options) is changed continuously, we can use Ito's Lemma to expand $dC = C(S+dS, t+dt) - C(S,t)$:

$$dC(S,t) = \frac{\partial C}{\partial S}dS + \frac{\partial C}{\partial t}dt + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 dt.$$

We determine Q_S and Q_C such as the risk factor is eliminated:

$$Q_S dS + Q_C \frac{\partial C}{\partial S} dS = 0.$$

We find out that the ratio of stocks to options must be instantaneously adjusted at the rate of - ∂C/∂S,

$$\frac{Q_C}{Q_S} = -\frac{\partial S}{\partial C}.$$

Normalizing $Q_S = 1$, the above equation shows that for each stock purchased we have to write (negative sing) ∂C/∂S options on it. Moreover, since the return on the equity on the hedge portfolio is certain, it must be equal to the risk free rate:

$$\frac{dV_H}{V_H} = r_f dt .$$

Thus, $dV_H = \frac{-\partial S}{\partial C}\left(\frac{\partial C}{\partial t}dt + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 dt\right)$ and solving for $\partial C/\partial t$ we reach:

$$\frac{\partial C}{\partial t} = r_f V_H\left(-\frac{\partial C}{\partial S}\right) - \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 .$$

Combining $V_H = S - C\left(1/\left(\partial C/\partial S\right)\right)$ with the above equation and rearranging, we reach to the following partial differential equation for the value of the option:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + r_f S\frac{\partial C}{\partial S} + \frac{\partial C}{\partial T} - r_f C = 0 ,$$

which is uniquely solved subject to a set of boundary conditions.

Having derived the Black and Scholes equation for the value of an option, we must next consider the boundary conditions yielding a unique solution to the partial differential equation. First, we are dealing with pricing a European call, $C(S,t)$, with exercise price K and expiry date T. At maturity day the value of the call is known with certainty and is the payoff, $C(S(T),T) = \max[0, S(T) - K]$. Moreover, if $S = 0$ then the call option is worthless even if there is a long time to expiry, $C(0,t) = 0$. Finally, as the stock price increases without bound, it becomes even more likely that the option will be exercised and the magnitude of the exercise price becomes less and less important. Thus, as $S \to \infty$ the value of the option become that of the asset, $C(S,t) \approx S$ as $S \to \infty$.

The boundary conditions to price a European put, denoted by $P(S,t)$, with exercise price K and expiry date T, claim that at maturity day the value of the put is known with certainty and is the payoff, $P(S(T),T) = \max[0, K - S(T)]$. If the stock price is zero the put price is the present value of the exercise price received at maturity day: $P(0,t) = \exp(-r_f(T-t))K$. Finally, as the asset price increases without bound the option is unlikely to be exercised: $P(S,t) \to 0$ as $S \to \infty$.

The unique solution of the partial differential equation, subject to the boundary conditions, yields the Black and Scholes Option Pricing Formula.

$$
\begin{aligned}
&C(S(t),t) = S(t)N(d_1) - Ke^{-r_f(T-t)}N(d_2) \\
&P(S(t),t) = Ke^{-r_f(T-t)}N(-d_2) - S(t)N(-d_1) \\
&d_1 = \frac{\ln\left(S(t)\big/K\right) + \left(r_f + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \\
&d_2 = d_1 - \sigma\sqrt{T-t}
\end{aligned}
$$

where $N(.)$ is the standard normal cumulative distribution function and $(T-t)$ is the time to maturity of the option.

Merton (1973b) extended the Black & Scholes model to allow for dividend yield. The model can be used to price European call and put options on a stock or stock index paying a known dividend yield equal to γ. Suppose that in time dt the underlying asset pays a dividend $\gamma S dt$. The asset price should fall by the amount of the dividend payment, thus the asset price distribution is:

$$dS(t) = (\mu - \gamma)S(t)dt + \sigma S(t)dB(t).$$

Proceeding exactly as before we reach to a partial differential equation of the form:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r_f - \gamma)S\frac{\partial C}{\partial S} + \frac{\partial C}{\partial T} - r_f C = 0,$$

which is uniquely solved subject to the same set of boundary conditions, except from the value of the option when the asset price increases without bound. As $S \to \infty$ the value of the call equals to the price of the asset without its dividend: $C(S,t) \approx S\exp(-\gamma(T-t))$. Adding a constant dividend yield the option pricing formula is:

$$
\begin{aligned}
&C(S(t),t) = S(t)e^{-\gamma(T-t)}N(d_1) - Ke^{-r_f(T-t)}N(d_2) \\
&P(S(t),t) = Ke^{-r_f(T-t)}N(-d_2) - S(t)e^{-\gamma(T-t)}N(-d_1) \\
&d_1 = \frac{\ln\left(S(t)\big/K\right) + \left(r_f - \gamma + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \\
&d_2 = d_1 - \sigma\sqrt{T-t}
\end{aligned}
$$

Note that in order to derive the option pricing formula we do assume nothing about investors' preferences. Both an economy consisting of risk-neutral investors and an economy consisting of risk-averse investors must yield the same price for the derivative security. Cox and Ross (1976) assume a risk neutral economy and define the price of the option as the expected value of its payoff discounted at the risk free rate:

$$C\left(\tilde{S}(T),t\right) = e^{-r_f(T-t)}E\left(\max\left[0, \tilde{S}(T) - K\right] | I_t\right).$$

The expectation is evaluated conditional to the information available at time t. The $\widetilde{S}(T)$ denotes the terminal stock price adjusted for risk neutrality. The procedure is applied to solve the conditional expectation is called risk-neutral pricing method and the solution yields to the Black and Scholes formula.

Cox et al. (1979) and Rendleman and Bartter (1979) independently derive the Binomial option pricing formula. They assume that the stock price follows a multiplicative binomial process over discrete period. At the limit, a binomial tree is equivalent to the continuous time Black and Scholes formula for pricing European options. The Binomial method provides solutions not only for a closed form European option pricing model but also for the more difficult American option problem where numerical solutions must be employed. To price European call and put options with τ days to maturity, the model is expressed as:

$$
\begin{aligned}
C(t) &= e^{-r_f \tau} \sum_{i=a}^{n} \left(\frac{n!}{i!(n-i)!} \right) p^i (1-p)^{n-i} \left(Su^i d^{n-i} - K \right) \\
P(t) &= e^{-r_f \tau} \sum_{i=0}^{a-1} \left(\frac{n!}{i!(n-i)!} \right) p^i (1-p)^{n-i} \left(K - Su^i d^{n-i} \right)
\end{aligned}
.
$$

The stock price can either increase by a fixed amount u with a probability p, or decrease by a fixed amount d with probability $1-p$. The number of time steps is n and a is the smallest nonnegative integer greater than $\ln\left(K/Sd^n\right)/\ln(u/d)$. The stock price at each node is set equal to $Su^i d^{j-i}$ for $i = 1,..., j$. The upward and downward jump size that the stock can take place at each time step $\Delta\tau = \tau/n$ is given by $u = \exp\left(\sigma\sqrt{\Delta\tau}\right)$ and $d = \exp\left(-\sigma\sqrt{\Delta\tau}\right)$ respectively, where n is the number of time steps. The probability of the stock price increasing at the next time step is $p = \left(\left(\exp\left(r_f - \gamma\right)\Delta\tau\right) - d\right)/(u-d)$ and the probability of going down is $1-p$.

Appendix 6.2.
Options Sensitivities

Delta, Lambda, Gamma, Theta, Vega and Rho comprise the pricing sensitivities and represent the key relationships between the individual characteristics of the option and the option price. The option sensitivities are the partial derivatives of the BS option price in relation to each individual factor that affects the price of the option. In the formulas following we omit the subscript symbol indicates the time, t, for notational simplicity.

Delta is the change in the option price for a given change in the stock price, that is, the hedge ratio.

$$\Delta_{CALL} = \frac{\partial C}{\partial S} = e^{-\gamma\tau} N(d_1) > 0$$

$$\Delta_{PUT} = \frac{\partial P}{\partial S} = e^{-\gamma\tau}\left(N(d_1) - 1\right) < 0$$

For example, a trader who buys one call option with Δ=0.6, and sells a different call option with Δ=0.4, has a net Δ=0.6-0.4=0.2. Thus, a $1 change in the stock price creates a +$0.2 increase in the combined option position.

Lambda or **Elasticity** measures the percentage change in the option price for a given percentage change in the stock price.

$$\Lambda_{CALL} = \frac{\partial C}{\partial S}\frac{S}{C} = e^{-\gamma\tau} N(d_1)\frac{S}{C} > 1$$

$$\Lambda_{PUT} = \frac{\partial P}{\partial S}\frac{S}{P} = e^{-\gamma\tau}\left(N(d_1) - 1\right)\frac{S}{C} < 0$$

A Lambda of 5 means that a 1% increase in the price of the stock causes a 5% increase in the price of the option. Leverage is an important characteristic of options that attracts speculators.

Rho measures the change in the option price for a given change in the risk free interest rate.

$$P_{CALL} = -\frac{\partial C}{\partial r_f} = \tau K e^{-r_f \tau} N(d_2) > 0$$

$$P_{PUT} = -\frac{\partial P}{\partial r_f} = -\tau K e^{-r_f \tau} N(-d_2) < 0$$

Option traders have only a minor interest in Pho, as an increase in risk free rate have only a minimal effect on the value of an option.

Theta is the change in the option price for a given a change in the time, until option expiration. As time to maturity decreases, it is normal to express the Theta as minus the partial derivative with respect to time.

$$\Theta_{CALL} = -\frac{\partial C}{\partial \tau} = -\frac{Se^{-\left(\gamma\tau + d_1^2/2\right)}\sigma}{2\sqrt{2\pi\tau}} + \gamma Se^{-\gamma\tau}N\left(d_1\right) - r_f Ke^{-r_f\tau}N\left(d_2\right)$$

$$\Theta_{PUT} = -\frac{\partial P}{\partial \tau} = -\frac{Se^{-\left(\gamma\tau + d_1^2/2\right)}\sigma}{2\sqrt{2\pi\tau}} - \gamma Se^{-\gamma\tau}N\left(-d_1\right) + r_f Ke^{-r_f\tau}N\left(-d_2\right)$$

Gamma measures the change in the delta for a given change in the stock price. Gamma is identical for call and put options.

$$\Gamma_{CALL,PUT} = \frac{\partial^2 C}{\partial S^2} = \frac{\partial^2 P}{\partial S^2} = \frac{e^{-\left(\frac{d_1^2}{2} + \gamma\tau\right)}}{S\sigma\sqrt{2\pi\tau}} > 0$$

Gamma is one measure of the effect of instability on the option position (the other is Vega). It shows the risk inherent in Delta. If Gamma is small, Delta is not sensitive to changes in the stock price. If Gamma is large, Delta is sensitive to stock price changes. If Gamma is 0.5 and the current Delta is 0, then an increase in the stock price of $1 causes the Delta to increase from 0 to 0.5. Now, the new Delta means that an increase in the stock price of $1 will now increase the option price by $0.5.

Vega is the change in the option price for a given change in the volatility of the stock. Vega is equal for call and put options.

$$V_{CALL,PUT} = \frac{\partial C}{\partial \sigma} = \frac{\partial P}{\partial \sigma} = \frac{Se^{-\left(\gamma\tau + d_1^2/2\right)}\sqrt{\tau}}{\sqrt{2\pi}} > 0$$

If V=18, then an increase in the annual standard deviation of the stock of 1% causes the option price to increase by $18. Traders often attempt to find out which options are cheaper or more expensive in terms of volatility than the market believes. Moreover, a relatively small change in the annual volatility of returns causes a relatively large change in the option price. Volatility is the only factor in BS formula that is not directly observable, so the traders forecast the future volatility to value options. Therefore, changes in implied volatility have a major impact on option prices. Strategies with positive Vega (buy an option) are profit when volatility increases and strategies with negative Vega (sell an option) are profit when volatility is stable or decreases. Both Gamma and Vega are dealing with volatility of the underlying asset, but they have a main difference. Vega indicates the sensitivity of our position to a change in the implied volatility of the stock. On the other hand, Gamma indicates the effect of the current volatility on the option price as the stock price changes.

