Chapter 1

Introduction

Statistical process control (SPC) is the collection of methods for recognizing special causes and bringing a process into a state of control and reducing variation about a target value. Statistical process control is a sector of statistics dealing with the effort of improving quality constantly. It uses many tools in order to achieve this goal. The first of them were introduced in 1920's (Shewhart charts). SPC is extensively used in industry to keep manufacturing processes under control. The need of monitoring specific processes led to its great development and improvement. Its tools are used in various fields of science such as industry, medicine, environment, economics, text analyses and informatics.

The most valuable tool of SPC is control charts. These charts give a graphical appearance of the process giving the ability to any manager with or without the knowledge of statistics to immediately understand if the process is under control or not. The wide use and popularity of control charts is a result of many reasons. First of all their proven ability to improve productivity, because the reduction of scrap and rework results in an increase of productivity, increase in production capacity measured in the number of good parts per hour and decrease in cost. Their effective prevention of defect items is also valuable. The use of control charts helps to keep the process under control. Finally, the diagnostic information of control charts is significant as it allows for changes in the process by experienced operators or engineers.

The research in the area of control charts is active for over eight decades now. Although someone would expect that there would be a decreasing interest in this area after all these years, we observe exactly the opposite. There is an increasing interest for this tool since it has proved its value in practice. Most of the deficiencies of control charts are under investigation and at the same time new problems need a solution in the quality field by the use of control charts. This thesis aims to investigate some of the characteristics of control charts and tries to give solutions to quality problems.

The outline of the thesis is the following. In Chapter 2, a review of the most known univariate and multivariate control charts is presented. The control charts presented are the Shewhart type for variables and attributes in both univariate and multivariate cases and the Cumulative Sum (CUSUM) and Exponentially Weighted Moving Average (EWMA) charts again for the univariate and multivariate cases. The main properties of these charts are also given. Chapter 3 deals with the estimation effect on control charts. A detailed review of the current status of the subject is given in the univariate and multivariate cases. Some new results on the estimation effect of the univariate control charts for dispersion are also presented. Chapter 4 considers the issue of non-normality in-control charts. The effect of non-normality on the univariate and multivariate Shewhart and EWMA control charts is presented. Additionally, new results are given in the case of the EWMA control charts for process dispersion under the presence of non-normality. Chapter 5 investigates the problem of interpreting a signal on a multivariate control chart. Several researchers have dealt with this problem and their results are presented. A new proposal for a chart that addresses this problem is given that is proved to have promising results. Measurement error effect on control charts is the subject of Chapter 6. The presence of measurement error is a factor that may affect the performance of a control chart. The different considerations of authors in the context of Shewhart control charts are outlined. The effect of such a problem in the EWMA case is examined thoroughly under the assumption of a specific model. Finally, in Chapter 7 some final thoughts and discussion for possible future research issues and generalizations are given for the different subjects that have been addressed in this thesis.

Chapter 2

Univariate and Multivariate Control Charts

2.1 Introduction

Control charts are one of the main tools of statistical process control. The literature on control charts is huge. In this chapter we try to present the main univariate and multivariate control charts along with their basic properties. We have to emphasize that the control charts presented in this chapter and their properties are by no means a detailed review of all charts.

In Section 2.2, we present the main characteristics of a control chart and a discussion of its evaluation using the most known measures. Univariate Shewhart Control Charts for data in subgroups and individual data for the mean and the variance are given in Section 2.3 for both variables and attributes. Cumulative Sum (CUSUM) charts for the mean and variance and their properties are described in Section 2.4. Exponentially Weighted Moving Average (EWMA) Control Charts are summarized in Section 2.5. Section 2.6 presents the multivariate Shewhart control charts for the process mean and dispersion for variables and attributes. Finally, Section 2.7 gives the multivariate CUSUM and EWMA control charts.

2.2 The fundamental characteristics of a control chart

When we have a production process there is usually a target value. We want our process to achieve this target for every product. However, in every process there is an inherent random variability. Therefore, no matter how good we design the whole procedure or how accurate our engines are, we expect to be close to the target value but not always on this value. The existence of this variability affects our process.

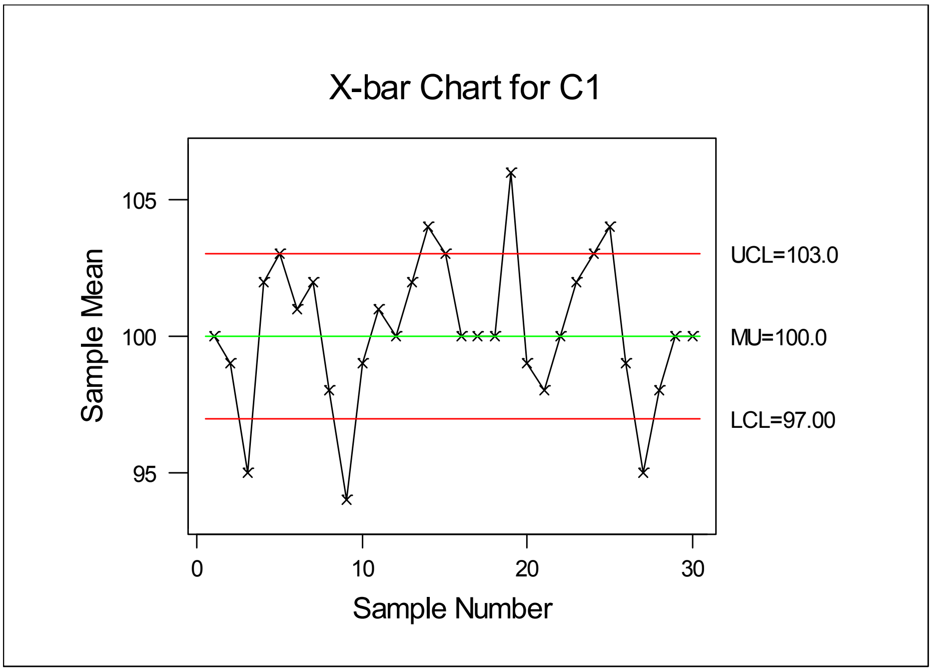
There are two different "versions" of this variability. The common cause (chance cause) variability is the natural variability every process experiences. Its existence is due to randomness as we can find purely random variability from one product to another. A process that operates with only common cause variability is said to be in-control. The special cause (assignable cause) variability is a result of factors that are not purely random. These factors cause heterogeneity in the process and as a result they affect it, leading to low quality product. A process that operates in the presence of special causes of variability is said to be out-of-control. This type of variability can be detected with control charts giving us the ability to remove its effect and therefore reduce the overall variability. As a result, removing special causes leads to an improvement of the quality of the product.

Common cause variability is the remainder of the variability after every component of special cause has been removed. In order to remove common cause variability we have to alter the process itself. However, a goal in today's industrial and technological world must be the continuous quality improvement. Under this perspective we have to stress that today's common cause can be a tomorrow's special cause. As the inspection process improves and the target for quality is constant we may be able to identify as special cause, a up to now identified common cause.

Special causes of variability can be divided in two different groups; transient special causes and persistent special causes. Transient special causes are those causes that affect a process for a short time until their reappearance in a future point in time. Persistent special causes are those causes that when they occur they stay in the process until they are detected and removed.

A control chart is a graphical representation of a characteristic of the process under investigation. It is used as the main tool to identify special causes of variability in a process. On the horizontal axis we have the number of the sample drawn from the process or the time that the sample was inspected. On the vertical axis we have the value of the characteristic measured for each sample or for the time of the horizontal axis. A straight line connects the successive points indicating the level of the characteristic in time or in successive samples. There are also three usually straight lines that stand for the upper control limit (UCL) the center line (CL) and the lower control limit (LCL). An example of a control chart is given in Figure 2.1.

Figure 2.1. A typical control chart



We assume that a process operates under control when the line connecting the sequence of points does not cross UCL or LCL. When a point is plotted outside these limits we assume that the process is in an out-of-control state and corrective actions must be taken in order to remove the assignable cause that led to this problem. The values of UCL and LCL are chosen usually in such a way that when the process is in-control the probability of a point plotting outside these limits is very small. However, there are some cases that even when all the points plot inside the control limits we characterize the process as being in an out-of-control state. Such cases are for example when we see a series of nine successive points plotting all above (below) the center line or when we see six successive points in a row steadily increasing or decreasing. We have to state here that the removal of any cause is not the objective of a control chart. A control chart simply indicates that an assignable cause may exist. It is the management's or the operator's job to act in order to get rid of the problem, if it exists.

In the literature, two distinct phases of control charting practice have been discussed (see, e.g. Woodall (2000)). In Phase I, charts are used for retrospectively testing whether the process was in-control when the first subgroups were being drawn. In this phase, the charts are used as aids to the practitioner, in bringing a process into a state of statistical control. Once this is accomplished, the control chart is used to define what is meant by statistical control. This is referred as the retrospective use of control charts. In general, there is a lot more going on in this phase than just charting some data. During this phase the practitioner is studying the process very intensively. The data collected are then analyzed in an attempt to answer the question "were the data collected from an in-control process?".

In Phase II, control charts are used for testing whether the process remains in-control when future subgroups are drawn. In this phase, the charts are used for monitoring the process for any change from an in-control state. At each sampling stage, the practitioner asks the question "has the state of process changed?". The meaning of in-control, in this phase, is usually determined by the values of the process parameters e.g., the mean and standard deviation for univariate continuously distributed variables. The values of the parameters are either given to the practitioner or they are estimated from the historical data known to be under control from Phase I. Note that in this phase the data is not taken as being from an in-control process unless the data provide evidence against no change in the process. Using these data to define what is meant by the process being in-control might lead to use an out-of-control process to define a state of statistical in-control. Woodall (2000) states that much work, process understanding and process improvement is often required in the transition from Phase I to Phase II.

In a control chart we have two objectives. Firstly, when a process is in-control, we want our chart to signal (false alarm) infrequently. In statistical terms we want the chart to operate with the planned probability of the statistic computed to plot outside the control limits if we are in-control. Secondly, when a process is out-of-control, we want the chart to signal as soon as possible. In statistical terms we want the probability of the statistic computed to plot in-control if we are out-of-control to be as small as possible. Different measures for evaluating the performance of a chart, concerning the previous two objectives, have been proposed. The most known measure is the average run length (ARL), which is based on the run length (RL) distribution. The number of observations (individual data), or samples (data in subgroups), needed for a control chart to signal is a run length or alternatively one observation of the RL distribution. The mean of the RL distribution is the ARL, which is actually the average number of observations needed for a control chart to signal. Page (1954) defined the average run length as follows: When the quality remains constant the average run length of a process inspection scheme is the expected number of articles sampled before action is taken. Ewan and Kemp (1960) gave a somewhat different definition; When the quality remains constant the average run length of an inspection scheme is the expected number of samples obtained before action is taken. Usually, along with the ARL, the standard deviation of the run length (SDRL) is computed. Alternatively, the ARL is expressed as the average number of observations to signal (ANOS). A measure similar to the ARL is the average time to signal (ATS), which is the average time needed for a control chart to signal and it is actually a product of the ARL and the sampling interval used in the case of fixed sampling.

From the preceding discussion we see that all these measures are related to the ARL. However the sole use of the ARL has been criticized (see, e.g. Barnard (1959), Bissell (1969), Woodall (1983) and Gan (1993b, 1994)).The disadvantage of the ARL is the skewness of the run length distribution in the out-of-control case and in non-normality and as a result the misleading conclusions one can draw based on the ARL. An alternative measure is the median run length (MRL), which is more credible since it is less affected by the skewness (see, e.g. Gan (1993b, 1994)).

A typical method of comparing control charts is based on the calculation of their average run length (ARL) (Woodall (1985)). Assume that independent random samples of size n are drawn successively from a process that measure the quality of a characteristic. Assume also that the sample means $\overline{x}_1, \overline{x}_2, ...$ are normally distributed with known variance σ^2/n. Consider as the objective of the control chart to keep the in-control mean equal to the target value μ_0. If $E(\overline{x}_i) = \mu, i = 1, 2, ...$ the parameter $\theta = \sqrt{n} \left|\mu - \mu_0\right| / \sigma$ denotes the shift in the mean measured in units of the standard error of the sample mean. We assume that any shift in the mean away from the target value occurs prior to the implementation of the control chart.

Let M_0, M_1 denote the in-control and out-of-control regions respectively. The in-control region M_0 contains all values of θ that correspond to acceptable shifts. Although "acceptable shifts" is an oxymoron, there is a meaningful explanation. When the shift in a process is very slight the attempt to adjust the process can lead to over-correction and introduce extra variability into the process. Duncan (1974) and Wetherill (1977) observe that low ARL values for small deviations from the target value is a drawback when some slack in the process is acceptable. The out-of-control region M_1 contains all values of θ for which a control procedure should give an out-of-control signal.

A control chart has an ARL value of at least L_0, when $\theta \in M_0$, and at most L_1, when $\theta \in M_1$. Consider two procedures A, B that are to be compared. If the ARL profile of A is above that of B for $\theta \in M_0$ and below that of B for $\theta \in M_1$, then procedure A is considered to be uniformly better than procedure B.

2.3 Univariate Shewhart Control Charts

The most known control charts are the Shewhart type control charts. They owe their name to Walter Shewhart who established them in his pioneering work in 1931. They are used to detect transient special causes in a process. This property is the result of the fact that Shewhart Control Charts are memoryless. In the following we present the Shewhart control charts for variables and attributes.

2.3.1 Control charts for the mean for data in subgroups

Assume that we have a variable that is normally distributed with mean μ and standard deviation σ. We assume that μ and σ are both known. Let $x_1, x_2, ..., x_n$ be a sample of n independent and identically distributed observations drawn from our production process. Then the average of this sample $\overline{x}$ is distributed as a normal variable with mean μ and standard deviation $\sigma/\sqrt{n}$. Therefore, we can use as control limits for each sample

$$\begin{aligned} UCL &= \mu + Z_{\alpha/2}\sigma/\sqrt{n} \\ LCL &= \mu - Z_{\alpha/2}\sigma/\sqrt{n} \end{aligned} \quad (2.1)$$

where UCL and LCL are the upper and lower control limits respectively, $Z_{\alpha/2}$ is the inverse of the normal cumulative distribution function for probability $\alpha/2$ and α is the probability that an in-control sample will plot outside these limits. If all the points (samples) plot inside the control limits we claim that we have an in-control process. This plot is a Phase II Shewhart chart for the mean.

However, in real world we usually do not know the values of μ and σ. Consequently, we have to estimate them. Therefore, the control limits in such a case will not be fixed numbers, but rather random variables. The control limits in this case for Phase I

Shewhart chart for the mean are

$$\begin{aligned}\widehat{UCL} &= \widehat{\mu} + k\widehat{\sigma}/\sqrt{n} \\ \widehat{LCL} &= \widehat{\mu} - k\widehat{\sigma}/\sqrt{n}\end{aligned}$$

where $\widehat{\mu}$ and $\widehat{\sigma}$ are the estimates for the mean and the standard deviation, respectively and k is a constant used to specify the width of the control limits usually taken to be equal to 3. If a point plots above $\widehat{UCL}$ or below $\widehat{LCL}$ we have an indication that this point (sample) is from an out-of-control process. Let $\overline{x}_1$, $\overline{x}_2$, ..., $\overline{x}_m$ be the sample means from samples each with n observations. Then, an estimate for the mean is $\widehat{\mu} = \overline{\overline{x}}$, the average of all the sample means. If the process is in-control this estimator is normally distributed with mean μ and variance $\sigma^2/(mn)$. For the standard deviation three different estimators have been proposed. The first one is based on the range. Let $R_1, R_2, ..., R_m$ denote the range for each of the m samples and $\overline{R}$ the average of these ranges. Then, a control charts' unbiased estimator is given by $\overline{R}/d_2$. The estimated control limits for the $\overline{X}$ chart are given by

$$\begin{aligned}\widehat{UCL} &= \overline{\overline{x}} + k\overline{R}/(d_2\sqrt{n}) \\ \widehat{LCL} &= \overline{\overline{x}} - k\overline{R}/(d_2\sqrt{n})\end{aligned} \tag{2.2}$$

where d_2 is the mean of the random variable R/σ and is a function of the sample size n. Details on the derivation of d_2 along with its values for different sample sizes can be found in textbooks, see e.g. Montgomery (2001).

A second version of the estimated control limits for the mean is based on a different unbiased estimator for the standard deviation. Let $S_1, S_2, ..., S_m$ denote the standard deviation for each of the m samples and $\bar{S} = \frac{1}{m}\sum_{i=1}^{m} S_i$ their average. An unbiased estimator for σ is $\overline{S}/c_4$ (see e.g. Ryan(2000)) where

$$c_4 = \left(\frac{2}{n-1}\right)^{1/2} \frac{\Gamma\left(n/2\right)}{\Gamma\left((n-1)/2\right)}$$

and $\Gamma(\cdot)$ stands for the gamma function, where the gamma function is defined as

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx, \ z > 0.$$

The control limits will be

$$\begin{aligned} \widehat{UCL} &= \overline{\overline{x}} + k\overline{S}/(c_4\sqrt{n}) \\ \widehat{LCL} &= \overline{\overline{x}} - k\overline{S}/(c_4\sqrt{n}) \end{aligned} \tag{2.3}$$

A third version for such type of limits is based on $\overline{U} = \sqrt{\frac{1}{m}\sum_{i=1}^{m} S_i^2}$, where $\overline{U}/c_{4,m}$ is an unbiased estimator of σ and

$$c_{4,m} = \frac{\sqrt{2}\Gamma((m(n-1)+1)/2)}{\sqrt{m(n-1)}\Gamma(m(n-1)/2}.$$

The control limits using this estimator will be

$$\begin{aligned} \widehat{UCL} &= \overline{\overline{x}} + k\overline{U}/(c_{4,m}\sqrt{n}) \\ \widehat{LCL} &= \overline{\overline{x}} - k\overline{U}/(c_{4,m}\sqrt{n}) \end{aligned}$$

It can be proved for the three different unbiased estimators that $Var\left(\overline{U}/c_{4,m}\right) \leq Var\left(\overline{S}/c_4\right) \leq Var\left(\overline{R}/d_2\right)$ (Derman and Ross (1995)). Therefore, a preferable estimator for σ is $\overline{U}/c_{4,m}$.

Yang and Hillier (1970) proposed a somewhat different Phase I control charts. The control limits for the statistic plotted at time i are not functions of the ith sample. One or more of the other $m-1$ samples are used to estimate μ_0 and σ_0. If $\widehat{\mu}_{(i)}$ and $\widehat{\sigma}_{(i)}$ are the estimators for the mean and standard deviation respectively when only the i sample is removed the control limits will be

$$\begin{aligned}\widehat{UCL} &= \widehat{\mu}_{(i)} + k\widehat{\sigma}_{(i)}/\sqrt{n} \\ \widehat{LCL} &= \widehat{\mu}_{(i)} - k\widehat{\sigma}_{(i)}/\sqrt{n}\end{aligned}$$

Champ and Chou (2003) compared the performance of the two Phase I control charts the one using the m samples and the other using $m-1$ samples and concluded that the one using all samples gives better results.

The ARL for Shewhart charts is given by

$$ARL = 1/\Pr(\text{a point plots outside the control limits}). \tag{2.4}$$

It has to be stressed though that this relationship holds in the case of known parameters. If the parameters are unknown and as a result they have to be estimated a different relationship holds. This matter is studied in detail in chapter 3.

Several authors have dealt with the Shewhart chart for the mean and have proposed improvements or modifications. For instance see Champ and Woodall (1987), Reynolds et al. (1988) and Quesenberry (1995a).

2.3.2 Control charts for the variability for data in subgroups

Assume that we have again a variable from a stable process that is normally distributed with mean μ and standard deviation σ comprising m samples of size n each. We assume that σ is known. In this process we want to keep the variability in-control. Then, it can be proved that the mean and standard deviation of the range of a sample from this process are $E(R) = d_2\sigma$ and $SD(R) = d_3\sigma$ where d_2 and d_3 are functions of the sample size n. Computation of d_2 and d_3 and values for different sample sizes can be found in textbooks, see e.g. Montgomery (2001). Then the Phase II control limits for the variability using the range will be

$$\begin{aligned} UCL &= d_2\sigma + kd_3\sigma \\ LCL &= d_2\sigma - kd_3\sigma \end{aligned} \tag{2.5}$$

The value of k is selected in the same way as in the chart for the mean. The most common value is 3. However, the selection in this case is actually approximating the 0.9973 probability limits for the mean when $\alpha = 0.0027$.

The usual design of the R-chart involves control limits that have equal tail probabilities (see, e.g. (2.5)). However, in such a case it is possible to have an interval (σ_1, σ_2) with $\sigma_1 < \sigma_2$ and for each σ in this interval $ARL(\sigma) > ARL(\sigma_0)$, where $ARL(\sigma_0)$ is the in control ARL. Such a chart is called a biased R chart. Champ (2001) showed how to design an ARL unbiased R control chart.

Another way to compute Phase II control limits for the variability is through the standard deviation. It can be proved that $E(S) = c_4\sigma$ and $SD(S) = \sigma\sqrt{1 - c_4^2}$. Then the control limits will be

$$\begin{aligned} UCL &= \left(c_4 + 3\sqrt{1 - c_4^2}\right)\sigma \\ LCL &= \left(c_4 - 3\sqrt{1 - c_4^2}\right)\sigma \end{aligned} \tag{2.6}$$

Usually, we do not know the value of σ and therefore we have to estimate it from past data. As in the case of the mean let $R_1, R_2, ..., R_m$ denote the range for each of the m samples and $\overline{R}$ the average of these ranges. An estimate based on the range as already mentioned is

$$\widehat{\sigma}_R = \frac{\overline{R}}{d_2}.$$

Then, the Phase I control limits are

$$\begin{aligned}\widehat{UCL} &= \left(1+\frac{3d_3}{d_2}\right)\bar{R} \\ \widehat{LCL} &= \left(1-\frac{3d_3}{d_2}\right)\bar{R}.\end{aligned} \tag{2.7}$$

A different estimate used is

$$\bar{S}=\frac{1}{m}\sum_{i=1}^{m}S_i$$

where m is the number of past samples used, $S_i^2=\frac{1}{n-1}\sum_{j=1}^{n}\left(X_j-\bar{X}_i\right)^2$ is the unbiased estimator of σ^2 and n is the sample size. However, we know that S is not an unbiased estimator of σ. It has been proved, as already mentioned, that an unbiased estimate of σ is $\bar{S}/c_4$ and that the standard deviation of S equals $\sigma\sqrt{1-c_4^2}$. The upper and lower control limits of the chart known as the Phase I S chart are

$$\begin{aligned}\widehat{UCL} &= \left(1+\frac{3}{c_4}\sqrt{1-c_4^2}\right)\bar{S} \\ \widehat{LCL} &= \left(1-\frac{3}{c_4}\sqrt{1-c_4^2}\right)\bar{S}\end{aligned} \tag{2.8}$$

Approaches making use of these limits are known as the three sigma approaches based on the normal approximation proposed by Shewhart in the early thirties. However, it is easy to prove that this approximation is not satisfactory since as is known

$$\frac{(n-1)S^2}{\sigma^2}\sim X_{n-1}^2 \tag{2.9}$$

Although this approximation is not accurate, it is usually used as a first check (see e.g. Ryan (2000), Klein (2000), Lowry, Champ and Woodall (1995)).

A modification of the control limits (2.6) and (2.8) based on property (2.9) uses probability limits in place of the three sigma limits (see e.g. Ryan(2000)). If the value of the standard deviation σ is known the Phase II control limits are

$$\begin{aligned} UCL &= \sigma\sqrt{\frac{\chi^2_{0.999}}{n-1}} \\ LCL &= \sigma\sqrt{\frac{\chi^2_{0.001}}{n-1}} \end{aligned} \tag{2.10}$$

In these limits, if the process variability operates in-control, the probability that the standard deviation of future subgroups will fall between them is 0.998, which is approximately equal to the 0.9973, the probability assumed when using the 3 sigma ones. If the true standard deviation is not known we use its unbiased estimate $\bar{S}/c_4$. The Phase I limits then become

$$\begin{aligned} \widehat{UCL} &= \frac{\bar{S}}{c_4}\sqrt{\frac{\chi^2_{0.999}}{n-1}} \\ \widehat{LCL} &= \frac{\bar{S}}{c_4}\sqrt{\frac{\chi^2_{0.001}}{n-1}}. \end{aligned} \tag{2.11}$$

Yang and Hillier (1970) proposed different Phase I control limits using the same way of thinking as in the case of the mean by excluding sample i from the calculation. Champ and Chou (2003) compared the performance of these different Phase I limits and concluded that the standard limits and the ones proposed by Yang and Hillier can be designed to be equivalent.

The ARL of the control charts for the variability of data in subgroups is given by the relationship (2.4) as this relationship is valid for all Shewhart charts with known parameters. More details on Shewhart charts for variability and related work can be found in Lowry, Champ and Woodall (1995), Klein (2000) and Sim(2000).

2.3.3 Control charts for individual data

Let $X_i, i = 1, .., n$ represent independent and identically distributed observations from a $N(\mu, \sigma^2)$ process. If the parameters μ and σ^2 are known, the Phase II X chart control limits are

$$
\begin{aligned}
UCL &= \mu + 3\sigma \\
LCL &= \mu - 3\sigma
\end{aligned}
$$

Usually, these parameters are not known and they have to be estimated. In this case, the variability is usually controlled using moving ranges. Nevertheless, Nelson (1982), Roes et al. (1993) and Rigdon et al. (1994) have recommended either against the use of the moving range chart or its use together with the classical X chart. Moreover, Sullivan and Woodall (1996a) showed that a moving range control chart does not contribute significantly to the identification of out-of-control situations. For these reasons we do not present it here. Therefore, the use of the X control chart for monitoring both the process mean and standard deviation is recommended. The Phase I control limits of the X control chart are

$$
\begin{aligned}
\widehat{UCL} &= \bar{X} + 3\widehat{\sigma} \\
\widehat{LCL} &= \bar{X} - 3\widehat{\sigma}
\end{aligned}
$$

where $\bar{X}$ is an unbiased estimate of the mean of the process and $\widehat{\sigma}$ is an estimate of the standard deviation σ of the process. Usually, the estimate of the standard deviation used is $\overline{MR}/d_2$ where $\overline{MR}$ denotes the average of the moving ranges and d_2 is the usual function of the sample size n used to make the estimator unbiased. However, Cryer and Ryan (1990) showed that a preferable estimate of σ is s/c_4 where c_4 is defined the same way as in the case of rational subgroups and s is the standard deviation of the observations. Sullivan and Woodall (1996a) proposed a Phase I control chart for independent observations that uses the log-likelihood function and is used to detect shifts in both the mean and the variance. This chart is shown to have better performance in comparison to the X chart or the combined X and MR chart. Moreover, it performs well for detecting sustained shifts in the distribution but not that well for outliers.

2.3.4 Control Charts for Attributes

When an item is produced or purchased it is inspected in order to identify if it satisfies a number of specifications. An item that does not satisfies those specifications is called a defective or a non-conforming item. These defectives lead to rework or they are characterized as scrap or second quality product. In any case we have a loss of money or working time or both. In order to avoid such products, control charts for the characteristics (attributes) have been developed (see, e.g. Woodall (1997), Ryan (2000) and Montgomery (2001)).

Assume that we have a random sample of n units and we inspect them for possible nonconforming items. The fraction nonconforming is defined as the ratio of the number of nonconforming items in a population to the total number of items in that population. Suppose the production is operating in a stable manner, such that the probability that any unit will not conform to specifications is p, and that successive units are produced independently. If d is the number of units of products that are nonconforming, then d has a binomial distribution with parameters n and p, that is

$$P(d = x) = \binom{n}{x} p^x (1 - p)^{n-x},\ x = 0, 1, 2, ..., n$$

where $E(d) = np$ and $V(d) = np(1 - p)$.

The sample fraction nonconforming is defined as the ratio of the number of nonconforming items in a sample to the total number of items in that sample that is

$$\widehat{p} = \frac{d}{n}$$

where $E(\widehat{p}) = p$ and $V(\widehat{p}) = p(1 - p)/n$.

If the true fraction nonconforming p in the production process is known or is a standard value specified by management, then the Phase II control limits for the p chart are defined as

$$\begin{aligned} UCL &= p + 3\sqrt{\frac{p(1-p)}{n}} \\ LCL &= p - 3\sqrt{\frac{p(1-p)}{n}} \end{aligned}$$

where the charting statistic is $\widehat{p}_i$, for sample i.

If the true fraction nonconforming p is not known, then it must be estimated from observed data. The usual procedure is to select m preliminary samples, each of size n. Then the average of these m individual sample fractions nonconforming is

$$\overline{p} = \frac{\sum_{i=1}^{m} d_i}{mn} = \frac{\sum_{i=1}^{m} \widehat{p}_i}{m}$$

and the Phase I control limits are defined as

$$\begin{aligned} \widehat{UCL} &= \overline{p} + 3\sqrt{\overline{p}(1-\overline{p})/n} \\ \widehat{LCL} &= \overline{p} - 3\sqrt{\overline{p}(1-\overline{p})/n} \end{aligned}$$

where the charting statistic is again $\widehat{p}_i$, for sample i.

For a constant sample size it is also possible to plot on a control chart the number of nonconforming units, rather than the fraction nonconforming. This chart is called the np control chart. If the true fraction nonconforming p in the production process is known or is a standard value specified by management, then the Phase II control limits are defined as

$$\begin{aligned} UCL &= np + 3\sqrt{np(1-p)} \\ LCL &= np - 3\sqrt{np(1-p)} \end{aligned}$$

where the charting statistic is $n\widehat{p}_i$, for each sample.

If the true fraction nonconforming p in the production process is not known, then the average of the m preliminary individual sample fractions nonconforming $\overline{p}$ is used and the Phase I control limits are defined as

$$\begin{aligned} \widehat{UCL} &= n\overline{p} + 3\sqrt{n\overline{p}(1-\overline{p})} \\ \widehat{LCL} &= n\overline{p} - 3\sqrt{n\overline{p}(1-\overline{p})} \end{aligned}$$

where the charting statistic is $n\widehat{p_i}$, for each sample. If we have a signal on a p chart we will have also one in an np chart because of the relation between the two charted statistics. Therefore, we may say that these charts are equivalent for a constant sample size.

Borror and Champ (2001) proposed Phase I charts for p and np charts based on the recommendation of Yang and Hillier (1970). Borror and Champ (2001) compared the false alarm rate performance of the standard and the new charts and concluded that the new chart has a higher false alarm rate. Additionally, the performance of the standard Phase I charts is not satisfactory. Therefore the practitioner should use such charts carefully, keeping in mind the possibility of larger number of false alarms than what should be expected from the design of the charts.

If we count the number of defects or nonconformities in a sampling unit then we can plot them in a control chart. This chart is used to control the total number of non-conformities in a unit. In such a chart we usually assume that the number of non-conformities in sample of constant size follows a Poisson distribution. If x is the number of nonconformities and $c>0$ is the parameter of the Poisson distribution, then

$$P(x) = \frac{e^{-x}c^{x}}{x!}, x = 0, 1, 2, ...$$

If the true value of c in the production process is known or is a standard value specified by management, then the Phase II control limits are defined as

$$\begin{aligned} UCL &= c + 3\sqrt{c} \\ LCL &= c - 3\sqrt{c} \end{aligned}$$

where the charting statistic is $\widehat{c}_i$ the number of nonconformities in sample i.

If the true value of c in the production process is not known, then the Phase I control limits are defined as

$$\begin{aligned} \widehat{UCL} &= \overline{c} + 3\sqrt{\overline{c}} \\ \widehat{LCL} &= \overline{c} - 3\sqrt{\overline{c}} \end{aligned}$$

where $\overline{c}$ is the average number of nonconformities in a preliminary sample of inspection units and it is used as an estimate of c. The charting statistic in this case is $\widehat{c}_i$, the number of nonconformities in sample i, again.

If we want to develop a control chart for a sample of n sampling units or for a sampling unit that is n times larger than the standard sampling unit, we may set up a control chart based on the average number of nonconformities per inspection unit. Specifically, let $u = c/n$, then since c is distributed as a Poisson random variable the Phase II control limits for this chart are

$$\begin{aligned} UCL &= u + 3\sqrt{\frac{u}{n}} \\ LCL &= u - 3\sqrt{\frac{u}{n}} \end{aligned}$$

in the case that u is known or is a standard value specified by management. If the true value of u is not known then the Phase I control limits will be

$$\widehat{UCL} = \overline{u} + 3\sqrt{\frac{\overline{u}}{n}}$$
$$\widehat{LCL} = \overline{u} - 3\sqrt{\frac{\overline{u}}{n}}$$

where $\overline{u}$ is the average number of nonconformities per inspection unit from a preliminary sample and it is used as an estimate of u.

If Y_i is the number of conforming items between the $(i-1)th$ and the ith nonconforming item from a stable process with the in-control probability of a nonconforming item be p_0 then this process is a sequence of independent Bernoulli trials with the same probability p_0. Therefore, Y_i+1 is a geometric random variable with parameter p_0. Then, the Phase II control limits for this chart are

$$UCL = \frac{\ln(\alpha/2)}{\ln(1-p_0)} - 1$$
$$LCL = \frac{\ln(1-\alpha/2)}{\ln(1-p_0)}$$

If p_0 is unknown the Phase I control limits are

$$\widehat{UCL} = \frac{\ln(\alpha/2)}{\ln(1-N/m)} - 1$$
$$\widehat{LCL} = \frac{\ln(1-\alpha/2)}{\ln(1-N/m)}$$

where $\widehat{p_0} = N/m$, N is the number of nonconforming items in a total of m items sampled.

In all the above control limits we can not accept a negative value. For this reason if the lower control limit is negative we set it equal to zero.

2.4 Cumulative Sum (CUSUM) Control Chart

CUSUM control charts were introduced by Page in 1954. They are used to identify persistent causes in a variable instead of Shewhart charts. This ability is attributed to the fact that they have a memory as they are based on successive sums of the observations minus a constant. Generally, we can say that CUSUM charts are able to detect small to moderate shifts whereas Shewhart charts are able to detect large shifts.

Let $x_1, x_2, ...x_n$ be n independent and identically distributed observations drawn from our production process and μ is the process mean target. Then, we define as the Phase II CUSUM control chart, the function

$$S_i = \sum_{i=1}^{n} (X_i - \mu)$$

plotted against the observation number. In the case of subgrouped data instead of each observation we have the corresponding sample mean.

A more usual way of calculating the CUSUM for an upward shift in mean is by the formulas

$$\begin{aligned} S_0^+ &= 0 \\ S_i^+ &= max(0, S_{i-1}^+ + (X_i - \mu) - k) \end{aligned}$$

where k is a constant called reference value. The CUSUM chart gives a signal if $S_i^+ > h$ where h is a value we choose to give the desired in-control ARL called decision interval. The corresponding CUSUM scheme for detecting downward shifts is

$$\begin{aligned} S_0^- &= 0 \\ S_i^- &= \min(0, S_{i-1}^- + (X_i - \mu) + k) \end{aligned}$$

and it signals if $S_i^- < -h$. There is a certain way to compute the values of k and h, which is related to the distribution of X_i's. Hawkins and Olwell's (1998) textbook is an excellent reference on this subject. We have to state here that in the case of standard normal data with $k = 3$ and $h = 0$ we end up with the classic Shewhart $\overline{X}$ chart for the mean. Moreover, in the case of subgrouped data we modify the preceding schemes and in place of each observation X_i we have the sample mean.

Koning and Does (2000) presented Phase I CUSUM control charts using recursive residuals. They showed that their chart has a better performance than the Likelihood Ratio chart of Sullivan and Woodall (1996a) and Q chart of Quesenberry (1995a).

In the following subsections for CUSUM charts we focus in the case of continuous distributed variables. The case of discrete distributed observations has been also examined see e.g., Lucas (1985), Gan (1993a) and Hawkins and Olwell (1998).

2.4.1 Optimality of the CUSUM

Assume that $x_1, x_2, ..$ are independent and identically distributed random variables that are observed sequentially. Let $x_1, x_2, .., x_{m-1}$ have (in-control) distribution function F_0 and $x_m, x_{m+1}, ..$ have (out-of-control) distribution function $F_1 \neq F_0$. The two distributions are known but the time of change m is assumed unknown.

Many schemes can detect such a change (e.g. Shewhart charts). These schemes are classified by the expected time until the process signals while it remains in-control (false alarm rate). Among all procedures with the same false alarm rates, the optimal procedure is the one that detects changes quicker. Or we could say that among all procedures with the same in-control expected number of samples until signal, the optimal procedure has the smallest expected time until it signals a change when the process shifts to the out-of-control state.

Moustakides (1986) proved that the CUSUM scheme was optimal in the above sense. Specifically, among all tests with the same in-control expected number of samples until signal, the CUSUM had the smallest out-of-control expected number of samples.

The optimality of the CUSUM is for detecting a shift to a single specific out-of-control distribution. The CUSUM that is optimal for detecting one particular shift is not optimal for detecting a different shift. For a different shift a different CUSUM will be optimal. However, while a CUSUM for detecting a shift of one standard deviation is optimal only for this shift, it performs nearly as well as the optimal CUSUM for all shifts that are not too far from one standard deviation.

2.4.2 Average Run Length (ARL)

Two different methods for the computation of the ARL have been developed; the integral equation method and the Markov chain approach. Page (1954) used integral equations for the computation of the ARL. Let the distribution function of a single score x be $F(x)$ and let $L(z)$ be the ARL of the one sided case. $L(0)$ stands for the ARL with an initial value of zero. Then, for $0 \leq z < h$

$$L(z) = 1 + L(0)F(-z) + \int_0^h f(x-z)L(x)dx$$

We may explain the above integral equation with the following description: the expected run length of a test which is now at z equals 1 (the next observation) plus the probability that the next observation will return the CUSUM to zero multiplied by the expected run length from $z = 0$ plus the integral over the probabilities that the CUSUM lands somewhere between zero and h multiplied by the respective expected run lengths from the new value of the CUSUM. Van Dobben de Bruyn (1968) gives a discussion on the derivation of this equation. Additionally, Wetherill (1977) gave an almost identical relationship but from a somewhat different way of thinking. Others that have dealt with the same problem are Ewan and Kemp (1960) and recently Champ, Rigdon and Scharnagl (2001) that give a general method for obtaining integral equations used in the evaluation of many control charts.

