ATHENS UNIVERSITY OF ECONOMICS AND BUSINESS

DEPARTMENT OF STATISTICS

ON SOME APPLICATIONS OF FRACTIONAL BROWNIAN MOTION TO INSURANCE AND FINANCE

By

Spyridon D. Vrontos

A THESIS
Submitted to the Department of Statistics
of the Athens University of Economics and Business
in partial fulfillment of the requirements for
the degree of PhD in Statistics

Athens, Greece
February 2005

ΟΙΚΟΝΟΜΙΚΟ ΠΑΝΕΠΙΣΤΗΜΙΟ ΑΘΗΝΩΝ

ΤΜΗΜΑ ΣΤΑΤΙΣΤΙΚΗΣ

ΕΦΑΡΜΟΓΕΣ ΤΗΣ ΚΛΑΣΜΑΤΙΚΗΣ ΚΙΝΗΣΗΣ BROWN ΣΤΑ ΑΣΦΑΛΙΣΤΙΚΑ ΚΑΙ ΣΤΑ ΧΡΗΜΑΤΟΟΙΚΟΝΟΜΙΚΑ

Σπυρίδων Δ. Βρόντος

ΔΙΑΤΡΙΒΗ

Που υποβλήθηκε στο Τμήμα Στατιστικής

του Οικονομικού Πανεπιστημίου Αθηνών

ως μέρος των απαιτήσεων για την απόκτηση

Διδακτορικού Διπλώματος στη Στατιστική

Αθήνα

Φεβρουάριος 2005

To my family

Acknowledgements

I am very grateful to my supervisor Nicholas E. Frangos, for his guidance, encouragement and financial support. Without his assistance this dissertation could not have been written.

I also would like to thank Athanassios N. Yannacopoulos, for his help, guidance and for many interesting discussions.

Finally, I would like to thank my parents and my brother for their help, encouragement, and financial support.

VITA

Spyridon D. Vrontos was born in Athens in 1973. He graduated from the Department of Statistics of the Athens University of Economics and Business in 1996. In 1999 he finished the M.Sc. Program of the Department of Statistics of the Athens University of Economics and Business. He has been awarded the Charles A. Hachemeister Prize by the Casualty Actuarial Society for the best 2001 ASTIN paper.

Publications - Working Papers

Design of optimal Bonus – Malus systems with a frequency and a severity component on an individual basis in automobile insurance. ASTIN Bulletin 31, 1, 2001, pp. 5-26. (with N.E. Frangos).

Ruin probability at a given time for a model with liabilities of the fractional Brownian motion type: A partial differential equation approach, (with N.E. Frangos and A.N. Yannacopoulos). (To appear in Scandinavian Actuarial Journal)

Insurance control for a simple model with liabilities of the fractional Brownian motion type, (with N.E. Frangos and A.N. Yannacopoulos).

Reinsurance Option Valuation when the Liabilities are of fractional Brownian motion type, (with N.E. Frangos and A.N. Yannacopoulos).

Reinsurance control in a model with liabilities of the fractional Brownian motion type, (with N.E. Frangos and A.N. Yannacopoulos).

Asset Liability Management using Derivatives under Long Range Dependence, (with N.E. Frangos and A.N. Yannacopoulos).

Abstract

Spyridon D. Vrontos

On Some Applications of Fractional Brownian motion to insurance and finance

February 2005

In this thesis we deal with some problems arising in the interplay of insurance and finance considering fractional Brownian motion to be the driving stochastic process.

We consider the problem of calculating the probability of ruin at a given time point when the claims are driven by fractional Brownian motion. We show that the probability of ruin of the firm can be expressed as the solution of a linear parabolic partial differential equation and we solve this partial differential equation analytically.

We formulate a simple problem of insurance control for liabilities of diffusion type driven by fractional Brownian motion. The problem reduces to a version of the fractional linear quadratic regulator. We prove that the solution to the control problem is given by the solution of a system of ordinary differential equations.

We study the valuation of a reinsurance policy as an option both for excess of loss and for proportional reinsurance when the liabilities of the insurance business are driven by fractional Brownian motion.

We consider the problem of reinsurance control when the liabilities of an insurance business are driven by fractional Brownian motion. We find the optimal reinsurance strategy that the insurance company must follow for a number of time periods in order to reach a desired capital target by minimizing a reasonable functional of the reinsurance cost and of the final capital target.