Consider the example of computing the theoretical option prices in section 6.2.3 of the 6th chapter. The option sensitivities for the call and put options are the following:

	Price	Delta	Lambda	Rho	Theta	Gamma	Vega
Call	1,867	0,337	10,833	4,591	-6,981	0,040	10,873
Put	6,326	-0,650	-6,169	-11,337	-4,847	0,040	10,873

Appendix 8

- **Figures 8.1-8.8. The Cumulative Density Function of the Minimum Component of a Tri-Variate Gamma Vector**

- **Tables 8.3-8.20. The Probability $(1-p)$ that the Minimum $X_{(1)}$ of a Trivariate Gamma Vector is Less than or Equal to ω_{1-p} for $2 \geq \omega_{1-p} \geq 50$, $5 \geq a \geq 50$**

Figure 8.1. The cumulative density function of the minimum component of a tri-variate gamma vector, for $60 \geq x \geq 0$, $100 \geq a > 0$, and $\rho_{1,2} = 5\%$, $\rho_{1,3} = 30\%$ and $\rho_{2,3} = 60\%$, the non-diagonal elements of C_{123}.

$$F_{X_{(1)}}(x;a,C_{123}) = 3F_{X_1}(x) - \sum_{i_1=1}^{2}\sum_{i_2=2}^{3} F_{X_{i_1},X_{i_2}}(x,x) + F_{X_1,X_2,X_3}(x,x,x)$$

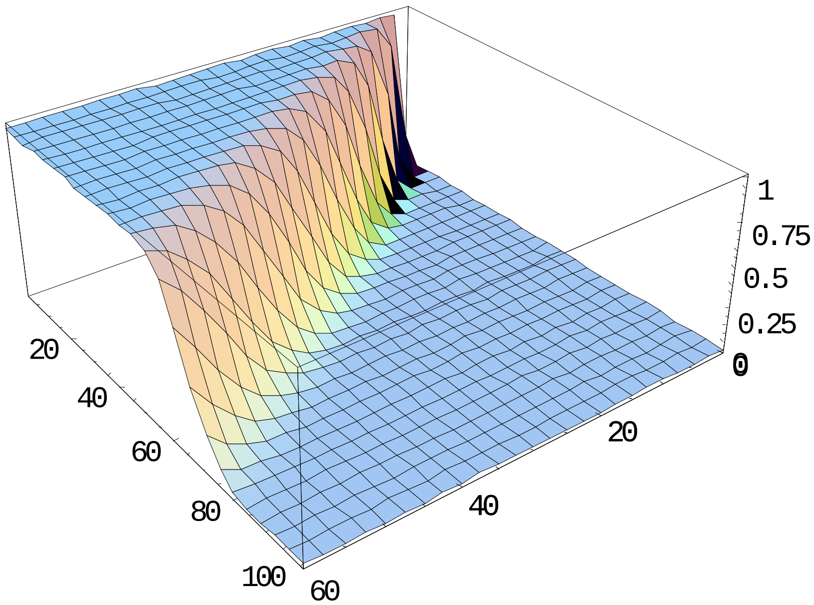


Figure 8.2. The cumulative density function of the minimum component of a tri-variate gamma vector, for $60 \geq x \geq 0$, $100 \geq a > 0$, and $\rho_{1,2} = 5\%$, $\rho_{1,3} = 60\%$ and $\rho_{2,3} = 95\%$, the non-diagonal elements of C_{123}.

$$F_{X_{(1)}}(x;a,C_{123}) = 3F_{X_1}(x) - \sum_{i_1=1}^{2}\sum_{i_2=2}^{3} F_{X_{i_1},X_{i_2}}(x,x) + F_{X_1,X_2,X_3}(x,x,x)$$

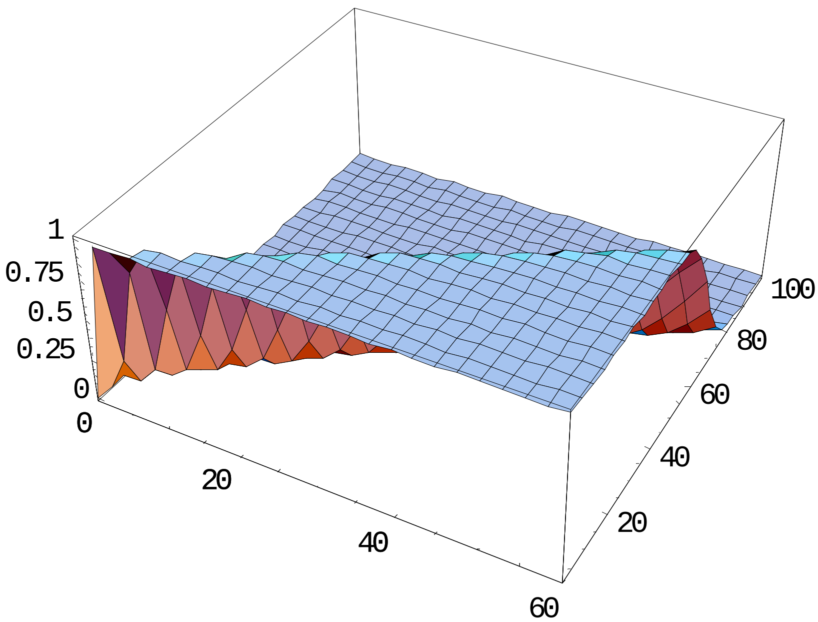


Figure 8.3. The cumulative density function of the minimum component of a tri-variate gamma vector, for $60 \geq x \geq 0$, $100 \geq a > 0$, and $\rho_{1,2} = 30\%$, $\rho_{1,3} = 30\%$ and $\rho_{2,3} = 30\%$, the non-diagonal elements of C_{123}.

$$F_{X_{(1)}}(x; a, C_{123}) = 3F_{X_1}(x) - \sum_{i_1=1}^{2}\sum_{i_2=2}^{3} F_{X_{i_1}, X_{i_2}}(x, x) + F_{X_1, X_2, X_3}(x, x, x)$$

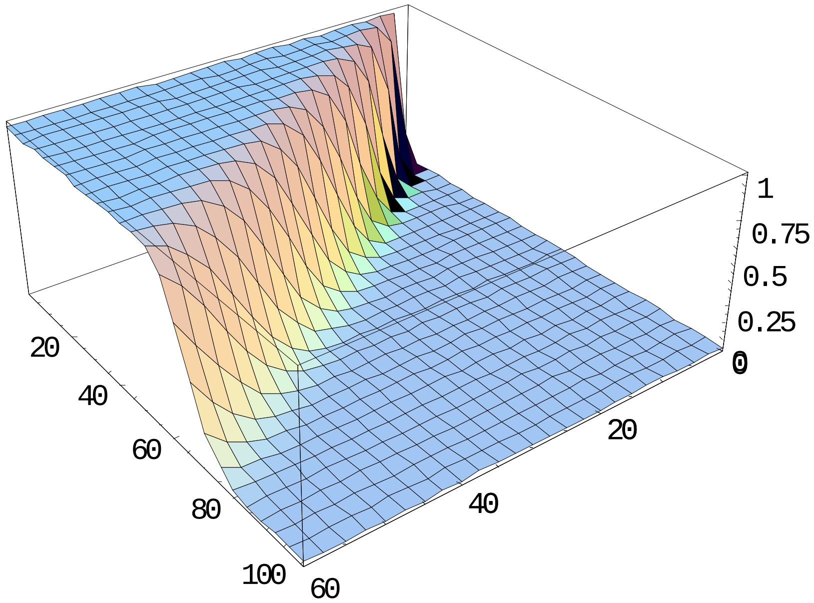


Figure 8.4. The cumulative density function of the minimum component of a tri-variate gamma vector, for $60 \geq x \geq 0$, $100 \geq a > 0$, and $\rho_{1,2} = 30\%$, $\rho_{1,3} = 60\%$ and $\rho_{2,3} = 95\%$, the non-diagonal elements of C_{123}.

$$F_{X_{(1)}}(x; a, C_{123}) = 3F_{X_1}(x) - \sum_{i_1=1}^{2}\sum_{i_2=2}^{3} F_{X_{i_1}, X_{i_2}}(x, x) + F_{X_1, X_2, X_3}(x, x, x)$$

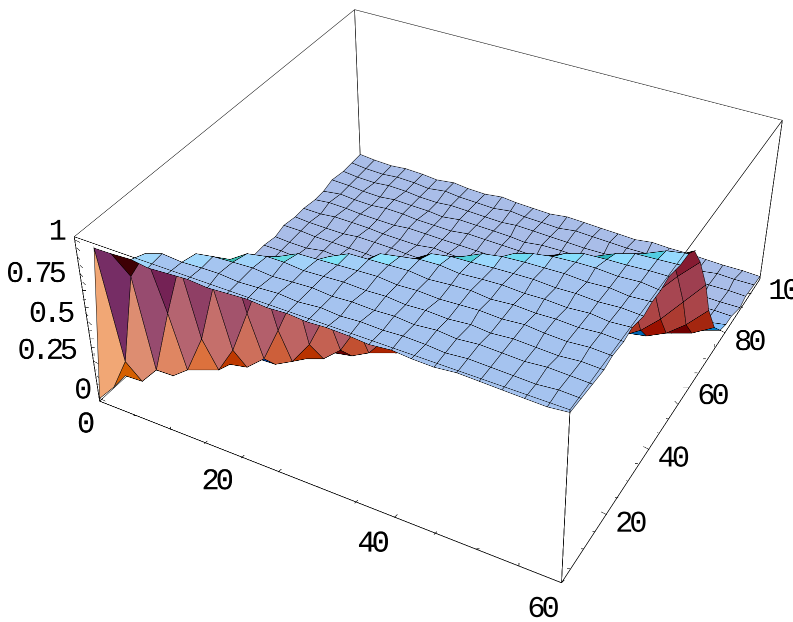


Figure 8.5. The cumulative density function of the minimum component of a tri-variate gamma vector, for $60 \geq x \geq 0$, $100 \geq a > 0$, and $\rho_{1,2} = 60\%$, $\rho_{1,3} = 60\%$ and $\rho_{2,3} = 60\%$, the non-diagonal elements of C_{123}.

$$F_{X_{(1)}}(x; a, C_{123}) = 3F_{X_1}(x) - \sum_{i_1=1}^{2}\sum_{i_2=2}^{3} F_{X_{i_1}, X_{i_2}}(x, x) + F_{X_1, X_2, X_3}(x, x, x)$$

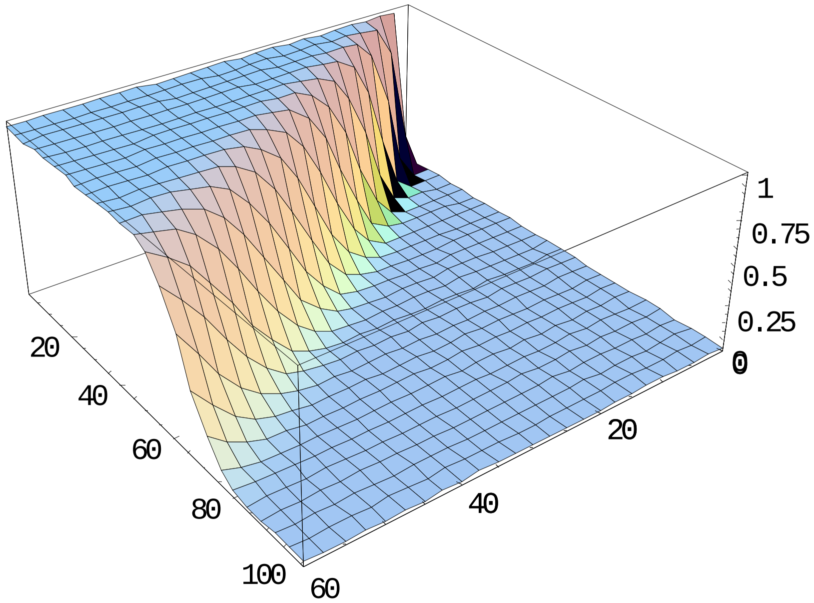


Figure 8.6. The cumulative density function of the minimum component of a tri-variate gamma vector, for $60 \geq x \geq 0$, $100 \geq a > 0$, and $\rho_{1,2} = 60\%$, $\rho_{1,3} = 60\%$ and $\rho_{2,3} = 95\%$, the non-diagonal elements of C_{123}.

$$F_{X_{(1)}}(x; a, C_{123}) = 3F_{X_1}(x) - \sum_{i_1=1}^{2}\sum_{i_2=2}^{3} F_{X_{i_1}, X_{i_2}}(x, x) + F_{X_1, X_2, X_3}(x, x, x)$$