The Markov Chain approach begins by approximating the problem of obtaining the average run length (ARL) and then obtains an exact solution to the approximate problem. The integral equation approach begins with the exact problem and finds an approximate solution to it. Champ and Rigdon (1991) compared integral and Markov chain approaches. They propose the integral equation as a preferable method when an integral equation can be found. On the other hand, there are situations, where only the Markov chain approach seems appropriate.

Brook and Evans (1972) were the first to propose the new method for computing the ARL based on a Markov chain. This method applies to both discrete and continuous variables. In the case of continuous variables let Z be the quality characteristic we want in-control, which is continuously distributed. Consider the one-sided case where we accumulate the deviations of Z from a reference value k and this procedure stops if we reach the upper decision boundary h or if the cumulative sum equals zero. A Markov process with continuous state space can represent this scheme.

Suppose that the Markov chain has $t+1$ states labeled $E_0, E_1, ...E_t$ where E_t is the absorbing state. The probability that the chain remains in the same state at the next step should correspond to the case where the cumulative sum does not change in value by more than a small amount say $0.5w$, meaning that the next value of Z does not differ from the reference value k by more than $0.5w$. The value of w determines the width of the grouping interval that is used to discretize the probability distribution of Z. This value must be carefully chosen because properties like average run length and percentage points are highly affected by the width of the decision interval. In order to avoid unwilling behavior a further restriction is the following; the probability of a jump from E_i to the absorbing state E_t should be equal to the probability that the cumulative sum for $(Z-k)$ jumps beyond the point h from a position in $(0, h)$ which corresponds approximately to the state E_i. Therefore

$$w = 2h/(2t-1) \tag{2.12}$$

The transition probabilities for the Markov chain are for $i = 0, 1, \ldots, t-1$ as follows

$$\begin{aligned}
P_{i0} &= pr(E_i \rightarrow E_0) = pr(Z - k \leq -iw + 0.5w) \\
P_{ij} &= pr(E_i \rightarrow E_j) = pr((j-i)w - 0.5w \leq Z - k \leq (j-i)w + 0.5w), 1 \leq j \leq t-1 \\
P_{it} &= pr(E_i \rightarrow E_t) = pr((t-i)w - 0.5w < Z - k)
\end{aligned}$$

Also, $pr(E_0 \rightarrow E_t) = pr(Z - k > h)$ for any w that satisfies relation (2.12). Let $p_r = pr(rw - 0.5w < Z - k \leq rw + 0.5w)$ and $F_r = pr(Z - k \leq rw + 0.5w)$ then the transition probability matrix $\mathbf{P}$ has the following form

$$\mathbf{P} = \begin{bmatrix}
F_0 & p_1 & p_2 & \ldots & p_j & \ldots & p_{t-1} & 1 - F_{t-1} \\
F_{-1} & p_0 & p_1 & \ldots & p_{j-1} & \ldots & p_{t-2} & 1 - F_{t-2} \\
\vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
F_{-i} & p_{1-i} & p_{2-i} & \ldots & p_{j-i} & \ldots & p_{t-1-i} & 1 - F_{t-1-i} \\
\vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
F_{1-t} & p_{2-t} & p_{3-t} & \ldots & p_{j-(t-1)} & \ldots & p_0 & 1 - F_0 \\
0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 1
\end{bmatrix}.$$

A relation that holds is $(\mathbf{I} - \mathbf{R})\boldsymbol{\mu}^{(s)} = s\mathbf{R}\boldsymbol{\mu}^{(s)}$, s=2,3,... where R is the matrix obtained from the transition probability matrix $\mathbf{P}$ by deleting the last row and column (those referring to the absorbing state E_t), $\mathbf{I}$ is the identity matrix and $\boldsymbol{\mu}^{(s)}$ is the vector of the sth factorial moments for the random variables $X_0, X_1, ..., X_{t-1}$. For s=1 the equation becomes $(\mathbf{I} - \mathbf{R})\boldsymbol{\mu} = \mathbf{1}$, where the vector $\mathbf{1}$ has each of its t elements equal to unity. The first element of the vector $\boldsymbol{\mu}$ gives the average run length for a CUSUM chart starting from zero and in general the ith element gives the mean of the run-length distribution when starting from state E_i, i=0,1,...,t-1. We have to state here that the above procedure, suitably modified (Brook and Evans (1972)), can be used for the computation of the ARL of discrete distributed observations also.

The two different computations of the ARL presented are for a one-sided scheme. In the case that we have a two-sided scheme Van Dobben de Bruyn (1968) showed that

$$\frac{1}{ARL} = \frac{1}{ARL^{+}} + \frac{1}{ARL^{-}},$$

where ARL^{+} is the ARL of an upward scheme and ARL^{-} is the ARL of a downward scheme.

The evaluation of CUSUM charts is usually done by the computation of the ARL. Two reasons for this are, firstly, that the computation of the run length distribution is difficult in most situations (see Page (1954), Ewan and Kemp (1960), Brook and Evans (1972), Woodall (1983,1984), Waldman (1986)) and secondly that the in-control run length distribution is approximately geometric, therefore it can be characterized by the ARL. On the contrary, the in-control run length distribution of a CUSUM chart is highly skewed and accordingly conjectures on the ARL can be misleading because the form of the run length distribution changes with a shift in the mean. Therefore the ARL is not a sufficient measure for the performance of the chart. Barnard (1959) and Bissell (1969) have criticized the use of the ARL only and they have proposed instead the simultaneous use of percentage points.

On the other hand, the median run length (MRL) is a quantity that we can rely on because it is more meaningful and more readily understood (see Gan (1994)). For example when the out-of-control MRL is 50, this means that half of all the run lengths are less than 50.

2.4.3 Fast Initial Response (FIR)

Lucas and Crosier (1982) extended the calculation of the average run length by using a head start value S_0 different than zero. The calculation of the ARL for several head start values showed that for a moderate value the in-control ARL has a small percentage decrease while the out-of-control ARL has a large percentage decrease. Therefore we may design a FIR CUSUM, with an almost equivalent in-control ARL and a smaller out-of-control ARL than a standard CUSUM, by increasing h (decision interval) slightly to compensate for the small decrease in ARL caused by the head start.

Lucas and Crosier (1982) recommended a head-start of $S_0 = h/2$. This recommendation is a result of the fact that a CUSUM scheme is a sequence of Wald tests (Page (1954)) with null hypothesis that the mean is zero and alternative hypothesis that the mean is $2k$.

The ARL computation of one-sided schemes is easily done using the proposed procedures of section 2.4.2. However, in the case of a two-sided FIR CUSUM scheme the computation is modified (Yashchin (1985)). Let H^+ and $-H^-$ denote the head starts of an upward and a downward CUSUM scheme, respectively. Also, let $A^+(s)$ and $A^-(s)$ denote the ARL of an upward and a downward CUSUM scheme with head starts s and $-s$, respectively. Then, the ARL of a two-sided FIR CUSUM is

$$ARL = \frac{A^+(H^+)A^-(0) + A^-(H^-)A^+(0) - A^+(0)A^-(0)}{A^+(0) + A^-(0)}.$$

This result holds if the following condition is satisfied

$$k^+ + k^- \geq \max\left(H^+ + H^- - \min(h^+, h^-), \left|h^+ - h^-\right|\right)$$

where k^+, k^- and h^+, h^- are the upward and downward reference values and decision intervals, respectively. This condition ensures that if the upward CUSUM signals the downward CUSUM will be at zero and vice-versa.

2.4.4 CUSUM control chart for process variability

A method for keeping in-control the process dispersion was developed by Chang and Gan (1995). Assume that the process mean is in-control and let $s_1^2, s_2^2, ...$ be successive sample variances observed from a process based on a sample of size n. The upper and lower CUSUM charts are obtained through the plotting of

$$C_i^+ = max(0, C_{i-1}^+ + y_i - k_C^+)$$

and

$$C_i^- = max(0, C_{i-1}^- + y_i + k_C^-)$$

against i respectively, for $i = 1, 2, \ldots$, where k_C^+, k_C^- are constants, $y_i = \log(s_i^2)$, $C_0^+ = u$ for $0 \leq u < h^+$ and $C_0^- = v$ for $-h^- \leq v < 0$. The upper CUSUM chart is used to detect increases in the variance and there is an out-of-control signal at the first i for which $C_i^+ > h^+$. The lower CUSUM chart is used to detect decreases in the variance and there is an out-of-control signal at the first i for which $C_i^- < -h^-$. In practice, we usually have to estimate the in-control variance because it is not known and this is done by taking samples from a process, which is assumed to be in-control.

The probability density function of $\log(s_i^2)$, when the measures of the quality characteristic are independent, identically and normally distributed, is

$$f(y) = \frac{\exp\left[ay - \exp\left(y\right)/\beta\right]}{\Gamma\left(a\right)\beta^a}, -\infty < y < \infty$$

where a=$(n-1)/2$, $\beta = 2\sigma^2/(n-1)$ and $\Gamma(a)$ is the gamma function. Let $H(u)$ be the ARL function of an upper CUSUM chart given that $C_0^+ = u$ where $0 \leq u \leq h^+$. Then, an approach similar to Page (1954) for the computation of the ARL is the following

$$H(u) = 1 + H(0)\Pr\left(\log\left(s^2\right) \leq k_C^+ - u\right) + \int_0^{h^+} f(x + k_C^+ - u)H(x)dx.$$

In practice, most of the processes are out-of-control at the beginning and a FIR CUSUM is recommended for a faster detection of this situation. As we have already said Lucas and Crosier (1982) have recommended using $h/2$ as the head start value for monitoring normal means. The distribution of $\log(s_i^2)$ is approximately normal, so $h^+/2$ and $-h^-/2$ are recommended here as head start values for the upper and lower CUSUM charts.

In order to design a CUSUM chart we have to determine the values of h and k. Chang and Gan (1995) provided tables for various values of sample sizes and for the out-of-control standard deviation which we want to detect quickly.

A CUSUM chart based on a larger sample size will be more sensitive than a CUSUM chart based on a smaller sample size for detecting changes in σ. For the two-sided case, the two one-sided CUSUM charts that are optimal for detecting a specific shift in either direction can be run simultaneously so as to detect changes in standard deviation in both directions.

The CUSUM chart described here is based on the assumption that the measures of the quality characteristic are independent, identically and normally distributed. For non-normally distributed observations the ARL values are different and especially for distributions with a tail larger than the normal one, the ARL tends to be small, so the false alarm rate is higher. When observations are positively serially correlated, then the CUSUM is less effective in detecting increases in σ, because the sample variance decreases as the serial correlation increases.

For other CUSUM charts developed for monitoring process variance see Yashchin (1994) and Srivastava (1997).

2.5 Exponentially Weighted Moving Average (EWMA) Control Chart

The EWMA chart was introduced by Roberts (1959) and it is used as the CUSUM chart to detect persistent shifts in a variable. Its ability is to signal faster than the Shewhart charts for small and moderate shifts but not that fast for large shifts. Generally, we can say that its performance is similar to the performance of the CUSUM chart.

In the following subsections we present the EWMA chart for continuous variables. The case of discrete variables has been studied by Gan (1990), Borror et al. (1998), Quesenberry (1995b) and Quesenberry (1995c).

2.5.1 EWMA Control Chart for the mean

Let the mean μ and standard deviation σ of a process to be known. The EWMA chart for individual observations is defined as

$$Z_i = \lambda x_i + (1-\lambda)Z_{i-1},\ Z_0 = \mu$$

where x_i is the observation at time $i = 1, 2, ...$, λ is a smoothing parameter that takes values between 0 and 1 and Z_0 is the initial value. When the value of λ is close to 0, the EWMA chart can detect small to moderate shifts in the process mean, when λ is close to unity the EWMA can detect large shifts in the process mean and when $\lambda = 1$ it is actually the $\overline{X}$ chart. As a starting value, instead of the in-control process mean, we can use the target value. The control limits of this chart are

$$\begin{aligned} UCL &= \mu + L\frac{\sigma}{\sqrt{n}}\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left[1-(1-\lambda)^{2i}\right]} \\ LCL &= \mu - L\frac{\sigma}{\sqrt{n}}\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left[1-(1-\lambda)^{2i}\right]}, \end{aligned} \tag{2.13}$$

where L is a constant used to specify the width of the control limits, μ is the mean of the process and $\frac{\sigma}{\sqrt{n}}\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left[1-(1-\lambda)^{2i}\right]}$ the standard deviation of Z_i when the process is in-control. In case the EWMA chart is used for some time, instead of control limits (2.13), we may use their limiting values

$$\begin{aligned} UCL &= \mu + L\frac{\sigma}{\sqrt{n}}\sqrt{\left(\frac{\lambda}{2-\lambda}\right)} \\ LCL &= \mu - L\frac{\sigma}{\sqrt{n}}\sqrt{\left(\frac{\lambda}{2-\lambda}\right)} \end{aligned} \tag{2.14}$$

since $\lim\limits_{i\to\infty}\left(\frac{\sigma^2}{n}\left(\frac{\lambda}{2-\lambda}\right)\left[1-(1-\lambda)^{2i}\right]\right) = \frac{\lambda\sigma^2}{(2-\lambda)n}$ (see e.g., Lucas and Saccucci (1990)). In this case, $\frac{\sigma}{\sqrt{n}}\sqrt{\lambda/(2-\lambda)}$ is the asymptotic standard deviation of Z_i. In the case of subgrouped data instead of a single observation we have the sample mean of the observations at time i. The control limits are correspondingly modified.

The main features of the EWMA chart are the same as the ones for the CUSUM except of the optimality. The computation of its run length distribution and the ARL can be done by the exact way using integral equations (Crowder(1987)). The ARL $L(u)$ of a two-sided EWMA chart for the mean given that the EWMA starts at u is computed through the relation

$$L(u) = 1 + \frac{1}{\lambda}\int_{-h}^{h} f\left(\frac{y-(1-\lambda)u}{\lambda}\right) L(y)dy$$

where y_i's are assumed to be independent, identically distributed observations with probability density function $f(\cdot)$, h is the upper control limit and $-h$ the lower control limit. This can be explained as follows; if for the first observation y_1, we have that $|(1-\lambda u)+\lambda y_1| > h$ then we have a signal. On the other hand, if this relation does not hold, the run length continues to move from $(1-\lambda u)+\lambda y_1$ and $L((1-\lambda u)+\lambda y_1)$ stands for the additional run length.

The approximation method of the Markov chain is the other alternative (Lucas and Saccucci (1990)). The ARL in this case is computed by

$$ARL = (\mathbf{I}-\mathbf{R})^{-1}\mathbf{1},$$

where $\mathbf{I}$ is the identity matrix, $\mathbf{1}$ is a vector of unities and $\mathbf{R}$ is a submatrix of the transition probability matrix $\mathbf{P}$, where

$$\mathbf{P} = \begin{bmatrix} \mathbf{R} & (\mathbf{I}-\mathbf{R})\,\mathbf{1} \\ \mathbf{0}^T & 1 \end{bmatrix}.$$

If p_{jk} is the probability that the control statistic goes from state j to state k then

$$p_{jk} = \Pr\left[\lambda^{-1}\left\{(S_k-\delta)-(1-\lambda)S_j\right\} < Y_i \le \lambda^{-1}\left\{(S_k+\delta)-(1-\lambda)S_j\right\}\right]$$

where S_k is the midpoint of the kth interval, $j = -m, ..., m$ and m is the number of states we will use. The larger the number of states the more accurate the computation will be.

The median run length and the fast initial response are properties that have been implemented and in the context of EWMA charts (see Lucas and Saccucci (1990) and Gan (1993b)). For further discussion of the EWMA charts and other modifications see Robinson and Ho (1978), Hunter (1986), Saccucci and Lucas (1990), Domangue and Patch (1991), Ingolfson and Sachs (1993), Steiner (1999).

2.5.2 EWMA control chart for process variability

Several publications dealing with the subject of keeping in-control the process variance using an EWMA chart have appeared in the literature like Wortham and Ringer (1971), Wortham (1972), Sweet (1986), Ng and Case (1989), Domangue and Patch (1991), Crowder and Hamilton (1992), Hamilton and Crowder (1992), MacGregor and Harris (1993), Acosta-Mejia and Pignatiello (2000). In this subsection we present schemes for sample size larger than unity. The schemes for $n = 1$ are investigated in Chapter 4.

The EWMA chart of squared deviations from target ($\mathrm{EWMA_S}$) was proposed by Wortham and Ringer (1971) for detecting a shift in the process standard deviation. The statistic of this chart is given by

$$S_i = \lambda(x_i - \mu_0)^2 + (1 - \lambda)S_{i-1},\ S_0 = \sigma_0^2,$$

where λ is a smoothing parameter that takes values between 0 and 1 and S_0 is the initial estimated value of the mean squared error. It can be proved (MacGregor and Harris (1993)) that under normality the quantity S_i/σ^2 is approximately distributed as $\chi^2(\nu)/\nu$ where the degrees of freedom ν depend on the parameter λ, the correlation of the x_i's and the degrees of freedom associated with the initial value. If we assume that the process mean is on target and the variance is σ_0^2 then the control limits of S_i are the $\alpha/2$ and $1 - \alpha/2$ percentiles of $\sigma_0^2\chi^2(\nu)/\nu$ distribution. In case of independent and normally distributed observations we may plot $\sqrt{S_i}$ and the corresponding control limits are

$$\begin{aligned} UCL &= \sigma_0\sqrt{\frac{\chi^2_{1-a/2}(\nu)}{\nu}} \\ LCL &= \sigma_0\sqrt{\frac{\chi^2_{a/2}(\nu)}{\nu}} \end{aligned}$$

However, the above statistic has the property to respond both to changes in mean and in variance. Therefore, a statistic that would plot out of the control limits only in the case of variance shifts is desirable. Sweet (1986) proposed the use of an estimate of the process mean in each step in time. Specifically, let μ_i denote an estimate of the process mean at time i. Then

$$S_i = \lambda(x_i - \mu_i)^2 + (1-\lambda)S_{i-1},\ S_0 = \sigma_0^2.$$

A usually used estimate for the mean is the EWMA statistic for the mean (Z_i). MacGregor and Harris (1993) computed control limits for this statistic which is usually addressed as the Exponentially Weighted Moving Variance (EWMV).

Crowder and Hamilton (1992) proposed a different control chart based on $\ln\left(\widehat{\sigma}_i^2\right)$. The scheme is

$$C_i = \max\left\{(1-\lambda)\, C_{i-1} + \lambda y_i, \ln\left(\sigma_0^2\right)\right\},$$

where $C_0 = \ln\left(\sigma_0^2\right)$, λ is the usual constant taking values between 0 and 1 and $y_i = \ln\left(\widehat{\sigma}_i^2\right)$. This statistic can be used to identify only upward shifts in the variance. The UCL of this chart in case of independent observations is given by

$$UCL = K\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left\{\frac{2}{n-1} + \frac{2}{(n-1)^2} + \frac{4}{3\,(n-1)^3} - \frac{16}{15\,(n-1)^5}\right\}},$$

where K is a constant chosen together with λ so as to achieve the desired ARL. If $L(u)$

is the ARL of this chart with u the starting value then

$$L(u) = 1 + L(0)F\left(\frac{-(1-\lambda)u}{\lambda}\right) + \frac{1}{\lambda}\int_0^{UCL} f\left(\frac{y-(1-\lambda)u}{\lambda}\right) L(y)dy,$$

where $F(x)$ and $f(x)$ are the cumulative distribution function and the probability distribution function of the log-gamma distribution respectively.

2.6 Multivariate Shewhart Control Charts

Multivariate Shewhart Control Charts are analogous to the univariate ones but they involve in the computations several variables instead of one. The Phase I and II charts discussion does not change in this case. Sparks (1992), Wierda (1994), Lowry and Montgomery (1995), Fuchs and Kenett (1998), Ryan (2000) and other statisticians and engineers agree with the definition of the Phases given in Section 2.2. However, Alt (1985) gives a somewhat different definition for the two distinct phases of control charting practice. In the following the definition of Section 2.2 is followed.

A crucial matter in Multivariate Shewhart Control Charts is the sample size n of each rational subgroup. As Lowry and Montgomery (1995) suggest, the appropriate use of a test statistic (X^2 or T^2) can be broken into four categories: 1) Phase I and $n = 1$, working with individual observations; 2) Phase I and $n > 1$, working with rational subgroups; 3) Phase II and $n = 1$, working with individual observations; 4) Phase II and $n > 1$, working with rational subgroups.

Mason and Young (2002) recently published a textbook for the implementation of multivariate statistical process control in the case of Shewhart charts that discusses in detail several subjects.

2.6.1 Control Charts for the Process Mean ($n > 1$)

Assume that the vector $\mathbf{x}$ follows a p-dimensional normal distribution, denoted as $N_p(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$, and that there are m samples of size $n > 1$ available from the process. A control chart can be based on the sequence of the following statistic

$$D_i^2 = n(\overline{\mathbf{x}}_i - \boldsymbol{\vartheta}_0)^t \mathbf{Z}_0^{-1} (\overline{\mathbf{x}}_i - \boldsymbol{\vartheta}_0),$$

where $\overline{\mathbf{x}}_i$ is the vector of the sample means of the ith rational subgroup, $\boldsymbol{\vartheta}_0$ and $\mathbf{Z}_0$ are the appropriate vector of means and the appropriate variance-covariance matrix in either Phase I or Phase II, respectively. The superscript t is used to define the transpose of a matrix. The D_i^2 statistic represents the Mahalanobis distance of any point from the target $\boldsymbol{\vartheta}_0$. Thus, if the value of the test statistic D_i^2 plots above the control limit (L_u), the chart signals a potential out-of-control process. Generally, control charts have both upper (L_u) and lower control limits (L_l). However, in this case only an upper control limit is meaningful, because extreme values of the D_i^2 statistic correspond to a point far away from the target $\boldsymbol{\vartheta}_0$, whereas small or zero values of the D_i^2 statistic correspond to points close to the target $\boldsymbol{\vartheta}_0$.

If $\boldsymbol{\vartheta}_0 = \overline{\overline{\mathbf{x}}}_0$, $\mathbf{Z}_0 = \overline{\mathbf{S}}$, $n > 1$ and $\overline{\mathbf{x}}_i$ is the mean of the ith observation then the $D_i^2/c_0(p,m,n)$ statistic follows an F distribution with p and $(mn - m - p + 1)$ degrees of freedom. Here $c_0(p,m,n) = [p(m-1)(n-1)]\,(mn - m - p + 1)^{-1}$, the parameter $\overline{\overline{\mathbf{x}}}$ is the overall sample mean vector and $\overline{\mathbf{S}}$ is the pooled sample variance-covariance matrix. Consequently, a multivariate Shewhart control chart for the process mean, with unknown parameters, has the following control limit

$$L_u = c_0(p,m,n) F_{1-a,p,mn-m-p+1}.$$

This control chart is called a Phase I T^2-chart**.** We must note that, for a Phase I T^2-chart the statement "if the process is in-control the probability of at least one of the D_i^2's being outside the control limits is a" does not hold. It does not hold because in this Phase the

D_i^2's are not independent (this is valid only for $i = 1$). In practical problems T^2-chart is typically recommended for the preliminary analysis of multivariate observations in process monitoring applications. Nedumaran and Pignatiello (2000) consider the issue of constructing retrospective T^2 control chart limits so as to control the overall probability of a false alarm at a specified value. Furthermore, Mason et. al. (2001) use the T^2-chart for monitoring batch processes in both Phase I and Phase II operations.

If $\boldsymbol{\vartheta}_0 = \overline{\overline{\mathbf{x}}}_0$, $\mathbf{Z}_0 = \overline{\mathbf{S}}$, $n > 1$ and $\overline{\mathbf{x}}_i$ is the mean of a future observation then the $D_i^2/c_1(p,m,n)$ statistic follows an F Distribution with p and $(mn - m - p + 1)$ degrees of freedom, where $c_1(p,m,n) = [p(m+1)(n-1)]\,(mn - m - p + 1)^{-1}$. Thus, a multivariate Shewhart control chart for the process mean, with unknown parameters, has the following control limit

$$L_u = c_1(p,m,n)F_{1-a,p,mn-m-p+1}.$$

This control chart is called a Phase II T^2-Chart.

If $\boldsymbol{\vartheta}_0 = \boldsymbol{\mu}_0$, $\mathbf{Z}_0 = \boldsymbol{\Sigma}_0$, $n > 1$ and $\overline{\mathbf{x}}_i$ is the mean of the ith observation then the D_i^2 statistic follows a X^2-distribution with p degrees of freedom. Therefore, a multivariate Shewhart control chart for the process mean, with known mean vector $\boldsymbol{\mu}_0$ and known variance-covariance matrix $\boldsymbol{\Sigma}_0$ has the upper control limit $L_u = X^2_{p,1-\alpha}$. This control chart is called a Phase II X^2-Chart.

The in-control ARL_0 of the multivariate Shewhart chart, when $\boldsymbol{\mu}_0$ and $\boldsymbol{\Sigma}_0$ are known, can be calculated as $ARL_0 = 1/\alpha$ where α is the probability that D_i^2 exceeds L_u. Furthermore, the out-of-control ARL_1 of the multivariate Shewhart chart depends on the mean vector and variance-covariance matrix only through the noncentrality parameter $\lambda^2(\boldsymbol{\mu}_1)$,

$$\lambda^2(\boldsymbol{\mu}_1) = n(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)^t \boldsymbol{\Sigma}_0^{-1}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0) = n\boldsymbol{\delta}^t\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\delta},$$

where $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_0 + \boldsymbol{\delta}$ is a specific out-of-control mean vector. Hence, it is possible to consider the ARL_1 as a function of $\lambda(\boldsymbol{\mu}_1)$, the square root of $\lambda^2(\boldsymbol{\mu}_1)$, and construct an ARL_1 curve by using the equation $ARL_1 = 1/\left(1-\beta\right)$, where β is the probability of the event "Procedure fails to diagnose an out-of-control situation". We have to note that the result that the ARL depends only on the noncentrality parameter is based on the assumptions that $\boldsymbol{\Sigma}_0$ is the known variance-covariance matrix and that random sampling is being done independently from a multivariate normal distribution.

The theory presented up to now considers the case of a pre-defined and fixed sample of size n. Jolayemi (1995) presented a power function model for determining sample sizes for the operation of a multivariate process control chart. Moreover, Aparisi (1996), gives a procedure for the construction of a control chart with adaptive sample sizes.

2.6.2 Control Charts for the Process Mean ($n = 1$)

For charts constructed using individual observations ($n = 1$), the test statistic for the ith individual observation has the form

$$D_i^2 = (\mathbf{x}_i - \boldsymbol{\vartheta}_0)^t \, \mathbf{Z}_0^{-1} \, (\mathbf{x}_i - \boldsymbol{\vartheta}_0) \, ,$$

where $\mathbf{x}_i$ is the ith observation, $i = 1, 2, ..., m$ following $N_p(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$, $\boldsymbol{\vartheta}_0$ and $\mathbf{Z}_0$ are the appropriate vector of means and the variance-covariance matrix in either Phase I or Phase II, respectively.

If $\boldsymbol{\vartheta}_0 = \overline{\mathbf{x}}_m$, $\mathbf{Z}_0 = \mathbf{S}_m$ and $\mathbf{x}_i$ is the ith individual observation then the $D_i^2/d_0(m)$ statistic follows a Beta distribution with $p/2$ and $(m - p - 1)$ degrees of freedom, where $d_0(m) = (m-1)^2 \, m^{-1}$. Thus, a multivariate Shewhart control chart for the process mean, with unknown parameters, has the following control limit (Tracy et al. (1992))

$$L_u = d_0(m) B_{1-\alpha/2, p/2, (m-p-1)/2},$$

where $\overline{\mathbf{x}}_m$ is the overall sample mean and $\mathbf{S}_m$ is the sample variance-covariance matrix. This control chart is called a Phase I T^2-Chart. Alternative estimators of the variance-covariance matrix has been proposed by Sullivan and Woodall (1996b) and Chou et al. (1999).

If $\boldsymbol{\vartheta}_0 = \overline{\mathbf{x}}_m$, $\mathbf{Z}_0 = \mathbf{S}_m$ and $\mathbf{x}_i$ is a future individual observation then the $D_i^2/d_1(m,p)$ statistic follows an F distribution with p and $(m-p)$ degrees of freedom, where $d_1(m,p) = p(m+1)(m-1)\left[m(m-p)\right]^{-1}$. Therefore, a multivariate Shewhart control chart for the process mean, with unknown parameters, has the following control limits (Tracy et al. (1992))

$$L_u = d_1(m,p)F_{1-\alpha,p,m-p}.$$

This control chart is called a Phase II T^2-Chart.

If $\boldsymbol{\vartheta}_0 = \boldsymbol{\mu}_0$, $\mathbf{Z}_0 = \boldsymbol{\Sigma}_0$ and $\mathbf{x}_i$ is the *ith* observation then the D_i^2 statistic follows a X^2-distribution with p degrees of freedom (Seber (1984)). Consequently, a multivariate Shewhart control chart for the process mean, with known mean vector $\boldsymbol{\mu}_0$ and known variance-covariance matrix $\boldsymbol{\Sigma}_0$, has upper control limit $L_u = X^2_{p,1-\alpha}$. This control chart is called a Phase II X^2-Chart.

2.6.3 Control Charts for Process Dispersion

In the following, multivariate control charts for controlling process dispersion are presented. In the previous two subsections, it was assumed that process dispersion remained constant and equal to $\boldsymbol{\Sigma}$. This assumption, is generally not true, and must be validated in practice. Process variability is summarized in the $p \times p$ variance-covariance matrix $\boldsymbol{\Sigma}$ which contains $p \times (p+1)/2$ parameters. There are two single-number quantities for measuring the overall variability of a set of multivariate data. The first one is the determinant of the variance-covariance matrix, $|\mathbf{S}|$, which is called the generalized variance. The square root of this quantity is proportional to the area or volume generated by a set of data. The second one is the trace of the variance-covariance matrix, $tr\mathbf{S}$, the sum of the variances of the variables. In this subsection, two different control charts for the process dispersion are presented since different statistics can be used to describe variability.

Assume that the vector $\mathbf{x}$ follows a $N_p(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$, and that there are m samples of size $n > 1$ available from the process. The first multivariate chart for the process dispersion can be based on the sequence of the following statistic

$$W_i = -pn + pn \ln n - n \ln \left[|\mathbf{A}_i| \, |\mathbf{\Sigma}_0|^{-1} \right] + trace \left(\mathbf{\Sigma}_0^{-1} \mathbf{A}_i \right)$$

for the ith sample, $i = 1, 2, ..., m$, where $\mathbf{A}_i = (n-1)\mathbf{S}_i$. The W_i statistic follows an asymptotic X^2-distribution with $p \times (p+1)/2$ degrees of freedom. Hence, a multivariate Shewhart control chart for process dispersion, with known mean vector $\boldsymbol{\mu}_0$ and known variance-covariance matrix $\mathbf{\Sigma}_0$ has the upper control limit $L_u = X^2_{p(p+1)/2, 1-\alpha}$. Therefore, if the value of the test statistic W_i plots above L_u, the chart signals a potential out-of-control process. This control chart is called a Phase II W chart.

The second chart is based on the sample generalized variance $|\mathbf{S}|$, where $\mathbf{S}$ is the $p \times p$ sample variance-covariance matrix. One approach in developing an $|\mathbf{S}|$-Chart is to utilize its distributional properties. Alt (1985) and Alt and Smith (1988) state that if there are two quality characteristics, then

$$\left[2(n-1) \, |\mathbf{S}|^{1/2} \right] |\mathbf{\Sigma}_0|^{-1/2} \text{ is distributed as a } X^2_{2n-4}.$$

Thus, the control limits for an $|\mathbf{S}|$-Chart are

$$\begin{aligned} L_u &= \left[|\mathbf{\Sigma}_0| \left(X^2_{2n-4, 1-\alpha/2} \right)^2 \right] [2(n-1)]^{-2} \\ L_l &= \left[|\mathbf{\Sigma}_0| \left(X^2_{2n-4, \alpha/2} \right)^2 \right] [2(n-1)]^{-2}, \end{aligned}$$

where L_u is the upper control limit and L_l is the lower control limit.

In a recent paper by Aparisi et al. (2001), the distribution of the $|\mathbf{S}|$-Chart is studied and suitable control limits are obtained for the case when there are more than two variables. Aparisi et al. (2001) propose the design of the $|\mathbf{S}|$ Chart with adaptive sample size to control process defined by two quality characteristics. Alt (1985) proposes a second approach in developing an $|\mathbf{S}|$-Chart by using only the first two moments of $|\mathbf{S}|$ and the property that most of the probability distribution of $|\mathbf{S}|$ is contained in the interval

$$E\left[|\mathbf{S}|\right] \pm 3\sqrt{V\left[|\mathbf{S}|\right]}.$$

Additionally, Alt and Smith (1988) propose a modification, the $|\mathbf{S}|^{1/2}$ Chart. Furthermore, Alt (1985) gives a proper unbiased estimator for $|\mathbf{\Sigma}_0|$, in order to define a Phase I control chart for controlling the process dispersion.

Although $|\mathbf{S}|$ is a widely used measure of multivariate variability, it is a relative simplistic scalar representation of a complex multivariate structure. Therefore, it can be misleading in some cases. Lowry and Montgomery (1995) present three sample covariance matrices for bivariate data that all have the same generalized variance and yet have distinctly different correlations. As a result, it is often desirable to provide more than the single number $|\mathbf{S}|$ as a summary of $\mathbf{S}$. The use of univariate dispersion charts as supplementary to a control chart for $|\mathbf{S}|$ is proposed by Alt (1985).

2.6.4 Multiattributes Control Charts

Patel (1973) was the first to deal with methods of quality control, when the p-dimensional observations are coming from a multivariate binomial or multivariate Poisson population. Specifically, Patel proposed a X^2 chart using an approximation to normality. Lu et al. (1998) developed a multivariate attribute control chart, called the MNP chart. The name of this chart stems from the fact that it is a straightforward extension of the univariate np chart. Let $\mathbf{p} = (p_1, p_2, ..., p_p)$ be the fraction nonconforming vector, $\mathbf{P}_0 = [\delta_{ij}]_{p \times p}$ the correlation matrix and $\mathbf{c} = (C_1, C_2, ..., C_p)$ the vector of counts of nonconforming units. Define

$$L = \sum_{i=1}^{p} \frac{C_i}{\sqrt{p_i}},$$

which is the weighted sum of the nonconforming units of all the quality characteristics in the sample. Since the nonconformance of a quality characteristic in one dimension may be more serious than in another dimension we want to take into account that information in the calculations. Montgomery (2001) suggested a statistic that uses this information.

Let $\mathbf{d} = (d_1, d_2, ..., d_p)$ denote the vector of the numbers of demerits, which indicates the severity of nonconformance in quality characteristics. Then the above statistic L can be extended as follows

$$L_D = \sum_{i=1}^{p} \frac{d_i C_i}{\sqrt{p_i}}.$$

For L_D Lu et al. (1998) proposed the following multivariate attribute chart

$$\begin{aligned} UCL &= n\sum_{j=1}^{p} d_j\sqrt{p_j} + 3\sqrt{n\left[\sum_{j=1}^{p} d_j^2(1-p_j) + 2\sum_{i<j}^{p}(d_i d_j \delta_{ij}\sqrt{(1-p_i)(1-p_j)})\right]} \\ LCL &= n\sum_{j=1}^{p} d_j\sqrt{p_j} - 3\sqrt{n\left[\sum_{j=1}^{p} d_j^2(1-p_j) + 2\sum_{i<j}^{p}(d_i d_j \delta_{ij}\sqrt{(1-p_i)(1-p_j)})\right]}. \end{aligned}$$

Given the values of the parameters, the control limits can be computed and the MNP chart can then be established using the above equation. If the real values are unknown, then they must be estimated. Furthermore, Lu et al. (1998) introduced a formula that can be used to calculate the appropriate sample size n of each rational subgroup and gave a procedure for the interpretation of an out-of-control signal.

Jolayemi (1999) proposed a multivariate attribute control chart (MACC), which is based on an approximation of the convolution of independent binomial variables and on an extension of the univariate np chart. When a process is monitored with respect to many independent attributes $X_1, X_2, ..., X_m$, each of which follows a binomial distribution, the distribution of the sum or the convolution of the number of defective items found in a sample of size n from the process, with respect to all m attributes, is well approximated by a binomial distribution with parameters mn and $\overline{p}_0$ (the mean of $p_1, p_2, ..., p_p$). Therefore, instead of plotting m different np charts we use a single one using the preceding approximation. The control limits of this chart are

$$\begin{aligned} UCL &= nm\overline{p}_0 + k\sqrt{nm\overline{p}_0(1-\overline{p}_0)} \\ LCL &= nm\overline{p}_0 - k\sqrt{nm\overline{p}_0(1-\overline{p}_0)}, \end{aligned}$$

where k (usually $k = 3$) is the constant that determines the width of the control limits and p_{0i}, i= 1, 2,..., m is the expected fraction defective produced with respect to attribute X_i when the process is in-control.

Therefore, a corrective action will be taken whenever the sum of the numbers of defective items found in a sample of size n, with respect to p attributes, exceeds an acceptance number c_p, where c_p is the largest integer less than or equal to the upper limit. The acceptance number c_p and the sample size n are given by the following equations

$$\begin{aligned} \sum_{i=0}^{c_p} \tbinom{nm}{i} \overline{p}_1^i (1-\overline{p}_1)^{nm-i} &= \beta \\ \sum_{i=0}^{c_p} \tbinom{nm}{i} \overline{p}_0^i (1-\overline{p}_0)^{nm-i} &= 1-\alpha, \end{aligned}$$

where $\overline{p}_1$ is the mean of p_{1i}, for $i = 1, 2, ..., p$, p_{1i} is the expected fraction defective produced with respect to attribute X_i when the process is out-of-control. The design of the MACC presupposes to solve the above equations with specified α and β values, to find the proper acceptance number c_p and the sample size n. Finally, Jolayemi (1999) gives a proper statistical procedure for the interpretation of an out-of-control signal.

2.7 Multivariate CUSUM and EWMA Control Charts

Multivariate Shewhart control charts use the information only from the current sample and they are relative insensitive to small and moderate shifts in the mean vector. Multivariate Cumulative Sum (CUSUM) and Exponentially Weighted Moving Average (EWMA) control charts are developed to overcome this problem.

The multivariate CUSUM and EWMA charts presented in the following subsections are Phase II control charts. Sullivan and Woodall (1998), recommend the use of multivariate CUSUM and EWMA charts for the preliminary analysis of multivariate observations.