We examine how one can use derivatives for asset allocation and how a pension fund can use derivatives in its asset allocation both for the case of a defined benefit pension scheme and a targeted money purchase scheme, when the stock price process is geometric fractional Brownian motion. A comparison with the case of Brownian motion is provided.

We review the methods that have been developed for the estimation of Hurst parameter and the detection of long range dependence and we apply these methods in data concerning stocks listed in the Athens Stock Exchange.

Περίληψη

Σπυρίδων Δ. Βρόντος

Εφαρμογές της κλασματικής κίνησης Brown στα ασφαλιστικά και στα χρηματοοικονομικά

Φεβρουάριος 2005

Σε αυτήν την διατριβή εξετάζουμε προβλήματα που άπτονται των ασφαλιστικών και των χρηματοοικονομικών θεωρώντας την κλασματική κίνηση Brown ως τον βασικό άξονα στον οποίο στηρίζεται η μοντελοποίηση. Τα προβλήματα αυτά θεωρούμε ότι έχουν ενδιαφέρον τόσο από θεωρητικής όσο και από πρακτικής απόψεως.

Εξετάζουμε το πρόβλημα της πιθανότητας καταστροφής σε μία δεδομένη στιγμή της ασφαλιστικής εταιρείας όταν οι υποχρεώσεις της εταιρείας μοντελοποιούνται με τη χρήση της κλασματικής κίνησης Brown. Δείχνουμε ότι η πιθανότητα της καταστροφής της εταιρίας μπορεί να εκφραστεί ως λύση μιας γραμμικής παραβολικής μερικής διαφορικής εξίσωσης και λύνουμε αυτήν την μερική διαφορική εξίσωση αναλυτικά.

Διατυπώνουμε ένα πρόβλημα ασφαλιστικού ελέγχου για την περίπτωση που οι υποχρεώσεις της εταιρείας μοντελοποιούνται με τη χρήση της κλασματικής κίνησης Brown. Βρίσκουμε τη βέλτιστη στρατηγική ελέγχου που η ασφαλιστική εταιρεία πρέπει να ακολουθήσει προκειμένου να επιτευχθεί ένας επιθυμητός οικονομικός στόχος και αποδεικνύουμε ότι η λύση στο πρόβλημα ελέγχου δίνεται από τη λύση ενός συστήματος των συνηθισμένων διαφορικών εξισώσεων.

Μελετάμε την τιμολόγηση ενός συμβολαίου αντασφάλισης τόσο στην περίπτωση της υπέρβασης ζημιάς καθώς και για την κατά αναλογία αντασφάλιση όταν οι υποχρεώσεις της εταιρείας μοντελοποιούνται με τη χρήση της κλασματικής κίνησης Brown χρησιμοποιώντας τεχνικές αποτίμησης παραγώγων χρηματοοικονομικών προϊόντων.

Εξετάζουμε το πρόβλημα του αντασφαλιστικού ελέγχου όταν οι υποχρεώσεις της εταιρείας μοντελοποιούνται με τη χρήση της κλασματικής κίνησης Brown. Βρίσκουμε τη βέλτιστη στρατηγική αντασφάλισης που η ασφαλιστική εταιρεία πρέπει να ακολουθήσει σε συγκεκριμένα χρονικά διαστήματα προκειμένου να επιτευχθεί ένας επιθυμητός οικονομικός στόχος.

Εξετάζουμε πώς κάποιος επενδυτής μπορεί να χρησιμοποιήσει τα παράγωγα στη διαχείριση χαρτοφυλακίου και επίσης πώς ένα

συνταξιοδοτικό ταμείο μπορεί να χρησιμοποιήσει τα παράγωγα στη διαχείριση του χαρτοφυλακίου του, όταν οι τιμές των μετοχών ακολουθούν την γεωμετρική κλασματική κίνηση Brown.

Παραθέτουμε τις μεθόδους που έχουν αναπτυχθεί για την εκτίμηση της παραμέτρου H του Hurst και εφαρμόζουμε αυτές τις μεθόδους σε δεδομένα από το Χρηματιστήριο Αξιών Αθηνών.

Contents

List of Tables

List of Figures

Introduction

In this thesis we deal with some problems arising in the interplay of insurance and finance. We consider fractional Brownian motion to be the driving stochastic process. In chapter one we give some of the main results that have been obtained for fractional Brownian motion. We discuss its definition, the most important properties it has, and some of the methods that have been developed for stochastic calculus for fractional Brownian motion.