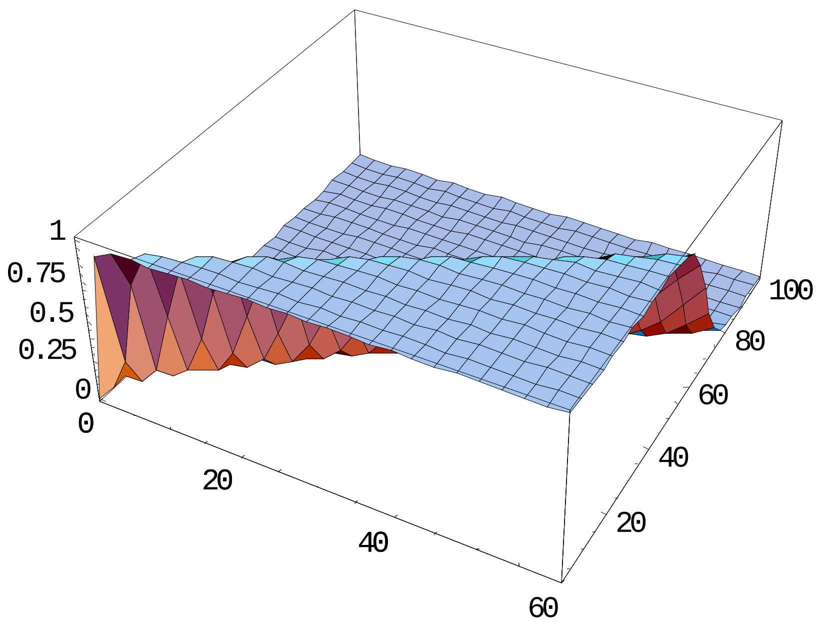


Figure 8.7. The cumulative density function of the minimum component of a tri-variate gamma vector, for $60 \geq x \geq 0$, $100 \geq a > 0$, and $\rho_{1,2} = 60\%$, $\rho_{1,3} = 95\%$ and $\rho_{2,3} = 95\%$, the non-diagonal elements of C_{123}.

$$F_{X_{(1)}}(x; a, C_{123}) = 3F_{X_1}(x) - \sum_{i_1=1}^{2} \sum_{i_2=2}^{3} F_{X_{i_1}, X_{i_2}}(x, x) + F_{X_1, X_2, X_3}(x, x, x)$$

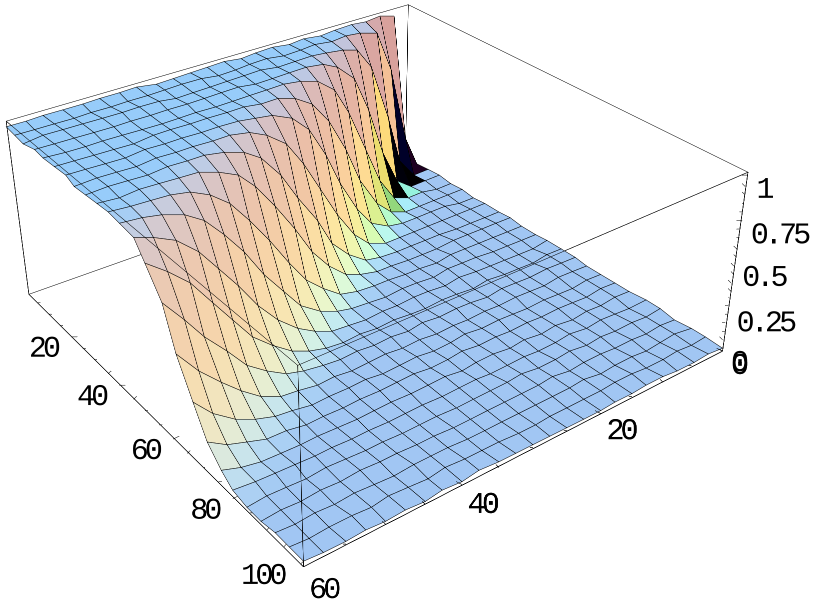


Figure 8.8. The cumulative density function of the minimum component of a tri-variate gamma vector, for $60 \geq x \geq 0$, $100 \geq a > 0$, and $\rho_{1,2} = 95\%$, $\rho_{1,3} = 95\%$ and $\rho_{2,3} = 95\%$, the non-diagonal elements of C_{123}.

$$F_{X_{(1)}}(x; a, C_{123}) = 3F_{X_1}(x) - \sum_{i_1=1}^{2} \sum_{i_2=2}^{3} F_{X_{i_1}, X_{i_2}}(x, x) + F_{X_1, X_2, X_3}(x, x, x)$$

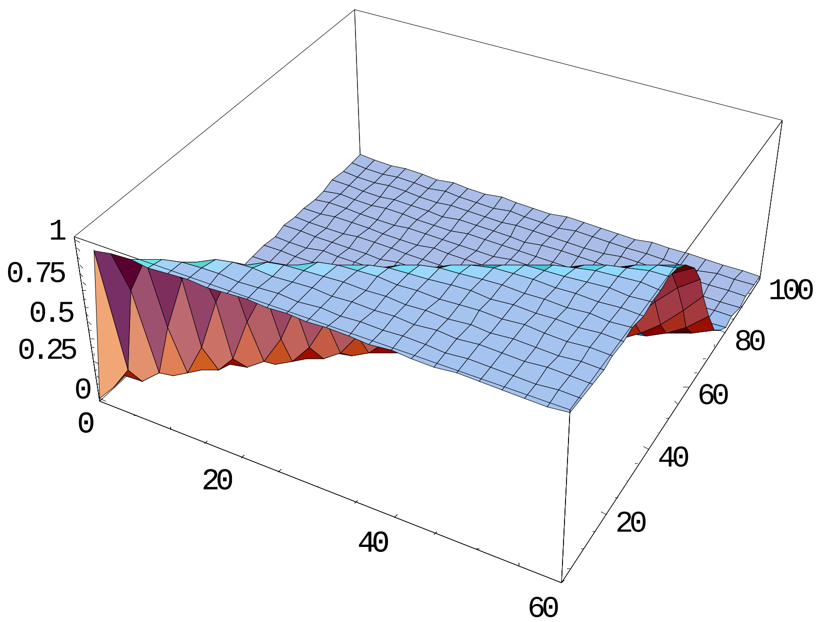
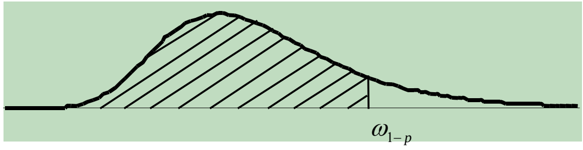


Table 8.3 depicts the probability $(1-p)$ that the minimum $X_{(1)}$ of a trivariate gamma vector is less than or equal to ω_{1-p} for $2 \geq \omega_{1-p} \geq 50$, $5 \geq a \geq 50$, and $\rho_{1,2} = 5\%$, $\rho_{1,3} = 5\%$ and $\rho_{2,3} = 5\%$, the non-diagonal elements of C_{123}.

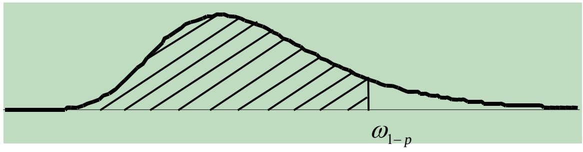
$$F_{X_{(1)}}\left(\omega_{1-p}; a, C_{123}\right) = P\left(X_{(1)} \leq \omega_{1-p}\right) = 1-p$$



	a									
ω_{1-p}	5	10	15	20	25	30	35	40	45	50
2	0.1497	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.7508	0.0242	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.9766	0.2311	0.0042	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8	0.9990	0.6314	0.0509	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	1.0000	0.9034	0.2299	0.0103	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
12	1.0000	0.9856	0.5394	0.0625	0.0021	0.0000	0.0000	0.0000	0.0000	0.0000
14	1.0000	0.9987	0.8137	0.2123	0.0150	0.0004	0.0000	0.0000	0.0000	0.0000
16	1.0000	0.9999	0.9499	0.4637	0.0654	0.0034	0.0001	0.0000	0.0000	0.0000
18	1.0000	1.0000	0.9908	0.7236	0.1910	0.0177	0.0007	0.0000	0.0000	0.0000
20	1.0000	1.0000	0.9988	0.8954	0.4001	0.0640	0.0045	0.0002	0.0000	0.0000
22	1.0000	1.0000	0.9999	0.9710	0.6389	0.1699	0.0190	0.0011	0.0000	0.0000
24	1.0000	1.0000	1.0000	0.9940	0.8293	0.3460	0.0604	0.0052	0.0002	0.0000
26	1.0000	1.0000	1.0000	0.9991	0.9375	0.5618	0.1502	0.0192	0.0013	0.0001
28	1.0000	1.0000	1.0000	0.9999	0.9822	0.7580	0.2998	0.0559	0.0056	0.0003
30	1.0000	1.0000	1.0000	1.0000	0.9960	0.8918	0.4927	0.1324	0.0187	0.0016
32	1.0000	1.0000	1.0000	1.0000	0.9993	0.9610	0.6860	0.2601	0.0510	0.0058
34	1.0000	1.0000	1.0000	1.0000	0.9999	0.9887	0.8371	0.4314	0.1164	0.0178
36	1.0000	1.0000	1.0000	1.0000	1.0000	0.9973	0.9298	0.6163	0.2259	0.0461
38	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9750	0.7769	0.3774	0.1023
40	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9926	0.8895	0.5506	0.1964
42	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9982	0.9537	0.7142	0.3299
44	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996	0.9836	0.8419	0.4898
46	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9951	0.9245	0.6515
48	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9987	0.9690	0.7891
50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9890	0.8880

Table 8.4 depicts the probability $(1-p)$ that the minimum $X_{(1)}$ of a trivariate gamma vector is less than or equal to ω_{1-p} for $2 \geq \omega_{1-p} \geq 50$, $5 \geq a \geq 50$, and $\rho_{1,2} = 5\%$, $\rho_{1,3} = 5\%$ and $\rho_{2,3} = 30\%$, the non-diagonal elements of C_{123}.

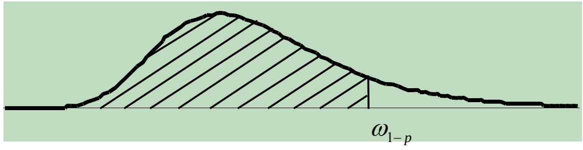
$$F_{X_{(1)}}\left(\omega_{1-p}; a, C_{123}\right) = P\left(X_{(1)} \leq \omega_{1-p}\right) = 1-p$$



	a									
ω_{1-p}	5	10	15	20	25	30	35	40	45	50
2	0.1491	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.7438	0.0242	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.9733	0.2296	0.0042	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8	0.9986	0.6250	0.0507	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	1.0000	0.8971	0.2283	0.0103	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
12	1.0000	0.9832	0.5338	0.0623	0.0021	0.0000	0.0000	0.0000	0.0000	0.0000
14	1.0000	0.9982	0.8063	0.2108	0.0150	0.0004	0.0000	0.0000	0.0000	0.0000
16	1.0000	0.9999	0.9452	0.4590	0.0652	0.0034	0.0001	0.0000	0.0000	0.0000
18	1.0000	1.0000	0.9891	0.7162	0.1897	0.0177	0.0007	0.0000	0.0000	0.0000
20	1.0000	1.0000	0.9984	0.8889	0.3962	0.0638	0.0045	0.0002	0.0000	0.0000
22	1.0000	1.0000	0.9998	0.9675	0.6321	0.1688	0.0189	0.0011	0.0000	0.0000
24	1.0000	1.0000	1.0000	0.9928	0.8219	0.3428	0.0602	0.0052	0.0002	0.0000
26	1.0000	1.0000	1.0000	0.9987	0.9322	0.5557	0.1493	0.0191	0.0013	0.0001
28	1.0000	1.0000	1.0000	0.9998	0.9796	0.7504	0.2972	0.0557	0.0056	0.0003
30	1.0000	1.0000	1.0000	1.0000	0.9951	0.8852	0.4875	0.1316	0.0187	0.0016
32	1.0000	1.0000	1.0000	1.0000	0.9990	0.9569	0.6787	0.2579	0.0508	0.0058
34	1.0000	1.0000	1.0000	1.0000	0.9998	0.9867	0.8297	0.4270	0.1158	0.0178
36	1.0000	1.0000	1.0000	1.0000	1.0000	0.9966	0.9243	0.6096	0.2242	0.0460
38	1.0000	1.0000	1.0000	1.0000	1.0000	0.9993	0.9718	0.7692	0.3736	0.1018
40	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9912	0.8829	0.5446	0.1950
42	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9976	0.9492	0.7067	0.3268
44	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9811	0.8346	0.4845
46	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9940	0.9188	0.6444
48	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9983	0.9654	0.7815
50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996	0.9872	0.8813

Table 8.5 depicts the probability $(1-p)$ that the minimum $X_{(1)}$ of a trivariate gamma vector is less than or equal to ω_{1-p} for $2 \geq \omega_{1-p} \geq 50$, $5 \geq a \geq 50$, and $\rho_{1,2} = 5\%$, $\rho_{1,3} = 5\%$ and $\rho_{2,3} = 60\%$, the non-diagonal elements of C_{123}.

