Density Estimates For A Fractional Elliptic System And Applications

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Acknowledgments

I would like to express my deepest gratitude to my supervisor, Professor Athanasios Yannacopoulos, for his invaluable guidance, unwavering support, and thoughtful feedback throughout the course of this thesis.

I am also profoundly thankful to Professor Nikolaos Alikakos for his mentorship, insightful mathematical discussions, and for introducing me to most of the core ideas that shaped this work.

I wish to thank the Department of Statistics at the Athens University of Economics and Business for providing a stimulating academic environment, as well as my colleagues for their constructive conversations and encouragement.

I would like to express my sincere gratitude to the members of my final examination committee for their time, careful reading of the thesis, and insightful comments that significantly improved the clarity and quality of this work. I am especially grateful to Professor Nikolaos Kavallaris (Department of Mathematics and Computer Sciences, Karlstad University, Sweden) for his deep and thorough corrections, precise remarks, and constructive suggestions, which helped refine both the presentation and several technical details of the manuscript.

Lastly, I am deeply grateful to my life companion, Galatoula Sofia, for her unwavering love, patience, and inspiration. Her presence gave me strength during the most demanding phases of this journey, and her support has been a cornerstone of my academic and personal growth.

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Chapter 1

Introduction

The study of interfaces in phase transition models has long been a central theme in the analysis of elliptic partial differential equations. In classical models, the energy functional typically takes the form:

$$J(u) = \int_\Omega \left(\frac{1}{2} |\nabla u(x)|^2 + W(u(x)) \right) dx, \tag{1.1}$$

where W is a multi-well potential and u represents the phase field. Minimizers of this energy capture transitions between different stable states of the system, and the interface is the region where u moves between different wells of W (see, e.g.,[29, 28, 4, 5, 6]).

A cornerstone result in this theory is the density estimates. Density estimates have been extensively studied in both local and nonlocal frameworks; see, for instance,[37, 38, 12, 14, 16]. In contrast, the vector-valued case, especially in the nonlocal setting, is considerably more delicate and less thoroughly explored. This is primarily due to the absence of a maximum principle and the richer geometric structure of the set of minimizers, which necessitates new analytical techniques and significant adaptations; see [2].

In the classical setting, it asserts that if a minimizer u deviates from a phase in a small ball—meaning $|u - a| > \lambda$ on a set of positive measure—then this deviation must persist in larger balls. More formally, let $a \in \{W = 0\}$ be a nondegenerate minimum and let $u : B_R(x_0) \to \mathbb{R}^m$

be a minimizer of J. Suppose for some $\lambda > 0$ and $\mu_0 > 0$ we have:

$$|\{x \in B_{r_0}(x_0) : |u(x) - a| > \lambda\}| \geq \mu_0, \tag{1.2}$$

for some fixed small $r_0 > 0$. Then there exists a constant $c = c(\mu_0, \lambda, W) > 0$ such that for all $r \geq r_0$,

$$|\{x \in B_r(x_0) : |u(x) - a| > \lambda\}| \geq cr^n. \tag{1.3}$$

In other words, the measure of the transition region does not shrink with scale; it occupies a fixed fraction of the space at every larger radius (see, e.g. [2, 38]).

This implies that phase transitions are not confined to isolated droplets; instead, once an interface appears, it must either vanish entirely or spread spatially. Physically, this encodes the idea that interfacial energy cannot be arbitrarily localized without cost.

A vivid real-world analogy is that of a droplet of oil placed in water. If the oil volume is too small, the surface tension dominates, and the droplet collapses. But if the oil droplet is sufficiently large, it becomes energetically favorable to maintain a stable interface. In this situation, the oil does not disappear—it spreads and persists. The density estimate mirrors this: once deviation from a phase begins and exceeds a certain energetic threshold, it cannot stay confined—it must grow or stabilize with positive density in all larger surrounding regions.

In contrast, fractional models replace the local Dirichlet energy with a nonlocal Gagliardo seminorm:

$$\mathcal{E}(u) = \frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy + \int_\Omega W(u(x)) \, dx, \tag{1.4}$$

for $s \in (0, 1)$. This double integral captures long-range interactions between points x and y across all of $\mathbb{R}^n$, with a kernel that decays like $|x - y|^{-n-2s}$ ([18, 25, 12]).

The parameter s controls the degree of nonlocality:

- As $s \to 1$, the kernel becomes more singular near $x = y$, and the operator approximates the classical Laplacian. The model becomes increasingly local, recovering the familiar behavior of sharp interfaces penalized by $|\nabla u|^2$.

- As $s \to 0$, the kernel decays slowly, and the model becomes highly nonlocal. Every point interacts almost equally with all others, and interfaces become diffuse with long algebraic tails.

A key insight lies in examining the integrand near the diagonal $x \approx y$:

$$\frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \approx |\nabla u(x)|^2 \cdot \frac{1}{|x - y|^{n+2s-2}},$$

which shows that for small $|x-y|$ the fractional energy has a Dirichlet-type local core. Heuristically this suggests the classical limit as $s \to 1^-$. Rigorously, by the Bourgain–Brezis–Mironescu formula,

$$\lim_{s \to 1^-} (1 - s) \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy = c_n \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx,$$

for an explicit constant $c_n > 0$ depending only on n (and the chosen normalization); see [10, 33, 18].

When s is close to 1, the fractional model retains sensitivity to sharp gradients. As x approaches y, the kernel's singularity strongly penalizes steep changes, mimicking classical local behavior. Conversely, when s is small, the decay of the kernel is slow, and long-range interactions dominate. In this regime, sharp transitions cost relatively less energy compared to wide-spanning.

To appreciate the distinction further, consider a localized perturbation—a small droplet of one phase inside another. In the classical setting, if the droplet is small, its interface energy becomes negligible as the radius shrinks, and it can persist as a harmless fluctuation (see, e.g. [26]). In the fractional setting, however, even small deviations have long-range energetic influence due to the global nature of the double integral. This results in a striking property: droplets cannot remain harmless unless their volume and interfacial structure are compatible with global energy constraints. The density estimate formalizes this intuition, ensuring that any nontrivial deviation from a phase cannot remain confined. It must either vanish or spread—an embodiment of rigidity.

This is vividly illustrated by physical systems such as crack propagation in brittle materials, where the evolution of a crack is not governed solely by the local stress at the tip, but also by remote stress fields. These interactions extend over large distances and are naturally captured by nonlocal descriptions, including fractional-type models. The resulting crack geometry exhibits nonlocal influence, in the sense that the relevant energy and driving quantities depend on the global configuration of the material, rather than only on purely local gradients. [21, 39, 24, 19]

1.1 Local and Nonlocal Operators as Generators of Stochastic Processes

Another powerful interpretation of both local and nonlocal operators arises from probability theory: they are the infinitesimal generators of stochastic processes ([7, 8, 35]).

In the classical case, the Laplacian Δ is the generator of Brownian motion. That is, if X_t is a standard Brownian motion in $\mathbb{R}^n$, then the infinitesimal generator $\mathcal{L}$ associated with the Markov semigroup $T_t f(x) = \mathbb{E}^x[f(X_t)]$ is:

$$\mathcal{L}f(x) = \lim_{t \to 0} \frac{\mathbb{E}^x[f(X_t)] - f(x)}{t} = \Delta f(x). \tag{1.5}$$

This operator governs the evolution of the expected value of observables along stochastic trajectories, and leads to the heat equation:

$$\partial_t u = \Delta u, \qquad u(0,x) = f(x), \tag{1.6}$$

with solution $u(t,x) = \mathbb{E}^x[f(X_t)]$.

In contrast, the fractional Laplacian $(-\Delta)^s$ for $s \in (0,1)$ is the generator of a symmetric $2s$-stable Lévy process X_t^s, whose paths exhibit jumps. The corresponding semigroup $T_t^s f(x) = \mathbb{E}^x[f(X_t^s)]$ satisfies:

$$\mathcal{L}_s f(x) = \lim_{t \to 0} \frac{\mathbb{E}^x[f(X_t^s)] - f(x)}{t} = -(-\Delta)^s f(x), \tag{1.7}$$

where $(-\Delta)^s$ is given by the singular integral:

$$(-\Delta)^s f(x) = C_{n,s} \text{ P.V.} \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x - y|^{n+2s}}\, dy. \tag{1.8}$$

This expression highlights the nonlocal nature of the operator: $f(x)$ interacts with values of $f(y)$ at all scales.

The Lévy process associated with this generator has independent and stationary increments, and its transition probabilities solve the fractional heat equation:

$$\partial_t u = -(-\Delta)^s u, \qquad u(0,x) = f(x), \tag{1.9}$$

with solution $u(t,x) = \mathbb{E}^x[f(X_t^s)]$.

A further refinement is provided by the Feynman–Kac formula, which links parabolic PDEs with potential terms to expectations over stochastic processes. In the classical case, for a bounded potential $V : \mathbb{R}^n \to \mathbb{R}$, the solution to the Schrödinger-type equation:

$$\partial_t u = \Delta u - V(x)u, \qquad u(0,x) = f(x), \tag{1.10}$$

is given by:

$$u(t,x) = \mathbb{E}^x\left[e^{-\int_0^t V(X_s)ds} f(X_t)\right], \tag{1.11}$$

where X_t is Brownian motion starting at x.

In the fractional setting, if X_t^s is a symmetric $2s$-stable Lévy process starting at x, then the Feynman–Kac representation for:

$$\partial_t u = -(-\Delta)^s u - V(x)u, \qquad u(0,x) = f(x), \tag{1.12}$$

is:

$$u(t,x) = \mathbb{E}^x\left[e^{-\int_0^t V(X_\sigma^s)d\sigma} f(X_t^s)\right]. \tag{1.13}$$

This formula elegantly connects the probabilistic behavior of Lévy flights with the analytic structure of nonlocal equations, and allows one to represent solutions of nonlocal Schrödinger-type problems in terms of expectations over jump processes.

These connections between differential operators and stochastic processes will be a recurring theme in this text. They serve not only as analytic tools but also offer a probabilistic perspective on phase transitions, energy concentration, and interface dynamics. Fractional operators have found roles in other areas as well: anomalous diffusion in porous media, biological transport, nonlocal image processing, and finance ([27, 17]). In all these settings, locality fails, and fractional models become indispensable.

1.2 Mathematical Preliminaries

In this section, we collect the definitions of the central analytic objects used throughout the study of fractional phase transitions. These include the fractional Laplacian, fractional Sobolev spaces, and the notion of minimality for energy functionals.

1.2.1 The Fractional Laplacian

For $s \in (0, 1)$ and a sufficiently smooth function $u : \mathbb{R}^n \to \mathbb{R}$, the fractional Laplacian of order s is defined as the singular integral:

$$(-\Delta)^s u(x) := C_{n,s} \text{ P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy, \tag{1.14}$$

where P.V. denotes the Cauchy principal value and $C_{n,s}$ is an explicit normalization constant given by:

$$C_{n,s} = \frac{2^{2s} s \Gamma\left(\frac{n}{2} + s\right)}{\pi^{n/2} \Gamma(1 - s)}. \tag{1.15}$$

Here Γ denotes the Gamma function, defined for $t > 0$ by

$$\Gamma(t) := \int_0^\infty x^{t-1} e^{-x} \, dx.$$

The operator $(-\Delta)^s$ is nonlocal: the value of $(-\Delta)^s u(x)$ depends on the behavior of u over the entire space $\mathbb{R}^n$.

In the literature, various definitions of the fractional Laplacian have been proposed, each suitable for different analytical and numerical purposes. Besides the definition used in this work, there are spectral, directional, and other characterizations. For a detailed review of these alternative definitions and their comparisons, see section 3.1 in Di Nezza et al. [18] or Lischke et al. [25].

1.2.2 Fractional Sobolev Spaces

The fractional Sobolev space $W^{s,2}(\mathbb{R}^n)$ or $H^s(\mathbb{R}^n)$ for $s \in (0,1)$ is defined as:

$$H^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dx\, dy < \infty \right\}. \tag{1.16}$$

It is a Hilbert space with norm:

$$\|u\|^2_{H^s(\mathbb{R}^n)} := \|u\|^2_{L^2(\mathbb{R}^n)} + [u]^2_{H^s(\mathbb{R}^n)}, \tag{1.17}$$

where the Gagliardo seminorm is:

$$[u]^2_{H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dx\, dy. \tag{1.18}$$

For bounded domains $\Omega \subset \mathbb{R}^n$, the space $H^s(\Omega)$ is defined via restriction and completion of smooth functions.

For detailed definition and properties of the fractional Sobolev space $W^{s,2}(\mathbb{R}^n)$ see section 2 in [18].

1.2.3 Minimality

Let $\Omega \subset \mathbb{R}^n$ be open and let $u : \mathbb{R}^n \to \mathbb{R}^m$ belong to an admissible function space such as $H^s(\mathbb{R}^n; \mathbb{R}^m)$. We say that u is a (local) minimizer of the energy functional:

$$\mathcal{E}(u; \Omega) = \frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dx\, dy + \int_{\Omega} W(u(x)) dx, \tag{1.19}$$

for Ω open, bounded, Lipschitz, if for every $v \in H^s(\mathbb{R}^n; \mathbb{R}^m)$, with $u = v$ in Ω^c

$$\mathcal{E}(u; \Omega) \leq \mathcal{E}(v; \Omega). \tag{1.20}$$

These concepts serve as the foundation for the variational and analytical theory developed throughout this thesis.

1.2.4 Heuristic Justification: From Nonlocal to Local Energies

To understand the asymptotic relation between nonlocal and local energies as $s \to 1^-$, we consider the approximation:

$$\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} \approx |\nabla u(x)|^2 \cdot \frac{1}{|x-y|^{n+2s-2}}.$$

This heuristic arises by performing a Taylor expansion of u around the point x, assuming sufficient regularity and that $|x-y|$ is small:

$$u(y) = u(x) + \nabla u(x) \cdot (y-x) + \mathcal{O}(|x-y|^2),$$

which implies

$$|u(x)-u(y)|^2 \approx |\nabla u(x) \cdot (x-y)|^2 = |\nabla u(x)|^2 |x-y|^2 \cos^2\theta,$$

where θ denotes the angle between $\nabla u(x)$ and $x-y$. Substituting into the original integrand, we obtain:

$$\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} \approx |\nabla u(x)|^2 \cdot \frac{\cos^2\theta}{|x-y|^{n+2s-2}}.$$

Averaging over directions, the $\cos^2\theta$ term contributes a constant factor. This approximation motivates the convergence of the fractional seminorm to the classical Dirichlet energy as $s \to 1^-$; see Di Nezza et al. [18, Section 2] and Caffarelli [15, Section 6] for rigorous discussions.

Detailed Derivation of the Approximation

We justify the approximation:

$$\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} \approx \frac{|\nabla u(x)|^2}{1} \cdot \frac{1}{|x-y|^{n+2s-2}},$$

which heuristically connects nonlocal energies to local gradient energies. Assume u is smooth (say, C^2) and $|x-y|$ is small.

Step 1: Taylor Expansion. Expand u at point x:

$$u(y) = u(x) + \nabla u(x) \cdot (y - x) + \frac{1}{2}(y - x)^T D^2 u(\xi)(y - x),$$

for some ξ on the line segment between x and y. Then:

$$u(y) - u(x) = \nabla u(x) \cdot (y - x) + \mathcal{O}(|x - y|^2).$$

Step 2: Square and Estimate. Square the difference:

$$|u(y) - u(x)|^2 = |\nabla u(x) \cdot (y - x)|^2 + \mathcal{O}(|x - y|^3).$$

We use the identity:

$$|\nabla u(x) \cdot (y - x)|^2 = |\nabla u(x)|^2 \cdot |x - y|^2 \cdot \cos^2 \theta,$$

where θ is the angle between $\nabla u(x)$ and $y - x$.

Hence,

$$\frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \approx \frac{|\nabla u(x)|^2 \cdot |x - y|^2 \cos^2 \theta}{|x - y|^{n+2s}} = |\nabla u(x)|^2 \cdot \frac{\cos^2 \theta}{|x - y|^{n+2s-2}}.$$

Step 3: Angular Averaging. To remove the angular dependence, we integrate over directions. From symmetry:

$$\int_{\mathbb{S}^{n-1}} \cos^2 \theta \, d\sigma(\omega) = \frac{1}{n} |\mathbb{S}^{n-1}|,$$

so on average,

$$\cos^2 \theta \mapsto \frac{1}{n}.$$

Conclusion. Therefore, in a neighborhood of x, the integrand in the Gagliardo seminorm:

$$\int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dy$$

can be locally approximated by:

$$|\nabla u(x)|^2 \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n+2s-2}} \, dy,$$

justifying the formal approximation up to a multiplicative constant.

Averaging Over Directions

To remove directional dependence, we average over all directions $\omega \in \mathbb{S}^{n-1}$. Let $v = \frac{\nabla u(x)}{|\nabla u(x)|}$, and write $y = x + r\omega$. Then:

$$\cos^2\theta = (v \cdot \omega)^2.$$

By rotational symmetry of the unit sphere:

$$\int_{\mathbb{S}^{n-1}} (v\cdot\omega)^2 \, d\sigma(\omega) = \int_{\mathbb{S}^{n-1}} \omega_1^2 \, d\sigma(\omega) = \cdots = \int_{\mathbb{S}^{n-1}} \omega_n^2 \, d\sigma(\omega).$$

Since $|\omega|^2 = \sum_{i=1}^n \omega_i^2 = 1$, we obtain:

$$\sum_{i=1}^n \int_{\mathbb{S}^{n-1}} \omega_i^2 \, d\sigma(\omega) = |\mathbb{S}^{n-1}| \Rightarrow \int_{\mathbb{S}^{n-1}} \omega_1^2 \, d\sigma(\omega) = \frac{1}{n}|\mathbb{S}^{n-1}|.$$

Thus,

$$\int_{\mathbb{S}^{n-1}} \cos^2\theta \, d\sigma(\omega) = \frac{1}{n}|\mathbb{S}^{n-1}|,$$

showing that $\cos^2\theta$ averages to a constant over directions. Hence, the approximation

$$\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} \approx \frac{|\nabla u(x)|^2}{|x-y|^{n+2s-2}}$$

holds up to a dimension-dependent constant, justifying the connection between nonlocal Gagliardo seminorms and local Dirichlet energies.

Why $\cos^2\theta$ Averages to a Constant

We elaborate on why the angular factor $\cos^2\theta$ contributes only a constant when averaged over directions on the sphere $\mathbb{S}^{n-1}$. Recall:

$$\cos^2\theta = \left(\frac{\nabla u(x)}{|\nabla u(x)|} \cdot \omega\right)^2,$$

where $\omega = \frac{y-x}{|y-x|}$ is a direction vector on the unit sphere, and θ is the angle between $\nabla u(x)$ and ω. To average $\cos^2\theta$ over all directions, we compute:

$$\int_{\mathbb{S}^{n-1}} \cos^2\theta \, d\sigma(\omega) = \int_{\mathbb{S}^{n-1}} (v\cdot\omega)^2 \, d\sigma(\omega),$$

where $v = \frac{\nabla u(x)}{|\nabla u(x)|}$ is a fixed unit vector.

By rotational symmetry of the sphere, the above integral is invariant under orthogonal transformations. Hence, the average must be the same for any direction v, and we may assume $v = e_1 = (1, 0, \dots, 0)$. Thus,

$$\int_{\mathbb{S}^{n-1}} (v \cdot \omega)^2 \, d\sigma(\omega) = \int_{\mathbb{S}^{n-1}} \omega_1^2 \, d\sigma(\omega).$$

Since $|\omega|^2 = 1$, we have $\sum_{i=1}^n \omega_i^2 = 1$, and by symmetry:

$$\int_{\mathbb{S}^{n-1}} \omega_1^2 \, d\sigma(\omega) = \int_{\mathbb{S}^{n-1}} \omega_2^2 \, d\sigma(\omega) = \cdots = \int_{\mathbb{S}^{n-1}} \omega_n^2 \, d\sigma(\omega).$$

Summing, we get:

$$\sum_{i=1}^{n} \int_{\mathbb{S}^{n-1}} \omega_i^2 \, d\sigma(\omega) = \int_{\mathbb{S}^{n-1}} \sum_{i=1}^{n} \omega_i^2 \, d\sigma(\omega) = \int_{\mathbb{S}^{n-1}} 1 \, d\sigma(\omega) = |\mathbb{S}^{n-1}|.$$

Therefore,

$$\int_{\mathbb{S}^{n-1}} \omega_1^2 \, d\sigma(\omega) = \frac{1}{n} |\mathbb{S}^{n-1}|,$$

so the average of $\cos^2 \theta$ over directions is a constant depending only on the dimension n.

This justifies why we can replace $\cos^2 \theta$ by a constant in the approximation of nonlocal energies by local gradients.

1.2.5 Definition: Semigroups and Their Role in Evolution Equations

A **strongly continuous semigroup** (also called a C_0-semigroup) on a Banach space X is a family of bounded linear operators $\{T(t)\}_{t \geq 0} \subset \mathcal{L}(X)$ satisfying:

- **Semigroup property:** $T(0) = I$ (the identity operator), and $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$.

- **Strong continuity:** For every $x \in X$, the map $t \mapsto T(t)x$ is continuous from $[0, \infty)$ to X.

Semigroups arise naturally in the study of evolution equations of the form

$$\frac{du}{dt} = Au, \quad u(0) = u_0,$$

where A is a (possibly unbounded) linear operator on X. When A generates a C_0-semigroup $\{T(t)\}$, the unique mild solution is given by

$$u(t) = T(t)u_0.$$

This framework provides a powerful analytic tool in the theory of linear partial differential equations, particularly parabolic problems like the heat equation. It also underpins probabilistic representations such as the Feynman–Kac formula. For instance, given a self-adjoint operator A generating a semigroup and a potential term $V(x)$, the perturbed operator $A + V(x)$ may still define a well-behaved evolution via semigroup theory, with probabilistic interpretations through stochastic processes.

For a detailed treatment of semigroup theory and its applications to PDEs and stochastic analysis, see [32] Chapters 1,4 and 5 or [7] Chapter 4.

1.2.6 Feynman–Kac Modeling and Applications

The Feynman–Kac formula establishes a deep connection between partial differential equations (PDEs), stochastic processes, and variational modeling. It provides a probabilistic representation of solutions to a class of linear parabolic PDEs, transforming deterministic problems into expectations over stochastic trajectories. This connection offers both theoretical insights and practical computational methods, particularly in physics, finance, and mathematical biology.

Mathematical Structure

Let $\{X_t\}_{t\geq 0}$ be a stochastic process with generator $\mathcal{L}$, for example, Brownian motion with $\mathcal{L} = \frac{1}{2}\Delta$. Then, the solution $u(x,t)$ to the parabolic PDE

$$\frac{\partial u}{\partial t} = \mathcal{L}u + V(x)u, \quad u(x,0) = u_0(x),$$

admits the representation

$$u(x,t) = \mathbb{E}_x \left[e^{\int_0^t V(X_s)\,ds} \cdot u_0(X_t) \right],$$

where $\mathbb{E}_x$ denotes the expectation over paths of X_t starting at $X_0 = x$.

This formula highlights the role of semigroups in stochastic analysis, with $u(x,t) = T_t u_0(x)$ where T_t is the Feynman–Kac semigroup associated to the operator $\mathcal{L}+V$. It also plays a central role in infinite-dimensional stochastic models, such as those used in financial mathematics and quantum field theory.

For detailed expositions and applications of the Feynman–Kac formula, see [23] Chapter 5 and [41].

Modeling Philosophy

In modeling terms, this representation allows one to interpret deterministic PDE evolution as the averaged behavior of a particle undergoing random motion (diffusion or jumps) while accumulating energy or cost dictated by the potential $V(x)$. This leads to a probabilistic understanding of:

- **Diffusion**: Governed by the operator $\mathcal{L}$ (e.g., local Laplacian or nonlocal fractional Laplacian).
- **Energy Landscape**: Encoded via the potential $V(x)$.
- **Terminal Reward**: Given by $u_0(X_t)$, interpreted as a payoff or boundary value.

Applications

- **Phase Transitions**: In Allen–Cahn-type equations and diffuse interface models, Feynman–Kac captures the stochastic dynamics of interfaces under potential-driven forces and noise.
- **Fractional PDEs**: When $\mathcal{L} = -(-\Delta)^s$, the associated process is a symmetric $2s$-stable Lévy process. This is crucial in modeling long-range interactions and anomalous diffusion.

- **Finance**: Used extensively in derivative pricing, such as the Black–Scholes model, where asset prices follow geometric Brownian motion.
- **Statistical Mechanics and Quantum Theory**: The path integral formulation for ground states, partition functions, and quantum observables often relies on Feynman–Kac structures.

Connection to Variational Methods

Although derived for linear equations, the Feynman–Kac structure is philosophically aligned with variational modeling: it emphasizes paths that minimize accumulated cost (akin to action minimization), provides Gibbs-type weightings, and connects to large deviations and metastability theory. It also offers a natural route to stochastic simulation methods, such as Monte Carlo techniques, for solving PDEs and exploring energy landscapes.

1.3 Novel Contributions

To the best of the author's knowledge, this is the first work to establish a full density estimate for vector-valued minimizers in the fractional energy setting. While the scalar case is well understood, the vectorial regime presents significant challenges due to the absence of a maximum principle and the geometric complexity of the set $W = 0$. A central novelty of this thesis lies in the adaptation and application of a maximum principle for nonlocal vector-valued systems (see also Alikakos work in [3], Chapter 4), which provides a crucial replacement for scalar tools and plays a key role in the analysis of interfaces.

Technically, the approach builds on the polar decomposition method developed in the local setting (see [2], Section 2.2), extending it to the nonlocal case where angular and radial components must be carefully disentangled due to long-range interactions. Moreover, a barrier construction—adapted from the scalar case in Savin and Valdinoci Lemma 3.1 in [38]—is incorporated into the vectorial framework to overcome the lack of comparison principles. These tools, in combination, enable the derivation of quantitative density estimates and offer new perspectives for analyzing nonlocal systems with multiple competing phases.

The structure of the thesis is as follows. Chapter 2 presents the essential preliminary material and foundational estimates, including the polar form of the energy and upper bounds that will be used throughout the analysis. Chapter 3 is devoted to establishing a maximum principle for the system, which plays a critical role in understanding qualitative properties of solutions. In Chapter 4, we develop refined density estimates, which are central to the study of interfaces and regularity phenomena in nonlocal variational problems. Each chapter builds on the previous one while remaining as self-contained as possible, aiming to balance technical depth with conceptual insight.

Chapter 2

Basic Facts

In this chapter, we introduce the polar decomposition of a vector-valued function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, following the approach developed in the local setting (see [2], Section 2.2). This decomposition plays a central role throughout the thesis, serving as a foundational tool in the analysis and facilitating the proofs of nearly all major results in this and subsequent chapters. We conclude the chapter by establishing the first significant estimate: an upper bound for the nonlocal energy functional

$$J(u;\Omega) = \frac{1}{2}\iint_{\mathbb{R}^{2n}\setminus(\Omega^c\times\Omega^c)} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega W(u(x))\,dx, \qquad s \in (0,1), \tag{2.1}$$

in analogy with the classical (local) case.

2.1 The polar form representation and its properties

The polar form for vector valued functions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, is defined as

$$u(x) = a + |u(x) - a|\mathbf{n}(x),$$

$$q^u(x) = |u(x) - a|,$$

$$\mathbf{n}(x) = \begin{cases} \frac{u(x)-a}{|u(x)-a|}, & \text{if} \quad u(x) \neq a, \\ \\ 0, & \text{if} \quad u(x) = a, \end{cases}$$

where $q^u : \Omega \subset \mathbb{R}^n \to \mathbb{R}_+$ is the polar part and $\mathbf{n} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is the angular part of u.

The polar form decomposition of the function u allows for a convenient decomposition of the energy functional J_Ω into a radial and a polar part.

Lemma 2.1.1. *The nonlocal energy functional J_Ω can be decomposed as*

$$J(u, \Omega) = J^\varrho(u, \Omega)(u) + J^\alpha(u, \Omega)(u) + J^p(u, \Omega)(u),$$

where $J^\varrho(u, \Omega)$ and $J^\alpha(u, \Omega)$ denote the radial and angular parts of the kinetic energy, respectively, and $J^p(u, \Omega)$ is the potential energy, namely

$$\begin{aligned} J^\varrho(u, \Omega) &:= \frac{1}{2} \iint_{\mathbb{R}^{2n} \setminus (\Omega^c \times \Omega^c)} \frac{\big(q^u(x) - q^u(y)\big)^2}{|x - y|^{n+2s}} \, dx \, dy \\ &= \frac{1}{2} \, q^u(\Omega, \Omega) + q^u(\Omega, \Omega^c), \\ J^\alpha(u, \Omega) &:= \iint_{\mathbb{R}^{2n} \setminus (\Omega^c \times \Omega^c)} \frac{q^u(x) q^u(y) \left(1 - n^u(x) \cdot n^u(y)\right)}{|x - y|^{n+2s}} \, dx \, dy, \\ J^p(u, \Omega) &:= \int_\Omega W(u) \, dx. \end{aligned}$$

Note that $J^\alpha(u, \Omega) \geq 0$ on account of the Cauchy-Schwartz inequality.

Proof. For any $x, y \in \mathbb{R}^{2n} \setminus (\Omega^c \times \Omega^c)$, it holds that

$$\begin{aligned}
|u(x)-u(y)|^2 &= |(u(x)-a)-(u(y)-a)|^2 \\
&= \big[(u(x)-a)-(u(y)-a)\big]\cdot\big[(u(x)-a)-(u(y)-a)\big] \\
&= |u(x)-a|^2+|u(y)-a|^2-2\,(u(x)-a)\cdot(u(y)-a) \\
&= (q^u(x))^2+(q^u(y))^2-2\,q^u(x)q^u(y)\,n^u(x)\cdot n^u(y) \\
&= (q^u(x)-q^u(y))^2+2\,q^u(x)q^u(y)\big(1-n^u(x)\cdot n^u(y)\big).
\end{aligned}$$

and the result follows by integration. □

The following lemma allows one to obtain appropriate modifications of a vector field which may be useful for comparison purposes.

Lemma 2.1.2 (Polar form)**.** *Let $u \in W^{s,p}(A;\mathbb{R}^m)$ and consider $f:\mathbb{R}\to\mathbb{R}_+$ of the form $f(s)=sg(s)$, where $g:\mathbb{R}\to[0,1]$ is a Lipschitz function*[1] *(the case $g(s)=1$ included).*

Define $\bar{u}(x) := a+(f\circ\rho)(x)\mathbf{n}(x) = a+\bar{\rho}(x)$, where we use the notation $\bar{\rho}(x)=f(\rho(x))$. Then, for every x, y it holds that

$$|\bar{u}(x)-\bar{u}(y)|^2 \geq |\bar{\rho}(x)-\bar{\rho}(y)|^2. \tag{2.2}$$

Moreover, for every x, y it holds that

$$|\bar{u}(x)-\bar{u}(y)|^2 \leq |u(x)-u(y)|^2. \tag{2.3}$$

The above imply that for any suitable $A, B \subset \mathbb{R}^n$ it holds that

$$I_2(\bar{u};A,B) \leq I_2(u;A,B). \tag{2.4}$$

where, in this proof we use the notation

$$I_p(w;A,B) := \int_A\int_B \frac{|w(x)-w(y)|^p}{|x-y|^{n+sp}}dxdy,$$

for $w \in W^{s,p}(A;\mathbb{R}^m)$

[1] f defined like that is a Lipschitz function of Lipschitz constant $Lip(f) \leq 1$.

Proof. We note the following identity

$$\begin{aligned} |\bar{u}(x)-\bar{u}(y)|^2 &= [(\bar{u}(x)-a)-(\bar{u}(y)-a]\cdot[(\bar{u}(x)-a)-(\bar{u}(y)-a)] \\ &= |\bar{u}(x)-a|^2+|\bar{u}(y)-a|^2-2(\bar{u}(x)-a)\cdot(\bar{u}(y)-a) \\ &= (\bar{\rho}(x)^2-\bar{\rho}(y))^2-2\bar{\rho}(x)\bar{\rho}(y)n(x)\cdot n(y) \\ &= (\bar{\rho}(x)-\bar{\rho}(y))^2+2\bar{\rho}(x)\bar{\rho}(y)[1-n(x)\cdot n(y)]. \end{aligned} \tag{2.5}$$

Then, we immediately deduce from (2.5) that

$$|\bar{u}(x)-\bar{u}(y)|^2 \geq |\bar{\rho}(x)-\bar{\rho}(y)|^2$$

which is (2.2). Moreover, (using the notation $\bar{\rho}(x) = f(\rho(x))$) we have from (2.5) that

$$|\bar{u}(x)-\bar{u}(y)|^2 = |f(\rho(x))-f(\rho(y))|^2+2f(\rho(x))f(\rho(y))[1-n(x)\cdot n(y)]$$

$$\leq |\rho(x)-\rho(y)|^2+2\rho(x)\rho(y)[1-n(x)\cdot n(y)] = |u(x)-u(y)|^2$$

with the inequality on the second line coming from the fact that f is a contraction (the Lipschitz constant ≤ 1).

Finally from the above calculation

$$|\bar{u}(x)-\bar{u}(y)|^2 \leq |u(x)-u(y)|^2,$$

and by integrating both sides we get

$$\int_A\int_B \frac{|\bar{u}(x)-\bar{u}(y)|^2}{|x-y|^{n+2s}}dxdy \leq \int_A\int_B \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy,$$

and so,

$$I_2(\bar{u};A,B) \leq I_2(u;A,B).$$

This proves (2.3). □

Lemma 2.1.3 (Minimality of $J^\varrho(u,\Omega)+J^p(u,\Omega)$)**.** *Let $u = a+q^un^u$ be a minimizer and let $v = a+q^vn^u$ be such that $q^u \leq q^v$ with $q^u = q^v$ on Ω^c. Then,*

$$J^\varrho(u,\Omega)+J^p(u,\Omega) \leq J^\varrho(v,\Omega)+J^p(v,\Omega).$$

Proof. Note that v is well defined since $q^v(x) = 0$ when $q^u(x) = 0$. Actually v is in the same Sobolev class as u.

By minimality of u,

$$J(u, \Omega) \leq J(v, \Omega).$$

Using the decomposition of Lemma 2.1.1 we have that

$$\begin{aligned}
&\frac{1}{2}q^u(\Omega, \Omega) + q^u(\Omega, \Omega^c) + J^p(u, \Omega) + \int\int_{\mathbb{R}^{2n}\setminus(\Omega^c\times\Omega^c)} \frac{(q^u(x)q^u(y))(1 - n^u(x)\cdot n^u(y))}{|x-y|^{n+2s}}dxdy \\
\leq &\frac{1}{2}q^v(\Omega, \Omega) + q^u(\Omega, \Omega^c) + J^p(v, \Omega) + \int\int_{\mathbb{R}^{2n}\setminus(\Omega^c\times\Omega^c)} \frac{(q^v(x)q^v(y))(1 - n^u(x)\cdot n^u(y))}{|x-y|^{n+2s}}dxdy,
\end{aligned}$$

which is rearranged as

$$\begin{aligned}
&\frac{1}{2}q^u(\Omega, \Omega) + q^u(\Omega, \Omega^c) + J^p(u, \Omega) - \left(\frac{1}{2}q^v(\Omega, \Omega) + q^u(\Omega, \Omega^c) + J^p(v, \Omega)\right) \\
&\leq \int\int_{\mathbb{R}^{2n}\setminus(\Omega^c\times\Omega^c)} \frac{(q^v(x)q^v(y) - q^u(x)q^u(y))(1 - n^u(x)\cdot n^u(y))}{|x-y|^{n+2s}}dxdy \leq 0,
\end{aligned}$$

since $q^v \leq q^u$ and $1 - n^u(x)\cdot n^u(y)) \geq 0$. Recalling the definition of $J^\varrho(u, \Omega)$ and $J^\varrho(v, \Omega)$ the result follows. □

2.2 The upper bound of the energy

In this section, we estimate an upper bound for the energy functional by applying the polar representation of the minimizer u. We begin by highlighting the analogue of this result in the classical local scalar case, specifically Lemma 1 in [13]. The extension to the local vector-valued setting is addressed in [2, Lemma 5.1], where a suitable adaptation is made using polar decomposition. Our goal is to carry out a similar strategy in the fractional (nonlocal) setting for general $s \in (0, 1)$. We recall the local vector case:

Theorem 2.2.1. *For the local system*

$$\Delta u - W_u(u) = 0$$

let u be a minimal solution of the system in $B_R(x_0)$. Under the same assumptions as in our case, for the potential W, and furthermore

$$|u(x) - \alpha| < 0,$$
$$|\nabla u(x)| < M$$

it holds

$$J_{B_R(x_0)}(u) \leq CR^{n-1}$$

for a constant C that depends on W and M (the upper bound of the minimizer $u(x)$) and for $R > 0$.