2.7.1 CUSUM Type Control Charts

The multivariate CUSUM control charts are distinguished in two major categories. In the first case, the direction of the shift (or shifts) is considered to be known whereas in the second the direction of the shift is considered to be unknown (directionally invariant schemes).

We first present the CUSUM control charts for which we assume that the direction of the shift (or shifts) is known. Woodall and Ncube (1985) described how a p-dimensional multivariate normal process, can be monitored by using p two or one-sided univariate CUSUM charts for the p original variables or using p two or one sided univariate CUSUM charts for the p principal components. This multiple univariate CUSUM scheme is called the MCUSUM. The MCUSUM gives an out-of-control signal whenever any of the univariate CUSUM charts gives an out-of-control signal. The ARL performance in a multivariate process, is studied in the cases of independent and dependent quality characteristics.

Healy (1987) uses the fact that CUSUM charts can be viewed as a series of sequential probability ratio tests, in order to develop a multivariate CUSUM chart. Let $\mathbf{x}_i$ be the ith observation, which comes from a $N_p(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$ with an in-control $p \times 1$ mean vector $\boldsymbol{\mu}_0$ and a known $p \times p$ common variance-covariance matrix $\boldsymbol{\Sigma}_0$. Denote $\boldsymbol{\mu}_1$ an out-of-control $p \times 1$ vector of means. The CUSUM for detecting a shift in $\boldsymbol{\mu}_0$ towards $\boldsymbol{\mu}_1$ may be written as

$$G_i^1 = \max\left[\left(G_{i-1}^1 + \mathbf{a}^t(\mathbf{x}_i - \boldsymbol{\mu}_0) - 0.5\lambda(\boldsymbol{\mu}_1)\right), 0\right],$$

where $\lambda(\boldsymbol{\mu}_1)$ is the square root of the noncentrality parameter and $\mathbf{a}^t = A/\lambda(\boldsymbol{\mu}_1)$ and $A = \left[(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)^t \boldsymbol{\Sigma}_0^{-1}\right]$. This CUSUM scheme signals when $G_i^1 \geq H$. For detecting a shift in the mean of a multivariate normal random variable, the CUSUM procedure reduces to a univariate normal procedure. That is, all the available theory for calculating ARL, H, G_0^1 for a univariate normal CUSUM can also be used for multivariate normal CUSUM.

A similar procedure is proposed for controlling the process dispersion. The CUSUM for detecting a change in the variance-covariance matrix may be written as

$$G_i^2 = \max\left[\left(G_i^2 + D_i^2 - K\right), 0\right],$$

where $\mathbf{\Sigma}_1 = C\mathbf{\Sigma}_0$ (C is a real constant), $K = p \log C \left(C/(C-1)\right)$ and

$$D_i^2 = \left(\mathbf{x}_i - \boldsymbol{\mu}_0\right)^t \mathbf{\Sigma}_0^{-1} \left(\mathbf{x}_i - \boldsymbol{\mu}_0\right).$$

This CUSUM scheme signals when $G_i^2 \geq H$. We could not find in the literature any proposal for an analogous charting procedure in the case that the mean vector and the variance-covariance matrix have to be estimated.

Hawkins (1991) introduced CUSUMs for regression adjusted variables based on the idea that the most common situation in practice is that departures from control have some known structure. In particular, it is assumed that the mean shifts with magnitude δ in only one variable.

Consider the multiple regression of X_j, the jth variable of $\mathbf{x}$ on all other variables of $\mathbf{x}$. Let Z_j be the regression residual when the jth variable is regressed on all other variables, rescaled to unit variance. This may be used to test the hypothesis that there is not a shift in the μ_j against the alternative that there is. The regression residual Z_j is given by

$$\mathbf{Z} = [diag(\mathbf{\Sigma}^{-1})]^{-1/2}\mathbf{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}_0)$$

whose null distribution is $N(0,1)$. Hawkins (1991, 1993) proposes to chart each Z_j using a CUSUM procedure because in general it is not known which of the p variables is out-of-control. For studying the p individual charts simultaneously, Hawkins (1991) proposed the following control charts

$$MCZ = \max(L_{i,j}^{+}, -L_{i,j}^{-}) \text{ and } ZNO = \sum_{j=1}^{p}(L_{i,j}^{+} + L_{i,j}^{-})^2,$$

where

$$L_{i,j}^{+} = \max\left(0, L_{i-1,j}^{+} + Z_{i,j} - k\right) \text{ and } L_{i,j}^{-} = \min\left(0, L_{i-1,j}^{-} + Z_{i,j} + k\right)$$

and $L_{0,j}^{+} = L_{0,j}^{-}$ $i = 1, 2, ..., m$. MCZ is the MCUSUM statistic introduced by Woodall and Ncube (1985) applied to the CUSUM for $\mathbf{Z}$. ZNO is the squared Euclidean norm of the resultant vectors of the CUSUM for upward and downward shifts in mean. The CUSUMs L^{+}, L^{-} test for shifts in location in the upward and downward directions, respectively. The plot of these CUSUMs on a common chart gives a powerful CUSUM control chart for location. An out-of-control signal occurs when any of these four CUSUMs exceeds the decision interval h. The values of h and k are selected as in any other CUSUM because this chart consists of separate random variables each following the $N(0,1)$ distribution. An out-of-control signal is indicated when MCZ and ZNO exceed a threshold value set to fix the in-control ARL. Hauck et al. (1999) applied multivariate statistical process monitoring and diagnosis with grouped regression-adjusted variables.

In the sequel, we present the directionally invariant CUSUM schemes. Crosier (1988) proposes two multivariate CUSUM schemes. The first CUSUM proposed by Crosier (1988) is a CUSUM of the scalars D_i, the square root of D_i^2 statistic, and is given by

$$G_i^3 = \max\left[\left(G_{i-1}^3 + D_i - K\right), 0\right]$$

where $G_0^3 \geq 0$ and $K \geq 0$. This scheme signals when $G_i^3 \geq H$, which is determined using the Markov chain approach. Crosier (1988) notes that a search for the optimal K produced a sequence that closely resemble the square root of the number of variables.

A similar CUSUM is proposed by Pignatiello and Runger (1990) defined as

$$G_i^4 = \max\left[0, G_{i-1}^4 + D_i^2 - k\right]$$

with $G_0^4 = 0$, and k chosen to be $0.5\lambda^2(\boldsymbol{\mu}_1) + p$, where p is the number of the variables. The process is out-of-control if G_i^4 exceeds an upper control limit H, which is determined using the Markov chain approach.

Crosier (1988) and Pignatiello and Runger (1990) found that ordinary one sided univariate CUSUMs based on successive values of D_i^2 or D_i statistic, respectively, do not have good ARL properties.

The second CUSUM proposed by Crosier (1988) is a CUSUM of vectors. A vector-valued scheme can be derived by replacing the scalar quantities of a univariate CUSUM scheme by vectors and is given by

$$G_i^5 = \left[\mathbf{g}_i^t \boldsymbol{\Sigma}_0^{-1} \mathbf{g}_i\right]^{1/2},$$

where

$$\mathbf{g}_i = \left\{ \begin{array}{cc} (\mathbf{g}_{i-1} + \mathbf{x}_i - \boldsymbol{\mu}_0)(1 - KC_i^{-1}) & \text{if } C_i > K \\ \mathbf{0} & \text{otherwise} \end{array} \right\}$$

and

$$C_i = \left[(\mathbf{g}_{i-1} + \mathbf{x}_i - \boldsymbol{\mu}_0)^t \, \boldsymbol{\Sigma}_0^{-1} \left(\mathbf{g}_{i-1} + \mathbf{x}_i - \boldsymbol{\mu}_0\right)\right]^{1/2}.$$

This scheme signals when $G_i^5 > H$, where H is chosen to provide a predefined in-control ARL, using simulation. Because of the fact that the ARL performance of this chart depends on the noncentrality parameter, Crosier (1988) recommends that $K = \lambda(\boldsymbol{\mu}_1)/2$ and $\mathbf{g}_0 = \mathbf{0}$. Both CUSUMs, as proposed by Crosier (1988) allow the use of recent enhancements in CUSUM schemes. Among the CUSUM schemes proposed by Crosier (1988) the vector-valued scheme has a better ARL performance than the scalar scheme.

The second CUSUM proposed by Pignatiello and Runger (1990) can be constructed by defining G_i^6 as

$$G_i^6 = \max\left\{\left[\mathbf{C}_i^t \boldsymbol{\Sigma}_0^{-1} \mathbf{C}_i\right]^{1/2} - kn_i, 0\right\},$$

where $G_0^6 = 0$, k is chosen to be $0.5\lambda(\boldsymbol{\mu}_1)$, $\boldsymbol{\mu}_1$ is a specified out-of-control mean, $\mathbf{C}_i$ equals

$$\mathbf{C}_i = \sum_{l=i-n_i+1}^{i} (\mathbf{x}_i - \boldsymbol{\mu}_0)$$

and n_i is the number of subgroups since the most recent renewal (i.e. zero value) of the CUSUM chart, formally defined as

$$n_i = \left\{ \begin{array}{c} n_{i-1} + 1, \text{ if } G_{i-1}^6 > 0 \\ 1, \text{otherwise} \end{array} \right\}.$$

This chart operates by plotting G_i^6 on a control chart with an upper control limit H which is again computed through simulation. If G_i^6 exceeds H then the process is out-of-control. Pignatiello and Runger (1990), proved that the ARL performance of the G_i^6 chart depends only on the square root of the noncentrality parameter and it is better in relation to G_i^5.

Ngai and Zhang (2001) gave a natural multivariate extension of the two-sided cumulative sum chart for controlling the process mean. Additionally, Chan and Zhang (2001) propose cumulative sum charts for controlling the covariance matrix.

2.7.2 Multivariate Exponentially Weighted Moving Average Charts

Let $\mathbf{x}_i^t$ be the ith p-dimensional observation. Also, assume that $\mathbf{x}_i$ follows a $N_p(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$ with a known variance-covariance matrix $\boldsymbol{\Sigma}$ and a known p-dimensional mean vector $\boldsymbol{\mu}_0$. A multivariate EWMA control chart is proposed by Lowry et al. (1992) as follows

$$\mathbf{z}_i = \mathbf{R}(\mathbf{x}_i - \boldsymbol{\mu}_0) + (\mathbf{I} - \mathbf{R})\,\mathbf{z}_{i-1} = \sum_{j=1}^{i} \mathbf{R}\,(\mathbf{I} - \mathbf{R})^{i-j}\,(\mathbf{x}_j - \boldsymbol{\mu}_0),\ i = 1, 2, 3, ...,$$

where $\mathbf{R} = diag(r_1, r_2, ..., r_p)$, $0 \leq r_k \leq 1$ for $k = 1,\ 2,\ 3, ...,\ p$, and $\mathbf{I}$ is the identity matrix. If there is no a-priori reason to weight past observations differently for the p quality characteristics being monitored, then $r_1 = r_2 = ... = r_p = r$. The initial value $\mathbf{z}_0$ is usually obtained equal to the in-control mean vector of the process. It is obvious that if $\mathbf{R} = \mathbf{I}$ then the multivariate EWMA control chart is equivalent to the T^2-Chart. The multivariate EWMA chart gives an out-of-control signal if $\mathbf{z}_i^t \mathbf{\Sigma}_{\mathbf{z}_i}^{-1} \mathbf{z}_i > h$ where $\mathbf{\Sigma}_{\mathbf{z}_i}$ is the variance-covariance matrix of $\mathbf{z}_i$. The value h is calculated via simulation to achieve a specified in-control ARL. The ARL performance of the multivariate EWMA control chart depends only on the noncentrality parameter. This means that the multivariate EWMA has the property of directional invariance. The variance-covariance matrix of $\mathbf{z}_i$ is calculated by the following formula

$$\mathbf{\Sigma}_{\mathbf{z}_i} = \sum_{j=1}^{i} Var\left[\mathbf{R}\left(\mathbf{I}-\mathbf{R}\right)^{i-j}\left(\mathbf{x}_j - \boldsymbol{\mu}_0\right)\right] = \sum_{j=1}^{i} \mathbf{R}\left(\mathbf{I}-\mathbf{R}\right)^{i-j} \mathbf{\Sigma} \left(\mathbf{I}-\mathbf{R}\right)^{i-j} \mathbf{R}$$

or in case that $r_1 = r_2 = ... = r_p = r$

$$\mathbf{\Sigma}_{\mathbf{z}_i} = \left(1 - (1-r)^{2i}\right) r / \left(2-r\right) \mathbf{\Sigma}.$$

An approximation of the variance-covariance matrix $\mathbf{\Sigma}_{\mathbf{z}_i}$ when i approaches infinity, is the following

$$\mathbf{\Sigma}_{\mathbf{z}_i} = \frac{r}{2-r} \mathbf{\Sigma}.$$

However, the use of the exact variance-covariance matrix of the multivariate EWMA, leads to a natural fast initial response for the multivariate EWMA chart.

In a univariate EWMA chart if the plotted statistic is on one side of the center line and a shift occurs on the other side the result is that the EWMA chart will need more observations until it signals. Such a problem is called inertia problem. Inertia problem may occur with the multivariate EWMA chart and the simultaneous use of a Shewhart type chart is proposed.

Lowry et al. (1992) studied the ARL of the multivariate EWMA. The ARL performance of the multivariate EWMA procedure depends only on $\boldsymbol{\mu}_0$ and $\boldsymbol{\Sigma}_0$ through the value of the noncentrality parameter. Since, the multivariate EWMA, the MCUSUM#1 and the vector CUSUM are all directionally invariant, these three charts can be compared to each other and to Hotelling's (1947) T^2-Chart. The comparison of these charts shows that the ARL performance of the multivariate EWMA is at least as good as those of vector-valued CUSUM and MCUSUM#1.

Rigdon (1995a, 1995b) gives an integral and a double integral equation for the calculation of in-control and out-of-control ARLs respectively. Moreover, Bodden and Rigdon (1999) developed a computer program for approximating the in-control ARL of the multivariate EWMA chart. Runger and Prabhu (1996) use a Markov chain approximation to determine the run length performance of the multivariate EWMA chart. This kind of analysis, can be applied whenever the multivariate control statistic can be modeled as a Markov chain and has the property of directional invariance. In addition, Prabhu and Runger (1997) provide recommendations for the selection of parameters for a multivariate EWMA chart. Molnau et. al. (2001) present a program that calculates the ARL for the multivariate EWMA when the values of the shift in the mean vector, the control limit and the smoothing parameter are known.

Kramer and Schmid (1997), proposed a generalization of the multivariate EWMA control scheme of Lowry et al. (1992) for multivariate time dependent observations. Sullivan and Woodall (1998) recommended the use of multivariate EWMA for the preliminary analysis of multivariate observations. Fasso (1999) developed a one-sided multivariate EWMA control chart, based on the restricted Maximum Likelihood Estimator.

Yumin (1996) proposed the construction of a multivariate EWMA using the principal components of the original variables. Runger et al. (1999) show how the shift detection capability of the multivariate EWMA can be significantly improved by transforming the original process variables to a lower-dimensional subspace through the use of the U transformation. The U transformation is similar to principal components transformation.

Margavio and Conerly (1995) developed two alternatives for the multivariate EWMA chart. The first of these alternatives is an arithmetic multivariate moving average while the second alternative is a truncated version of the multivariate EWMA.

Chapter 3

Estimation Effect in Control Charts

3.1 Introduction

A feature that may affect the performance of a control chart is the estimation effect. In this chapter we present the current status of research of this field and some new results. In Section 3.2, we present the case on the estimation effect issue in univariate and multivariate Shewhart charts. New results on the effect of estimation on the values of average run length (ARL) and standard deviation of the run length (SDRL) of the S chart with three sigma and probability limits in the case of subgroups are also presented. Corresponding results for the X chart for individual observations are also presented. In Section 3.3 we refer to the estimation effect in the EWMA chart.

3.2 Estimation Effect in Univariate and Multivariate Shewhart Charts

The estimation effect issue in Shewhart charts was investigated by many authors. Proschan and Savage (1960) considered the effect of the number of samples and sample size on the performance of the $\overline{X}$ chart in terms of the probability of the mean plotting outside the control limits if we are in-control when the average range or a pooled estimate of the variance is used as an estimate of the process variability. They provided values for the number of samples needed for keeping stable the probability of the mean plotting outside the control limits if we are in-control for given values of the sample size.

Table 3.1.Correlation for several values of m and n

	n			
m	5	10	20	50
5	0.46581	0.37055	0.30735	0.25370
10	0.30362	0.22741	0.18158	0.14528
20	0.17898	0.12829	0.09986	0.07833
30	0.12689	0.08935	0.06886	0.05362
50	0.08020	0.05560	0.04249	0.03288
100	0.04178	0.02859	0.02171	0.01671
200	0.02133	0.01450	0.01097	0.00843
500	0.00864	0.00585	0.00442	0.00339
1000	0.00434	0.00293	0.00221	0.00170

However, they did not take into account the dependence between the event that the sample mean of sample i exceeds UCL and the event that the sample mean of another sample j exceeds UCL. Therefore, there results are of limited use. Hillier (1969) dealt with the problem of estimated control limits in the case of $\overline{X}$ and R chart. He provided a method of evaluating the probability of the mean plotting outside the control limits in the case of the $\overline{X}$ with the range R used to compute the process variability. This method did not consider the dependence issue as the method of Proschan and Savage (1960), consequently we can not base the design of our chart on these results.

Ghosh et al. (1981) gave formulas for the computation of the run length distribution in the case of the $\overline{X}$ chart with unknown variance. Quesenberry (1993) examined the effect of estimation of the process mean and standard deviation on the control limits of the Shewhart chart for the mean for both rational subgroups and individual observations.

Table 3.2. ARL and SDRL values for the S (three sigma) chart when $n = 5$

	σ_1^2/σ_0^2									
	1		1.2		1.4		1.6		1.8	
m	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
5	$4 \cdot 10^5$	$1 \cdot 10^5$	2223.5	$7 \cdot 10^4$	594.02	$2 \cdot 10^4$	105.38	1353.3	37.41	143.16
10	2200.1	$3 \cdot 10^4$	310.65	2288.7	86.99	330.39	39.29	104.72	21.06	45.34
20	551.16	1699.7	139.42	297.08	54.88	95.14	27.47	41.75	16.48	21.70
30	415.06	840.14	112.55	182.48	48.74	74.75	25.79	34.47	15.52	18.54
50	346.68	545.72	101.62	134.96	43.32	55.43	23.36	27.19	14.73	16.19
100	298.59	407.05	91.09	106.99	40.75	44.42	22.35	23.56	14.20	14.55
200	276.08	318.09	85.28	93.97	39.28	41.12	21.55	22.18	13.93	13.92
500	262.29	275.14	85.20	88.07	38.55	39.24	21.75	21.79	13.94	13.52
1000	253.76	258.84	84.37	87.67	37.32	37.06	20.97	20.66	13.59	13.14
∞	249.31	248.81	82.44	81.94	37.72	37.21	21.22	20.71	13.69	13.18

He proved that

$$Corr(\overline{X}_i - \widehat{UCL}, \overline{X}_j - \widehat{LCL}) = \frac{Var(\widehat{UCL})}{Var(\overline{X}_i - \widehat{UCL})} = \left[1 + m\left(1 + \frac{9\,(1 - c_4^2)}{c_4^2}\right)^{-1}\right]^{-1},$$

which means that there is a correlation between the events $\overline{X}_i - \widehat{UCL}$ and $\overline{X}_j - \widehat{LCL}$. He concluded that $\overline{X}$ chart requires about $400/(n-1)$ samples for estimating the parameters in order for the estimated control limits to behave as the theoretical ones, where n is the subgroup size. In the case of individual observations he showed that 300 observations are needed for the estimated control limits to behave as the theoretical ones. The control chart he used is the X chart with the variability estimated by the moving range.

Chen (1997) extended this work by using three different estimators of the standard deviation in the $\overline{X}$ chart case. Let $X_{ij}, i = 1, 2, ...m$ and $j = 1, 2, ..., n$ represent data from a period known to operate in-control and let $Y_{ij}, i = 1, 2, ...$ and $j = 1, 2, ..., n$ represent

Table 3.3. ARL and SDRL values for the S (three sigma) chart when $n = 10$

	σ_1^2/σ_0^2									
	1		1.2		1.4		1.6		1.8	
m	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
5	606.61	1064.81	236.14	634.06	78.31	263.87	29.43	112.67	14.39	41.18
10	538.65	919.10	145.57	329.89	45.83	99.37	19.21	31.64	10.17	13.87
20	461.44	725.80	106.92	175.82	33.86	48.22	15.85	19.75	9.04	10.34
30	430.50	626.79	95.59	137.72	32.34	40.07	15.02	17.08	8.59	9.14
50	389.91	510.09	88.05	106.54	30.29	33.98	14.38	15.65	8.37	8.58
100	359.35	411.69	80.89	88.11	28.79	30.81	13.85	14.24	8.26	8.16
200	344.38	367.08	78.19	82.25	28.46	28.96	13.43	13.31	7.97	7.58
500	334.53	340.97	76.10	76.60	27.45	27.27	13.52	13.14	8.06	7.65
1000	334.56	337.96	75.88	75.93	27.31	27.01	13.50	13.04	7.98	7.48
∞	331.17	330.67	75.66	75.16	27.52	27.01	13.47	12.96	8.00	7.48

current or future data. Also, let $X_{ij} \backsim N(\mu, \sigma^2)$ and $Y_{ij} \backsim N(\mu + \alpha\sigma, b^2\sigma^2)$ with α, b constants. Since $\overline{\overline{X}} \backsim N(\mu, \sigma^2/(mn))$ and $\overline{Y}_i \backsim N(\mu + \alpha\sigma, b^2\sigma^2/n)$ for given $\overline{\overline{X}} = \overline{\overline{x}}$ and given $\widehat{\sigma}$ we have

$$P(\overline{Y}_i < \widehat{LCL} \text{ or } \overline{Y}_i > \widehat{UCL}|\overline{\overline{x}}, \widehat{\sigma}) = 1 - \Phi\left(\frac{z}{b\sqrt{m}} + \frac{3}{b}w - \frac{\alpha}{b}\sqrt{n}\right) + \Phi\left(\frac{z}{b\sqrt{m}} - \frac{3}{b}w - \frac{\alpha}{b}\sqrt{n}\right),$$

where $z = (\overline{\overline{x}} - \mu) / (\sigma/\sqrt{mn})$ and $w = \widehat{\sigma}/\sigma$. Then, the ARL is computed through the following relation

$$ARL = \int_{-\infty}^{+\infty}\int_{0}^{+\infty} \frac{1}{P(\overline{Y}_i < \widehat{LCL} \text{ or } \overline{Y}_i > \widehat{UCL}|\overline{\overline{x}}, \widehat{\sigma})} \frac{1}{\sqrt{2\pi}} \exp\left(-0.5z^2\right) f(w) dz dw,$$

where $f(w)$ is calculated for three different estimators of σ. For a detailed discussion on the different estimators of σ, see Vardeman (1999).

Table 3.3.(continued) ARL and SDRL values for the S (three sigma) chart when $n = 10$

	σ_1^2/σ_0^2							
	0.2		0.4		0.6		0.8	
m	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
5	24.01	37.43	306.28	520.37	1019.6	1274.5	1136.2	1433.6
10	21.04	25.88	254.28	377.34	1071.7	1253.2	1316.9	1514.7
20	19.77	22.62	230.60	275.74	1079.4	1207.5	1472.6	1603.4
30	19.15	20.62	223.33	249.71	1056.5	1155.1	1569.2	1656.9
50	18.47	19.15	218.20	229.50	1047.3	1106.2	1644.7	1686.9
100	18.21	18.10	210.57	215.40	1037.5	1061.3	1696.2	1729.9
200	17.95	17.93	205.32	205.49	1023.1	1027.8	1744.5	1746.7
500	18.20	17.79	205.59	203.14	1009.1	1022.0	1773.9	1785.0
1000	17.53	17.28	206.96	204.95	1006.7	1007.9	1768.3	1773.9
∞	17.90	17.39	206.06	205.56	1011.7	1011.2	1777.2	1776.7

Nedumaran and Pignatiello (2001) developed new control limits for the $\overline{X}$ chart taking into account the estimation effect. Specifically, let $\overline{X}_i$ be the average of a future subgroup, $\overline{V}$ be the average variance of the m initial in-control subgroups and $T_i = \frac{\overline{X}_i - \overline{\overline{X}}}{\sqrt{\frac{m+1}{mn}}\sqrt{\overline{V}}}$. Then, $(T_{m+1}, T_{m+2}, ..., T_{m+k})$ has a positively equicorrelated multivariate t distribution with correlation $1/(m+1)$, where k is a specified number of future subgroups. If $P\left(\widehat{LCL} \leq \overline{X}_i \leq \widehat{UCL}\right) = 1 - \gamma$, $i = m+1, m+2, ..., m+k$ then γ must be equal to the run length distribution percentile when we have true limits, for the estimated limits to have equivalent performance with the true ones. Then $\gamma = P[RL \leq k] = 1 - (1-\alpha)^k$ where α is the probability of a false alarm for a single subgroup. If $P\left[\max_{m+1 \leq i \leq m+k} |T_i| \leq h'_{\gamma,m,k,\nu}\right] = 1 - \gamma$ where $\nu = m(n-1)$, we have that $P\left[\overline{\overline{X}} - h'_{\gamma,m,k,\nu}\sqrt{\frac{m+1}{mn}}\sqrt{\overline{V}} \leq \overline{X}_i \leq \overline{\overline{X}} + h'_{\gamma,m,k,\nu}\sqrt{\frac{m+1}{mn}}\sqrt{\overline{V}}\right] = 1-\gamma$. Consequently, the control limits are

$$\begin{aligned}\widehat{UCL} &= \overline{\overline{X}} + h'_{\gamma,m,k,\nu}\sqrt{\frac{m+1}{mn}}\sqrt{\overline{V}}\\ \widehat{LCL} &= \overline{\overline{X}} - h'_{\gamma,m,k,\nu}\sqrt{\frac{m+1}{mn}}\sqrt{\overline{V}}.\end{aligned}$$

Table 3.4. ARL and $SDRL$ values for the S (three sigma) chart when $n = 20$

	σ_1^2/σ_0^2									
	1		1.2		1.4		1.6		1.8	
m	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
5	332.72	444.02	121.96	244.63	32.29	78.86	11.46	27.03	5.54	10.33
10	362.96	457.01	92.99	166.56	23.32	45.79	8.71	11.78	4.62	5.39
20	371.24	439.25	75.00	115.39	19.63	25.36	7.87	8.77	4.32	4.20
30	372.32	430.13	68.53	86.90	18.23	21.84	7.67	8.17	4.29	4.12
50	362.66	403.51	63.80	76.74	17.52	18.73	7.49	7.60	4.24	3.97
100	364.01	393.80	60.17	65.57	17.01	17.34	7.36	7.15	4.11	3.58
200	359.00	374.30	59.56	60.31	16.61	16.45	7.15	6.82	4.11	3.59
500	355.18	358.14	59.11	59.61	16.36	16.15	7.13	6.70	4.09	3.56
1000	353.23	353.28	57.59	57.23	16.26	15.79	7.15	6.66	4.08	3.59
∞	356.50	356.00	57.37	56.87	16.39	15.88	7.15	6.63	4.07	3.53

In the case of the attributes charts p and c with estimated control limits Braun (1999) computed the run length distributions. If W is the run length until the next signal we have that

$$P(W \leq w) = 1 - \sum_x P\left(\sum_{j=1}^m X_j = x\right)\left(1 - P\left(F_1 | \sum_{j=1}^m X_j = x\right)\right)^w,$$

where F_i is the event that the ith new observation is outside the estimated control limits.

In the case of the c chart we have that

$$P\left(F_1|\sum_{j=1}^{m}X_j=x\right)=1-\sum_{j=\left[x/m-3\sqrt{x/m}\right]+1}^{\left[x/m+3\sqrt{x/m}\right]}\frac{e^{-bc}\left(bc\right)^j}{j!},x=0,1,2,...$$

and in the case of the p chart

$$P\left(F_1|\sum_{j=1}^{m}X_j=x\right)=1-\sum_{j=\left[nx/m-3\sqrt{nx/m(1-x/m)}\right]+1}^{\left[nx/m+3\sqrt{nx/m(1-x/m)}\right]}\binom{n}{j}\left(bp\right)^j\left(1-bp\right)^{n-j},$$

where $x=0,1/n,2/n,...,(mn-1)/n,mn/n$. In the case of the c chart $\sum_{j=1}^{m}X_j$ is distributed as a Poisson random variable with mean mc therefore $P\left(\sum_{j=1}^{m}X_j\right)=\frac{e^{-mc}(mc)^x}{x!}$, $x=0,1,2,...$ In the case of the p chart $n\sum_{j=1}^{m}X_j$ is distributed as a Binomial random variable with parameters mn and p that is $P\left(\sum_{j=1}^{m}X_j\right)=\binom{mn}{nx}p^{nx}\left(1-p\right)^{(m-x)n}$, $x=0,1/n,2/n,...,(mn-1)/n,mn/n$. Finally, the ARL is equal to

$$ARL=\sum_x P\left(\sum_{j=1}^{m}X_j=x\right)P\left(F_1|\sum_{j=1}^{m}X_j=x\right)^{-1}$$

Braun (1999) showed that, as for variables control charts, the estimation effect can be serious.

Yang et al. (2002) examined the case of the Geometric chart with estimated control limits. The run length distribution in this case is equal to

$$P\left(R\leq r;p,p_0\right)=\sum_{n=0}^{m}\left[1-\alpha\left(n\right)\right]^{r-1}\alpha\left(n\right)\binom{m}{n}p_0^n\left(1-p_0\right)^{m-n},$$

where $\alpha\left(n\right)=\left(1-p\right)^{\ln(\alpha/2)[\ln(1-n/m)]}-\left(1-p\right)^{\ln(1-\alpha/2)[\ln(1-n/m)]}+1$, p_0 is the fraction nonconforming, m is the sample size and n is the number of nonconforming items. The

ARL in this case is equal to

$$ARL = \sum_{n=0}^{m} \frac{1}{\alpha(n)} \binom{m}{n} p_0^n (1-p_0)^{m-n}.$$

Yang et al. (2002) showed that the effect on the alarm probability is significant even when the sample size is very large e.g. 10000. Despite that fact, the ARL is not affected that seriously, unless we have a small sample size and a large process improvement.

Nedumaran and Pigniatiello (1999) investigated the estimation effect on the T^2 control charts. They proposed that the number of subgroups needed for the estimated control limits to behave as the theoretical ones must be between $800p/3(n-1)$ and $400p/(n-1)$, where p is the number of variables and n is the sample size. Moreover, they gave an exact procedure for the construction of the T^2 control charts when we estimate the parameters so as to perform similar to the ones with known parameters.

Table 3.4. (continued) ARL and $SDRL$ values for the S(three sigma) chart when $n = 20$

	σ_1^2/σ_0^2							
	0.2		0.4		0.6		0.8	
m	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
5	1.32	0.75	11.92	18.83	111.01	190.51	383.43	457.13
10	1.28	0.64	10.03	12.23	90.20	127.11	423.04	463.05
20	1.26	0.60	9.21	9.96	80.28	94.68	442.70	473.45
30	1.26	0.58	8.97	9.20	78.03	88.22	444.96	471.63
50	1.24	0.54	8.90	8.59	75.70	80.02	451.20	469.84
100	1.25	0.57	8.68	8.20	73.42	75.19	450.90	455.92
200	1.23	0.54	8.70	8.27	73.69	74.61	447.50	446.43
500	1.24	0.55	8.55	8.05	73.62	73.39	441.19	441.96
1000	1.24	0.56	8.54	8.14	72.08	71.77	445.81	448.49
∞	1.24	0.54	8.56	8.04	72.91	72.41	449.79	449.29

Woodall and Montgomery (1999) emphasized the need for much more research in this area since it is proved that more data than usually recommended is needed for the control charts to behave as expected from theory. In the same paper, Woodall and Montgomery state that much work has been done concerning the control of the process mean but not that much for the process dispersion.

Table 3.5. ARL and SDRL values for the S (three sigma) chart when $n = 50$

	σ_1^2/σ_0^2									
	1		1.2		1.4		1.6		1.8	
m	*ARL*	*SDRL*	*ARL*	*SDRL*	*ARL*	*SDRL*	*ARL*	*SDRL*	*ARL*	*SDRL*
5	263.03	325.32	59.79	125.84	9.49	18.56	3.23	3.96	1.86	1.57
10	304.52	359.74	44.11	76.18	7.69	9.98	2.89	2.87	1.73	1.23
20	328.25	365.28	36.59	49.56	6.91	7.54	2.83	2.52	1.69	1.15
30	340.23	369.51	33.55	39.88	6.65	6.77	2.76	2.37	1.68	1.11
50	345.02	369.81	32.36	35.89	6.64	6.61	2.72	2.24	1.67	1.09
100	355.17	366.97	30.64	31.98	6.37	6.11	2.7	2.2	1.67	1.08
200	357.85	364.35	30.75	30.97	6.39	6.06	2.67	2.09	1.67	1.06
500	362.32	358.59	30.32	30.28	6.38	5.87	2.65	2.1	1.67	1.06
1000	356.30	352.76	30.62	29.97	6.29	5.89	2.67	2.08	1.67	1.05
∞	365.96	365.46	30.23	29.72	6.35	5.83	2.67	2.11	1.66	1.04

Chen (1998) deals with the run length properties of the R, s and s^2 control charts in the case that σ is estimated. Let $X_{ij}, i = 1, 2, ...m$ and $j = 1, 2, ..., n$ denote historically in-control data and $Y_{ij}, i = 1, 2, ...$ and $j = 1, 2, ..., n$ represent current or future data. Let $X_{ij} \backsim f((x-\mu)/\sigma)/\sigma$ and $Y_{ij} \backsim f((y-\mu)/(b\sigma))/(b\sigma)$ with b constant, μ, σ the process mean and standard deviation respectively and $f(\cdot)$ the form of the known density function. Denote $U = \widehat{\sigma}/\sigma$, where $\widehat{\sigma}$ is an estimate of σ calculated from the historical data set and $U \backsim h(u; m, n)$. Let $T_i = \widehat{\sigma}_i/\sigma$, where $\widehat{\sigma}_i$ is an estimate of $b\sigma$ using Y_{ij}. Also, denote $G(t;b,n) = P(T_i \leq t)$. Then, if L_n and U_n are the constants multiplied with σ for the known lower and upper control limits case respectively, we have that

$$P(\widehat{\sigma}_i < L_n\widehat{\sigma} \text{ or } \widehat{\sigma}_i > U_n\widehat{\sigma}|\widehat{\sigma}) = G(L_n u/b;b,n) + 1 - G(U_n u/b;b,n) = l(u;b,n),$$

where $u = \widehat{\sigma}/\sigma$. Then, the ARL is computed through the following relation

$$ARL = \int_0^{+\infty} \frac{1}{l(u;b,n)} h(u;m,n)du.$$

Table 3.5. (continued) ARL and SDRL values for the S (three sigma) chart when $n = 50$

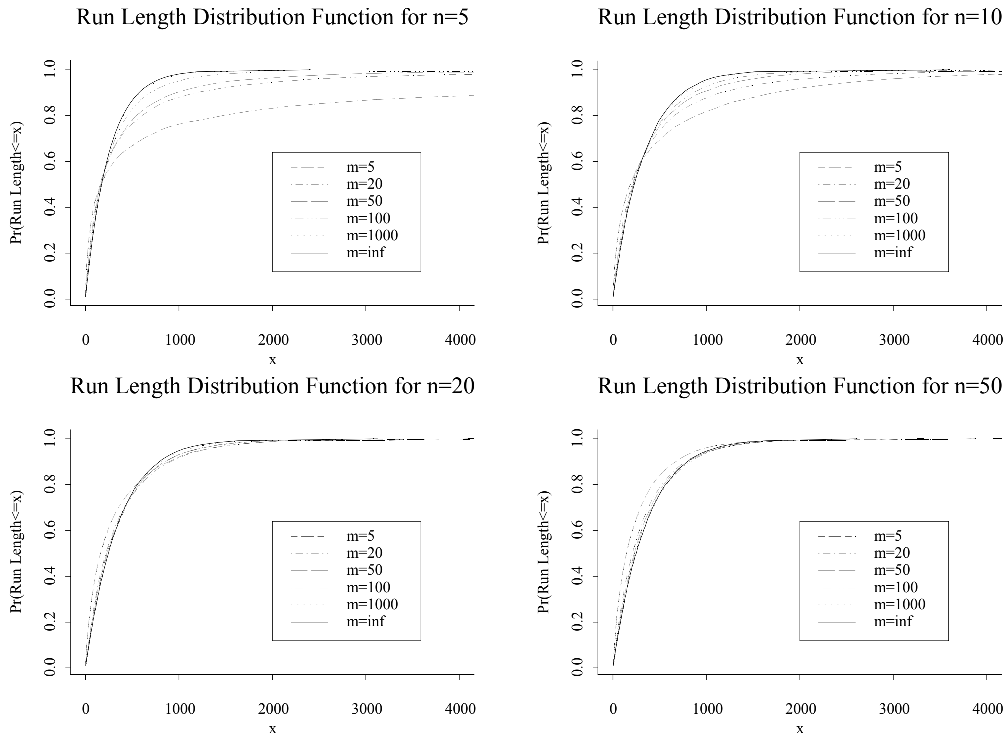
	σ_1^2/σ_0^2							
	0.2		0.4		0.6		0.8	
m	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
5	1	0	1.25	0.66	8.68	13.45	124.56	199.43
10	1	0	1.23	0.56	7.20	8.36	110.20	171.69
20	1	0	1.21	0.51	6.80	7.10	97.84	128.55
30	1	0	1.20	0.50	6.55	6.53	93.47	110.48
50	1	0	1.20	0.48	6.51	6.38	89.64	98.31
100	1	0	1.20	0.48	6.44	6.04	85.98	91.30
200	1	0	1.19	0.47	6.37	5.91	85.92	88.10
500	1	0	1.18	0.47	6.34	5.92	85.47	85.74
1000	1	0	1.18	0.47	6.26	5.82	85.82	85.86
∞	1	0	1.19	0.48	6.28	5.76	84.25	83.75

Maravelakis, Panaretos and Psarakis (2002) examine the effect of estimation of the process parameters on the control limits of charts for process dispersion by extending the results of Chen (1998) for both rational subgroups and individual observations. In sections 3.2.1-3.2.4 we present this work.