In chapter two we consider the problem of calculating the probability of ruin at a given time point when the claims are driven by fractional Brownian motion. We propose a model for an insurance business facing liabilities presenting long term correlations. The long term correlations are modelled with the use of a fractional brownian motion with Hurst exponent H. The insurance firm invests in an interest account which is assumed to be deterministic. It is shown that the cash balance process of the firm satisfies an Ornstein-Uhlenbeck stochastic differential equation driven by fractional Brownian motion. Using the recently developed tools of fractional stochastic calculus we show that the probability of ruin of the firm can be expressed as the solution of a linear parabolic partial differential equation. We solve this partial differential equation analytically and we provide an exact expression for the ruin probability in terms of error functions, valid for all times. Using this exact expression one may derive asymptotc results using standard techniques. Finally, the partial differential equation allows an efficient numerical treatment of the problem which may be used as an alternative to Monte-Carlo type simulations.

In chapter three we formulate a simple problem of insurance control for liabilities of diffusion type driven by fractional Brownian motion. The problem reduces to a version of the fractional linear quadratic regulator. We consider the same model for an insurance firm as we did in the previous chapter but now we assume that the firm may control its cash balance process by asking its customers for an input having the meaning of extra premium. The input is considered as a control parameter which allows the firm to reach a desired capital target at a specified time. The cash balance equation is then a controlled fractional Ornstein-Uhlenbeck equation. The control is chosen in such a manner as to minimize a reasonable functional of the final capital target and of a weight function of the cost of the input. We use the method of completion of the squares and the method of maximum principle. We prove that the solution to the control problem is given by the solution of a system of ordinary differential equations.

In chapter four we study the valuation of a reinsurance policy both for excess of loss and for proportional reinsurance when the liabilities of the insurance business are driven by fractional Brownian motion.

In chapter five we consider the problem of reinsurance when the liabilities of an insurance business are driven by fractional Brownian motion. We assume that the insurance company uses a proportional reinsurance scheme. We find the optimal reinsurance strategy that the insurance company must follow for a number of time periods in order to reach a desired capital target by minimizing a reasonable functional of the reinsurance cost and of the final capital target. We show that in the case of linear reinsurance cost the minimization problem reduces to a quadratic programming problem. We also examine the case of non linear reinsurance cost.

In chapter six we examine how a pension fund can use derivatives in its asset allocation both for the case of a defined benefit pension scheme and a targeted money purchase scheme, when the stock price process is exponential fractional Brownian motion. A comparison with the case of Brownian motion is provided.

In chapter seven we review the methods that have been developed for the estimation of Hurst parameter and the detection of long range dependence and we apply these methods in data concerning stocks listed in the Athens Stock Exchange.

Chapter 1

Fractional Brownian Motion

1.1 Preliminaries

In this chapter we give the definition of fractional Brownian motion, some of its most important properties, some of the representations of fractional Brownian motion that have appeared in the literature and some of the approaches that have been used for stochastic calculus with respect to fractional Brownian motion.

Definition 1 *A continuous centered Gaussian process* $\{W_t, t \geq 0\}$, $W_0 = 0$ *, is a fractional Brownian motion with Hurst parameter* $H \in (0,1)$ *if its covariance function is given by*

$$E[W_t^H W_s^H] = \frac{1}{2} Var(W_1^H)(t^{2H} + s^{2H} - |t - s\ |^{2H}),$$

for all $t, s \geq 0$.

We will consider the normalised version of this process by taking $Var(W_1^H) = 1$.

For $H = 1/2$, from fractional Brownian motion we take the standard Brownian motion. Fractional Brownian motion was introduced by Kolmogorov (1940) within a Hilbert space framework. Following Kolmogorov, Mandelbrot and Van Ness (1968) studied fractional Brownian motion and they established the folowing stochastic integral representation of fractional Brownian motion in terms of a Brownian motion on the whole real line:

$$W_t^H = \frac{1}{C_1(H)} \int_R [((t-s)^+)^{H-\frac{1}{2}} - ((-s)^+)^{H-\frac{1}{2}}] dW_s, \tag{1.1}$$

where $\{W(A), A \text{ Borel subset of } R\}$, is a Brownian measure on R and

$$C_1(H) = (\int_0^\infty ((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}) ds + \frac{1}{2H})^{\frac{1}{2}}.$$