Now, we prove the analog theorem for our case:

Theorem 2.2.2 (Upper bound)**.** *Let u be a minimizer in B_{R+2}. Then*

$$\begin{aligned} J^{\varrho}(u, B_R) + J^{p}(u, B_R) &= \frac{1}{2} q^u(B_R, B_R) + q^u(B_R, B_R^c) + \int_{B_R} W(u)dx \\ &\leq \Psi(R) = \begin{cases} CR^{n-1} & \text{if} \quad s \in (1/2, 1) \\ CR^{n-1} \ln R & \text{if} \quad s = 1/2 \\ CR^{n-2s} & \text{if} \quad s \in (0, 1/2) \end{cases} \end{aligned}$$

for an appropriate constant C independent of R.

Remark 2.2.3. We observe that the case $1/2 < s < 1$, yields the same result as in the classical case. This is not surprising, while $s \to 1$ we are expecting that the operators $(-\Delta)^s()$ and $\Delta()$ exhibit similar properties

Remark 2.2.4. In particular the fundamental estimate along with the positivity of the potential yields the bounds $\int_{B_R \cap \{|u-a|>c\}} W(u)dx < \Psi(R)$ and $\int_{B_R \cap \{|u-a|<c\}} W(u)dx < \Psi(R)$.

Proof. We break the proof in 7 steps.

Step 1. Express the minimizer u in polar form as $u = a + q^u n^u$. Let $\|u\|_{L^\infty(B_{R+2})} \leq M$, and construct the following comparison functions

$$q^\psi(x) = M \ \min\{(|x| - R - 1)_+, 1\}$$
$$\psi = a + q^\psi n^u,$$
$$v = a + q^v n^u, \ \ q^v = \min\{q^\psi, q^u\}.$$

Clearly by construction $q^\psi = 0$ in B_{R+1}, so that $\psi = a$ in B_{R+1}, while $v = u$ in B^c_{R+2}, so that v and u coincide on the complement of B_{R+2}. Hence since u is a minimizer in B_{R+2} we have that

$$J(u, B_{R+2}) \leq J(v, B_{R+2})$$

Moreover, by Lemma 2.1.3 it holds that

$$J^\varrho(u, B_{R+2}) + J^p(u, B_{R+2}) \leq J^\varrho(v, B_{R+2}) + J^p(v, B_{R+2}). \tag{2.6}$$

Step 2. We claim that we may estimate $J^\varrho(u, B_R) + J^p(u, B_R)$ in terms of

$$J^\varrho(u, B_R) + J^p(u, B_R) \leq J^\varrho(\psi, B_{R+2}) + J^p(\psi, B_{R+2}) + q^u(B_R, B^c_{R+1}). \tag{2.7}$$

All 3 terms on the RHS of the above inequality can be easily estimated, the first two do not involve the minimizer and are only expressed in terms of the known test function ψ, while the third one concerns the radial kinetic energy over two disconnected sets so that it can be easily controlled by the L^∞ a priori bounds of the minimizer.

To show claim (2.7) we work as follows:

$$
\begin{aligned}
J^{\varrho}(u,B_R)+J^{p}(u,B_R) &= \frac{1}{2}q^u(B_R,B_R)+q^u(B_R,B_R^c)+\int_{B_R}W(u)dx\\
&= \frac{1}{2}q^u(B_R,B_R)+q^u(B_R,B_{R+1}\setminus B_R)\\
&\quad +q^u(B_R,B_{R+1}^c)+\int_{B_R}W(u)dx\\
\le \frac{1}{2}q^u(B_R,B_R)+q^u(B_R,B_{R+1}\setminus B_R)&+\frac{1}{2}q^u(B_{R+1}\setminus B_R,B_{R+1}\setminus B_R)\\
&\quad +q^u(B_R,B_{R+1}^c)+\int_{B_R}W(u)dx\\
&= \frac{1}{2}q^u(B_{R+1},B_{R+1})+q^u(B_R,B_{R+1}^c)+\int_{B_R}W(u)dx\\
&\le \frac{1}{2}q^u(B_{R+1},B_{R+1})+\int_{B_{R+1}}W(u)dx+q^u(B_R,B_{R+1}^c).
\end{aligned}
\tag{2.8}
$$

We now claim that

$$
\begin{aligned}
\frac{1}{2}q^u(B_{R+1},B_{R+1})+\int_{B_{R+1}}W(u)dx &\le J^{\varrho}(\psi,B_{R+2})+J^{p}(\psi,B_{R+2})\\
&= \frac{1}{2}q^{\psi}(B_{R+2},B_{R+2})+q^{\psi}(B_{R+2},B_{R+2}^c)+\int_{B_{R+2}}W(\psi)dx.
\end{aligned}
\tag{2.9}
$$

Combining (2.8) with (2.9) will immediately lead to the estimate (2.7). So it suffices to show (2.9).

Step 3. Proof of (2.9). To show (2.9) we need to use the minimality of u in B_{R+2} and in particular (2.6). In order to estimate the LHS of (2.9), which involves the minimizer u in B_{R+1} by a quantity which involves only the test function ψ in B_{R+2}, we will define the set

$$
A=\{x\in B_{R+2}\ :\ q^u(x)=q^{\psi}(x)\},\quad B_{R+1}\subset A\subset B_{R+2}.
$$

By the definitions of $q^v=\min\{q^u,q^{\psi}\}$, and v, we have that

$$
q^v=\begin{cases} q^{\psi}, & \text{on} \quad A,\\ q^u & \text{on} \quad B_{R+2}\setminus A,\end{cases}
\quad \text{and} \quad
v=\begin{cases} \psi, & \text{on} \quad A,\\ u & \text{on} \quad B_{R+2}\setminus A.\end{cases}
\tag{2.10}
$$

Using the decomposition $B_{R+2} = A \cup (B_{R+2} \setminus A)$, we write

$$
\begin{aligned}
J^{\varrho}(u, B_{R+2}) + J^{p}(u, B_{R+2}) &= \frac{1}{2} q^{u}(B_{R+2}, B_{R+2}) + q^{u}(B_{R+2}, B^{c}_{R+2}) + \int_{B_{R+2}} W(u)dx \\
&= \frac{1}{2} q^{u}(A, A) + \underbrace{q^{u}(A, B_{R+2} \setminus A)} + \frac{1}{2} q^{u}(B_{R+2} \setminus A, B_{R+2} \setminus A) \\
+ \underbrace{q^{u}(A, B^{c}_{R+2})} &+ q^{u}(B_{R+2} \setminus A, B^{c}_{R+2}) + \int_{A} W(u)dx + \int_{B_{R+2} \setminus A} W(u)dx \\
&= \frac{1}{2} q^{u}(A, A) + q^{u}(A, A^{c}) + \frac{1}{2} q^{u}(B_{R+2} \setminus A, B_{R+2} \setminus A) \\
&+ q^{u}(B_{R+2} \setminus A, B^{c}_{R+2}) + \int_{A} W(u)dx + \int_{B_{R+2} \setminus A} W(u)dx.
\end{aligned}
\tag{2.11}
$$

We perform the same decomposition for v, which yields

$$
\begin{aligned}
& J^{\varrho}(v, B_{R+2}) + J^{p}(v, B_{R+2}) \\
&= \frac{1}{2} q^{v}(A, A) + q^{v}(A, A^{c}) + \frac{1}{2} q^{v}(B_{R+2} \setminus A, B_{R+2} \setminus A) + \\
& q^{v}(B_{R+2} \setminus A, B^{c}_{R+2}) + \int_{A} W(v)dx + \int_{B_{R+2} \setminus A} W(v)dx \\
&= \frac{1}{2} q^{\psi}(A, A) + q^{v}(A, A^{c}) + \frac{1}{2} q^{u}(B_{R+2} \setminus A, B_{R+2} \setminus A) \\
&+ q^{u}(B_{R+2} \setminus A, B^{c}_{R+2}) + \int_{A} W(\psi)dx + \int_{B_{R+2} \setminus A} W(u)dx,
\end{aligned}
\tag{2.12}
$$

where we also used (2.10).

Using (2.6) and (2.11)-(2.12) we see that

$$
\begin{aligned}
0 &\le \Big(J^{\varrho}(v, B_{R+2}) + J^{p}(v, B_{R+2}) \Big) - \Big(J^{\varrho}(u, B_{R+2}) + J^{p}(u, B_{R+2}) \Big) \\
&= \frac{1}{2} q^{\psi}(A, A) - \frac{1}{2} q^{u}(A, A) + \underbrace{q^{v}(A, A^{c})} - q^{u}(A, A^{c}) + \int_{A} (W(\psi) - W(u))dx,
\end{aligned}
\tag{2.13}
$$

which is almost the required inequality (2.9) except for the underbraced term, and the fact that the required quantities are on A rather than on B_{R+1}.

To control the term $q^{v}(A, A^{c})$, note that for any $x \in A$, $y \in A^{c}$,

$$
q^{v}(x) = q^{\psi}(x) \le q^{u}(x), \;\; q^{v}(y) = q^{u}(y) \le q^{\psi}(y),
$$

so that

$$q^v(x) - q^v(y) \le q^u(x) - q^u(y),$$
$$q^v(y) - q^v(x) \le q^\psi(y) - q^\psi(x),$$

which in turn implies

$$|q^v(x) - q^v(y)| \le \max\{|q^u(x) - q^u(y)|, |q^\psi(x) - q^\psi(y)|\}$$

and upon squaring (and fortifying the resulting inequality)

$$|q^v(x) - q^v(y)|^2 \le |q^u(x) - q^u(y)|^2 + |q^\psi(x) - q^\psi(y)|^2.$$

We divide by the singular kernel and integrate over all $x \in A$, $y \in A^c$ to obtain

$$q^v(A, A^c) \le q^u(A, A^c) + q^\psi(A, A^c). \tag{2.14}$$

Combining (2.13) with (2.14) we conclude that

$$\frac{1}{2}q^u(A, A) + \int_A W(u)dx \le \frac{1}{2}q^\psi(A, A) + q^\psi(A, A^c) + \int_A W(\psi)dx.$$

Since $B_{R+1} \subset A \subset B_{R+2}$ we easily see that

$$\begin{aligned} &\frac{1}{2}q^u(B_{R+1}, B_{R+1}) + \int_{B_{R+1}} W(u)dx \\ &\le \frac{1}{2}q^u(A, A) + \int_A W(u)dx \le \frac{1}{2}q^\psi(A, A) + q^\psi(A, A^c) + \int_A W(\psi)dx \\ &\le \frac{1}{2}q^\psi(B_{R+2}, B_{R+2}) + q^\psi(B_{R+1}, B^c_{R+2}) + \int_{B_{R+2}} W(\psi)dx, \end{aligned}$$

which is the required estimate (2.9).

Having established the estimate (2.7),

$$J^\varrho(u, B_R) + J^p(u, B_R) \le J^\varrho(\psi, B_{R+2}) + J^p(\psi, B_{R+2}) + q^u(B_R, B^c_{R+1}),$$

we strive to establish bounds for each of two terms $J^\varrho(\psi, B_{R+2}) + J^p(\psi, B_{R+2})$ and $q^u(B_R, B^c_{R+1})$ separately.

We will show that both terms can be estimated in terms of integrals of the function

$$d(x) = \max\{(R+1-|x|), 1\},$$

over the balls B_R and B_{R+2}.

Step 4. We start by estimating the last term and claim that

$$q^u(B_R, B^c_{R+1}) \le C \int_{B_R} d(x)^{-2s} dx. \tag{2.15}$$

Note that when $x \in B_R$ and $y \in B^c_{R+1}$, we have that

$$|z| = |x-y| \ge |y| - |x| \ge R+1-|x| = d(x). \tag{2.16}$$

Since u is bounded, so is q^u so that

$$\begin{aligned} q^u(B_R, B^c_{R+1}) = \int_{B_R}\int_{B^c_{R+1}} \frac{|q^u(x)-q^u(y)|^2}{|x-y|^{n+2s}} dxdy &\le C \int_{B_R}\int_{B^c_{R+1}} \frac{1}{|x-y|^{n+2s}} dxdy \\ &\overset{z=x-y,(2.16)}{=} C\int_{B_R} dx \int_{|z|=|x-y|\ge d(x)} |z|^{-(n+2s)} dz \\ &\overset{\text{polar coord.}}{=} C' \int_{B_R} dx \int_{d(x)}^{\infty} \rho^{-(n+2s)} \rho^{n-1} d\rho \\ &= C'' \int_{B_R} d(x)^{-2s} dx, \end{aligned}$$

which upon redefining the constant is (2.15).

Step 5. We now estimate $J^\varrho(\psi, B_{R+2})$ and show that

$$\begin{aligned} J^\varrho(\psi, B_{R+2}) \le \int_{B_{R+2}} d(x)^{-2s}(x) dx = \int_{B_R} d(x)^{-2s} dx + \int_{B_{R+2}\setminus B_R} d(x)^{-2s} dx \\ \le \int_{B_R} d(x)^{-2s} dx + CR^{n-1}. \end{aligned} \tag{2.17}$$

Indeed,

$$\begin{aligned} J^\varrho(\psi, B_{R+2}) = \frac{1}{2} q^\psi(B_{R+2}, B_{R+2}) + q^\psi(B_{R+2}, B^c_{R+2}) &\le q^\psi(B_{R+2}, B_{R+2}) + q^\psi(B_{R+2}, B^c_{R+2}) \\ = q^\psi(B_{R+2}, \mathbb{R}^n) &= \int_{B_{R+2}} \int_{\mathbb{R}^n} \frac{|q^\psi(x)-q^\psi(y)|^2}{|x-y|^{n+2s}} dxdy \\ = \int_{B_{R+2}} \left\{ \int_{|x-y|\le d(x)} \frac{|q^\psi(x)-q^\psi(y)|^2}{|x-y|^{n+2s}} dy + \int_{|x-y|\ge d(x)} \frac{|q^\psi(x)-q^\psi(y)|^2}{|x-y|^{n+2s}} dy \right\} \end{aligned} \tag{2.18}$$

We now estimate each term in the inner integral in (2.18) separately in terms of the function $d(x)$. The second inner integral is well behaved since $|x-y|$ is bounded away from zero. In fact, since q^ψ is bounded,

$$\begin{aligned}\int_{|x-y|\geq d(x)} \frac{|q^\psi(x)-q^\psi(y)|^2}{|x-y|^{n+2s}}dy &\leq C\int_{|x-y|\geq d(x)} \frac{1}{|x-y|^{n+2s}}dy \\ &= C'\int_{d(x)}^{\infty} \rho^{-n-2s}\rho^{n-1}d\rho = C''d(x)^{-2s}.\end{aligned} \tag{2.19}$$

The estimate of the first term in the inner integral in (2.18) is more delicate as $|x-y|$ may become 0. For that we need to obtain a Lipschitz like estimate for q^ψ for $|x-y|\leq d(x)$. We claim that

$$|q^\psi(x)-q^\psi(y)| \leq M\frac{|x-y|}{d(x)}, \text{ for } |x-y|\leq d(x). \tag{2.20}$$

This easy to see that if $|x|\leq R$ then

$$|y| = |x-y+x| \leq |x-y|+|x| \leq d(x)+|x| \leq R+1$$

so that $q^\psi(x)=q^\psi(y)=0$ and (2.20) holds trivially. If $|x|\geq R$, then $d(x)=1$ and (2.20) holds since q^ψ by construction is a Lipschitz function of Lipschitz constant M. Using (2.20) we have that

$$\begin{aligned}\int_{|x-y|\leq d(x)} \frac{|q^\psi(x)-q^\psi(y)|^2}{|x-y|^{n+2s}}dy &\leq \frac{C}{d(x)^2}\int_{|x-y|\leq d(x)} \frac{|x-y|^2}{|x-y|^{n+2s}}dy \\ &= \frac{C'}{d(x)^2}\int_0^{d(x)} \rho^{-n-2s+2}\rho^{n-1}d\rho = C''d(x)^{-2s}.\end{aligned} \tag{2.21}$$

Combining (2.19) and (2.21) with (2.18) we have (redefining constants)

$$J^\varrho(\psi, B_{R+2}) = \frac{1}{2}q^\psi(B_{R+2}, B_{R+2}) + q^\psi(B_{R+2}, B^c_{R+2}) \leq C\int_{B_{R+2}} d(x)^{-2s}dx, \tag{2.22}$$

which is the first inequality in (2.17). Since $d(x)=1$ in $B_{R+2}\setminus B_R$ we can see that

$$\int_{B_{R+2}} d(x)^{-2s}dx \leq |B_{R+2}\setminus B_R| \leq CR^{n-1},$$

and (2.17) follows.

Step 6 We can similarly estimate the potential term

$$J^p(\psi, B_{R+2}) = \int_{B_{R+2}} W(\psi)dx = \int_{B_{R+1}} W(\psi)dx + \int_{B_{R+2}\setminus B_{R+1}} W(\psi)dx$$
$$= \int_{B_{R+2}\setminus B_{R+1}} W(\psi)dx \leq C'|B_{R+2} \setminus B_{R+1}| \leq CR^{n-1}, \tag{2.23}$$

where we used the fact that in $B_{R+2} \setminus B_{R+1}$, $\psi = a$ and $W(a) = 0$.

Combining (2.17) with (2.23) we obtain the estimate

$$J^\varrho(\psi, B_{R+2}) + J^p(\psi, B_{R+2}) + q^u(B_R, B^c_{R+1}) \leq C \int_{B_R} d(x)^{-2s}dx + CR^{n-1},$$

for a suitable constant C, so that by the estimate (2.7) we have that

$$J^\varrho(u, B_R) + J^p(u, B_R) \leq C \int_{B_R} d(x)^{-2s}dx + CR^{n-1}. \tag{2.24}$$

Step 7. As the final step we estimate the integral $\int_{B_R} d(x)^{-2s}dx$.

By the definition of $d(x) = R + 1 - |x|$ in B_R we see that

$$\int_{B_R} d(x)^{-2s}dx = \int_{B_R} (R + 1 - |x|)^{-2s}dx = C \int_0^R (R + 1 - \rho)^{-2s}\rho^{n-1}d\rho$$
$$\overset{t=\rho/(R+1)}{=} C(R + 1)^{n-2s} \int_0^{1-\frac{1}{R+1}} t^{n-1}(1 - t)^{-2s}dt \tag{2.25}$$
$$\leq C(R + 1)^{n-2s} \int_0^{1-\frac{1}{R+1}} (1 - t)^{-2s}dt \leq (R + 1)^{n-2s}\Phi(R),$$

where

$$\Phi(R) := \begin{cases} C & \text{if} \quad s \in (0, \frac{1}{2}) \\ C \ln R & \text{if} \quad s = \frac{1}{2}, \\ CR^{2s-1} & \text{if} \quad s \in (\frac{1}{2}, 1) \end{cases}$$

We define $\Psi(R) = (R + 1)^{n-2s}\Phi(R)$.

Combining (2.24) with (2.25) and since for any $s \in (0, 1)$, it holds that $R^{n-1} \leq C'\Psi(R)$ for a suitable constant C' we have upon redefining constants that

$$J^\varrho(u, B_R) + J^p(u, B_R) \leq \Psi(R),$$

which is the required result. □

Chapter 3

A Maximum Principle

3.1 Introduction

Consider the nonlocal functional

$$J_A(u) = \frac{1}{4}\int\int_{\mathbb{R}^{2n}\setminus(A^c\times A^c)} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy + \int_A W(u(x))dx, \quad s\in(0,1), \tag{3.1}$$

among all functions $u : A \subset \mathbb{R}^n \to \mathbb{R}^m$, such that $u \in W^{s,2}(A;\mathbb{R}^m)$, with $u = g$ (given) on $\mathbb{R}^n\setminus A$, for a given open bounded set $A \subset \mathbb{R}^n$, where $|\cdot|$ denotes the Euclidean distance (in $\mathbb{R}^n$ or $\mathbb{R}^m$) and $W : \mathbb{R}^m \to \mathbb{R}$ is a continuous non negative potential with $a \in \{W = 0\}$, satisfying the hypothesis $\exists r_0 > 0, : \forall\, \xi \in \mathbb{R}^m, |\xi| = 1\ (0, r_0] \ni r \mapsto W(a + r\xi),$ is non decreasing, with $W(a + r_0\xi) > 0$.

We would like to allow solutions of

$$(-\Delta)^s u = -W_u(u), \tag{3.2}$$

which are defined in an open set $\mathcal{O} \subset \mathbb{R}^n$, generally bounded, with $\mathcal{O} = \mathbb{R}^n$ an option, and which are minimizers in the sense that they minimize, for each $A \subset \mathcal{O}$, open, bounded, Lipschitz, the energy J_A among all test functions v with $v = u$ on A^c:

Definition 3.1.1. The map $u \in W^{s,2}_{loc}(\mathcal{O};\mathbb{R}^m)$ is called a minimizer of the non-local functional J_A, in the sense of De Giorgi if $J_A(u) \leq J_A(v)$ for any $v = u$ on A^c, for all A open, bounded, Lipschitz with $A \subset \mathcal{O}$.

Clearly, if $\mathcal{O}$ is open, bounded, Lipschitz, then the notion of minimizer in the sense of De Giorgi coincides with the usual definition of minimizer. This concept is particularly useful for $\mathcal{O} = \mathbb{R}^n$, where it is often the case that the functional becomes infinite for the object of interest.

We recall that $(-\Delta)^s$ is the fractional Laplacian operator defined (up to a constant $C(n,s)$) by

$$[(-\Delta)^s u](x) = PV\left(\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}dy\right) = -\frac{1}{2}\int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}dy,$$

for $s \in (0,1)$.

We will show the following maximum principle for minimizers of the non-local functional J_A:

Theorem 3.1.2. *Let $A \subset \mathbb{R}^n$ be open bounded and with Lipschitz boundary, $v \in W^{s,2}(A;\mathbb{R}^m) \cap L^\infty(A;\mathbb{R}^m)$, $s > 1/2$, be a minimizer of J_A and $a \in \{W = 0\}$*
Assume that $|v(x) - a| \leq r$ on $\partial A \cup A^c$, for $0 < 2r \leq r_0$. Then,

(i) $|v(x) - a| \leq r$ on A.

(ii) If moreover, $u \mapsto W_u(u)$ is Lipschitz, and $|v(\hat{x}) - a| = r$ at an interior point $\hat{x} \in A$, then $v(x) = const$ in the connected component of $\hat{x}$ in A.

We note that this is a different from the usual maximum principle in major ways. First the usual maximum principle is a calculus fact and applies to all solutions, not just to minimizers, while the theorem above does not even apply to local minimizers as noted in [30].

3.2 Some useful lemmas

We recall the definition of the fractional Sobolev space $W^{s,p}(A;\mathbb{R})$, $s \in (0,1)$, consisting of functions $u : A \to \mathbb{R}$ such that the Gagliardo seminorm

$$[u]_{s,p} = \left(\int_A\int_A \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dxdy\right)^{1/2} < \infty,$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. This space can be turned into a Banach space with the norm $\|u\|_{W^{s,p}(A)} = (\|u\|^p_{L^p(A)} + [u]^p_{s,p})^{1/p}$, and has a Hilbert space structure if $p = 2$. For detailed information concerning fractional Sobolev spaces and their properties see e.g. [18]. The corresponding fractional Sobolev spaces for vector valued functions $u : \mathbb{R}^n \to \mathbb{R}^m$ can be defined similarly, replacing the absolute value in the numerator of the Gagliardo seminorm by the Euclidean norm in $\mathbb{R}^m$.

We will use the following simplified notation. For any $u \in W^{s,p}(A;\mathbb{R}^m) \cap W^{s,p}(B;\mathbb{R}^m)$, define

$$I_p(u; A, B) := \int_A \int_B \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dxdy,$$

where by $|\cdot|$ we denote the Euclidian distance on $\mathbb{R}^n$ or $\mathbb{R}^m$ depending on the context. In case $A = B$ this quantity coincides with the Gagliardo seminorm. We will focus here on the case $p = 2$.

3.2.1 The minimization problem and weak formulation of the nonlocal system

In this section we briefly recall some information concerning the minimization problem (4.2) and its connection with weak solutions of the nonlocal elliptic system (3.2).

We first note that by symmetry the functional J_A admits the equivalent form

$$J_A(u) = \frac{1}{4} \int_A \int_A \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dxdy + \frac{1}{2} \int_A \int_{A^c} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dxdy + \int_A W(u(x))dx.$$

The first two terms correspond to a nonlocal version of the kinetic energy, henceforth denoted by $J_\Omega^{(c)}$ whereas the last term corresponds to the potential energy, henceforth denoted by $J^\rho(u, \Omega)$.

Assume sufficient smoothness of the potential W. By perturbing u to $u + \epsilon v$, where v is any function in $W^{s,2}(A)$, vanishing on A^c we see that

$$\frac{1}{\epsilon}(J_A(u + \epsilon v) - J_A(u)) = \int_A \int_A \frac{(u(x) - u(y)) \cdot (v(x) - v(y))}{|x - y|^{n+2s}} dxdy +$$
$$2 \int_A \int_{A^c} \frac{(u(x) - u(y)) \cdot (v(x) - v(y))}{|x - y|^{n+2s}} dxdy + \int_A W_u(u(x))v(x)dx + O(\epsilon),$$

which by the properties of v becomes

$$\frac{1}{\varepsilon}\big(J_A(u+\varepsilon v) - J_A(u)\big) = \frac{1}{2}\iint_{A\times A} \frac{(u(x)-u(y))\cdot(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy + \int_A W_u(u(x))\cdot v(x)\,dx + O(\varepsilon).$$

Since u is a minimum, taking the above in the limit as $\epsilon \to 0$ for any arbitrary v in $W^{s,2}(A;\mathbb{R}^m)$, vanishing on A^c , we conclude that a minimizer is a solution of

$$\frac{1}{2}\int_A\int_A \frac{(u(x)-u(y))\cdot(v(x)-v(y))}{|x-y|^{n+2s}}dxdy + \int_A W_u(u(x))v(x)dx = 0, \quad \forall\, v \in W^{s,2}(A;\mathbb{R}^m),\ v = 0 \text{ on } A^c,$$

which is the weak form of (3.2).

A more detailed derivation of the above, follows:

Lemma 3.2.1 (First variation with $C^{1,1}$ potential)**.** *Let $A \subset \mathbb{R}^n$ be open, $s \in (0,1)$, and for $u : \mathbb{R}^n \to \mathbb{R}^m$ set*

$$J_A(u) := \frac{1}{4}\iint_{A\times A} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \frac{1}{2}\iint_{A\times A^c} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_A W(u(x))\,dx.$$

Fix $u \in W^{s,2}(\mathbb{R}^n;\mathbb{R}^m)$ and $v \in H_0^s(A;\mathbb{R}^m)$ (so $v = 0$ on A^c), and set $u_\varepsilon := u + \varepsilon v$. Assume $W \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^m)$ on a neighborhood of the segment $\{u + t\varepsilon v :\ t \in [0,1]\}$. Then for every ε,

$$\frac{J_A(u_\varepsilon) - J_A(u)}{\varepsilon} = \frac{1}{2}\iint_{A\times A} \frac{(u(x)-u(y))\cdot\big(v(x)-v(y)\big)}{|x-y|^{n+2s}}\,dx\,dy + \iint_{A\times A^c} \frac{(u(x)-u(y))\cdot v(x)}{|x-y|^{n+2s}}\,dx\,dy + \int_A W_u(u(x))\cdot v(x)\,dx \ + \ R_\varepsilon, \tag{3.3}$$

where

$$|R_\varepsilon| \le C\,\varepsilon\Big(\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \|v\|^2_{L^2(A)}\Big), \tag{3.4}$$

with $C > 0$ depending only on n, s and on a local Lipschitz constant of ∇W along the above segment. In particular, letting $\varepsilon \to 0$ in (3.3) yields the first variation formula.

Moreover, since $v = 0$ on A^c, one may rewrite the kinetic contribution in whole-space form:

$$\frac{1}{2}\iint_{A\times A} \cdots \ + \ \iint_{A\times A^c} \cdots = \frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x)-u(y))\cdot\big(v(x)-v(y)\big)}{|x-y|^{n+2s}}\,dx\,dy,$$

and therefore

$$\lim_{\varepsilon\to 0}\frac{J_A(u+\varepsilon v)-J_A(u)}{\varepsilon}=\frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{(u(x)-u(y))\cdot\big(v(x)-v(y)\big)}{|x-y|^{n+2s}}\,dx\,dy+\int_A W_u(u)\cdot v\,dx.$$

Proof. For the kinetic part over $A\times A$, expand the square:

$$\frac{1}{\varepsilon}\cdot\frac{1}{4}\iint_{A\times A}\big(|u_\varepsilon(x)-u_\varepsilon(y)|^2-|u(x)-u(y)|^2\big)\frac{dx\,dy}{|x-y|^{n+2s}}=\frac{1}{2}\iint_{A\times A}\frac{(u(x)-u(y))\cdot\big(v(x)-v(y)\big)}{|x-y|^{n+2s}}\,dx\,dy+\frac{\varepsilon}{4}\iint_{A\times A}\frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy.$$

For the kinetic part over $A\times A^c$, since $v(y)=0$ for $y\in A^c$,

$$\frac{1}{\varepsilon}\cdot\frac{1}{2}\iint_{A\times A^c}\big(|u_\varepsilon(x)-u_\varepsilon(y)|^2-|u(x)-u(y)|^2\big)\frac{dx\,dy}{|x-y|^{n+2s}}=\iint_{A\times A^c}\frac{(u(x)-u(y))\cdot v(x)}{|x-y|^{n+2s}}\,dx\,dy+\frac{\varepsilon}{2}\iint_{A\times A^c}\frac{|v(x)|^2}{|x-y|^{n+2s}}\,dx\,dy.$$

For the potential, Taylor's theorem (mean value form) gives

$$W(u+\varepsilon v)=W(u)+\varepsilon\,W_u(u)\cdot v+\frac{\varepsilon^2}{2}\,v^\top W_{uu}\big(u+\theta\varepsilon v\big)v,\qquad \theta=\theta(x,\varepsilon)\in(0,1),$$

hence

$$\frac{1}{\varepsilon}\int_A\big(W(u+\varepsilon v)-W(u)\big)\,dx=\int_A W_u(u)\cdot v\,dx+\frac{\varepsilon}{2}\int_A v^\top W_{uu}\big(u+\theta\varepsilon v\big)v\,dx.$$

If $\|W_{uu}\|\le L$ along the segment $\{u+t\varepsilon v:\ t\in[0,1]\}$, then

$$\left|\frac{\varepsilon}{2}\int_A v^\top W_{uu}(\cdot)\,v\,dx\right|\le\frac{\varepsilon L}{2}\,\|v\|^2_{L^2(A)}.$$

Collecting the three contributions yields (3.3) with

$$|R_\varepsilon|\le\frac{\varepsilon}{4}\iint_{A\times A}\frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy+\frac{\varepsilon}{2}\iint_{A\times A^c}\frac{|v(x)|^2}{|x-y|^{n+2s}}\,dx\,dy+\frac{\varepsilon L}{2}\,\|v\|^2_{L^2(A)},$$

which implies (3.4) (using the zero extension of v to identify the full H^s-seminorm). Letting $\varepsilon\to 0$ gives the first variation formula. The whole-space rewriting follows from symmetry of the kernel and $v=0$ on A^c. □

Remark 3.2.2. Note that when u is regular enough then

$$\frac{1}{2}\int_A\int_A \frac{(u(x)-u(y))\cdot(v(x)-v(y))}{|x-y|^{n+2s}}dxdy = \frac{1}{2}PV\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(u(x)-u(y))\cdot v(x)}{|x-y|^{n+2s}}dxdy+$$
$$\frac{1}{2}PV\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(u(y)-u(x))\cdot v(y)}{|x-y|^{n+2s}}dxdy = \frac{1}{2}\int_{\mathbb{R}^n}PV\left(\int_{\mathbb{R}^n}\frac{(u(y)-u(x))}{|x-y|^{n+2s}}dy\right)\cdot v(x)dx+$$
$$\frac{1}{2}\int_{\mathbb{R}^n}PV\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(u(x)-u(y))}{|x-y|^{n+2s}}dx\right)\cdot v(y)dy = \frac{1}{2}\int_{\mathbb{R}^n}(-\Delta)^s u(x)v(x)dx+$$
$$\frac{1}{2}\int_{\mathbb{R}^n}(-\Delta)^s u(y)v(y)dy = \int_{\mathbb{R}^n}(-\Delta)^s u(x)\cdot v(x)dx,$$

where we used the symmetry of the kernel. This brings the weak form (3.6) into a more habitual form used for local operators (see e.g. [34] or [30]).

Up to a multiplication of the kinetic energy by an appropriate constant, a suitable redefinition of the scaling constant in front of the fractional Laplacian and the potential we may consider the following simple form for the energy functional as

$$J_A(u) = \frac{1}{2}\int\int_{\mathbb{R}^{2n}\setminus(A^c\times A^c)}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy + \int_A W(u(x))dx, \;\; s\in(0,1),$$

whose minimizers in the sense of De Giorgi for sufficiently smooth potentials satisfy the nonlocal elliptic system,

$$(-\Delta)^s u = -W_u(u), \tag{3.5}$$

with prescribed behaviour $u = g$ on A^c, with weak form

$$\int_A\int_A\frac{(u(x)-u(y))\cdot(v(x)-v(y))}{|x-y|^{n+2s}}dxdy + \int_A W_u(u(x))v(x)dx = 0,$$
$$\forall\, v\in W^{s,2}(A;\mathbb{R}^m),\;\; v=0 \text{ on } A^c,$$

where by rescaling we omit the factor 1/2 in the weak form.

Corollary 3.2.3 (Weak Euler–Lagrange equation)**.** *If u is a minimizer of J_A with exterior condition on A^c, then for all $v\in H_0^s(A;\mathbb{R}^m)$,*

$$\frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{(u(x)-u(y))\cdot\big(v(x)-v(y)\big)}{|x-y|^{n+2s}}\,dx\,dy + \int_A W_u(u)\cdot v\,dx \;=\; 0,$$

i.e. $(-\Delta)^s u + W_u(u) = 0$ in A in the distributional sense (with the normalization consistent with J_A), and u attains the prescribed data on A^c.

3.2.2 A weak maximum principle for the scalar fractional Laplacian

The following weak maximum principle holds for the fractional Laplacian.

Lemma 3.2.4 (Lemma 4.6 [30])**.** *Let $v \in W^{s,2}(A)$ satisfy $(-\Delta)^s v \le 0$ in the weak sense, with $v \le 0$ in $\mathbb{R}^n \setminus A$. Then, $v \le 0$ in $\mathbb{R}^n$.*

This can be extended for the operator $(-\Delta)^s + c^2 I$. The extension is straightforward, we provide the proof for completeness and for the ease of the reader.