3.2.1 The S (Three Sigma) Control Chart

Assume that we have the control limits (2.6) and their estimated counterparts in (2.8). Let A_i denote the event that the ith sample standard deviation S_i exceeds UCL or is exceeded by LCL. Then, since S_i and S_j are independent for $i \neq j$, the sequence of trials A_i and A_j are independent meaning that they constitute a sequence of Bernoulli trials.

Figure 3.1. Empirical Run Length Distribution Functions for the 3 sigma chart



The mean and standard deviation of the run length distribution, ARL and $SDRL$ respectively, of this process is that of a geometric distribution given by the following formulas

$$
\begin{aligned}
ARL &= \frac{1}{1-\beta} \\
SDRL &= \frac{\sqrt{\beta}}{1-\beta}
\end{aligned}
\tag{3.1}
$$

where $\beta = 1 - \Pr(A_i) = \Pr(LCL \leq S_i \leq UCL)$.

Assume now that we are in the case when the true value of the standard deviation is not known, which is the most usual case. Let B_i denote the event that the ith sample standard deviation S_i exceeds $\widehat{UCL}$ or is exceeded by $\widehat{LCL}$.

Table 3.6.Correlation for several values of m and n

	n			
m	5	10	20	50
5	0.51095	0.39568	0.32137	0.26032
10	0.34314	0.24663	0.19144	0.14964
20	0.20710	0.14066	0.10585	0.08087
30	0.14831	0.09839	0.07315	0.05541
50	0.09460	0.06145	0.04521	0.03400
100	0.04965	0.03170	0.02313	0.01729
200	0.02545	0.01611	0.01170	0.00872
500	0.01034	0.00650	0.00471	0.00351
1000	0.00520	0.00326	0.00236	0.00176

The formulas (3.1) for ARL and $SDRL$ are not valid any more because the events B_i and B_j are not independent for $i \neq j$. We can prove that $E(\widehat{UCL}) = UCL$ and $Var(\widehat{UCL}) = \left(1 + \frac{3}{c_4}\sqrt{1 - c_4^2}\right)^2 \sigma^2 \frac{(1-c_4^2)}{m}$ and using these relations we prove after some calculations that

$$Cov(S_i - \widehat{UCL}, S_j - \widehat{LCL}) = Var(\widehat{UCL}) = \left(1 + \frac{3}{c_4}\sqrt{1 - c_4^2}\right)^2 \sigma^2 \frac{(1 - c_4^2)}{m}$$

and

$$Var(S_i - \widehat{UCL}) = \left[1 + \frac{\left(1 + \frac{3}{c_4}\sqrt{1 - c_4^2}\right)^2}{m}\right] \sigma^2(1 - c_4^2)$$

Therefore, the correlation between the random variables $S_i - \widehat{UCL}$ and $S_j - \widehat{LCL}$ is

$$Corr(S_i - \widehat{UCL}, S_j - \widehat{LCL}) = \frac{Var(\widehat{UCL})}{Var(S_i - \widehat{UCL})} = \frac{\left(1 + \frac{3}{c_4}\sqrt{1 - c_4^2}\right)^2}{m + \left(1 + \frac{3}{c_4}\sqrt{1 - c_4^2}\right)^2}$$

Table 3.7. ARL and $SDRL$ values for the S (probability limits) chart when $n = 5$

	σ_1^2/σ_0^2									
	1		1.2		1.4		1.6		1.8	
m	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
5	359.97	463.12	267.35	405.54	173.40	312.06	111.11	231.57	71.17	173.0
10	401.46	491.51	268.52	395.19	154.68	263.77	83.88	161.01	47.93	102.77
20	441.09	495.15	254.39	350.22	127.40	199.04	64.92	106.92	36.69	58.40
30	462.04	509.78	247.68	320.05	115.34	164.84	58.02	84.90	33.35	49.45
50	472.24	504.56	239.29	295.97	108.19	137.80	52.48	65.23	30.29	35.47
100	489.90	512.64	229.28	262.50	99.08	115.37	49.79	54.68	28.81	31.03
200	498.35	505.20	221.61	240.21	94.66	102.45	48.20	50.97	27.67	29.10
500	500.93	505.59	216.74	223.58	93.45	95.24	46.06	46.00	28.00	27.60
1000	497.73	503.09	213.01	217.36	92.29	94.50	47.12	47.08	27.31	26.70
∞	500.02	499.52	214.74	214.24	91.78	91.28	46.51	46.01	27.33	26.82

It is obvious that the correlation is a function of m and n only. In Table 3.1 we present values of the correlation for combinations of m and n. From this Table we see that as the sample size and the number of samples increases the correlation decreases. For small or moderate sample size ($n \leq 20$) we need 200 samples for the correlation to be negligible. However, for larger sample size the value $m = 50$ is suitable.

In order to examine the values of the first two moments of the run length distribution, we performed a simulation study based on various numbers of samples and various sample sizes. In particular the number of samples and samples sizes considered were $m = 5$, 10, 20, 30, 50, 100, 200, 500, 1000 and $n = 5$, 10, 20, 50. For every combination of m and

n we simulated m samples of size n from a $N(\mu, \sigma_0^2)$ distribution and computed $\widehat{UCL}$ and $\widehat{LCL}$. Then, we simulated samples from a $N(\mu, \sigma_1^2)$ distribution until we obtained a value above $\widehat{UCL}$ or below $\widehat{LCL}$. The number of samples simulated up to the one that lead to a value outside the control limits constitutes one observation of the run length distribution. This procedure was repeated 10000 times in order to get estimates of the values of ARL and $SDRL$. The results are presented in Tables $3.2 - 3.5$.

Table 3.7.(continued) ARL and $SDRL$ values for the S (probability limits) chart when $n = 5$

	σ_1^2/σ_0^2							
	0.2		0.4		0.6		0.8	
m	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
5	59.28	92.90	207.50	279.98	367.15	426.84	423.91	494.16
10	51.33	60.99	188.80	229.67	383.62	419.59	478.62	506.71
20	49.25	53.77	178.80	195.31	381.22	406.88	535.06	558.84
30	47.36	50.56	174.26	182.63	378.10	395.01	551.71	561.82
50	47.06	47.90	172.37	175.45	374.69	387.19	572.90	579.90
100	46.45	46.53	170.21	172.41	369.25	373.72	588.21	585.37
200	44.99	44.60	169.92	169.76	369.89	371.74	595.42	594.84
500	45.64	45.22	168.09	168.67	364.29	363.97	604.03	601.67
1000	45.64	44.65	165.86	166.04	364.05	369.30	598.0	601.84
∞	45.09	44.59	167.40	166.90	366.87	366.37	597.91	597.41

From Tables 3.2 through 3.5 certain conclusions are drawn. We see that we have results for both upward and downward shifts when $n > 5$ but only for upward when $n = 5$. This happens because for $n \leq 5$ the lower control limit is set to zero. Therefore, it can never be crossed. For upward shifts as m increases the ARL and $SDRL$ values decrease and approach their theoretical values. For downward shifts as m increases the same thing happens for $n = 50$. For $n = 10, 20$ the ARL and $SDRL$ values do not follow a specific trend. In the in-control state we also do not have a clear pattern for either ARL or $SDRL$ values. What we can say in every case is that ARL and $SDRL$ values behave in the same way.

Table 3.8. ARL and $SDRL$ values for the S (probability limits) chart when $n = 10$

	σ_1^2/σ_0^2									
	1		1.2		1.4		1.6		1.8	
m	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
5	341.44	422.99	217.14	329.46	110.54	218.04	52.43	130.38	25.60	62.05
10	391.03	456.05	208.08	307.21	86.61	155.83	36.61	67.22	17.72	28.51
20	428.95	469.37	194.55	257.65	70.14	106.07	28.50	39.48	14.96	18.23
30	448.41	480.41	187.90	234.75	65.03	88.79	27.33	33.58	14.20	16.15
50	464.28	481.37	178.40	209.85	60.27	72.62	25.81	28.64	13.61	14.78
100	479.05	488.20	169.77	184.28	56.35	61.10	24.52	25.62	13.16	13.69
200	484.86	493.03	166.70	176.09	54.73	56.70	24.26	24.50	12.81	12.66
500	490.54	489.97	161.32	164.74	52.91	53.41	24.02	24.08	13.11	12.82
1000	492.16	480.65	161.60	161.87	53.81	53.11	23.60	23.23	12.70	12.38
∞	500.05	499.55	161.99	161.48	53.44	52.94	23.46	22.95	12.74	12.23

As m increases the ARL is getting closer to the theoretical value faster than the $SDRL$. Moreover, as n increases the theoretical values, in the in-control state, approach the ones from a normal distribution, which are $ARL = 370.4$ and $SDRL = 369.9$. The same of course happens and for the out-of-control states.

If we use this type of chart for identifying shifts in process dispersion we have to use samples of size n at least 20, for minimizing the effect of estimating S. If n is less than this value the practitioner will face an increased number of false alarms. The effect of estimation is also severe for $m \leq 20$, especially in the in-control state and for small shifts.

For values $30 \leq m \leq 50$ the effect is moderate and for values of 100 or larger the effect is small enough. A last point we have to make is that when we have small downward shifts for $n \leq 20$ the ARL and $SDRL$ values are larger than the corresponding in-control values. This result is also confirmed by Klein (2000). Consequently, in such cases special care must be given and it is better to use control charts for small shifts like CUSUM and EWMA.

Table 3.8. (continued) ARL and $SDRL$ values for the S (probability limits) chart when $n = 10$

	σ_1^2/σ_0^2							
	0.2		0.4		0.6		0.8	
m	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
5	5.34	7.41	46.09	81.74	182.01	262.85	339.52	406.60
10	4.63	4.92	38.28	48.93	162.80	206.09	378.42	422.41
20	4.40	4.27	35.18	40.33	154.72	181.67	396.79	417.98
30	4.36	4.10	34.06	37.31	147.37	159.98	400.78	424.06
50	4.20	3.76	33.36	35.08	144.77	156.54	401.66	421.77
100	4.27	3.81	32.66	34.24	139.43	140.95	402.89	413.00
200	4.26	3.73	32.78	33.25	137.34	137.24	400.48	403.36
500	4.17	3.64	32.44	31.76	136.91	133.59	405.11	405.09
1000	4.21	3.63	31.95	31.10	133.69	132.26	398.98	400.52
∞	4.23	3.70	32.13	31.62	136.47	135.97	400.85	400.35

In Figure 3.1 we present the empirical run length distribution functions (ERL) for $n = 5, 10, 20, 50$. In each Figure we plot six different lines representing the ERL functions for $m = 5, 20, 50, 100, 1000$ and the theoretical run length distribution (inf). It is obvious that as m increases the ERL approaches the theoretical run length distribution. Moreover, as n increases the ERL's for the m values approach the theoretical run length distribution faster.

3.2.2 The S (Probability Limits) Control Chart

Consider the control limits 2.10 and 2.11. In the same way of thinking as in the case of three sigma limits we can prove that $Var(\widehat{UCL}) = [\sigma^2(1-c_4^2)\chi^2_{0.999}]/[(n-1)c_4^2 m]$ and consequently

$$Cov(S_i - \widehat{UCL}, S_j - \widehat{LCL}) = Var(\widehat{UCL}) = \frac{\sigma^2(1-c_4^2)\chi^2_{0.999}}{(n-1)c_4^2 m}.$$

Moreover,

$$Var(S_i - \widehat{UCL}) = \sigma^2(1-c_4^2)\left[1 + \frac{\chi^2_{0.999}}{(n-1)c_4^2 m}\right]$$

Table 3.9. ARL and $SDRL$ values for the S (probability limits) chart when $n = 20$

	σ_1^2/σ_0^2									
	1		1.2		1.4		1.6		1.8	
m	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
5	327.65	381.36	170.43	279.59	56.40	125.43	18.47	47.23	8.07	17.04
10	379.94	415.88	154.94	241.56	40.09	71.48	13.24	19.53	6.45	8.01
20	421.10	434.89	135.60	194.87	33.08	46.55	11.89	14.66	5.93	6.32
30	442.13	451.34	126.95	170.03	30.45	37.64	11.46	12.67	5.81	6.0
50	461.32	467.99	117.98	139.49	29.17	32.83	11.09	11.54	5.69	5.50
100	476.40	478.77	113.42	126.66	27.69	28.82	10.97	10.94	5.57	5.26
200	486.38	486.97	109.86	115.12	27.35	27.82	10.50	10.20	5.50	5.17
500	485.13	488.22	108.31	108.90	26.81	26.19	10.43	10.12	5.41	4.89
1000	494.29	488.10	106.57	108.41	26.63	25.79	10.30	9.78	5.48	4.91
∞	500.01	499.51	106.64	106.14	26.67	26.17	10.42	9.91	5.46	4.93

and finally

$$Corr(S_i - \widehat{UCL}, S_j - \widehat{LCL}) = \frac{Var(\widehat{UCL})}{Var(S_i - \widehat{UCL})} = \frac{\chi^2_{0.999}}{\chi^2_{0.999} + (n-1)c_4^2 m}.$$

Table 3.9.(continued) ARL and $SDRL$ values for the S (probability limits) chart when $n = 20$

	σ_1^2/σ_0^2							
	0.2		0.4		0.6		0.8	
m	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
5	1.17	.51	7.0	10.62	56.41	103.25	245.11	324.91
10	1.14	.43	6.11	7.0	45.91	69.83	247.16	304.65
20	1.13	.40	5.71	5.85	40.82	47.93	233.76	273.54
30	1.13	.39	5.44	5.32	40.07	44.78	228.37	256.46
50	1.12	.36	5.49	5.21	38.59	41.06	223.43	242.44
100	1.12	.37	5.36	4.95	37.85	39.05	219.40	225.87
200	1.11	.36	5.40	4.92	37.99	38.81	217.81	217.89
500	1.11	.36	5.36	4.87	37.41	36.61	213.94	209.40
1000	1.12	.37	5.29	4.80	37.69	37.00	213.70	213.52
∞	1.12	.36	5.29	4.77	37.44	36.94	215.93	215.43

As in the case of three sigma limits this correlation depends only on m and n. In Table 3.6 we calculated the correlation for various combinations of m and n. From this Table we conclude again that as the sample size and the number of samples increases the correlation decreases. The recommendation for sample sizes and number of samples is the same as in the case of three sigma limits.

We computed the ARL and $SDRL$ values for several values of m and n via simulation along the same lines as in the three sigma limits. The number of samples and sample sizes considered were $m = 5, 10, 20, 30, 50, 100, 200, 500, 1000$ and $n = 5, 10, 20, 50$. The results are presented on Tables 3.7 – 3.10. From Tables 3.7 through 3.10 we deduce the following points. For upward shifts as m increases the ARL and $SDRL$ values generally decrease and approach their theoretical values. For downward shifts as m increases the same thing happens for $n = 20,\ 50$. For $n = 5,\ 10$ the ARL and $SDRL$ values do not follow a specific pattern. In the in-control state the ARL and $SDRL$ values increase until they get close to their theoretical values, which is in accordance with the results of Chen (1998). As an overall conclusion we can say that the ARL and $SDRL$ values behave in the same way except that as m increases the ARL is getting closer to the theoretical value faster than the $SDRL$.

Table 3.10. ARL and $SDRL$ values for the S (probability limits) chart when $n = 50$

	σ_1^2/σ_0^2									
	1		1.2		1.4		1.6		1.8	
m	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
5	320.32	380.78	93.28	184.90	13.83	29.14	4.02	5.49	2.10	1.90
10	369.19	410.82	70.91	122.34	10.73	15.32	3.61	4.02	1.93	1.49
20	411.62	433.18	58.04	85.60	9.50	10.86	3.37	3.19	1.93	1.42
30	431.22	447.76	53.56	68.63	8.96	9.54	3.42	3.10	1.90	1.36
50	452.27	459.10	50.77	58.78	8.96	8.95	3.28	2.85	1.89	1.30
100	472.90	472.99	48.14	50.64	8.62	8.59	3.25	2.79	1.88	1.32
200	482.50	481.24	47.71	48.24	8.51	8.24	3.25	2.75	1.86	1.29
500	493.58	498.61	47.47	48.19	8.60	8.11	3.24	2.69	1.85	1.23
1000	490.32	499.04	47.59	47.66	8.56	8.10	3.23	2.66	1.86	1.27
∞	500.01	499.51	47.23	46.73	8.52	8.01	3.22	2.67	1.86	1.27

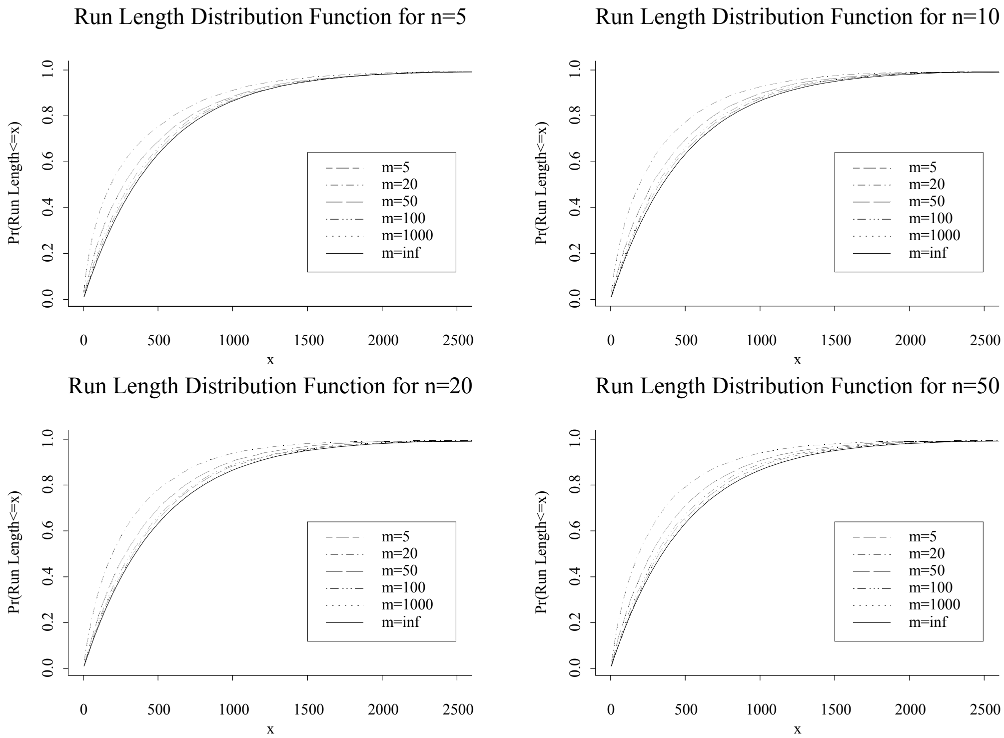
When we are in-control we need at least $m = 200$, otherwise the practitioner will face many false alarms whereas the value of n is not equally important. In the out-of-control situations the value of n is important for minimizing the effect of estimating S. Specifically, when $\sigma_1^2/\sigma_0^2 = 1.2$ the ARL values for $n = 5, 10, 20, 50$ are $239.29, 178.40, 117.98$ and 50.77 respectively. Therefore, we observe a dramatic reduction as n becomes larger. A similar situation occurs for downward shifts. Consequently, large values of n, larger than 20, are recommended. The effect of estimation is severe for $m \leq 20$, especially for small shifts. For values $30 \leq m \leq 50$ the effect is moderate and for values of 100 or larger the effect is small enough. When we have small downward shifts for $n = 5$, and for $n = 10$ when $m \leq 10$, the ARL and $SDRL$ values are larger than the corresponding in-control values. In such a situation it is better to use control charts for detecting small shifts like CUSUM and EWMA charts.

Table 3.10. (continued) ARL and $SDRL$ values for the S (probability limits) chart when $n = 50$

	σ_1^2/σ_0^2							
	0.2		0.4		0.6		0.8	
m	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
5	1	0	1.21	.62	7.47	11.58	107.80	188.75
10	1	0	1.19	.50	6.19	6.96	92.22	141.22
20	1	0	1.18	.47	5.86	5.90	79.17	102.14
30	1	0	1.16	.45	5.69	5.50	74.86	87.60
50	1	0	1.16	.45	5.65	5.46	71.97	78.71
100	1	0	1.15	.42	5.64	5.22	69.87	73.40
200	1	0	1.16	.43	5.62	5.18	69.61	69.69
500	1	0	1.15	.42	5.49	4.95	68.90	70.23
1000	1	0	1.15	.42	5.51	5.06	69.27	70.27
∞	1	0	1.16	.43	5.48	4.96	68.04	67.54

In Figure 3.2 we present the empirical run length distribution functions (ERL) for $n = 5, 10, 20, 50$. In each Figure we plot six different lines representing the ERL functions for $m = 5, 20, 50, 100, 1000$ and the theoretical run length distribution (inf). We see that as m increases the ERL approaches the theoretical run length distribution. Also, an increasing n value causes the ERL's for the m values to approach the theoretical run length distribution faster.

Figure 3.2. Empirical Run Length Distribution Functions for the probability limits chart



3.2.3 The X Chart for Monitoring Process Dispersion

Consider the control limits of Section 2.3.3. In order to assess the effect of the number of observations on the control limits of the X chart we performed a simulation study. The results are presented in Table 3.11. For each value in the Table, we simulated N values from a $N(\mu, \sigma_0^2)$ distribution, we computed the $\widehat{UCL}$ and $\widehat{LCL}$ and subsequently we generated values from a $N(\mu, \sigma_1^2)$ distribution until we obtained a value above $\widehat{UCL}$ or below $\widehat{LCL}$. The number of samples simulated up to the one that was outside the control limits constitutes one observation on the run length. This procedure was repeated 32000 times in order to get estimates of the values of ARL and $SDRL$.

Table 3.11. ARL and $SDRL$ values for the X control chart

	σ_1^2/σ_0^2									
	1		1.2		1.4		1.6		1.8	
N	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$	ARL	$SDRL$
30	986.31	5024.83	315.36	1058.44	147.93	439.79	84.36	187.50	53.74	98.54
50	614.94	1565.0	229.95	476.60	116.69	200.50	69.61	107.23	47.23	66.81
75	503.75	948.78	202.02	318.54	105.18	150.77	64.51	84.15	43.99	54.87
100	467.07	770.60	190.53	274.54	100.73	131.39	61.98	75.26	42.78	50.48
200	413.88	518.65	173.68	205.96	93.86	105.77	58.63	63.56	40.67	42.81
300	398.94	476.34	167.79	187.69	92.76	100.47	57.93	61.37	41.26	42.29
500	387.38	429.45	167.90	179.39	90.34	93.58	56.80	58.96	39.69	40.54
1000	379.32	401.55	162.96	168.50	89.12	91.10	57.03	57.78	39.90	39.85
2000	372.64	383.71	162.70	166.87	89.45	89.41	56.35	55.82	39.62	39.17
∞	370.40	369.90	162.08	161.58	89.05	88.55	56.48	55.98	39.45	38.95

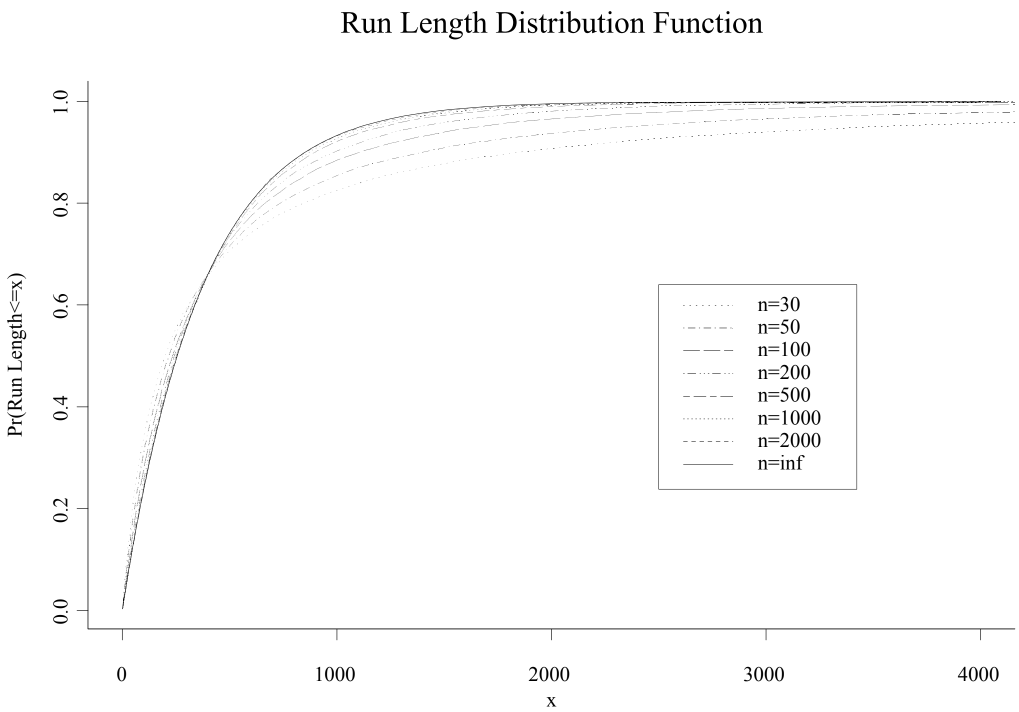
From Table 3.11 we see that we do not have results for downward shifts. This happens because a decreasing standard deviation will never cause a value below the lower control limit. The simulation reveals that the ARL and $SDRL$ values decrease until they approach their theoretical values. We need at least 300 observations to minimize the effect of estimation in the control limits of the X chart.

In Figure 3.3 we present the empirical run length distribution function (ERL) for $n = 30, 50, 100, 200, 500, 1000, 2000$ and the theoretical run length distribution (inf). The result is that as n increases the ERL approaches the theoretical run length distribution.

3.2.4 Discussion

In the rational subgroups case we propose larger n values than usual and someone may report that this is a problem. However, Woodall and Montgomery (1999) remarked that in industry now there are large data sets available in contrast to the past. Therefore, such values for the sample size should not be a problem, generally. On the other hand, if for some special applications this still remains a problem, the practitioner should keep in mind the great influence on the estimated control chart performance displayed on the tables of this work.

Figure 3.3. Empirical Run Length Distribution Functions for the X chart



3.3 Estimation Effect in the EWMA Chart

Jones et al. (2001), considered the problem of estimating the parameters of the EWMA chart in the normal case. They proved that if the random variable T is the run length of the EWMA chart, then the ARL of such a chart is given by

$$ARL = E\left[T|\gamma, \delta, u\right] = \int_{-\infty}^{\infty}\int_{0}^{\infty} M\left(w, z_0, \gamma, \delta, u\right) f_w\left(w\right)\phi\left(z_0\right) dwdz_0,$$

where $M\left(w, z_0, \gamma, \delta, u\right) = 1 + \frac{w}{r\gamma}\int_{-h}^{h} M\left(w, z_0, \gamma, \delta, v\right)\phi\left(\frac{w}{r\gamma}\left[v - \left(1-r\right)u\right] - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right) dv$, $\gamma = \sigma/\sigma_0$, $\delta = \left(\mu - \mu_0\right)/\left(\sigma_0/\sqrt{n}\right)$, w, z_0 are specific values of the random variables $W = \widehat{\sigma}_0/\sigma_0$, $Z_0 = \sqrt{m}\frac{\left(\widehat{\mu}_0 - \mu_0\right)}{\left(\sigma_0/\sqrt{n}\right)}$ and u is the starting value of the EWMA. Also, μ_0, σ_0 are the in-control mean and standard deviation, $\widehat{\mu}_0$, $\widehat{\sigma}_0$ are their estimates respectively and μ, σ are the mean and standard deviation at time t. Additionally, the second moment of T is given by

$$E\left[T^2|\gamma, \delta, u\right] = \int_{-\infty}^{\infty}\int_{0}^{\infty} M_2\left(w, z_0, \gamma, \delta, u\right) f_w\left(w\right)\phi\left(z_0\right) dwdz_0,$$

where $M_2\left(w, z_0, \gamma, \delta, u\right) = 1 + \frac{2w}{r\gamma}\int_{-h}^{h} M\left(w, z_0, \gamma, \delta, v\right)\phi\left(\frac{w}{r\gamma}\left[v - \left(1-r\right)u\right] - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right) dv + \frac{w}{r\gamma}\int_{-h}^{h} M_2\left(w, z_0, \gamma, \delta, v\right)\phi\left(\frac{w}{r\gamma}\left[v - \left(1-r\right)u\right] - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right) dv$. The SDRL can be computed by $SDRL = \sqrt{E\left[T^2\right] - \left(E\left[T\right]\right)^2}$. Jones et al. (2001) concluded that in both in-control and out-of-control cases the process's run length performance is affected. In particular, the estimation effect results in more false alarms and generally leads to a reduction of the ability of the chart to detect process shifts.

Additionally, Jones (2002) developed a procedure for designing an EWMA chart with estimated parameters. Using this procedure a practitioner is able to design an EWMA chart to have the desirable performance. The steps of this method are

Step 1. Identify the desired in-control ARL of the chart

Step 2. Determine the subgroup size n and number of subgroups m that will be used to estimate the parameters of the in-control process. Obtain a reference sample of m

subgroups, of n observations each

Step 3. Ensure that the reference sample is representative of the in-control state of the process. Estimate the parameters according to $\widehat{\mu}_0 = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}$ and $\widehat{\sigma}_0 = \frac{S_p}{c_{4,m}}$ where $S_p = \sqrt{\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\left(X_{ij}-\overline{X}_{i.}\right)^2}{m(n-1)}}$ and $c_{4,m} = \frac{\sqrt{2}\Gamma\left(\frac{m(n-1)+1}{2}\right)}{\sqrt{m(n-1)}\Gamma\left(\frac{m(n-1)}{2}\right)}$.

Step 4. Select the smoothing constant λ.

Step 5. Using λ from Step 4, identify the constant L that produces an EWMA chart with the desired in-control ARL.

Chapter 4

Non-Normality in Control Charts

4.1 Introduction

In control charting methodology an assumption often used to determine statistical properties is that the data are normally distributed. However, it can be shown that this assumption is critical for the performance of the control charts. In Section 4.2, we present the non-normality effect in Univariate and Multivariate Shewhart Charts. In Section 4.3 the ascription under non-normality in univariate and multivariate EWMA Charts is given. The EWMA control charts for dispersion are investigated in detail and results on their performance are given together with some recommendations.

4.2 Non-Normality in Univariate and Multivariate Shewhart Charts

The usual way of constructing the Shewhart charts is by assuming normality for the underlying characteristic. In the case of nonnormality if we know the exact distribution of the characteristic plotted we may construct the corresponding probability limits without a problem. The case that appears to be the most difficult is when we do not have a normally distributed characteristic and the probability density function of this characteristic is not known. Then, we have two alternatives; either use nonparametric control charts (see Chakraborti et al. (2001)) or use the existing theory developed for a normally distributed variable. For this second case, Burr (1967) examined the effect of nonnormality on the often used constants in the Shewhart control charts and concluded that they are robust to the assumption of normality except in cases of extremely non-normal distributions. Additionally, Schilling and Nelson (1976) surveyed on the effect of non-normality on the control limits of the $\overline{X}$ chart. They found that usually a sample of size 5 is enough to ensure the robustness to normality of the control limits. Yourstone and Zimmer (1992) proposed the use of the generalized Burr distribution for determining non-symmetrical limits for a control chart for sample averages. They focused on the effect of non-normality measured by the skewness and kurtosis on the ARL values. They concluded that a large skewness or kurtosis in the original data will result in sizeable large skewness or kurtosis values for the sample averages. Therefore, the practitioner should consider non-symmetrical control charts. Janacek and Meikle (1997) proposed the use of control charts of medians in the case of non-normal data. They assumed that at the beginning the process is in-control and we collect a reference sample of size N. Then, we take samples of size n to check if the process remains in-control in terms of location. Let B be the number of members of the test sample less than k_q where $F_X(k_q) = q$. If the distribution of the reference sample $F_X(x)$ is unknown and $\widehat{m}$ is the sample median, then

$$P\left(B=b\right)=\frac{\binom{j+b-1}{b}\binom{N+n-j-b}{n-b}}{\binom{N+n}{n}}.$$

It can be proved that

$$P\left(x_{(j)}<\widehat{m}<x_{(N-j+1)}\right)=1-2\sum_{b=[n/2]+1}^{n}\frac{\binom{j+b-1}{b}\binom{N+n-j-b}{n-b}}{\binom{N+n}{n}}$$

and this relationship can be used to construct suitable control limits. As Janacek and Meikle (1997) indicate their proposed approach is very reliable when we have non-normal data, but when used with normal data there is a loss of power.

The nonnormality effect on the T^2 control charts have been studied by many authors such as Chase and Bulgren (1971), Mardia (1974, 1975), Everitt (1979), Bauer (1981), Tiku and Singh (1982) and Srivastava and Awan (1982). They proved through simulation that this statistic is affected by nonnormal distributions and especially in the case of the highly skewed ones.

4.3 Non-Normality in Univariate and Multivariate EWMA Charts

The assumption of normality in the EWMA chart has drawn the attention of researchers in the last years. In subsection 4.3.1 we present the recent results on this field for the EWMA chart for the mean in univariate and multivariate cases. Moreover, some new results (Maravelakis et al. (2003)) about the robustness to normality of the EWMA charts for dispersion are given in subsections 4.3.2-4.3.5.

4.3.1 The EWMA control charts for monitoring the process mean

The EWMA is a popular chart for detecting small to moderate shifts and because of another characteristic. As Montgomery (2001) states "It is almost a perfectly nonparametric (distribution free) procedure". Borror et al. (1999), examined the ARL performance of the EWMA chart for the mean in non-normal cases when the parameters of the process are known and concluded in the same result for certain values of the smoothing parameter. They proposed that an EWMA chart with smoothing parameter equal to 0.05 is very effective in the case of nonnormality. Its in-control value is very close to the one for the normal case. Furthermore, it does not lose its ability to detect fast an out-of-control situation. However, as the value of the smoothing parameter increases the performance of the chart under nonnormality is not that good. Recently, Stoumbos and Sullivan (2002) and Testik et al. (2003) extended the work of Borror et al. (1999) to the multivariate case of the EWMA chart. They concluded that a properly designed multivariate EWMA control chart is robust to the non-normality assumption. In particular, Stoumbos and Sullivan (2002) showed that for up to five dimensions a value of the smoothing parameter in the range $[0.02, 0.05]$ is enough to preserve performance as in the multinormality case. However, when we have more than five dimensions a value of 0.02 or less is needed for the MEWMA chart to behave as under multinormality.

4.3.2 The EWMA control charts for monitoring the process dispersion

Let μ_0 and σ_0 denote the in-control values of the process parameters that are either known or estimated from a very large sample taken when the process is assumed to be in-control. We want to detect any shifts of the dispersion in the process using EWMA charts that are known to be efficient for detecting small to moderate shifts in the parameters. For the remaining of this study we assume that we have independent and identically distributed data with sample size unity and also that we are in the prospective setting (Phase II) where the estimates or the parameter values are used to monitor the process. In the case of rational subgroups the central limit theorem applies and therefore the non normality issue does not bother us as much.

Several publications dealing with the subject of detecting shifts in the dispersion using an EWMA type chart have appeared in the literature (see, e.g. Domangue and Patch (1991), MacGregor and Harris (1993), Acosta-Mejia and Pignatiello (2000)). Our main concern is to detect increases in the process dispersion. We have to stress though, that detecting decreases in the dispersion is equally important because they indicate an improvement in the process. Nevertheless, it is not probable that a reduction in the process standard deviation, or variance, will occur without a corrective action. Therefore, when an attempt to improve the quality of a process is taking place, the time that this possible change occurs is known. A control chart is one of the tools to check for possible reduction in the variance before and after the corrective action. However, the main use of a control chart is to detect persistent or sudden shifts in a process at unknown times.

Table 4.1 In-control ARL, MRL and SDRL values for upward shifts $\lambda = 0.05$

		WR	SR	HO	DP1	DP2
	h	2.876	2.604	2.436	2.1492	2.495
N(μ,σ^2)	ARL	370.4	370.4	370.4	370.4	370.4
	MRL	260	260	264	259	257
	SDRL	361.3	358.1	353.6	361.8	368.3
G(4,1)	ARL	151.3	304.2	444.1	490.5	181.2
	MRL	106	213	312	340	124
	SDRL	148.0	296.3	431.7	486.5	183.0
G(3,1)	ARL	133.1	290.6	473.2	535.2	162.6
	MRL	93	205	331	372	111
	SDRL	131.0	283.0	461.9	532.6	166.0
G(2,1)	ARL	112.4	267.5	522.5	641.5	140.3
	MRL	79	187	365	444	95
	SDRL	110.0	262.1	511.3	640.5	144.1
G(1,1)	ARL	84.1	225.3	659.4	1048.1	111.8
	MRL	59	158	461	723	75
	SDRL	82.7	220.7	647.9	1056.6	116.1
G(0.5,1)	ARL	67.8	185.8	840.3	2449.9	94.8
	MRL	47	130	583	1679	63
	SDRL	66.8	184.3	837.1	2489.1	99.9

The EWMA chart of squared deviations from target (EWMA_S) was proposed by Wortham and Ringer (1971) for detecting a shift in the process standard deviation. The statistic of this chart is given by

$$S_i = \lambda(x_i - \mu_0)^2 + (1-\lambda)\max(S_{i-1}, \sigma_0^2),\ S_0 = \sigma_0^2,$$

Table 4.1 (continued) In-control ARL, MRL and SDRL values for upward shifts $\lambda = 0.1$

		WR	SR	HO	DP1	DP2
	h	3.432	2.916	2.628	2.409	3.094
N(μ,σ^2)	ARL	370.4	370.4	370.4	370.4	370.4
	MRL	259	257	260	259	258
	SDRL	365.9	360.8	359.2	363.6	367.4
G(4,1)	ARL	129.7	237.0	380.8	421.1	147.2
	MRL	91	166	265	293	102
	SDRL	127.7	231.7	374.1	418.8	147.3
G(3,1)	ARL	114.3	218.0	382.1	437.2	130.7
	MRL	79	152	267	304	90
	SDRL	113.2	214.5	373.8	433.7	131.3
G(2,1)	ARL	95.6	191.6	388.3	472.1	111.8
	MRL	66	133	271	328	77
	SDRL	94.8	188.9	382.1	469.3	112.7
G(1,1)	ARL	72.5	150.6	393.3	569.5	87.0
	MRL	51	105	273	396	60
	SDRL	71.2	148.3	388.3	570.5	88.2
G(0.5,1)	ARL	59.2	120.2	399.4	816.4	73.1
	MRL	41	83	278	564	50
	SDRL	58.6	119.1	395.1	822.3	74.7

where λ is a smoothing parameter that takes values between 0 and 1 and S_0 is the initial value. The above statistic is one-sided and it is defined in a way to detect only upward shifts. This happens because, whenever S_i is less than σ_0^2, we set it equal to its starting value. The control limit of this chart is

$$UCL = \sigma_0^2 + h_S \sigma_0^2 \sqrt{\left(\frac{2\lambda}{2-\lambda}\right)},$$

Table 4.1 (continued) In-control ARL, MRL and SDRL values for upward shifts $\lambda = 0.2$

		WR	SR	HO	DP1	DP2
	h	4.112	3.215	2.742	2.584	3.821
N(μ,σ^2)	ARL	370.4	370.4	370.4	370.4	370.4
	MRL	256	257	257	259	258
	SDRL	368.9	363.4	363.3	366.4	368.8
G(4,1)	ARL	113.4	171.8	281.2	319.7	121.9
	MRL	79	120	196	221	84
	SDRL	112.6	169.0	277.9	318.3	121.4
G(3,1)	ARL	99.7	154.3	263.7	310.5	107.5
	MRL	69	107	184	216	75
	SDRL	98.9	153.1	260.7	308.2	107.5
G(2,1)	ARL	83.5	131.4	240.8	296.4	91.3
	MRL	58	92	167	205	63
	SDRL	82.7	129.5	238.0	294.4	91.3
G(1,1)	ARL	64.1	100.6	205.5	279.1	70.7
	MRL	45	70	144	194	49
	SDRL	63.3	99.3	202.6	277.7	70.9
G(0.5,1)	ARL	52.5	81.0	179.6	291.4	59.4
	MRL	36	57	125	201	41
	SDRL	51.8	80.3	178.3	293.2	59.4

where h_S is a constant used to specify the width of the control limit. Note that σ_0^2 would be the mean and $\sigma_0^2\sqrt{2\lambda/(2-\lambda)}$ would be the asymptotic standard deviation of S_i if the reset was not used. However, the control limit is not modified in order to resemble the form of an asymptotic EWMA control limit (Reynolds and Stoumbos (2001)).