It is clear from the covariance function of fractional Brownian motion that

$$E\{[W_t^H]^2\} = \frac{1}{2}(t^{2H} + t^{2H} - |t - t\,|^{2H}) = t^{2H}.$$

Fractional Brownian motion has the following self-similarity property: For any constant $\alpha > 0$, the processes $\{W_{at}^H\}$ and $\{\alpha^H W_t^H\}$ have the same distribution. This property is a direct consequence of the fact that the covariance function of fractional Brownian motion is homogeneous of order 2. This is obvious by taking:

$$\begin{aligned} E[W_{at}^H W_{as}^H] &= \frac{1}{2}\{(at)^{2H} + (as)^{2H} - |at - as\,|^{2H}\} = \\ &= a^{2H} E[W_t^H W_s^H] \\ &= E[(a^H W_t^H)(a^H W_s^H)]. \end{aligned}$$

Since all processes are centered Gaussian, this equality in covariance implies that $\{W_{at}^H\} \stackrel{d}{=} \{a^H W_t^H\}$. The variance of the increment of the process in an interval $[s,t]$ can be found to be

$$
\begin{aligned}
& E[|W_t^H - W_s^H|^2] \\
= \;& E[(W_t^H)^2] + E[(W_s^H)^2] - 2E[W_t^H W_s^H] \\
= \;& t^{2H} + s^{2H} - t^{2H} - s^{2H} + |t-s|^{2H} \\
= \;& |t-s|^{2H}.
\end{aligned}
$$

This implies that fractional Brownian motion has stationary increments. One can also show that

$$
\begin{aligned}
& E[(W_{t+h}^H - W_h^H)(W_{s+h}^H - W_h^H)] \\
= \;& E[W_{t+h}^H W_{s+h}^H] - E[W_{t+h}^H W_h^H] - E[W_h^H W_{s+h}^H] + E[W_h^H W_h^H] \\
= \;& \frac{1}{2}\{[(t+h)^{2H} + (s+h)^{2H} - |t-s|^{2H}] - \\
& [(t+h)^{2H} + h^{2H} - t^{2H}] \\
& -[h^{2H} + (s+h)^{2H} - s^{2H}] + h^{2H} \\
= \;& E[W_t^H W_s^H],
\end{aligned}
$$

concluding that

$$\{W_{t+h}^H - W_h^H\} \stackrel{d}{=} \{W_t^H\}.$$

FBM is characterized by the Hurst exponent H which determines the sign of the covariance of the past and the future increments. When $H > \frac{1}{2}$ the covariance is positive and when $H < \frac{1}{2}$ the covariance is negative. This is derived from the convexity/concavity of the power functions. For $H = \frac{1}{2}$, the covariance can be written as $E[W_t^{\frac{1}{2}} W_s^{\frac{1}{2}}] = t \wedge s$, and the process as we have already said is reduced to a standard fractional Brownian motion. In this case the increments of the process in disjoint intervals are independent. However for $H \neq \frac{1}{2}$ the increments are not independent. The covariance between two increments $W^H_{t+h} - W^H_t$ and $W^H_{s+h} - W^H_s$, where $s + h \leq t$, and $t - s = nh$ is

$$\begin{aligned}\rho_H(n) &= \frac{1}{2}h^{2H}((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}) \\ &\approx h^{2H}H(2H-1)n^{2H-2} \to 0\end{aligned}$$

as $n \to \infty$. Therefore:

i) if $H > \frac{1}{2}$, $\rho_H(n) > 0$ and $\sum_{n=1}^{\infty} \rho_H(n) = \infty$.

ii) if $H < \frac{1}{2}$, $\rho_H(n) < 0$ and $\sum_{n=1}^{\infty} |\rho_H(n)| < \infty$.

In case i) two increments of the form $W^H_{t+h} - W^H_t$ and $W^H_{t+2h} - W^H_{t+h}$ are positively correlated and the process presents aggregation behavior. In case ii) these increments are negatively correlated and we say that there is intermittency.

For more details on fractional Brownian motion and self-similar processes we refer to Embrechts and Maejima (2003), Nualart (2003), and Samorodnitsky and Taqqu (1994).

1.1.1 Construction of fractional Brownian Motion

We have already seen the Mandelbrot - Van Ness representation of fractional Brownian motion (1.1). We mention here and some other representations of fractional Brownian motion that have appeared in the literature.