Lemma 3.2.5. *Let $v \in W^{s,2}(A)$ satisfy $(-\Delta)^s v + c^2 v \le 0$ in the weak sense, with $v \le 0$ in $\mathbb{R}^n \setminus A$. Then, $v \le 0$ in $\mathbb{R}^n$.*

Proof. The weak form of the inequality is

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}}dxdy + c^2\int_{\mathbb{R}^n} v(x)\psi(x)dx \le 0,$$

for any test function $\psi \in W^{s,2}(A)$, $\psi \ge 0$ with $\psi = 0$ on $\mathbb{R}^n \setminus A$, (see Remark 3.2.2, with c^2 possibly replaced by $2c^2$, which is immaterial for the argument here). Using as test function v^+ and the simplified notation

$$\langle v, \psi\rangle_s = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}}dxdy, \quad \|v\|_s = \sqrt{\langle v, v\rangle_s},$$

and $\langle\cdot,\cdot\rangle$ for the standard L^2 inner product with $\|v\| = \sqrt{\langle v, v\rangle}$, we express the weak form as

$$\langle v, v^+\rangle_s + c^2\langle v, v^+\rangle \le 0. \tag{3.6}$$

Express $v = v^+ - v^-$ and note that

$$\begin{aligned}\langle v, v^+\rangle_s + c^2\int_{\mathbb{R}^n} v(x)v^+(x)dx &= \|v^+\|_s^2 - \langle v^-, v^+\rangle_s + c^2\|v^+, v^+\|^2 - \langle v^+, v^-\rangle \\ &= \|v^+\|_s^2 + c^2\|v^+, v^+\|^2 + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{v^+(x)v^-(y) + v^+(y)v^-(x)}{|x-y|^n}dxdy \ge \\ &\qquad \|v^+\|_s^2 + c^2\|v^+\|^2,\end{aligned}$$

since $v^+v^- = 0$ and $v^+(x)v^-(y) \ge 0$. Combining this estimate with (3.6) yields the required result. $\square$

3.2.3 A polar form representation and its properties

We will also use the following representation for vector valued functions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$,

$$u(x) = a + |u(x) - a|\mathbf{n}(x),$$

$$\rho(x) = |u(x) - a|,$$

$$\mathbf{n}(x) = \begin{cases} \frac{u(x)-a}{|u(x)-a|}, & \text{if} \quad u(x) \neq a, \\ 0, & \text{if} \quad u(x) = a, \end{cases}$$

where $\rho : \Omega \subset \mathbb{R}^n \to \mathbb{R}_+$ is the polar part and $\mathbf{n} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is the angular part of u. One can easily see that for all suitable x, y

$$|u(x) - u(y)|^2 = |\rho(x) - \rho(y)|^2 + 2\rho(x)\rho(y)(1 - \mathbf{n}(x) \cdot \mathbf{n}(y)) \geq |\rho(x) - \rho(y)|^2, \tag{3.7}$$

where we used the Cauchy-Schwarz inequality $\mathbf{n}(x) \cdot \mathbf{n}(y) \leq 1$.

The following lemma allows one to obtain appropriate modifications of a vector field which may be useful for comparison purposes.

Lemma 3.2.6 (Polar form). *Let $u \in W^{s,p}(A; \mathbb{R}^m)$ and consider $f : \mathbb{R} \to \mathbb{R}_+$ of the form $f(s) = sg(s)$, where $g : \mathbb{R} \to [0,1]$ is a Lipschitz function*[1] *(the case $g(s) = 1$ included).*

Define $\bar{u}(x) := a + (f \circ \rho)(x)\mathbf{n}(x) = a + \bar{\rho}(x)$, where we use the notation $\bar{\rho}(x) = f(\rho(x))$. Then, for every x, y it holds that

$$|\bar{u}(x) - \bar{u}(y)|^2 \geq |\bar{\rho}(x) - \bar{\rho}(y)|^2. \tag{3.8}$$

Moreover, for every x, y it holds that

$$|\bar{u}(x) - \bar{u}(y)|^2 \leq |u(x) - u(y)|^2. \tag{3.9}$$

The above imply that for any suitable $A, B \subset \mathbb{R}^n$ it holds that

$$I_2(\bar{u}; A, B) \leq I_2(u; A, B).$$

[1] f defined like that is a Lipschitz function of Lipschitz constant $Lip(f) \leq 1$.

Proof. We note the following identity

$$
\begin{aligned}
|\bar{u}(x)-\bar{u}(y)|^2 &= [(\bar{u}(x)-a)-(\bar{u}(y)-a]\cdot[(\bar{u}(x)-a)-(\bar{u}(y)-a)] \\
&= |\bar{u}(x)-a|^2+|\bar{u}(y)-a|^2-2(\bar{u}(x)-a)\cdot(\bar{u}(y)-a) \\
&= (\bar{\rho}(x)^2-\bar{\rho}(y))^2-2\bar{\rho}(x)\bar{\rho}(y)n(x)\cdot n(y) \\
&= (\bar{\rho}(x)-\bar{\rho}(y))^2+2\bar{\rho}(x)\bar{\rho}(y)[1-n(x)\cdot n(y)].
\end{aligned}
\tag{3.10}
$$

Then, we immediately deduce from (3.10) that

$$
|\bar{u}(x)-\bar{u}(y)|^2 \geq |\bar{\rho}(x)-\bar{\rho}(y)|^2
$$

which is (3.8). Moreover, (using the notation $\bar{\rho}(x)=f(\rho(x))$) we have from (3.10) that

$$
\begin{aligned}
|\bar{u}(x)-\bar{u}(y)|^2 &= |f(\rho(x))-f(\rho(y))|^2+2f(\rho(x))f(\rho(y))[1-n(x)\cdot n(y)] \\
&\leq |\rho(x)-\rho(y)|^2+2\rho(x)\rho(y)[1-n(x)\cdot n(y)] = |u(x)-u(y)|^2
\end{aligned}
$$

with the inequality on the second line coming from the fact that f is a contraction. This proves (3.9).

The last claim follows easily from the above upon integration. □

3.3 A "continuity" lemma for functions in fractional Sobolev spaces

In the fractional framework the statement "$f \leq \hat{r}$ on ∂A" is not automatic for $f \in W^{s,2}(A)$, and it becomes meaningful only once a trace is available. When $s > \frac{1}{2}$ and ∂A is Lipschitz, the trace operator $\mathrm{Tr} : W^{s,2}(A) \to W^{s-\frac{1}{2},2}(\partial A)$ is well defined and continuous, hence boundary inequalities can be imposed in a rigorous way and are stable under the truncations used below. The content of Lemma 3.3.1 is then a simple but crucial "no-jump" property: if a function is bounded by $\hat{r}$ on the boundary and exceeds a larger level $\hat{s}$ on a set of positive measure inside, it must also occupy the intermediate strip $\{\hat{r} < f \leq \hat{s}\}$ on a set of positive measure. Equivalently, for $s > \frac{1}{2}$ true step–function transitions across a codimension-one interface are excluded in connected domains, so one cannot jump from the boundary level directly to a higher interior level without paying infinite fractional seminorm.

We provide a “continuity” lemma for functions in fractional Sobolev spaces in the spirit of the corresponding result for functions in $W^{1,2}(A)$ provided in Lemma 4.1 in [3].

Lemma 3.3.1 (Continuity of functions in fractional Sobolev spaces)**.** *Let $A \subset \mathbb{R}^n$ be an open, bounded, and connected domain with Lipschitz boundary and assume that $f \in W^{s,2}(A;\mathbb{R})$, $s > 1/2$, satisfies*

$$\begin{aligned} &f \le \hat{r} \text{ on } \partial A, \\ &|A \cap \{\hat{s} < f\}| > 0 \text{ for some } \hat{r} < \hat{s}. \end{aligned} \tag{3.11}$$

Then $|A \cap \{\hat{r} < f \le \hat{s}\}| > 0$.

Proof. In order to show this result we need to recall two basic facts. The first one is a trace result (see e.g. Theorem 9.2.1 [1]), according to which if $s > 1/2$ and A is as above, then the trace map $W^{s,2}(A;\mathbb{R}) \to W^{s-1/2,2}(\partial A;\mathbb{R})$ is bounded and linear (hence continuous). The second fact is that if v belongs to $W^{s',2}(A;\mathbb{R})$, $2s' > 1$, then v is constant. This follows from the basic fact

$$\int_{B_R}\int_{B_R} \frac{|f(x)-f(y)|}{|x-y|^{n+1}} dxdy < \infty \;\Rightarrow\; f = \text{constant},$$

where $B_R = \{x \in \mathbb{R}^n \mid |x| \le R\}$, together with the connectedness of A. We refer to Proposition 1 in [11].

Following [3] (see Lemma 4.3) we define 3 distinct sets

$$\begin{aligned} E_1 &:= A \cap \{f \le \hat{r}\}, \\ E_2 &:= A \cap \{\hat{r} < f \le \hat{s}\}, \\ E_3 &:= A \cap \{\hat{s} < f\}. \end{aligned}$$

Our aim is to prove that under the conditions above, $|E_2| > 0$.

We also introduce the cutoff functions

$$\sigma(s) := \begin{cases} f(x) & x \in E_1 \cup E_2, \\ \hat{s} & x \in E_3, \end{cases}$$

and

$$\tau(s) := \begin{cases} \hat{r} & x \in E_1, \\ f(x) & x \in E_2, \\ \hat{s} & x \in E_3, \end{cases}$$

Suppose on the contrary that $|E_2| = 0$. This implies that τ is a step function, and since by construction $\tau \in W^{s,2}(A;\mathbb{R})$ - by the second basic fact in the beginning of the proof - τ is a constant, which is in contradiction with $\hat{r} < \hat{s}$. Hence $|E_2| > 0$. The proof is complete. $\square$

Remark 3.3.2. We note that $W^{s,2}(A,\mathbb{R})$, $2s < 1$, A as above **does** support step functions. This suggests that the hypotheses in the replacement lemma are optimal. Indeed it is easily checked that

$$\int_E \int_{\Omega\setminus E} \frac{dxdy}{|x-y|^q} < \infty, \;\; q < n+1, \tag{3.12}$$

for any smooth $E \subset \Omega$, $\Omega \subset \mathbb{R}^n$, connected and bounded. To see this verify (3.12) for $n = 1$ and then construct examples in higher dimensions by splitting a connected smooth set with a co-dimension 1 flat interface [11].

Remark 3.3.3. Using Theorem B.1 in [9] we see that if A is a connected open set in $\mathbb{R}^n$ and $s \in (0,\infty)$, $p \in (1,\infty)$ are such that $sp \geq 1$ (including $s = 1$ and $p = 1$) then we may not have a step function of the form $f(x) = c_1\mathbf{1}_\omega(x) + c_2\mathbf{1}_{A\setminus\omega}(x)$, $\omega \subset A$, measurable, such that $f \in W^{s,p}(A)$. For if we had, (i.e. if $c_1 \neq c_2$, say $c_1 > c_2$ without loss of generality) then $\hat{f} = \frac{1}{c_1-c_2}(f - c_2) \in W^{s,p}(A;\mathbb{Z})$ with $\hat{f}(x) = 1\,\mathbf{1}_\omega(x) + 0\,\mathbf{1}_{A\setminus\omega})(x)$ which contradicts Theorem B.1, Appendix B in [9] (see Remark 3.3.2).

3.4 A fundamental replacement lemma

The Replacement Lemma is the basic comparison tool that replaces scalar maximum-principle arguments in the vector-valued setting. Under the boundary control $|u - a| \leq r$ on $A^c \cup \partial A$, one constructs a competitor $\bar{u}$ by keeping the angular part of u fixed and truncating only its radial part $\rho = |u - a|$ toward the well a, i.e. $\bar{u} = a + f(\rho)\nu$ with a suitable Lipschitz cutoff f. In the "monotonicity" regime of the potential this truncation yields a strict gain in the interaction (kinetic) energy, while the potential does not increase; outside that regime the gain is obtained instead from the potential energy by forcing $\bar{u}$ to hit the well on a set of positive measure. The continuity lemma is precisely what guarantees that this positive-measure set exists: if $|A \cap \{\rho > r_0\}| > 0$ and $\rho \le r_0$ on ∂A, then Lemma 3.3.1 forces the intermediate layer $\{r_0 < \rho \le r_0 + \varepsilon\}$ to have positive measure, which is where the strict potential decrease is produced. Thus the continuity lemma provides the missing "transition region", and the replacement construction turns it into a strict energy improvement, contradicting minimality.

Lemma 3.4.1 (Replacement Lemma)**.** *Let $A \subset \mathbb{R}^n$ be an open bounded Lipschitz domain, and $u \in W^{s,2}(A;\mathbb{R}^m) \cap L^\infty(A;\mathbb{R}^m)$, $s > \frac{1}{2}$, and such that*

(i) $|u(x) - a| \le r$ *on* $A^c \cup \partial A$, $0 < 2r \le r_0$,

(ii) $|A \cap \{|u(x) - a| > r\}| > 0$.

Then, there exists $\bar{u} \in W^{s,2}(A;\mathbb{R}^m) \cap L^\infty(A;\mathbb{R}^m)$, such that

(i) $\bar{u} = u$ *on* $A^c \cup \partial A$,

(ii) $|\bar{u}(x) - a| \le r$ *on* A,

(iii) $J_A(\bar{u}) < J_A(u)$.

Proof. We will explicitly construct the required function $\bar{u}$. We will consider separately two cases, (a) $\rho(x) \le r_0$ and (b) $|A \cap \{\rho(x) > r_0\}| > 0$.

Case (a). In this case we will construct a $\bar{u}$, such that $J_A^{(c)}(\bar{u}) < J_A^{(c)}(u)$, where

$$J_A^{(c)}(u) := \frac{1}{2} I_2(u; A, A) + I_2(u; A, A^c).$$

Let $f(s) = \min(s, r)$, and $g(s) = \frac{1}{s}\min(s, r)$, and consider $\bar{u}(x) = a + f(\rho(x))\mathbf{n}(x)$, where $u(x) = a + \rho(x)\mathbf{n}(x)$. This is a perturbation of u, keeping the angular part the same, while modifying the modulus, in such a way that $u = \bar{u}$ whenever $|u(x) - a| \le r$, whereas $\bar{u}(x) =$

$a + r\mathbf{n}(x)$ whenever $|u(x) - a| > r$. By Lemma 3.2.6 it holds that $J^{(c)}_{\Omega}(\bar{u}) \leq J^{(c)}_{\Omega}(u)$. We will show that the inequality is strict.

Suppose not. Then,

$$0 = J^{(c)}_A(\bar{u}) - J^{(c)}_A(u) = \frac{1}{2}(I_2(\bar{u}; A, A) - I_2(u; A, A)) + (I_2(\bar{u}; A, A^c) - I_2(u; A, A^c)). \quad (3.13)$$

We will study each term separately. Let us note that for any $A_1, A_2 \subset \mathbb{R}^n$ it holds that

$$\begin{aligned} I_2(\bar{u}; A_1, A_2) - I_2(u; A_1, A_2) &= \int_{A_1}\int_{A_2} \frac{1}{|x-y|^{n+2s}}(|\bar{u}(x) - \bar{u}(y)|^2 - |u(x) - u(y)|^2)dxdy \\ &= \int_{A_1}\int_{A_2} \frac{1}{|x-y|^{n+2s}}\Big\{\big[|\bar{\rho}(x) - \bar{\rho}(y)|^2 - |\rho(x) - \rho(y)|^2\big] \\ &\quad +2(\bar{\rho}(x)\bar{\rho}(y) - \rho(x)\rho(y))(1 - \mathbf{n}(x)\cdot\mathbf{n}(y))\Big\}dxdy \\ &\leq \int_{A_1}\int_{A_2} \frac{1}{|x-y|^{n+2s}}\big[|\bar{\rho}(x) - \bar{\rho}(y)|^2 - |\rho(x) - \rho(y)|^2\big]dxdy, \end{aligned}$$

since $\bar{\rho}(x) = f(\rho(x)) \leq \rho(x)$ (and similarly for y) so that $\bar{\rho}(x)\bar{\rho}(y) - \rho(x)\rho(y) \leq 0$ (while by the Cauchy-Schwarz inequality $1 - \mathbf{n}(x)\cdot\mathbf{n}(y) \geq 0$).

Setting $A_1 = A$ and $A_2 = A^c$, and keeping in mind that when $y \in A^c$ it holds that $\bar{\rho}(y) = \rho(y)$, we have

$$\begin{aligned} I_2(\bar{u}; A, A^c) - I_2(u; A, A^c) &\leq \int_A\int_{A^c} \frac{1}{|x-y|^{n+2s}}\big[|\bar{\rho}(x) - \bar{\rho}(y)|^2 - |\rho(x) - \rho(y)|^2\big]dxdy \\ &= \int_A\int_{A^c} \frac{1}{|x-y|^{n+2s}}(\bar{\rho}(x) - \rho(x))(\bar{\rho}(x) + \rho(x) - 2\rho(y))dxdy \\ &= \int_{A\cap\{\rho(x)\geq r\}}\int_{A^c} \frac{1}{|x-y|^{n+2s}}(\bar{\rho}(x) - \rho(x))(\bar{\rho}(x) + \rho(x) - 2\rho(y))dxdy \\ &\quad + \int_{A\cap\{\rho(x)< r\}}\int_{A^c} \frac{1}{|x-y|^{n+2s}}(\bar{\rho}(x) - \rho(x))(\bar{\rho}(x) + \rho(x) - 2\rho(y))dxdy \\ &= \int_{A\cap\{\rho(x)\geq r\}}\int_{A^c} \frac{1}{|x-y|^{n+2s}}(\bar{\rho}(x) - \rho(x))(\bar{\rho}(x) + \rho(x) - 2\rho(y))dxdy \\ &= \int_{A\cap\{\rho(x)\geq r\}}\int_{A^c} \frac{1}{|x-y|^{n+2s}}(r - \rho(x))(r + \rho(x) - 2\rho(y))dxdy \leq 0, \end{aligned}$$

since on $A \cap \{\rho(x) \geq r\}$ it holds that $r - \rho(x) \leq 0$, and $r + \rho(x) - 2\rho(y) \geq 0$ as $\rho(y) \leq r$ for $y \in A^c$. Hence,

$$I_2(\bar{u}; A, A^c) - I_2(u; A, A^c) \leq 0. \quad (3.14)$$

Next we set $A_1 = A$ and $A_2 = A$. Then,

$$\begin{aligned}
I_2(\bar{u}; A, A) - I_2(u; A, A) &\leq \int_A \int_A \frac{1}{|x-y|^{n+2s}} \left[|\bar{\rho}(x) - \bar{\rho}(y)|^2 - |\rho(x) - \rho(y)|^2\right] dxdy \\
&= \int_{A\cap\{\rho(x)\leq r\}} \int_{A\cap\{\rho(y)\leq r\}} \frac{1}{|x-y|^{n+2s}} \left[|\bar{\rho}(x) - \bar{\rho}(y)|^2 - |\rho(x) - \rho(y)|^2\right] dxdy \\
&+ \int_{A\cap\{\rho(x)> r\}} \int_{A\cap\{\rho(y)> r\}} \frac{1}{|x-y|^{n+2s}} \left[|\bar{\rho}(x) - \bar{\rho}(y)|^2 - |\rho(x) - \rho(y)|^2\right] dxdy \\
&+2 \int_{A\cap\{\rho(x)\leq r\}} \int_{A\cap\{\rho(y)> r\}} \frac{1}{|x-y|^{n+2s}} \left[|\bar{\rho}(x) - \bar{\rho}(y)|^2 - |\rho(x) - \rho(y)|^2\right] dxdy \\
&= - \int_{A\cap\{\rho(x)> r\}} \int_{A\cap\{\rho(y)> r\}} \frac{1}{|x-y|^{n+2s}} |\rho(x) - \rho(y)|^2 dxdy \\
&+2 \int_{A\cap\{\rho(x)\leq r\}} \int_{A\cap\{\rho(y)> r\}} \frac{1}{|x-y|^{n+2s}} \left[|\rho(x) - r|^2 - |\rho(x) - \rho(y)|^2\right] dxdy \\
&\leq - \int_{A\cap\{\rho(x)> r\}} \int_{A\cap\{\rho(y)> r\}} \frac{1}{|x-y|^{n+2s}} |\rho(x) - \rho(y)|^2 dxdy,
\end{aligned}$$

since

$$\begin{aligned}
|\rho(x)^2 - r|^2 - |\rho(x) - \rho(y)|^2 = (\rho(y) - r)(\rho(x) - r + \rho(x) - \rho(y)) \leq 0, \\
\text{on } (A \cap \{\rho(x) \leq r\}) \times (A \cap \{\rho(y) > r\}).
\end{aligned}$$

Hence,

$$\begin{aligned}
I_2(\bar{u}; A, A) - I_2(u; A, A) \leq \\
- \int_{A\cap\{\rho(x)> r\}} \int_{A\cap\{\rho(y)> r\}} \frac{1}{|x-y|^{n+2s}} |\rho(x) - \rho(y)|^2 dxdy.
\end{aligned} \tag{3.15}$$

Combining (3.13), (3.14) and (3.15), we conclude that

$$0 \leq - \int_{A\cap\{\rho(x)> r\}} \int_{A\cap\{\rho(y)> r\}} \frac{1}{|x-y|^{n+2s}} |\rho(x) - \rho(y)|^2 dxdy,$$

which implies that

$$\rho(x) - \rho(y) = 0, \;\; a.e. \text{ on } (A \cap \{\rho(x) > r\}) \times (A \cap \{\rho(y) > r\}). \tag{3.16}$$

We claim that (3.16) implies that

$$\bar{\rho}(x) = \rho(x), \quad \text{a.e. in } A. \tag{3.17}$$

To see this we reason as follows:

We first note that, upon defining $\psi := \bar{\rho} - \rho$ we have

$$\psi(x) - \psi(y) := (\bar{\rho}(x) - \rho(x)) - (\bar{\rho}(y) - \rho(y)) = 0, \text{ a.e on } \rho(x) > r,\ \rho(y) > r, \tag{3.18}$$

since for x, y as above it holds that $\bar{\rho}(x) = \bar{\rho}(y) = r$ (by the definition of $\bar{\rho}$) and $\rho(x) = \rho(y)$ by (3.16). Moreover,

$$\psi(x) - \psi(y) := (\bar{\rho}(x) - \rho(x)) - (\bar{\rho}(y) - \rho(y)) = 0, \text{ a.e on } \rho(x) \leq r,\ \rho(y) \leq r, \tag{3.19}$$

as for x, y as above $\bar{\rho}(x) = \rho(x)$ and $\bar{\rho}(y) = \rho(y)$.

Combining (3.18) and (3.19) we conclude that ψ is constant on $\{\rho > r\}$ and constant on $\{\rho \leq r\}$, hence ψ is a step function of the form

$$\psi(x) = \begin{cases} c_1 & \text{on} \quad \{x \ : \ \rho(x) < r\} \\ c_2 & \text{on} \quad \{x \ : \ \rho(x) \geq r\} \end{cases}$$

for suitable constants c_1, c_2.

Since by assumption $u \in W^{s,2}(A; \mathbb{R}^m)$ we also have by (3.7) that $\rho \in W^{s,2}(A; \mathbb{R})$ and since $\bar{\rho}$ is obtained from ρ by a Lipschitz transformation, we also have that $\bar{\rho} \in W^{s,2}(A; \mathbb{R})$. Hence, $\psi \in W^{s,2}$. Assuming, without loss of generality that $c_1 > c_2$ we may define $\bar{\psi} = \frac{1}{c_1 - c_2}(\psi - c_2)$ which is a step function and $\bar{\psi} \in W^{s,2}(A; \mathbb{R})$. This leads to the construction of a function $\bar{\psi} \in W^{s,2}(A; \mathbb{Z})$ which (see Remark 3.3.3) must necessarily be a constant, hence $c_1 = c_2$. This implies that $\psi = \rho - \bar{\rho}$ must be constant on A, and since (by the construction of $\bar{\rho}$) we have that $\psi = \rho - \bar{\rho} = 0$ on $A \cap \{\rho < r\}$ it must be that $\psi = \rho - \bar{\rho} = 0$ on the whole of A, hence $\rho = \bar{\rho}$ on A, as stated in (3.17).

Having established (3.17) we see that $\bar{\rho}(x) = \rho(x)$ a.e. in A implies that $|A \cap \{\rho(x) > r\}| = 0$, which contradicts the assumption. Therefore,

$$J_A^{(c)}(\bar{u}) < J_A^{(c)}(u). \tag{3.20}$$

Since we are in the monotonicity region of the potential,

$$W(\bar{u}(x)) = W(a + \bar{\rho}(x)\mathbf{n}(x)) \leq W(a + \rho(x)\mathbf{n}(x)) = W(u(x)),$$

and upon integration we have that

$$J^p_A(\bar{u}) := \int_A W(\bar{u}(x))dx \leq J^p(u,\Omega)(u) := \int_A W(u(x))dx. \tag{3.21}$$

Upon combining (3.20) and (3.21) it follows that $J_A(\bar{u}) < J_A(u)$.

Case (b). In case $|A \cap \{\rho(x) > r_0\}| > 0$, then we can no longer rely on the monotonicity of the potential. To construct $\bar{u}$ in this case we need to use an alternative perturbation of u, such that $J^p_A(\bar{u}) < J^p_A(u)$, i.e. we will gain the strict inequality from the potential energy rather than the kinetic energy term.

Following [3] we define the perturbation function $f(s) = sg(s) = \min(s,r)a(s)$, where

$$a(s) = \begin{cases} 1 & \text{if} \quad s \leq r, \\ \frac{2r-s}{r} & \text{if} \quad r \leq s \leq 2r, \\ 0 & \text{if} \quad s > 2r, \end{cases}$$

and consider $\bar{u}(x) = a + f(\rho(x))\mathbf{n}(x)$, instead of $u(x) = a + \rho(x)\mathbf{n}(x)$. Note that, using the notation $\bar{\rho}(x) = f(\rho(x))$, whenever $\rho(x) \leq r$ we have that $\bar{\rho}(x) = \rho(x)$, hence $\bar{u}(x) = u(x)$. On the other hand, whenever $\rho(x) \geq 2r$, then $\bar{\rho}(x) = 0$, hence $\bar{u}(x) = a$, and $W(\bar{u}(x)) = W(a) = 0$. Based on this observation we will show the strict inequality for the potential energy. Note that by Lemma 3.2.6 we have that $J^{(c)}_A(\bar{u}) \leq J^{(c)}_A(u)$.

Choose $\epsilon > 0$ such that $W(u) > 0$ on $r_0 \leq \rho(x) \leq r_0 + \epsilon$. Define the sets

$$E_1 := A \cap \{\rho \leq r_0\},$$
$$E_2 := A \cap \{r_0 < \rho \leq r_0 + \epsilon\},$$
$$E_3 := A \cap \{\rho > r_0 + \epsilon\},$$

which form a partition of A. We claim that $|E_2| > 0$.

We prove this claim using the "continuity" Lemma 3.3.1. Indeed, in general $|E_2| \geq 0$, so that if $|E_2| = 0$, then since (by assumption) $|A \cap \{\rho > r_0\}| > 0$, it must necessarily hold that $|E_3| > 0$. On $\partial A \cup A^c$ it holds that $\rho \leq r < 2r < r_0 < r_0 + \epsilon$ (all inequalities but the first - which is by assumption - are trivial), so that applying Lemma 3.3.1 for $\hat{r} = r_0$ and $\hat{s} = r_0 + \epsilon$, we conclude that $|A \cap \{r_0 < \rho \leq r_0 + \epsilon\}| = |E_2| > 0$. Hence $|E_2| > 0$.

By the definition of E_2 and the choice of ϵ, we have that $W(u(x)) > 0$ for $x \in E_2$.

We now consider the partition of A,

$$\begin{aligned} E_1' &:= A \cap \{\rho \leq r\}, \\ E_2' &:= A \cap \{r < \rho \leq 2r\}, \\ E_3' &:= A \cap \{\rho > 2r\}, \end{aligned}$$

where $2r < r_0$. Then,

(i) On E_1', it holds that $\bar{u} = u$, so that $W(\bar{u}) = W(u)$.

(ii) On E_2', since we are in the monotonicity region of the potential, and by the definition of the perturbation for $r < \rho \leq 2r$, it holds that $\alpha(\rho) < 1$,

$$W(\bar{u}(x)) = W(a + r\alpha(\rho(x))\mathbf{n}(x)) \leq W(a + r\mathbf{n}(x)) \leq W(a + \rho(x)\mathbf{n}(x)) = W(u(x)),$$

where the inequalities follow by the monotonicity of the potential.

(iii) On E_3', by the choice of the perturbation we have that $\bar{u}(x) = a$, so that

$$0 = W(\bar{u}(x)) \leq W(u(x)),$$

since $W \geq 0$. Moreover, $E_3' = E_2 \cup (E_3' \setminus E_2)$, and as shown $|E_2| > 0$ with $0 = W(\bar{u}(x)) < W(u(x))$, a.e. in E_2.

We therefore have for the potential energy

$$\begin{aligned} J_A^p(\bar{u}) &= \int_{E_1'} W(\bar{u}(x))dx + \int_{E_2'} W(\bar{u}(x))dx + \int_{E_3' \setminus E_2} W(\bar{u}(x))dx + \int_{E_2} W(\bar{u}(x))dx \\ &< \int_{E_1'} W(u(x))dx + \int_{E_2'} W(u(x))dx + \int_{E_3' \setminus E_2} W(u(x))dx + \int_{E_2} W(u(x))dx = J_A^p(u), \end{aligned}$$

where the strict inequality follows by the contribution of the integral over E_2.

The proof is complete. $\square$

3.5 A maximum principle

Theorem 3.5.1. *Let $A \subset \mathbb{R}^n$ be open bounded and with Lipschitz boundary, and $v \in W^{s,2}(A;\mathbb{R}^m) \cap L^\infty(A;\mathbb{R}^m)$, $s > \frac{1}{2}$, be a minimizer of $J_A := J_A^{(c)} + J_A^p$.*

Assume that $|v(x) - a| \leq r$ on $\partial A \cup A^c$, for $0 < 2r \leq r_0$,where r_0 is the radius of (uniform) radial monotonicity around the well a. Then,

(i) $|v(x) - a| \leq r$ on A.

(ii) If moreover, $u \mapsto W_u(u)$ is Lipschitz, and $|v(\hat{x}) - a| = r$ at an interior point $\hat{x} \in A$, then $v(x) = const$ in the connected component of $\hat{x}$ in A.

Proof. Suppose for simplicity that A is connected.

(i) Suppose on the contrary that $|v(x) - a| \leq r$ on A does not hold. Then, $|A \cap \{|v(x) - a| > r\}| > 0$, so by the replacement Lemma 3.4.1 there exists $\bar{v} \in W^{s,2}(A;\mathbb{R}^m) \cap L^\infty(A;\mathbb{R}^m)$, such that

$$\begin{aligned} \bar{v} = v, &\text{ on } \partial A \cap A^c, \\ |\bar{v}(x) - a| \leq r &\text{ on } A, \\ J_A(\bar{v}) &< J_A(v). \end{aligned}$$

But this contradicts minimality of v, hence $|v(x) - a| \leq r$ on A.

(ii) Assume that $|v(\hat{x}) - a| = r$ at an interior point $\hat{x} \in A$. Note that under the assumptions made, $|v - a|^2 \in W^{s,2}(A;\mathbb{R}) \cap L^\infty(A;\mathbb{R})$.

We calculate $-[(-\Delta)^s |v - a|^2] = -\sum_{i=1}^m [(-\Delta)^s |v_i - a_i|^2](x)$, using the representation for the fractional Laplacian

$$-[(-\Delta)^s \phi_i](x) = \frac{1}{2} \int_{\mathbb{R}^n} \frac{\phi_i(x+y) - \phi_i(x-y) - 2\phi_i(x)}{|x-y|^{n+2s}} dy,$$

for $\phi_i(x) = |v_i(x) - a_i|^2$.

We work component wise, and express the numerator of the fraction in the integral as

$$\begin{aligned}
&\phi_i(x+y) - \phi_i(x-y) - 2\phi_i(x) \\
&= \big[(v_i(x+y)-a_i)^2 - (v_i(x)-a_i)^2\big] + \big[(v_i(x-y)-a_i)^2 - (v_i(x)-a_i)^2\big] \\
&= \big(v_i(x+y) - v_i(x)\big)^2 + \big(v_i(x-y) - v_i(x)\big)^2 \\
&\quad + 2\,(v_i(x)-a_i)\,\big(v_i(x+y) + v_i(x-y) - 2v_i(x)\big).
\end{aligned}$$

We multiply by $|x-y|^{-(n+2s)}$, integrate over all y, and then add over all $i = 1, \cdots, m$ to obtain

$$\begin{aligned}
-[(-\Delta)^s|v-a|^2](x) = \frac{1}{2}\int_{\mathbb{R}^n} \frac{|v(x+y)-v(x)|^2}{|x-y|^{n+2s}}dy + \frac{1}{2}\int_{\mathbb{R}^n} \frac{|v(x-y)-v(x)|^2}{|x-y|^{n+2s}}dy \\
-2(v(x)-a)\cdot[(-\Delta)^s v](x),
\end{aligned}$$

and since v is a minimizer and satisfies the Euler- Lagrange equation $(-\Delta)^s v = -W_v(v)$ we obtain

$$\begin{aligned}
-[(-\Delta)^s|v-a|^2](x) = \frac{1}{2}\int_{\mathbb{R}^n} \frac{|v(x+y)-v(x)|^2}{|x-y|^{n+2s}}dy + \frac{1}{2}\int_{\mathbb{R}^n} \frac{|v(x-y)-v(x)|^2}{|x-y|^{n+2s}}dy \\
+2(v(x)-a)W_v(v(x)) \geq 0,
\end{aligned} \tag{3.22}$$

where the positivity of the last term comes from the properties of the potential.

Consider now the function $\varphi = r^2 - |v-a|^2 \in W^{s,2}(A;\mathbb{R}) \cap L^\infty(A,\mathbb{R})$. By (3.22) we have that $(-\Delta)^s\varphi \geq 0$, while by assumption $\varphi \geq 0$ on A^c. Then, by the strong maximum principle for the fractional Laplacian (see Theorem 2.3.3 in [12], see also Theorem 1.1 in [22]), $\varphi > 0$ unless φ vanishes identically. But since $\varphi(\hat{x}) = 0$, it must hold that $\varphi \equiv 0$ in A, hence $|v(x)-a|^2 = r^2$ for every $x \in A$, and $v = const$ (once more by invoking (3.22) along with the fact that since $|v-a|^2 = const$, $(-\Delta)^s|v-a|^2 = 0$). □

Remark 3.5.2. Note that Theorem 3.5.1(i) holds even in the case of singular potentials, where the minimizer may not be characterized in terms of the nonlocal Euler-Lagrange equation (3.2).

3.6 Applications

3.6.1 A Liouville type theorem

Theorem 3.6.1. *Let a be a minimum of $W \geq 0$, and $u : \mathbb{R}^n \to \mathbb{R}^m$ a minimizer in the sense of De Giorgi. We further assume that $u \to a$, as $|x| \to \infty$. Then, $u \equiv a$.*

Proof. Since $u \to a$ as $|x| \to \infty$, for every $\epsilon > 0$ there exists R_ϵ, such that $|u(x) - a| < \epsilon$ for $|x| \geq R_\epsilon$. We apply Theorem 3.5.1 to $A = \{x \ : \ |x| \leq R\}$ for $R \geq R_\epsilon$, to obtain that $|u(x) - a| < \epsilon$ for $|x| \leq R$, hence $u = a$, by the fact that ϵ is arbitrary. □

3.6.2 Uniqueness for minimization problems

This constitutes a generalization for the fractional case of Theorem 4.3 of [3].

Theorem 3.6.2. *Let $W : \mathbb{R}^m \to \mathbb{R}$, $W \geq 0$, $W \in C^2$ and $a \in \mathbb{R}^m$ such that $W(a) = 0$, with the extra property that $\xi^T W_{uu}(u)\xi \geq c^2|\xi|^2$ for every $\xi \in \mathbb{R}^m$ and $|u - a| \leq r_0$ where r_0 is the radius of (uniform) radial monotonicity around the well a.*

Let $A \subset \mathbb{R}^n$ be open and bounded with Lipschitz boundary and consider minimizers of J_A within the class of functions $\mathcal{L}_g = \{u \in W^{s,2}(A; \mathbb{R}^m),\ u = g \text{ on } A^c\}$, for given function g.