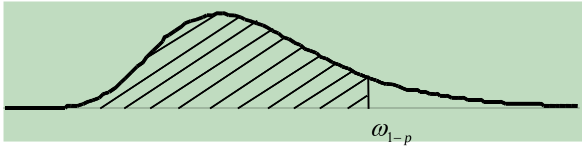
$$F_{X_{(1)}}\left(\omega_{1-p}; a, C_{123}\right) = P\left(X_{(1)} \leq \omega_{1-p}\right) = 1-p$$



	a									
ω_{1-p}	5	10	15	20	25	30	35	40	45	50
2	0.1461	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.7199	0.0240	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.9631	0.2227	0.0042	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8	0.9972	0.6024	0.0500	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.9999	0.8771	0.2211	0.0103	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
12	1.0000	0.9756	0.5137	0.0612	0.0021	0.0000	0.0000	0.0000	0.0000	0.0000
14	1.0000	0.9966	0.7823	0.2042	0.0148	0.0004	0.0000	0.0000	0.0000	0.0000
16	1.0000	0.9997	0.9301	0.4415	0.0640	0.0034	0.0001	0.0000	0.0000	0.0000
18	1.0000	1.0000	0.9835	0.6916	0.1839	0.0175	0.0007	0.0000	0.0000	0.0000
20	1.0000	1.0000	0.9970	0.8683	0.3812	0.0626	0.0044	0.0002	0.0000	0.0000
22	1.0000	1.0000	0.9996	0.9562	0.6087	0.1638	0.0187	0.0011	0.0000	0.0000
24	1.0000	1.0000	0.9999	0.9885	0.7981	0.3300	0.0591	0.0052	0.0002	0.0000
26	1.0000	1.0000	1.0000	0.9976	0.9155	0.5344	0.1450	0.0189	0.0013	0.0001
28	1.0000	1.0000	1.0000	0.9996	0.9711	0.7256	0.2863	0.0547	0.0056	0.0003
30	1.0000	1.0000	1.0000	0.9999	0.9919	0.8642	0.4685	0.1280	0.0185	0.0016
32	1.0000	1.0000	1.0000	1.0000	0.9981	0.9437	0.6543	0.2488	0.0499	0.0058
34	1.0000	1.0000	1.0000	1.0000	0.9996	0.9803	0.8061	0.4104	0.1127	0.0176
36	1.0000	1.0000	1.0000	1.0000	0.9999	0.9942	0.9067	0.5866	0.2165	0.0452
38	1.0000	1.0000	1.0000	1.0000	1.0000	0.9985	0.9615	0.7445	0.3593	0.0992
40	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9863	0.8618	0.5234	0.1886
42	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9958	0.9348	0.6819	0.3145
44	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9989	0.9731	0.8111	0.4655
46	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9903	0.9006	0.6205
48	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9969	0.9538	0.7568
50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9991	0.9810	0.8601

Table 8.6 depicts the probability $(1-p)$ that the minimum $X_{(1)}$ of a trivariate gamma vector is less than or equal to ω_{1-p} for $2 \geq \omega_{1-p} \geq 50$, $5 \geq a \geq 50$, and $\rho_{1,2} = 5\%$, $\rho_{1,3} = 5\%$ and $\rho_{2,3} = 95\%$, the non-diagonal elements of C_{123}.

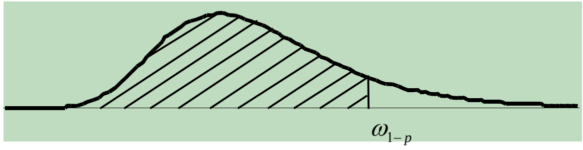
$$F_{X_{(1)}}(\omega_{1-p}; a, C_{123}) = P(X_{(1)} \leq \omega_{1-p}) = 1-p$$



	a									
ω_{1-p}	5	10	15	20	25	30	35	40	45	50
2	0.1317	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.6599	0.0227	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.9395	0.1943	0.0041	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8	0.9936	0.5434	0.0456	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.9995	0.8320	0.1922	0.0098	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
12	1.0000	0.9569	0.4584	0.0550	0.0020	0.0000	0.0000	0.0000	0.0000	0.0000
14	1.0000	0.9924	0.7275	0.1773	0.0140	0.0004	0.0000	0.0000	0.0000	0.0000
16	1.0000	0.9990	0.8948	0.3908	0.0572	0.0033	0.0001	0.0000	0.0000	0.0000
18	1.0000	0.9999	0.9692	0.6338	0.1596	0.0164	0.0007	0.0000	0.0000	0.0000
20	1.0000	1.0000	0.9932	0.8208	0.3353	0.0558	0.0043	0.0002	0.0000	0.0000
22	1.0000	1.0000	0.9987	0.9290	0.5514	0.1422	0.0175	0.0011	0.0000	0.0000
24	1.0000	1.0000	0.9998	0.9777	0.7437	0.2889	0.0526	0.0050	0.0002	0.0000
26	1.0000	1.0000	1.0000	0.9943	0.8761	0.4796	0.1260	0.0176	0.0013	0.0001
28	1.0000	1.0000	1.0000	0.9987	0.9503	0.6681	0.2497	0.0488	0.0054	0.0003
30	1.0000	1.0000	1.0000	0.9998	0.9836	0.8157	0.4171	0.1113	0.0171	0.0015
32	1.0000	1.0000	1.0000	1.0000	0.9954	0.9120	0.5965	0.2164	0.0446	0.0055
34	1.0000	1.0000	1.0000	1.0000	0.9988	0.9644	0.7517	0.3630	0.0982	0.0163
36	1.0000	1.0000	1.0000	1.0000	0.9997	0.9877	0.8652	0.5301	0.1879	0.0405
38	1.0000	1.0000	1.0000	1.0000	0.9999	0.9963	0.9364	0.6871	0.3160	0.0866
40	1.0000	1.0000	1.0000	1.0000	1.0000	0.9990	0.9741	0.8127	0.4695	0.1635
42	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9907	0.9003	0.6238	0.2753
44	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9970	0.9534	0.7568	0.4148
46	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9991	0.9808	0.8578	0.5631
48	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9929	0.9256	0.6995
50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9976	0.9655	0.8106

Table 8.7 depicts the probability $(1-p)$ that the minimum $X_{(1)}$ of a trivariate gamma vector is less than or equal to ω_{1-p} for $2 \geq \omega_{1-p} \geq 50$, $5 \geq a \geq 50$, and $\rho_{1,2} = 5\%$, $\rho_{1,3} = 30\%$ and $\rho_{2,3} = 30\%$, the non-diagonal elements of C_{123}.

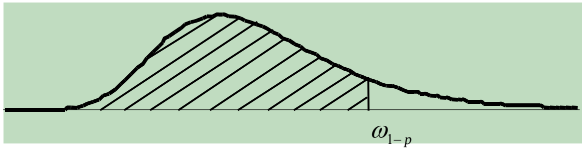
$$F_{X_{(1)}}(\omega_{1-p}; a, C_{123}) = P(X_{(1)} \leq \omega_{1-p}) = 1-p$$



	a									
ω_{1-p}	5	10	15	20	25	30	35	40	45	50
2	0.1485	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.7370	0.0241	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.9699	0.2281	0.0042	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8	0.9981	0.6188	0.0506	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.9999	0.8908	0.2267	0.0103	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
12	1.0000	0.9807	0.5284	0.0621	0.0021	0.0000	0.0000	0.0000	0.0000	0.0000
14	1.0000	0.9977	0.7990	0.2094	0.0150	0.0004	0.0000	0.0000	0.0000	0.0000
16	1.0000	0.9998	0.9403	0.4545	0.0650	0.0034	0.0001	0.0000	0.0000	0.0000
18	1.0000	1.0000	0.9872	0.7089	0.1885	0.0177	0.0007	0.0000	0.0000	0.0000
20	1.0000	1.0000	0.9980	0.8824	0.3924	0.0636	0.0045	0.0002	0.0000	0.0000
22	1.0000	1.0000	0.9997	0.9638	0.6254	0.1678	0.0189	0.0011	0.0000	0.0000
24	1.0000	1.0000	1.0000	0.9914	0.8146	0.3397	0.0600	0.0052	0.0002	0.0000
26	1.0000	1.0000	1.0000	0.9984	0.9268	0.5498	0.1484	0.0191	0.0013	0.0001
28	1.0000	1.0000	1.0000	0.9997	0.9768	0.7429	0.2946	0.0555	0.0056	0.0003
30	1.0000	1.0000	1.0000	1.0000	0.9940	0.8786	0.4824	0.1309	0.0186	0.0016
32	1.0000	1.0000	1.0000	1.0000	0.9987	0.9526	0.6716	0.2558	0.0507	0.0058
34	1.0000	1.0000	1.0000	1.0000	0.9998	0.9846	0.8224	0.4226	0.1152	0.0178
36	1.0000	1.0000	1.0000	1.0000	1.0000	0.9958	0.9186	0.6030	0.2224	0.0459
38	1.0000	1.0000	1.0000	1.0000	1.0000	0.9990	0.9684	0.7617	0.3700	0.1013
40	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9896	0.8762	0.5387	0.1936
42	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9970	0.9445	0.6993	0.3237
44	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9993	0.9785	0.8273	0.4794
46	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9928	0.9130	0.6374
48	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9979	0.9616	0.7739
50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9851	0.8746

Table 8.8 depicts the probability $\left(1-p\right)$ that the minimum $X_{(1)}$ of a trivariate gamma vector is less than or equal to ω_{1-p} for $2 \geq \omega_{1-p} \geq 50$, $5 \geq a \geq 50$, and $\rho_{1,2} = 5\%$, $\rho_{1,3} = 30\%$ and $\rho_{2,3} = 60\%$, the non-diagonal elements of C_{123}.

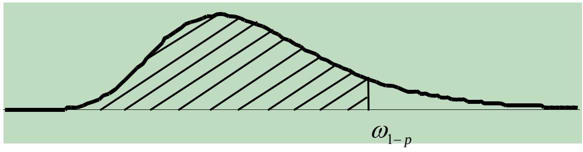
$$F_{X_{(1)}}\left(\omega_{1-p}; a, C_{123}\right) = P\left(X_{(1)} \leq \omega_{1-p}\right) = 1-p$$



	a									
ω_{1-p}	5	10	15	20	25	30	35	40	45	50
2	0.1455	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.7139	0.0239	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.9596	0.2213	0.0042	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8	0.9966	0.5968	0.0499	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.9998	0.8710	0.2197	0.0103	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
12	1.0000	0.9728	0.5088	0.0610	0.0021	0.0000	0.0000	0.0000	0.0000	0.0000
14	1.0000	0.9959	0.7755	0.2028	0.0148	0.0004	0.0000	0.0000	0.0000	0.0000
16	1.0000	0.9995	0.9252	0.4374	0.0638	0.0034	0.0001	0.0000	0.0000	0.0000
18	1.0000	1.0000	0.9813	0.6849	0.1827	0.0175	0.0007	0.0000	0.0000	0.0000
20	1.0000	1.0000	0.9963	0.8619	0.3777	0.0624	0.0044	0.0002	0.0000	0.0000
22	1.0000	1.0000	0.9994	0.9523	0.6026	0.1628	0.0187	0.0011	0.0000	0.0000
24	1.0000	1.0000	0.9999	0.9868	0.7912	0.3272	0.0589	0.0052	0.0002	0.0000
26	1.0000	1.0000	1.0000	0.9970	0.9100	0.5290	0.1442	0.0189	0.0013	0.0001
28	1.0000	1.0000	1.0000	0.9994	0.9680	0.7186	0.2840	0.0545	0.0056	0.0003
30	1.0000	1.0000	1.0000	0.9999	0.9905	0.8578	0.4638	0.1273	0.0184	0.0016
32	1.0000	1.0000	1.0000	1.0000	0.9976	0.9392	0.6477	0.2469	0.0498	0.0058
34	1.0000	1.0000	1.0000	1.0000	0.9995	0.9779	0.7991	0.4064	0.1122	0.0176
36	1.0000	1.0000	1.0000	1.0000	0.9999	0.9931	0.9010	0.5805	0.2149	0.0451
38	1.0000	1.0000	1.0000	1.0000	1.0000	0.9981	0.9578	0.7374	0.3559	0.0987
40	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996	0.9844	0.8553	0.5180	0.1873
42	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9949	0.9299	0.6751	0.3117
44	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9986	0.9701	0.8041	0.4608
46	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996	0.9888	0.8948	0.6140
48	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9963	0.9497	0.7496
50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9989	0.9786	0.8535

Table 8.9 depicts the probability $(1-p)$ that the minimum $X_{(1)}$ of a trivariate gamma vector is less than or equal to ω_{1-p} for $2 \geq \omega_{1-p} \geq 50$, $5 \geq a \geq 50$, and $\rho_{1,2} = 5\%$, $\rho_{1,3} = 30\%$ and $\rho_{2,3} = 95\%$, the non-diagonal elements of C_{123}.