We will see first a representation that uses fractional integrals and fractional derivatives following Bender (2003). Let $(\Omega, \mathcal{F}, P)$ be a probability space that carries a two-sided Brownian motion B. For $a, b \in R$ define the indicator function as

$$1_{(a,b)}(t) = \left\{ \begin{array}{ll} 1, & if\ a \leqslant t \leqslant b, \\ -1 & if\ b \leqslant t \leqslant a \\ 0 & \text{otherwise} \end{array} \right\}. \quad (1.2)$$

Furthermore let

$$K_H := \Gamma(H+\frac{1}{2})\left(\int_0^\infty ((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}})ds + \frac{1}{2H}\right)^{-\frac{1}{2}},$$

and define the operator

$$M_\pm^H f := \left\{ \begin{array}{ll} K_H D_\pm^{-(H-\frac{1}{2})} f, & 0 < H < \frac{1}{2}, \\ f, & H = \frac{1}{2} \\ K_H I_\pm^{H-\frac{1}{2}} f & \frac{1}{2} < H < 1. \end{array} \right\}. \tag{1.3}$$

Here $I_\pm^\alpha$, $0 < \alpha < 1$, is the fractional integral of Weyl's type, defined by:

$$(I_-^a f)(x) := \frac{1}{\Gamma(a)} \int_x^\infty f(t)(t-x)^{a-1} dt = \frac{1}{\Gamma(a)} \int_0^\infty f(x+t) t^{a-1} dt, \tag{1.4}$$

$$(I_-^a f)(x) := \frac{1}{\Gamma(a)} \int_{-\infty}^x f(t)(x-t)^{a-1} dt = \frac{1}{\Gamma(a)} \int_0^\infty f(x-t) t^{a-1} dt, \tag{1.5}$$

if the integrals exist for all $x \in R$. For $\alpha \in (0,1)$ and $\in > 0$, define

$$(D_{\pm,\varepsilon}^a f)(x) = \frac{1}{\Gamma(1-\alpha)} \int_\varepsilon^\infty \frac{f(x) - f(x \mp t)}{t^{\alpha+1}} dt.$$

Then the fractional derivatives of Marchaud's type are given by

$$(D_\pm^a f) := \lim_{\varepsilon \to 0} (D_{\pm,\varepsilon}^a f),$$

if the limit exists in $L^p(R)$ fo some p. With these definitions we have the following theorem.

Theorem 2 *(Bender 2003). For $0 < H < 1$, let operators $M_\pm^H$ be defined by 1.3. Then $M_-^H 1(0,t) \in L^2(R)$ and a fractional Brownian motion B^H is given by a continuous version of the Wiener integral*

$$\int_R (M_-^H(0,t))(s) dB_s. \tag{1.6}$$

It is possible also to establish a spectral representation of fractional Brownian motion, see for more Samorodnitsky and Taqqu (1994):

$$W_t^H = \frac{1}{C2(H)} \int\limits_R \frac{e^{its}-1}{is} |s|^{\frac{1}{2}-H} d\widetilde{W}_s, \tag{1.7}$$

where $\widetilde{W} = W^1 + iW^2$ is a complex Gaussian measure on R such that $W^1(A) = W^1(-A), W^2(A) = -W^2(A)$, and $E(W^1(A)^2) = \frac{|A|}{2}$ and

$$C_2(H) = (\frac{\pi}{H\Gamma(2H)\sin(H\pi)})^{\frac{1}{2}}.$$

Norros, Valkeila and Virtamo (1999) have proved the following integral representation of fractional Brownian motion in terms of Brownian motion.

$$W_t^H = \int\limits_0^t z(t,s) dW_s$$

where

$$z(t,s) = c_H \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - \left(H-\frac{1}{2}\right) s^{\frac{1}{2}-H} \int\limits_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right], \tag{1.8}$$

and

$$c_H = \sqrt{\frac{2H\Gamma\left(\frac{3}{2}-H\right)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}}$$

1.1.2 Sample Path Properties

We say that a stochastic process $\{X(t), 0 \leq t \leq T\}$ is Holder continuous of order $\gamma \in (0,1)$ if

$$P\{\omega \in \Omega : \sup_{\substack{0 < t - s < h(\omega) \\ s,t \in [0,T]}} \frac{|X(t,\omega) - X(s,\omega)|}{|t-s|^{\gamma}} \leq \delta\} = 1$$

where h is an almost surely positive random variable and $\delta > 0$ is an appropriate constant.