If $|g - a| \leq r$ on A^c, $0 < 2r < r_0$, then there exists a unique minimizer of J_A within the class $\mathcal{L}_g$.

Proof. Assume not, and let u, v be two distinct minimizers within the same class. Applying Theorem 3.5.1 we see that

$$\begin{aligned} |u - a| &\leq r \leq \frac{r_0}{2}, \ \text{ on } \ A, \\ |v - a| &\leq r \leq \frac{r_0}{2}, \ \text{ on } \ A. \end{aligned} \tag{3.23}$$

We now estimate

$$\begin{aligned} W(u) - W(v) - W_u(u) \cdot (v - u) &= \int_0^1 (W_u(s(v-u) + u) - W_u(u)) \cdot (v - u) ds \\ &= \int_0^1 s \int_0^1 W_{uu}(ts(v-u) + u)(v - u) \cdot (v - u) dt ds \geq \frac{1}{2} c^2 |v - u|^2, \end{aligned}$$

where in the above, we first estimated the difference $W_u(s(v-u)+u) - W_u(u)$ in terms of the integral of $t \mapsto W_{uu}(ts(v-u)+u)(v-u)$ and then used the fact that

$$
\begin{aligned}
|ts(v-u)+u-a| = |ts((v-a)-(u-a))+(u-a)| = |ts(v-a)+(1-ts)(u-a)| \\
\leq \frac{r_0}{2} + \frac{r_0}{2} = r_0,
\end{aligned}
$$

(which is true on account of (3.23)) along with the properties of the potential.

The Euler-Lagrange equation for the functional J_A can be expressed as

$$
\begin{aligned}
&\int_A \int_A \frac{(u(x)-u(y))(\eta(x)-\eta(y))}{|x-y|^{n+2s}}\, dx\, dy \\
&\quad + 2\int_A \int_{A^c} \frac{(u(x)-u(y))(\eta(x)-\eta(y))}{|x-y|^{n+2s}}\, dx\, dy + \int_A W_u(u)\, \eta\, dx = 0,
\end{aligned}
\tag{3.24}
$$

for every η, vanishing on A^c.

The above follows easily by choosing any η such that it vanishes on A^c and expressing

$$
\begin{aligned}
\frac{1}{\epsilon}(J_A(u+\epsilon\eta) - J_A(u)) = \int_A \int_A \frac{(u(x)-u(y))(\eta(x)-\eta(y))}{|x-y|^{n+2s}} dxdy + \\
2\int_A \int_{A^c} \frac{(u(x)-u(y))(\eta(x)-\eta(y))}{|x-y|^{n+2s}} dxdy + \int_A W_u(u+\epsilon v)\eta dx + 0(\epsilon),
\end{aligned}
$$

and passing to the limit as $\epsilon \to 0$.

Since $J_A(u) = J_A(v)$ we see that

$$
\begin{aligned}
0 = J_A(v) - J_A(u) = \frac{1}{2}\int_A \int_A \frac{|v(x)-v(y)|^2 - |u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy \\
+ \int_A \int_{A^c} \frac{|v(x)-v(y)|^2 - |u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy + \int_A (W(v) - W(u)) dx.
\end{aligned}
\tag{3.25}
$$

For any $\xi_1, \xi_2 \in \mathbb{R}^m$, we have that

$$
\begin{aligned}
|\xi_1|^2 - |\xi_2|^2 = (\xi_1 - \xi_2)\cdot(\xi_1+\xi_2) = \\
(\xi_1 - \xi_2)\cdot(\xi_1 - \xi_2 + 2\xi_2) = |\xi_1 - \xi_2|^2 + 2(\xi_1 - \xi_2)\cdot \xi_2,
\end{aligned}
$$

and applying the above for $\xi_1 = v(x) - v(y)$ and $\xi_2 = u(x) - u(y)$, we obtain

$$
\begin{aligned}
|v(x)-v(y)|^2 - |u(x)-u(y)|^2 = |v(x) - u(x) - (v(y) - u(y))|^2 \\
+2(u(x)-u(y))\cdot(v(x)-v(y)-u(x)+u(y)).
\end{aligned}
$$

We substitute the above into (3.25) to obtain (after rearranging)

$$
\begin{aligned}
0 = J_A(v) - J_A(u) =& \frac{1}{2}\int_A\int_A \frac{|v(x)-u(x)-(v(y)-u(y))|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&+ \int_A\int_{A^c} \frac{|v(x)-u(x)-(v(y)-u(y))|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&+ \int_A\int_A \frac{(u(x)-u(y))\cdot(v(x)-v(y)-u(x)+u(y))}{|x-y|^{n+2s}}\,dx\,dy \\
&+ 2\int_A\int_{A^c} \frac{(u(x)-u(y))\cdot(v(x)-v(y)-u(x)+u(y))}{|x-y|^{n+2s}}\,dx\,dy \\
&+ \int_A (W(v)-W(u))\,dx.
\end{aligned}
\tag{3.26}
$$

We see that the third and fourth term in the above can be eliminated via the Euler-Lagrange equation (3.24), using as test function $\eta = v - u$, to yield

$$
\begin{aligned}
0 = J_A(v) - J_A(u) &= \frac{1}{2}\int_A\int_A \frac{|v(x)-u(x)-(v(y)-u(y))|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad + \int_A\int_{A^c} \frac{|v(x)-u(x)-(v(y)-u(y))|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad + \int_A \big(W(v)-W(u)-W_u(u)\,(v-u)\big)\,dx \\
&\geq \frac{1}{2}c^2\int_A |v-u|^2\,dx.
\end{aligned}
$$

from which follows that $u = v$ in A. □

In a similar fashion we may obtain the following comparison result.

Proposition 3.6.3. *Under the same assumptions on W, a, A and u as in Theorem 3.6.2 it holds that*

$$|u(x)-a|^2 \leq \phi(x)r^2, \ \text{on } A,$$

where ϕ is the solution to

$$-(-\Delta)^s\phi = c^2\phi, \quad \text{on } A \tag{3.27}$$

$$\phi = 1, \quad \text{on } A^c. \tag{3.28}$$

Proof. As in the proof of Theorem 3.6.2 we have that $|u(x) - a| \leq C$ on A. By the non degeneracy assumption of the potential, near the minimum a, it holds that

$$W_u(u) \cdot (u - a) = (W_u(u) - W_u(a)) \cdot (u - a) \geq \frac{1}{2} c^2 |u - a|^2.$$

Working as above (see proof of Theorem 3.5.1) we obtain that

$$\begin{aligned} -[(-\Delta)^s |u - a|^2](x) = \frac{1}{2} \int_{\mathbb{R}^n} \frac{|u(x+y) - u(x)|^2}{|x - y|^{n+2s}} \, dy + \frac{1}{2} \int_{\mathbb{R}^n} \frac{|u(x-y) - u(x)|^2}{|x - y|^{n+2s}} \, dy \\ + 2(v(x) - a) \, W_v(v(x)) \geq c^2 |u(x) - a|^2, \end{aligned} \tag{3.29}$$

so that the function Ψ, defined by $\Psi(x) := |u(x) - a|^2$, satisfies the inequality

$$-(-\Delta)^s \Psi \geq c^2 \Psi, \tag{3.30}$$

with $\Psi \leq r^2$ on A and on A^c.

Let ϕ be a solution of (3.27), and consider $\Phi = r^2 \phi$ which solves (3.27) with boundary condition $\Phi = r^2$ on A^c. Combining (3.30) with the above, we obtain

$$\begin{aligned} (-\Delta)^2 (\Psi - \Phi) \leq -c^2 (\Psi - \Phi), \text{ on } A, \\ \Psi - \Phi \leq 0, \ A^c. \end{aligned}$$

Using the comparison principle in Lemma 3.2.5 we conclude the stated result. □

Chapter 4

Density Estimates

4.1 Introduction

In this chapter we study bounded minimizers of the fractional vectorial system

$$(-\Delta)^s u = -W_u(u) \quad \text{in } \mathbb{R}^n,$$

and prove quantitative *density estimates* for their transition region. The aim is to develop a nonlocal, vector-valued analogue of the classical density theory for phase transition models, in the spirit of Alikakos–Fusco [2], where the Caffarelli–Córdoba estimates [13] were extended to systems in the local setting.

To recall the local picture, one considers solutions of

$$\Delta u - W_u(u) = 0 \quad \text{in } D \subset \mathbb{R}^n,$$

as critical points or minimizers of the energy

$$J_D(u) := \int_D \Big(\tfrac{1}{2}|\nabla u(x)|^2 + W(u(x))\Big)\, dx,$$

where $u : D \to \mathbb{R}^m$ and $W \geq 0$ is a multi-well potential. For bounded minimizers, one has the basic energy growth bound

$$\int_{B_R(x_0)} \Big(\tfrac{1}{2}|\nabla u(x)|^2 + W(u(x))\Big)\, dx \;\leq\; CR^{n-1}.$$

In the scalar case ($m = 1$), the competition between bulk and interfacial energy largely dictates the structure of minimizers. In contrast, for systems ($m \geq 2$) the geometry of the zero set $\{W = 0\}$ plays a central role: the relative position of wells and the possible ways different phases meet in space are encoded in this set and strongly influence the behaviour of entire solutions.

In the vector Allen–Cahn model with finitely many global minima, as well as in related systems such as Ginzburg–Landau and phase separation models, entire solutions are closely linked to geometric objects. In singular limits, interfaces connect to minimal surfaces and, in genuinely vectorial settings, to Plateau complexes and singular minimizing cones organizing interfaces in a hierarchical way (vertices, edges, higher-dimensional faces, etc.); see, e.g., [2, 3].

The main result of Alikakos–Fusco [2] is a density estimate for vector-valued minimizers of the local energy. Assuming that a is an isolated zero of W (so $W(a) = 0$ and a is isolated in $\{W = 0\}$), they prove that if the transition set

$$\{x \in B_R(x_0) : |u(x) - a| > \lambda\}$$

has positive measure at some fixed scale, then it occupies a uniformly positive proportion of every larger ball. More precisely, for

$$0 < \lambda < \operatorname{dist}\big(a, \{W = 0\} \setminus \{a\}\big),$$

if

$$|B_1(x_0) \cap \{|u - a| > \lambda\}| \geq \mu_0 > 0,$$

then there exists $C = C(\mu_0, \lambda, \|u\|_{L^\infty}) > 0$ such that

$$|B_R(x_0) \cap \{|u - a| > \lambda\}| \;\geq\; CR^n, \qquad R \geq 1.$$

In diffuse–interface language, this means that once a nontrivial interface is present, the transition region cannot disappear at large scales: it persists with a definite density. In the scalar case, the Caffarelli–Córdoba density estimates [13] sharpen the connection between Allen–Cahn and minimal surfaces and played an important role in rigidity and symmetry results (see Savin [36]).

Further extensions in related variational settings are due, among others, to Farina–Valdinoci, Savin–Valdinoci, and Sire–Valdinoci; see, e.g., [20, 37, 38, 42].

A key technical feature of [2] is a polar-type decomposition around a well a,

$$u(x) = a + q_u(x)\,\nu_u(x),$$

where $q_u(x) := |u(x) - a|$ is the radial part and $\nu_u(x) \in \mathbb{S}^{m-1}$ is the angular part. The argument is driven by competitors that modify only the modulus q_u (inside a ball) while keeping the direction ν_u fixed. This isolates the radial contribution to the energy and allows one to combine energy comparison with coarea and isoperimetric inequalities to propagate lower bounds on the size of the transition set. Among the consequences are optimal lower bounds of order R^{n-1} for the energy of non-constant solutions and Liouville-type rigidity results for minimizers [2].

Here we develop an analogous theory for minimizers of the fractional energy

$$J(u,\Omega) := \frac{1}{2}\iint_{\mathbb{R}^{2n}\setminus(\Omega^c\times\Omega^c)} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega W(u(x))\,dx, \qquad s\in(0,1),$$

in the vector-valued setting. Compared with the local case, the nonlocal Gagliardo seminorm couples values of u across all length scales, so energy localization is delicate, and purely local PDE arguments are not available. The density mechanism therefore has to be implemented directly at the level of the nonlocal energy, via carefully designed radial competitors and fractional functional inequalities.

We work under structural assumptions on the potential W parallel to those in [2]. We assume $W \geq 0$, $W(a) = 0$, and that a is an isolated point of the zero set $\{W = 0\}$, together with a quantitative lower growth bound near a of the form

$$W(a+\rho\nu) \;\gtrsim\; \rho^\alpha,$$

for some $\alpha \in (0,2]$ and all $\nu \in \mathbb{S}^{m-1}$. The exponent α describes the order of contact of W with its minimum at a: $\alpha = 2$ corresponds to a nondegenerate quadratic well, while $\alpha \in (0,2)$ allows for sharper wells and (possibly) reduced regularity of W at a. In the nonlocal setting, the quantitative form of the density bounds depends on the interplay between α and the fractional parameter s through the relevant fractional Sobolev inequalities and the scaling of the energy.

The proof follows the general scheme of [2], with substantial adaptations to the fractional framework. In Section 4.2 we derive a fractional energy comparison inequality of the form

$$J^{\varrho}(u-\sigma, B_r) \leq \int_{B_r\cap\{q^u>q^h=q^\sigma\}} (W(\sigma)-W(u))\,dx + 2\int_{B_r\cap\{q^u>q^h\}} (q^u-q^h)\big[-(-\Delta)^s q^h\big]\,dx \quad (4.1)$$

where σ is a competitor obtained by truncating the modulus q^u via a suitable profile q^h. The construction of q^h (and hence σ) is carried out in Section 4.3, treating separately the regimes $\alpha = 2$, $1 < \alpha < 2$, and $0 < \alpha < 1$, reflecting the different behaviour of W near the well. The inequality (4.1) has a unified formal structure, while the estimates extracted from it depend quantitatively on s and α.

Sections 4.4, 4.5, and 4.6 contain the main density theorems in these three regimes. In each case, we combine the polar decomposition, fractional Sobolev inequalities, and suitable energy growth bounds to obtain a lower bound on the measure of the transition region $\{|u-a| > \lambda\}$ in large balls, provided it is nontrivial at some reference scale. Section 4.7 explains how to treat the range $s \geq \frac{1}{2}$ by refining the use of the energy growth bounds and implementing an iteration in the fractional parameter. Finally, Section 4.8 presents applications analogous to those in [2], including pointwise control and a Liouville-type rigidity result for entire minimizers.

In this way, the chapter provides a nonlocal density theory for fractional vector-valued minimizers, offering a counterpart to the classical results of Caffarelli–Córdoba and Alikakos–Fusco [13, 2] and to their nonlocal extensions [37, 38, 42].

Remark 4.1.1 (Classical local model and motivation)**.** The heuristic computation below is adapted from [3, Section 5.1] (see also [2]) and is included only for motivation and completeness. It describes the sharp–interface limit in the local case: a minimal surface $\Sigma^{n-1} = \partial D$ partitions a ball $B_r(x)$ into two phases, and the coarea and isoperimetric inequalities yield a uniform density bound for the volume of one phase inside $B_r(x)$, once the interface Σ^{n-1} is present.

In the diffuse–interface setting, the perimeter of D is replaced by the energy of a transition layer and the sharp interface by the region where $|u-a|$ is away from the wells. The nonlocal inequalities of Section 4.2 are precisely the fractional, vector-valued counterparts of this classical computation. For a complete and rigorous treatment of the local theory we refer the reader to [3, Section 5.1]; the rest of this chapter is devoted to establishing the corresponding density estimates for minimizers of the fractional energy.

Consider a minimal surface $\Sigma^{n-1} = \partial D$, Let $x \in \Sigma^{n-1}$. The surface Σ^{n-1} partitions the ball $B_r(x)$ into two parts, D_r and D_r^c. Let $V(r) = \mathcal{L}^n(D_r)$, $A(r) = \mathcal{H}^{n-1}(\Sigma^{n-1} \cap B_r)$, $\mathcal{H}^n$ the n-dimensional Hausdorff measure, $\mathcal{L}^n$ the n-dimensional Lebesque measure and S_r the spherical cup bounding D_r. Consider the following formal computation:

$$\begin{aligned} V(r) &\leq C[\mathcal{H}^{n-1}(\Sigma^{n-1} \cap B_r) + \mathcal{H}^{n-1}(S_r)]^{\frac{n}{n-1}}, && \text{by the isoperimetric inequality,} \\ &\leq C[2\mathcal{H}^{n-1}(S_r)]^{\frac{n}{n-1}}, && \text{by minimality since } \partial(\Sigma^{n-1} \cap \partial B_r) = \partial S_r, \\ &\leq C[V'(r)]^{\frac{n}{n-1}}, && \text{by the coarea formula .} \end{aligned}$$

From the last inequality, it follows that

$$V(r) \geq Cr^n, \quad C = C(n), \quad \forall r > 0.$$

The above estimate shows that both the sets D and D^c have a uniformly positive density at each x, for all radii r ranging from 0 to $+\infty$. The behaviour as $r \to 0$ is related to the fine structure of the singular set and leads to singularity and stratification results for minimal surfaces and their diffuse counterparts. On the other hand, the behaviour as $r \to \infty$ is connected with rigidity phenomena of Bernstein type. In the classical setting, Bernstein-type theorems assert that entire minimal graphs with suitable growth or stability assumptions must in fact be affine (so that the corresponding minimal surface is a hyperplane). In the diffuse–interface language this translates into one-dimensional symmetry and classification results for entire minimizers: if an interface has uniform density and does not "tilt" at infinity, then the solution must reduce to a planar profile, and no more complicated global geometry is possible.

4.2 A basic energy comparison inequality

We begin with the derivation of (4.1).The proof of this estimate is due to Savin and Valdinoci (see [38]). We consider minimizers of the nonlocal energy functional

$$J(u,\Omega) = \frac{1}{2}\int\int_{\mathbb{R}^{2n}\setminus(\Omega^c\times\Omega^c)} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}}dxdy + \int_\Omega W(u(x))dx, \ \ s\in(0,1), \tag{4.2}$$

among all functions $u:\Omega\subset\mathbb{R}^n\to\mathbb{R}^m$, such that $u\in W^{s,2}(\Omega;\mathbb{R}^m)$, with $u=g$ (given) on $\mathbb{R}^n\setminus\Omega$, for a given open bounded set $\Omega\subset\mathbb{R}^n$, where $|\cdot|$ denotes the Euclidean distance (in $\mathbb{R}^n$ or $\mathbb{R}^m$) and $W:\mathbb{R}^m\to\mathbb{R}$ is a continuous and positive potential with a zero at $a\in\mathbb{R}^m$, satisfying the hypothesis

$$\exists\, r_0>0, \ : \ \forall\,\xi\in\mathbb{R}^m, \ |\xi|=1 \ (0,r_0]\ni r\mapsto W(a+r\xi), \text{ is non decreasing with } W(a+r_0\xi)>0.$$

We will moreover consider cases in which, sufficiently close to any point where the potential vanishes, its local behaviour is as follows:

$$W(a+\rho\xi)\sim\rho^\alpha, \ \ \alpha\in(0,2],$$

Moreover, we demand for the potential:

Assumption 4.2.1 (Local behaviour of the potential for the case $\alpha=2$)**.** The potential W is C^2 in a neighbourghood of a and there exist constants $q_0>0$, $c_0>0$, $c_0'>0$ such that

$$c_0\le\xi^T W_{uu}(u)\xi\le c_0', \ \forall\, u: \ |u-a|\le q_0, \ \forall\,\xi: \ |\xi|=1.$$

Assumption 4.2.2 (Local behaviour of the potential for the case $1<\alpha<2$,)**.** The potential W is differentiable in a neighborhood of a and there exist constants $\rho_0>0$, C^* and C_*, independent of α such that

$$\alpha C_*\rho^{\alpha-1}\ge\frac{d}{d\rho}W(a+\rho\xi)\ge\alpha C^*\rho^{\alpha-1}, \ \ \forall\,\rho\in(0,\rho_0], \ \forall\,\xi\in\mathbb{R}^m \ : \ |\xi|=1.$$

Assumption 4.2.3 (Local behaviour of the potential for the case $0<\alpha<1$)**.** The potential W is differentiable in a neighborhood of a and there exist constants $\rho_0>0$, C^*, independent of α such that

$$\frac{d}{d\rho}W(a+\rho\xi)\ge\alpha C^*\rho^{\alpha-1}, \ \ \forall\,\rho\in(0,\rho_0], \ \forall\,\xi\in\mathbb{R}^m \ : \ |\xi|=1.$$

Throughout this chapter we will use the notation $u(A, B)$ for the integral

$$u(A, B) := \int_A \int_B \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} dxdy.$$

Using this notation we can express the energy J as

$$J(u, \Omega) = \frac{1}{2} u(\Omega, \Omega) + u(\Omega, \Omega^c) + \int_\Omega W(u(x))dx.$$

***Claim* 4.2.4.** Let u be a minimizer with $u = \alpha + q^u n$ the polar form of u. We introduce the vector maps $h = \alpha + q^h n$ and $\sigma = \alpha + q^\sigma n$, with $q^\sigma = \min\{q^u, q^h\}$, where $q^h \in W^{2,s}(B_r) \cap L^\infty(B_r)$ is a suitable radial function (the test function), that will be defined later and $q^\sigma = q^u$ on B_r^c. The minimality of the radial part of the energy

$$J(u - \sigma, \Omega) = J^\rho(u - \sigma, \Omega) + J^\alpha(u - \sigma, \Omega) + J^p(u - \sigma, \Omega),$$

leads to the estimate

$$J^\varrho(u - \sigma, B_r) \leq \int_{B_r \cap \{q^u > q^h = q^\sigma\}} (W(\sigma) - W(u))dx + 2 \int_{B_r \cap \{q^u > q^h\}} (q^u - q^h)[-(-\Delta)^s q^h]dx.$$

Proof. Note that $u = \sigma$ in B_r^c so that $u - \sigma$ and $q^u - q^\sigma$ vanish in B_r^c. Then, using the simplified notation $D = (\mathbb{R}^n \setminus B_r^c) \times (\mathbb{R}^n \setminus B_r^c)$ we have that

$$J^\varrho(u - \sigma, B_r) = \int\int_D \frac{|(q^u - q^\sigma)(x) - (q^u - q^\sigma)(y)|^2}{|x - y|^{n+2s}} dxdy.$$

Using the identity $|a - b|^2 + b^2 - a^2 = 2b(b - a)$ for $a = q^u(x) - q^u(y)$ and $b = q^\sigma(x) - q^\sigma(y)$, we obtain that

$$J^\varrho(u - \sigma, B_r) + J^\varrho(\sigma, B_r) - J^\varrho(u, B_r) = \int\int_D \frac{[(q^u - q^\sigma)(x) - (q^u - q^\sigma)(y)]\,(q^\sigma(y) - q^\sigma(x))}{|x - y|^{n+2s}} dxdy$$

$$= \int\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{[(q^u - q^\sigma)(x) - (q^u - q^\sigma)(y)]\,(q^\sigma(y) - q^\sigma(x))}{|x - y|^{n+2s}} dxdy,$$

where we used the fact that $q^u - q^\sigma = 0$ on B_r^c.

Using symmetry arguments,

$$\iint_{B_r\times B_r} \frac{\left[(q^u-q^\sigma)(x)-(q^u-q^\sigma)(y)\right](q^\sigma(y)-q^\sigma(x))}{|x-y|^{n+2s}}\,dx\,dy$$
$$= 2\iint_{B_r\times B_r} \frac{(q^u-q^\sigma)(x)\,(q^\sigma(y)-q^\sigma(x))}{|x-y|^{n+2s}}\,dx\,dy,$$
$$\iint_{B_r\times B_r^c} \frac{\left[(q^u-q^\sigma)(x)-(q^u-q^\sigma)(y)\right](q^\sigma(y)-q^\sigma(x))}{|x-y|^{n+2s}}\,dx\,dy$$
$$= 2\iint_{B_r\times B_r^c} \frac{(q^u-q^\sigma)(x)\,(q^\sigma(y)-q^\sigma(x))}{|x-y|^{n+2s}}\,dx\,dy.$$

Combining the above estimates

$$J^\varrho(u-\sigma,B_r)+J^\varrho(\sigma,B_r)-J^\varrho(u,B_r)=$$

$$2\int\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{(q^u-q^\sigma)(x)(q^\sigma(y)-q^\sigma(x))}{|x-y|^{n+2s}}dxdy=$$

$$2\int_{\mathbb{R}^n}(q^u-q^\sigma)(x)\left(\int_{\mathbb{R}^n}\frac{q^\sigma(y)-q^\sigma(x)}{|x-y|^{n+2s}}dy\right)dx=$$

$$2C_L\int_{\mathbb{R}^n}(q^u-q^\sigma)(x)[-(-\Delta)^s q^\sigma](x)dx,$$

where C_L is the normalizing constant used in the definition of the fractional Laplacian. Without loss of generality we will take $C_L=1$. By the definition of $q^\sigma=\min\{q^u,q^h\}$, we see that this integral is non vanishing only when $q^\sigma=q^h$, and this leads to the estimate

$$J^\varrho(u-\sigma,B_r)\le J^\varrho(u,B_r)-J^\varrho(\sigma,B_r)+2\int_{B_r\cap\{q^u>q^\sigma\}}(q^u-q^h)[-(-\Delta)^s q^h]dx, \tag{4.3}$$

where adding and subtracting the integral of $W(u)-W(\sigma)$ over B_r and using the radial part minimality leads to (4.1). □

4.3 Test Functions

In this section, we build the test function for each case is needed for the proof the Density Theorem. The constructions are inspired by the work of Savin and Valdinoci in [38] for the scalar case $\alpha = 2$. The test function plays the role of the comparison map q^h we defined above. We first construct an appropriate barrier function for each case. The idea is to find solutions of the inequality $-(-\Delta)^s w \leq c_o w^\alpha$ (in the case $0 < \alpha < 1$ we are looking for solutions of the inequality $-(-\Delta)^s w \leq c_o (w + \beta)^\alpha$ for a fixed constant β), which decay suitably fast in balls of of sufficiently large radius, for any $0 < \alpha < 2$.

4.3.1 The test function for the case $\alpha = 2$.

Lemma 4.3.1. *For any $\tau > 0$ there exists $C_\tau \geq 1$ (depending on n, s, τ) such that for every $r \geq C_\tau$, there exists $w \in C(\mathbb{R}^n; \mathbb{R}_+)$ that satisfies the differential inequality*

$$\begin{aligned} -(-\Delta)^s w \leq \tau w, \ \textit{in } B_r, \\ w = 1, \ \textit{in } B_r^c. \end{aligned} \tag{4.4}$$

Moreover, the function w, satisfies the bounds

$$\frac{1}{2C_\tau}(r + 1 - |x|)^{-2s} \leq w \leq \frac{C_\tau}{2}(r + 1 - |x|)^{-2s}. \tag{4.5}$$

Proof. In Lemma 3.1 in [38] it is shown that there exists a function $w_0 \in C(\mathbb{R}^n, [-1 + Cr^{-2s}, 1])$ such that

$$\begin{aligned} -(-\Delta)^s w_0 \leq \tau(1 + w_0), \ \text{in } B_r \\ w_0 = 1, \ \text{in } B_r^c, \end{aligned} \tag{4.6}$$

which satisfies the bounds

$$\frac{1}{C}(r + 1 - |x|)^{-2s} \leq 1 + w_0 \leq C(r + 1 - |x|)^{-2s},$$

for some $C > 0$.

This function is negative so it cannot be used as a test function for our purpose. We need to adapt this function so that it is positive, so that it can represent a barrier function for the modulus of a vector valued function. To this end define

$$w = \frac{w_0 + 1}{2},$$

which clearly satisfies (4.4) and the related bound (4.5). Note that w is positive. □

4.3.2 The test function for the case $1 < \alpha < 2$.

Lemma 4.3.2 (Construction of a barrier function I)**.** *For every $c_o > 0$ and $\nu > 0$, there exists a function $\hat{w}$ satisfying the inequality*

$$\begin{aligned} -(-\Delta)^s \hat{w}(x) \leq c_o \hat{w}(x)^\nu, \ \forall x \in B_R, \\ \hat{w}(x) = M, \ x \in B_R^c, \end{aligned} \tag{4.7}$$

and the bounds

$$\underline{C}(R+1-|x|)^{-2s/\nu} \leq \hat{w}(x) \leq \bar{C}(R+1-|x|)^{-2s/\nu}, \ x \in B_R, \tag{4.8}$$

for suitable strictly positive constants $\underline{C}$ and $\bar{C}$.

Proof. We will explicitly construct such a function, by appropriately modifying Lemma 3.1 in [38].

We fix a large $r \geq 1$ and a $T > 0$. These are as yet unspecified, but it will become clear towards the end of the proof that the values of r and T are related to R and c_o.

Step 1 . Set $r_0 = r - T$ let $g(t) = t^{-\gamma}$ for $\gamma > 0$ to be defined later on, (it turns out towards the end of the proof that $\gamma = \frac{2s}{\nu}$) and define the function

$$h(t) = \begin{cases} \min\{1, g(t) - g(r-r_0) - g'(r-r_0)(t-(r-r_0))\}, & \text{for} \quad t \leq r - r_0 = T, \\ 0, & \text{for} \quad t > r - r_0 = T. \end{cases}$$

Note that h is continuous. We define further,

$$v(x) = \begin{cases} h\big(r - |x|\big) & \text{for} \quad x \in B_r, \\ 1 & \text{for} \quad x \in B_r^c. \end{cases} \tag{4.9}$$

This is a continuous radially non decreasing function. Moreover, by the definition of h we see that

$$v(x) = \begin{cases} 0 & x \in B_{r-T}, \\ \min\{1, g\big((r-|x|)\big) - g(r-r_0) + g'(r-r_0)(r-r_0-|x|)\}, & x \in B_r \setminus B_{r-T}, \\ 1 & x \in B_r^c. \end{cases}$$

We will construct a solution for (4.7) by a proper rescaling and linear transformation of the function v, as

$$w(x) = A_o v(\frac{x}{S_o}) + B, \tag{4.10}$$

for appropriate choice of the constants A_o, S_o and B. Depending on the choice of constants we may either obtain a test function which vanishes on an appropriately chosen ball B_{R-R_0} or one that does not vanish anywhere but achieves some upper and lower bounds displaying a suitable power law decay. We will also set $r_0 = T = r/2$ (eventhough other choices are possible). However, to facilitate the understanding of the rationale for this choice (as well as the construction of alternative test functions if needed) we will leave these constants in their general form for a few more steps before choosing them in their final form.

Step 2 . We establish some useful upper and lower bounds for the function h,

$$h(t) \geq t^{-\gamma} - (1+\gamma)(r-r_0)^{-\gamma}, \text{ if } t \leq r - r_0, \text{ and } h(t) < 1, \tag{4.11}$$

$$\min\{1, t^{-\gamma}\} \leq h(t) + (1+\gamma)(r-r_0)^{-\gamma}, \tag{4.12}$$

$$h(t) \leq t^{-\gamma} + \gamma(r-r_0)^{-\gamma}, \;\; t \in (0, \infty). \tag{4.13}$$

We first establish (4.11). Since $t \leq r - r_0$ and $h(t) < 1$ we have that

$$h(t) = g(t) - g(r-r_0) - g'(r-r_0)(t-(r-r_0) \geq g(t) - g(r-r_0) - |g'(r-r_0)|(r-r_0) = t^{-\gamma} - (1+\gamma)(r-r_0)^{-\gamma},$$

since $g'(r-r_0) \leq 0$.

We now establish (4.12). We consider 3 cases.

(i) $t \leq r - r_0$ and $h(t) < 1$. Then, (4.11) implies that

$$h(t) + (1+\gamma)(r-r_0)^{-\gamma} \geq t^{-\gamma} \geq \min\{1, t^{-\gamma}\},$$

which is (4.12).

(ii) $t \leq r - r_0$ and $h(t) = 1$. Then,

$$1 < t^{-\gamma} - (r-r_0)^{-\gamma} + \gamma\,(r-r_0)^{-\gamma-1}\,(t-(r-r_0) \leq t^{-\gamma},$$

hence $\min\{1, t^{-\gamma}\} = 1$. Then,

$$\min\{1, t^{-\gamma}\} = 1 \le 1 + (1+\gamma)(r - r_0)^{-\gamma} = h(t) + (1+\gamma)(r - r_0)^{-\gamma},$$

which is (4.12).

(iii) $t > r - r_0$ so that $h(t) = 0$. Then, $t^{-\gamma} < (r - r_0)^{-\gamma} < 1$ for r suitably large, so that

$$\min\{1, t^{-\gamma}\} = t^{-\gamma} < (r - r_0)^{-\gamma} < (1+\gamma)(r - r_0)^{-\gamma} = h(t) + (1+\gamma)(r - r_0)^{-\gamma},$$

which is (4.12).

Finally we establish (4.13). The inequality is clearly true for $t > r - r_0$ so we focus on the case $t \le r - r_0$. For such t we have (taking into account that $g'(r - r_0) \le 0$ so that $-g'(r - r_0) = |g'(r - r_0)|$) that

$$\begin{aligned} h(t) &\le g(t) - g(r - r_0) + |g'(r - r_0)|(t - (r - r_0)) \le g(t) + |g'(r - r_0)|t \\ &\le g(t) + |g'(r - r_0)|(r - r_0) = t^{-\gamma} + \gamma(r - r_0)^{-\gamma}, \end{aligned}$$

which is (4.13).

Step 3 . The bounds in Step 2 provide the following bounds for v,

$$v(x) \ge (r - |x|)^{-\gamma} - (1+\gamma)(r - r_0)^{-\gamma}, \text{ if } |x| \ge r_0 = r - T, \text{ and } v(x) < 1, \tag{4.14}$$

$$\min\left\{1, (r - |x|)^{-\gamma}\right\} \le v(x) + (1+\gamma)(r - r_0)^{-\gamma}, \tag{4.15}$$

$$v(x) \le (r - |x|)^{-\gamma} + \gamma(r - r_0)^{-\gamma}, \ \ x \in B_r. \tag{4.16}$$

These bounds are immediate by setting $t = (r - |x|)$ in the bounds in Step 2.

Step 4 . The function v has bounded second derivative for any ball centered at any $x \in B_r$, of suitable radius. In particular,

$$\|D^2 v\|_{L^\infty(B_{\rho(x)}(x))} \le C_1 (r - |x|)^{-\gamma-2}, \text{ for } \rho(x) = \frac{r - |x|}{2}, \ \forall\, x \in B_r. \tag{4.17}$$

We first note that $v(x) = 0$ and $D^2 v(x) = 0$ for any $x \in B_{r/2}$, so the estimate trivially holds in this case.

Consider now any $y \in B_{\rho(x)}(x) \cap B^c_{r/2}$. Then,

$$|y| \le |y - x| + |x| \le \frac{r - |x|}{2} + |x| = \frac{r + |x|}{2} \implies \\ r - |y| \ge \frac{r - |x|}{2} \ge 0 \implies |r - |y|| = r - |y| \le |y|, \tag{4.18}$$

the last estimate following since $y \in B^c_{r/2}$.

Then,

$$D^2 v(y) = (\frac{\partial^2 v}{\partial y_j \partial y_\ell}), \quad j, \ell = 1, \cdots, n$$

$$\frac{\partial^2 v}{\partial y_j \partial y_\ell} = \nu_0^2 h''(\nu_0(r - |y|)) \frac{y_j y_\ell}{|y|^2} + \nu_0 h'(r - |y|) \frac{w_{j\ell}}{|y|^3},$$

$$w_{j\ell} = \begin{cases} y_j y_\ell & j \ne \ell, \\ -\sum_{k \ne j} y_k^2 & j = \ell. \end{cases}$$

hence taking the Euclidean norm

$$|D^2 v(y)| \le C \left(|h''(r - |y|)| + \frac{h'(r - |y|)}{|y|} \right) \le C' \max \left\{ |h''(r - |y|)|, \frac{|h'(r - |y|)|}{|y|} \right\} \\ \le C' \max \left\{ |h''(r - |y|)|, \frac{|h'(r - |y|)|}{r - |y|} \right\} \le C''(r - |y|)^{-\gamma - 2} \le C_1 (r - |x|)^{-\gamma - 2},$$

for suitable constants C , C', C'' and $C_1 = 2^{\gamma+2} C''$, where we also used (4.18). Taking the supremum the bound (4.17) follows.