As Stoumbos and Reynolds (2000) indicate, when the normality assumption is questionable for the observations, the $EWMA_S$ statistic does not converge quickly to normality

Table 4.2. Out-of-control ARL, MRL and SDRL values for upward shifts $\lambda = 0.05$

Shift		1.2					1.4				
		WR	SR	HO	DP1	DP2	WR	SR	HO	DP1	DP2
$N(\mu, \sigma^2)$	ARL	113.3	116.2	126.1	114.4	100.8	55.6	58.4	65.9	58.6	48.8
	MRL	81	84	92	82	72	41	44	50	44	37
	SDRL	105.5	105.1	113.1	104.5	94.2	48.2	48.5	54.2	48.7	42.5
G(4,1)	ARL	66.0	111.0	167.3	171.9	68.8	36.2	54.5	80.2	77.6	35.5
	MRL	47	80	120	121	48	27	40	59	56	26
	SDRL	62.6	103.0	155.7	164.8	67.5	32.9	47.8	70.2	70.0	33.3
G(3,1)	ARL	64.4	115.4	185.2	193.6	68.9	37.9	59.9	92.0	91.5	37.9
	MRL	46	82	132	136	48	27	44	67	66	27
	SDRL	61.7	108.2	174.0	187.4	68.1	35.1	53.8	82.1	84.7	36.1
G(2,1)	ARL	61.1	119.3	214.1	237.9	67.6	39.2	67.0	111.9	116.8	40.6
	MRL	43	85	152	166	46	28	48	81	83	28
	SDRL	58.6	113.4	203.1	233.4	68.0	36.8	61.2	101.9	110.5	39.5
G(1,1)	ARL	54.6	121.3	294.0	393.6	64.8	39.1	76.8	164.0	196.6	43.3
	MRL	39	86	206	271	44	28	55	117	137	29
	SDRL	52.7	116.5	284.7	395.9	66.4	37.6	72.8	154.8	194.3	43.8
G(0.5,1)	ARL	49.4	117.1	420.7	910.3	62.7	38.5	82.9	252.4	444.3	46.0
	MRL	35	82	293	623	41	27	58	177	304	31
	SDRL	48.2	114.3	413.2	933.2	65.9	37.3	80.2	245.3	454.7	47.8

because it is a weighted average of squared deviations. For this reason they propose an EWMA chart of the absolute deviations from target ($EWMA_V$), adjusted for detecting only upward shifts. The statistic of this chart is

$$V_i = \lambda\,|x_i - \mu_0| + (1-\lambda)\max(V_{i-1}, \sigma_0\sqrt{2/\pi}),\ V_0 = \sigma_0\sqrt{2/\pi},$$

where V_0 is the initial value. The above statistic, as in the case of the EWMA_S statistic, is one-sided and can detect only upward shifts. The control limit of this chart is

$$UCL = \sigma_0\sqrt{2/\pi} + h_V\sigma_0\sqrt{1-(2/\pi)}\sqrt{\lambda/\left(2-\lambda\right)},$$

where h_V is a constant specifying the width of the control limit. We have to mention that $\sigma_0\sqrt{2/\pi}$ would be the mean and $\sigma_0\sqrt{1-(2/\pi)}\sqrt{\lambda/\left(2-\lambda\right)}$ would be the asymptotic standard deviation of V_i if the reset was not used. Again, the control limit is not modified and therefore it does not resemble the form of the standard EWMA control limit.

Hawkins and Olwell (1998, p.82) suggested a different statistic for monitoring individual readings for scale changes. Specifically, they recommended the use of the differences $(X_n - \mu_0)$ CUSUMming the square root of their absolute values. In our case, and since we use an EWMA type chart, Maravelakis et al. (2003) introduced such a control chart. Let $H = \sqrt{|x_i - \mu_0|}$, where x_i are our observations. It can be shown that if X is normally distributed ($\text{N}(\mu_0,\, \sigma_0^2)$) then

$$f(h;\sigma_0^2) = \frac{4h}{\sigma_0\sqrt{2\pi}}\exp\left(-\frac{h^4}{2\sigma_0^2}\right), 0 \le h$$

with

$$\begin{aligned} E(h) &= \int_0^\infty \frac{4h^2}{\sigma\sqrt{2\pi}}\exp\left(-\frac{h^4}{2\sigma^2}\right)dh = \frac{2^{3/4}\sigma^{1/2}}{\sqrt{2\pi}}\int_0^\infty \frac{2h^2}{\sigma^{3/2}2^{-1/4}}\exp\left(-\frac{h^4}{2\sigma^2}\right)dh = \\ &= \frac{2^{3/4}\sigma^{1/2}}{\sqrt{2\pi}}\int_0^\infty \left(\frac{h^4}{2\sigma^2}\right)^{-1/4}\exp\left(-\frac{h^4}{2\sigma^2}\right)d\left(\frac{h^4}{2\sigma^2}\right) = \left(2^{3/4}\right)\Gamma\left(3/4\right)\sqrt{\sigma_0/2\pi} \end{aligned}$$

and

$$E(h^2) = \int_0^\infty \frac{4h^3}{\sigma\sqrt{2\pi}} \exp\left(-\frac{h^4}{2\sigma^2}\right) dh = \frac{2\sigma^2}{\sigma\sqrt{2\pi}} \int_0^\infty \frac{4h^3}{2\sigma^2} \exp\left(-\frac{h^4}{2\sigma^2}\right) dh =$$
$$= \frac{2\sigma^2}{\sigma\sqrt{2\pi}} \int_0^\infty \left(\frac{h^4}{2\sigma^2}\right)^0 \exp\left(-\frac{h^4}{2\sigma^2}\right) d\left(\frac{h^4}{2\sigma^2}\right) = \frac{2\sigma^2\Gamma(1)}{\sigma\sqrt{2\pi}} = \sigma\sqrt{\frac{2}{\pi}}.$$

Table 4.2 (continued) Out-of-control ARL, MRL and SDRL values for upward shifts $\lambda = 0.05$

Shift		1.6					1.8				
		WR	SR	HO	DP1	DP2	WR	SR	HO	DP1	DP2
N(μ, σ^2)	ARL	34.9	37.7	43.2	38.5	30.8	25.0	27.5	32.2	28.9	22.3
	MRL	27	30	34	30	24	20	22	26	23	18
	SDRL	28.5	28.9	32.8	29.6	25.2	19.5	19.7	22.7	20.7	17.5
G(4,1)	ARL	23.4	32.9	46.6	44.2	22.4	16.8	22.8	31.3	29.4	15.8
	MRL	18	25	35	33	17	13	18	25	23	12
	SDRL	20.5	27.0	37.9	37.1	20.1	14.2	17.7	23.5	22.9	13.6
G(3,1)	ARL	25.4	37.4	55.0	53.2	24.8	18.7	26.4	37.6	35.7	18.0
	MRL	19	28	41	39	18	14	20	29	27	13
	SDRL	22.8	31.8	46.1	46.3	22.8	16.1	21.3	29.6	29.2	15.9
G(2,1)	ARL	27.8	43.7	69.5	69.4	27.7	21.2	31.6	48.1	47.1	20.8
	MRL	20	32	51	50	20	16	24	36	35	15
	SDRL	25.4	38.6	60.7	62.9	26.3	19.0	26.8	40.0	40.7	19.2
G(1,1)	ARL	30.2	54.3	105.7	117.9	32.1	24.4	41.1	75.4	80.1	25.3
	MRL	22	39	76	82	22	18	30	55	57	18
	SDRL	28.6	50.1	97.4	114.1	31.8	22.8	37.2	67.8	75.1	24.6
G(0.5,1)	ARL	31.5	63.0	170.1	260.8	36.2	26.9	50.5	123.9	173.1	29.8
	MRL	22	45	120	178	24	19	36	88	118	20
	SDRL	30.4	60.4	164.2	265.8	37.3	25.6	47.9	117.7	175.3	30.2

Table 4.2 (continued) Out-of-control ARL, MRL and SDRL values for upward shifts $\lambda = 0.1$

	Shift	1.2					1.4				
		WR	SR	HO	DP1	DP2	WR	SR	HO	DP1	DP2
$N(\mu, \sigma^2)$	ARL	124.1	123.3	131.8	123.2	113.0	60.7	61.7	68.2	62.2	54.6
	MRL	88	88	94	88	80	44	45	50	45	39
	SDRL	119.8	116.6	123.5	116.2	109.1	56.3	55.6	60.7	55.6	50.5
G(4,1)	ARL	60.5	94.9	147.9	156.7	62.8	34.3	48.7	71.9	73.6	34.2
	MRL	43	67	105	110	44	25	35	52	53	24
	SDRL	58.9	91.0	140.8	152.0	61.8	32.4	45.0	66.2	68.6	32.8
G(3,1)	ARL	58.5	95.4	157.1	171.3	61.8	35.4	52.4	81.1	84.4	35.9
	MRL	41	67	111	120	43	25	37	58	60	25
	SDRL	56.8	91.8	151.1	167.4	61.1	33.7	49.0	75.2	79.7	34.8
G(2,1)	ARL	54.8	94.4	171.7	195.0	59.8	36.0	55.9	94.3	102.0	37.4
	MRL	38	66	120	136	42	25	40	67	72	26
	SDRL	53.5	91.5	165.8	191.4	59.5	34.5	52.7	88.5	98.1	36.6
G(1,1)	ARL	48.3	89.0	199.0	260.6	54.8	35.3	59.6	119.8	146.0	38.5
	MRL	34	62	139	180	38	25	42	85	102	26
	SDRL	47.2	87.0	194.2	259.9	55.1	34.2	57.3	114.8	143.9	38.5
G(0.5,1)	ARL	43.5	81.7	230.3	400.3	51.6	34.6	60.6	151.6	237.8	39.3
	MRL	30	57	161	276	35	24	42	106	164	27
	SDRL	42.8	80.3	226.4	404.6	52.5	33.7	59.4	147.7	240.1	39.9

Then,

$$Var(h) = E(h^2) - [E(h)]^2 = \sigma_0 \left(\sigma \sqrt{\frac{2}{\pi}} - \sqrt{2} \frac{\Gamma^2\left(3/4\right)}{\pi} \right)$$

and the EWMA$_{\text{H}}$ chart is based on the statistic

$$\begin{aligned} H_i &= \lambda\sqrt{|x_i-\mu_0|}+(1-\lambda)\max\left(H_{i-1},\left(2^{3/4}\right)\Gamma\left(3/4\right)\sqrt{\sigma_0/2\pi}\right), \\ H_0 &= \left(2^{3/4}\right)\Gamma\left(3/4\right)\sqrt{\sigma_0/2\pi} \end{aligned}$$

where H_0 is the initial value. The control limit of this chart is

$$UCL = \left(2^{3/4}\right)\Gamma\left(3/4\right)\sqrt{\sigma_0/2\pi}+h_H\sqrt{\sigma_0\left(\left(2/\sqrt{2\pi}\right)-\sqrt{2}\Gamma^2\left(3/4\right)/\pi\right)\lambda/\left(2-\lambda\right)},$$

where h_H is a constant specifying the width of the control limit. The mean of H_i is $\left(2^{3/4}\right)\Gamma\left(3/4\right)\sqrt{\sigma_0/2\pi}$ and $\sqrt{\sigma_0\left(\left(2/\sqrt{2\pi}\right)-\sqrt{2}\Gamma^2\left(3/4\right)/\pi\right)\lambda/\left(2-\lambda\right)}$ is the asymptotic standard deviation of H_i if the reset is not used. The control limit in this case also is not modified to keep the form of a standard EWMA control limit.

Domangue and Patch (1991) introduced the omnibus EWMA control charts. The statistic used in these charts is $Z_i = (X_i-\mu_0)/\sigma_0$ and the proposed EWMA$_{\text{A}}$ scheme is

$$A_i = \lambda\left|Z_i\right|^{\alpha}+(1-\lambda)A_{i-1},$$

where the starting value A_0 is set by the practitioner and it is usually equal to the asymptotic mean of A_i. Two different schemes were proposed by Domangue and Patch, one with $a = 0.5$ and the second with $a = 2$. When we have independent samples from a normal process with mean μ_0 and standard deviation σ_0 Domangue and Patch (1991) showed that the asymptotic mean and variance of A_i for the scheme with $a = 1/2$ are $E(A_i) = \left(\sqrt{2}/\pi\right)^{1/2}\Gamma\left(3/4\right)$ and $Var(A_i) = \frac{\sqrt{2}\lambda}{(2-\lambda)\pi}\left[\sqrt{\pi}-\Gamma^2\left(3/4\right)\right]$. In the case of $a = 2$ they proved that $E(A_i) = 1$ and $Var(A_i) = \frac{2r}{(2-r)}$. Then, the control limit in each case is

$$UCL = E(A_i)+h_A Var(A_i)^{1/2}$$

where h_A is a constant specifying the width of the control limit and either of the schemes signal whenever $A_i \geq UCL$. We have to note here that these schemes can signal only upward because of the way they are constructed. Moreover, as Domangue and Patch indicate these schemes are sensitive to increases in dispersion.

For all the above schemes we observe that they are vulnerable to shifts in the mean apart from the dispersion. Therefore a signal of these charts might be the result of a change in the mean. This deficiency can be resolved by using the moving range (Hawkins and Olwell (1998, p.82)) or by calculating at each point in time (observation) an estimate of the mean (MacGregor and Harris (1993)). However, the use of either of these techniques might lead to other problems such as dependence of the observations and since they involve cumbersome calculations, they are not considered here.

4.3.3 Methods of evaluating control charts performance and their computation

In the context of EWMA charts there are three ways of computing the previously stated measures of performance. The integral equation method, the Markov chain method and a simulation study (see e.g., Brook and Evans (1972), Lucas and Saccucci (1990) and Domangue and Patch (1991)). The integral equation method is an accurate method but it can not be computed in all cases. The Markov chain method can be implemented in the cases that the former method can not, but we need to discretise the continuity of the process using many steps. The simulation study is easy in the implementation and, when using a large number of iterations, the results are very accurate. In the following calculations simulation is used and we repeat the simulation 200001 times for each entry in the tables.

In order to study the effect of non-normality in the performance of the EWMA charts for dispersion we used the same types of distributions as in Borror et al. (1999) and Stoumbos and Reynolds (2000); symmetric and skewed ones. Specifically, we simulated observations in the skewed case from the Gamma(a, b) distribution with probability density function

$$f(x;\alpha,b)=\left\{\begin{array}{cc}\frac{b^{\alpha}}{\Gamma(\alpha)}x^{\alpha-1}\exp\left(-bx\right) & x>0\\ 0, & x\leq 0\end{array}\right\},$$

Table 4.2 (continued) Out-of-control ARL, MRL and SDRL values for upward shifts $\lambda = 0.1$

Shift		1.6					1.8				
		WR	SR	HO	DP1	DP2	WR	SR	HO	DP1	DP2
$N(\mu,\sigma^2)$	ARL	37.3	38.5	43.4	39.7	33.6	26.1	27.4	31.4	28.8	23.6
	MRL	27	29	32	30	25	20	21	24	22	18
	SDRL	33.4	32.9	36.6	33.4	29.9	22.5	22.2	25.2	22.9	20.2
G(4,1)	ARL	22.4	29.8	42.1	41.9	21.9	16.1	20.6	28.2	27.5	15.5
	MRL	16	22	31	31	16	12	16	21	21	11
	SDRL	20.5	26.3	36.9	37.2	20.3	14.3	17.4	23.3	23.2	14.0
G(3,1)	ARL	24.1	33.4	49.4	49.9	24.0	17.9	23.6	33.6	33.3	17.4
	MRL	17	24	36	36	17	13	18	25	25	13
	SDRL	22.4	30.1	44.0	45.4	22.7	16.3	20.6	28.6	29.0	16.0
G(2,1)	ARL	25.9	37.9	59.8	62.7	26.4	19.9	27.7	42.1	42.9	19.9
	MRL	18	27	43	45	19	14	20	31	31	14
	SDRL	24.4	34.9	54.6	58.7	25.4	18.6	25.0	37.3	39.0	18.7
G(1,1)	ARL	27.6	43.7	81.3	94.0	29.2	22.6	34.0	59.9	66.6	23.4
	MRL	20	31	58	66	20	16	24	43	47	16
	SDRL	26.4	41.4	77.1	91.6	28.9	21.4	31.7	55.5	63.6	22.8
G(0.5,1)	ARL	28.6	47.6	108.9	157.4	31.9	24.4	39.3	83.7	113.1	26.8
	MRL	20	34	76	108	22	17	28	59	78	18
	SDRL	27.7	46.1	105.6	158.4	32.3	23.5	37.7	80.7	112.9	26.8

where the mean is α/b and the variance is α/b^2. In the remaining of the chapter we set b equal to unity without loss of generality. Under this condition as α increases the gamma distribution approaches the normal. In the symmetric case we simulated observations from the t(k) distribution with probability density function

$$f(x;k) = \frac{\Gamma\left(\left(k+1\right)/2\right)}{\sqrt{\pi}\Gamma\left(k/2\right)} \frac{1}{\left(\left(x^2/k\right)+1\right)^{(k+1)/2}}, -\infty < x < \infty,$$

Table 4.2 (continued) Out-of-control ARL, MRL and SDRL values for upward shifts $\lambda = 0.2$

Shift		1.2					1.4				
		WR	SR	HO	DP1	DP2	WR	SR	HO	DP1	DP2
$N(\mu, \sigma^2)$	ARL	136.7	133.4	137.7	132.0	128.8	69.1	67.0	71.1	67.6	63.5
	MRL	95	94	97	93	90	49	48	51	48	45
	SDRL	134.5	129.8	133.1	128.0	127.1	67.1	63.6	66.6	63.4	61.4
G(4,1)	ARL	56.0	76.9	116.3	128.7	57.6	33.1	42.1	59.8	63.8	32.9
	MRL	39	54	82	90	40	23	30	42	45	23
	SDRL	54.9	74.7	113.1	126.5	56.8	32.0	40.1	56.7	61.2	32.0
G(3,1)	ARL	53.5	75.2	118.5	133.7	55.6	33.5	43.9	65.0	70.7	34.0
	MRL	37	53	83	93	39	24	31	46	50	24
	SDRL	52.8	73.5	115.3	131.4	55.2	32.3	42.2	62.0	68.4	33.4
G(2,1)	ARL	49.6	71.4	119.6	140.2	52.3	33.5	45.4	70.2	79.4	34.5
	MRL	35	50	84	97	36	24	32	49	56	24
	SDRL	48.7	69.9	116.4	138.7	51.9	32.6	44.0	67.5	77.1	34.0
G(1,1)	ARL	43.6	64.0	118.1	151.0	46.8	32.5	45.4	77.7	94.6	34.3
	MRL	31	45	83	105	33	23	32	55	66	24
	SDRL	42.7	62.6	115.8	150.3	46.8	31.6	44.1	75.3	93.6	33.9
G(0.5,1)	ARL	39.5	57.8	117.1	174.9	43.4	31.5	44.6	83.7	118.3	34.3
	MRL	28	40	81	121	30	22	31	59	82	24
	SDRL	38.8	57.2	115.6	175.8	43.4	30.9	43.6	82.1	118.5	34.2

where k are the degrees of freedom, the mean is 0 and the variance is $k/(k-2)$. The t distribution is symmetric about 0 but it has more probability in the tails than the normal. Moreover, as the degrees of freedom increase, the t distribution approaches the normal.

In the simulation algorithm, the parameter values we simulated from, are α=0.5, 1, 2, 3, 4 and b=1 in the gamma case, and k= 4, 6, 8, 10, 20, 30, 40, 50 in the t distribution case. The steps of the algorithm are the following

Step 1. Set the values of μ_0 and σ_0

Step 2. Set the values of λ and the constants specifying the width of the control limits (h_S, h_V, h_H, h_A) and calculate the control limits.

Step 3. Generate a value from gamma(α,1) [from a t(k) distribution] and calculate the appropriate statistic in each case.

Step 4. Repeat Step 3 until the statistic computed crosses the upper control limit and record the sample this happens.

Step 5. Repeat Steps 3 to 4 200001 times.

Step 6. Obtain estimates of the ARL and SDRL values.

Step 7. Sort the 200001 values and set observation 100001 equal to the MRL.

Evidently, the above algorithm is used for calculating the in-control ARL, MRL and SDRL values. For the out-of-control cases Step 3 is properly modified. In Step 1, the in-control mean when we are in the gamma case is equal to α/b and the variance is α/b^2. When we have a t distribution the in-control mean is 0 and the variance is $k/(k-2)$. The values under the normal distribution are calculated also in each case for studying the non-normality effect. The values of λ chosen are 0.05, 0.1 and 0.2 which are the usually chosen values for studying the non-normality effect (see e.g., Borror et al. (1999), Stoumbos and Reynolds (2000), Reynolds and Stoumbos (2001)). The values of (h_S, h_V, h_H, h_A) are chosen in a way that under normality they give the same in-control value for ARL approximately 370.4. Also, in all the cases, results are displayed for asymptotic control limits. Finally, all the out-of-control computations performed in this chapter are made under the assumption of immediate occurrence of the shift at the beginning of the process.

Table 4.2 (continued) Out-of-control ARL, MRL and SDRL values for upward shifts $\lambda = 0.2$

Shift		1.6					1.8				
		WR	SR	HO	DP1	DP2	WR	SR	HO	DP1	DP2
N(μ, σ^2)	ARL	42.0	41.4	44.9	42.2	38.6	29.1	28.8	31.6	29.9	26.8
	MRL	30	30	32	31	28	21	21	23	22	19
	SDRL	40.0	38.1	40.9	38.3	36.7	27.1	25.7	27.7	26.2	24.7
G(4,1)	ARL	22.0	26.6	36.1	37.5	21.6	15.8	18.6	24.3	24.9	15.5
	MRL	16	19	26	27	15	11	14	18	18	11
	SDRL	20.8	24.6	33.1	34.8	20.6	14.7	16.8	21.5	22.5	14.5
G(3,1)	ARL	23.2	29.0	40.7	43.2	23.2	17.3	20.9	28.4	29.5	17.1
	MRL	16	21	29	31	16	12	15	21	21	12
	SDRL	22.1	27.3	37.9	40.9	22.3	16.3	19.2	25.7	27.1	16.2
G(2,1)	ARL	24.5	31.6	47.0	51.4	24.9	19.1	23.9	33.8	36.2	19.1
	MRL	17	22	33	36	17	14	17	24	26	14
	SDRL	23.6	30.0	44.0	49.3	24.2	18.1	22.3	31.2	34.0	18.3
G(1,1)	ARL	25.7	34.6	55.8	66.0	26.6	21.1	27.7	43.0	49.2	21.6
	MRL	18	24	39	46	19	15	20	30	35	15
	SDRL	24.9	33.5	53.3	64.8	26.2	20.4	26.5	40.7	47.5	21.1
G(0.5,1)	ARL	26.4	36.2	64.6	87.2	28.1	22.6	30.6	52.0	67.6	24.1
	MRL	19	25	45	61	20	16	21	37	47	17
	SDRL	25.7	35.3	63.1	86.6	27.8	21.9	29.7	50.5	67.4	23.8

4.3.4 Results

In Tables 4.1 through 4.4, we have the results of the EWMA charts for the dispersion for the five different charts (EWMA$_S$, EWMA$_V$, EWMA$_H$ and EWMA$_A$ for α=1/2 and α=2). We have results for three combinations of λ and the corresponding h_S, h_V, h_H

and h_A values. In the second row of table 4.1 we have the five different h_S, h_V, h_H and h_A values, which are calculated so as to give under normality an in-control value of ARL equal to 370.4. The same values for these h constants are used in Tables 4.2, 4.3 and 4.4 and for this reason they are not displayed. The third column (WR) in each Table is the ARL, MRL and SDRL values for the $\mathrm{EWMA_S}$, the fourth column (SR) is for the $\mathrm{EWMA_V}$, the fifth column (HO) is for the $\mathrm{EWMA_H}$ and the sixth (DP1) and seventh (DP2) columns are for the $\mathrm{EWMA_A}$ with α=0.5 and α=2 respectively.

In Table 4.1, the results for the in-control case for the gamma distribution are displayed and in Table 4.3 the corresponding ones for the t distribution (ARL(0)). In Table 4.2, we have the results in the out-of-control case for the Gamma distribution and in Table 4.4 the corresponding ones for the t distribution (ARL(1)). In each Table we have computed additionally the ARL, MRL and SDRL values for the normal distribution to identify the non normality effect. The shift in the out-of-control cases is in the in-control process variance, whose value is set at the first Step of the algorithm, by multiplying it with 1.2, 1.4, 1.6 and 1.8.

The conclusions drawn from these tables are the following. When the process is in-control, the $\mathrm{EWMA_H}$ chart for $\lambda = 0.1$ has a satisfactory non normality performance. Additionally, the $\mathrm{EWMA_A}$ chart when $a = 0.5$ for $\lambda = 0.2$ gives also results comparable to the normal ones when we are in-control. One also concludes that the other charts are much less efficient regarding non normality for every combination of the smoothing parameter and the process parameters presented. Most of the times they lead to a larger number of false alarms than the nominal. However, the $\mathrm{EWMA_H}$ and $\mathrm{EWMA_A}$ for $a = 0.5$ can give for certain parameters, very large ARL values. As the value of α in the gamma case and k in the t-distribution case, become larger so does the ARL and MRL for $\mathrm{EWMA_S}$, $\mathrm{EWMA_V}$ and $\mathrm{EWMA_A}$ for $a = 2$. On the other hand, the ARL and MRL values for the $\mathrm{EWMA_H}$ and $\mathrm{EWMA_A}$ for $\alpha = 0.5$ decrease when $\lambda = 0.05$, $\lambda = 0.1$ and increase for $\lambda = 0.2$.

In the out-of-control cases, as the shift increases the non normality effect decreases.

Table 4.3. In-control ARL, MRL and SDRL values for upward shifts $\lambda = 0.05$

		WR	SR	HO	DP1	DP2
$N(\mu, \sigma^2)$	ARL	370.4	370.4	370.4	370.4	370.4
	MRL	260	260	264	259	257
	SDRL	361.3	358.1	353.6	361.8	368.3
t_4	ARL	112.6	271.0	792.9	2208	147.4
	MRL	79	189	549	1515	100
	SDRL	110.8	267.4	787.3	2251	151.4
t_6	ARL	138.6	297.8	585.2	946.5	170.9
	MRL	97	209	410	653	117
	SDRL	135.8	290.6	573.1	953.5	173.4
t_8	ARL	165.6	318.3	589.9	695.3	195.3
	MRL	117	224	412	481	134
	SDRL	161.4	310.5	580.7	695.6	197.8
t_{10}	ARL	186.0	329.9	476.8	591.8	216.2
	MRL	131	231	336	411	149
	SDRL	180.9	321.1	462.4	588.2	218.0
t_{20}	ARL	252.3	352.6	416.0	456.7	276.4
	MRL	178	249	292	318	192
	SDRL	244.8	341.9	402.2	452.8	274.8
t_{30}	ARL	285.8	358.8	401.2	424.1	303.1
	MRL	200	252	282	296	211
	SDRL	279.8	346.4	389.3	417.6	301.4
t_{40}	ARL	302.5	361.5	390.6	409.7	319.5
	MRL	213	254	275	285	222
	SDRL	293.3	349.5	377.5	404.3	317.4
t_{50}	ARL	314.9	366.4	387.7	400.8	327.6
	MRL	221	256	272	279	227
	SDRL	307.0	356.5	374.1	395.1	324.8

Table 4.3 (continued) In-control ARL, MRL and SDRL values for upward shifts $\lambda = 0.1$

		WR	SR	HO	DP1	DP2
$N(\mu, \sigma^2)$	ARL	370.4	370.4	370.4	370.4	370.4
	MRL	259	257	260	259	258
	SDRL	365.9	360.8	359.2	363.6	367.4
t_4	ARL	97.7	187.4	441.5	882.4	116.4
	MRL	68	131	307	609	80
	SDRL	96.1	185.5	438.1	890.9	117.5
t_6	ARL	120.9	219.9	409.6	590.9	140.1
	MRL	84	153	288	410	97
	SDRL	119.6	217.4	399.3	590.8	140.2
t_8	ARL	145.2	247.1	400.9	508.8	163.9
	MRL	101	173	279	354	114
	SDRL	143.0	243.0	395.4	506.8	164.1
t_{10}	ARL	167.7	269.7	394.1	470.4	185.0
	MRL	117	189	275	326	128
	SDRL	165.0	264.5	388.0	467.3	184.9
t_{20}	ARL	233.3	316.5	380.6	412.9	250.2
	MRL	163	222	267	287	173
	SDRL	230.0	310.3	372.7	408.4	249.3
t_{30}	ARL	270.6	334.3	378.3	397.1	283.2
	MRL	190	234	264	277	196
	SDRL	264.6	326.7	371.1	392.2	283.8
t_{40}	ARL	291.7	341.8	375.0	390.1	301.0
	MRL	205	239	262	272	210
	SDRL	286.2	336.0	367.5	385.8	298.9
t_{50}	ARL	305.1	348.1	373.9	386.9	314.3
	MRL	213	243	263	269	218
	SDRL	301.8	341.1	363.4	381.4	312.6

Table 4.3 (continued) In-control ARL, MRL and SDRL values for upward shifts $\lambda = 0.2$

		WR	SR	HO	DP1	DP2
$N(\mu, \sigma^2)$	ARL	370.4	370.4	370.4	370.4	370.4
	MRL	256	257	257	259	258
	SDRL	368.9	363.4	363.3	366.4	368.8
t_4	ARL	86.7	130.1	238.3	383.0	96.0
	MRL	60	91	166	266	67
	SDRL	86.4	128.7	235.7	383.1	95.8
t_6	ARL	109.2	159.2	265.3	353.8	118.0
	MRL	76	111	186	245	82
	SDRL	108.0	158.1	261.3	354.3	117.5
t_8	ARL	131.2	187.8	283.9	349.7	140.9
	MRL	91	131	198	243	98
	SDRL	130.8	186.1	278.8	347.4	140.6
t_{10}	ARL	152.1	212.3	302.6	353.4	162.0
	MRL	106	148	211	246	112
	SDRL	150.3	210.0	298.0	352.3	162.5
t_{20}	ARL	220.0	275.9	332.2	361.4	229.3
	MRL	154	192	232	251	159
	SDRL	217.8	273.1	328.0	357.4	228.7
t_{30}	ARL	257.1	303.7	347.4	365.5	264.7
	MRL	178	212	243	255	184
	SDRL	255.4	299.4	343.2	362.4	262.3
t_{40}	ARL	279.3	321.1	351.7	365.5	285.6
	MRL	195	223	245	255	198
	SDRL	276.2	318.7	346.0	362.8	284.0
t_{50}	ARL	294.7	327.3	355.1	366.1	298.9
	MRL	205	228	247	254	207
	SDRL	292.6	323.0	351.1	362.4	298.0

Table 4.4. Out–of-control ARL, MRL and SDRL values for upward shifts $\lambda = 0.05$

	Shift	1.2					1.4				
		WR	SR	HO	DP1	DP2	WR	SR	HO	DP1	DP2
$N(\mu, \sigma^2)$	ARL	113.3	116.2	126.1	114.4	100.8	55.6	58.4	65.9	58.6	48.8
	MRL	81	84	92	82	72	41	44	50	44	37
	SDRL	105.5	105.1	113.1	104.5	94.2	48.2	48.5	54.2	48.7	42.5
t_4	ARL	75.5	159.5	417.6	930.3	91.0	59.5	116.2	283.6	556.6	68.4
	MRL	53	112	291	638	62	42	82	199	381	46
	SDRL	73.6	155.8	410.6	946.7	92.9	57.7	113.1	277.1	569.0	69.6
t_6	ARL	74.9	140.2	275.0	386.2	83.4	53.3	93.2	177.0	231.1	57.1
	MRL	53	99	193	267	57	38	66	126	160	39
	SDRL	72.0	134.5	265.4	386.8	83.8	50.9	88.0	168.2	230.8	56.5
t_8	ARL	77.1	134.6	232.9	284.5	82.2	52.1	85.2	147.5	170.7	54.0
	MRL	55	96	165	198	57	37	61	105	119	38
	SDRL	73.8	127.8	223.4	281.8	81.8	49.4	79.2	138.0	167.2	52.6
t_{10}	ARL	78.7	131.2	212.2	244.2	82.5	51.6	81.4	133.4	147.1	52.4
	MRL	56	93	151	170	57	37	59	96	103	37
	SDRL	75.2	124.4	201.6	240.1	81.3	48.5	75.3	123.4	141.9	50.9
t_{20}	ARL	83.6	126.2	180.6	187.6	84.6	50.7	75.3	112.3	114.9	50.3
	MRL	60	90	129	132	59	37	55	81	82	36
	SDRL	79.1	118.1	169.5	180.7	82.1	47.0	68.1	102.0	108.1	47.5
t_{30}	ARL	86.1	125.1	171.6	175.4	85.8	50.7	73.6	106.6	107.1	49.6
	MRL	61	90	123	123	60	37	54	77	77	35
	SDRL	81.2	116.9	159.2	168.4	83.2	46.7	66.1	96.1	99.9	47.0
t_{40}	ARL	87.1	124.9	168.0	169.1	86.1	50.5	72.8	103.9	104.0	49.5
	MRL	62	89	120	120	61	37	53	76	74	36
	SDRL	82.2	116.2	156.1	161.0	83.2	46.8	65.0	93.5	96.8	46.7
t_{50}	ARL	87.7	124.1	165.7	165.6	86.5	50.7	72.1	102.4	101.9	49.2
	MRL	63	89	119	117	61	37	53	75	73	35
	SDRL	82.5	115.3	153.6	157.5	83.2	46.6	64.6	91.8	94.5	46.0

Table 4.4 (continued) Out–of-control ARL, MRL and SDRL values for upward shifts $\lambda = 0.05$

	Shift	1.6					1.8				
		WR	SR	HO	DP1	DP2	WR	SR	HO	DP1	DP2
$N(\mu, \sigma^2)$	ARL	34.9	37.7	43.2	38.5	30.8	25.0	27.5	32.2	28.9	22.3
	MRL	27	30	34	30	24	20	22	26	23	18
	SDRL	28.5	28.9	32.8	29.6	25.2	19.5	19.7	22.7	20.7	17.5
t_4	ARL	50.7	95.1	220.0	397.5	57.1	45.6	81.9	183.3	311.6	50.1
	MRL	36	67	155	273	39	32	58	129	213	34
	SDRL	48.8	91.5	213.2	406.4	57.6	43.8	78.8	176.8	318.2	50.4
t_6	ARL	43.3	72.1	133.6	166.5	45.3	37.3	60.5	110.0	132.8	38.6
	MRL	31	51	95	116	31	27	44	79	93	27
	SDRL	41.0	67.7	125.3	164.5	44.6	35.1	56.1	102.2	129.8	37.6
t_8	ARL	40.9	65.0	109.9	124.2	41.8	34.8	53.5	90.0	99.8	35.1
	MRL	29	47	79	87	29	25	39	65	70	25
	SDRL	38.3	59.8	101.2	120.4	40.3	32.3	48.5	81.9	95.4	33.6
t_{10}	ARL	39.7	61.2	99.0	107.5	39.7	33.6	50.3	81.0	86.7	33.3
	MRL	29	44	71	76	28	25	37	59	62	24
	SDRL	37.0	55.8	90.1	102.0	37.9	31.0	45.2	72.9	81.4	31.5
t_{20}	ARL	37.9	55.3	83.6	84.2	37.2	31.3	44.8	67.9	68.4	30.7
	MRL	28	41	61	60	27	23	33	50	49	22
	SDRL	34.8	49.0	74.3	78.0	34.8	28.5	39.2	59.4	62.1	28.5
t_{30}	ARL	37.5	53.4	79.0	79.0	36.4	30.8	43.7	64.3	63.9	30.1
	MRL	27	39	58	57	26	23	32	48	47	22
	SDRL	34.1	47.2	69.4	72.2	33.8	27.7	37.9	55.4	57.2	27.8
t_{40}	ARL	37.2	52.7	77.1	76.2	36.2	30.5	43.0	62.9	61.8	29.5
	MRL	27	39	57	55	26	22	32	47	45	22
	SDRL	33.8	46.1	67.5	69.1	33.5	27.5	37.2	54.1	55.0	27.1
t_{50}	ARL	37.2	52.4	76.2	75.0	36.1	30.4	42.6	61.9	60.8	29.4
	MRL	27	39	56	55	26	22	32	46	45	21
	SDRL	33.7	45.8	66.6	67.6	33.3	27.3	36.8	52.9	54.1	26.9