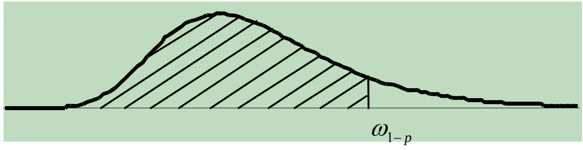
$$F_{X_{(1)}}(\omega_{1-p}; a, C_{123}) = P(X_{(1)} \leq \omega_{1-p}) = 1-p$$



	a									
ω_{1-p}	5	10	15	20	25	30	35	40	45	50
2	0.1313	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.6554	0.0227	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.9363	0.1934	0.0041	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8	0.9926	0.5390	0.0455	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.9994	0.8267	0.1912	0.0098	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
12	1.0000	0.9539	0.4546	0.0548	0.0020	0.0000	0.0000	0.0000	0.0000	0.0000
14	1.0000	0.9913	0.7217	0.1764	0.0140	0.0004	0.0000	0.0000	0.0000	0.0000
16	1.0000	0.9987	0.8901	0.3876	0.0570	0.0033	0.0001	0.0000	0.0000	0.0000
18	1.0000	0.9998	0.9665	0.6282	0.1588	0.0164	0.0007	0.0000	0.0000	0.0000
20	1.0000	1.0000	0.9921	0.8152	0.3326	0.0556	0.0043	0.0002	0.0000	0.0000
22	1.0000	1.0000	0.9985	0.9248	0.5464	0.1415	0.0174	0.0011	0.0000	0.0000
24	1.0000	1.0000	0.9997	0.9753	0.7377	0.2867	0.0525	0.0050	0.0002	0.0000
26	1.0000	1.0000	1.0000	0.9934	0.8710	0.4752	0.1254	0.0176	0.0013	0.0001
28	1.0000	1.0000	1.0000	0.9985	0.9467	0.6622	0.2479	0.0486	0.0054	0.0003
30	1.0000	1.0000	1.0000	0.9997	0.9816	0.8099	0.4133	0.1108	0.0171	0.0015
32	1.0000	1.0000	1.0000	0.9999	0.9946	0.9073	0.5909	0.2149	0.0445	0.0055
34	1.0000	1.0000	1.0000	1.0000	0.9986	0.9613	0.7457	0.3598	0.0978	0.0163
36	1.0000	1.0000	1.0000	1.0000	0.9997	0.9861	0.8597	0.5251	0.1867	0.0404
38	1.0000	1.0000	1.0000	1.0000	0.9999	0.9956	0.9322	0.6811	0.3133	0.0863
40	1.0000	1.0000	1.0000	1.0000	1.0000	0.9987	0.9714	0.8067	0.4650	0.1625
42	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9894	0.8952	0.6180	0.2731
44	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9965	0.9497	0.7507	0.4109
46	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9989	0.9787	0.8521	0.5577
48	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9919	0.9211	0.6934
50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9972	0.9624	0.8045

Table 8.10 depicts the probability $(1-p)$ that the minimum $X_{(1)}$ of a trivariate gamma vector is less than or equal to ω_{1-p} for $2 \geq \omega_{1-p} \geq 50$, $5 \geq a \geq 50$, and $\rho_{1,2} = 5\%$, $\rho_{1,3} = 60\%$ and $\rho_{2,3} = 60\%$, the non-diagonal elements of C_{123}.

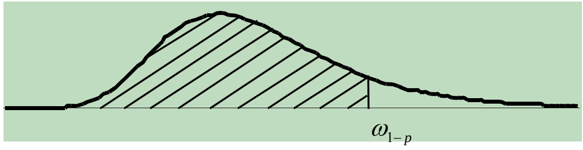
$$F_{X_{(1)}}\left(\omega_{1-p}; a, C_{123}\right) = P\left(X_{(1)} \leq \omega_{1-p}\right) = 1 - p$$



	a									
ω_{1-p}	5	10	15	20	25	30	35	40	45	50
2	0.1428	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.6934	0.0238	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.9493	0.2151	0.0042	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8	0.9947	0.5769	0.0492	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.9996	0.8523	0.2131	0.0102	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
12	1.0000	0.9644	0.4909	0.0599	0.0020	0.0000	0.0000	0.0000	0.0000	0.0000
14	1.0000	0.9937	0.7535	0.1968	0.0147	0.0004	0.0000	0.0000	0.0000	0.0000
16	1.0000	0.9991	0.9101	0.4217	0.0626	0.0034	0.0001	0.0000	0.0000	0.0000
18	1.0000	0.9999	0.9746	0.6627	0.1773	0.0174	0.0007	0.0000	0.0000	0.0000
20	1.0000	1.0000	0.9943	0.8422	0.3641	0.0612	0.0044	0.0002	0.0000	0.0000
22	1.0000	1.0000	0.9989	0.9403	0.5815	0.1581	0.0185	0.0011	0.0000	0.0000
24	1.0000	1.0000	0.9998	0.9815	0.7691	0.3156	0.0578	0.0051	0.0002	0.0000
26	1.0000	1.0000	1.0000	0.9952	0.8933	0.5098	0.1401	0.0187	0.0013	0.0001
28	1.0000	1.0000	1.0000	0.9989	0.9585	0.6959	0.2741	0.0535	0.0056	0.0003
30	1.0000	1.0000	1.0000	0.9998	0.9863	0.8376	0.4467	0.1239	0.0182	0.0015
32	1.0000	1.0000	1.0000	1.0000	0.9961	0.9253	0.6255	0.2386	0.0489	0.0057
34	1.0000	1.0000	1.0000	1.0000	0.9990	0.9703	0.7770	0.3914	0.1093	0.0174
36	1.0000	1.0000	1.0000	1.0000	0.9998	0.9897	0.8834	0.5596	0.2079	0.0443
38	1.0000	1.0000	1.0000	1.0000	1.0000	0.9969	0.9464	0.7145	0.3429	0.0963
40	1.0000	1.0000	1.0000	1.0000	1.0000	0.9992	0.9783	0.8348	0.4989	0.1814
42	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9922	0.9149	0.6524	0.3005
44	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9975	0.9609	0.7819	0.4436
46	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9993	0.9839	0.8766	0.5923
48	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9941	0.9371	0.7266
50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9980	0.9711	0.8329

Table 8.11 depicts the probability $(1-p)$ that the minimum $X_{(1)}$ of a trivariate gamma vector is less than or equal to ω_{1-p} for $2 \geq \omega_{1-p} \geq 50$, $5 \geq a \geq 50$, and $\rho_{1,2} = 5\%$, $\rho_{1,3} = 60\%$ and $\rho_{2,3} = 95\%$, the non-diagonal elements of C_{123}.

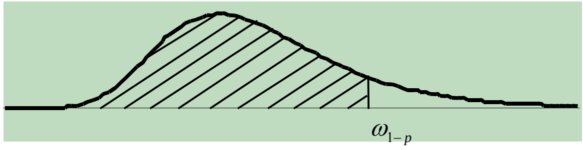
$$F_{X_{(1)}}\left(\omega_{1-p}; a, C_{123}\right) = P\left(X_{(1)} \leq \omega_{1-p}\right) = 1-p$$



	a									
ω_{1-p}	5	10	15	20	25	30	35	40	45	50
2	0.1297	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.6396	0.0226	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.9281	0.1892	0.0041	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8	0.9907	0.5227	0.0450	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.9991	0.8111	0.1864	0.0098	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
12	0.9999	0.9462	0.4397	0.0540	0.0020	0.0000	0.0000	0.0000	0.0000	0.0000
14	1.0000	0.9885	0.7031	0.1717	0.0139	0.0004	0.0000	0.0000	0.0000	0.0000
16	1.0000	0.9981	0.8769	0.3744	0.0561	0.0033	0.0001	0.0000	0.0000	0.0000
18	1.0000	0.9998	0.9594	0.6092	0.1545	0.0163	0.0007	0.0000	0.0000	0.0000
20	1.0000	1.0000	0.9894	0.7983	0.3212	0.0547	0.0043	0.0002	0.0000	0.0000
22	1.0000	1.0000	0.9978	0.9133	0.5282	0.1377	0.0173	0.0011	0.0000	0.0000
24	1.0000	1.0000	0.9996	0.9692	0.7188	0.2769	0.0516	0.0050	0.0002	0.0000
26	1.0000	1.0000	0.9999	0.9909	0.8558	0.4585	0.1221	0.0174	0.0013	0.0001
28	1.0000	1.0000	1.0000	0.9977	0.9367	0.6426	0.2396	0.0478	0.0053	0.0003
30	1.0000	1.0000	1.0000	0.9995	0.9764	0.7922	0.3985	0.1080	0.0170	0.0015
32	1.0000	1.0000	1.0000	0.9999	0.9925	0.8939	0.5718	0.2079	0.0438	0.0055
34	1.0000	1.0000	1.0000	1.0000	0.9979	0.9527	0.7265	0.3468	0.0954	0.0162
36	1.0000	1.0000	1.0000	1.0000	0.9995	0.9817	0.8436	0.5069	0.1808	0.0397
38	1.0000	1.0000	1.0000	1.0000	0.9999	0.9938	0.9204	0.6613	0.3021	0.0843
40	1.0000	1.0000	1.0000	1.0000	1.0000	0.9981	0.9642	0.7886	0.4483	0.1576
42	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9858	0.8806	0.5983	0.2634
44	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9950	0.9395	0.7312	0.3958
46	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9984	0.9726	0.8352	0.5388
48	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9889	0.9080	0.6734
50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9959	0.9536	0.7860

Table 8.12 depicts the probability $(1-p)$ that the minimum $X_{(1)}$ of a trivariate gamma vector is less than or equal to ω_{1-p} for $2 \geq \omega_{1-p} \geq 50$, $5 \geq a \geq 50$, and $\rho_{1,2} = 5\%$, $\rho_{1,3} = 95\%$ and $\rho_{2,3} = 95\%$, the non-diagonal elements of C_{123}.

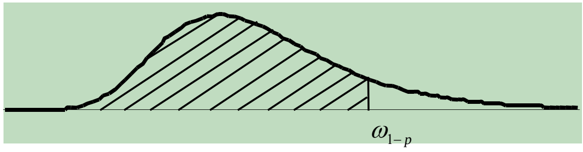
$$F_{X_{(1)}}\left(\omega_{1-p}; a, C_{123}\right) = P\left(X_{(1)} \leq \omega_{1-p}\right) = 1-p$$



	a									
ω_{1-p}	5	10	15	20	25	30	35	40	45	50
2	0.1223	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.5895	0.0216	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.9171	0.1695	0.0040	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8	0.9904	0.4708	0.0417	0.0007	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.9992	0.7790	0.1635	0.0094	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
12	1.0000	0.9369	0.3898	0.0492	0.0020	0.0000	0.0000	0.0000	0.0000	0.0000
14	1.0000	0.9851	0.6594	0.1493	0.0132	0.0004	0.0000	0.0000	0.0000	0.0000
16	1.0000	0.9976	0.8546	0.3281	0.0507	0.0032	0.0001	0.0000	0.0000	0.0000
18	1.0000	0.9998	0.9487	0.5602	0.1339	0.0153	0.0007	0.0000	0.0000	0.0000
20	1.0000	1.0000	0.9853	0.7644	0.2789	0.0492	0.0042	0.0002	0.0000	0.0000
22	1.0000	1.0000	0.9969	0.8942	0.4778	0.1191	0.0162	0.0010	0.0000	0.0000
24	1.0000	1.0000	0.9996	0.9586	0.6767	0.2388	0.0463	0.0048	0.0002	0.0000
26	1.0000	1.0000	1.0000	0.9870	0.8281	0.4090	0.1056	0.0162	0.0013	0.0001
28	1.0000	1.0000	1.0000	0.9969	0.9190	0.5955	0.2056	0.0429	0.0051	0.0003
30	1.0000	1.0000	1.0000	0.9995	0.9670	0.7567	0.3512	0.0935	0.0158	0.0015
32	1.0000	1.0000	1.0000	1.0000	0.9891	0.8693	0.5223	0.1779	0.0393	0.0053
34	1.0000	1.0000	1.0000	1.0000	0.9971	0.9368	0.6847	0.3026	0.0828	0.0151
36	1.0000	1.0000	1.0000	1.0000	0.9994	0.9738	0.8125	0.4571	0.1544	0.0358
38	1.0000	1.0000	1.0000	1.0000	0.9999	0.9910	0.8980	0.6151	0.2614	0.0733
40	1.0000	1.0000	1.0000	1.0000	1.0000	0.9975	0.9504	0.7516	0.3996	0.1345
42	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9793	0.8525	0.5497	0.2265
44	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9927	0.9195	0.6893	0.3491
46	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9978	0.9610	0.8019	0.4892
48	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9837	0.8824	0.6278
50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9941	0.9362	0.7479

Table 8.13 depicts the probability $(1-p)$ that the minimum $X_{(1)}$ of a trivariate gamma vector is less than or equal to ω_{1-p} for $2 \geq \omega_{1-p} \geq 50$, $5 \geq a \geq 50$, and $\rho_{1,2} = 30\%$, $\rho_{1,3} = 30\%$ and $\rho_{2,3} = 30\%$, the non-diagonal elements of C_{123}.