<u>Step 5 .</u> From the bound (4.17) we may obtain the alternative bound

$$\|D^2 v\|_{L^\infty(\{v<1\})} \le C_1 K^{-\gamma-2}, \tag{4.19}$$

for a suitable constant K. This bound guarantees that $v \in W^{2,\infty}(B_r)$.

To obtain (4.19) note that

$$\{v < 1\} \subset B_{r-K}, \tag{4.20}$$

for a suitable $K < T$. Then if $x \in \{v < 1\}$ we have that $|x| \le r - K$ so that $r - |x| > K$ and (4.19) follows from (4.17).

Hence, it only remains to prove (4.20). To this end, observe that $x \in \{v < 1\}$ implies that $h(r-|x|) < 1$. We consider two cases.

(i) If $|x| < r_0 = r - T$ then $v(x) = h(r-|x|) = 0 < 1$. But then $|x| < r_0 = R - T < r - K$ as long as $K < T$ which is (4.20).

(ii) If $|x| \geq r_0 - r - T$ and $v(x) < 1$ then by the lower bound (4.44)

$$1 > v(x) \geq (r-|x|)^{-\gamma} - (1+\gamma)(r-r_0)^{-\gamma} \implies (r-|x|)^{-\gamma} < 1 + (1+\gamma)(r-r_0)^{-\gamma} \implies$$
$$|x| < r - (1+(1+\gamma)(r-r_0)^{-\gamma})^{-1/\gamma} \leq r - K,$$

as long as

$$K \leq (1+(1+\gamma)(r-r_0)^{-\gamma})^{-1/\gamma} = (1+(1+\gamma)T^{-\gamma})^{-1/\gamma},$$

which is (4.20).

Step 6 . We now consider the action of the fractional Laplacian on v and obtain the bound

$$\left|\int_{\mathbb{R}^n} \frac{v(y)-v(x)}{|x-y|^{n+2s}} dy\right| \leq C_3\{(r-|x|)^{-\gamma-2s} + (r-|x|)^{-2s}\}, \ \forall x \in B_r, \tag{4.21}$$

for a suitable constant C_3.

Since $v \in W^{2,\infty}(B_{\rho(x)}(x))$, for $\rho(x) = \frac{r-|x|}{2}$ and for any $x \in B_r$ by Step 4 (see estimate (4.17)), we can use the result of Lemma 6.13 in Palatucci et al [31] according to which if $\psi \in L^\infty(\mathbb{R}^n) \cap W^{2,\infty}(B_\rho(x))$ we have that

$$\left|\int_{\mathbb{R}^n} \frac{\psi(y)-\psi(x)}{|x-y|^{n+2s}} dy\right| \leq \frac{\omega_{n-1}}{(1-s)s}(\|D^2\psi\|_{L^\infty(B_\rho(x))}\,\rho^{2(1-s)} + \|\psi\|_{L^\infty(\mathbb{R}^n)}\,\rho^{-2s}).$$

Applying the above for $\psi = v$ and $\rho = \rho(x) = \frac{r-|x|}{2}$, and using the bound (4.17) for $\|D^2 v\|_{L^\infty(B_\rho(x))}$ we obtain that

$$\begin{aligned}\left|\int_{\mathbb{R}^n} \frac{v(y)-v(x)}{|x-y|^{n+2s}} dy\right| &\leq \frac{\omega_{n-1}}{(1-s)s}\left(\|D^2 v\|_{L^\infty(B_\rho(x))}\left(\frac{r-|x|}{2}\right)^{2(1-s)} + \|\psi\|_{L^\infty(\mathbb{R}^n)}\left(\frac{r-|x|}{2}\right)^{-2s}\right)\\ &\overset{(4.17)}{\leq} \frac{\omega_{n-1}}{(1-s)s}\left(C_1(r-|x|)^{-\gamma-2}\left(\frac{r-|x|}{2}\right)^{2(1-s)} + \|\psi\|_{L^\infty(\mathbb{R}^n)}\left(\frac{r-|x|}{2}\right)^{-2s}\right),\end{aligned}$$

which is (4.21) for a suitable C_3.

Step 7 . We now obtain an alternative bound for the action of the fractional Laplacian on v as

$$\left|\int_{\mathbb{R}^n} \frac{v(y)-v(x)}{|x-y|^{n+2s}}dy\right| \leq C_4(K^{-\gamma-2}+1), \ \ \forall\, x \in B_r, \tag{4.22}$$

for a suitable constant C_4, depending on s and n.

This follows from Lemma 6.14 in Palatucci et al [31] according to which if $\psi \in L^\infty(\mathbb{R}^n)$ is continuous, radial and radially non decreasing with

$$\sup_{\mathbb{R}^n} \psi = \max_{\mathbb{R}^n} \psi = M,$$

and $\psi \in W^{2,\infty}(\{\psi < M\})$ then,

$$\left|\int_{\mathbb{R}^n} \frac{\psi(y)-\psi(x)}{|x-y|^{n+2s}}dy\right| \leq \frac{\omega_{n-1}}{(1-s)s}(\|D^2\psi\|_{L^\infty(\{\psi<M\})} + \|\psi\|_{L^\infty(\mathbb{R}^n)}).$$

Applying this result to $\psi = v$ and $M = 1$, and using estimate (4.19) from Step 5 we obtain (4.22).

Step 8 . We now have two alternative bounds for $-(-\Delta)^s v$, bound (4.21) and (4.22). The first bound is very tight when $|x| \to 0$ (towards the center of the ball) and trivial (blows up to infinity) when $|x| \to r$, while the second bound is uniform over the whole ball. We may thus combine them into a single bound as

$$-(-\Delta)^s v(x) \leq C_0' \min\{(r-|x|)^{-\gamma-2s} + (r-|x|)^{-2s}, 1\}, \ \ \forall x \in B_r. \tag{4.23}$$

Noting that for a suitable constant C it holds

$$\min\{(r-|x|)^{-\gamma-2s} + (r-|x|)^{-2s}, 1\} \leq C \min\{(r-|x|)^{-2s}, 1\}, \tag{4.24}$$

so combining (4.23) with (4.24) we eventually get that

$$-(-\Delta)^s v(x) \leq C_0' C \min\{(r-|x|)^{-2s}, 1\}. \tag{4.25}$$

Therefore, v as constructed above satisfies

$$\begin{aligned} -(-\Delta)^s v(x) \leq C_0' C \min\{(r-|x|)^{-2s}, 1\}, \ \ x \in B_r, \\ v(x) = 0, \ \ x \in B_{r-T}, \\ v(x) = 1, \ \ x \in B_r^c. \end{aligned} \tag{4.26}$$

Step 9 . At this point (and from now onwards) we set

$$r_0 = T = r/2$$

and consider all the bounds we obtained in step 3, as well as the boundary conditions for v for this choice. This yields a function v satisfying (see (4.26))

$$\begin{aligned} -(-\Delta)^s v(x) \leq C_0' C \min\{(r-|x|)^{-2s}, 1\}, \quad & \forall\, x \in B_r \\ v(x) = 0, \quad & x \in B_{r/2}, \\ v(x) = 1, \quad & x \in B_r^c. \end{aligned} \tag{4.27}$$

Moreover, v satisfies the following bounds (see (4.11), (4.12) and (4.13))

$$v(x) \geq (r-|x|)^{-\gamma} - (1+\gamma)2^\gamma r^{-\gamma}, \text{ if } |x| \geq r/2, \text{ and } v(x) < 1, \tag{4.28}$$

$$\min\left\{1, (r-|x|)^{-\gamma}\right\} \leq v(x) + (1+\gamma)2^\gamma r^{-\gamma}, \tag{4.29}$$

$$v(x) \leq (r-|x|)^{-\gamma} + \gamma 2^\gamma r^{-\gamma}, \; x \in B_r. \tag{4.30}$$

Step 10 . Set $\kappa = \gamma/2s$ and note that for a suitable constant C',

$$\min\{(r-|x|)^{-2s}, 1\} \leq C' \left(v(x) + (1+\gamma)\left(\frac{r}{2}\right)^{-\gamma} \right)^{1/\kappa}. \tag{4.31}$$

To show (4.31) consider two separate cases (i) $|x| > r-1$ and (ii) $|x| \leq r-1$.

In case (i) we have that $r - |x| < 1$ so that $(r-|x|)^{-2s} > 1$ and $(r-|x|)^{-\gamma} > 1$. Hence,

$$1 = \min\{1, (r-|x|)^{-2s}\} = \min\{1, (r-|x|)^{-\gamma}\} \overset{(4.29)}{\leq} v(x) + (1+\gamma)2^\gamma r^{-\gamma},$$

and since $\kappa > 0$ the above implies that

$$1 = \min\{1, (r-|x|)^{-2s}\} \leq (v(x) + (1+\gamma)2^\gamma r^{-\gamma})^{1/\kappa},$$

which is the required inequality (4.31).

In case (ii) we have that $r-|x|\geq 1$ so that $(r-|x|)^{-2s}\leq 1$ and $(r-|x|)^{-\gamma}\leq 1$. Then,

$$\min\{(r-|x|)^{-2s},1\}=(r-|x|)^{-2s}, \quad \text{and} \quad \min\{(r-|x|)^{-\gamma},1\}=(r-|x|)^{-\gamma}.$$

Hence, (4.29) becomes

$$(r-|x|)^{-\gamma}\leq v(x)+(1+\gamma)2^{\gamma}r^{-\gamma} \implies (r-|x|)^{-2s}\leq (v(x)+(1+\gamma)2^{\gamma}r^{-\gamma})^{\frac{2s}{\gamma}},$$

which is the required inequality (4.31).

Step 11 . Combining (4.27) with (4.31), setting $C_0=C_0'CC'$ and choosing $\gamma=\frac{2s}{\nu}$ we see that v satisfies the inequality

$$\begin{aligned}-(-\Delta)^s v(x)\leq C_0(\beta+v(x))^{\nu}, &\quad \forall\, x\in B_r,\\ v(x)=0,\ \ x\in B_{r/2},&\\ v(x)=1,\ \ x\in B_r^c,&\end{aligned} \tag{4.32}$$

where

$$\beta=(\gamma+1)\,(r-T)^{-\gamma}=\left(\frac{2s}{\nu}+1\right)(r-T)^{-2s/\nu}. \tag{4.33}$$

Note that β can be chosen as small as we wish if r is chosen sufficiently large.

We are already very close to the desired test function and are now ready to rescale suitably $\hat{v}$ as described in (4.10).

Step 12 . We first define $\hat{v}(x):=v(x)+\beta$ and note that due to the translation invariance property of the fractional Laplacian $\hat{v}$ satisfies the inequality

$$\begin{aligned}-(-\Delta)^s\hat{v}(x)\leq C_0\hat{v}(x)^{\nu},\ \forall\, x\in B_r,&\\ \hat{v}(x)=\beta,\ x\in B_{r/2},&\\ \hat{v}(x)=1+\beta,\ x\in B_r^c,&\end{aligned} \tag{4.34}$$

i.e. $\hat{v}$ no longer vanishes in $B_{r/2}$ but achieves very low values ($\beta\sim r^{-2s/\nu}\to 0$ for large r).

Define $\hat{w}$ by

$$\hat{w}(x)=A_o\hat{v}\left(\frac{x}{S_o}\right),$$

for A_o and S_o to be determined.

We first note that

$$-(-\Delta)^s \hat{w}(x) = A_o \int_{\mathbb{R}^n} \frac{\hat{v}(y/S_o) - \hat{v}(x/S_o)}{|x-y|^{n+2s}} dy \overset{y'=y/S_o}{=} A_o \int_{\mathbb{R}^n} \frac{\hat{v}(y') - \hat{v}(x/S_o)}{|x - S_o y'|^{n+2s}} S_o^n dy'$$

$$= A_o S_o^{-2s} \int_{\mathbb{R}^n} \frac{\hat{v}(y') - \hat{v}(x/S_o)}{|x/S_o - y'|^{n+2s}} dy' = A_o S_o^{-2s} [-(-\Delta)^s \hat{v}] \left(\frac{x}{S_o}\right) \overset{(4.34)}{\leq} A_o S_o^{-2s} C_0 \hat{v}(x/S_o))^\nu$$

$$= A_o^{1-\nu} S_o^{-2s} C_0 (1 + \bar{w}(x))^{\frac{2s}{\gamma}}.$$

Hence $\hat{w}$ satisfies the inequality

$$\begin{aligned} -(-\Delta)^s \hat{w}(x) &\leq A_o^{1-\nu} S_o^{-2s} C_0 \hat{w}(x)^\nu, \ \ x \in B_R, \\ \hat{w}(x) &= (1+\beta) A_o, \ \ x \in B_R^c, \end{aligned} \tag{4.35}$$

for $R = r S_o$.

Choosing

$$A_o = \frac{M}{1+\beta} \quad \text{and} \quad S_o = (c_o A_o^{\nu-1} C_0^{-1})^{-1/2s} = c_o^{-1/2s} \left(\frac{M}{1+\beta}\right)^{(1-\nu)/2s} C_0^{1/2s}, \ \ R = S_o r, \tag{4.36}$$

we see that $\hat{w}$ satisfies the inequality

$$\begin{aligned} -(-\Delta)^s \hat{w}(x) &\leq c_o \hat{w}(x)^\nu, \ \forall\, x \in B_R, \\ \hat{w}(x) &= M \frac{\beta}{1+\beta}, \ x \in B_{R/2}, \\ \hat{w}(x) &= M, \ x \in B_R^c, \end{aligned} \tag{4.37}$$

which is the stated result.

Step 13 . We now obtain detailed upper and lower bounds for $\hat{w}$.

We first claim the following upper bound for $\hat{w}$:

$$\hat{w}(x) \leq \bar{C}(R + 1 - |x|)^{-\gamma}, \ \ \forall\ x \in B_R.$$

If $|x| < R/2$ this is definitely true, however, we will show that it also holds for all x.

Concerning the upper bound from (4.30) we obtain that

$$\hat{v}(x) \leq (r - |x|)^{-\gamma} + (1+2\gamma) \left(\frac{r}{2}\right)^{-\gamma} \implies$$

$$\hat{w}(x) \leq A_o (r - |x|/S_o)^{-\gamma} + A_o (1+2\gamma) \left(\frac{r}{2}\right)^{-\gamma} = A_o S_o^\gamma \left\{ (R - |x|)^{-\gamma} + (1+2\gamma) \left(\frac{R}{2}\right)^{-\gamma} \right\}.$$

By the definition of $\hat{w}$ and since $v \leq 1$ we have that $\hat{w} \leq A_o = \frac{M}{1+\beta}$. Hence, by the above we have that

$$\hat{w}(x) \leq A_o \min\left\{ S_o^\gamma (R-|x|)^{-\gamma} + (1+2\gamma) S_o^\gamma \left(\frac{R}{2}\right)^{-\gamma}, 1 \right\} \leq \bar{C}(R+1-|x|)^{-\gamma},$$

for a suitable constant $\bar{C} > 0$.

For the lower bound consider (4.29) which yields for $\hat{w}$,

$$\hat{w}(x) \geq A_o \min\{1, S_o^\gamma (R-|x|)^{-\gamma}\} \geq \underline{C}(R+1-|x|)^{-\gamma},$$

for a suitable $\underline{C} > 0$.

The proof is complete. □

4.3.3 The test function for the case $0 < \alpha < 1$.

Lemma 4.3.3 (Construction of a barrier function II)**.** *For every $\nu > 0$, there exists a function $\hat{w}$ satisfying the inequality*

$$\begin{aligned} -(-\Delta)^s \hat{w}(x) \leq C_o (\hat{w}(x) + \beta)^\nu, \ \forall x \in B_r, \\ \hat{w}(x) = M, \ x \in B_r^c, \\ \hat{w}(x) = 0, \ x \in B_{r-T}, \end{aligned} \tag{4.38}$$

for a fixed constant T, where $C_o = C_o(s,\nu,n)$, $\beta = \beta(s,\nu,T)$ and M the upper bound of u.

Proof. Step 1 . Let $g(t) = t^{-\gamma}$ for $\gamma > 0$ to be defined later on, (it turns out towards the end of the proof that $\gamma = \frac{2s}{\nu}$) and for fixed $T > T^*$, where T^*, also, a fixed constant that we develop through the proof, define the function

$$h(t) = \begin{cases} \min\{1, g(t) - g(T) - g'(T)(t-T))\}, & \text{for} \quad t \leq T, \\ 0, & \text{for} \quad t > T. \end{cases}$$

Note that h is continuous. We define further,

$$v(x) = \begin{cases} h\big(r - |x|\big) & \text{for} \quad x \in B_r \\ 1 & \text{for} \quad x \in B_r^c. \end{cases} \tag{4.39}$$

This is a continuous radially non decreasing function. Moreover, by the definition of h we see that

$$v(x) = \begin{cases} 0 & x \in B_{r-T}, \\ \min\{1, g\big(r - |x|\big) - g(T) + g'(T)(r - T - |x|)\}, & x \in B_r \setminus B_{r-T}. \\ 1 & x \in B_r^c. \end{cases}$$

We will construct a solution for (4.38) by a simple linear transformation of the function v, as

$$\hat{w}(x) = Mv(x). \tag{4.40}$$

Step 2 . We establish some useful upper and lower bounds for the function h,

$$h(t) \geq t^{-\gamma} - T^{-\gamma}, \text{ if } t \leq T, \text{ and } h(t) < 1, \tag{4.41}$$

$$\min\{1, t^{-\gamma}\} \leq h(t) + T^{-\gamma}, \tag{4.42}$$

$$h(t) \leq t^{-\gamma} + \gamma T^{-\gamma}, \ \ t \in (0, \infty). \tag{4.43}$$

We first establish (4.41). Since $t \leq T$ and $h(t) < 1$ we have that

$$\begin{aligned} h(t) = g(t) - g(T) - g'(T)(t - T) \geq \\ g(t) - g(T) - |g'(T)|(t - T) = \\ t^{-\gamma} - T^{-\gamma} - |-\gamma T^{-\gamma-1}|(t - T) = \\ t^{-\gamma} - T^{-\gamma} - \gamma T^{\gamma-1}(t - T) = \\ t^{-\gamma} - T^{-\gamma}(1 + \gamma T^{-1}(t - T)) \geq \\ t^{-\gamma} - T^{-\gamma}, \end{aligned}$$

since $g'(r - T) \leq 0$ and $\gamma T^{-1}(t - T) \leq 0$.

We now establish (4.42). We consider 3 cases.

(i) $t \leq T$ and $h(t) < 1$. Then, (4.41) implies that

$$h(t) + T^{-\gamma} \geq t^{-\gamma} \geq \min\{1, t^{-\gamma}\},$$

which is (4.42).

(ii) $t \leq T$ and $h(t) = 1$. Then,

$$1 < t^{-\gamma} - T^{-\gamma} + \gamma T^{-\gamma-1}(t - T) \leq t^{-\gamma},$$

hence $\min\{1, t^{-\gamma}\} = 1$. Then,

$$\min\{1, t^{-\gamma}\} = 1 \leq 1 + T^{-\gamma} = h(t) + T^{-\gamma},$$

which is (4.42).

(iii) $t > T$ so that $h(t) = 0$. Then, $t^{-\gamma} < T^{-\gamma} < 1$ for $T > 1$ (extra restriction for T) , so that

$$\min\{1, t^{-\gamma}\} = t^{-\gamma} < T^{-\gamma} = h(t) + T^{-\gamma},$$

which is again (4.42).

Finally we establish (4.43). The inequality is clearly true for $t > T$ since $h(t) = 0$, so we focus on the case $t \leq T$. For such t we have (taking into account that $g'(T) \leq 0$ so that $-g'(T) = |g'(T)|$)

$$\begin{aligned} h(t) &\leq g(t) - g(T) + |g'(T)|(t - T) \\ &\leq g(t) + |g'(T)|(t - T) \\ &\leq g(t) + |g'(T)|T = t^{-\gamma} + \gamma T^{-\gamma}, \end{aligned}$$

which is (4.43).

Step 3 . The bounds in Step 2 provide the following bounds for v,

$$v(x) \geq (r - |x|)^{-\gamma} - T^{-\gamma}, \text{ if } |x| \geq r - T, \text{ and } v(x) < 1, \tag{4.44}$$

$$\min\left\{1, (r - |x|)^{-\gamma}\right\} \leq v(x) + T^{-\gamma}, \tag{4.45}$$

$$v(x) \leq (r - |x|)^{-\gamma} + \gamma T^{-\gamma}, \ x \in B_r. \tag{4.46}$$

These bounds are immediate by setting $t = (r - |x|)$ in the bounds in Step 2.

Remark 4.3.4. Step 4 is the first time we need to alter the proof from the previous cases. We take cases for the dimension n to point the difference.

Step 4 . The function v has bounded second derivative for any ball centered at any $x \in B_r$, of suitable radius. In particular,

$$\|D^2 v\|_{L^\infty(B_{\rho(x)}(x))} \leq C_1(\gamma)(r-|x|)^{-\gamma-2}, \text{ for } \rho(x) = \frac{r-|x|}{2}, \ \forall x \in B_r, \tag{4.47}$$

for C_1 a constant that depends only on γ.

We first note that $v(x) = 0$ and $D^2 v(x) = 0$ for any $x \in B_{r-T}$, so the estimate trivially holds in this case.

Consider now any $y \in B_{\rho(x)}(x) \cap B^c_{r-T}$, then,

$$\begin{aligned} |y| \leq |y-x| + |x| \leq \frac{r-|x|}{2} + |x| = \frac{r+|x|}{2} \implies \\ r - |y| \geq \frac{r-|x|}{2} \geq 0 \end{aligned} \tag{4.48}$$

Remark 4.3.5. we skip for now the last estimate (4.46).

Now, case (i), $n = 1$.

We estimate $u''(y)$:

$$\begin{aligned} |u''(y)| = |g''(r-|y|)| = \gamma(\gamma+1)|(r-|y|)|^{-\gamma-2}\frac{y^2}{|y|^2} = \gamma(\gamma+1)(r-|y|)^{-\gamma-2} \leq \\ 2^{\gamma+2}\gamma(\gamma+1)(r-|x|)^{-\gamma-2}. \end{aligned} \tag{4.49}$$

Now, we can proceed to step (5) with no further modifications.

case (ii), $n > 1$.

Using,

$$D^2v(y) = (\frac{\partial^2 v}{\partial y_j \partial y_\ell}), \;\; j,\ell = 1,\cdots,n$$
$$\frac{\partial^2 v}{\partial y_j \partial y_\ell} = h''((r-|y|))\frac{y_j y_\ell}{|y|^2} + h'(r-|y|)\frac{w_{j\ell}}{|y|^3},$$
$$w_{j\ell} = \left\{ \begin{array}{cc} y_j y_\ell & j \neq \ell, \\ -\sum_{k\neq j} y_k^2 & j = \ell, \end{array} \right\}$$

we get

$$|D^2v(y)| \leq |h''(r-|y|)| + \frac{|h'(r-|y|)|}{|y|} = \gamma(\gamma+1)(r-|y|)^{-\gamma-2} + \gamma\frac{(r-|y|)^{-\gamma-1}}{|y|} \tag{4.50}$$

To move further, we need the inequality:

$|r-|y|| = r-|y| \leq |y|$.

It is true that $r-|y|>0$ for $y \in B_{\rho(x)}(x) \cap B^c_{r-T}$.

We need :

$$r-|y| \leq r-T \leq |y|. \tag{4.51}$$

The above inequality is true for $T \leq |y|$. To get $r-T<|y|$, for the particular choice of T, we choose r such that

$$T < r-T \Rightarrow 2T < r, \tag{4.52}$$

and so, from above:

$$|D^2v(y)| \leq \gamma(\gamma+1)(r-|y|)^{-\gamma-2} + \gamma\frac{(r-|y|)^{-\gamma-1}}{|y|} \leq$$
$$\gamma(\gamma+2)(r-|y|)^{-\gamma-2} = C_1(\gamma)(r-|y|)^{-\gamma-2}.$$

Now, we can proceed to step (5).

Remark 4.3.6. The construction of the test function should begin by first fixing a $T>T^*$, where T^* a constant independent of R and will be defined completely during the proof of the Density Theorem, and then set $r > max\{2T,1\}$.

Step 5 . From the bound (4.47) we may obtain the alternative bound

$$\|D^2 v\|_{L^\infty(\{v<1\})} \leq C_2(\gamma) K^{-\gamma-2}, \tag{4.53}$$

for a constant $K$and $C_2(\gamma)$ a constant that depends only on γ. This bound guarantees that $v \in W^{2,\infty}(B_r)$.

To obtain (4.53) note that

$$\{v < 1\} \subset B_{r-K}, \tag{4.54}$$

for a fixed constant $K < T$. Then if $x \in \{v < 1\}$ we have that $|x| \leq r - K$ so that $r - |x| \geq K$ and (4.53) follows from (4.47).

Hence, it only remains to prove (4.54). To this end, observe that $x \in \{v < 1\}$ implies that $h(r - |x|) < 1$. We consider two cases.

(i) If $|x| < r - T$ then $v(x) = h(r - |x|) = 0 < 1$. But then $|x| < r - T < r - K$ as long as $K < T$ which is (4.54).

(ii) If $|x| \geq r - T$ and $v(x) < 1$ then by the lower bound (4.44)

$$\begin{aligned} 1 > v(x) \geq (r-|x|)^{-\gamma} - T^{-\gamma} \implies (r-|x|)^{-\gamma} < 1 + T^{-\gamma} \implies \\ |x| < r - (1+T^{-\gamma})^{-1/\gamma} \leq r - K, \end{aligned}$$

as long as

$$K \leq (1 + T^{-\gamma})^{-1/\gamma},$$

which is (4.54).

Step 6 . We now consider the action of the fractional Laplacian on v and obtain the bound

$$\left| \int_{\mathbb{R}^n} \frac{v(y) - v(x)}{|x-y|^{n+2s}} dy \right| \leq C_3(\gamma, n, s)\{(r-|x|)^{-\gamma-2s} + (r-|x|)^{-2s}\}, \ \forall x \in B_r, \tag{4.55}$$

for a constant C_3 that depends on γ, n and s

Since $v \in W^{2,\infty}(B_{\rho(x)}(x))$, for $\rho(x) = \frac{r-|x|}{2}$ and for any $x \in B_r$ by Step 4 (see estimate (4.47)), we can use the result of Lemma 6.13 in Palatucci et al [31] according to which if $\psi \in L^\infty(\mathbb{R}^n) \cap W^{2,\infty}(B_\rho(x))$ we have that

$$\left| \int_{\mathbb{R}^n} \frac{\psi(y) - \psi(x)}{|x-y|^{n+2s}} dy \right| \leq \frac{\omega_{n-1}}{(1-s)s} (\|D^2\psi\|_{L^\infty(B_\rho(x))}\, \rho^{2(1-s)} + \|\psi\|_{L^\infty(\mathbb{R}^n)}\, \rho^{-2s}).$$

Applying the above for $\psi = v$ and $\rho = \rho(x) = \frac{r-|x|}{2}$, and using the bound (4.47) for $\|D^2 v\|_{L^\infty(B_\rho(x))}$ we obtain that

$$\left|\int_{\mathbb{R}^n} \frac{v(y)-v(x)}{|x-y|^{n+2s}}dy\right| \leq \frac{\omega_{n-1}}{(1-s)s}\left(\|D^2 v\|_{L^\infty(B_\rho(x))}\left(\frac{r-|x|}{2}\right)^{2(1-s)} + \|v\|_{L^\infty(\mathbb{R}^n)}\left(\frac{r-|x|}{2}\right)^{-2s}\right)$$
$$\overset{(4.47)}{\leq} \frac{\omega_{n-1}}{(1-s)s}\left(C_1(\gamma)(r-|x|)^{-\gamma-2}\left(\frac{r-|x|}{2}\right)^{2(1-s)} + \|v\|_{L^\infty(\mathbb{R}^n)}\left(\frac{r-|x|}{2}\right)^{-2s}\right)$$

$$= \frac{\omega_{n-1}}{(1-s)s}\left(C_1(\gamma)\left(\frac{1}{2}\right)^{2(1-s)}(r-|x|)^{-\gamma-2}(r-|x|)^{2(1-s)} + \left(\frac{1}{2}\right)^{-2s}(r-|x|)^{-2s}\right)$$
$$\leq \frac{\omega_{n-1}}{(1-s)s}\max\left\{C_1(\gamma)\left(\frac{1}{2}\right)^{2(1-s)}, \left(\frac{1}{2}\right)^{-2s}\right\}\left((r-|x|)^{-\gamma-2}(r-|x|)^{2(1-s)} + (r-|x|)^{-2s}\right)$$
$$= C_3(\gamma)\left((r-|x|)^{-\gamma-2s} + (r-|x|)^{-2s}\right)$$

which is (4.55).

Step 7 . We now obtain an alternative bound for the action of the fractional Laplacian on v as

$$\left|\int_{\mathbb{R}^n} \frac{v(y)-v(x)}{|x-y|^{n+2s}}dy\right| \leq C_4(K^{-\gamma-2}+1), \quad \forall\, x \in B_r, \tag{4.56}$$

for a suitable constant C_4, depending on s and n.

This follows from Lemma 6.14 in Palatucci et al [31] according to which if $\psi \in L^\infty(\mathbb{R}^n)$ is continuous, radial and radially non decreasing with

$$\sup_{\mathbb{R}^n} \psi = \max_{\mathbb{R}^n} \psi = M,$$

and $\psi \in W^{2,\infty}(\{\psi < M\})$ then,

$$\left|\int_{\mathbb{R}^n} \frac{\psi(y)-\psi(x)}{|x-y|^{n+2s}}dy\right| \leq \frac{\omega_{n-1}}{(1-s)s}(\|D^2\psi\|_{L^\infty(\{\psi<M\})} + \|\psi\|_{L^\infty(\mathbb{R}^n)}).$$

Applying this result to $\psi = v$ and $M = 1$, and using estimate (4.53) from Step 5 we obtain (4.56).

Step 8 . We now have two alternative bounds for $-(-\Delta)^s v$, bound (4.55) and (4.56). The first bound is very tight when $|x| \to 0$ (towards the center of the ball) and trivial (blows up to infinity) when $|x| \to r$, while the second bound is uniform over the whole ball.

We can refine bound (4.55) more:

$$
\begin{aligned}
-(-\Delta)^s v(x) &\leq C_3(\gamma)\left((r-|x|)^{-\gamma-2s} + (r-|x|)^{-2s}\right) \\
&\leq C_3(\gamma)(r-|x|)^{-2s}\left((r-|x|)^{-\gamma} + 1\right) \\
&\leq C_3(\gamma)(r-|x|)^{-2s}\left(1 + T^{-\gamma} + 1\right) \\
&= C_3(\gamma)(r-|x|)^{-2s}\left(2 + T^{-\gamma}\right) \\
&= C_5(n,s,\gamma,T)(r-|x|)^{-2s}.
\end{aligned}
$$

The last inequality is true, because we choose

$$T < r - T,$$

and $C_5(n,s,\gamma,T)$ a constant that depends on n, s, γ and T.

Now, we combine the above refined bound with (4.56) :

$$
\begin{aligned}
-(-\Delta)^s v(x) &\leq \min\{C_5(r-|x|)^{-2s}, C_4(K^{-\gamma-2}+1)\}, \ \ \forall x \in B_r, &(4.57)\\
&\leq \max\{C_3, C_4(K^{-\gamma-2}+1)\}\min\{(r-|x|)^{-2s}, 1\} &(4.58)\\
&= C_o \min\{(r-|x|)^{-2s}, 1\}, &(4.59)
\end{aligned}
$$

where C_o a constant that depends on n, s, γ, T and K.

Therefore, v as constructed above satisfies

$$
\begin{aligned}
-(-\Delta)^s v(x) &\leq C_0 \min\{(r-|x|)^{-2s}, 1\}, \ \ x \in B_r, \\
v(x) &= 0, \ \ x \in B_{r-T}, \\
v(x) &= 1, \ \ x \in B_r^c.
\end{aligned} \tag{4.60}
$$

Step 9 . Now, we set $\kappa = \gamma/2s$ and we prove:

$$\min\{(r-|x|)^{-2s}, 1\} \leq \left(v(x) + T^{-\gamma}\right)^{1/\kappa}. \tag{4.61}$$

To show (4.61) consider two separate cases (i) $|x| > r-1$ and (ii) $|x| \le r-1$.

In case (i) we have that $r - |x| < 1$ so that $(r-|x|)^{-2s} > 1$ and $(r-|x|)^{-\gamma} > 1$. Hence,

$$1 = \min\{1, (r-|x|)^{-2s}\} = \min\{1, (r-|x|)^{-\gamma}\} \le v(x) + T^{-\gamma},$$

and since $\kappa > 0$ the above implies that

$$1 = \min\{1, (r-|x|)^{-2s}\} \le (v(x) + T^{-\gamma})^{1/\kappa},$$

which is the required inequality (4.61).

In case (ii) we have that $r - |x| \ge 1$ so that $(r-|x|)^{-2s} \le 1$ and $(r-|x|)^{-\gamma} \le 1$. Then,

$$\min\{(r-|x|)^{-2s}, 1\} = (r-|x|)^{-2s}, \quad \text{and} \quad \min\{(r-|x|)^{-\gamma}, 1\} = (r-|x|)^{-\gamma}.$$

Hence,

$$(r-|x|)^{-\gamma} \le v(x) + T^{-\gamma} \implies (r-|x|)^{-2s} \le (v(x) + T^{-\gamma})^{\frac{2s}{\gamma}},$$

which is the required inequality (4.61).

Step 10 . Combining Step 8 with (4.61), and choosing $\gamma = \frac{2s}{\nu}$ we see that v satisfies the inequality

$$\begin{aligned} -(-\Delta)^s v(x) &\le C_o(\beta + v(x))^{\nu}, \quad \forall\, x \in B_r, \\ v(x) &= 0, \;\; x \in B_{r-T}, \\ v(x) &= 1, \;\; x \in B_r^c, \end{aligned} \tag{4.62}$$

where

$$\beta = T^{-\gamma} = T^{-2s/\nu}. \tag{4.63}$$

Note that β is a fixed constant.

We observe, by defining,

$$q^h(x) = \hat{w}(x) = Mv(x),$$

that $q^h = 0$ in B_{r-T} and $q^h = M$ in B_r^c.

We note that this test function is compatible with the proof scheme of Alikakos–Fusco [2] for the case $0 < \alpha < 2$ in the local case. □

Using the function w defined in the previous Lemmata, we construct the positive valued function $q^h := Mw$, where M is the upper bound for the minimizer ($\|u\|_{L^\infty} \leq M$), which has the property that $q^h = M$ on B_r^c. Using q^h we define $q^\sigma = \min\{q^u, q^h\}$. Clearly, $q^\sigma \leq q^u$ and $q^\sigma = q^u$ on B_r^c. Upon expressing the minimizer in polar form $u = a + q^u n^u$, we define

$$h = a + q^h n^u,$$
$$\sigma = a + q^\sigma n^u.$$

It can be seen that $\sigma = u$ on B_r^c. Using Lemma 2.1.3 it holds that

$$J^\varrho(\sigma, B_r) + \int_{B_r} W(\sigma)dx \geq J^\varrho(u, B_r) + \int_{B_r} W(u)dx. \tag{4.64}$$

4.4 The Density Theorem for the case $\alpha = 2$

Theorem 4.4.1. *Let $s < 1/2$ and assume that the potential satisfies Assumption 4.2.1, $\mathcal{O}$ is open and $u : \mathcal{O} \subset \mathbb{R}^n \to \mathbb{R}^m$ is minimal in the De Giorgi sense. Then, for any $\mu_0 > 0$, and any $\lambda \in (0, d_0)$ where $d_0 = \min_z\{|a - z| \,:\, z \neq a,\ W(z) = 0\} > 0$, the condition*

$$|B_{r_0}(x_0) \cap \{|u - a| > \lambda\}| \geq \mu_0$$

implies that

$$|B_{r_0}(x_0) \cap \{|u - a| > \lambda\}| \geq Cr^n, \ \ \text{for } r \geq r_0,$$

as long as $B_r(x_0) \subset \mathcal{O}$ where $C = C(W, \mu_0, \lambda, r_0, M)$.