Table 4.4 (continued) Out–of-control ARL, MRL and SDRL values for upward shifts $\lambda = 0.1$

	Shift	1.2					1.4				
		WR	SR	HO	DP1	DP2	WR	SR	HO	DP1	DP2
$N(\mu, \sigma^2)$	ARL	124.1	123.3	131.8	123.2	113.0	60.7	61.7	68.2	62.2	54.6
	MRL	88	88	94	88	80	44	45	50	45	39
	SDRL	119.8	116.6	123.5	116.2	109.1	56.3	55.6	60.7	55.6	50.5
t_4	ARL	67.4	116.8	254.6	449.8	76.3	53.8	88.6	182.3	300.3	59.6
	MRL	47	82	178	310	53	38	62	127	207	41
	SDRL	66.0	114.7	250.4	454.0	76.7	52.5	86.9	179.2	302.4	59.6
t_6	ARL	68.1	110.8	202.3	270.4	73.4	49.3	76.0	134.3	171.4	52.3
	MRL	48	78	142	187	51	35	54	95	119	36
	SDRL	66.3	107.6	197.1	269.9	73.5	48.0	73.0	129.5	170.6	51.7
t_8	ARL	70.7	110.2	184.7	224.3	75.2	48.5	72.1	118.7	139.7	50.4
	MRL	50	78	130	156	52	34	51	84	98	35
	SDRL	69.1	106.5	178.6	222.0	74.2	46.7	68.8	113.3	136.8	49.4
t_{10}	ARL	73.0	111.1	175.9	203.6	76.4	48.5	70.1	111.2	126.4	49.5
	MRL	51	79	124	142	53	34	50	79	89	35
	SDRL	70.9	107.2	169.2	200.5	75.6	46.7	66.6	105.4	122.7	48.3
t_{20}	ARL	79.7	112.9	160.8	172.5	81.4	48.4	67.2	99.1	105.4	48.8
	MRL	56	80	113	121	57	34	48	71	75	34
	SDRL	77.2	107.9	154.0	167.5	80.0	46.1	63.1	92.9	101.1	47.0
t_{30}	ARL	82.9	114.0	156.5	164.2	83.5	48.7	66.6	95.6	100.3	48.6
	MRL	58	81	111	116	59	35	48	68	71	34
	SDRL	80.5	109.1	149.4	159.0	81.4	46.1	62.4	89.3	95.7	47.0
t_{40}	ARL	84.3	114.9	154.2	160.8	85.0	49.2	66.0	93.7	97.3	48.5
	MRL	60	81	109	113	60	35	47	67	69	34
	SDRL	81.2	109.9	147.1	155.3	83.0	47.0	61.9	87.3	92.6	46.9
t_{50}	ARL	85.3	115.4	153.2	158.8	85.4	49.2	66.0	93.2	96.6	48.6
	MRL	60	82	109	112	60	35	47	67	68	34
	SDRL	82.6	110.4	145.2	153.8	83.7	47.1	61.9	87.0	91.7	46.8

Table 4.4 (continued) Out–of-control ARL, MRL and SDRL values for upward shifts $\lambda = 0.1$

Shift		1.6					1.8				
		WR	SR	HO	DP1	DP2	WR	SR	HO	DP1	DP2
$N(\mu, \sigma^2)$	ARL	37.3	38.5	43.4	39.7	33.6	26.1	27.4	31.4	28.8	23.6
	MRL	27	29	32	30	25	20	21	24	22	18
	SDRL	33.4	32.9	36.6	33.4	29.9	22.5	22.2	25.2	22.9	20.2
t_4	ARL	46.3	73.8	145.8	229.1	50.4	41.6	64.8	124.2	189.9	45.1
	MRL	33	52	102	158	35	29	46	87	131	31
	SDRL	45.3	71.7	142.4	231.2	50.5	40.4	62.8	120.9	191.9	45.0
t_6	ARL	40.4	59.9	102.9	127.7	42.0	35.1	51.0	85.5	103.8	36.2
	MRL	28	42	73	89	29	25	36	61	72	25
	SDRL	39.0	57.7	98.3	126.2	41.2	33.7	48.6	80.8	101.7	35.4
t_8	ARL	38.4	55.2	89.6	103.4	39.3	32.8	46.4	73.4	84.0	33.4
	MRL	27	39	64	73	28	23	33	52	59	24
	SDRL	37.0	52.3	84.4	100.2	38.4	31.3	43.8	68.8	80.7	32.3
t_{10}	ARL	37.6	52.8	82.9	92.8	38.3	31.8	44.0	67.6	75.0	32.0
	MRL	27	38	59	65	27	23	32	49	53	23
	SDRL	35.9	49.7	77.8	89.1	37.0	30.3	41.1	62.7	71.5	30.9
t_{20}	ARL	36.5	49.2	72.7	77.0	36.1	30.0	40.4	59.0	62.2	29.8
	MRL	26	35	52	55	26	21	29	43	44	21
	SDRL	34.4	45.5	67.0	72.7	34.6	28.3	37.1	53.9	58.2	28.4
t_{30}	ARL	36.2	48.5	70.2	73.1	35.9	29.8	39.2	56.8	59.2	29.4
	MRL	26	35	51	52	25	21	28	41	42	21
	SDRL	34.3	45.0	64.7	68.5	34.2	27.8	36.0	51.5	55.1	27.9
t_{40}	ARL	36.2	47.9	68.9	71.4	35.5	29.5	38.9	55.7	57.3	29.1
	MRL	26	35	50	51	25	21	28	40	41	21
	SDRL	34.2	44.2	63.0	67.0	33.9	27.6	35.8	50.3	53.2	27.5
t_{50}	ARL	35.9	47.7	68.3	70.4	35.5	29.4	38.7	55.0	56.5	28.8
	MRL	26	34	49	50	25	21	28	40	41	21
	SDRL	33.9	44.0	62.4	65.9	33.8	27.5	35.3	49.6	52.4	27.4

Table 4.4 (continued) Out–of-control ARL, MRL and SDRL values for upward shifts $\lambda = 0.2$

	Shift	1.2					1.4				
		WR	SR	HO	DP1	DP2	WR	SR	HO	DP1	DP2
N(μ, σ^2)	ARL	136.7	133.4	137.7	132.0	128.8	69.1	67.0	71.1	67.6	63.5
	MRL	95	94	97	93	90	49	48	51	48	45
	SDRL	134.5	129.8	133.1	128.0	127.1	67.1	63.6	66.6	63.4	61.4
t_4	ARL	61.2	87.0	149.0	223.9	65.8	49.4	68.1	112.3	162.1	52.7
	MRL	43	61	104	155	46	35	48	79	113	37
	SDRL	60.3	85.8	146.8	224.1	65.6	48.6	66.7	110.2	161.8	52.4
t_6	ARL	62.7	86.3	136.7	175.8	66.0	46.2	60.9	94.3	117.5	47.9
	MRL	44	60	96	122	46	32	43	66	82	33
	SDRL	61.8	84.7	133.9	174.8	65.5	45.4	59.5	91.5	116.4	47.4
t_8	ARL	66.1	88.8	135.1	161.9	68.8	45.9	59.5	88.3	104.7	47.3
	MRL	46	62	95	113	48	32	42	62	73	33
	SDRL	64.9	87.2	131.3	160.1	68.1	45.0	57.8	85.5	102.9	46.6
t_{10}	ARL	68.8	91.0	133.8	156.3	71.5	46.2	59.5	85.7	98.4	47.1
	MRL	48	64	94	109	50	32	42	61	69	33
	SDRL	67.5	89.0	130.2	154.6	70.5	45.1	57.7	82.3	96.4	46.2
t_{20}	ARL	76.9	98.1	133.6	147.4	78.0	47.2	58.9	81.5	88.9	47.5
	MRL	54	69	94	103	54	33	42	58	63	33
	SDRL	75.4	95.6	129.8	144.9	77.1	46.0	56.7	78.0	86.2	46.6
t_{30}	ARL	80.6	100.7	134.1	144.6	81.0	47.9	59.0	80.1	86.6	47.9
	MRL	56	70	94	101	56	34	41	57	61	34
	SDRL	79.2	98.7	130.4	141.2	80.0	46.7	56.8	76.8	84.3	46.9
t_{40}	ARL	82.4	102.2	133.9	143.1	82.6	48.0	59.0	79.4	85.7	48.2
	MRL	57	72	94	100	58	34	42	56	60	34
	SDRL	81.0	99.4	129.7	139.8	81.6	46.9	56.9	76.3	83.2	47.2
t_{50}	ARL	83.4	103.5	133.9	143.3	83.9	48.4	58.9	79.3	85.3	47.9
	MRL	58	73	94	100	58	34	42	56	60	34
	SDRL	82.3	100.6	129.6	140.2	83.1	47.1	56.5	75.8	82.9	46.9

Table 4.4 (continued) Out–of-control ARL, MRL and SDRL values for upward shifts $\lambda = 0.2$

	Shift	1.6					1.8				
		WR	SR	HO	DP1	DP2	WR	SR	HO	DP1	DP2
$N(\mu, \sigma^2)$	ARL	42.0	41.4	44.9	42.2	38.6	29.1	28.8	31.6	29.9	26.8
	MRL	30	31	32	31	28	21	21	23	22	19
	SDRL	40.0	38.1	40.9	38.3	36.7	27.1	25.7	27.7	26.2	24.7
t_4	ARL	42.6	57.3	92.6	130.4	45.1	38.6	51.2	80.4	111.2	40.5
	MRL	30	40	65	91	31	27	36	56	77	28
	SDRL	42.0	56.2	90.4	130.2	44.9	37.8	50.1	78.4	111.4	40.1
t_6	ARL	38.0	49.2	73.7	90.4	39.1	33.2	42.3	62.3	75.5	33.9
	MRL	27	35	52	63	27	23	30	44	53	24
	SDRL	37.1	47.9	71.4	89.0	38.5	32.3	40.8	59.9	74.3	33.4
t_8	ARL	36.7	46.4	67.3	79.1	37.4	31.5	39.3	55.9	65.1	31.8
	MRL	26	33	47	55	26	22	28	40	46	22
	SDRL	35.7	45.0	64.9	77.3	36.5	30.5	37.9	53.5	63.4	31.1
t_{10}	ARL	36.2	45.3	64.6	73.6	36.7	30.5	38.1	53.3	60.3	30.8
	MRL	26	32	46	52	26	22	27	38	42	22
	SDRL	35.1	43.6	61.8	71.8	36.0	29.4	36.5	50.5	58.5	30.1
t_{20}	ARL	35.5	43.4	59.7	65.6	35.6	29.3	35.5	48.4	52.7	29.4
	MRL	25	31	43	46	25	21	25	34	37	21
	SDRL	34.3	41.6	56.5	63.6	34.7	28.2	33.8	45.8	50.5	28.4
t_{30}	ARL	35.4	43.0	58.3	63.0	35.2	29.1	35.0	46.9	50.9	29.0
	MRL	25	30	41	45	25	21	25	34	36	20
	SDRL	34.3	41.0	55.2	60.6	34.3	28.0	33.3	44.0	48.8	28.0
t_{40}	ARL	35.3	42.5	57.9	62.1	35.1	29.0	34.6	46.5	50.0	28.7
	MRL	25	30	41	44	25	21	25	33	36	20
	SDRL	34.0	40.3	54.6	59.8	34.2	27.8	32.8	43.5	47.5	27.7
t_{50}	ARL	35.3	42.6	57.2	61.3	35.1	28.7	34.6	46.1	49.4	28.6
	MRL	25	30	41	43	25	20	25	33	35	20
	SDRL	34.2	40.5	54.1	58.6	34.1	27.6	32.7	43.2	47.2	27.7

Note that a direct comparison of the different schemes is not possible because they do not have the same in-control ARL or MRL values. We observe that the out-of-control ARL performance of the charts that have a good in-control one, is far from the normal. They do not give that fast an out-of-control signal, therefore they lose the ability of the EWMA charts to identify an out-of-control situation for small shifts quickly.

Consequently, the $EWMA_H$ and $EWMA_A$ (for α=0.5) charts are a very good choice when normality is questionable for specific values of the smoothing parameter λ when our process is in-control. In the out-of-control cases they give disappointing results. The $EWMA_S$ and $EWMA_A$ for α=2 charts are not recommended since their performance in both in-control and out-of-control situations is far from the normal. The $EWMA_V$ chart does not perform well for skewed distributions but in the symmetric case the results are better for small values of λ. Generally we can say that none of the presented schemes is robust to the normality assumption.

4.3.5 Discussion

The research for the non-normality effect of the EWMA control charts for process dispersion was conducted in a way for the results drawn, to hold for data coming from any distribution without a need to know this distribution. However, in the particular case that we know explicitly the distribution our data are coming from, we may use a transformation of the data to the normal. Such a transformation is given in Hawkins and Olwell (p. 163, 1998) and it has been used also by Quesenberry (1995a) and Chen et al. (2001) in the context of EWMA charts. This transformation not only achieves approximate normality but also independence of the resulting data.

Chapter 5

Identification of the Out-of-Control Variable When a Multivariate Control Chart Signals

5.1 Introduction

Multivariate control charts are a powerful tool in Statistical Process Control for identifying an out-of-control process. Woodall and Montgomery (1999) emphasized the need for much more research in this area since most of the processes involve a large number of variables that are correlated. As Jackson (1991) notes, any multivariate quality control procedure should fulfill four conditions 1) Single answer to the question "Is the process in-control?" 2) An overall probability for the event "Procedure diagnoses an out-of-control state erroneously" must be specified 3) The relationship among the variables must be taken into account and 4) Procedures should be available to answer the question "If the process is out-of-control, what is the problem?". The last question has proven to be an interesting subject for many researchers in the last years. Woodall and Montgomery (1999) state that although there is difficulty in interpreting the signals from multivariate control charts more work is needed on data reduction methods and graphical techniques.

In this chapter we present the available solutions for the problem of identification and additionally we propose a new method based on principal components analysis (PCA), for detecting the out-of-control variable, or variables, when a multivariate control chart for individual observations signals. Section 5.2 describes the use of univariate control charts for solving the above stated problem, whereas Section 5.3 gives the use of an elliptical control region. In Section 5.4 a $T^{\mathbf{2}}$ decomposition is presented. Section 5.5 summarizes the methods based on principal components analysis. A presentation of the new method, is given in the Section 5.6 with some interesting points and discussion on the performance and application of the new method. Moreover, a comparative study evaluates the performance of the proposed method in relation to the existing methods that use PCA. Finally, graphical techniques that attempt to solve the problem under investigation are presented in Section 5.7.

5.2 The Use of Univariate Control Charts

5.2.1 Univariate Control Charts with Standard Control Limits

The use of p univariate control charts, gives a first evidence for which of the p variables are responsible for an out-of-control signal. However, there are some problems in using p univariate control charts in place of X^2-Chart. These problems are that, the overall probability of the mean plotting outside the control limits if we are in-control is not controlled and the correlations among the variables are ignored. The problem of ignoring the correlations among the variables cannot be solved. The problem of controlling the overall probability of the mean plotting outside the control limits if we are in-control can be solved by using p univariate control charts with Bonferroni limits.

5.2.2 Using Univariate Control Charts with Bonferroni Control Limits

The use of Bonferroni control limits, was proposed by Alt (1985). Bonferroni control limits can be used to investigate which of the p variables are responsible for an out-of-control signal. Using the Bonferroni method the following control limits are established

$$\begin{aligned} UCL &= \mu_i + Z_{1-\alpha/2p}\frac{\sigma_i}{\sqrt{n}} \\ LCL &= \mu_i - Z_{1-\alpha/2p}\frac{\sigma_i}{\sqrt{n}}. \end{aligned}$$

Thus, p individual control charts can be constructed, each with probability of the mean plotting outside the control limits if we are in-control, equal to α/p and not α. However, as Alt (1985) states, this does not imply that p univariate control charts should be used in place of X^2-Chart.

5.2.3 Hayter and Tsui's Interpretation Method

Hayter and Tsui (1994) extended the idea of Bonferroni type control limits by giving a procedure for exact simultaneous control intervals for each of the variable means. The control procedure operates as follows. For a known variance-covariance matrix $\mathbf{\Sigma}$ and a chosen probability of the mean plotting outside the control limits if we are in-control α, the experimenter first evaluates the critical point $C_{R,\alpha}$ where $\mathbf{R}$ is the correlation matrix obtained from $\mathbf{\Sigma}$. Then, following any observation $\mathbf{x}^t = (X_1, X_2, ..., X_p)$, the experimenter constructs confidence intervals

$$(X_i - \sigma_i C_{R,\alpha}, X_i + \sigma_i C_{R,\alpha})$$

for each of the p variables. This procedure ensures that an overall probability of the mean plotting outside the control limits if we are in-control α is achieved. The process is considered to be in-control as long as each of these confidence intervals contains the respective standard value μ_{0i}. The process is considered to be out-of-control if any of these confidence intervals does not contain the respective standard value μ_{0i}. The variable or variables whose confidence intervals do not contain μ_{0i}, are identified as those responsible for the out-of-control signal.

This procedure signals when

$$M = max\left|X_i - \mu_{0i}\right|/\sigma_i > C_{R,\alpha}.$$

Hayter and Tsui (1994) give guidance and various tables for choosing the critical point $C_{R,\alpha}$.

5.3 Using an Elliptical Control Region

The second method uses an elliptical control region. This method is discussed by Alt (1985) and Jackson (1991) and can be applied only in the special case of two quality characteristics.

The simplest case in multivariate statistics is when the vector $\mathbf{x} = \binom{x_1}{x_2}$ has a bivariate normal distribution, where X_i is distributed normally with mean μ_i, standard deviation σ_i, i= 1, 2 and ρ is the correlation coefficient between the two variables. In this case an elliptical control region, can be constructed. This elliptical region is centered at $\boldsymbol{\mu}_0^t = (\mu_1, \mu_2)$ and can be used in place of the X^2-chart. All points lying on the ellipse would have the same value of X^2. While, X^2-Chart gives a signal every time the process is out-of-control, the elliptical region is useful in indicating which variable led to the out-of-control signal.

Therefore, a $100\,(1-\alpha)\,\%$ elliptical control region can be constructed by applying the following equation as given by Jackson (1991)

$$Q = \left\{\frac{1}{1-\rho^2} \times \left[\left(\frac{X_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{X_2-\mu_2}{\sigma_2}\right)^2 - \frac{2\rho\,(X_1-\mu_1)\,(X_2-\mu_2)}{\sigma_1\sigma_2}\right]\right\} = X^2_{2,1-\alpha}.$$

A unique ellipse is defined for given values of $\mu_1, \mu_2, \sigma_1, \sigma_2, \rho$ and α. Points on the perimeter of the ellipse may be determined by setting X_1 equal to some constant and solving the resulting quadratic equation for X_2.

Mader et al. (1996) presented the use of the elliptical control region for power supply calibration.

5.4 Using T^2 Decomposition

The third method is the use of T^2 decomposition, which is proposed by Mason, Tracy and Young (1995,1997). The main idea of this method (MYT) is to decompose the T^2 statistic into independent parts, each of which reflects the contribution of an individual variable. This method is developed for the case of individual observations, but according to the authors it can be applied also with a few modification for the case of rational subgroups.

In this section we also present, the methodologies of Roy (1958), Murphy (1987), Doganaksoy et al. (1991), Hawkins (1991, 1993), Timm (1996) and Runger and Montgomery (1996), which are included in the MYT partitioning of T^2.

5.4.1 Mason, Tracy and Young's T^2 Decomposition

Mason et al. (1995) presented the following interpretation method of an out-of-control signal. The T^2 statistic can be broken down or decomposed into p orthogonal components. One form of the MYT decomposition is given by

$$T^2 = T_1^2 + T_{2\cdot 1}^2 + T_{3\cdot 1,2}^2 + ... + T_{p\cdot 1,2,...,p-1}^2 = T_1^2 + \sum_{j=1}^{p-1} T_{j\cdot 1,2,...,j-1}^2.$$

The first term of this decomposition, T_1^2, is an unconditional Hotelling's T^2 for the first variable of the observation vector $\mathbf{x}$,

$$T_1^2 = \left(\frac{X_1 - \overline{X}_1}{s_1} \right)^2,$$

where $\overline{X}_1$ and s_1 is the mean and standard deviation of variable X_1, respectively.

The general form of the other terms, referred to as conditional terms, is given as

$$T_{j\cdot 1,2,\ldots,j-1}^2 = \frac{(X_j - \overline{X}_{j\cdot 1,2,\ldots,j-1})^2}{s_{j\cdot 1,2,\ldots,j-1}^2}, \text{ for } j = 1, 2, \ldots, p,$$

where

$$\overline{X}_{j\cdot 1,2,\ldots,j-1} = \overline{X}_j + \mathbf{b}_j^t \left(\mathbf{X}_i^{(j-1)} - \overline{\mathbf{X}}^{(j-1)} \right)$$

and $\mathbf{X}_i^{(j-1)}$ is the $(j-1)th$ vector excluding the jth variable, $\overline{X}_j$ is the sample mean of the jth variable, $\mathbf{b}_j = [\mathbf{S}_{\mathbf{xx}}^{-1} \mathbf{S}_{X\mathbf{x}}]$ is a $(j-1)th$ dimensional vector estimating the regression coefficients of the jth variable regressed on the first $(j-1)$ variables,

$$s_{j\cdot 1,2,\ldots,j-1}^2 = s_X^2 - \mathbf{S}_{X\mathbf{x}}^t \mathbf{S}_{\mathbf{xx}}^{-1} \mathbf{S}_{X\mathbf{x}}$$

and

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_{\mathbf{xx}} & \mathbf{S}_{X\mathbf{x}} \\ \mathbf{S}_{X\mathbf{x}}^t & s_X^2 \end{bmatrix}.$$

Consequently, the $T_{j\cdot 1,2,\ldots,j-1}^2$ value is the square, of the jth variable adjusted by the estimates of the mean and standard deviation of the conditional distribution of X_j given $X_1, X_2, \ldots, X_{j-1}$ and its exact distribution is as follows

$$T_{j\cdot 1,2,\ldots,j-1}^2 \sim \frac{n+1}{n} F(1, n-1).$$

Thus, this statistic can be used to check whether the jth variable is conforming to the relationship with other variables as established by the historical data set, since the adjusted observation is more sensitive to changes in the covariance structure.

The ordering of the p components is not unique and the one given above represents only one of the possible $p!$ different ordering of these components. Each ordering generates the same overall T^2 value, but provides a distinct partitioning of T^2 into p orthogonal terms. If we exclude redundancies, there are $p \times 2^{p-1}$ distinct components among the $p \times p!$ possible terms that should be evaluated for potential contribution to signal.

Similarly, the p unconditional T^2 terms based on squaring a univariate t statistic can be computed and then be compared to the appropriate F distribution. Moreover, the distances $D_i = T^2 - T_i^2$ can be computed and also be compared to the F distribution.

5.4.2 Mason, Tracy and Young's Out-of-Control Variable Selection Algorithm

The following is a sequential computational scheme that has the potential of further reducing the computations to a reasonable number when the overall T^2 signals, as was proposed by Mason, Tracy and Young (1997).

Step 0: Conduct a T^2 test with a specified nominal confidence level α. If an out-of-control condition is signaled then continue with the step 1.

Step 1: Compute the individual T^2 statistic for every component of the $\mathbf{x}$ vector. Remove variables whose observations produce a significant T_i^2. The observations on these variables are out of individual control and it is not necessary to check how they relate to the other observed variables. With significant variables removed we have a reduced set of variables. Check the subvector of the remaining k variables of a signal. If you do not receive a signal we have located the source of the problem.

Step 2: Optional: Examine the correlation structure of the reduced set of variables. Remove any variable having a very weak correlation (0.3 or less) with all the other variables. The contribution of a variable that falls in this category is measured by the T_i^2 component.

Step 3: If a signal remains in the subvector of k variables not deleted, compute all $T_{i \cdot j}^2$ terms. Remove from the study all pairs of variables, (X_i, X_j), that have a significant $T_{i \cdot j}^2$

term. This indicates that something is wrong with the bivariate relationship. When this occurs it will further reduce the set of variables under consideration. Examine all removed variables for the cause of the signal. Compute the T^2 terms for the remaining subvector. If no signal is present, the source of the problem is with the bivariate relationships and those variables that were out of individual control.

Step 4: If the subvector of the remaining variables still contains a signal, compute all $T^2_{i \cdot j,k}$ terms. Remove any triple, (X_i, X_j, X_k), of variables that show significant results and check the remaining subvector for a signal.

Step 5: Continue computing the higher order terms in this fashion until there are no variables left in the reduced set. The worst case situation is that all unique terms will have to be computed.

Generally, the T^2 statistic associated with an observation from a multivariate problem is a function of the residuals taken from a set of linear regressions among the various process variables. These residuals are contained in the conditional T^2 terms of the orthogonal decomposition of the statistic. Mason and Young (1999) showed that a large residual in one of these fitted models can be due to incorrect model specification. By improving the model specification at the time that the historical data set is constructed, it may be possible to increase the sensitivity of the T^2 statistic to signal detection. Also, they showed that the resulting regression residual, can be used to improve the sensitivity of the T^2 statistic to small but consistent process shifts, using plots that are similar to cause-selecting charts.

The productivity of an industrial processing unit often depends on equipment that changes over time. These changes may not be stable, and, in many cases, may appear to occur in stages. Although changes in the process levels within each stage may appear insignificant, they can be substantial when monitored across the various stages. Standard process control procedures do not perform well in the presence of these step-like changes, especially when the observations from stage to stage are correlated. Mason et al. (1996) present an alternative control procedure for monitoring a process under these conditions, which is based on a double decomposition of Hotelling's T^2 statistic.

5.4.3 Doganaksoy, Faltin and Tucker's Out-of-Control Variable Selection Algorithm

The method that is presented in this subsection was proposed by Doganaksoy et al. (1991). The main idea of this method is the use of the univariate t ranking procedure and it is based on p unconditional T^2 terms. The statistic used is

$$t = \frac{\overline{X}_{i,new} - \overline{X}_{i,ref}}{\sqrt{s_{ii}\left(\frac{1}{n_{new}} + \frac{1}{n_{ref}}\right)}},$$

where $\overline{X}_{i,new}$ is the mean of the new sample, $\overline{X}_{i,ref}$ is the mean of the reference sample, s_{ii} is the estimate of the variance of the ith variable from the reference sample, n_{new} is the size of the new sample and n_{ref} is the size of the reference sample. The steps of this algorithm are the following

Step 1: Conduct a T^2 test with a specified nominal significance level α. If an out-of-control condition is signalled then continue with step 2.

Step 2: For each variable calculate the smallest significance level α^{ind} that would yield an individual confidence interval for $(\mu_{i,ref} - \mu_{i,new})$ that contains zero, where $\mu_{i,new}$ and $\mu_{i,ref}$ are the mean vectors of the populations from which the reference and new samples are drawn, respectively. For this α^{ind}, let t be the calculated value of the univariate t statistic for a variable and $T(t, d)$ be the cumulative distribution function of the t distribution with d degrees of freedom. Then $\alpha^{ind} = [2T(t; n_{ref} - 1) - 1]$.

Step 3: Plot α^{ind} for each variable on a 0-1 scale. Note that variables with larger α^{ind} values are the ones with relatively larger univariate t statistic values which require closer investigation as possible being among those components which have undergone a change. If indications of highly suspect variables are desired then continue.

Step 4: Compute the confidence interval α^{bonf} that yields the desired nominal confidence interval α^{sim} of the Bonferroni type simultaneous confidence intervals for $\mu_{i,ref} - \mu_{i,new}$. Here, $\alpha^{bonf} = [(p + \alpha^{sim} - 1)/p]$.

Step 5: Components having $\alpha^{ind} > \alpha^{bonf}$ are classified as being those which are most likely to have changed.

Furthermore, the authors give guidance for the choice of the α^{sim}.

5.4.4 Murphy's Out-of-Control Variable Forward Selection Algorithm (FSA)

This method is proposed by Murphy (1987). It is a subcase of the T^2 decomposition method, which was proposed by Mason et al. (1995) and stems from the field of discriminant analysis. It uses the overall T^2 value and compares it to a T_*^2 value based on a subset of variables.

The diagnostic approach is triggered by an out-of-control signal from a T^2-Chart. Murphy (1987) partitioned the sample mean vector $\overline{\mathbf{x}}$ into two subvectors $\overline{\mathbf{x}}_{*1}$ and $\overline{\mathbf{x}}_{*2}$, where the p_1 dimensional vector $\overline{\mathbf{x}}_{*1}$ is the subset of the $p = p_1 + p_2$ variables, which is suspect for the out-of-control signal. Then

$$T_p^2 = n\left(\overline{\mathbf{x}} - \boldsymbol{\mu}_0\right)^t \boldsymbol{\Sigma}_0^{-1} \left(\overline{\mathbf{x}} - \boldsymbol{\mu}_0\right)$$

is the full squared distance and

$$T_*^2 = n\left(\overline{\mathbf{x}}_{*1} - \boldsymbol{\mu}_{01}\right)^t \boldsymbol{\Sigma}_{01}^{-1} \left(\overline{\mathbf{x}}_{*1} - \boldsymbol{\mu}_{01}\right)$$

is the reduced distance corresponding to the subset of the p variables that is suspect for the out-of-control signal.

Finally, the following difference is calculated

$$D = T_p^2 - T_*^2.$$

It is proved that, under the null hypothesis, D follows a Chi-Square distribution with p_1 degrees of freedom and the subvector $\overline{\mathbf{x}}_{*1}$ follows a p_1-dimensional distribution with mean $\boldsymbol{\mu}_{01}$ and variance-covariance matrix $\boldsymbol{\Sigma}_{01}$. Murphy (1987) gave a forward selection algorithm.

The steps of this algorithm are the following; For each $\overline{\mathbf{x}} = \left[\overline{X}_1, ..., \overline{X}_p\right]$,

Step 1: Conduct a T^2 test with a specified nominal significance level α. If an out-of-control condition is signalled then continue with step 2.

Step 2: Calculate the p individual $T_1^2(X_i)$, equivalent to looking at p individual charts, and calculate the p differences $D_{p-1}(i) = \left[T_p^2 - T_1^2(X_i)\right]$. Choose the $min\left(D_{p-1}(i)\right) = D_{p-1}(r)$ and test this minimum difference.

If $D_{p-1}(r)$ is not significant then the rth variable only requires attention.

If $D_{p-1}(r)$ is significant then continue with step 3.

Step 3: Calculate the $p-1$ differences $D_{p-2}(r,j) = \left[T_p^2 - T_2^2(X_r, X_j)\right]$, $1 \leq r,\, j \leq p$ and $r \neq j$. Choose the $min\left(D_{p-2}(r,j)\right) = D_{p-2}(r,s)$ and test this minimum difference.

If $D_{p-2}(r,s)$ is not significant then the rth and the sth variables only require attention.

If $D_{p-2}(r,s)$ is significant then continue with step 4.

Step 4: Similar to step 3.

Step .: Similar to steps 3,4.

Step p: If the final $D_{p-(p-1)}$ test is significant then all p variables will require attention.

Murphy (1987) recommends that in tests of $D_{p-i} \backsim X_{p-i}^2$ a significance level in the interval $0.1 \leq \alpha \leq 0.2$ be used. From a practical point of view the applicability of this approach is severely limited when the number of quality or process variables is moderately large.

5.4.5 Chua and Montgomery's Out-of-Control Variable Selection Method

The method that is presented in this subsection was proposed by Chua and Montgomery (1992). This method, like the method of Murphy (1987), uses the overall T^2 value and compares it to a T_*^2 value based on a subset of variables. It is quite similar to Murphy's method, but uses a backward selection algorithm (BSA). The operations of the control scheme can be described as follows;

1) A multivariate observation is fed into the multivariate EWMA control chart.

2) If the observation is in-control, the control operation loops back to the beginning and checks the next observation. Otherwise, it checks the number of process variables.

3) If the number of process variables is greater than five, bypass the BSA and use the hyper-plane method directly. Otherwise, feed the observation into BSA.

4) The BSA will select the out-of-control variable set and feed it into the hyper-plane method.

5) The hyper-plane method will generate the necessary elliptical control charts for diagnosis.

6) Based on the diagnosis, corrective actions are then taken and the control operation loops back to the beginning and checks for the next observation.

Chua and Montgomery (1992) describe all the needed actions for the application of their method. The algorithm divides the p variables into two subsets of size p_1 and p_2, where $p_1 + p_2 = p$. Also, the sample covariance matrix is divided into two parts with p_1 and p_2 variables respectively. For all the p variables the T_p^2 is calculated. Similarly, for the p_1 variables the T_{p1}^2 is calculated. Then a difference value $D = T_p^2 - T_{p1}^2$ is calculated, which is distributed as a Chi-Square distribution with p_2 degrees of freedom.

The steps of this algorithm are the following; For each $\overline{\mathbf{x}} = \left[\overline{X}_1, ..., \overline{X}_p\right]$,

Step 1: Conduct a T^2 test with a specified nominal confidence level α. If an out-of-control condition is signalled then continue with step 2.

Step 2: Perform T_{p-1}^2 tests and calculate the differences D_{p-1}=T_p^2-T_{p-1}^2. If D_{p-1}

is not significant then the union of variable sets for all sets which are denoted as not significant is the new criterion. If $D_{p-1}(r)$ is significant then keep the criterion from step 1, denote the $min\,(D_{p-1}) = D(p)$ and continue to the next step.

Step 3: Perform T^2_{p-2} tests and calculate the differences D_{p-2}=T^2_p-T^2_{p-2}. If D_{p-2} is not significant then the union of variable sets for all sets which are denoted as not significant is the new criterion. If D_{p-2} is significant then keep the criterion from step 2, denote the $min\,(D_{p-2}) = D(p-1)$ and continue to the next step.

Step 4: Similar to step 3.

Step .: Similar to step 3, 4.

Step p: Perform T^2_1 tests and calculate the differences D_1=T^2_p-T^2_1. If D_1 is not significant then the union of variable sets for all sets which are denoted as not significant is the new criterion. If D_1 is significant then keep the criterion from step p-1, denote the $min\,(D_1) = D(1)$ and continue to the next step.

Step p+1: If all the tests from steps 2 to p are significant, exit. All p variables are out-of-control. If $D(i) \leq D(j)$, where $i < j$, $i = 1, 2, ..., p-1$, and $j = i+1, ..., p$ and the variable set for $D(i)$ is a subset of the variable set for $D(j)$, then the variable set for $D(i)$ is the out-of-control variable set. Otherwise, increment i by 1 and repeat the condition check.

The hyper-plane method is an extension of the elliptical control chart. Chua and Montgomery (1992) present the hyper-plane method graphically. The mathematical background of the hyper-ellipsoid method follows.

The equation for the hyper-ellipsoid control region is $(\mathbf{x}-\overline{\mathbf{x}})\mathbf{S}^{-1}(\mathbf{x}-\overline{\mathbf{x}}) = T^2_{1-\alpha}$. Obviously, if the left-hand side of the equation is less than $T^2_{1-\alpha}$, the observation will be inside the hyper-ellipsoid, and thus in-control. On the other-hand, if it is greater than $T^2_{1-\alpha}$, the observation will be outside the hyper-ellipsoid, and thus out-of-control.

Moreover, the vector equation of a hyper-plane is $\mathbf{d}\cdot(\mathbf{x}-\mathbf{b}) = 0$, where $\mathbf{d}$ is the direction vector of the hyperplane, $\mathbf{x}$ is the observation which lies in the hyperplane. In order to obtain a hyperplane which has a direction vector parallel to one of the co-ordinate axes, it is necessary that the direction vector $\mathbf{d}$ be $\mathbf{d}^t = (0, 0, ..., i, 0, 0, 0)$, where $i = 1$ denotes the desired co-ordinate axis. To cut the hyper-ellipsoid with the hyperplane, it is equivalent to solve the p-1 simultaneous equations

$$\begin{aligned} (\mathbf{x}-\overline{\mathbf{x}})^t\mathbf{S}^{-1}(\mathbf{x}-\overline{\mathbf{x}}) &= T^2_{1-\alpha} \\ \mathbf{d}(\mathbf{x}-\mathbf{b}) &= 0 \end{aligned}$$

for $i = 1, 2, 3, ..., p$, excluding the two targeted control variables.

After the substitution, the remaining equation is a second-degree polynomial equation with two unknown variables. In fact, the idea here is to reduce the $p-1$ simultaneous equations into a second-degree polynomial equation. Then the polynomial equation is diagonalized, so that the cross-product term is eliminated. The remaining equation is further reduced to the standard equation of an ellipse by completing the squares. Under normal conditions, the hyperplane method will provide $p(p-1)/2$ elliptical control charts.

5.4.6 Roy's Interpretation Algorithm

Another method based on the T^2 is the step-down procedure of Roy (1958). It assumes that there is an a priori ordering among the means of the p variables and it sequentially tests subsets using this ordering to determine the sequence. The test statistic has the form

$$F_j = \frac{T_j^2 - T_{j-1}^2}{1 + \left[T_{j-1}^2 / (n-1)\right]},$$

where the T_j^2 represents the unconditional T^2 for the first j variables in the chosen group. In the setting of a multivariate control chart, F_j would be the charting statistic, which under the null has the following distribution

$$\frac{(n_f - 1)\, j}{n_f - j} F\left(p_j, n_f - j\right).$$

This procedure can be considered as an alternative to the regular T^2 chart and not only as supplement. Moreover, it can be shown that the enumerator of F_j is a conditional T^2 value

$$T_j^2 - T_{j-1}^2 = T_{j\cdot 1,2,...,j-1}.$$

5.4.7 Timm's Interpretation Algorithm

Another method based on the T^2 is the step-down procedure of Timm (1996), using Finite Intersections Tests (FIT). It assumes that there is an a priori ordering among the means of the p variables and it sequentially tests subsets using this ordering to determine the sequence.

Although T^2 is optimal for finding a general shift in the mean vector, it is not optimal for shifts that occur for some subset of variables, a variable a time. Timm (1996) states that when this occurs, the optimal procedure is to utilize a finite intersection test.