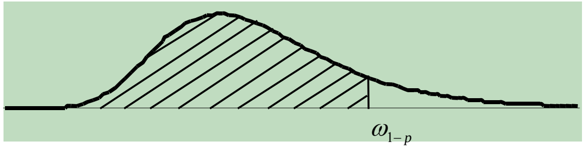
$$F_{X_{(1)}}\left(\omega_{1-p}; a, C_{123}\right) = P\left(X_{(1)} \leq \omega_{1-p}\right) = 1-p$$



	a									
ω_{1-p}	5	10	15	20	25	30	35	40	45	50
2	0.1479	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.7297	0.0241	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.9658	0.2266	0.0042	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8	0.9974	0.6123	0.0505	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.9999	0.8839	0.2252	0.0103	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
12	1.0000	0.9776	0.5229	0.0619	0.0021	0.0000	0.0000	0.0000	0.0000	0.0000
14	1.0000	0.9969	0.7914	0.2079	0.0149	0.0004	0.0000	0.0000	0.0000	0.0000
16	1.0000	0.9997	0.9349	0.4499	0.0648	0.0034	0.0001	0.0000	0.0000	0.0000
18	1.0000	1.0000	0.9850	0.7015	0.1872	0.0177	0.0007	0.0000	0.0000	0.0000
20	1.0000	1.0000	0.9973	0.8755	0.3886	0.0634	0.0045	0.0002	0.0000	0.0000
22	1.0000	1.0000	0.9996	0.9597	0.6186	0.1667	0.0189	0.0011	0.0000	0.0000
24	1.0000	1.0000	1.0000	0.9897	0.8070	0.3366	0.0598	0.0052	0.0002	0.0000
26	1.0000	1.0000	1.0000	0.9978	0.9210	0.5439	0.1476	0.0191	0.0013	0.0001
28	1.0000	1.0000	1.0000	0.9996	0.9737	0.7353	0.2920	0.0553	0.0056	0.0003
30	1.0000	1.0000	1.0000	0.9999	0.9927	0.8716	0.4773	0.1302	0.0186	0.0016
32	1.0000	1.0000	1.0000	1.0000	0.9983	0.9479	0.6643	0.2537	0.0505	0.0058
34	1.0000	1.0000	1.0000	1.0000	0.9997	0.9822	0.8148	0.4183	0.1146	0.0177
36	1.0000	1.0000	1.0000	1.0000	0.9999	0.9948	0.9126	0.5964	0.2207	0.0457
38	1.0000	1.0000	1.0000	1.0000	1.0000	0.9987	0.9647	0.7540	0.3664	0.1008
40	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9877	0.8693	0.5328	0.1922
42	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9963	0.9395	0.6919	0.3207
44	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9990	0.9756	0.8198	0.4743
46	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9913	0.9068	0.6304
48	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9973	0.9575	0.7662
50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9993	0.9828	0.8676

Table 8.14 depicts the probability $(1-p)$ that the minimum $X_{(1)}$ of a trivariate gamma vector is less than or equal to ω_{1-p} for $2 \geq \omega_{1-p} \geq 50$, $5 \geq a \geq 50$, and $\rho_{1,2} = 30\%$, $\rho_{1,3} = 30\%$ and $\rho_{2,3} = 60\%$, the non-diagonal elements of C_{123}.

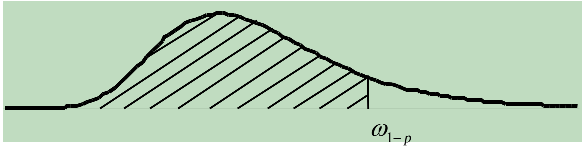
$$F_{X_{(1)}}\left(\omega_{1-p}; a, C_{123}\right) = P\left(X_{(1)} \leq \omega_{1-p}\right) = 1-p$$



	a									
ω_{1-p}	5	10	15	20	25	30	35	40	45	50
2	0.1449	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.7063	0.0239	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.9544	0.2199	0.0042	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8	0.9955	0.5904	0.0497	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.9997	0.8635	0.2181	0.0102	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
12	1.0000	0.9689	0.5034	0.0608	0.0021	0.0000	0.0000	0.0000	0.0000	0.0000
14	1.0000	0.9947	0.7676	0.2015	0.0148	0.0004	0.0000	0.0000	0.0000	0.0000
16	1.0000	0.9993	0.9190	0.4330	0.0636	0.0034	0.0001	0.0000	0.0000	0.0000
18	1.0000	0.9999	0.9783	0.6776	0.1815	0.0175	0.0007	0.0000	0.0000	0.0000
20	1.0000	1.0000	0.9953	0.8546	0.3741	0.0622	0.0044	0.0002	0.0000	0.0000
22	1.0000	1.0000	0.9992	0.9474	0.5960	0.1618	0.0187	0.0011	0.0000	0.0000
24	1.0000	1.0000	0.9999	0.9845	0.7835	0.3242	0.0587	0.0052	0.0002	0.0000
26	1.0000	1.0000	1.0000	0.9962	0.9037	0.5233	0.1433	0.0189	0.0013	0.0001
28	1.0000	1.0000	1.0000	0.9992	0.9641	0.7111	0.2816	0.0543	0.0056	0.0003
30	1.0000	1.0000	1.0000	0.9999	0.9887	0.8505	0.4590	0.1266	0.0184	0.0016
32	1.0000	1.0000	1.0000	1.0000	0.9969	0.9338	0.6407	0.2449	0.0496	0.0058
34	1.0000	1.0000	1.0000	1.0000	0.9993	0.9748	0.7915	0.4023	0.1116	0.0175
36	1.0000	1.0000	1.0000	1.0000	0.9998	0.9917	0.8945	0.5742	0.2133	0.0449
38	1.0000	1.0000	1.0000	1.0000	1.0000	0.9976	0.9534	0.7298	0.3525	0.0983
40	1.0000	1.0000	1.0000	1.0000	1.0000	0.9994	0.9819	0.8480	0.5125	0.1860
42	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9938	0.9243	0.6678	0.3089
44	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9981	0.9665	0.7965	0.4560
46	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9868	0.8882	0.6073
48	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9953	0.9449	0.7420
50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9985	0.9756	0.8463

Table 8.15 depicts the probability $(1-p)$ that the minimum $X_{(1)}$ of a trivariate gamma vector is less than or equal to ω_{1-p} for $2 \geq \omega_{1-p} \geq 50$, $5 \geq a \geq 50$, and $\rho_{1,2} = 30\%$, $\rho_{1,3} = 30\%$ and $\rho_{2,3} = 95\%$, the non-diagonal elements of C_{123}.

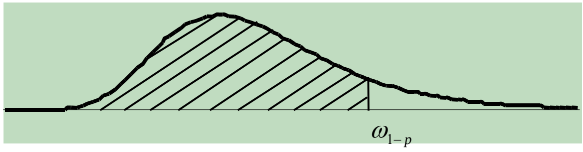
$$F_{X_{(1)}}\left(\omega_{1-p}; a, C_{123}\right) = P\left(X_{(1)} \leq \omega_{1-p}\right) = 1-p$$



	a									
ω_{1-p}	5	10	15	20	25	30	35	40	45	50
2	0.1306	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.6473	0.0227	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.9292	0.1920	0.0041	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8	0.9906	0.5326	0.0454	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.9991	0.8180	0.1899	0.0098	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
12	0.9999	0.9486	0.4495	0.0547	0.0020	0.0000	0.0000	0.0000	0.0000	0.0000
14	1.0000	0.9892	0.7133	0.1752	0.0140	0.0004	0.0000	0.0000	0.0000	0.0000
16	1.0000	0.9982	0.8828	0.3836	0.0568	0.0033	0.0001	0.0000	0.0000	0.0000
18	1.0000	0.9997	0.9622	0.6208	0.1578	0.0164	0.0007	0.0000	0.0000	0.0000
20	1.0000	1.0000	0.9902	0.8072	0.3294	0.0554	0.0043	0.0002	0.0000	0.0000
22	1.0000	1.0000	0.9979	0.9187	0.5400	0.1406	0.0174	0.0011	0.0000	0.0000
24	1.0000	1.0000	0.9996	0.9717	0.7297	0.2841	0.0524	0.0050	0.0002	0.0000
26	1.0000	1.0000	0.9999	0.9918	0.8637	0.4698	0.1247	0.0175	0.0013	0.0001
28	1.0000	1.0000	1.0000	0.9979	0.9414	0.6547	0.2458	0.0485	0.0054	0.0003
30	1.0000	1.0000	1.0000	0.9995	0.9786	0.8022	0.4089	0.1103	0.0171	0.0015
32	1.0000	1.0000	1.0000	0.9999	0.9932	0.9008	0.5842	0.2133	0.0444	0.0055
34	1.0000	1.0000	1.0000	1.0000	0.9981	0.9569	0.7379	0.3561	0.0974	0.0163
36	1.0000	1.0000	1.0000	1.0000	0.9995	0.9836	0.8525	0.5191	0.1854	0.0403
38	1.0000	1.0000	1.0000	1.0000	0.9999	0.9945	0.9265	0.6736	0.3104	0.0859
40	1.0000	1.0000	1.0000	1.0000	1.0000	0.9983	0.9677	0.7991	0.4599	0.1615
42	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9874	0.8886	0.6111	0.2707
44	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9955	0.9448	0.7430	0.4065
46	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9986	0.9755	0.8449	0.5515
48	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996	0.9902	0.9151	0.6860
50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9964	0.9581	0.7970

Table 8.16 depicts the probability $(1-p)$ that the minimum $X_{(1)}$ of a trivariate gamma vector is less than or equal to ω_{1-p} for $2 \geq \omega_{1-p} \geq 50$, $5 \geq a \geq 50$, and $\rho_{1,2} = 30\%$, $\rho_{1,3} = 60\%$ and $\rho_{2,3} = 60\%$, the non-diagonal elements of C_{123}.