Proof. The proof proceeds in 11 steps.

Step 1 . The minimality of the radial part of the energy (4.64) leads to the estimate

$$J^\varrho(u - \sigma, B_r) \leq \int_{B_r \cap \{q^u > q^h = q^\sigma\}} (W(\sigma) - W(u))dx + 2\int_{B_r \cap \{q^u > q^h\}} (q^u - q^h)[-(-\Delta)^s q^h]dx \tag{4.65}$$

The proof of this estimate follows closely the derivation in [12] or [38] and is only sketched here for completeness. Note that $u = \sigma$ in B_r^c so that $u - \sigma$ and $q^u - q^\sigma$ vanish in B_r^c. Then, using the simplified notation $B = (\mathbb{R}^n \setminus B_r^c) \times (\mathbb{R}^n \setminus B_r^c)$ we have that

$$J^{\varrho}(u-\sigma, B_r) = \int\int_D \frac{|(q^u - q^{\sigma}(x) - (q^u - q^{\sigma}(y)|^2}{|x-y|^{n+2s}} dxdy.$$

Using the identity $|a-b|^2 + b^2 - a^2 = 2b(b-a)$ for $a = q^u(x) - q^u(y)$ and $b = q^{\sigma}(x) - q^{\sigma}(y)$, we obtain that

$$\begin{aligned} J^{\varrho}(u-\sigma, B_r) + J^{\varrho}(\sigma, B_r) - J^{\varrho}(u, B_r) &= \int\int_D \frac{[(q^u - q^{\sigma})(x) - (q^u - q^{\sigma})(y)]\,(q^{\sigma}(y) - q^{\sigma}(x))}{|x-y|^{n+2s}} dxdy \\ &= \int\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{[(q^u - q^{\sigma})(x) - (q^u - q^{\sigma})(y)]\,(q^{\sigma}(y) - q^{\sigma}(x))}{|x-y|^{n+2s}} dxdy, \end{aligned}$$

where we used the fact that $q^u - q^{\sigma} = 0$ on B_r^c.

Using symmetry arguments,

$$\begin{aligned} &\int\int_{B_r \times B_r} \frac{[(q^u - q^{\sigma})(x) - (q^u - q^{\sigma})(y)]\,(q^{\sigma}(y) - q^{\sigma}(x))}{|x-y|^{n+2s}} dxdy \\ &\qquad = 2\int\int_{B_r \times B_r} \frac{(q^u - q^{\sigma})(x)(q^{\sigma}(y) - q^{\sigma}(x))}{|x-y|^{n+2s}} dxdy, \\ &\int\int_{B_r \times B_r^c} \frac{[(q^u - q^{\sigma})(x) - (q^u - q^{\sigma})(y)]\,(q^{\sigma}(y) - q^{\sigma}(x))}{|x-y|^{n+2s}} dxdy \\ &\qquad = 2\int\int_{B_r \times B_r^c} \frac{(q^u - q^{\sigma})(x)(q^{\sigma}(y) - q^{\sigma}(x))}{|x-y|^{n+2s}} dxdy. \end{aligned}$$

Combining the above estimates

$$\begin{aligned} &J^{\varrho}(u-\sigma, B_r) + J^{\varrho}(\sigma, B_r) - J^{\varrho}(u, B_r) = \\ &2\int\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(q^u - q^{\sigma})(x)(q^{\sigma}(y) - q^{\sigma}(x))}{|x-y|^{n+2s}} dxdy = \\ &2\int_{\mathbb{R}^n} (q^u - q^{\sigma})(x) \left(\int_{\mathbb{R}^n} \frac{q^{\sigma}(y) - q^{\sigma}(x)}{|x-y|^{n+2s}} dy \right) dx = \\ &2C_L \int_{\mathbb{R}^n} (q^u - q^{\sigma})(x)[-(-\Delta)^s q^{\sigma}](x) dx, \end{aligned}$$

where C_L is the normalizing constant used in the definition of the fractional Laplacian. Without loss of generality we will take $C_L = 1$. By the definition of $q^{\sigma} = \min\{q^u, q^h\}$, we see that this integral is non vanishing only when $q^{\sigma} = q^h$, and this leads to the estimate

$$J^{\varrho}(u-\sigma, B_r) \le J^{\varrho}(u, B_r) - J^{\varrho}(\sigma, B_r) + 2\int_{B_r \cap \{q^u > q^{\sigma}\}} (q^u - q^h)[-(-\Delta)^s q^h] dx, \tag{4.66}$$

where adding and subtracting the integral of $W(u) - W(\sigma)$ over B_r and using the radial part minimality lemma (2.1.3) leads to (4.65).

Step 2 . We establish the inequality

$$\begin{aligned}\bar{C}\left(\int_{B_r}(q^u - q^\sigma)^{2n/(n-2s)}\right)^{(n-2s)/n} \leq & \int_{B_r\cap\{q^u>q^h=q^\sigma\}}(W(\sigma)-W(u))dx \\ & +2\int_{B_r\cap\{q^u>q^h\}}(q^u-q^h)[-(-\Delta)^s q^h]dx.\end{aligned} \tag{4.67}$$

We recall the fractional Sobolev inequality

$$\|v\|_{L^{np/(n-sp)}(\mathbb{R}^n} \leq C\left(\int\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|v(x)-v(y)|^p}{|x-y|^{n+sp}}dxdy\right)^{1/p}, \tag{4.68}$$

valid for any $s \in (0,1)$, $p \in (1, n/s)$ and $u \in C_0^\infty(\mathbb{R}^n)$ (and by extension to any $u \in W^{s,p}(\mathbb{R}^n)$), see Theorem 2.2.1 [12]. Using (4.68) for $p = 2$ with (4.1), combined with the fact that $q^u - q^\sigma = 0$ on B_r^c, we conclude (4.67).

Step 3 . Choose $q^h = Mw$, where w is as in Lemma 4.3.1, with $\tau > 0$ to be determined shortly. Clearly, $q^h > 0$ and satisfies

$$\begin{aligned}-(-\Delta)^s q^h \leq \tau q^h, &\text{ in } B_r,\\ q^h = M &\text{ in } B_r^c.\end{aligned} \tag{4.69}$$

By Lemma 4.3.1, q^h satisfies the bounds

$$\frac{M}{2C_\tau}(r+1-|x|)^{-2s} \leq q^h \leq \frac{MC_\tau}{2}(r+1-|x|)^{-2s}.$$

For $x \in B_{r-T}$ the above bound yields that

$$q^h(x) \leq \frac{MC_\tau}{2}(T+1)^{-2s} \leq \frac{MC_\tau}{2}T^{-2s}.$$

Clearly by definition

$$q^\sigma = \min(q^h, q^u) \leq q^h \leq \frac{MC_\tau}{2}T^{-2s}.$$

We set a $\lambda' \in (0, \lambda)$ (say $\lambda' = \frac{\lambda}{2}$) and choose $T > 0$ such that

$$\frac{MC_\tau}{2}T^{-2s} \leq \lambda - \lambda'. \tag{4.70}$$

Then,

$$q^u - q^\sigma = q^u - q^h > \lambda', \text{ in } B_{r-T} \cap \{q^u > \lambda\}. \tag{4.71}$$

Indeed, q^σ by definition can either be q^u or q^h. On $B_{r-T} \cap \{q^u > \lambda\}$ we have that $q^u > \lambda$, whereas

$$q^\sigma = \min\{q^u, q^h\} \leq q^h \leq \frac{MC_\tau}{2} T^{-2s} \leq \lambda - \lambda' < \lambda,$$

so that $q^u > q^\sigma$ and hence $q^\sigma = q^h$. Therefore, on this set

$$q^u - q^\sigma = q^u - q^h > \lambda - q^h > \lambda - (\lambda - \lambda') = \lambda',$$

and the claim (4.71) is established. Moreover, by(4.71) it follows that

$$B_{r-T} \cap \{q^u > \lambda\} \cap \{q^u - q^h \leq \lambda'\} = \emptyset. \tag{4.72}$$

Step 4 . We establish the estimate

$$\left(\int_{B_r} (q^u - q^\sigma)^{2n/(n-2s)} dx \right)^{(n-2s)/n} \geq (\lambda')^2 \, |B_{r-T} \cap \{q^u > \lambda\}|^{(n-2s)/n}, \tag{4.73}$$

that will be used along with the previous estimate (4.67).

To see this note that

$$\begin{aligned}
&\left(\int_{B_r} (q^u - q^\sigma)^{2n/(n-2s)} \right)^{(n-2s)/n} dx \geq \left(\int_{B_{r-T}} (q^u - q^\sigma)^{2n/(n-2s)} dx \right)^{(n-2s)/n} = \\
&\quad \Bigg(\int_{B_{r-T} \cap \{q^u > \lambda\} \cap \{q^u - q^h \leq \lambda'\}} (q^u - q^\sigma)^{2n/(n-2s)} dx + \\
&\qquad \int_{B_{r-T} \cap \{q^u > \lambda\} \cap \{q^u - q^h > \lambda'\}} (q^u - q^\sigma)^{2n/(n-2s)} dx \Bigg)^{(n-2s)/n} = \\
&\quad \left(\int_{B_{r-T} \cap \{q^u > \lambda\} \cap \{q^u - q^h > \lambda'\}} (q^u - q^\sigma)^{2n/(n-2s)} dx \right)^{(n-2s)/n} \geq (\lambda')^2 \, |B_{r-T} \cap \{q^u > \lambda\}|^{(n-2s)/n},
\end{aligned}$$

where we used (4.72). Hence (4.73) is established.

Step 5 . We now consider the RHS of (4.67) and try to establish an upper bound for the potential difference term. We claim that

$$\int_{B_r}(W(\sigma)-W(u))dx = \int_{B_r\cap\{q^h<q^u\}}(W(h)-W(u))dx \leq \\ \frac{c_0}{2}\int_{B_r\cap\{q^h<q^u\}\cap\{q^u\leq\lambda\}}((q^h)^2-(q^u)^2)dx + \int_{B_r\cap\{q^h<q^u\}\cap\{q^u>\lambda\}}(W(h)-W(u))dx \tag{4.74}$$

This requires use of the assumption 4.2.1 on the potential function.

Before proving (4.74) we note that since $q^\sigma = \min\{q^u, q^h\}$ we have that

$$\begin{aligned}\int_{B_r}(W(\sigma)-W(u))dx &= \int_{(B_r\cap\{q^\sigma=q^u\})\cup\{q^\sigma<q^u\}}(W(\sigma)-W(u))dx \\ &= \int_{B_r\cap\{q^h<q^u\}}(W(h)-W(u))dx\end{aligned}$$

This comes from the fact that

$$B_r = \{q^\sigma < q^u\}\cup(B_r\cap\{q^\sigma = q^u\}) \text{ and } \{q^\sigma<q^u\} = \{q^h<q^u\},$$

which follows by the definition of q^σ and σ (q^σ may take either the value q^h or q^u).

We now as in [3], under the assumption that $\lambda \leq q_0$ (see Assumption 4.2.1 for the definition of q_0) approximate the potential difference with a quadratic approximation. On $\{q^h < q^u\}\cap\{q^u\leq\lambda\}$ it holds that $q^u \leq q_0$ so we have that

$$W(u)-W(h) \geq \frac{c_0}{2}((q^u)^2-(q^h)^2), \text{ on } \{q^h<q^u\}\cap\{q^u\leq\lambda\}. \tag{4.75}$$

This follows from Assumption 4.2.1 since

$$W(u)-W(h) = W(a+q^un^u)-W(a+q^hn^u) = \int_{q^h}^{q^u}\left[\int_0^q \frac{d^2\Psi}{d\bar{s}^2}d\bar{s}\right]dq,$$

where $\Psi(\bar{s}) := W(a+sn^u)$. Since

$$\frac{d^2\Psi}{d\bar{s}^2}d\bar{s} = (n^u)^T D_{uu}W(u)n^u \geq c_0,\ \forall\ \bar{s}\in(0,q_0),$$

by Assumption 4.2.1 the estimate (4.75) follows by integration.

Hence, using (4.75) we have for the potential term

$$\int_{B_r\cap\{q^h<q^u\}\cap\{q^u\leq\lambda\}}(W(h)-W(u))dx \leq \frac{c_0}{2}\int_{B_r\cap\{q^h<q^u\}\cap\{q^u\leq\lambda\}}((q^u)^2-(q^h)^2)dx. \tag{4.76}$$

Hence, using (4.76) we have that

$$\int_{B_r\cap\{q^h<q^u\}}(W(h)-W(u))dx =$$
$$\int_{B_r\cap\{q^h<q^u\}\cap\{q^u\leq\lambda\}}(W(h)-W(u))dx+\int_{B_r\cap\{q^h<q^u\}\cap\{q^u>\lambda\}}(W(h)-W(u))dx$$
$$\leq\frac{c_0}{2}\int_{B_r\cap\{q^h<q^u\}\cap\{q^u\leq\lambda\}}((q^u)^2-(q^h)^2)dx+\int_{B_r\cap\{q^h<q^u\}\cap\{q^u>\lambda\}}(W(h)-W(u))dx,$$

which is (4.74).

Step 6 . We now consider an upper bound for the second term on the RHS of (4.67), in terms of

$$2\int_{B_r\cap\{q^u>q^h\}}(q^u-q^h)[-(-\Delta)^s q^h]dx\leq$$
$$2\tau\int_{B_r\cap\{q^u>q^h\}\cap\{q^u\leq\lambda\}}q^h(q^u-q^h)dx+2\tau\int_{B_r\cap\{q^u>q^h\}\cap\{q^u>\lambda\}}q^h(q^u-q^h)dx. \tag{4.77}$$

This follows easily by recalling that q^h satisfies (4.69) so that since we are only interested in the subset of B_r where $q^u-q^h\geq 0$, the above follows by using the inequality in (4.69) and by splitting $B_r\cap\{q^u>q^h\}=(B_r\cap\{q^u>q^h\}\cap\{q^u\leq\lambda\})\cup(B_r\cap\{q^u>q^h\}\cap\{q^u>\lambda\})$.

Step 7 . We establish the estimate

$$\bar{C}\,(\lambda')^2\,|B_{r-T}\cap\{q^u>\lambda\}|^{(n-2s)/n}\leq$$
$$\int_{B_r\cap\{q^h<q^u\}\cap\{q^u>\lambda\}}(W(h)-W(u))dx+2\tau\int_{B_r\cap\{q^u>q^h\}\cap\{q^u>\lambda\}}q^h(q^u-q^h)dx. \tag{4.78}$$

This estimate contains only the contributions from the subsets of B_r on which $q^u>\lambda$, hence can be used for the estimation of the Lebesgue measure $|B_r\cap\{q^u>\lambda\}|$.

To prove (4.78) we work as follows: Combining the estimates (4.67) and (4.73) as well as the estimates (4.67), (4.74) and (4.115) we arrive at the estimate

$$\begin{aligned} \bar{C}\,(\lambda')^2\,|B_{r-T}\cap\{q^u>\lambda\}|^{(n-2s)/n} \leq \\ \frac{c_0}{2}\int_{B_r\cap\{q^h<q^u\}\cap\{q^u\leq\lambda\}}((q^h)^2-(q^u)^2)dx+\int_{B_r\cap\{q^h<q^u\}\cap\{q^u>\lambda\}}(W(h)-W(u))dx+ \\ 2\tau\int_{B_r\cap\{q^u>q^h\}\cap\{q^u\leq\lambda\}}q^h(q^u-q^h)dx+2\tau\int_{B_r\cap\{q^u>q^h\}\cap\{q^u>\lambda\}}q^h(q^u-q^h)dx \\ =\int_{B_r\cap\{q^h<q^u\}\cap\{q^u\leq\lambda\}}\left[\frac{c_0}{2}((q^h)^2-(q^u)^2)+2\tau q^h(q^u-q^h)\right]dx+ \\ \int_{B_r\cap\{q^h<q^u\}\cap\{q^u>\lambda\}}(W(h)-W(u))dx+2\tau\int_{B_r\cap\{q^u>q^h\}\cap\{q^u>\lambda\}}q^h(q^u-q^h)dx. \end{aligned} \tag{4.79}$$

We now concentrate at the contributions on the RHS of (4.79) on $B_r\cap\{q^h<q^u\}\cap\{q^u\leq\lambda\}$. We look only at the integrands and collect like terms which are

$$\begin{aligned} \frac{c_0}{2}((q^h)^2-(q^u)^2)+2\tau q^h(q^u-q^h) = \\ (q^h-q^u)\left[\frac{c_0}{2}(q^h+q^u)-2\tau q^h\right] \overset{\frac{c_0}{2}=2\tau}{=} \frac{c_0}{2}(q^h-q^u)q^u\leq 0 \ \text{ on } B_r\cap\{q^h<q^u\}. \end{aligned} \tag{4.80}$$

Hence,

$$\int_{B_r\cap\{q^h<q^u\}\cap\{q^u\leq\lambda\}}\left[\frac{c_0}{2}((q^h)^2-(q^u)^2)+2\tau q^h(q^u-q^h)\right]dx\leq 0,$$

and combining this with (4.79) we arrive at (4.78). Note that from now on we set $2\tau=\frac{c_0}{2}$.

Step 8 . Our strategy now is to estimate the two terms on the RHS of (4.78) in terms of $|B_r\cap\{q^u>\lambda\}|$. Since q^h does not vanish identically on a subset of B_r we need to break B_r into spherical shells on which we may use estimates of q^h to get some control over the integrals. Following [3] we set $r=r_0+pT$ and define $r_j=r_0+jT$, $j=1,\cdots,p$. We adopt the notation $B_0=B_{r_0}$, $B_j=B_{r_j}$, $j=1,\cdots,p$, and note that $B_p=B_r$, while $B_{p-1}=B_{r-T}$. It is easy to see that $B_r=B_0\cup(B_1\setminus B_0)\cup\cdots\cup(B_p\setminus B_{p-1})$, with the corresponding spherical shells being disjoint sets. Moreover, we define

$$\begin{aligned} \omega_0=|B_0\cap\{q^u>\lambda\}|, \\ \omega_j=|(B_j\setminus B_{j-1})\cap\{q^u>\lambda\}|,\ \ j=1,\cdots,p, \end{aligned} \tag{4.81}$$

and note that once we have estimates from below for the quantities ω_0,ω_j, $j=1,\cdots,p$ we can use them to estimate from below the quantity of interest which is $|B_r\cap\{q^u>\lambda\}|=\sum_{j=0}^p\omega_j$.

Step 9 . We claim that the estimate (4.78) leads to an inequality for ω_j of the form

$$C^{\diamondsuit}\left(\sum_{j=0}^{p-1}\omega_j\right)^{(n-2s)/n} \leq \sum_{j=0}^{p-1}(p-j)^{-2s}\omega_j + \omega_p \tag{4.82}$$

for a suitable constant $C^{\diamondsuit}$.

We consider first the LHS of (4.78). Note that $|B_{r-T} \cap \{q^u > \lambda\}| = \sum_{j=0}^{p-1}\omega_j$, so that

$$|B_{r-T} \cap \{q^u > \lambda\}|^{(n-2s)/n} = \left(\sum_{j=0}^{p-1}\omega_j\right)^{(n-2s)/n}. \tag{4.83}$$

We now consider the RHS of (4.78), expressed as

$$I := \int_{B_r\cap\{q^h<q^u\}\cap\{q^u>\lambda\}}\Big[(W(h)-W(u)) + 2\tau q^h(q^u-q^h)\Big]dx.$$

Since $B_r = B_0 \cup (B_1 \setminus B_0) \cup \cdots \cup (B_p \setminus B_{p-1})$ and adopting the notation

$$\begin{aligned} A_0 &= B_0 \cap \{q^h < q^u\} \cap \{q^u > \lambda\}, \\ A_j &= (B_j \setminus B_{j-1}) \cap \{q^h < q^u\} \cap \{q^u > \lambda\}, \qquad j = 1, \cdots, p. \end{aligned}$$

we have that

$$\begin{aligned} I^1 &:= \int_{B_r\cap\{q^h<q^u\}\cap\{q^u>\lambda\}}\Big[(W(h)-W(u)) + 2\tau q^h(q^u-q^h)\Big]\,dx \\ &= \sum_{j=0}^{p}\int_{A_j}\Big[(W(h)-W(u)) + 2\tau q^h(q^u-q^h)\Big]\,dx =: \sum_{j=0}^{p} I_j. \end{aligned}$$

We now estimate each of the above integrals separately.

For $j = p$, we have that $x \in A_p := (B_p \setminus B_{p-1}) \cap \{q^h < q^u\} \cap \{q^u > \lambda\} \subset (B_p \setminus B_{p-1}) \cap \{q^u > \lambda\}$, so that $|x| > r-T$. There we may only use the upper bounds for the potential and the minimizer that hold in general (not near a, as in Assumption 4.2.1). We then have, for the integrand in I_p that

$$\begin{aligned} (W(h)-W(u)) + 2\tau q^h(q^u-q^h) &\leq W(h) + 2\tau q^h q^u \\ &\leq W(h) + 2\tau (q^u)^2 \\ &\leq W_M + 2\tau M^2, \qquad \text{in } A_p. \end{aligned}$$

so that

$$I_p \le (W_M + 2\tau M^2)|A_p| \le (W_M + 2\tau M^2)|(B_p \setminus B_{p-1}) \cap \{q^u > \lambda\}| = (W_M + 2\tau M^2)\omega_p.$$

We now consider the terms for $j = 0, 1, \cdots, p-1$. In all these cases we are within B_{r-T} so that

$$q^h \le \frac{C_\tau M}{2}(T+1)^{-2s} \le \frac{C_\tau M}{2} T^{-2s} \le \lambda - \lambda' < \lambda < q_0.$$

That means that for all these terms we may use the detailed local estimates for the potential, provided by Assumption 4.2.1 i.e., that

$$W(h) \le \frac{c_0'}{2}(q^h)^2.$$

Moroever, in $B_j \setminus B_{j-1}$, $j = 1, \cdots, p-1$ we have that $r_0 + (j-1)T < |x| < r_0 + jT$ so that

$$r + 1 - |x| > r + 1 - r_0 - jT = 1 + (p-j)T \ge (p-j)T,$$

i.e.

$$(r + 1 - |x|)^{-2s} \le T^{-2s}(p-j)^{-2s},$$

hence by Lemma 4.3.1

$$q^h \le \frac{C_\tau M}{2} T^{-2s}(p-j)^{-2s}. \tag{4.84}$$

We will use these estimates to get sharper upper bounds for the integrals I_j, $j = 1, \cdots, p-1$, than those obtained for I_p.

For $j = 1, \cdots, p-1$ we have that $x \in A_j := (B_j \setminus B_{j-1}) \cap \{q^h < q^u\} \cap \{q^u > \lambda\} \subset (B_j \setminus B_{j-1}) \cap \{q^u > \lambda\}$, and we estimate the integrands in I_j as

$$\begin{aligned}
\big(W(h) - W(u)\big) + 2\tau q^h(q^u - q^h) &\le W(h) + 2\tau q^h q^u \\
&\le \frac{c_0'}{2}(q^h)^2 + 2\tau q^h q^u \\
&\le \left(\frac{c_0'}{2} q^u + 2\tau q^u\right) q^h \\
&\le \left(\frac{c_0'}{2} + 2\tau\right) \frac{C_\tau M^2}{2} T^{-2s}(p-j)^{-2s}, \qquad \text{in } A_j.
\end{aligned}$$

The above estimates yield

$$\begin{aligned} I_j &\le \left(\frac{c_0'}{2}+2\tau\right)\frac{C_\tau M^2}{2}\,T^{-2s}(p-j)^{-2s}\,|A_j| \\ &\le \left(\frac{c_0'}{2}+2\tau\right)\frac{C_\tau M^2}{2}\,T^{-2s}(p-j)^{-2s}\,\big|(B_j\setminus B_{j-1})\cap\{q^u>\lambda\}\big| \\ &= \left(\frac{c_0'}{2}+2\tau\right)\frac{C_\tau M^2}{2}\,T^{-2s}(p-j)^{-2s}\,\omega_j, \qquad j=1,\cdots,p-1. \end{aligned}$$

Finally, we consider the $j=0$ term. In A_0 we are in $B_0=B_{r_0}$, i.e., $|x|\le r_0$, so that $(r+1-|x|)^{-2s}<(r+1-r_0)^{-2s}$, and using the upper bound for q^h given in (4.5) we conclude that

$$q^h \le \frac{C_\tau M}{2}(r+1-r_0)^{-2s} = \frac{C_\tau M}{2}(1+pT)^{-2s} \le \frac{C_\tau M}{2}T^{-2s}p^{-2s}.$$

Working as before we have that

$$I_0 \le \left(\frac{c_0'}{2}+2\tau\right)\frac{C_\tau M^2}{2}T^{-2s}p^{-2s}\omega_0.$$

We now combine the above estimates to get the inequality

$$\begin{aligned} \bar{C}(\lambda')^2\left(\sum_{j=0}^{p-1}\omega_j\right)^{(n-2s)/n} &\le \left(\frac{c_0'}{2}+2\tau\right)\frac{C_\tau M^2}{2}\,T^{-2s}\left[p^{-2s}\omega_0+\sum_{j=1}^{p-1}(p-j)^{-2s}\omega_j\right]+(W_M+2\tau M^2)\,\omega_p \\ &= \left(\frac{c_0'}{2}+2\tau\right)\frac{C_\tau M^2}{2}\,T^{-2s}\sum_{j=0}^{p-1}(p-j)^{-2s}\omega_j+(W_M+2\tau M^2)\,\omega_p. \end{aligned}$$

Setting

$$C_M=\max\left\{\left(\frac{c_0'}{2}+2\tau\right)\frac{C_\tau M^2}{2}T^{-2s}, W_M+2\tau M^2\right\}$$

and choosing

$$C^{\diamondsuit}=\frac{\bar{C}(\lambda')^2}{C_M},$$

we arrive at the inequality (4.82).

Step 10 . We now establish that the following lower bounds for ω_p arise from the inequality (4.82),

$$\omega_{p-1}\ge c^* p^{n-1},\ \ p=1,\cdots,\ \ \text{if}\ \ s<1/2, \tag{4.85}$$

for a suitable constant c^* to be determined. In fact it suffices to show that the above estimate (4.85) holds for all $p > p^*$ for a suitable $p^* \in \mathbb{N}$.

We will prove this claim by induction.

Clearly the claim holds for $p = 1$, by assumption.

Assume that it holds for $p-1$. We will show that it also holds for p, using (4.82).

We start by the LHS of (4.82). By the induction hypothesis,

$$\sum_{j=0}^{p-1} \omega_j \geq c^* \sum_{j=1}^{p} j^{n-1} \geq c^* \int_0^p s^{n-1} ds = \frac{c^*}{n-1} p^n,$$

so that for the LHS of (4.82) we have

$$C^{\diamondsuit} \left(\frac{c^*}{n-1} \right)^{(n-2s)/n} p^{n-2s} \leq C^{\diamondsuit} \left(\sum_{j=0}^{p-1} \omega_j \right)^{(n-2s)/n}.$$

Concerning the RHS of (4.82) we start with the elementary upper bounds for ω_j, $j = 1, \cdots, p-1$,

$$\omega_j = |(B_j \setminus B_{j-1}) \cap \{q^u > \lambda\}| \leq |B_j \setminus B_{j-1}| \leq \gamma_n (r_0 + jT)^{n-1} T \leq c_* T^n p^{n-1},$$

where γ_n is the volume of the unit sphere in $\mathbb{R}^n$ and c_* is a constant that depends on r_0 and n. Using this estimate we have that

$$\begin{aligned} \sum_{j=0}^{p-1} (p-j)^{-2s} \omega_j + \omega_p &\leq c_* T^n p^{n-1} \sum_{j=0}^{p-1} (p-j)^{-2s} + \omega_p \\ &= c_* T^n p^{n-1} \sum_{j=1}^{p} j^{-2s} + \omega_p. \end{aligned}$$

Since $s < \frac{1}{2}$ the series $\sum_{j=1}^{\infty} j^{-2s}$ does not converge. We may estimate the series by the corresponding integral and argue the existence of a constant $\hat{c}$, depending on s such that

$$\sum_{j=1}^{p} j^{-2s} \leq \hat{c} p^{1-2s}$$

Then,

$$\sum_{j=0}^{p-1} (p-j)^{-2s} \omega_j + \omega_p \leq c_* \hat{c} T^n p^{n-2s} + \omega_p.$$

Putting all these estimates in (4.82) we find that

$$C^{\diamondsuit}\left(\frac{c^*}{n-1}\right)^{(n-2s)/n} p^{n-2s} \leq c_*\hat{c}T^n p^{n-2s} + \omega_p, \tag{4.86}$$

which if solved in terms of ω_p can provide a lower bound of the form $\omega_p \geq c^* p^{(n-2s)/n}$ as long as c^* is chosen appropriately.

More specifically for any $T > 0$ such that (4.70) holds, we must first choose a c^* such that

$$C^{\diamondsuit}\left(\frac{c^*}{n-1}\right)^{(n-2s)/n} > c_*\hat{c}T^n.$$

Without loss of generality let us choose

$$C^{\diamondsuit}\left(\frac{c^*}{n-1}\right)^{(n-2s)/n} = 2c_*\hat{c}T^n \implies c^* = (n-1)\left(\frac{2c_*\hat{c}T^n}{C^{\diamondsuit}}\right)^{n/(n-2s)}. \tag{4.87}$$

Then, (4.86) yields

$$\begin{aligned}\omega_p \geq c_*\hat{c}T^n p^{n-2s} = c_*\hat{c}T^n p^{1-2s} p^{n-1} &\overset{(p\geq p^*,\ 2s<1)}{\geq} c_*\hat{c}T^n (p^*)^{1-2s} p^{n-1} \\ &\geq c_*\hat{c}T^n (p^*)^{1-2s}\left(\frac{p}{p+1}\right)^{n-1}(p+1)^{n-1} \\ &\overset{\frac{p}{p+1}\geq 1/2}{\geq} 2c_*\hat{c}\left(\frac{T}{2}\right)^n (p^*)^{1-2s}(p+1)^{n-1} \\ &\geq c^*(p+1)^{n-1}, \quad p \geq p^*,\end{aligned}$$

as long as c^* is chosen such that

$$c^* < 2c_*\hat{c}\left(\frac{T}{2}\right)^n (p^*)^{1-2s}.$$

Note that (4.88) and (4.87) are compatible if

$$(n-1)\left(\frac{2c_*\hat{c}T^n}{C^{\diamondsuit}}\right)^{n/(n-2s)} = c^* < 2c_*\hat{c}\left(\frac{T}{2}\right)^n (p^*)^{1-2s}, \tag{4.88}$$

as long as $p^* \in \mathbb{N}$ is chosen large enough, and in particular

$$p^* = [(n-1)^{\frac{1}{1-2s}} 2^{n-1} (C^{\diamondsuit})^{-\frac{n}{(n-2s)(1-2s)}} (c_*\hat{c})^{\frac{2s}{(n-2s)(1-2s)}} T^{\frac{2sn}{(n-2s)(1-2s)}}] + 1. \tag{4.89}$$

This proves the validity of (4.85) for all $p \geq p^*$ with p^* depending on T (and ultimately on λ) as well as the constants of the potential term. The fact that $p \geq p^*$ implies that the density estimate will hold for balls with sufficiently large radius r.

Step 11 . Using the lower bound for ω_p derived in step 10, for $s < 1/2$, and working as in [3] we can obtain the required density result. For the sake of completeness we repeat the arguments in [3].

Let $k^* := [\frac{r-r_0}{T}]$ be the integer part of $\frac{r-r_0}{T}$. Then,

$$\begin{aligned}|B_r \cap \{q^u > \lambda\}| \geq |B_{r_0+k^*T} \cap \{q^u > \lambda\}| &= \sum_{j=1}^{k^*+1} \omega_{j-1} \\ &\geq c^* \sum_{j=p^*}^{k^*+1} j^{n-1} \\ &\geq \frac{c^*}{n}(k^*+1)^n = \frac{c^*}{nT^n}(k^*T+T)^n \geq \frac{c^*}{nT^n} r^n.\end{aligned}$$

which is the required result. □

Remark 4.4.2. The above estimates seem to blow up for $n = 1$, however the theorem must hold true also for this case. One way to see this is to look for more refined asymptotic behaviour in (4.85) of the form $\omega_p > c^* p^\gamma$ for a suitable $\gamma > 0$ to be determined. We can rework Step 10 with this choice and we see that the argument works as long as γ satisfies

$$\frac{(\gamma+1)(n-2s)}{n} \geq n-2s, \text{ and } \frac{(\gamma+1)(n-2s)}{n} = \gamma.$$

The second condition gives that

$$\gamma = \frac{n-2s}{2s},$$

which when substituted in the first condition makes the first condition equivalent to

$$\begin{aligned}\frac{n-2s}{2s} > n-2s \;\Leftrightarrow\; n-2s > 2sn-(2s)^2 \;\Leftrightarrow\; 2s(2s-1) > n(2s-1) \\ \Leftrightarrow 2s < n\end{aligned}$$

which is true.

So in this case we establish that $\omega_p \geq c^* p^\gamma$ which completes the induction step.

That implies that $\omega_p \geq c^* p^{\frac{n-2s}{2s}}$. Note that since $\frac{n-2s}{2s} \geq n-1$ this is compatible with the estimate in [3] that $\omega_p \geq c^* p^{n-1}$. Moroever, using this refined estimate Step 11 works for $n = 1$ as well.

4.5 The Density Theorem for the case $1 < \alpha < 2$

Theorem 4.5.1. *Let $s < 1/2$ and assume that the potential satisfies Assumption 4.2.2, $\mathcal{O}$ is open and $u : \mathcal{O} \subset \mathbb{R}^n \to \mathbb{R}^m$ is minimal in the De Giorgi sense. Then, for any $\mu_0 > 0$, and any $\lambda \in (0, d_0)$ where $d_0 = \min_z \{|a - z| \,:\, z \neq a,\ W(z) = 0\} > 0$, the condition*

$$|B_{r_0}(x_0) \cap \{|u - a| > \lambda\}| \geq \mu_0$$

implies that

$$|B_{r_0}(x_0) \cap \{|u - a| > \lambda\}| \geq C r^n, \ \ \text{for } r \geq r_0,$$

as long as $B_r(x_0) \subset \mathcal{O}$ where $C = C(W, \mu_0, \lambda, r_0, M)$.

Proof. The proof has certain steps in common with the proof of Theorem 4.4.1 for the case of $\alpha = 2$.