A process is in-control if each hypothesis

$$H_i : \mu_i = \mu_{0i}, \qquad i = 1, 2, ..., p$$

or equivalently the intersection of the H_i

$$H_0 : \bigcap_{i=1}^{p} H_i$$

is simultaneously true, where μ_i is the mean of the ith variable. To test each hypothesis, we may use the FIT procedure. To construct a FIT of H_0, we may define a likelihood ratio test statistic for each elementary hypothesis H_i and determine the joint distribution of the test statistics. Let the test statistics be defined as

$$Z_i^2 = \frac{(X_i - \mu_{0i})^2}{\sigma_{ii}}.$$

The joint distribution of Z_i^2 follows a non-central multivariate X^2 distribution with 1 degree of freedom. The known variance-covariance matrix $\mathbf{\Sigma} = \sigma^2\mathbf{\Omega}$, where $\mathbf{\Omega} = [p_{ij}]$ is the correlation matrix of the accompanying multivariate normal. A multivariate quality control process is in-control if $Z_i^2 < X^2_{p,1-\alpha}$ where $P(Z_i^2 < X^2_{p,1-\alpha}/H_0) = 1-\alpha$ and $X^2_{p,1-\alpha}$ is the $1-\alpha$ percentage value of the multivariate X^2 distribution with 1 degree of freedom. The process is out-of-control if $maxZ_i^2 > X^2_{p,1-\alpha}$, where $P(maxZ_i^2 > X^2_{p,1-\alpha}) = \alpha$. Since a non-central multivariate X^2 distribution with 1 degree of freedom is a special case of the multivariate t distribution with infinite degrees of freedom, the process may alternatively be judged as out-of-control if $P(max\,|T_i| = |X_i - \mu_{0i}|\,/\sqrt{\sigma_{ii}} > T^2_{p,1-\alpha}) = \alpha$. Hence,

$$P(|T_i| \leq T^2_{p,1-\alpha}/H_0) = 1 - \alpha.$$

The $1-\alpha$ level simultaneous confidence sets are easily established for each variable

$$X_i - T^2_{p,1-\alpha}\sqrt{\sigma_{ii}} \leq \mu_i \leq X_i + T^2_{p,1-\alpha}\sqrt{\sigma_{ii}}$$

The process is said to be out-of-control if the confidence sets d_i do not contain μ_{0i} for $i = 1, 2, ..., p$. Timm (1996) gave a step down FIT procedure for the case that the variance-covariance matrix $\mathbf{\Sigma}$ is unknown.

5.4.8 Contributors to a Multivariate SPC Chart Signal

The contribution of a variable to a control signal can be measured by the minimum value of the chi-squared statistic that can be obtained by changing a single variable. Variables that incur large changes are important to the signal. Runger et al. (1996) propose the following diagnostics.

Let $\mathbf{x}$ be the p dimensional vector of an observation, with known mean vector $\mathbf{0}$ and known variance-covariance matrix $\mathbf{\Sigma}$. Let $\mathbf{e}_i$ be the unit vector in the direction of the ith coordinate axis. To measure the contribution of X_i to X^2, we determine c_i to minimize

$$\left(\mathbf{x} - \frac{c_i\mathbf{e}_i}{\sqrt{\mathbf{e}_i^t\mathbf{\Sigma}^{-1}\mathbf{e}_i}}\right)^t \mathbf{\Sigma}^{-1} \left(\mathbf{x} - \frac{c_i\mathbf{e}_i}{\sqrt{\mathbf{e}_i^t\mathbf{\Sigma}^{-1}\mathbf{e}_i}}\right)$$

The expression $\mathbf{e}_i^t\mathbf{\Sigma}^{-1}\mathbf{e}_i$ is a scale factor that is used so that c_i can be interpreted as a measure of Euclidean distance. This is discussed further below. If c_i is large relative to the others c_i's, then a large modification to X_i is required to minimize X^2 and this indicates that X_i is an important contributor to X^2.

Also, let

$$\mathbf{z} = \mathbf{\Sigma}^{-1/2}\mathbf{x} \tag{5.1}$$

and

$$\mathbf{v}_i = \mathbf{\Sigma}^{-1/2}\frac{\mathbf{e}_i}{\sqrt{\mathbf{e}_i^t\mathbf{\Sigma}^{-1}\mathbf{e}_i}}.$$

Then, $\mathbf{v}_i$ is a unit vector in the direction $\mathbf{\Sigma}^{-1/2}\mathbf{e}_i$. Now, (5.1) can be written as

$$(\mathbf{z} - c_i\mathbf{v}_i)^t(\mathbf{z} - c_i\mathbf{v}_i)$$

and c_i is interpreted as the Euclidean distance along the unit vector $\mathbf{v}_i$ that minimizes X^2. The value of c_i that minimizes (5.1) can be determined as the slope of the regression of $\mathbf{z}$ on $\mathbf{v}_i$. Therefore,

$$c_i = \mathbf{v}_i^t\mathbf{z} = \frac{\mathbf{e}_i^t\mathbf{\Sigma}^{-1}\mathbf{x}}{\sqrt{\mathbf{e}_i^t\mathbf{\Sigma}^{-1}\mathbf{e}_i}}$$

and obviously c_i is proportional to $\mathbf{e}_i^t\mathbf{\Sigma}^{-1}\mathbf{x}$.

Interestingly, c_i equals the recommended control statistic to detect a shift of the process mean along the vector $\mathbf{e}_i$ as described by Healy (1987), Pignatiello and Runger (1990), and Hawkins (1993). A simple multivariate control is to compare the relative magnitudes of c_i^2 for $i = 1, 2, 3, ..., p$. If c_i^2 is large, the conclusion is that the component measurement X_i is distant from the bulk of the historical data and X_i is a primarily contributor to X^2. Geometrically, c_i^2 is the length of the orthogonal projection of the vector $\mathbf{z}$ onto $\mathbf{v}_i$. That is

$$c_i^2 = \|\mathbf{P}_p\mathbf{z}\|,$$

where the orthogonal projection matrix onto the vector $\mathbf{v}_i$ is denoted as $\mathbf{P}_p$. The geometric analogies are facilitated by considering $\mathbf{Z}$ and c_i because, after transforming to these variables, X^2 is just the squared Euclidean distance of $\mathbf{Z}$ from the zero vector.

A second approach to the problem is the following. Consider the minimum value of X^2 that is obtained when X_i is changed by c_i. Define

$$X_i^2 = \left(\mathbf{x} - \frac{c_i \mathbf{e}_i}{\sqrt{\mathbf{e}_i^t \mathbf{\Sigma}^{-1} \mathbf{e}_i}}\right)^t \mathbf{\Sigma}^{-1} \left(\mathbf{x} - \frac{c_i \mathbf{e}_i}{\sqrt{\mathbf{e}_i^t \mathbf{\Sigma}^{-1} \mathbf{e}_i}}\right) = (\mathbf{z} - c_i \mathbf{v}_i)^t (\mathbf{z} - c_i \mathbf{v}_i). \tag{5.2}$$

Then, (5.2) can be interpreted as the residual sum of squares in a regression model. Therefore,

$$X_i^2 = \mathbf{z}^t (\mathbf{I} - \mathbf{P}_i) \mathbf{z} = \mathbf{z}^t \mathbf{z} - \mathbf{z}^t \mathbf{v}_i \mathbf{v}_i^t \mathbf{z} = \mathbf{z}^t \mathbf{z} - c_i^2. \tag{5.3}$$

Thus, X_i is a major contributor to X^2, if the value of X^2 can be substantially reduced by a modification to X_i. Consequently, if the metric $D_i = X^2 - X_i^2$ is large then X_i is a major contributor to X^2. Since $X^2 = \mathbf{x}_i^t \mathbf{\Sigma}^{-1} \mathbf{x}_i = \mathbf{z}^t \mathbf{z}$ and because of relation (5.3) we have that $D_i = c_i^2$ therefore the metric D_i is equivalent to c_i^2. Finally, a third approach to develop a diagnostic is the union-intersection principle of multivariate hypothesis tests.

5.4.9 Cause-Selecting Control Chart

Wade and Woodall (1993) consider a two step process in which the steps are not independent. The first step of a cause-selecting control chart is to chart variable X_1 (first step of the process) and then monitor the outgoing quality X_2 (second step of the process) after adjusting for the incoming quality. This method uses a relation between X_2 and X_1, where a simple regression model appears to be very useful. To be more precise, a chart for X_1 and a chart for $Z = X_2 - \widehat{X}_2$, where $\widehat{X}_2$ is the estimate for X_2 based on the regression line, are used. Thus, the Z_i's are independent normal random variables if X_1 and X_2 are normally distributed variables. If controllable assignable causes are present in the process the distribution of Z_i's shifts from normality for some values of i.

5.4.10 Hawkins' Interpretation Method

Hawkins (1991, 1993), as we have already mentioned, defined a set of regression-adjusted variables in which each variable was regressed on all others. In a first approach Hawkins defined the set of regression-adjusted variables using the vector $\mathbf{Z} = \mathbf{\Sigma}^{-1}(\mathbf{X} - \boldsymbol{\mu})$. If we replace $\boldsymbol{\mu}$ and $\mathbf{\Sigma}$ with their sample estimates and expanding the left-hand side of the previous relation Hawkins showed that the jth component of $\mathbf{Z}$ is equal to

$$Z_j = \frac{T_{j \cdot 1,2,\ldots,j-1,j+1,\ldots,P}}{s_{j \cdot 1,2,\ldots,j-1,j+1,\ldots,P}}.$$

Then, Z_j is the standardized residual when the jth variable is regressed on the remaining $p-1$ variables in $\mathbf{X}$. This regression statistic is useful in the interpretation of a T^2 signal because its value is directly connected with the value of the T^2 statistic although it is only one of the several different conditional T^2 values.

An additional approach presented by Hawkins is based on the decomposition of the T_j^2 statistics, using the standardized residuals from the jth variable on the first $j-1$ variables. This is defined by $\mathbf{Y} = \mathbf{C}(\mathbf{X} - \boldsymbol{\mu})$ where $\mathbf{C}$ is the Cholesky lower triangular root of $\mathbf{\Sigma}^{-1}$. The jth component of vector $\mathbf{Y}$ is given by

$$Y_j = \frac{T_{j \cdot 1,2,\ldots,j-1}}{s_{j \cdot 1,2,\ldots,j-1}}$$

As in the first approach Y_j is one of the conditional T^2 values.

5.4.11 Minimax Control Chart

Sepuldveda and Nachlas (1997) presented a new control chart. This chart is called the Minimax control chart because it is based on monitoring the maximum standardized sample mean and the minimum standardized sample mean of samples taken from a multivariate process. It is assumed that the data are normally distributed and that the variance-covariance matrix is known and constant over time.

Samples of size n are taken from a p-dimensional process. The process is assumed to have sudden shifts in the mean such that the new mean μ_{1i}, $i = 1, 2, ..., p$, for the ith changing variable is given by $\mu_{1i} = \mu_{0i} + \delta_i \sigma_{0i}$ and consequently $\delta_i = (\mu_{1i} - \mu_{0i})/\sigma_{0i}$. Whenever $\delta_i = 0$, the process is said to be in-control. To decide whether the process is in-control or not, the minimax control chart is used as a method to test in each sample $H_0 : \boldsymbol{\delta} = \mathbf{0}$ against $H_1 : \boldsymbol{\delta} \neq \mathbf{0}$.

The principal idea behind the minimax control chart is to standardize all p means and to monitor the maximum Z^{mx} and the minimum Z^{mn} of those standardized sample means. Note that Timm (1996) monitors only the maximum. Let

$$\begin{aligned} Z^{mn} &= \min(Z_i),\ i = 1, 2, 3, ..., p \\ Z^{mx} &= \max(Z_i),\ i = 1, 2, 3, ..., p, \end{aligned}$$

where

$$Z_i = \frac{\overline{X}_i - \mu_{0i}}{\sigma_{ii}/\sqrt{n}}.$$

Therefore, by monitoring the maximum and the minimum standardized sample mean, an out-of-control signal is directly connected with the corresponding out-of-control variable. Sepuldveda and Nachlas (1997), also discussed the statistical properties and the ARL performance of the minimax control chart.

5.5 Using Principal Components

Another method used for the identification problem is principal components analysis (PCA). This method was first proposed by Jackson (1991) and further discussed later by Pignatiello and Runger (1990), Kourti and MacGregor (1996). A $p \times p$ symmetric, non-singular matrix, such as the variance-covariance matrix $\boldsymbol{\Sigma}$, may be reduced to a diagonal matrix $\mathbf{L}$ by premultiplying and postmultiplying it by a particular orthonormal matrix $\mathbf{U}$ such that $\mathbf{U}^\intercal \boldsymbol{\Sigma} \mathbf{U} = \mathbf{L}$. The diagonal elements of $\mathbf{L}$, $l_1, l_2, ..., l_p$ are the characteristic roots, or eigenvalues of $\mathbf{\Sigma}$. The columns of $\mathbf{U}$ are the characteristic vectors, or eigenvectors of $\mathbf{\Sigma}$. Based on the previous result the method of PCA was developed (see e.g., Jackson (1991)). The PCA transforms p correlated variables $x_1, x_2, ..., x_p$ into p new uncorrelated ones. The main advantage of this method is the reduction of dimensionality. Since the first two or three PCs usually explain the majority of the variability in a process, they can be used for interpretation purposes instead of the whole set of variables.

5.5.1 Jackson's Approach

Principal components can be used to investigate which of the p variables in a multivariate control chart are responsible for an out-of-control signal. The most common practice is to use the first k most significant principal components, if a T^2 control chart gives an out-of-control signal, for further investigation.

The basic idea is that the first k principal components can be physically interpreted, and named. Consequently, if the T^2 chart gives an out-of-control signal and, for example, the second principal component chart also gives an out-of-control signal, then from the interpretation of this component, a direction to the variables which are suspect to be out-of-control can be deduced (Jackson (1991)).

The practice just mentioned transforms and considers the variables as a set of attributes. The discovery of the assignable cause of the problem, with this method, demands a further knowledge of the process itself, from the practitioner. The basic problem is that the principal components do not always have a physical interpretation.

5.5.2 Bivariate Control Chart for Paired Measurements

As we already said one of the ways to use PCA in the problem under question is to chart them. However, if the components are not easily interpreted the problem remains. Tracy et al. (1995) expanded previous work and provided an interesting bivariate setting in which the principal components have meaningful interpretations. When monitoring a process with paired measurements (for example two testing laboratories) on a single sample, the principal components of the correlation matrix actually represent the characteristics of interest for process control. The correlation coefficient between the original variables is the only additional information needed to describe the condition of the process.

5.5.3 Kourti's and MacGregor's Approach

Kourti and MacGregor (1996) provided a newer approach based on principal components analysis. The T^2 statistic is expressed in terms of normalized principal components scores of the multinormal variables. When an out-of-control signal is received, the normalized scores with high values are detected, and contribution plots are used to find the variables responsible for the signal. A contribution plot indicates how each variable involved in the calculation of that score contributes to it. The computation of variable contributions showed that principal components can actually have physical interpretation. This approach is particularly applicable to large ill-conditioned data sets, due to the use of principal components.

5.6 A New Method

Let $\mathbf{x}_i = (x_{1i}, x_{2i}, ..., x_{pi})^\intercal$ denote the observation (vector) i for the p variables of a process. Assume that $\mathbf{x}_i$ follows a p-dimensional normal distribution $N_p(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$, where $\boldsymbol{\mu}_0$ is the vector $(p \times 1)$ of known means and $\boldsymbol{\Sigma}_0$ is the known $(p \times p)$ variance-covariance matrix. We want to keep this process under control. For this purpose we use a X^2 control chart given by the formula $X_i^2 = (\mathbf{x}_i - \boldsymbol{\mu}_0)^\intercal \boldsymbol{\Sigma}_0^{-1} (\mathbf{x}_i - \boldsymbol{\mu}_0)$. If the value of this statistic plots above $X^2_{p,1-a}$ we get an out-of-control signal, where $X^2_{p,1-a}$ is the chi-square distribution with p degrees of freedom and a is the probability of plotting outside the control limits if we are in-control. The next problem is to detect which variable is the one that caused the problem.

The typical form of a PCA model is the following: $Z_k = u_{1k}X_1 + u_{2k}X_2 + u_{3k}X_3 + ... +$

$u_{pk}X_p$, where Z_k is the k PC, $(u_{1k}, u_{2k}, u_{3k}, ..., u_{pk})^{\intercal}$ is the corresponding k eigenvector and $X_1, ..., X_p$ are the process variables. The score for vector $\mathbf{x}_i$ in PC k is $Y_{ki} = u_{1k}x_{1i} + u_{2k}x_{2i} + ... + u_{pk}x_{pi}$. Assuming that the process variables follow a multivariate normal distribution the PCs are also normally distributed.

Our purpose is to use PCA, when we have an out-of-control signal in the X^2 control chart, to identify the variable or variables that are responsible. For this objective two different methodologies are developed one for the case that the covariance matrix has only positive correlations and the second one for the case that we have both positive and negative ones (Maravelakis, Bersimis, Panaretos, Psarakis (2002)).

5.6.1 Covariance matrix with positive correlations

Assume that using one of the existing methods for choosing PCs (see, e.g. Jackson (1991)), Runger and Alt (1996)) we choose $d \leq p$ significant PCs. The proposed method in this case is based on ratios of the form

$$r_{ki} = \frac{(u_{k1} + u_{k2} + ... + u_{kd})x_{ki}}{Y_{1i} + Y_{2i} + ... + Y_{di}}, \tag{5.4}$$

where x_{ki} is the ith value of variable X_k, Y_{ji}, $j = 1, .., d$ is the score of the ith vector of observations in the jth PC (Bersimis (2001)). In this ratio, the numerator corresponds to the sum of the contributions of variable X_k in the first d PCs in observation (vector) i, whereas the denominator is the sum of scores of observation (vector) i in the first d PCs. Since we have assumed that the variables follow a multivariate normal distribution the ratios are ratios of two correlated normal variables.

The rationale of this method is to compute the impact of each of the p variables on the out-of-control signal by using its contribution to the total score. It is obvious that the use of only the first d PCs excludes pieces of information. However, a multivariate chart is used when there is at least moderate and usually large correlation between the variables. Under such circumstances the first d PCs account for the largest part of the process variability. The main disadvantage of using PCs in process control, as reported by many authors (see e.g., Runger and Alt (1996), Kourti and McGregor (1996)), is the lack of physical interpretation. The proposed method eliminates much of that criticism.

Since we have a ratio of two correlated normals its distribution can be computed using the analytical result of Hinkley (1969). Specifically, if X_1, X_2 are normally distributed random variables with means μ_i, variances σ_i^2 and correlation coefficient ρ the distribution function of $R = X_1/X_2$ is given by the formula

$$F(r) = L\left\{\frac{\mu_1 - \mu_2 r}{\sigma_1\sigma_2\alpha(r)}; -\frac{\mu_2}{\sigma_2}; \frac{\sigma_2 r - \rho\sigma_1}{\sigma_1\sigma_2\alpha(r)}\right\} + L\left\{\frac{\mu_2 r - \mu_1}{\sigma_1\sigma_2\alpha(r)}; \frac{\mu_2}{\sigma_2}; \frac{\sigma_2 r - \rho\sigma_1}{\sigma_1\sigma_2\alpha(r)}\right\}, \qquad (5.5)$$

where $L(h;k;\gamma) = \frac{1}{2\pi\sqrt{1-\gamma^2}} \int_h^\infty \int_k^\infty \exp\left\{-\frac{x^2 - 2\gamma xy + y^2}{2(1-\gamma^2)}\right\} dxdy$ is the standard bivariate normal integral.

However, the proposed method has a correlation problem since the ratios of different variables are interrelated. A simulation study presented in Section 5.6.4 is implemented to test the effect on the performance of the proposed procedure. In the following we present the method as a stepwise procedure.

- Step 1. Calculate the X^2 statistic for the incoming observation. If we get an out-of-control signal continue with Step 2.

- Step 2. Calculate ratios for all the variables using relation (5.4). Calculate as many ratios for each variable as the number of observations from the beginning of the process. If the proposed process is not used for the first time, calculate as many ratios for each variable, as the number of observations from the last out-of-control signal till the out-of-control signal we end up with in Step 1. Alternatively, calculate ratios for only the (last) observation that caused the out-of-control signal (see Section 5.6.4).

- Step 3. Plot the ratios for each variable in a control chart. Compute the a and $1-a$ percentage points of distribution (5.5) with suitable parameters and use them as lower control limit (LCL) and upper control limit (UCL), respectively.

- Step 4. Observe which variable, or variables, issue an out-of-control signal
- Step 5. Fix the problem and continue with Step 1.

In the case where all the variables are positively correlated as Jackson (1991) indicates, the first PC is a weighted average of all the variables. Consequently, we can use only this PC for inferential purposes.

5.6.2 Covariance matrix with positive and negative correlations

In this case we propose the computation of ratios of the form

$$r_{ki}^{*} = \frac{(u_{k1} + u_{k2} + ... + u_{kd})x_{ki}}{\overline{Y}_1 + \overline{Y}_2 + ... + \overline{Y}_d}, \tag{5.6}$$

where x_{ki} is the ith value of variable X_k and $\overline{Y}_j$, $j = 1, ..., d$ is the score of the jth PC using, in place of each X_k, their in-control values. The subscript d stands for the number of significant principal components as in the preceding case.

These ratios are the sum of the contributions of variable X_k in the first d PCs in observation (vector) i, divided by the sum of the in-control scores of the first d PCs. Since the denominator of this statistic is constant we actually compute the effect of each of the p variables on the out-of-control signal. The numerator of the ratios is normally distributed, as already stated, whereas the denominator is just a constant. Therefore the ratios (5.6) are normally distributed.

Since the variables are correlated the statistic proposed in (5.6) for the k different variables may exhibit a correlation problem. As in the previous case a simulation study is presented in Section 5.6.4 to test for the effect of the correlation on the control limits performance of the proposed procedure. The proposed method in steps is as follows:

- Step 1. Calculate the X^2 statistic for the incoming observation. If we get an out-of-control signal continue with Step 2.

- Step 2. Calculate ratios for all the variables using relation (5.6). Calculate as many ratios for each variable as the number of observations from the beginning of the process. If the proposed process is not used for the first time, calculate as many ratios for each variable, as the number of observations from the last out-of-control signal till the out-of-control signal we end up with in Step 1. Alternatively, calculate ratios for only the (last) observation that caused the out-of-control signal (see Section 5.6.4).

- Step 3. Plot the ratios for each variable in a control chart. Compute the a and $1 - a$ percentage points of the normal distribution with suitable parameters and use them as LCL and UCL, respectively.

- Step 4. Observe which variable, or variables, issue an out-of-control signal

- Step 5. Fix the problem and continue with Step 1.

We have to mention that this procedure can not be applied when we have standardized values since the denominator of the ratios in (5.6) equals zero.

5.6.3 Illustrative examples

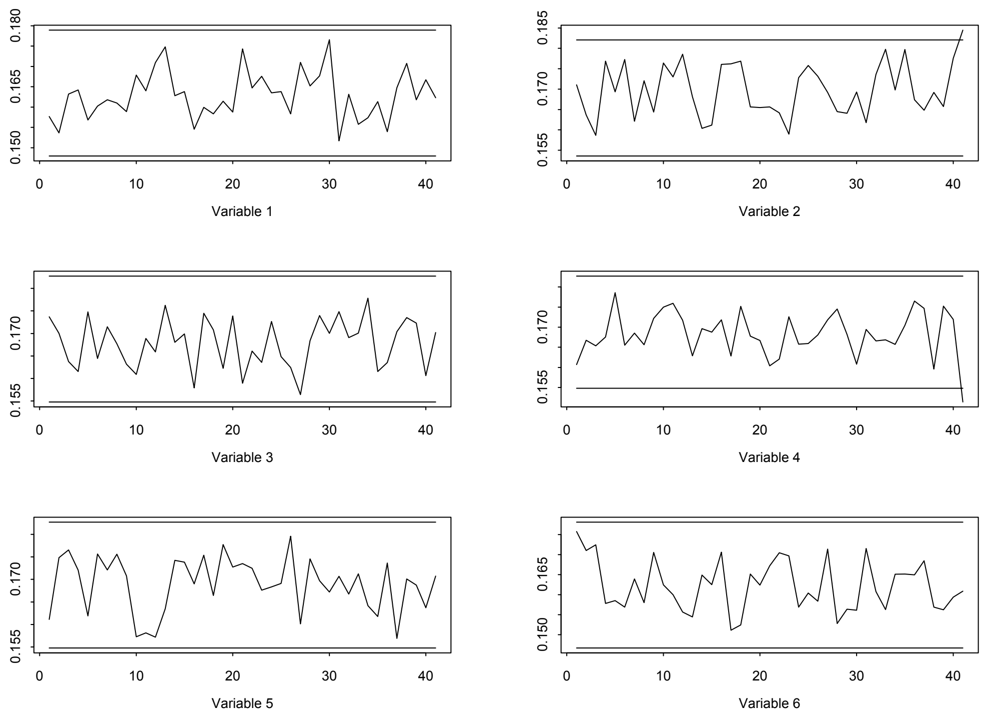
Two examples, one for each case are presented in the sequel.

Example 1. Assume that we have a process with known covariance matrix

$$\begin{bmatrix} 100 & & & & & \\ 70 & 100 & & & & \\ 80 & 80 & 100 & & & \\ 75 & 85 & 75 & 100 & & \\ 75 & 80 & 80 & 80 & 100 & \\ 75 & 72 & 75 & 75 & 75 & 100 \end{bmatrix}$$

and in-control vector of means $(100, 100, 100, 100, 100, 100)^\intercal$. We simulated 40 in-control observations from a multivariate normal distribution with the preceding parameters. Then, we simulated out-of-control ones with the same covariance matrix but now with vector of means $(100, 115, 100, 85, 100, 100)^\intercal$, until we get an out-of-control signal in the X^2 test. The shift is 1.5σ in the means of variables 2 and 4. We get a signal on the first out-of-control observation and we plot each of the variables in a control chart (Figure 5.1) with the control limits from distribution (5.5) using $a = 0.05$. Note that we used the average root method for simplicity and we ended up with one significant principal component (see, e.g. Jackson (1991)). It is obvious from Figure 5.1 that the out-of-control variables were identified and additionally the direction of the shift was also revealed.

Figure 5.1. Control charts for positive covariance matrix

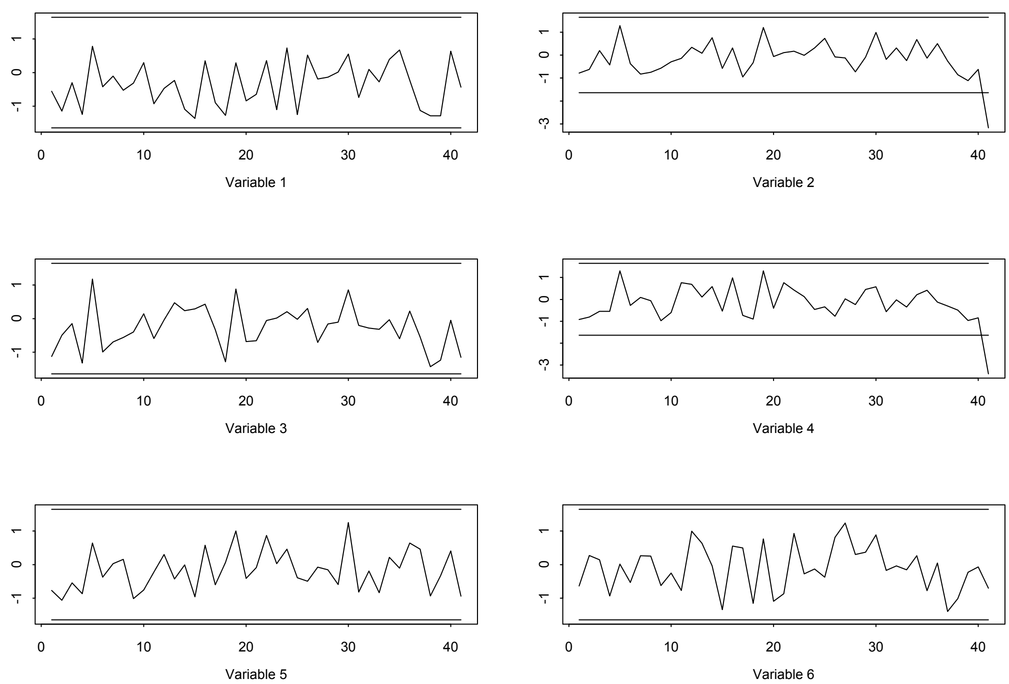


Example 2. Assume that we have a process with known covariance matrix

$$\begin{bmatrix} 100 & & & & & \\ -70 & 100 & & & & \\ 80 & -80 & 100 & & & \\ 75 & -85 & 75 & 100 & & \\ 75 & -80 & 80 & 80 & 100 & \\ 75 & -72 & 75 & 75 & 75 & 100 \end{bmatrix}$$

and in-control vector of means $(100, 100, 100, 100, 100, 100)^{\intercal}$. We simulated 40 in-control observations from a multivariate normal distribution with the same parameters as in the previous example. Then, we simulated out-of-control ones with vector of means

Figure 5.2. Control charts for positive negative covariance matrix



$(100, 115, 100, 85, 100, 100)^{\intercal}$, the same covariance matrix till we get an out-of-control signal in the X^2 test. The shift is again 1.5σ in the means of variables 2 and 4. We plot each of the variables in a control chart (Figure 5.2) with the control limits from a normal distribution using $a = 0.05$. As in example 1, we have one significant PC using the average root method again (see, e.g. Jackson (1991)). We have to indicate that in Figure 5.2 the ratios are standardized and the control limits are properly modified. However, it is not necessary to do this when using this technique. From Figure 5.2, we deduce that the out-of-control variables were identified but the direction of the shift was not.

5.6.4 Further Investigation

One may observe that the ratios in both methods are interrelated. This fact may affect the control limits of the charts. In order to examine this possible correlation we performed a simulation study. In particular, we simulated 100000 in-control ratios from the known covariance matrices and vector of means of the two examples and we computed the theoretical control limits as proposed in Section 5.6.3 with $a = 0.05$. Then, we checked if each ratio is in or out of these limits and recorded it. We used this information in order to approximate the probability of plotting outside the control limits if we are in-control of our limits and compare it with the theoretical one. The results are presented in Table 5.1.

Table 5.1. Probability of plotting outside the control limits if we are in-control

Variable	1	2	3	4	5	6
example 1	5040	5006	5007	4968	5000	5096
example 2	4941	4983	4898	4882	4906	5005

It should be noted that although the two examples are specific cases, a large number of other cases revealed the same performance. Consequently, we may draw the conclusion that the interrelation does not affect either of the proposed processes. However, we have to point out that after careful examination the procedure proposed for positive correlations can be used only when we have positive values for the in-control means. If a process does not have positive in-control means, which is a rare event, we may use instead the statistic (5.6) that is not affected in any case.

Another point which has to be checked is the performance of the processes in identifying the out-of-control variables and the direction of the shift. For this purpose a simulation study was conducted. Using the covariance matrices and the in-control means of the examples we computed in each case the theoretical control limits. Then, we simulated observations from the out-of-control mean vector used in the examples until we got a signal from the X^2 test using $a = 0.05$. Next, we computed the ratios for each variable and plotted them in a chart with the corresponding control limits. We checked each ratio if it is in or out of the control limits for every variable and recorded which variable, or variables, have given an out-of-control signal and in which direction. We repeated the whole process 10000 times and the results are presented in Tables 5.2 and 5.3. In the first row of the tables (U), we have the number of times the generated ratios crossed the UCL for each of the variables, in the second row (L) we have the corresponding number of times the generated ratios crossed the LCL for each of the variables and in the last line (Total) we have the number of times the generated ratios crossed UCL or LCL for each of the variables. One may observe that in some variables there is an inconsistency, since the sum of rows U and L does not equal the total. This happens because in one iteration we may generate more than one observations (vectors) until we get an out-of-control signal in the X^2 test -although this test is sensitive for such shifts- and after computing the ratios for each variable it is possible that for one variable the first ratio crosses UCL and the second ratio crosses LCL. Therefore, we record one value in row U and one in row L but only one in row total.

Table 5.2. Out of control performance for example 1

Variable	1	2	3	4	5	6
U	283	9371	228	0	275	233
L	240	0	299	9429	231	245
Total	523	9371	527	9429	506	478

From Table 5.2, we observe that the statistic used is very informative since it is able to detect the out-of-control variables with very high precision and also to identify the direction of the shift with absolute success. This kind of behavior is similar in other examples also, keeping in mind the limitation about the positive in-control means. Note also that the total times the other variables gave a false signal almost coincides with the type I error rate of the constructed limits.

Table 5.3. Out of control performance for example 2

Variable	1	2	3	4	5	6
U	364	1	357	2	366	350
L	353	4627	394	4603	357	371
Total	715	4628	748	4605	717	720

In Table 5.3, we see that the statistic used detected the out-of-control situation but not with the same degree of success as in the previous case. Moreover, the direction of the shift was not identified. The false alarm rate of the in-control variables is not significantly different from the theoretical one.

We already said that the X^2 test is sensitive in the sense that it gives a quick signal when we have out-of-control observations. In order to examine the ability of the last generated observation (the one that gives the signal on the X^2 test) to identify the shifted variable we checked their performance on the previous simulation study. The results are displayed in Tables 5.4 and 5.5.

Table 5.4. Out of control performance of the last observation for example 1

Variable	1	2	3	4	5	6
U	283	9370	228	0	275	233
L	240	0	299	9429	231	245
Total	523	9370	527	9429	506	478

From Table 5.4, we conclude that the performance of the statistic (5.4) is almost totally explained by the ratios of the last observation meaning that the ratios of the last observation are sufficient to draw a conclusion about the out-of-control variable. On the other hand, from Table 5.5 we observe that this does not happen for the statistic (5.6). One may argue that since the X^2 test is sensitive we produce a small number of observations in each iteration hence the performance in both cases is a result of this fact. When we have small shifts, where we produce more observations, the last observation is not that informative.

Table 5.5. Out of control performance of the last observation for example 2

Variable	1	2	3	4	5	6
U	333	1	316	2	324	315
L	312	4279	343	4251	310	333
Total	645	4280	659	4253	634	648

The proposed procedures are valid under the assumption of known variance-covariance matrix. However, this is not a case usually met in practice. Tracy et al. (1992) examined the performance of multivariate control charts for individual observations when the covariance matrix is known and unknown. They showed that the test statistics used in either case perform the same for a number of observations that depends on the variables involved. This number of observations is small enough, for instance when we have five variables we need 100 observations (vectors) for the two statistics to give a close number.

As Woodall and Montgomery (1999) note in today's industry we have huge data sets therefore such a number of observations should not be a problem. From this number of observations we estimate the mean vector and the covariance matrix used in our process. Although the control limits computed using the procedures developed in subsections 5.6.1 and 5.6.2 will not be exact under the estimation process, we expect them to have a satisfactory performance if we use the required number of observations.

5.6.5 A Comparison

As we already stated in section 5.5, the competitive methods that use principal components for the specific problem are Jackson's (1991), Tracy et al.'s (1995) and Kourti and MacGregor's (1996). Kourti and MacGregor (1996) provide an improved method in relation to Jackson's (1991) and Tracy et al. (1995) have the disadvantage that their method is applied to a bivariate case. Therefore, the rival of the proposed method is the one by Kourti and MacGregor (1996).

In order to check the performance of the two antagonistic methods we perform a simulation study. We apply the method of Kourti and MacGregor (1996) in the data of examples 1 and 2 of Section 5.6.3. As Kourti and MacGregor (1996) propose, we use Bonferroni limits on the normalized scores and we calculate the contributions of the variables with the same sign as the score, since contributions of the opposite sign does not add anything to the score, in fact they make it smaller. In the paper of Kourti and MacGregor (1996) there is not a specific rule on how to decide whether a contribution is significant or not. Since in the two examples of Section 5.6.3 we have two variables shifted, we choose to record the first and the second larger contributions in each iteration of the simulation study if they exist. The simulation study was conducted 10000 times in order to have a credible estimate of the ability of Kourti and MacGregor's (1996) method to identify the out-of-control variables. In Tables 5.6 and 5.7 we have the results of this simulation for examples 1 and 2 respectively.

Table 5.6. Performance of Kourti and MacGregor's method for example 1

Variable	1	2	3	4	5	6
Largest contribution	2	5204	16	4717	0	0
Second largest contribution	1511	2864	1865	3655	30	10
Total	1513	8068	1881	8372	30	12

Table 5.7. Performance of Kourti and MacGregor's method for example 2

Variable	1	2	3	4	5	6
Largest contribution	23	191	10	2	51	2
Second largest contribution	64	63	47	74	19	12
Total	87	254	57	76	70	14

From the results in Table 5.6 for Example 1, we observe that the method of Kourti and MacGregor does not succeed in identifying the out-of-control variables as many times as our proposed method does (see Tables 5.2 and 5.4). Moreover, the inherent inability of the method to point if there is an upward or a downward shift is also present. Things are even worse in Table 5.7 for example 2. Specifically, the method of Kourti and MacGregor leads to recordable contributions very rarely, a fact that may lead the practitioner to assume that the signal on the multivariate chart is due to the probability of plotting outside the control limits if we are in-control. The ability of the new method to operate more effectively is obvious (see Tables 5.3 and 5.5).