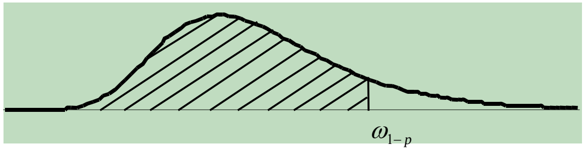
$$F_{X_{(1)}}\left(\omega_{1-p}; a, C_{123}\right) = P\left(X_{(1)} \leq \omega_{1-p}\right) = 1-p$$



	a									
ω_{1-p}	5	10	15	20	25	30	35	40	45	50
2	0.1421	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.6839	0.0237	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.9419	0.2136	0.0042	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8	0.9927	0.5697	0.0490	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.9993	0.8430	0.2116	0.0102	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
12	0.9999	0.9590	0.4852	0.0598	0.0020	0.0000	0.0000	0.0000	0.0000	0.0000
14	1.0000	0.9917	0.7446	0.1954	0.0147	0.0004	0.0000	0.0000	0.0000	0.0000
16	1.0000	0.9986	0.9024	0.4171	0.0624	0.0034	0.0001	0.0000	0.0000	0.0000
18	1.0000	0.9998	0.9703	0.6548	0.1761	0.0173	0.0007	0.0000	0.0000	0.0000
20	1.0000	1.0000	0.9926	0.8336	0.3605	0.0610	0.0044	0.0002	0.0000	0.0000
22	1.0000	1.0000	0.9984	0.9340	0.5747	0.1571	0.0185	0.0011	0.0000	0.0000
24	1.0000	1.0000	0.9997	0.9781	0.7605	0.3126	0.0576	0.0051	0.0002	0.0000
26	1.0000	1.0000	1.0000	0.9938	0.8857	0.5040	0.1393	0.0187	0.0013	0.0001
28	1.0000	1.0000	1.0000	0.9985	0.9533	0.6879	0.2718	0.0533	0.0056	0.0003
30	1.0000	1.0000	1.0000	0.9997	0.9835	0.8293	0.4419	0.1232	0.0182	0.0015
32	1.0000	1.0000	1.0000	0.9999	0.9949	0.9187	0.6183	0.2367	0.0487	0.0057
34	1.0000	1.0000	1.0000	1.0000	0.9986	0.9661	0.7686	0.3874	0.1087	0.0174
36	1.0000	1.0000	1.0000	1.0000	0.9997	0.9875	0.8758	0.5532	0.2064	0.0442
38	1.0000	1.0000	1.0000	1.0000	0.9999	0.9959	0.9408	0.7064	0.3396	0.0958
40	1.0000	1.0000	1.0000	1.0000	1.0000	0.9988	0.9749	0.8267	0.4934	0.1802
42	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9904	0.9081	0.6449	0.2978
44	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9967	0.9562	0.7737	0.4388
46	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9990	0.9811	0.8691	0.5855
48	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9926	0.9312	0.7186
50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9974	0.9671	0.8249

Table 8.17 depicts the probability $(1-p)$ that the minimum $X_{(1)}$ of a trivariate gamma vector is less than or equal to ω_{1-p} for $2 \geq \omega_{1-p} \geq 50$, $5 \geq a \geq 50$, and $\rho_{1,2} = 30\%$, $\rho_{1,3} = 60\%$ and $\rho_{2,3} = 95\%$, the non-diagonal elements of C_{123}.

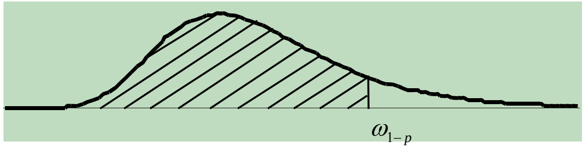
$$F_{X_{(1)}}\left(\omega_{1-p}; a, C_{123}\right) = P\left(X_{(1)} \leq \omega_{1-p}\right) = 1-p$$



	a									
ω_{1-p}	5	10	15	20	25	30	35	40	45	50
2	0.1283	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.6269	0.0225	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.9155	0.1873	0.0041	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8	0.9867	0.5140	0.0448	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.9983	0.7977	0.1848	0.0098	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
12	0.9998	0.9374	0.4333	0.0538	0.0020	0.0000	0.0000	0.0000	0.0000	0.0000
14	1.0000	0.9848	0.6915	0.1704	0.0139	0.0004	0.0000	0.0000	0.0000	0.0000
16	1.0000	0.9970	0.8659	0.3697	0.0559	0.0033	0.0001	0.0000	0.0000	0.0000
18	1.0000	0.9995	0.9525	0.5996	0.1535	0.0162	0.0007	0.0000	0.0000	0.0000
20	1.0000	0.9999	0.9861	0.7870	0.3176	0.0545	0.0043	0.0002	0.0000	0.0000
22	1.0000	1.0000	0.9966	0.9042	0.5205	0.1369	0.0173	0.0011	0.0000	0.0000
24	1.0000	1.0000	0.9993	0.9634	0.7083	0.2742	0.0515	0.0050	0.0002	0.0000
26	1.0000	1.0000	0.9999	0.9880	0.8457	0.4524	0.1215	0.0174	0.0013	0.0001
28	1.0000	1.0000	1.0000	0.9966	0.9289	0.6333	0.2375	0.0477	0.0053	0.0003
30	1.0000	1.0000	1.0000	0.9991	0.9715	0.7820	0.3936	0.1075	0.0169	0.0015
32	1.0000	1.0000	1.0000	0.9998	0.9900	0.8848	0.5637	0.2063	0.0437	0.0055
34	1.0000	1.0000	1.0000	1.0000	0.9969	0.9461	0.7167	0.3429	0.0950	0.0161
36	1.0000	1.0000	1.0000	1.0000	0.9991	0.9776	0.8339	0.5001	0.1796	0.0397
38	1.0000	1.0000	1.0000	1.0000	0.9998	0.9917	0.9123	0.6522	0.2990	0.0839
40	1.0000	1.0000	1.0000	1.0000	0.9999	0.9972	0.9585	0.7788	0.4427	0.1566
42	1.0000	1.0000	1.0000	1.0000	1.0000	0.9992	0.9823	0.8717	0.5903	0.2610
44	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9932	0.9323	0.7219	0.3912
46	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9976	0.9677	0.8259	0.5317
48	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9992	0.9860	0.8999	0.6646
50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9944	0.9473	0.7767

Table 8.18 depicts the probability $(1-p)$ that the minimum $X_{(1)}$ of a trivariate gamma vector is less than or equal to ω_{1-p} for $2 \geq \omega_{1-p} \geq 50$, $5 \geq a \geq 50$, and $\rho_{1,2} = 30\%$, $\rho_{1,3} = 95\%$ and $\rho_{2,3} = 95\%$, the non-diagonal elements of C_{123}.

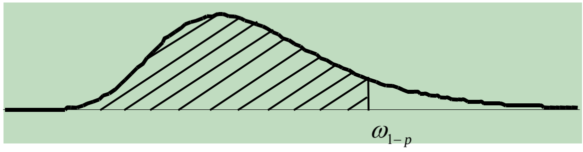
$$F_{X_{(1)}}\left(\omega_{1-p}; a, C_{123}\right) = P\left(X_{(1)} \leq \omega_{1-p}\right) = 1-p$$



	a									
ω_{1-p}	5	10	15	20	25	30	35	40	45	50
2	0.1166	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.5634	0.0213	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.8863	0.1678	0.0040	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8	0.9808	0.4573	0.0411	0.0007	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.9970	0.7512	0.1624	0.0093	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
12	0.9997	0.9190	0.3814	0.0488	0.0020	0.0000	0.0000	0.0000	0.0000	0.0000
14	1.0000	0.9768	0.6378	0.1485	0.0131	0.0004	0.0000	0.0000	0.0000	0.0000
16	1.0000	0.9948	0.8340	0.3225	0.0504	0.0032	0.0001	0.0000	0.0000	0.0000
18	1.0000	0.9992	0.9344	0.5442	0.1333	0.0152	0.0007	0.0000	0.0000	0.0000
20	1.0000	0.9999	0.9784	0.7440	0.2752	0.0491	0.0041	0.0002	0.0000	0.0000
22	1.0000	1.0000	0.9941	0.8769	0.4662	0.1188	0.0161	0.0010	0.0000	0.0000
24	1.0000	1.0000	0.9987	0.9476	0.6583	0.2363	0.0463	0.0048	0.0002	0.0000
26	1.0000	1.0000	0.9998	0.9810	0.8103	0.4007	0.1055	0.0162	0.0013	0.0001
28	1.0000	1.0000	1.0000	0.9941	0.9047	0.5799	0.2039	0.0429	0.0051	0.0003
30	1.0000	1.0000	1.0000	0.9985	0.9578	0.7395	0.3454	0.0935	0.0158	0.0015
32	1.0000	1.0000	1.0000	0.9997	0.9837	0.8535	0.5097	0.1768	0.0393	0.0053
34	1.0000	1.0000	1.0000	0.9999	0.9946	0.9249	0.6688	0.2984	0.0828	0.0150
36	1.0000	1.0000	1.0000	1.0000	0.9985	0.9659	0.7965	0.4473	0.1537	0.0358
38	1.0000	1.0000	1.0000	1.0000	0.9996	0.9863	0.8843	0.6009	0.2585	0.0734
40	1.0000	1.0000	1.0000	1.0000	0.9999	0.9952	0.9401	0.7363	0.3921	0.1341
42	1.0000	1.0000	1.0000	1.0000	1.0000	0.9986	0.9723	0.8379	0.5375	0.2245
44	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996	0.9886	0.9074	0.6750	0.3435
46	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9959	0.9519	0.7873	0.4790
48	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9987	0.9774	0.8691	0.6148
50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996	0.9905	0.9254	0.7339

Table 8.19 depicts the probability $(1-p)$ that the minimum $X_{(1)}$ of a trivariate gamma vector is less than or equal to ω_{1-p} for $2 \geq \omega_{1-p} \geq 50$, $5 \geq a \geq 50$, and $\rho_{1,2} = 60\%$, $\rho_{1,3} = 60\%$ and $\rho_{2,3} = 60\%$, the non-diagonal elements of C_{123}.

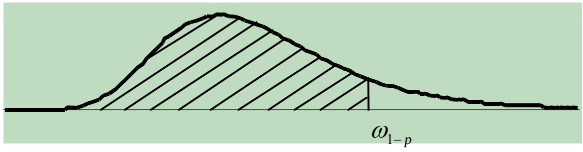
$$F_{X_{(1)}}\left(\omega_{1-p}; a, C_{123}\right) = P\left(X_{(1)} \leq \omega_{1-p}\right) = 1-p$$



	a									
ω_{1-p}	5	10	15	20	25	30	35	40	45	50
2	0.1392	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.6612	0.0235	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.9267	0.2076	0.0042	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8	0.9886	0.5495	0.0483	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.9985	0.8211	0.2054	0.0101	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
12	0.9998	0.9468	0.4678	0.0588	0.0020	0.0000	0.0000	0.0000	0.0000	0.0000
14	1.0000	0.9872	0.7213	0.1897	0.0146	0.0004	0.0000	0.0000	0.0000	0.0000
16	1.0000	0.9974	0.8839	0.4022	0.0613	0.0034	0.0001	0.0000	0.0000	0.0000
18	1.0000	0.9995	0.9603	0.6324	0.1712	0.0172	0.0007	0.0000	0.0000	0.0000
20	1.0000	0.9999	0.9885	0.8117	0.3479	0.0599	0.0044	0.0002	0.0000	0.0000
22	1.0000	1.0000	0.9971	0.9186	0.5542	0.1528	0.0183	0.0011	0.0000	0.0000
24	1.0000	1.0000	0.9994	0.9698	0.7374	0.3019	0.0566	0.0051	0.0002	0.0000
26	1.0000	1.0000	0.9999	0.9902	0.8663	0.4857	0.1357	0.0185	0.0013	0.0001
28	1.0000	1.0000	1.0000	0.9972	0.9405	0.6651	0.2627	0.0524	0.0055	0.0003
30	1.0000	1.0000	1.0000	0.9993	0.9766	0.8075	0.4258	0.1201	0.0180	0.0015
32	1.0000	1.0000	1.0000	0.9998	0.9918	0.9019	0.5967	0.2291	0.0479	0.0057
34	1.0000	1.0000	1.0000	1.0000	0.9974	0.9554	0.7457	0.3735	0.1061	0.0172
36	1.0000	1.0000	1.0000	1.0000	0.9993	0.9818	0.8559	0.5334	0.2000	0.0435
38	1.0000	1.0000	1.0000	1.0000	0.9998	0.9933	0.9264	0.6835	0.3276	0.0936
40	1.0000	1.0000	1.0000	1.0000	1.0000	0.9977	0.9659	0.8049	0.4754	0.1748
42	1.0000	1.0000	1.0000	1.0000	1.0000	0.9993	0.9856	0.8904	0.6228	0.2875
44	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9945	0.9438	0.7509	0.4229
46	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9980	0.9736	0.8489	0.5647
48	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9994	0.9886	0.9157	0.6957
50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9955	0.9566	0.8031

Table 8.20 depicts the probability $(1-p)$ that the minimum $X_{(1)}$ of a trivariate gamma vector is less than or equal to ω_{1-p} for $2 \geq \omega_{1-p} \geq 50$, $5 \geq a \geq 50$, and $\rho_{1,2} = 60\%$, $\rho_{1,3} = 60\%$ and $\rho_{2,3} = 95\%$, the non-diagonal elements of C_{123}.