Steps 1 and 2 are identical and are not repeated. We therefore establish the estimate

$$\begin{aligned} \bar{C}\left(\int_{B_r} (q^u - q^\sigma)^{2n/(n-2s)}\right)^{(n-2s)/n} \leq & \int_{B_r \cap \{q^u > q^h = q^\sigma\}} (W(\sigma) - W(u)) dx \\ & + 2\int_{B_r \cap \{q^u > q^h\}} (q^u - q^h)[-(-\Delta)^s q^h] dx. \end{aligned} \tag{4.90}$$

Step 3′ . We now choose q^h to be the solution of the nonlocal differential inequality

$$\begin{aligned} -(-\Delta)^s q^h(x) \leq \tau_0 q^h(x)^\nu, \ \forall\, x \in B_r, \\ q^h(x) = M, \ x \in B_r^c, \end{aligned} \tag{4.91}$$

for τ_0 and ν to be determined shortly. By Lemma 4.3.3 such a test function exists for $\nu > 0$ and satisfies the bounds

$$\underline{C}(r + 1 - |x|)^{-2s/\nu} \leq q^h(x) \leq \bar{C}(r + 1 - |x|)^{-2s/\nu}, \ \ x \in B_r, \ \ \nu > 0,$$

for suitable constants $\underline{C}$ and $\bar{C}$. For uniformity of notation with Theorem 4.4.1 we rewrite these constants as

$$\underline{C} = \frac{M}{2C_\tau}, \;\; \bar{C} = \frac{MC_\tau}{2}, \;\; \sigma(\nu) = \nu,$$

so that we express the bounds of Lemma 4.3.3 for q^h as

$$\frac{M}{2C_\tau}(r+1-|x|)^{-2s/\nu r} \leq q^h \leq \frac{MC_\tau}{2}(r+1-|x|)^{-2s/\nu r}. \tag{4.92}$$

For $x \in B_{r-T}$ the above bound yields that

$$q^h(x) \leq \frac{MC_\tau}{2}(T+1)^{-2s/\nu r} \leq \frac{MC_\tau}{2}T^{-2s/\nu r}.$$

Clearly by definition

$$q^\sigma = \min(q^h, q^u) \leq q^h \leq \frac{MC_\tau}{2}T^{-2s/\nu r}.$$

We set a $\lambda' \in (0, \lambda)$ (say $\lambda' = \frac{\lambda}{2}$) and choose $T > 0$ such that

$$\frac{MC_\tau}{2}T^{-2s/\nu r} \leq \lambda - \lambda'.$$

Then,

$$q^u - q^\sigma = q^u - q^h > \lambda', \;\text{ in } B_{r-T} \cap \{q^u > \lambda\}. \tag{4.93}$$

Indeed, q^σ by definition can either be q^u or q^h. On $B_{r-T} \cap \{q^u > \lambda\}$ we have that $q^u > \lambda$, whereas

$$q^\sigma = \min\{q^u, q^h\} \leq q^h \leq \frac{MC_\tau}{2}T^{-2s/\nu r} \leq \lambda - \lambda' < \lambda,$$

so that $q^u > q^\sigma$ and hence $q^\sigma = q^h$. Therefore, on this set

$$q^u - q^\sigma = q^u - q^h > \lambda - q^h > \lambda - (\lambda - \lambda') = \lambda',$$

and the claim (4.93) is established. Moreover, by(4.93) it follows that

$$B_{r-T} \cap \{q^u > \lambda\} \cap \{q^u - q^h \leq \lambda'\} = \emptyset. \tag{4.94}$$

Step 4 is identical and allows us to establish the estimate

$$\left(\int_{B_r}(q^u-q^\sigma)^{2n/(n-2s)}dx\right)^{(n-2s)/n} \geq (\lambda')^2\,|B_{r-T}\cap\{q^u>\lambda\}|^{(n-2s)/n}, \tag{4.95}$$

that will be used along with the previous estimate (4.90).

Step 5′ . We now consider the RHS of (4.90) and try to establish an upper bound for the potential difference term. We claim that

$$\begin{aligned}\int_{B_r}(W(\sigma)-W(u))dx &= \int_{B_r\cap\{q^h<q^u\}}(W(h)-W(u))dx\\ \leq C^*\int_{B_r\cap\{q^h<q^u\}\cap\{q^u\leq\lambda\}}((q^h)^\alpha-(q^u)^\alpha)dx &+ \int_{B_r\cap\{q^h<q^u\}\cap\{q^u>\lambda\}}(W(h)-W(u))dx\end{aligned} \tag{4.96}$$

This requires use of the assumption 4.2.2 on the potential function.

Before proving (4.96) we note that since $q^\sigma=\min\{q^u,q^h\}$ we have that

$$\begin{aligned}\int_{B_r}\big(W(\sigma)-W(u)\big)\,dx &= \int_{(B_r\cap\{q^\sigma=q^u\})\cup\{q^\sigma<q^u\}}\big(W(\sigma)-W(u)\big)\,dx\\ &= \int_{B_r\cap\{q^h<q^u\}}\big(W(h)-W(u)\big)\,dx.\end{aligned}$$

This comes from the fact that

$$B_r=\{q^\sigma<q^u\}\cup(B_r\cap\{q^\sigma=q^u\}) \text{ and } \{q^\sigma<q^u\}=\{q^h<q^u\},$$

which follows by the definition of q^σ and σ (q^σ may take either the value q^h or q^u).

We now as in [3], under the assumption that $\lambda\leq q_0$ (see Assumption 4.2.2 for the definition of q_0) approximate the potential difference with a local power approximation. On $\{q^h<q^u\}\cap\{q^u\leq\lambda\}$ it holds that $q^u\leq q_0$ and $q^\sigma=\min(q^h,q^u)\leq q^u$ so we have that

$$\begin{aligned}\frac{d}{d\rho}W(a+\rho\xi)\geq\alpha C^*\rho^{\alpha-1} \implies & \int_{q^\sigma}^{q^u}\frac{d}{d\rho}W(a+\rho\xi)d\rho\geq\int_{q^\sigma}^{q^u}\alpha C^*\rho^{\alpha-1}d\rho\\ \implies & W(u)-W(\sigma)\geq C^*[\,(q^u)^\alpha-(q^\sigma)^\alpha\,]\end{aligned} \tag{4.97}$$

Hence, using (4.97) we have for the potential term

$$\int_{B_r\cap\{q^h<q^u\}\cap\{q^u\leq\lambda\}}(W(h)-W(u))dx\leq\frac{c_0}{2}\int_{B_r\cap\{q^h<q^u\}\cap\{q^u\leq\lambda\}}((q^h)^\alpha-(q^u)^\alpha)dx. \tag{4.98}$$

Hence, using (4.98) we have that

$$\begin{aligned}\int_{B_r\cap\{q^h<q^u\}}\big(W(h)-W(u)\big)\,dx &= \int_{B_r\cap\{q^h<q^u\}\cap\{q^u\le\lambda\}}\big(W(h)-W(u)\big)\,dx\\ &\quad+\int_{B_r\cap\{q^h<q^u\}\cap\{q^u>\lambda\}}\big(W(h)-W(u)\big)\,dx\\ &\le \frac{c_0}{2}\int_{B_r\cap\{q^h<q^u\}\cap\{q^u\le\lambda\}}\big((q^h)^\alpha-(q^u)^\alpha\big)\,dx\\ &\quad+\int_{B_r\cap\{q^h<q^u\}\cap\{q^u>\lambda\}}\big(W(h)-W(u)\big)\,dx.\end{aligned}$$

which is (4.96).

Step 6′ . We now substitute estimates (4.95) and (4.96) into (4.90) which leads to the estimate

$$\begin{aligned}\bar{C}\left(\int_{B_r}(q^u-q^\sigma)^{2n/(n-2s)}\right)^{(n-2s)/n} &\le C^*\int_{B_r\cap\{q^h<q^u\}\cap\{q^u\le\lambda\}}\big((q^h)^\alpha-(q^u)^\alpha\big)\,dx\\ &\quad+\int_{B_r\cap\{q^h<q^u\}\cap\{q^u>\lambda\}}\big(W(h)-W(u)\big)\,dx\\ &\quad+2\int_{B_r\cap\{q^u>q^h\}}(q^u-q^h)\big[-(-\Delta)^s q^h\big]\,dx.\end{aligned}\tag{4.99}$$

By the choice of q^h as in (4.91) this yields

$$\begin{aligned}&\bar{C}\left(\int_{B_r}(q^u-q^\sigma)^{2n/(n-2s)}\right)^{(n-2s)/n}\le\\ &C^*\int_{B_r\cap\{q^h<q^u\}\cap\{q^u\le\lambda\}}((q^h)^\alpha-(q^u)^\alpha)dx+\int_{B_r\cap\{q^h<q^u\}\cap\{q^u>\lambda\}}(W(h)-W(u))dx\\ &+2\tau_0\int_{B_r\cap\{q^u>q^h\}\cap\{q^u\le\lambda\}}(q^u-q^h)(q^h)^\nu dx+2\int_{B_r\cap\{q^u>q^h\}\cap\{q^u\le\lambda\}}(q^u-q^h)[-(-\Delta)^s q^h]dx\\ &=\underbrace{C^*\int_{B_r\cap\{q^h<q^u\}\cap\{q^u\le\lambda\}}((q^h)^\alpha-(q^u)^\alpha)dx+2\tau_0\int_{B_r\cap\{q^u>q^h\}\cap\{q^u\le\lambda\}}(q^u-q^h)(q^h)^\nu dx}_{I}\\ &+\underbrace{\int_{B_r\cap\{q^h<q^u\}\cap\{q^u>\lambda\}}(W(h)-W(u))dx+2\int_{B_r\cap\{q^u>q^h\}\cap\{q^u\le\lambda\}}(q^u-q^h)[-(-\Delta)^s q^h]dx}_{II}\\ &=: I+II,\end{aligned}$$

where I contains the contributions from $q^u\le\lambda$, and II contains the contribution from $q^u>\lambda$, i.e.,

$$I := C^* \int_{B_r\cap\{q^h<q^u\}\cap\{q^u\le\lambda\}} ((q^h)^\alpha - (q^u)^\alpha)dx + 2\tau_0 \int_{B_r\cap\{q^u>q^h\}\cap\{q^h\le\lambda\}} (q^u-q^h)(q^h)^\nu dx,$$

and

$$II := \int_{B_r\cap\{q^h<q^u\}\cap\{q^u>\lambda\}} (W(h)-W(u))dx + 2\int_{B_r\cap\{q^u>q^h\}\cap\{q^u>\lambda\}} (q^u-q^h)[-(-\Delta)^s q^h]dx.$$

We first consider the term I.

We choose

$$\alpha = \nu+1, \;\; \nu > 0$$

and we have that

$$\begin{aligned} I = &\int_{B_r\cap\{q^h<q^u\}\cap\{q^u\le\lambda\}} \Big[C^*(q^h)^\alpha - 2\tau_0 (q^h)^{\nu+1}\Big]dx + \int_{B_r\cap\{q^h<q^u\}\cap\{q^u\le\lambda\}} \Big[2\tau_0 q^u (q^h)^\nu - C^*(q^u)^\alpha\Big]dx \le \\ &\int_{B_r\cap\{q^h<q^u\}\cap\{q^u\le\lambda\}} \Big[C^*(q^h)^\alpha - 2\tau_0 (q^h)^{\nu+1}\Big]dx + \int_{B_r\cap\{q^h<q^u\}\cap\{q^u\le\lambda\}} \Big[2\tau_0 (q^u)^{\nu+1} - C^*(q^u)^\alpha\Big]dx, \end{aligned} \tag{4.100}$$

where for the second integral we have used the obvious estimate

$$q^h < q^u \implies (q^h)^\nu < (q^u)^\nu, \;\text{ since }\; \nu > 0.$$

Note that in term I we use the form of the inequality $-(-\Delta)^s q^h \le \tau_0 (q^h)^\nu$ whereas in term II we temporarily leave the fractional Laplacian term of the test function as it is and we will bound it in the next step accordingly.

Choosing $\nu + 1 = \alpha$ and $2\tau_0 = C^*$ we see from (4.100) that

$$I \le 0,$$

hence term I can be omitted from the right hand side of (4.99) which yields

$$\begin{aligned} \bar{C}\left(\int_{B_r} (q^u - q^\sigma)^{2n/(n-2s)}\right)^{(n-2s)/n} &\le II \\ &= \int_{B_r\cap\{q^h<q^u\}\cap\{q^u>\lambda\}} (W(h)-W(u))\,dx \\ &\quad + 2\int_{B_r\cap\{q^u>q^h\}\cap\{q^u>\lambda\}} (q^u-q^h)\big[-(-\Delta)^s q^h\big]\,dx. \end{aligned} \tag{4.101}$$

Hence term I can be ommited from the right hand side of the above inequalities and we only need to consider term II.

Step $7'$. Combining the observations we obtained in step $6'$, i.e. (4.101) with (4.95) we obtain the estimate

$$\bar{C}\,(\lambda')^2\left|B_{r-T}\cap\{q^u>\lambda\}\right|^{(n-2s)/n}\le\int_{B_r\cap\{q^h<q^u\}\cap\{q^u>\lambda\}}(W(h)-W(u))\,dx \\ +\int_{B_r\cap\{q^u>q^h\}\cap\{q^u>\lambda\}}\left[-(-\Delta)^s q^h\right](q^u-q^h)\,dx. \tag{4.102}$$

This estimate contains only the contributions from the subsets of B_r on which $q^u>\lambda$, hence can be used for the estimation of the Lebesgue measure $|B_r\cap\{q^u>\lambda\}|$. Note, this is identical to what we obtained in the $\alpha=2$ case in which case $\nu=1$ (i.e., estimate (4.78)).

Step 8 is identical as in the $\alpha=2$ case. It is only repeated here for convenience (to recall the definitions or the relevant quantities).

Step 8 . Our strategy now is to estimate the two terms on the RHS of (4.78) in terms of $|B_r\cap\{q^u>\lambda\}|$. Since q^h does not vanish identically on a subset of B_r we need to break B_r into spherical shells on which we may use estimates of q^h to get some control over the integrals. Following [3] we set $r=r_0+pT$ and define $r_j=r_0+jT$, $j=1,\cdots,p$. We adopt the notation $B_0=B_{r_0}$, $B_j=B_{r_j}$, $j=1,\cdots,p$, and note that $B_p=B_r$, while $B_{p-1}=B_{r-T}$. It is easy to see that $B_r=B_0\cup(B_1\setminus B_0)\cup\cdots\cup(B_p\setminus B_{p-1})$, with the corresponding spherical shells being disjoint sets. Moreover, we define

$$\omega_0=\left|B_0\cap\{q^u>\lambda\}\right|, \\ \omega_j=\left|(B_j\setminus B_{j-1})\cap\{q^u>\lambda\}\right|,\qquad j=1,\cdots,p. \tag{4.103}$$

and note that once we have estimates from below for the quantities ω_0,ω_j, $j=1,\cdots,p$ we can use them to estimate from below the quantity of interest which is $|B_r\cap\{q^u>\lambda\}|=\sum_{j=0}^{p}\omega_j$.

Step $9'$. We claim that the estimate (4.102) leads to an inequality for ω_j of the form

$$C^{\diamondsuit}\left(\sum_{j=0}^{p-1}\omega_j\right)^{(n-2s)/n}\le\sum_{j=0}^{p-1}(p-j)^{-2s}\omega_j+\omega_p \tag{4.104}$$

for a suitable constant $C^{\diamondsuit}$.

We consider first the LHS of (4.102). Note that $|B_{r-T} \cap \{q^u > \lambda\}| = \sum_{j=0}^{p-1} \omega_j$, so that

$$|B_{r-T} \cap \{q^u > \lambda\}|^{(n-2s)/n} = \left(\sum_{j=0}^{p-1} \omega_j \right)^{(n-2s)/n}. \tag{4.105}$$

We now consider the RHS of (4.102).

We will consider the case $0 < \nu < 1$ (corresponding to $1 < \alpha < 2$).

We use the inequality for q^h according to which $-[-\Delta]^s q^h \le (q^h)^\nu$, and substitutite it in the RHS of (4.102) which is now expressed as

$$I := \int_{B_r \cap \{q^h < q^u\} \cap \{q^u > \lambda\}} \Big[(W(h) - W(u)) + 2\tau_0 (q^h)^\nu (q^u - q^h) \Big] dx.$$

Since $B_r = B_0 \cup (B_1 \setminus B_0) \cup \cdots \cup (B_p \setminus B_{p-1})$ and adopting the notation

$$A_0 = B_0 \cap \{q^h < q^u\} \cap \{q^u > \lambda\},$$
$$A_j = (B_j \setminus B_{j-1}) \cap \{q^h < q^u\} \cap \{q^u > \lambda\}, \qquad j = 1, \cdots, p.$$

we have that

$$\begin{aligned} I^1 &:= \int_{B_r \cap \{q^h < q^u\} \cap \{q^u > \lambda\}} \Big[(W(h) - W(u)) + 2\tau_0 (q^h)^\nu (q^u - q^h) \Big] dx \\ &= \sum_{j=0}^{p} \int_{A_j} \Big[(W(h) - W(u)) + 2\tau_0 (q^h)^\nu (q^u - q^h) \Big] dx =: \sum_{j=0}^{p} I_j. \end{aligned}$$

We now estimate each of the above integrals separately.

For $j = p$, we have that $x \in A_p := (B_p \setminus B_{p-1}) \cap \{q^h < q^u\} \cap \{q^u > \lambda\} \subset (B_p \setminus B_{p-1}) \cap \{q^u > \lambda\}$, so that $|x| > r - T$. There we may only use the upper bounds for the potential and the minimizer that hold in general (not near a, as in Assumption 4.2.2). We then have, for the integrand in I_p that

$$\begin{aligned} (W(h) - W(u)) + 2\tau_0 (q^h)^\nu (q^u - q^h) &\le W(h) + 2\tau_0 (q^h)^\nu q^u \\ &\le W(h) + 2\tau_0 (q^u)^\nu q^u \\ &\le W(h) + 2\tau_0 (q^u)^\alpha \\ &\le W_M + 2\tau_0 M^\alpha, \qquad \text{in } A_p. \end{aligned}$$

so that

$$I_p \leq (W_M + 2\tau_0 M^\alpha)|A_p| \leq (W_M + 2\tau_0 M^\alpha)|(B_p \setminus B_{p-1}) \cap \{q^u > \lambda\}| = (W_M + 2\tau_0 M^\alpha)\omega_p.$$

We now consider the terms for $j = 0, 1, \cdots, p-1$. In all these cases we are within B_{r-T} so that

$$q^h \leq \frac{C_\tau M}{2}(T+1)^{-2s} \leq \frac{C_\tau M}{2} T^{-2s} \leq \lambda - \lambda' < \lambda < q_0.$$

That means that for all these terms we may use the detailed local estimates for the potential, provided by Assumption 4.2.2 i.e., that

$$W(h) \leq \frac{c_0'}{2}(q^h)^\alpha. \tag{4.106}$$

Moroever, in $B_j \setminus B_{j-1}$, $j = 1, \cdots, p-1$ we have that $r_0 + (j-1)T < |x| < r_0 + jT$ so that by Lemma 4.3.3 (see estimates (4.92))

$$q^h \leq \frac{C_\tau M}{2}[(p-j)T]^{-2s/\nu} = \frac{C_\tau M}{2} T^{-2s/\nu}(p-j)^{-2s/\nu},$$

hence taking the ν power we obtain that,

$$(q^h)^\nu \leq \left(\frac{C_\tau M}{2}\right)^\nu T^{-2s}(p-j)^{-2s}.$$

We will use these estimates to get sharper upper bounds for the integrals I_j, $j = 1, \cdots, p-1$, than those obtained for I_p.

For $j = 1, \cdots, p-1$ we have that $x \in A_j := (B_j \setminus B_{j-1}) \cap \{q^h < q^u\} \cap \{q^u > \lambda\} \subset (B_j \setminus B_{j-1}) \cap \{q^u > \lambda\}$, and we estimate the integrands in I_j as

$$\begin{aligned}
\big(W(h) - W(u)\big) + 2\tau_0 (q^h)^\nu (q^u - q^h) &\leq W(h) + 2\tau_0 (q^h)^\nu q^u \\
&\leq \frac{c_0'}{2}(q^h)^\alpha + 2\tau_0 (q^h)^\nu q^u = \frac{c_0'}{2}(q^h)^{\nu+1} + 2\tau_0 (q^h)^\nu q^u \\
&= \left(\frac{c_0'}{2} q^h + 2\tau_0 q^u\right)(q^h)^\nu \\
&\leq \left(\frac{c_0'}{2} q^u + 2\tau_0 q^u\right)(q^h)^\nu \\
&\leq \left(\frac{c_0'}{2} + 2\tau_0\right)\left(\frac{C_\tau M}{2}\right)^\nu T^{-2s}(p-j)^{-2s}, \qquad \text{in } A_j.
\end{aligned}$$

The above estimates yield

$$
\begin{aligned}
I_j &\le \left(\frac{c_0'}{2}+2\tau_0\right)\left(\frac{C_\tau M}{2}\right)^{\nu} M\,T^{-2s}(p-j)^{-2s}\,|A_j| \\
&\le \left(\frac{c_0'}{2}+2\tau_0\right)\left(\frac{C_\tau M}{2}\right)^{\nu} M\,T^{-2s}(p-j)^{-2s}\,\big|(B_j\setminus B_{j-1})\cap\{q^u>\lambda\}\big| \\
&= \left(\frac{c_0'}{2}+2\tau_0\right)\left(\frac{C_\tau M}{2}\right)^{\nu} M\,T^{-2s}(p-j)^{-2s}\,\omega_j, \qquad j=1,\cdots,p-1.
\end{aligned}
$$

Finally, we consider the $j=0$ term. In A_0 we are in $B_0=B_{r_0}$, i.e., $|x|\le r_0$, so that $(r+1-|x|)^{-2s/\nu} < (r+1-r_0)^{-2s/\nu}$, and using the upper bound for q^h given in (4.5) we conclude that

$$
q^h \le \frac{C_\tau M}{2}(r+1-|x|)^{-2s/\nu} \le \frac{C_\tau M}{2}(r+1-r_0)^{-2s/\nu} = \frac{C_\tau M}{2}(1+pT)^{-2s/\nu} \le \frac{C_\tau M}{2}T^{-2s}p^{-2s/\nu}.
$$

Working as before we have that

$$
I_0 \le \left(\frac{c_0'}{2}+2\tau_0\right)\left(\frac{C_\tau M}{2}\right)^{\nu} MT^{-2s}p^{-2s}\omega_0.
$$

We now combine the above estimates to get the inequality

$$
\begin{aligned}
\bar{C}(\lambda')^2\Bigg(\sum_{j=0}^{p-1}\omega_j\Bigg)^{(n-2s)/n} &\le \left(\frac{c_0'}{2}+2\tau_0\right)\left(\frac{C_\tau M}{2}\right)^{\nu} M\,T^{-2s}\sum_{j=0}^{p-1}(p-j)^{-2s}\omega_j \\
&\quad + (W_M+2\tau_0 M^{\alpha})\,\omega_p.
\end{aligned}
$$

Setting

$$
C_M = \max\left\{\left(\frac{c_0'}{2}+2\tau_0\right)\left(\frac{C_\tau M}{2}\right)^{\nu} MT^{-2s}, W_M+2\tau_0 M^{\alpha}\right\}
$$

and choosing

$$
C^{\diamondsuit} = \frac{\bar{C}(\lambda')^2}{C_M},
$$

we arrive at the inequality (4.104).

Note that (4.104) is identical to the relevant inequality (4.82) we obtained for the $\alpha=2$ case.

The steps 10 and 11 are identical as those of Theorem 4.4.1 for the $\alpha=2$ case. The proof is complete. $\square$

4.6 The Density Theorem for the case $0 < \alpha < 1$

Theorem 4.6.1. *Let $s < 1/2$ and assume that the potential satisfies Assumption 4.2.3, $\mathcal{O}$ is open and $u : \mathcal{O} \subset \mathbb{R}^n \to \mathbb{R}^m$ is minimal in the De Giorgi sense. Then, for any $\mu_0 > 0$, and any $\lambda \in (0, d_0)$ where $d_0 = \min_z\{|a - z| \, : \, z \neq a, \; W(z) = 0\} > 0$, the condition*

$$|B_{r_0}(x_0) \cap \{|u - a| > \lambda\}| \geq \mu_0$$

implies that

$$|B_r(x_0) \cap \{|u - a| > \lambda\}| \geq Cr^n, \;\; \text{for } r \geq r_0,$$

as long as $B_r(x_0) \subset \mathcal{O}$ where $C = C(W, \mu_0, \lambda, r_0, M)$.

Remark 4.6.2 (Relation with the local case)**.** The initial goal of the proof is to construct the analogue of the variational inequality (2.33) in [2], namely

$$c(\lambda)\big(V(r-T)^{\frac{n-2s}{n}} + A(r-T)\big) \leq \big(V(r) - V(r-T)\big) + \big(A(r) - A(r-T)\big), \tag{4.111}$$

where

$$V(r) = L^n\big(B_r \cap \{q^u > \lambda\}\big), \tag{4.112}$$

and

$$A(r) = \int_{B_r \cap \{q^u < \lambda\}} W(u)\, dx, \tag{4.113}$$

for $T > T^*$, the constant arising in the construction of the test function.

Our inequality should be viewed as its nonlocal counterpart for fractional energies, with the exponent $\frac{n-2s}{n}$ reflecting the fractional scaling.

Remark 4.6.3. The first steps of the proof are independent of s.

Proof. Step 1 . First we control the RHS of the inequality

$$\begin{aligned} J^{\varrho}(u - \sigma, B_r) \leq & \int_{B_r \cap \{q^u > q^h = q^\sigma\}} (W(\sigma) - W(u))\, dx \\ & + 2 \int_{B_r \cap \{q^u > q^h\}} (q^u - q^h) \big[- (-\Delta)^s q^h \big]\, dx. \end{aligned} \tag{4.107}$$

We split the domain $B_r \cap \{q^u > q^h\}$ in:

$$B_r \cap \{q^u > q^h\} = \{B_{r-T} \cap \{q^u > q^h\}\} \cup \{(B_r \setminus B_{r-T}) \cap \{q^u > q^h\}\}. \tag{4.108}$$

We further split $B_{r-T} \cap \{q^u > q^h\}$ in:

$$B_{r-T} \cap \{q^u > q^h\} \cap \{q^u \leq \lambda\} := A_1 \tag{4.109}$$

and

$$B_{r-T} \cap \{q^u > q^h\} \cap \{q^u > \lambda\} := A_2 \tag{4.110}$$

and both in A_1, A_2 it holds

$$q^h = 0 < q^u,$$
$$W(\sigma) = W(a) = 0.$$

Step 2 .In A_1, for the quantities inside the integrals in 4.1 we have:

$$\begin{aligned} W(\sigma) - W(u) + 2(q^u - q^h)\big[-(-\Delta)^s q^h\big] &= -W(u) + 2q^u\big[-(-\Delta)^s q^h\big] \\ &\leq -\frac{W(u)}{2} - \frac{C^*(q^u)^\alpha}{2} + 2q^u M C_o(\hat{w} + \beta)^\nu \\ &= -\frac{W(u)}{2} - \frac{C^*(q^u)^\alpha}{2} + 2q^u M C_o \beta^\nu. \end{aligned}$$

since $q^u = \hat{w} = 0$ in B_{r-T}.

In the above calculation, we also used that from the hypotheses of the potential and for $q^u \leq \lambda \leq \rho_0$ it follows that

$$W(u) = \int_0^{q^u} W_u(\alpha + s\mathbf{n}^u)\mathbf{n}^u ds \geq C^*(q^u)^\alpha.$$

Now, we claim that for suitable $T > T_1$ (will be defined shortly) we have

$$-\frac{C^*(q^u)^\alpha}{2} + 2q^u M C_o \beta^\nu \leq 0.$$

Proof.

$$-\frac{C^*(q^u)^\alpha}{2}+2q^uMC_o\beta^\nu =$$
$$(q^u)^\alpha((q^u)^{1-\alpha}MC_o\beta^\nu-\frac{C^*}{2})\leq$$
$$(q^u)^\alpha((M^{1-\alpha}MC_o\beta^\nu-\frac{C^*}{2}).$$

We recall that $\beta=T^{-\gamma}=T^{-2s/\nu}$, hence, :

$$M^{1-\alpha}MC_o\beta^\nu-\frac{C^*}{2}\leq 0\Rightarrow$$
$$M^{2-\alpha}C_o\beta^\nu-\frac{C^*}{2}\leq 0\Rightarrow$$
$$\beta^\nu\leq\frac{C^*}{2M^{2-\alpha}C_o}\Rightarrow$$
$$T^{-2s}\leq\frac{C^*}{2M^{2-\alpha}C_o}\Rightarrow$$
$$T\geq(\frac{C^*}{2M^{2-\alpha}C_o})^{-\frac{1}{2s}}:=T_1.$$

□

Step 3 . In A_2, first we set $W_m=inf_{\lambda<|u-a|\leq M}$ W(u), and for the quantities inside the integrals in 4.1 we have:

$$W(\sigma)-W(u)+2(q^u-q^h)[-(-\Delta)^sq^h]=$$
$$-W(u)+2q^u[-(-\Delta)^sq^h]\leq$$
$$-\frac{W(u)}{2}-\frac{W_m}{2}+2q^uMC_o(\hat{w}+\beta)^\nu=$$

$$-\frac{W(u)}{2}-\frac{W_m}{2}+2q^uMC_o\beta^\nu\leq$$
$$-\frac{W(u)}{2}-\frac{W_m}{2}+2M^2C_o\beta^\nu.$$

Arguing as before, we claim the existence of T_2, such that for $T>T_2$ it holds:

$$-\frac{W_m}{2}+2M^2C_o\beta^\nu\leq 0.$$

Proof.

$$
\begin{aligned}
-\frac{W_m}{2} + 2M^2 C_o \beta^\nu \leq 0 \Rightarrow \\
\beta^\nu \leq \frac{W_m}{2M^2 C_o} \Rightarrow \\
T^{-2s} \leq \frac{W_m}{2M^2 C_o} \Rightarrow \\
T \geq (\frac{W_m}{2M^2 C_o})^{-\frac{1}{2s}} := T_2.
\end{aligned}
$$

Finally, we choose $T \geq max\{T_1, T_2\} := T^*$. □

And, so, in $B_{r-T} \cap \{q^u > q^h\}$:

$$
\begin{aligned}
&\int_{B_{r-T}\cap\{q^u>q^h\}} \big(W(\sigma) - W(u)\big)\, dx + 2\int_{B_{r-T}\cap\{q^u>q^h\}} (q^u - q^h)\big[-(-\Delta)^s q^h\big]\, dx \\
&= \int_{A_1} \big(W(\sigma) - W(u)\big)\, dx + 2\int_{A_1} (q^u - q^h)\big[-(-\Delta)^s q^h\big]\, dx \\
&\quad + \int_{A_2} \big(W(\sigma) - W(u)\big)\, dx + 2\int_{A_2} (q^u - q^h)\big[-(-\Delta)^s q^h\big]\, dx \\
&\leq -\int_{A_1} \frac{W(u)}{2}\, dx - \int_{A_2} \frac{W(u)}{2}\, dx \\
&= -\int_{B_{r-T}\cap\{q^u>0\}} \frac{W(u)}{2}\, dx = -\int_{B_{r-T}} \frac{W(u)}{2}\, dx \\
&\leq -\int_{B_{r-T}\cap\{q^u<\lambda\}} \frac{W(u)}{2}\, dx = -\frac{1}{2}A(r-T).
\end{aligned}
$$

Step 4 . Now, we split the domain $(B_r \setminus B_{r-T}) \cap \{q^u > q^h\}$ in

$$
(B_r \setminus B_{r-T}) \cap \{q^h < q^u < \lambda\} := D_1
$$

and

$$
(B_r \setminus B_{r-T}) \cap \{q^u > q^h\} \cap \{q^u > \lambda\} := D_2.
$$

For the first term on the RHS in 4.1 using the monotonicity of the potential in $q^u < \lambda$ (assuming $\lambda \leq \rho_0$) becomes

$$
\int_{D_1} W(\sigma) - W(u) dx \leq 0,
$$

while in D_2 and by setting $W_M = max_{|u-a|\le M} W(u)$:

$$\int_{D_2} \big(W(\sigma) - W(u)\big)\, dx \le \int_{D_2} W(\sigma)\, dx$$
$$\le W_M\, L^n\big((B_r \setminus B_{r-T}) \cap \{q^u > \lambda\}\big) = W_M\big(V(r) - V(r-T)\big).$$

since $D_2 \subseteq B_r \setminus B_{r-T} \cap \{q^u > \lambda\}$.

The second term on the RHS in 4.1, assuming, furthermore, that $\lambda \le min\{\rho_0, 1\}$ and by the fact that $D_1 \subseteq (B_r \setminus B_{r-T}) \cap \{q^u < \lambda\}$, becomes

$$\begin{aligned}
2\int_{D_1} (q^u - q^h)\big[-(-\Delta)^s q^h\big]\, dx &\le 2\int_{D_1} q^u M C_o (q^h + \beta)^\nu\, dx \\
&\le 2\int_{D_1} q^u M^*\, dx \\
&\le 2\int_{D_1} (q^u)^\alpha M^*\, dx \\
&\le \frac{2M^*}{C^*} \int_{D_1} W(u)\, dx \\
&\le \frac{2M^*}{C^*}\big(A(r) - A(r-T)\big).
\end{aligned}$$

where $M^* = C_o M (M+\beta)^\nu$.

Remark 4.6.4. We note that the estimate obtained above,

$$2\int_{D_1} (q_u - q_h)\,[-(-\Delta)^s q_h]\, dx \ \le\ \frac{2M^*}{C^*}\big(A(r) - A(r-T)\big),$$

with

$$M^* = C_0 M (M+\beta)^\nu,$$

is the only place in the proof where the exponent ν coming from the barrier inequality

$$-(-\Delta)^s q_h \ \le\ C_0 (q_h + \beta)^\nu,$$

enters the argument. In order to treat in a unified way all exponents $0 < \alpha < 2$, one would like to relate ν quantitatively to the growth exponent α of W near the well, so that $(q_h + \beta)^\nu$ can be absorbed by the term $W(u) \sim |u-a|^\alpha$ in a uniform way. However, with our present construction of q_h in Section 4.3 the exponent ν is essentially fixed (it depends only on s and the geometry), and the constant M^* then develops a strong dependence on ν which prevents us from obtaining an estimate that is uniform in $\alpha \in (0,2)$.

For this reason the above estimate is sufficient to handle the case $0 < \alpha < 1$, but does not allow us to incorporate the case $1 < \alpha < 2$ into the same scheme. A more flexible construction of the barrier, allowing a precise coupling between ν and α, would remove this technical restriction, but requires a finer analysis of $-(-\Delta)^s q_h$ and is left for future work.

Step 5 . In D_2,

$$
\begin{aligned}
2\int_{D_2} (q^u - q^h)\big[-(-\Delta)^s q^h\big]\,dx &\le 2\int_{D_2} MM^*\,dx \\
&\le 2MM^*\,L^n\big((B_r \setminus B_{r-T}) \cap \{q^u > \lambda\}\big) \\
&= 2MM^*\big(V(r) - V(r-T)\big).
\end{aligned}
$$

since $D_2 \subseteq B_r \setminus B_{r-T} \cap \{q^u > \lambda\}$ and M^* as above.

Step 6 . Collecting all the previous estimates, 4.1 becomes

$$
\begin{aligned}
J^{\varrho}(u-\sigma, B_r) &\le -\frac{1}{2}A(r-T) + W_M\big(V(r) - V(r-T)\big) + \frac{2M^*}{C^*}\big(A(r) - A(r-T)\big) \\
&\quad + 2MM^*\big(V(r) - V(r-T)\big) \\
&\le -\frac{1}{2}A(r-T) + \big(W_M + 2MM^*\big)\big(V(r) - V(r-T)\big) + \frac{2M^*}{C^*}\big(A(r) - A(r-T)\big) \\
&\le -\frac{1}{2}A(r-T) + C^{**}\Big(\big(V(r) - V(r-T)\big) + \big(A(r) - A(r-T)\big)\Big).
\end{aligned}
\tag{4.111}
$$

where $C^{**} = max\{W_M + 2MM^*, \frac{2M^*}{C^*}\}$.