Theorem 3 *(Embrechts and Maejima, 2003) Fractional Brownian motion* $\{W_t^H, t \geq 0\}$, $0 < H < 1$, *has a modification, the sample paths of which are Holder continuous of order* $\beta \in [0, H)$.

Theorem 4 *(Vervaat, 1985) Supposse* $\{X(t)\}$ *is* H *self-similar with stationary increments. If* $H \leq 1$ *and* $P\{X(t) = tX(1)\} = 0$,*then the sample paths of* $\{X(t)\}$ *have infinite variation almost surely, on all compact intervals.*

Corollary 5 *(Embrechts and Maejima, 2003) Sample paths of fractional Brownian motion have nowhere bounded variation.*

Since fractional Brownian motion is H self-similar with stationary increments, $0 < H < 1$, it has nowhere bounded variation from the theorem of Vervaat (1985).

Let us assume that $H > \frac{1}{2}$ and consider the partition of $[0,t]$, $t_0 = 0 < t_1 < ... < t_n = t$. A partition will be identified with the set of pairs of consecutive dividing points, i.e.

$$\Delta = \{(t_0, t_1), (t_1, t_2), ..., (t_{n-1}, t_n) \,|t_0 = 0 < t_1 < ... < t_n = t\}$$

Define

$$|\Delta_n| := \max\{|t_j - t_{j-1}|, 1 \leq j \leq n\},$$

and consider sequences of partitions Δ_n with $\lim_{n\to\infty} |\Delta_n| \to 0$.

Theorem 6 *(Lin 1995). The quadratic variation of W^H is a zero process. Namely,*

$$\sum_{\mu,\nu\in\Delta_n}(W_\nu^H - W_\mu^H)^2 \to 0$$

in probability as $n\to\infty$.

Lemma 7 *(Rogers 1997). Fix $p>0$. Then $V_{n,p}\equiv \sum_{t_j\in\Delta^n}|W_{t_{j+1}}^H - W_{t_j}^H|^p \to \left\{\begin{array}{c} 0, \text{ if } pH>1 \\ +\infty, \text{ if } pH<1 \end{array}\right\}$ in the sense of convergence in probability.*

Theorem 8 *(Lin 1995). Fractional Brownian motion is not a semimartingale.*

Proof: Following Lin (1995) we have the following. If W^H were a semimartingale we would have a Doob-Meyer decomposition and thus

$$W_t^H = M_t + V_t$$

where M is a continuous local martingale and V is a finite variation process with $M_0 = V_0 = 0$. Denoting by $[Z,Z]$ the quadratic variation of a semimartingale Z we would have that

$$\begin{aligned} 0 &= \left[W^H, W^H\right]_t = [M+V, M+V]_t = \\ &= [M,M]_t + 2\,[M,V]_t + [V,V]_t \\ &= [M,M]_t \end{aligned}$$

By the Burkholder-Gundy-Davis inequality, M is itself a zero process and hence $W^H = V$ has finite variation. This is impossible since almost all sample paths of W^H have Hausdorff dimension $2-H>1$. A proof of this theorem appears also in Liptser and Shiryaev (1989) and furthermore in Rogers (1997).

Let us also note here that fractional Brownian motion is not a Markov process.

1.2 Stochastic calculus with respect to FBM

In this section we review some fundamental results of fractional stochastic calculus. As we have said fractional Brownian motion is not a semimartingale. This presents problems in the definition of a stochastic integral and a stochastic calculus with respect to fractional Brownian motion, as we may not apply the standard theory of stochastic integration over a semimartingale to define a stochastic integral over fractional Brownian motion. More precisely we cannot obtain a fully satisfactory theory when integrating over all predictable processes (predictable with respect to the filtration generated by fractional Brownian motion), as a consequence of the Bichteler - Dellacherie theorem, see for example Bichteler (1981), Protter (1990), and Dellacherie and Meyer (1982). Several approaches to this subject has been proposed, we will mention here briefly the approaches of Lin (1995), Shiryaev (1998), Duncan, Hu and Pasik-Duncan (2000), Elliott and van der Hoek (2003) and Bender (2003).