5.6.6 A Graphical Technique

The charts proposed in Sections 5.6.1 and 5.6.2 are Shewhart type. Therefore, they have the ability to identify large shifts quickly but they are not that good for small shifts. An alternative way to plot these statistics is as a Cumulative Sum (CUSUM) chart. The definition of the CUSUM chart for detecting upward shifts is

$$\begin{aligned} S_0^+ &= 0 \\ S_n^+ &= \max(0, S_{n-1}^+ + (X_{pn} - k)), \end{aligned}$$

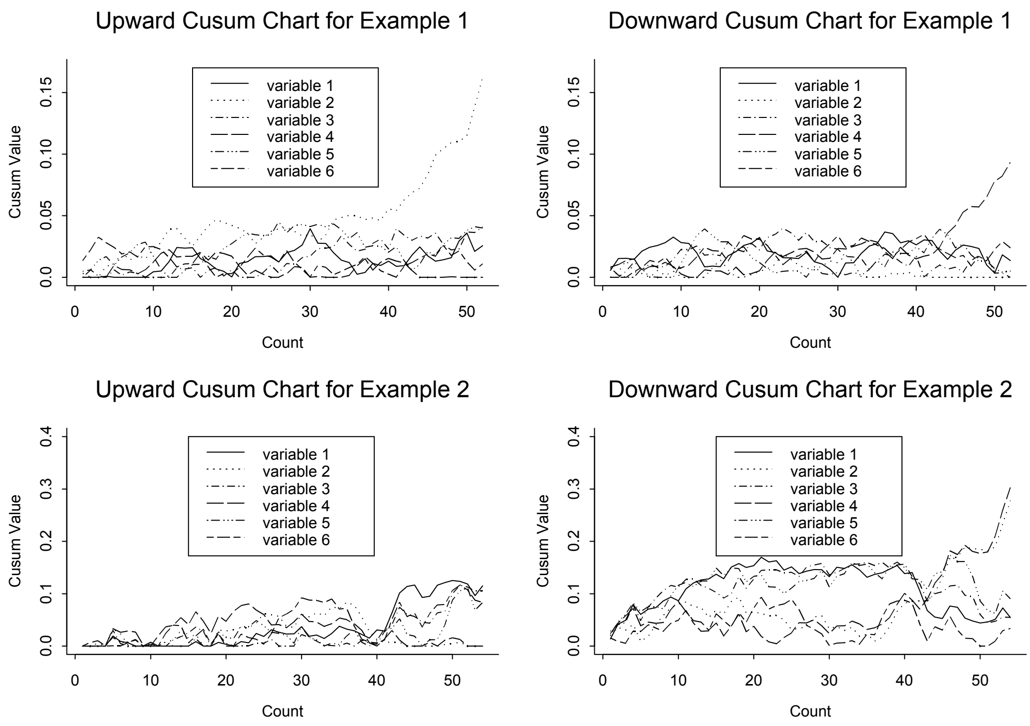
where X_{pn} is the nth observation of variable X_p and k is called the reference value. The corresponding CUSUM chart for detecting downward shifts is

$$\begin{aligned} S_0^- &= 0 \\ S_n^- &= \max(0, S_{n-1}^- + (k - X_{pn})), \end{aligned}$$

In the usual concept of CUSUM charts we evaluate an optimal value of k depending on the distributional assumption (see Hawkins (1992) and Hawkins and Olwell (1998)). This value of k is used along with the value h, which is the control limit, to characterize the ARL performance of a CUSUM chart. In our case the application of this theory for CUSUM charts is cumbersome due to the underlying distribution. However, we can use the previously defined statistics for upward and downward shifts as a graphical technique solely. The only quantity remaining unknown is the value k we have to use. A straightforward selection for k is to use in each case of statistics (5.4) and (5.6) their in-control counterparts. Specifically, for statistic (5.4) we use in place of each x_{ki} its in-control value both in the numerator and the denominator. A similar action takes place in statistic (5.6) but only in the numerator this time.

To study the performance of these statistics in practice we applied them to the examples of Section 5.6.3. We used the same 40 in-control observations but now we generated out-of-control ones with the same covariance matrix as in Section 5.6.3, and vector of means $(100, 105, 100, 95, 100, 100)^\intercal$, in both examples until we get an out-of-control signal in the X^2 test. The shift is 0.5σ in the means of variables 2 and 4. In example 1, we simulated 12 observations till the out-of-control signal and 14 in example 2. We computed the CUSUM values for the 52 and 54 values for all six variables in examples 1 and 2 respectively, and we plotted them in the chart given in Figure 5.3.

Figure 5.3. Control charts for CUSUM values



From Figure 5.3 we easily deduce that the charts give a clear indication of the out-of-control variables in both examples. As in the Shewhart type charts of Section 5.6.3, the statistic (5.4) used in example 1 detected also the direction of the shift something that did not happen with statistic (5.6) in example 2. We have to mention here that the effectiveness of the CUSUM as a graphical device for shifts less than 0.5σ is questionable. However, it is an easily interpreted method that can give an indication.

Summarizing, we note that the charts proposed are an easily applied alternative to most of the existing methods since the computational effort is diminished. Furthermore, we try to give an answer to the problem under a control charting perspective giving operational control limits or design strategies that are not difficult for a practitioner to apply.

5.7 Graphical Techniques

5.7.1 Multivariate Profile Charts

Fuchs and Benjamini (1994) presented a method (multivariate profile chart) for simultaneous control of a process and interpretation of an out-of-control signal. The multivariate profile chart (MP chart) is a scatterplot with symbols. Specifically, symbols are used for data of individual variables whereas the location of the symbol on the scatterplot is used for providing information about the group. Each group of observations is displayed by one symbol and this symbol of the profile plot enables the user to get a clear view of the size and the sign of each variable from its reference value.

The first step in the construction of an MP chart is to draw a horizontal base line for each symbol and then calculate sequentially a bar for every variable. This bar plots either above (below) the base line if the deviation is positive (negative). The size of the bar depends on the size of the deviation. Let $d_{ij} = \frac{\overline{x}_{ij} - m_i}{v_i}$, where $\overline{x}_{ij} = 1/n \sum_l x_{ijl}$, v_i is a scale factor, m_i is the ith reference value and x_{ijl} is the value of the lth observation on the ith variable in the jth subgroup. Then, the size of the bar is proportional to d_{ij} up to value of 4. If the standardized deviation exceeds 2 the corresponding bar is painted gray. If the standardized deviation exceeds 3 the corresponding bar is painted black. If all variables means are equal to their standard values the symbol is actually the baseline.

The symbols are plotted in sequential order along a horizontal time axis. In the vertical axis the location of each symbol is determined by the multivariate deviation of the sample mean vector from the standard mean vector as measured by T^2. The critical value of this axis equal to 0.997 is at a symbol size distance from the top and it is chosen to be equivalent to three standard deviations control limit of the univariate control charts. Additionally, a dashed line runs across the chart at the $T^2_{0.997}$ level. The critical value corresponding to 0 is placed half a symbol size distance from the bottom.

A symbol's baseline, which is placed horizontally, is located at the appropriate vertical location when it is less than the T^2 critical value. Those observations with a T^2 value higher than the critical value are placed at the top of the chart stack at the ceiling and completely beyond the dashed line.

The numerical value of the T^2 statistic is not as informative as in the univariate charts the deviations from standard values that are measured by the standard errors. For this reason Fuchs and Benjamini (1994) present on the vertical scale of the chart, the corresponding tail probabilities instead of the actual T^2 values.

If the process is in-control and at the nominal values, the chart appears as a simple horizontal line. If the process is out-of-control, the MP chart gives three visual warnings. The first one is that the symbol for the group that gives an out-of-control signal is located above the line of the observations that are under control. Secondly, as the size of the deviations gets large so does the size of the symbol. The last one is that the symbol gets darker, attracting more visual attention. All three visual warnings hold regardless if we have an upward or a downward signal. Even if the shift is constantly appearing in one direction the visual warnings will be there.

Finally, the MP chart can be used to identify easily the cause of a shift. Since all the individual variables are displayed on a common scale within a group, the MP chart gives us the ability to detect visually a change in their interrelationships. Additionally, when they deviate from their standard, our attention is drawn by the behavior of individual variables, by the height and darkness of the corresponding bars.

5.7.2 Dynamic Biplots

Sparks et al. (1997) presented a graphical method for monitoring multivariate process data based on the Gabriel biplot. This method uses reduction to two dimensions for identifying the in or out-of-control state. However, we use all the data for the decision if we are in or out-of-control even though the method is a dimension reduction one. This method can be used as a control chart and also if we are out-of-control to detect the reason that led to this problem. In particular with this approach a practitioner is able to detect changes in location, variation, and correlation structure.

Chapter 6

Measurement Error Effect in Control Charts

6.1 Introduction

A problem faced in the context of control charts generally is the measurement error variability. This problem is the result of the inability to measure accurately the variable of interest X. The use of imprecise measurement devices affects the ability of control charts to detect an out-of-control situation. Moreover, the variable of interest may be related through a covariate with the measurement system used.

Section 6.2 presents the research in Shewhart control charts with measurement error in the univariate and multivariate case. A simple linear model together with a model with covariates are given. In Section 6.3 the research work up to now on the effect of the measurement error in the EWMA chart is described along with some new results. Specifically, a covariate model is assumed and investigated together with a detailed examination of some factors that may affect the performance of the EWMA chart.

6.2 Shewhart Control Charts and Measurement Error

The effect of measurement error on Shewhart control charts was studied by several authors. Bennet (1954) examined the effect on the Shewhart chart for the mean using the model $Y = X + \varepsilon$, where X is the actual value of the variable and Y is the measurement we have because of the random error ε. It is additionally assumed that variables Y and X are normally distributed but with different values for the variance. Specifically, variable Y has a larger variance than X because its variance comprises both the variances of X and ε. Bennet (1954) proposed that if the variance due to the measurement error is smaller than the variance due to the process it can be overlooked. Moreover, he investigated the measurement error effect on the Shewhart chart for the mean. Abraham (1977) used the same model as Bennet (1954) and considered the effect of not accurate measurements on the process variation.

6.2.1 Measurement error effect on the joined $\overline{X}$-R and $\overline{X}$-S control charts

Kanazuka (1986) examined the effect of the measurement error on the process variance of the joined $\overline{X}$ and R chart assuming the model of Bennet (1954). He showed that the power of the $\overline{X}$-R chart is given by

$$P_{\overline{X}-R} = 1 - (1 - P_{\overline{X}})(1 - P_R),$$

where

$$P_{\overline{X}} = \Phi\left\{\sqrt{\frac{1+r^2}{k^2+r^2}}\left(-3 + \frac{d\sqrt{n}}{\sqrt{1+r^2}}\right)\right\} + \Phi\left\{\sqrt{\frac{1+r^2}{k^2+r^2}}\left(-3 - \frac{d\sqrt{n}}{\sqrt{1+r^2}}\right)\right\}$$

and

$$P_R = \Pr\left(R \geq D_2\sqrt{\sigma_p^2+\sigma_M^2}\right) + \Pr\left(R \leq D_1\sqrt{\sigma_p^2+\sigma_M^2}\right),$$

$d = (\mu' - \mu)/\sigma_p$, $k^2 = \sigma_p'^2/\sigma_p^2$, $r^2 = \sigma_M^2/\sigma_p^2$, Φ stands for the standard normal distribution and D_1, D_2 are control chart factors that depend on the sample size n. Moreover, μ, μ' denote the in and out-of-control mean respectively, $\sigma_p^2, \sigma_p'^2$ are the in and out-of-control process variance and σ_M^2 is the measurement error variance. In order to compute the probabilities in the formula for the power of the R chart Kanazuka proposed the use of the table given in Pearson (1941). Kanazuka (1986) noted that the power of the chart to detect a shift in the vector (μ, σ^2) is diminished and proposed the use of larger sample sizes to increase this power. Mittag (1995) and Mittag and Stemann (1998) examined the effect of measurement error on the joined $\overline{X}$-S control chart assuming the model of Bennet (1954). Mittag (1995) considered the effect at the onset of a process whereas Mittag and Stemann (1998) considered also the effect in the case of subsequent error occurrence.

Mittag and Stemann (1998) proved that the power function of the $\overline{X}$-S control chart in the case of immediate error occurrence (at the beginning of the implementation of $\overline{X}$-S control chart) is given by

$$G^e(\delta;\varepsilon) = 1 - H^e_{\overline{X}}(\delta;\varepsilon)H^e_S(\varepsilon), \tag{6.1}$$

where

$$H^e_{\overline{X}}(\delta;\varepsilon) = 1 - \left[\Phi\left(\frac{-z_{1-\alpha_1/2}(1+\tau^2)^{1/2}+\delta n^{1/2}}{(\varepsilon^2+\tau^2)^{1/2}}\right) + \Phi\left(\frac{-z_{1-\alpha_1/2}(1+\tau^2)^{1/2}-\delta n^{1/2}}{(\varepsilon^2+\tau^2)^{1/2}}\right)\right],$$

and

$$H^e_S(\varepsilon) = Ch\left(\frac{1+\tau^2}{\varepsilon^2+\tau^2}\chi^2_{n-1;1-\alpha_2}|n-1\right),$$

α_1, α_2 are the probabilities of a false signal on the $\overline{X}$ and S charts respectively, ε equals σ/σ_0, δ equals $(\mu-\mu_0)/\sigma_0$, where μ_0,σ_0 are the process parameters target values. The symbol Φ stands for the standard normal distribution again, z_ω denotes its ω quantile, Ch is the central chi-squared distribution and $\chi^2_{n-1;\omega}$ represents its ω quantile with $n-1$ degrees of freedom. In the case of subsequent error occurrence they proved that the power function of the $\overline{X}$-S control chart is given by relationship (6.1) as in the immediate error occurrence but now with

$$H^e_{\overline{X}}(\delta;\varepsilon) = 1 - \left[\Phi\left(\frac{-z_{1-\alpha_1/2}+\delta n^{1/2}}{(\varepsilon^2+\tau^2)^{1/2}}\right) + \Phi\left(\frac{-z_{1-\alpha_1/2}-\delta n^{1/2}}{(\varepsilon^2+\tau^2)^{1/2}}\right)\right]$$

and

$$H^e_S(\varepsilon) = Ch\left(\frac{\chi^2_{n-1;1-\alpha_2}}{\varepsilon^2+\tau^2}|n-1\right).$$

In both cases we have a reduced ability of the $\overline{X}$-S control chart to identify when a process is out-of-control.

6.2.2 Model with Covariates

Linna and Woodall (2001) extended the model considered by Bennet (1954) assuming one with covariates examining the effect on the $\overline{X}$ and S^2 control charts. Specifically, they considered the model $Y = A + BX + \varepsilon$, where X is a normally distributed variable with mean μ and variance σ_p^2 and ε is normally distributed with mean 0 and variance σ_m^2. If the mean of X shifts to some value μ' then the probability of a signal on the Shewhart chart for the mean is

$$1 - \Phi\left(3 + \frac{(\mu-\mu')\sqrt{n}}{\sqrt{\sigma_p^2+\sigma_m^2/B^2}}\right) + \Phi\left(-3 + \frac{(\mu-\mu')\sqrt{n}}{\sqrt{\sigma_p^2+\sigma_m^2/B^2}}\right).$$

The probability of a signal on the Shewhart chart for the variance if the characteristic X shifts from σ_p^2 to $\sigma_p^{2'}$ is

$$1 - \Pr\left(\frac{B^2\sigma_p^2+\sigma_m^2}{B^2\sigma_p^{2'}+\sigma_m^2}\chi^2_{a/2,n-1} < Q < \frac{B^2\sigma_p^2+\sigma_m^2}{B^2\sigma_p^{2'}+\sigma_m^2}\chi^2_{1-a/2,n-1}\right),$$

where Q is a chi-square random variable with $n-1$ degrees of freedom. They proved that in both charts the effect of the measurement error variance and the value B on their power is significant. Linna and Woodall among others, proposed the use of multiple measurements per item as a solution to this problem. If we assume that we take k successive independent measurements on each of n items then the variance of the sample mean of the subgroup is $\frac{B^2\sigma_p^2}{n}+\frac{\sigma_m^2}{nk}$, indicating that as k increases this variance decreases. In the case of linearly increasing variance we assume that ε is normally distributed with mean 0 and variance $C+D\mu^{'}$ where C and D are assumed known constants. The probability of a signal in this case is

$$1-\Phi\left(\frac{\left(\mu-\mu^{'}\right)\sqrt{n}+3\sqrt{\sigma_p^2+C/B^2+D\mu/B^2}}{\sqrt{\sigma_p^2+C/B^2+D\mu'/B^2}}\right)$$
$$+\Phi\left(\frac{\left(\mu-\mu^{'}\right)\sqrt{n}-3\sqrt{\sigma_p^2+C/B^2+D\mu/B^2}}{\sqrt{\sigma_p^2+C/B^2+D\mu'/B^2}}\right).$$

In this case, both charts for the mean and the variance are affected.

Linna, Woodall and Busby (2001) examined the same model with covariates in the multivariate case, in the case of the X^2 chart. In particular, let $\mathbf{Y}_i=\mathbf{A}+\mathbf{B}\mathbf{X}_i+\boldsymbol{\varepsilon}_i$, $i=1,\ 2,...$ where $\mathbf{A}$ is a $p\times 1$ vector of constants, $\mathbf{B}$ is an invertible $p\times p$ matrix of constants and $\boldsymbol{\varepsilon}_i$ is a $p\times 1$ normal random vector independent of $\mathbf{X}$ with a mean vector of zeroes and variance covariance matrix $\boldsymbol{\Sigma}_m$. Linna, Woodall and Busby proved that multivariate control charts under measurement error effect can detect in a more powerful way shift in one direction than in other. Moreover, they have shown that multivariate control charts are affected and this effect can be very serious because of the loss of the directional invariance property.

6.3 EWMA Charts and Measurement Error

Stemann and Weihs (2001) were the first to investigate the effect of measurement error on the EWMA chart. They considered the EWMA-X-S chart, as they name it, which is a combination of the EWMA charts for the mean and the standard deviation. Specifically, they showed through simulation that the ARL behavior of this chart is affected by the measurement error. The model assumed is the one proposed by Bennet (1954). They checked both the cases of the presence of measurement error in the beginning of the process and subsequently during production. However, the model with covariates proposed by Linna and Woodall (2001) was not investigated. This model is investigated in the case of measurement error for the EWMA chart for the mean in Maravelakis et al. (2004) and is presented in the following subsections.

6.3.1 The EWMA Chart Using Covariates

Assume again that we have a process where the true value of the characteristic X under investigation is normally distributed with mean μ and variance σ^2 when the process is in-control. However, we are not able to observe this true value but rather a value Y, which is related to X with the formula $Y = A + BX + \varepsilon$, where A and B are constants and ε is the random error distributed independently of X as a normal random variable with mean zero and variance σ_m^2. We assume here that all model parameters are known.

From the formula relating Y and X it is straightforward that Y is normally distributed with mean $A + B\mu$ and variance $B^2\sigma^2 + \sigma_m^2$. We need to construct an EWMA chart for the measured quantity Y since in this way we can keep under control the variable X. Assume that at each sampling point we collect n values of Y, we compute the mean of these observations $\overline{Y_i}$ and we compute the EWMA statistic z_i using the formula

$$\begin{aligned} z_i &= \lambda\overline{Y_i} + (1-\lambda)z_{i-1}, \\ z_0 &= A + B\mu \end{aligned}$$

where $\overline{Y_i}$ is the mean of the observations collected at time $i = 1, 2, ...$ and λ is the smoothing parameter.

The control limits are

$$\begin{aligned} UCL &= A + B\mu + L\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left[1-(1-\lambda)^{2i}\right]\frac{(B^2\sigma^2+\sigma_m^2)}{n}} \\ LCL &= A + B\mu - L\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left[1-(1-\lambda)^{2i}\right]\frac{(B^2\sigma^2+\sigma_m^2)}{n}}, \end{aligned} \tag{6.2}$$

where L is a constant used to specify the width of the control limits and $A + B\mu$ and $\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left[1-(1-\lambda)^{2i}\right]\frac{(B^2\sigma^2+\sigma_m^2)}{n}}$ are the mean and standard deviation of Z_i respectively, when the process is in-control. In case the EWMA chart is used for some time, instead of the control limits (6.2), we may use their limiting values

$$\begin{aligned} UCL &= A + B\mu + L\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\frac{(B^2\sigma^2+\sigma_m^2)}{n}} \\ LCL &= A + B\mu - L\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\frac{(B^2\sigma^2+\sigma_m^2)}{n}}, \end{aligned} \tag{6.3}$$

(see e.g., Lucas and Saccucci (1990)). In this case $\sqrt{(\lambda/(2-\lambda))\left((B^2\sigma^2+\sigma_m^2)/n\right)}$ is the asymptotic standard deviation of Z_i.

6.3.2 Multiple measurements

In order to decrease the measurement error effect, a technique that is suggested by Linna and Woodall (2001) is to take more than one measurements in each sampled unit. Taking more than one measurements and averaging them leads to a more precise measurement. Moreover, the variance of the measurement error component in the average of the multiple observations becomes smaller as the number of multiple measurements increases. Therefore, ideally if the number of multiple measurements becomes infinite the variance of the measurement error component will become zero. Consequently, the larger the number of multiple measurements the better, keeping in mind always the additional cost and time needed for these observations. We must understand also that in the absence of measurement error multiple measurements will not contribute to the control charting methodology anything (in fact they will add the cost of measuring the extra observations).

In the case of sufficient number of multiple measurements we can assume that our process actually operates without measurement error. However, the cost of extra measurements and the time are factors that can not be overlooked. Therefore, a careful examination of these factors in the specific application we are working on is essential. We have to stress that the measurement error variance has to be large enough and the two factors small enough for the extra observations to have a practical value.

In order to compute the EWMA statistic we assume that at each sampling point we collect k measurements for each of n observations of Y, we compute the overall mean of these observations $\overline{\overline{Y_i}}$ and we compute the EWMA statistic q_i using the formula

$$\begin{aligned} q_i &= \lambda\overline{\overline{Y_i}} + (1-\lambda)q_{i-1}, \\ q_0 &= A + B\mu \end{aligned}$$

where $\overline{\overline{Y_i}}$ is the mean of the observations collected at time $i = 1, 2, ...$, λ is a smoothing parameter that takes values between 0 and 1 and q_0 is the initial value. Moreover, we assume that the k observations collected at the same sampling unit are independent. If $k = 1$ we face the measurement error case discussed in Section 6.3.1.

It is straightforward to prove (Linna and Woodall (2001)) that the variance of the overall mean is $\frac{B^2\sigma^2}{n} + \frac{\sigma_m^2}{nk}$. Therefore, the control limits are

$$\begin{aligned} UCL_q &= A + B\mu + L\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left[1-(1-\lambda)^{2i}\right]\left(\frac{B^2\sigma^2}{n} + \frac{\sigma_m^2}{nk}\right)} \quad (6.4) \\ LCL_q &= A + B\mu - L\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left[1-(1-\lambda)^{2i}\right]\left(\frac{B^2\sigma^2}{n} + \frac{\sigma_m^2}{nk}\right)}, \end{aligned}$$

where L is a constant used to specify the width of the control limits and $A + B\mu$ and $\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left[1-(1-\lambda)^{2i}\right]\left(\frac{B^2\sigma^2}{n}+\frac{\sigma_m^2}{nk}\right)}$ are the mean and standard deviation of q_i respectively, when the process is in-control. In case the EWMA chart is used for some time, instead of the control limits (6.4), we may use their limiting values

$$\begin{aligned} UCL_q &= A + B\mu + L\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left(\frac{B^2\sigma^2}{n}+\frac{\sigma_m^2}{nk}\right)} \\ LCL_q &= A + B\mu - L\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left(\frac{B^2\sigma^2}{n}+\frac{\sigma_m^2}{nk}\right)}. \end{aligned} \tag{6.5}$$

6.3.3 Linearly increasing variance

Although the model with covariates considered assumes constant variance it is not unlikely to have a model with variance that depends on the mean level of the process. Montgomery and Runger (1994) and Linna and Woodall (2001) refer to practical problems indicating situations where this phenomenon occurs in industry.

We assume that the variance changes linearly with variable X. The model we use is again $Y = A + BX + \varepsilon$ with the same assumptions as in Section 6.3.1, except that this time ε is distributed as a normal variable with mean 0 and variance $C + D\mu$. As in Section 6.3.1 all model parameters are assumed known. From the relation between Y and X we deduce that Y is normally distributed with mean $A + B\mu$ and variance $B^2\sigma^2 + C + D\mu$. The EWMA statistic will be exactly the same as in Section 6.3.1.

It can be shown that the control limits of the EWMA statistic are

$$\begin{aligned} UCL_l &= A + B\mu + L\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left[1-(1-\lambda)^{2i}\right]\left(\frac{B^2\sigma^2+C+D\mu}{n}\right)} \\ LCL_l &= A + B\mu - L\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left[1-(1-\lambda)^{2i}\right]\left(\frac{B^2\sigma^2+C+D\mu}{n}\right)}, \end{aligned} \tag{6.6}$$

where L is again a constant used to specify the width of the control limits and $A + B\mu$ and $\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left[1-(1-\lambda)^{2i}\right]\left(\frac{B^2\sigma^2+C+D\mu}{n}\right)}$ are the mean and standard deviation of the

EWMA statistic respectively, when the process is in-control. When the EWMA chart is used for a suitable number of points in time, instead of the control limits (6.6), we can use their limiting values

$$\begin{aligned} UCL_l &= A + B\mu + L\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left(\frac{B^2\sigma^2 + C + D\mu}{n}\right)} \\ LCL_l &= A + B\mu - L\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left(\frac{B^2\sigma^2 + C + D\mu}{n}\right)}. \end{aligned} \tag{6.7}$$

6.3.4 ARL computations

In order to compute the probability density function, the cumulative distribution function and the first moment of the run length distribution of the EWMA chart for the mean we may approximate it as a discrete Markov Chain by dividing the distance between the control limits in 2m+1 states each of which has width 2δ (see, e.g. Brook and Evans (1972)). We say that the statistic Z_i remains in state j as long as $S_j - \delta < Z_i \leq S_j + \delta$ where $-m \leq j \leq m$ and S_j is the midpoint in the jth interval. When Z_i crosses the control limits we say that it is in the absorbing state. On the other hand when the process is in-control we say that it is in a transient state.

The transition probability matrix for the EWMA chart for the mean is computed as

$$\mathbf{P} = \begin{bmatrix} \mathbf{R} & (\mathbf{I}-\mathbf{R})\mathbf{1} \\ \mathbf{0}^T & 1 \end{bmatrix},$$

where $\mathbf{R}$ is a submatrix containing the transient states, $\mathbf{I}$ is a $(t \times t)$ identity matrix and $\mathbf{1}$ is a $(t \times 1)$ vector of unities. The $jkth$ element of the submatrix $\mathbf{R}$ is given by $p_{jk} = P\left[S_j - \delta < \lambda y_i + (1-\lambda)S_j \leq S_j + \delta\right]$. In the case of the normal distribution with the model with covariates of our case the probabilities are given by

$$p_{jk} = \Phi\left[\frac{(S_k+\delta) - (1-\lambda)S_j - \lambda\left(A + B\mu\right)}{\lambda\sqrt{\left(B^2\sigma^2 + \sigma_m^2\right)/n}}\right] - \Phi\left[\frac{(S_k-\delta) - (1-\lambda)S_j - \lambda\left(A + B\mu\right)}{\lambda\sqrt{\left(B^2\sigma^2 + \sigma_m^2\right)/n}}\right].$$

When we have multiple measurements the probabilities are

$$p_{jk} = \Phi\left[\frac{(S_k+\delta)-(1-\lambda)S_j-\lambda\left(A+B\mu\right)}{\lambda\sqrt{\left(\frac{B^2\sigma^2}{n}+\frac{\sigma_m^2}{nk}\right)}}\right]-\Phi\left[\frac{(S_k-\delta)-(1-\lambda)S_j-\lambda\left(A+B\mu\right)}{\lambda\sqrt{\left(\frac{B^2\sigma^2}{n}+\frac{\sigma_m^2}{nk}\right)}}\right]$$

and in the case of linearly increasing variance the probabilities are

$$p_{jk} = \Phi\left[\frac{(S_k+\delta)-(1-\lambda)S_j-\lambda\left(A+B\mu\right)}{\lambda\sqrt{\left(\frac{B^2\sigma^2+C+D\mu}{n}\right)}}\right]-\Phi\left[\frac{(S_k-\delta)-(1-\lambda)S_j-\lambda\left(A+B\mu\right)}{\lambda\sqrt{\left(\frac{B^2\sigma^2+C+D\mu}{n}\right)}}\right].$$

Let τ denote the run length of the EWMA, then $P\left(\tau \leq t\right) = \left(\mathbf{I}-\mathbf{R}^t\right)\mathbf{1}$ and therefore $P\left(\tau = t\right) = \left(\mathbf{R}^{t-1}-\mathbf{R}^t\right)\mathbf{1}$ for $t \geq 1$. The ARL can be computed using the formula $E(\tau) = \sum_{i=1}^{\infty} tP\left(\tau = t\right) = (\mathbf{I}-\mathbf{R}^{-1})\mathbf{1}$.

6.3.5 Effect of the measurement error

In the context of EWMA charts as we have already said there are three ways of computing the ARL. The integral equation method, the Markov chain method and simulation. Here, we use the Markov Chain method in all the computations.

In Table 6.1, we can see the ARL results of the covariate model for different values of the ratio σ_m^2/σ^2 when B=1. The in-control ARL value is the same for all combinations in order to achieve a fair comparison. From the table we see that there is an increasing effect on the out-of-control ARL as the ratio of σ_m^2/σ^2 increases. This result is similar to the one in Linna and Woodall (2001). In Table 6.2, we can see the ARL results of the covariate model for different values of B. The results are displayed for the same parameters as in Table 6.1 when σ_m^2/σ^2=1. We observe that as the value of B increases the effect on the ARL diminishes. This result is again in accordance with Linna and Woodall (2001). Furthermore, in both Tables 6.1 and 6.2 the effect of the measurement error on the ARL values lessens as the shift increases. We have to state also that A does not affect the ARL performance in this study.

Table 6.1. ARL for the covariate model for different values of σ_m^2/σ^2

Shift	No Error	0.1	0.2	0.3	0.5	1
0	370.22	370.27	370.27	370.27	370.27	370.26
0.5	41.13	45.22	49.26	53.23	60.96	79.06
1	10.25	11.21	12.18	13.16	15.15	20.26
1.5	5.18	5.57	5.96	6.36	7.16	9.20
2	3.46	3.69	3.91	4.13	4.57	5.67
2.5	2.65	2.80	2.94	3.09	3.37	4.08
3	2.19	2.29	2.40	2.50	2.71	3.22

Table 6.2. ARL for the covariate model for different values of B

Shift	No Error	1	2	3	5
0	370.22	370.26	370.27	370.26	370.27
0.5	41.13	79.06	51.25	45.67	42.78
1	10.25	20.26	12.67	11.31	10.63
1.5	5.18	9.20	6.16	5.61	5.33
2	3.46	5.67	4.02	3.71	3.55
2.5	2.65	4.08	3.01	2.81	2.71
3	2.19	3.22	2.45	2.31	2.23

In Table 6.3, we can see the ARL results for the covariate model with multiple measurements for different values of σ_m^2/σ^2 when $k = 5$ and $B = 1$. It is obvious that if the practitioner has the ability to take five measurements in each unit then for values of σ_m^2/σ^2 less than 0.3 we may say that the process operates actually without measurement error. For values larger than 0.3 the effect is seriously reduced in comparison to the $k = 1$ case, which corresponds to the results in Table 6.1, even for $\sigma_m^2/\sigma^2 = 1$.

Table 6.3. ARL for multiple measurements k=5, B=1 for different values of σ_m^2/σ^2

Shift	No Error	0.1	0.2	0.3	0.5	1
0	370.22	370.26	370.26	370.27	370.27	370.27
0.5	41.13	41.96	42.78	43.59	45.22	49.26
1	10.25	10.44	10.63	10.82	11.21	12.18
1.5	5.18	5.25	5.33	5.41	5.57	5.96
2	3.46	3.51	3.55	3.60	3.69	3.91
2.5	2.65	2.68	2.71	2.74	2.80	2.94
3	2.19	2.21	2.23	2.25	2.29	2.40

Table 6.4. ARL for multiple measurements k=5, σ_m^2/σ^2=1 for different values of B

Shift	No Error	1	2	3	5
0	370.22	370.27	370.26	370.28	370.27
0.5	41.13	49.26	43.18	42.05	41.46
1	10.25	12.18	10.73	10.46	10.33
1.5	5.18	5.96	5.37	5.26	5.21
2	3.46	3.91	3.57	3.51	3.48
2.5	2.65	2.94	2.72	2.68	2.66
3	2.19	2.40	2.24	2.21	2.20

Table 6.4 presents the results in the case of multiple measurements for different values of B. We see that as the value of B increases the effect on the ARL diminishes. This result is in accordance with the results in Table 6.2. Moreover, in Table 6.5 we have results in the case of multiple measurements for different k values. As the value of k increases the measurement error effect lessens. However, since cost and time needed for the extra measurements are important factors, the practitioner will have to do a trade-off

between these two concerns and the measurement error he can put up with. We have to stress here that the results displayed refer to the worst case, since we choose $B = 1$ and $\sigma_m^2/\sigma^2 = 1$, that correspond to the most affected combination. Therefore, one may conclude that the results in the other cases will be even better.

Table 6.5. ARL for multiple measurements for different values of k

Shift	No Error	5	10	20	50
0	370.22	370.27	370.27	370.26	370.26
0.5	41.13	49.26	45.22	43.18	41.96
1	10.25	12.18	11.21	10.73	10.44
1.5	5.18	5.96	5.57	5.37	5.25
2	3.46	3.91	3.69	3.57	3.51
2.5	2.65	2.94	2.80	2.72	2.68
3	2.19	2.40	2.29	2.24	2.21

Table 6.6. ARL for linearly increasing variance for different values of D

Shift	No Error	1	2	3	5
0	370.22	370.27	370.28	370.27	370.28
0.5	41.13	231.40	282.70	306.34	328.76
1	10.25	102.95	161.16	198.80	244.30
1.5	5.18	50.14	90.14	122.03	168.52
2	3.46	28.10	53.72	76.89	115.28
2.5	2.65	17.79	34.49	50.85	80.43
3	2.19	12.39	23.73	35.38	57.77

The results in the case of linearly increasing variance are displayed on Tables 6.6 and 6.7. In Table 6.6 we have the ARL values when $B = 1$, $C = 0$, $\sigma_m^2/\sigma^2 = 1$ for different values of D. We see that even for small values of D there is a more serious effect than in the no error case. Additionally, as the value of D increases, this effect is getting larger. This result is expected because as D increases so does the variance of the error component in the model. In this special case of measurement error, extra precaution is needed because the ability of the EWMA chart to detect fast small shifts is canceled out. Consequently, serious distortion factors may go undetected for a long time costing a lot in money, time and credibility.

Table 6.7 presents the ARL results when $B = 1$, $D = 1$ and $\sigma_m^2/\sigma^2 = 1$ for different values of C. In analogy to Table 6.6, increasing values of C cause an increasing measurement error effect on the ARL. However, this effect is not of the same magnitude as the effect of D. This result is also expected since D is multiplied by the mean μ, thus increasing faster the error variance as D increases whereas C is just added to this variance.

Table 6.7. ARL for linearly increasing variance for different values of C

Shift	No Error	0	1	2	3
0	370.22	370.27	370.29	370.27	370.27
0.5	41.13	231.40	239.08	245.96	252.14
1	10.25	102.95	110.13	116.95	123.44
1.5	5.18	50.14	54.53	58.84	63.06
2	3.46	28.10	30.72	33.34	35.95
2.5	2.65	17.79	19.43	21.09	22.75
3	2.19	12.39	13.49	14.59	15.71

In all the computations we used 211 states for the Markov Chain method. The values of the constants are $\lambda = 0.25$ and $L = 2.898$. In order to detect small shifts fast the λ value usually used is 0.1 or less. However, such small values are not able to detect small to moderate shifts and this is the reason for choosing this particular value of λ. Note also that in all the cases the control limits used are the ones with the limiting values.

Chapter 7

Conclusions

In this thesis we have presented and studied the main univariate and multivariate control charts. Our scope was to deal with specific problems in the context of control charts. The first problem under consideration was the estimation effect on the control charts performance. This issue was investigated in the case of the univariate Shewhart charts for dispersion for both subgrouped and individual data. Specific recommendations are given for the number of samples or the number of observations needed to estimate accurately the parameters in order for the control chart to behave as in the theoretical case of known parameters. A second problem is nonnormality and how it affects the performance of a control chart. It was investigated on the EWMA charts for the dispersion when we have individual observations. A new EWMA type chart for the process dispersion is given that is proved to be robust up to nonnormality when we are in-control.

The identification of the out-of-control variable when a multivariate control chart signals is another problem examined. A new method is presented that computes probability limits that indicate with the desired probability the out-of-control variable or variables. This method has proven to be a competitive alternative to the existing procedures. Finally, the effect of measurement error on the performance of control charts was considered. A model with covariates for the EWMA chart is presented. This model is examined in some cases and it is proved that the presence of measurement error seriously affects the ability of a control chart to identify the out-of-control situation.

Some thoughts for further research are given in the following discussion. The estimation effect is one of the issues that practitioners have to face. Although until recently the guidelines in the design of a control chart were talking about a number of subgroups needed, recent research proved that this recommendation is deceiving. Generally, a larger number of samples is needed. Although in today's industry there are usually large data sets, there are still processes that the recommendation for e.g. 100 samples for the estimation of the process parameters is a very large number and usually impossible to obtain because of the time needed for these observations or because of the money we have to spend. Therefore, the control limits have to be adjusted in a way that they will take into account the estimation effect. Nedumaran and Pignatiello (2001) have given such a solution for the $\overline{X}$ control chart and Jones (2002) proposes a way to design the EWMA chart for this case. Additional work on the other control charts has to be done.

Most of the control charting methodology has been implemented under the normality assumption. However, most of the times this assumption is not valid. All the charts are affected in the case of nonnormality but to a different extent. The EWMA chart for the mean has proved to be less affected when properly designed. The EWMA chart for the dispersion proposed in Maravelakis et al. (2003) is less disturbed by the nonnormality issue when we are in-control. Nevertheless, an EWMA chart for dispersion that can be robust in terms of nonnormality for both in and out-of-control is needed.

In the multivariate case, the nonnormality problem is even more challenging. But if normality is cumbersome in the univariate case, then in a multivariate environment the situation is even more dramatic. Recently, Stoumbos and Sullivan (2002) and Testik et. al. (2003) proved the robustness of the multivariate EWMA chart for the vector of means. However, there is a lot more work that has to be done in the field. A general technique able to detect efficiently, under any distribution, the out-of-control situation is needed. Moreover, since a process might involve both continuous and discrete characteristics another problem is to find a control charting methodology that will consider both of them.

A very important problem in the multivariate control charts is to identify the out-of-control variable or variables when a multivariate control chart signals. This problem has generated many different opinions in the last decade. Although this problem is thoroughly investigated under the multinormality assumption most of the proposed solutions are mathematically complicated or time consuming. Consequently, new procedures that will overcome these disadvantages are needed. Graphical techniques are such techniques.

Measurement error is a factor that can affect seriously the performance of a control chart. The literature up to now investigates this problem under the assumptions of normality, independence, known parameters and predefined additive relationship of the true and measured variables. All of these assumptions have to be reconsidered. Moreover, the estimation effect in other types of control charts is an open problem.

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