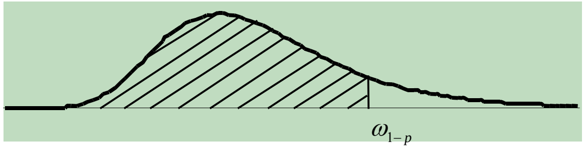
$$F_{X_{(1)}}\left(\omega_{1-p}; a, C_{123}\right) = P\left(X_{(1)} \leq \omega_{1-p}\right) = 1-p$$



	a									
ω_{1-p}	5	10	15	20	25	30	35	40	45	50
2	0.1253	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.6031	0.0223	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.8943	0.1844	0.0041	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8	0.9822	0.4931	0.0443	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.9962	0.7733	0.1805	0.0097	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
12	0.9995	0.9232	0.4167	0.0532	0.0020	0.0000	0.0000	0.0000	0.0000	0.0000
14	1.0000	0.9770	0.6675	0.1662	0.0138	0.0004	0.0000	0.0000	0.0000	0.0000
16	1.0000	0.9942	0.8459	0.3563	0.0552	0.0033	0.0001	0.0000	0.0000	0.0000
18	1.0000	0.9989	0.9387	0.5776	0.1499	0.0161	0.0007	0.0000	0.0000	0.0000
20	1.0000	0.9998	0.9789	0.7645	0.3067	0.0538	0.0043	0.0002	0.0000	0.0000
22	1.0000	1.0000	0.9938	0.8861	0.5011	0.1338	0.0171	0.0011	0.0000	0.0000
24	1.0000	1.0000	0.9984	0.9513	0.6854	0.2652	0.0508	0.0049	0.0002	0.0000
26	1.0000	1.0000	0.9996	0.9817	0.8249	0.4356	0.1189	0.0172	0.0013	0.0001
28	1.0000	1.0000	0.9999	0.9940	0.9126	0.6114	0.2301	0.0471	0.0053	0.0003
30	1.0000	1.0000	1.0000	0.9982	0.9610	0.7598	0.3793	0.1053	0.0168	0.0015
32	1.0000	1.0000	1.0000	0.9995	0.9844	0.8656	0.5435	0.2002	0.0431	0.0055
34	1.0000	1.0000	1.0000	0.9999	0.9944	0.9317	0.6943	0.3307	0.0932	0.0160
36	1.0000	1.0000	1.0000	1.0000	0.9982	0.9685	0.8129	0.4819	0.1746	0.0392
38	1.0000	1.0000	1.0000	1.0000	0.9994	0.9869	0.8948	0.6305	0.2888	0.0824
40	1.0000	1.0000	1.0000	1.0000	0.9998	0.9950	0.9458	0.7569	0.4265	0.1526
42	1.0000	1.0000	1.0000	1.0000	1.0000	0.9982	0.9744	0.8520	0.5697	0.2525
44	1.0000	1.0000	1.0000	1.0000	1.0000	0.9994	0.9890	0.9166	0.6998	0.3770
46	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9956	0.9565	0.8048	0.5127
48	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9984	0.9791	0.8816	0.6431
50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9994	0.9907	0.9331	0.7549

Table 8.21 depicts the probability $(1-p)$ that the minimum $X_{(1)}$ of a trivariate gamma vector is less than or equal to ω_{1-p} for $2 \geq \omega_{1-p} \geq 50$, $5 \geq a \geq 50$, and $\rho_{1,2} = 60\%$, $\rho_{1,3} = 95\%$ and $\rho_{2,3} = 95\%$, the non-diagonal elements of C_{123}.

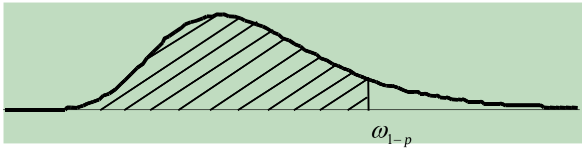
$$F_{X_{(1)}}\left(\omega_{1-p}; a, C_{123}\right) = P\left(X_{(1)} \leq \omega_{1-p}\right) = 1-p$$



	a									
ω_{1-p}	5	10	15	20	25	30	35	40	45	50
2	0.1176	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.4204	0.0200	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.8393	0.2360	0.0037	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8	0.9028	0.3163	0.0435	0.0007	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.9663	0.7353	0.1663	0.0093	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
12	0.9972	0.8391	0.3099	0.0527	0.0019	0.0000	0.0000	0.0000	0.0000	0.0000
14	1.0000	0.9428	0.6177	0.1456	0.0131	0.0004	0.0000	0.0000	0.0000	0.0000
16	1.0000	0.9728	0.8747	0.2840	0.0529	0.0032	0.0001	0.0000	0.0000	0.0000
18	1.0000	0.9989	0.9094	0.5227	0.1305	0.0153	0.0007	0.0000	0.0000	0.0000
20	1.0000	0.9999	0.9464	0.7544	0.2524	0.0504	0.0041	0.0002	0.0000	0.0000
22	1.0000	0.9999	0.9844	0.8637	0.4459	0.1167	0.0163	0.0010	0.0000	0.0000
24	1.0000	0.9999	0.9985	0.9142	0.6531	0.2220	0.0471	0.0048	0.0002	0.0000
26	1.0000	1.0000	0.9998	0.9590	0.8030	0.3826	0.1041	0.0164	0.0013	0.0001
28	1.0000	1.0000	0.9997	0.9893	0.8727	0.5682	0.1946	0.0434	0.0051	0.0003
30	1.0000	1.0000	0.9999	0.9977	0.9300	0.7318	0.3299	0.0926	0.0159	0.0015
32	1.0000	1.0000	1.0000	0.9993	0.9701	0.8261	0.4960	0.1706	0.0397	0.0053
34	1.0000	1.0000	1.0000	0.9996	0.9909	0.8953	0.6580	0.2857	0.0823	0.0152
36	1.0000	1.0000	1.0000	0.9999	0.9971	0.9444	0.7749	0.4337	0.1496	0.0361
38	1.0000	1.0000	1.0000	1.0000	0.9989	0.9771	0.8545	0.5873	0.2483	0.0731
40	1.0000	1.0000	1.0000	1.0000	0.9995	0.9917	0.9147	0.7188	0.3795	0.1314
42	1.0000	1.0000	1.0000	1.0000	0.9998	0.9969	0.9563	0.8097	0.5226	0.2165
44	1.0000	1.0000	1.0000	1.0000	0.9999	0.9987	0.9813	0.8805	0.6593	0.3323
46	1.0000	1.0000	1.0000	1.0000	1.0000	0.9994	0.9923	0.9306	0.7624	0.4643
48	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9968	0.9651	0.8417	0.5992
50	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9986	0.9841	0.9014	0.7126

Table 8.22 depicts the probability $(1-p)$ that the minimum $X_{(1)}$ of a trivariate gamma vector is less than or equal to ω_{1-p} for $2 \geq \omega_{1-p} \geq 50$, $5 \geq a \geq 50$, and $\rho_{1,2} = 95\%$, $\rho_{1,3} = 95\%$ and $\rho_{2,3} = 95\%$, the non-diagonal elements of C_{123}.

$$F_{X_{(1)}}\left(\omega_{1-p}; a, C_{123}\right) = P\left(X_{(1)} \leq \omega_{1-p}\right) = 1-p$$



	a									
ω_{1-p}	5	10	15	20	25	30	35	40	45	50
2	0.1185	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.5480	0.0231	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.8673	0.1715	0.0042	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8	0.9615	0.4650	0.0463	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.9894	0.7322	0.1752	0.0101	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
12	0.9981	0.8835	0.4000	0.0558	0.0020	0.0000	0.0000	0.0000	0.0000	0.0000
14	0.9998	0.9512	0.6324	0.1657	0.0145	0.0004	0.0000	0.0000	0.0000	0.0000
16	1.0000	0.9838	0.7967	0.3471	0.0582	0.0034	0.0001	0.0000	0.0000	0.0000
18	1.0000	0.9959	0.8963	0.5522	0.1521	0.0170	0.0007	0.0000	0.0000	0.0000
20	1.0000	0.9992	0.9523	0.7162	0.3029	0.0571	0.0044	0.0002	0.0000	0.0000
22	1.0000	0.9999	0.9825	0.8337	0.4844	0.1378	0.0182	0.0011	0.0000	0.0000
24	1.0000	1.0000	0.9948	0.9091	0.6444	0.2653	0.0542	0.0051	0.0002	0.0000
26	1.0000	1.0000	0.9988	0.9568	0.7695	0.4260	0.1238	0.0184	0.0013	0.0001
28	1.0000	1.0000	0.9998	0.9831	0.8597	0.5800	0.2330	0.0505	0.0055	0.0003
30	1.0000	1.0000	1.0000	0.9944	0.9215	0.7075	0.3751	0.1108	0.0179	0.0015
32	1.0000	1.0000	1.0000	0.9985	0.9621	0.8072	0.5216	0.2051	0.0464	0.0057
34	1.0000	1.0000	1.0000	0.9997	0.9845	0.8804	0.6491	0.3307	0.0989	0.0171
36	1.0000	1.0000	1.0000	0.9999	0.9946	0.9328	0.7541	0.4681	0.1807	0.0423
38	1.0000	1.0000	1.0000	1.0000	0.9984	0.9673	0.8361	0.5943	0.2918	0.0880
40	1.0000	1.0000	1.0000	1.0000	0.9996	0.9862	0.8980	0.7019	0.4193	0.1594
42	1.0000	1.0000	1.0000	1.0000	0.9999	0.9949	0.9430	0.7902	0.5429	0.2576
44	1.0000	1.0000	1.0000	1.0000	1.0000	0.9984	0.9720	0.8597	0.6515	0.3749
46	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996	0.9879	0.9131	0.7438	0.4945
48	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9954	0.9518	0.8194	0.6032
50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9985	0.9763	0.8797	0.6978

Glossary

American Option : An option which can be exercised at any time prior to expiration.

At the Money : An option whose exercise price is equal to the current price of the underlying contract. On listed option exchanges the term is more commonly used to refer to the option whose exercise price is closest to the current price of the underlying contract.

Buy/Write : The purchase of an underlying contract together with the sale of a call option on that contract.

Call Option : A contract between a buyer and a seller whereby the buyer acquires the right, but not to the obligation, to purchase a specified underlying contract at a fixed price on or before a specified date. The seller of the call option assumes the obligation of delivering the underlying contract should the buyer wish to exercise his option.

Delta (Δ) : The sensitivity of an option's theoretical value to a change in the price of the underlying contract.

Delta Neutral : A position where the sum total of all the positive and negative deltas adds up to approximately zero.

Exercise : The process by which the holder of an option notifies the seller of his intention to take delivery of the underlying contract, in the case of a call, or to make delivery of the underlying contract, in the case of a put, at the specified exercise price.

Exercise Price : The price at which the underlying contract will be delivered in the event an option is exercised.

Expiration (Expiry) : The date and time after which an option may no longer be exercised.

European Option : An option which may only be exercised at expiration.

Fair Value : Theoretical value.

Gamma (Γ) : The sensitivity of an option's delta to a change in the price of the underlying contract.

Hedge Ratio : Delta.

Implied Volatility : Assuming all other inputs are known, the volatility which would have to be input into a theoretical pricing model in order to yield a theoretical value identical to the price of the option in the marketplace.

In the Money : A call (put) option whose exercise price is lower (higher) than the current price of the underlying contract.

Intrinsic Value : The amount by which an option is in the money. Out of the money options have no intrinsic value.

Long : A position resulting the purchase of a contract. The term is also used to describe a position, which will theoretical increase (decrease) in value should the price of the underlying contract rise (fall). Note that a long (short) put position is a short (long) market position.

Omega (Ω) : The Greek letter sometimes used to denote an option's elasticity.

Out of the Money : An option which currently has no intrinsic value. A call (put) is out of the money if its exercise price is more (less) than the current price of the underlying contract.

Put Option : A contract between a buyer and a seller whereby the buyer acquires the right, but not the obligation , to sell a specified underlying contract at a fixed price on or before a specified date. The seller of the put option assumes the obligation of taking delivery of the underlying contract should the buyer wish to exercise his option.

Rho (ρ) : The sensitivity of an option's theoretical value to change in interest rates.

Series : All options with the same underlying contract, same exercise price, and same expiration date.

Short : A position resulting from the sale of a contract. The term is also used to describe a position which will theoretically increase (decrease) in value should the price of the underlying contract fall (rise). Note that a short (long) put position is a long (short) market position.

Sigma (σ) : The commonly used notation for standard deviation. Since volatility is usually expressed as a standard deviation, the same notation is often used to denote volatility.

Straddle : A long (short) call and a long (short) put, where both options have the same underlying contract, the same expiration date, and the same exercise price.

Strike Price (Strike) : Exercise price.

Tau (τ) : The commonly used notation for the amount of time remaining to expiration.

Theoretical Value : An option value generated by a mathematical model given certain prior assumptions about the terms of the option, the characteristics of the underlying contract, and prevailing interest rates.

Theta (θ) : The sensitivity of an option's theoretical value to a change in the amount of time remaining to expiration.

Underlying : The instrument to be delivered in the event an option is exercised.

Vega : The sensitivity of an option's theoretical value to a change in volatility.

Volatility : The degree to which the price of an underlying instrument tends to fluctuate over time.

Write : Sell an option.

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