Step 7 . Regarding the LHS of 4.1, it has been established through the fractional Sobolev inequality that

$$
J^{\varrho}(u-\sigma, B_r) \ge C\left(\int_{B_R} (q^u - q^{\sigma})^{2n/(n-2s)}dx\right)^{(n-2s)/n},
$$

and, since, $q^\sigma = q^h = 0$ in B_{r-T} we get

$$\begin{aligned} C\left(\int_{B_R} (q^u - q^\sigma)^{\frac{2n}{n-2s}}\, dx\right)^{\frac{n-2s}{n}} &\geq C\left(\int_{B_{r-T}\cap\{q^u>\lambda\}} (q^u - q^\sigma)^{\frac{2n}{n-2s}}\, dx\right)^{\frac{n-2s}{n}} \\ &= C\left(\int_{B_{r-T}\cap\{q^u>\lambda\}} (q^u)^{\frac{2n}{n-2s}}\, dx\right)^{\frac{n-2s}{n}} \\ &\geq C\left(\int_{B_{r-T}\cap\{q^u>\lambda\}} \lambda^{\frac{2n}{n-2s}}\, dx\right)^{\frac{n-2s}{n}} \\ &\geq \lambda^2\, L^n\big(B_{r-T}\cap\{q^u>\lambda\}\big)^{\frac{n-2s}{n}} = \lambda^2\big(V(r-T)\big)^{\frac{n-2s}{n}}. \end{aligned}$$

From this and 4.111, 4.1 becomes

$$\begin{aligned} \lambda^2\,(V(r-T))^{(n-2s)/n} + \frac{1}{2}A(r-T) \leq C^{**}((V(r)-V(r-T)) + (A(r)-A(r-T))) \Rightarrow \\ c(\lambda)(V^{\frac{n-2s}{n}}(r-T) + A(r-T)) \leq (V(r)-V(r-T)) + (A(r)-A(r-T)) \end{aligned} \tag{4.112}$$

where $c(\lambda) = \frac{min\{\lambda^2,\frac{1}{2}\}}{C^{**}}$.

Step 8 . We will prove that 4.111 leads to

$$V(r) + A(r) \geq Cr^n,$$

for a constant C that will be defined during the proof.

***Claim* 4.6.5.** There exist $C' := C'(\lambda, \mu_0)$ such that

$$V(kT) + A(kT) \geq C' k^n, \tag{4.113}$$

for $k \geq k_0$, k a positive integer.

Proof. We will prove the claim by induction.

From the hypotheses of the theorem we know that $V(r_0) \geq \mu_0$. We set $r_0 = k_0 T$, for k_0 a constant such that $k_0 T > 1$ and $k_0^{2s} c(\lambda) > 2^{n+1}$, where $c(\lambda) = \frac{min\{\lambda^2,\frac{1}{2}\}}{C^{**}}$ the constant in 4.112. In addition, we choose C' such that $\mu_0 \geq C' k_0^n$ and we have

$$V(r_0) + A(r_0) = V(k_0 T) + A(k_0 T) \geq V(k_o T) \geq \mu_0 \geq C' k_0^n$$

and so 4.113 holds for $k = k_0$.

Now, we set $r - T = \bar{k}T$ and we assume that 4.113 holds for $k = \bar{k}$:

$$V(\bar{k}T) + A(\bar{k}T) \geq C^{'}\bar{k}^n.$$

For the induction step and for $k = \bar{k} + 1 \left(r - T = \bar{k}T \Rightarrow r = (\bar{k}+1)T\right)$, 4.112 becomes

$$V((\bar{k}+1)T) + A((\bar{k}+1)T) \geq V(\bar{k}T) + A(\bar{k}T) + c(\lambda)\left(V(\bar{k}T)^{\frac{n-2s}{n}} + A(\bar{k}T)\right). \quad (4.114)$$

Now, we argue that from the inequality

$$V(\bar{k}T) + A(\bar{k}T) \geq C^{'}\bar{k}^n,$$

we conclude that either

$$V(\bar{k}T) \geq \frac{1}{2}C^{'}\bar{k}^n,$$

or

$$A(\bar{k}T) \geq \frac{1}{2}C^{'}\bar{k}^n.$$

If $A(\bar{k}T) \geq \frac{1}{2}C^{'}\bar{k}^n$ then

$$c(\lambda)A(\bar{k}T) \geq c(\lambda)\frac{1}{2}C^{'}\bar{k}^n \geq \frac{1}{2}C^{'}\bar{k}^n\frac{2^{n+1}}{k_0^{2s}} \geq \frac{1}{2}C^{'}\bar{k}^n\frac{2^{n+1}}{\bar{k}^{2s}} \geq 2^n C^{'}\bar{k}^{n-2s},$$

else

$$c(\lambda)V(\bar{k}T)^{\frac{n-2s}{n}} \geq c(\lambda)\left(\frac{1}{2}C^{'}\bar{k}^n\right)^{\frac{n-2s}{n}} = c(\lambda)\left(\frac{1}{2}C^{'}\right)^{\frac{n-2s}{n}}\bar{k}^{n-2s}.$$

Now, regarding the constant C', we further demand

$$c(\lambda)\left(\frac{1}{2}C^{'}\right)^{\frac{n-2s}{n}} \geq 2^n C^{'} \Rightarrow C^{'} \leq \left(\frac{c(\lambda)}{2^{\frac{n-2s}{n}}2^n}\right)^{\frac{n}{2s}} := c_1$$

and, finally, we choose

$$C' \leq min\{c_1, \frac{\mu_0}{k_0^n}\}.$$

So,

$$c(\lambda)V(\bar{k}T)^{\frac{n-2s}{n}} \geq 2^n C' \bar{k}^{n-2s}.$$

Therefore from 4.114 we get

$$V((\bar{k}+1)T) + A((\bar{k}+1)T) \geq C^{'}\bar{k}^n + 2^n C^{'}\bar{k}^{n-2s} \geq C^{'}\bar{k}^n + 2^n C^{'}\bar{k}^{n-1} \geq C^{'}(\bar{k}+1)^n.$$

Remark 4.6.6. Since $s \leq \frac{1}{2}$ it holds $n - 2s \geq n - 1$.

The induction is proved and so the claim.

<u>Step 9 .</u> □

The final step is to prove that

$$V(kT) \geq C'k^n,$$

where $r = kT$.

From 4.113 and the fundamental estimate (upper bound of the energy) for the case $s < \frac{1}{2}$:

$$A(kT) \leq Cr^{n-2s},$$

we have:

$$V(kT) \geq C^{'}k^n - Cr^{n-2s} = C^{'}\frac{r^n}{T^n} - Cr^{n-2s} = \left(\frac{C^{'}}{T^n} - Cr^{-2s}\right) r^n \geq \frac{C^{'}}{2T^n} r^n,$$

as long as

$$\frac{C^{'}}{T^n} - Cr^{-2s} \geq \frac{C^{'}}{2T^n} \Rightarrow r \geq (\frac{C^{'}}{2CT^n})^{-\frac{1}{2s}}.$$

The proof for the case $s < 1/2$ is completed. □

Remark 4.6.7. For the case $s = \frac{1}{2}$, the difference is the upper bound of energy:

$$A(kT) \leq Cr^{n-1} ln r,$$

and so, the final step becomes

$$V(kT) \geq C^{'}k^n - Cr^{n-1}lnr = C^{'}\frac{r^n}{T^n} - Cr^{n-1}lnr = \left(\frac{C^{'}}{T^n} - Cr^{-1}lnr\right)r^n \geq \frac{C^{'}}{2T^n}r^n,$$

as long as

$$\frac{C^{'}}{T^n} - Cr^{-1}lnr \geq \frac{C^{'}}{2T^n} \Rightarrow \frac{r}{lnr} \geq (\frac{C^{'}}{2CT^n})^{-1}.$$

The existence of such r that satisfies the last inequality is justified by the fact that the function $f(r) = \frac{r}{lnr}$ is increasing for $r \geq e$ and also $f(r) \to \infty$ while $r \to \infty$.

Finally, for the case $s > \frac{1}{2}$ the upper bound of the energy is:

$$A(kT) \leq Cr^{n-1}.$$

We observe that this upper bound is the same as in the classical case and we can direct calculate the final step:

$$V(kT) \geq C^{'}k^n - Cr^{n-1} = C^{'}\frac{r^n}{T^n} - Cr^{n-1} = \left(\frac{C^{'}}{T^n} - Cr^{-1}\right)r^n \geq \frac{C^{'}}{2T^n}r^n.$$

The above calculation is valid if

$$\frac{C^{'}}{T^n} - Cr^{-1} \geq \frac{C^{'}}{2T^n} \Rightarrow r \geq \left(\frac{C^{'}}{2CT^n}\right)^{-1}.$$

4.7 The proof for $s > 1/2$

In Sections 4.4–4.6 we established the density estimates in the regime $s < \frac{1}{2}$, relying crucially on the lower bound (4.86), which is a fractional Sobolev-type estimate for the interaction term. In this section we treat the complementary range $s > \frac{1}{2}$. The overall structure of the proof is essentially the same as in the case $s < \frac{1}{2}$; the genuinely new ingredient is the *lower* bound used in place of (4.86).

More precisely, in Step 5′ we invoke the interaction estimate of Savin and Valdinoci (see, for instance, [37, 38]), which yields

$$L(a_r, d_r) + |b_r| \geq c_0|a_r|^{\frac{n-1}{n}}, \qquad s > \tfrac{1}{2},$$

see (4.128). This inequality provides a geometric lower control on the interaction term in terms of the measure of the transition region, and, once combined with the preceding steps, leads to the inequality (4.134) with exponent $(n-1)/n$ on the left-hand side. The subsequent combinatorial argument and the final use of the energy upper bound proceed exactly as in the $s < \frac{1}{2}$ case and yield the desired density estimate with the correct scaling.

We first carry out the argument in detail under the nondegeneracy assumption $\alpha = 2$, and then indicate the modifications required for the critical case $s = \frac{1}{2}$. Finally, Remark 4.7.1 explains how the same scheme can be extended to the case $0 < \alpha < 2$, at the expense of adjusting the growth assumptions on W and the corresponding energy bounds; for brevity we only record the statement there and omit the full details.

4.7.1 $\alpha = 2$

Proof. We will label the steps by a number and a prime in order to make the easiest comparison with the proof in the case $s < 1/2$.

Step $1'$. The minimality of the radial part of the energy leads to the estimate

$$J^{\varrho}(u-\sigma, B_r) \leq \int_{B_r \cap \{q^u > q^h = q^\sigma\}} (W(\sigma) - W(u))\, dx + 2\int_{B_r \cap \{q^u > q^h\}} (q^u - q^h)[-(-\Delta)^s q^h]\, dx \tag{4.115}$$

This exactly the Step 1 in the $s < 1/2$ case.

Step $2'$. Choosing a test function q^h, and q^σ, T and λ, λ' exactly as in Step 3 in the $s < 1/2$ case, we obtain from (4.115) the estimate

$$J^{\varrho}(u-\sigma, B_r) \leq \int_{B_r \cap \{q^u > q^h = q^\sigma\}} (W(\sigma) - W(u)) dx + 2\tau \int_{B_r \cap \{q^u > q^h\}} q^h(q^u - q^h) dx. \tag{4.116}$$

The only difference here is that we do not fully specify λ' at this stage, but simply constraint it so that $\lambda' \in (0, \lambda)$. Its precise value will be determined at a later step.

Step $3'$. We now consider the LHS of (4.116) first and try to obtain a suitable lower bound for this term.

Following [38] we define the sets

$$a_r = B_r \cap \{q^u - q^h \geq \lambda'\},$$
$$b_r = B_r \cap \left\{\frac{\lambda'}{2} < q^u - q^h < \lambda'\right\},$$
$$d_r = (a_r \cup b_r)^c = B_r^c \cup \left(B_r \cap \left\{q^u - q^h \leq \frac{\lambda'}{2}\right\}\right).$$

Moreover it is easy to see that

$$B_{r-T} \cap \{q^u > \lambda\} \subset a_r. \tag{4.117}$$

Indeed, by the choice of T, for any $x \in B_{r-T} \cap \{q^u > \lambda\}$ it holds that $q^h(x) < \lambda - \lambda'$ and if additionaly $q^u(x) > \lambda$ then $(q^u - q^h)(x) > \lambda'$, hence $x \in a_r$.

We also have that

$$|a_r| \geq |B_{r-T} \cap \{q^u > \lambda\}| \geq |B_{r_0} \cap q^u > \lambda\}| \geq \mu > 0,$$

with the last bound coming by hypothesis.

We now consider

$$\begin{aligned}
\int_{a_r}\int_{d_r} \frac{\big|(q^u - q^\sigma)(x) - (q^u - q^\sigma)(y)\big|^2}{|x-y|^{n+2s}}\,dx\,dy &= \int_{a_r}\int_{B_r^c} \frac{\big|(q^u - q^\sigma)(x) - (q^u - q^\sigma)(y)\big|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad + \int_{a_r}\int_{B_r \cap \{q^u - q^h < \frac{\lambda'}{2}\}} \frac{\big|(q^u - q^\sigma)(x) - (q^u - q^\sigma)(y)\big|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\leq \int_{B_r}\int_{B_r^c} \frac{\big|(q^u - q^\sigma)(x) - (q^u - q^\sigma)(y)\big|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad + \int_{B_r}\int_{B_r} \frac{\big|(q^u - q^\sigma)(x) - (q^u - q^\sigma)(y)\big|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&= J^\varrho(u - \sigma; B_r).
\end{aligned}$$

hence,

$$J^\varrho(u - \sigma; B_r) \geq \int_{a_r}\int_{d_r} \frac{|(q^u - q^\sigma)(x) - (q^u - q^\sigma)(y)|^2}{|x-y|^{n+2s}} dxdy.$$

On the other hand, if $y \in d_r$, then

$$(q^u - q^\sigma)(y) \leq \frac{\lambda'}{2}.$$

Indeed, be definition q^σ is either q^u or q^h. If $q^\sigma = q^u$ the inequality is trivially true. If $q^\sigma = q^h$ then $q^u - q^\sigma = q^u - q^h \le \frac{\lambda'}{2}$ in d_r.

By the same argument, on a_r, we have that $q^u - q^\sigma$ is either 0 or q^h.

Hence, by the triangle inequality, for $x \in a_r$, $y \in d_r$ we have,

$$|(q^u - q^\sigma)(x) - (q^u - q^\sigma)(y)| \ge (q^u - q^\sigma)(x) - (q^u - q^\sigma)(y)$$

$$\ge \begin{cases} -(q^u - q^\sigma)(y) \ge \frac{\lambda'}{2} & \text{if} \quad q^\sigma(x) = q^u(x) \\ (q^u - q^\sigma)(x) - (q^u - q^\sigma)(y) \ge \lambda' - \frac{\lambda'}{2} = \frac{\lambda'}{2} & \text{if} \quad q^\sigma(x) = q^h(x), \end{cases}$$

so that

$$|(q^u - q^\sigma)(x) - (q^u - q^\sigma)(y)| \ge \frac{\lambda'}{2}, \quad \forall\, x \in a_r,\ y \in d_r. \tag{4.118}$$

Upon using the notation

$$L(A, B) := \int_A \int_B \frac{1}{|x-y|^{n+2s}} dxdy,$$

we see using (4.118) that

$$J^\varrho(u - \sigma; B_r) \ge \int_{a_r} \int_{d_r} \frac{|(q^u - q^\sigma)(x) - (q^u - q^\sigma)(y)|^2}{|x-y|^{n+2s}} dxdy \ge \left(\frac{\lambda'}{2}\right)^2 L(a_r, d_r). \tag{4.119}$$

<u>Step 4$'$.</u> Combining (4.116) and (4.119) we have that

$$\left(\frac{\lambda'}{2}\right)^2 L(a_r, d_r) \le \int_{B_r \cap \{q^u > q^h = q^\sigma\}} (W(\sigma) - W(u))dx + 2\tau \int_{B_r \cap \{q^u > q^h\}} q^h(q^u - q^h)dx. \tag{4.120}$$

This will be the starting point for our estimations.

<u>Step 5$'$.</u> Arguing as in [38] we have that

$$L(a_r, d_r) + |b_r| \ge c_0 |a_r|^{\frac{n-1}{n}}, \quad s > 1/2, \quad c_0 > 0, \tag{4.121}$$

with c_0 possibly depending on μ.

Adding $\left(\frac{\lambda'}{2}\right)^2 |b_r|$ on both sides of (4.120) and using (4.121) we arrive at the estimate

$$\begin{aligned} c_0 \left(\frac{\lambda'}{2}\right)^2 |a_r|^{\frac{n-1}{n}} &\le \left(\frac{\lambda'}{2}\right)^2 \big(L(a_r, d_r) + |b_r|\big) \\ &\le \int_{B_r \cap \{q^u > q^h = q^\sigma\}} (W(\sigma) - W(u))\, dx + 2\tau \int_{B_r \cap \{q^u > q^h\}} q^h(q^u - q^h)\, dx \\ &\quad + \left(\frac{\lambda'}{2}\right)^2 |b_r|. \end{aligned} \tag{4.122}$$

By further combining (4.117) with (4.122) we have that

$$c_0\left(\frac{\lambda'}{2}\right)^2|B_{r-T}\cap\{q^u>\lambda\}|^{\frac{n-1}{n}} \leq \int_{B_r\cap\{q^u>q^h=q^\sigma\}}(W(\sigma)-W(u))dx \\ +2\tau\int_{B_r\cap\{q^u>q^h\}} q^h(q^u-q^h)dx+\left(\frac{\lambda'}{2}\right)^2|b_r|. \tag{4.123}$$

The LHS term in (4.123) is exactly what we want and all the terms on the RHS of (4.123) - except the last one - are in such form that can be manipulated easily so that the induction proof of the case $s>1/2$ can be used. Luckilly the last term is multiplied by $\left(\frac{\lambda'}{2}\right)^2$, where $\lambda'\in(0,\lambda)$ is as yet unspecified, so that it can be chosen small enough for its contrubution to be suppressed by the negative terms of the above. The negative term over which we have control is the term $-2\tau(q^h)^2$ so this will be our candidate.

We express the RHS of (4.123) as

$$\begin{aligned} I &:= \int_{B_r\cap\{q^u>q^h=q^\sigma\}}(W(\sigma)-W(u))\,dx+2\tau\int_{B_r\cap\{q^u>q^h\}} q^h(q^u-q^h)\,dx+\left(\frac{\lambda'}{2}\right)^2|b_r| \\ &= \int_{B_r\cap\{q^u>q^h=q^\sigma\}}(W(\sigma)-W(u))\,dx+\tau\int_{B_r\cap\{q^u>q^h\}} q^h(q^u-q^h)\,dx \\ &\quad +\tau\int_{B_r\cap\{q^u>q^h\}} q^hq^u\,dx+\left\{-\tau\int_{B_r\cap\{q^u>q^h\}}(q^h)^2\,dx+\left(\frac{\lambda'}{2}\right)^2|b_r|\right\}, \end{aligned} \tag{4.124}$$

and (given τ - which is still undetermined) concentrate in choosing λ' so that the term in the curly brackets in (4.124) is negative.

To this end we need to use lower bounds for the test function q^h. By Lemma 4.3.1, for any $x\in B_r$, we have that

$$q^h\geq\frac{M}{2C_\tau}(r+1-|x|)^{-2s}\geq\frac{M}{2C_\tau}(r+1)^{-2s},$$

so that

$$-\tau(q^h)^2\leq-\tau\left(\frac{M}{2C_\tau}\right)^2(r+1)^{-4s}.$$

Then, we have for the term in the curly brackets, that

$$\begin{aligned} \left\{-\tau\int_{B_r\cap\{q^u>q^h\}}(q^h)^2dx+\left(\frac{\lambda'}{2}\right)^2|b_r|\right\} &\leq -\tau\left(\frac{M}{2C_\tau}\right)^2(r+1)^{-4s}|B_r\cap\{q^u>q^h\}|+\left(\frac{\lambda'}{2}\right)^2|b_r| \\ &\leq\left\{-\tau\left(\frac{M}{2C_\tau}\right)^2(r+1)^{-4s}+\left(\frac{\lambda'}{2}\right)^2\right\}|b_r|\leq 0, \end{aligned}$$

as long as λ' is chosen so that

$$\lambda' < \sqrt{\tau}\frac{M}{C_\tau}(r+1)^{-2s}.$$

So, choosing

$$\lambda' = \min\left\{\sqrt{\tau}\frac{M}{C_\tau}(r+1)^{-2s}, \lambda\right\},$$

from (4.124) we obtain the estimate

$$\begin{aligned} I &\leq \int_{B_r\cap\{q^u>q^h=q^\sigma\}}(W(\sigma)-W(u))\,dx + \tau\int_{B_r\cap\{q^u>q^h\}}q^h(q^u-q^h)\,dx \\ &\quad+\tau\int_{B_r\cap\{q^u>q^h\}}q^hq^u\,dx \\ &\leq \int_{B_r\cap\{q^u>q^h=q^\sigma\}}(W(\sigma)-W(u))\,dx + 2\tau\int_{B_r\cap\{q^u>q^h\}}q^h(q^u-q^h)\,dx. \end{aligned}$$

Combining the above with (4.123) we conclude that

$$c_0\left(\frac{\lambda'}{2}\right)^2|B_{r-T}\cap\{q^u>\lambda\}|^{\frac{n-1}{n}} \leq \int_{B_r\cap\{q^u>q^h=q^\sigma\}}(W(\sigma)-W(u))dx + 2\tau\int_{B_r\cap\{q^u>q^h\}}q^h(q^u-q^h)dx. \tag{4.125}$$

<u>Step 6′</u>. Working in exactly the same way as in steps 5, 6 and 7 of the $s > 1/2$ case we obtain from (4.125) the estimate

$$\begin{aligned} c_0\left(\frac{\lambda'}{2}\right)^2\big|B_{r-T}\cap\{q^u>\lambda\}\big|^{\frac{n-1}{n}} &\leq \int_{B_r\cap\{q^h<q^u\}\cap\{q^u>\lambda\}}(W(h)-W(u))\,dx \\ &\quad+2\tau\int_{B_r\cap\{q^u>q^h\}\cap\{q^u>\lambda\}}q^h(q^u-q^h)\,dx. \end{aligned} \tag{4.126}$$

which is the analogue of (4.78)

<u>Step 7′</u>. Using the same notation (see (4.103)) and construction as in Step 8 of the $s < 1/2$ case we obtain following verbatim the calculations in Step 9 of the $s < 1/2$ case the estimate

$$C^\diamondsuit\left(\sum_{j=0}^{p-1}\omega_j\right)^{(n-1)/n} \leq \sum_{j=0}^{p-1}(p-j)^{-2s}\omega_j + \omega_p, \tag{4.127}$$

for a suitable constant $C^\diamondsuit$. Note the difference in the exponent on the LHS of this inequality with the corresponding result for the $s < 1/2$ case in (4.82).

Step 8$'$. We claim that (4.127) establishes a lower bound

$$\omega_{p-1} \geq c^* p^{n-1}, \;\; p = 1, \cdots, \tag{4.128}$$

for a suitable constant c^* to be determined.

This is proved by induction as in Step 10 of the $s < 1/2$ case.

We start by the LHS of (4.82). By assumption it holds for $p = 1$. Assume that it holds for $p-1$. We intend to show that it also holds for p.

By the induction hypothesis,

$$\sum_{j=0}^{p-1} \omega_j \geq c^* \sum_{j=1}^{p} j^{n-1} \geq c^* \int_0^p s^{n-1} ds = \frac{c^*}{n} p^n,$$

so that for the LHS of (4.82) we have

$$C^{\diamondsuit} \left(\frac{c^*}{n} \right)^{(n-1)/n} p^{n-1} \leq C^{\diamondsuit} \left(\sum_{j=0}^{p-1} \omega_j \right)^{(n-1)/n}.$$

Concerning the RHS of (4.82) we start with the elementary upper bounds for ω_j, $j = 1, \cdots, p-1$,

$$\omega_j = |(B_j \setminus B_{j-1}) \cap \{q^u > \lambda\}| \subset |B_j \setminus B_{j-1}| \leq \gamma_n (r_0 + jT)^{n-1} T \leq c_* T^n p^{n-1},$$

where γ_n is the volume of the unit sphere in $\mathbb{R}^n$ and c_* is a constant that depends on r_0 and n. Using this estimate we have that

$$\begin{aligned}
\sum_{j=0}^{p-1} (p-j)^{-2s} \omega_j + \omega_p &\leq c_* T^n p^{n-1} \sum_{j=0}^{p-1} (p-j)^{-2s} + \omega_p \\
&= c_* T^n p^{n-1} \sum_{j=1}^{p} j^{-2s} + \omega_p \\
&\leq c_* T^n p^{n-1} \sum_{j=1}^{\infty} j^{-2s} + \omega_p \\
&= \Lambda(s)\, c_* T^n p^{n-1} + \omega_p.
\end{aligned}$$

so that solving the inequality for ω_p we see that the claim is true for p, for appropriate choice of c_*.

Step $9'$. The lower bound (4.128) leads to the conclusion that $|B_r \cap \{q^u > \lambda\}| > c^* r^n$ in exactly the same way as in Step 11 of the $s > 1/2$ case.

The proof is complete.

4.7.2 The $s = 1/2$ case

For the $s = 1/2$ case we may follow a combination of the proofs of the $s > 1/2$ and $s < 1/2$ case. We briefly check the main changes. Here again we may not use the Sobolev type inequality we used in the $s > 1/2$ case, so we proceed as in steps $1'$, $2'$, $3'$ and $4'$ of the $s < 1/2$ case and conclude (4.120). The main difference here is that (4.121) is replaced by (see [38])

$$L(a_r, d_r) + |b_r| \geq c_0 |a_r|^{\frac{n-1}{n}} (\ln |a_r| + 1), \quad s = 1/2, \quad c_0 > 0. \tag{4.129}$$

The remainder of Step $5'$ proceed unchanged so that (4.125) is replaced by

$$c_0 |a_r|^{\frac{n-1}{n}} (\ln |a_r| + 1) \leq \int_{B_r \cap \{q^u > q^h = q^\sigma\}} (W(\sigma) - W(u)) dx + 2\tau \int_{B_r \cap \{q^u > q^h\}} q^h (q^u - q^h) dx. \tag{4.130}$$

Recall also that $|a_r| \geq |B_{r-T} \cap \{q^u > \lambda\}|$.

Steps $6'$ and $7'$ proceed without any changes concerning the RHS of the estimates leading to

$$C^\diamondsuit |a_r|^{\frac{n-1}{n}} \ln |a_r| \leq C^\diamondsuit |a_r|^{\frac{n-1}{n}} (\ln |a_r| + 1) \leq \sum_{j=0}^{p-1} (p-j)^{-1} \omega_j + \omega_p, \tag{4.131}$$

for a suitable constant $C^\diamondsuit$, which replaces (4.127).

We may now prove by induction that (4.131) implies that $\omega_{p-1} \geq c_* p^{n-1}$, $p = 1, \cdots$, for a suitable c_*. Note that the induction hypothesis and the fact that $|a_r| \geq |B_{r-T} \cap \{q^u > \lambda\}|$ implies

$$|a_r| \geq \sum_{j=1}^{p-1} \omega_j \geq \frac{c^*}{n} p^n,$$

so that

$$\ln |a_r| \geq c' \ln p.$$

. Combining the above we have that the LHS of (4.131) satisfies

$$\frac{c^*}{n-1}p^n(c'\ln p+1) \le LHS = C^{\diamondsuit}|a_r|^{\frac{n-1}{n}}(\ln|a_r|+1).$$

Now for the RHS we have, using the trivial upper bound $\omega_j \le \gamma_n p^{n-1}$, that

$$RHS = \sum_{j=0}^{p-1}(p-j)^{-1}\omega_j + \omega_p \le c''p^{n-1}\sum_{j=0}^{p-1}(p-j)^{-1} + \omega_p \le c'''p^{n-1}\ln p + \omega_p.$$

Comparing the LHS and RHS estimates we can see that for suitable constant c^*, we obtain from (4.131) that

$$\omega_p \ge c^* p^{n-1}\ln p \ge c^* p^{n-1},$$

for suitable c^* and large enough p ($p > 3$). The induction hypothesis is complete and the lower bound is established. Then the proof concludes as in Step 11 or Step $9'$. □

Remark 4.7.1. For the cases $0 < \alpha < 2$, we will follow the same strategy as in the case $\alpha = 2$. The complete details will be presented in a future work.

4.8 Applications

4.8.1 Pointwise Estimates

In this subsection we show how the density estimate forces a quantitative pointwise control of minimizers near a given well. Roughly speaking, if a minimizer u is close to a minimum $a \in \{W = 0\}$ at one point and the ball around that point is sufficiently large, then u must in fact stay uniformly close to a at the center. In the local theory this type of conclusion is often obtained by combining Lipschitz bounds with the density estimate; in the present nonlocal setting we cannot assume that u is Lipschitz, so we replace this ingredient by an interior Hölder regularity result for fractional systems (see, for instance, [40, Section 5]).

By coupling this regularity with the Density Theorem, we derive a quantitative radius r_q such that the smallness of $|u - a|$ in a ball of radius r_q forces $|u(x_0) - a| < q$ at its center. The structure of the argument is closely inspired by the corresponding pointwise estimate for local vector minimizers in [2, Section 3], and yields a nonlocal analogue of those results in the spirit of the classical case.

Theorem 4.8.1. *Assume that W satisfies the assumptions as in the Density Theorem and $u : \mathcal{O} \subset \mathbb{R}^n \to \mathbb{R}^m$ is minimal in the De Giorgi sense. Let $Z_a := \{W = 0\} \setminus \{a\}$ and suppose either $Z_a = \varnothing$ or $d_0 := \inf_{x\in\mathcal{O}} \mathrm{dist}\big(u(x), Z_a\big) > 0$. Then, given $q \in (0, M)$ (with $M = \|u\|_{L^\infty}$), there exists a radius $r_q > 0$* depending only on n, s, W, M *(and on d_0 when $Z_a \neq \varnothing$) such that*

$$B_{r_q}(x_0) \subset \mathcal{O} \quad \Longrightarrow \quad |u(x_0) - a| < q.$$

Proof. We will prove this theorem by contradiction. Given $q \in (0, M)$, assume

$$|u(x_0) - a| \geq q.$$

Interior Hölder regularity gives $|u(x) - u(x_0)| \leq C_0|x - x_0|^{\alpha_0}$ on $B_{r_0}(x_0)$ for some $0 < \alpha_0 < 2s$. Then the (Hölder) continuity of u implies that the condition

$$|B_{r_0}(x_0) \cap \{|u - a| > \lambda\}| \geq \mu_0,$$

in the Density Theorem, is satisfied for

$$r_0 := (\frac{\lambda}{2C_0})^{1/\alpha_0}, \qquad \mu_0 := \mathcal{L}^n(B_{r_0}(x_0)), \qquad \lambda = q/2.$$

Then

$$\mathcal{L}^n\Big(B_{r_0}(x_0) \cap \big\{|u - a| \geq \tfrac{\lambda}{2}\big\}\Big) \;\geq\; \mu_0.$$

Indeed, for any $x \in B_{r_0}(x_0)$ we have, by the Hölder estimate $|u(x) - u(x_0)| \leq C_0|x - x_0|^{\alpha_0}$,

$$|u(x) - u(x_0)| \;\leq\; C_0\, r_0^{\alpha_0} = C_0 \left(\tfrac{\lambda}{2C_0}\right) = \tfrac{\lambda}{2}.$$

Hence

$$|u(x) - a| \;\geq\; |u(x_0) - a| - |u(x) - u(x_0)| \;\geq\; \lambda - \tfrac{\lambda}{2} = \tfrac{\lambda}{2},$$

so the entire ball $B_{r_0}(x_0)$ is contained in the level set $\{|u-a| \geq \lambda/2\}$. Therefore

$$\mathcal{L}^n\big(B_{r_0}(x_0) \cap \{|u-a| \geq q/2\}\big) = \mathcal{L}^n\big(B_{r_0}(x_0)\big) = \mu_0.$$

The Density Theorem implies

$$|B_r(x_0) \cap \{|u-a| > q/2\}| \geq C_1 r^n, \text{ for } r \geq r_0.$$

Now, we define

$$\bar{w} := \begin{cases} \min\limits_{|z-a| \in [q/2, M]} W(z), & Z_a = \varnothing, \\ \min\limits_{\substack{|z-a| \in [q/2, M] \\ d(z, Z_a) \geq d_0}} W(z), & Z_a \neq \varnothing. \end{cases}$$

Also, we recall the Upper Bound of the energy:

$$\Psi(R) = \begin{cases} Cr^{n-1} & \text{if } s \in (1/2, 1), \\ Cr^{n-1} \ln r & \text{if } s = 1/2, \\ Cr^{n-2s} & \text{if } s \in (0, 1/2). \end{cases}$$

Hence, from the above and for $s \in (1/2, 1)$ we get

$$\bar{w} C_1 r^n \leq J_u(B_{r_0}) \leq Cr^{n-1},$$

which can not be true for

$$r > \frac{C}{\bar{w} C_1}.$$

For $s = 1/2$ in the same spirit:

$$\bar{w} C_1 r^n \leq J_u(B_{r_0}) \leq Cr^{n-1} \ln r,$$

which can not be true for

$$\frac{r}{\ln r} > \frac{C}{\bar{w} C_1}.$$

For $s \in (0, 1/2)$ we argue as above, and we get:

$$\bar{w} C_1 r^n \leq J_u(B_{r_0}) \leq Cr^{n-2s},$$

which can not be true for

$$r > (\frac{C}{\bar{w}C_1})^{\frac{1}{2s}}.$$

Therefore if we set $r_q = \frac{2C}{\bar{w}C_1}$ (first case). then $B_{r_q}(x_0) \subset \mathcal{O}$ contradicts

$$|u(x_0) - a| \geq q.$$

The conclusion for the other two cases is the same. □

4.8.2 A Liouville type result

As a final application of the density and pointwise estimates, we establish a Liouville-type rigidity result for entire minimizers. The idea is that if a global minimizer $u : \mathbb{R}^n \to \mathbb{R}^m$ stays trapped in a sufficiently small neighbourhood of a single well a at all large scales, then the combination of minimality, the Density Theorem, and the pointwise estimate of Section 4.8.1 forces u to coincide identically with a in $\mathbb{R}^n$. This is the nonlocal vector-valued analogue of the Liouville theorems for phase transition models proved in the local case by Alikakos and Fusco [2, Theorem 3.2] (see also [3, Corollary 5.2]): under appropriate structural assumptions on W, bounded entire minimizers with "no room to develop an interface at infinity" must be constant.

Theorem 4.8.2. *Assume that W and u satisfy the hypotheses of Theorem 4.8.1 and take $\mathcal{O} = \mathbb{R}^n$. Then $u \equiv a$ in $\mathbb{R}^n$.*

Proof. Pick an arbitrary $q \in (0, M)$. Because the entire space is the domain, $B_{r_q}(x) \subset \mathbb{R}^n$ for every $x \in \mathbb{R}^n$, where r_q is the radius given by the above theorem. Hence

$$|u(x) - a| \ < \ q, \qquad \forall\, x \in \mathbb{R}^n.$$

Since the choice of $q \in (0, M)$ is *arbitrary*, we can let $q \to 0^+$ and obtain $|u(x) - a| = 0$ for every $x \in \mathbb{R}^n$; that is, $u(x) = a$ everywhere.

□

4.9 Conclusion and outlook

This thesis develops a quantitative theory for vector-valued minimizers of fractional (nonlocal) phase-transition energies. The main point is that, despite the lack of a scalar comparison principle and the long-range nature of the interaction term, one can still extract robust geometric information from variational arguments combined with tools adapted to the nonlocal setting. In particular, the polar decomposition framework and the ensuing separation into radial and angular contributions provide a systematic way to analyze minimizers near a well of the potential and to convert energetic control into pointwise and measure-theoretic estimates.

The results yield structural energy bounds, a maximum-principle mechanism tailored to vector systems, and density estimates describing how minimizers occupy phases at large scales. These density bounds have qualitative consequences, including pointwise control near a well and Liouville-type rigidity for entire minimizers: under suitable trapping near a single well, a global minimizer must coincide identically with that well.

Several natural directions for further investigation emerge from this work. These include sharpening decay and quantitative convergence rates near a well, developing a more systematic stability theory for admissible competitor classes under limiting procedures, and pursuing a stratification theory for nonlocal vectorial energies aimed at describing the geometry and possible singular structures of interfaces in genuinely vector-valued regimes.

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