1.2.1 Lin (1995)

Denote by $\mathcal{E}_t$ the step functions from $[0, t]$ to R. For any

$$\phi(s) = \sum_{i=1}^{n} \alpha_i 1_{(t_i, t_{i+1}]}(s)$$

$\phi(s) \in \mathcal{E}_t$, we define

$$\int_0^t \phi(s) dW_s^H = \sum_{i=1}^{n} \alpha_i (W_{t_{i+1}}^H - W_{t_i}^H)$$

The stochastic integral is defined in terms of the L^2 limit of the above sum.

1.2.2 Shiryaev (1998)

Shiryaev (1998) considers the stochastic integral

$$\int_0^t f(W_u^H)dW_u^H$$

in the case of $H \in \left(\frac{1}{2}, 1\right)$, and takes $f = f(x)$, $x \in R$, to be a function that belongs to the class C^1. For

$$F(x) = F(0) + \int_0^x f(y)dy$$

by Taylor's formula with remainder in the integral form it is

$$F(x) = F(y) + f(y)(x - y) + \int_y^x f^{'}(u)(x - u)du.$$

Then for each sequence $T^n \equiv \{t_m^n, m \geq 1\}, n \geq 1$, of times $t_m^n, (0 = t_1^n, \leq t_2^n \leq ...)$ it is

$$\begin{aligned} F(W_t^H) - F(W_0^H) &= \sum_m \left[F(W_{t\wedge t_{m+1}^n}^H) - F(W_{t\wedge t_m^n}^H)\right] = & (1.9) \\ &= \sum_m f(W_{t\wedge t_{m+1}^n}^H)(W_{t\wedge t_{m+1}^n}^H - W_{t\wedge t_m^n}^H) + R_t^n, & (1.10) \end{aligned}$$

where

$$R_t^n = \sum_m \int_{W_{t\wedge t_m^n}^H}^{W_{t\wedge t_{m+1}^n}^H} f^{'}(u)(W_{t\wedge t_{m+1}^n}^H - u)du.$$

Clearly

$$P(\sup_{0\leq u\leq t} |f'(W_u^H)| < \infty) = 1$$

and because for $H \in \left(\frac{1}{2}, 1\right)$

$$P - \lim_n \sum_m |W^H_{t\wedge t^n_{m+1}} - W^H_{t\wedge t^n_m}|^2 = 0,$$

one can obtain

$$|R^n_t| \le \frac{1}{2} \sup_{0\le u\le t} |f'(W^H_u)| \cdot \sum_m |W^H_{t\wedge t^n_{m+1}} - W^H_{t\wedge t^n_m}|^2 \xrightarrow{P} 0.$$

The left-hand side of (1.9) is independent of n and $R^n_t \xrightarrow{P} 0$. So

$$P - \lim_n \sum_m f'(W^H_{t\wedge t^n_m}) \left(W^H_{t\wedge t^n_{m+1}} - W^H_{t\wedge t^n_m}\right)$$

exists and it is denoted by

$$\int_0^t f(W^H_u) dW^H_u$$

and called as the stochastic integral with respect to the fractional Brownian motion $W^H = \left\{W^H_u\right\}_{u\le t}, H \in \left(\frac{1}{2}, 1\right), f \in C^1$. The arguments given also prove that (P-a.s.)

$$F(W^H_t) - F(W^H_0) = \int_0^t f(W^H_u) dW^H_u$$

which as mentioned by Shiryaev (1998) can be regarded as an analogue of Ito's formula, for a fractional Brownian motion. Note that this formula leads to

$$E\left[\int_0^t f(W^H_u) dW^H_u\right] \neq 0$$

in contrast with the corresponding Ito integral for the case of Brownian motion where the above expectation is zero. Shiryaev (1998) used this type of integral to construct a fractional Black-Scholes market and he showed that it admits arbitrage opportunities.

1.2.3 Duncan, Hu and Pasik-Duncan (2000)

In the paper of Duncan, Hu and Pasik-Duncan (2000) a stochastic integral over fractional Brownian motion of Hurst exponent $1/2 < H < 1$ has been defined, having some properties that have similarities with the corresponding properties of the Ito stochastic integral over Brownian motion.

We summarize here the basic results they have derived. The stochastic integral $\int_0^t f_s dW_s^H$ over deterministic functions f is defined to provide a zero mean, Gaussian random variable with variance

$$\int_0^\infty \int_0^\infty f_s f_t \phi(s,t) ds dt$$

where

$$\phi(s,t) = H(2H-1) \mid s-t \mid^{2H-2} .$$

The stochastic integral $\int_0^t F_s dW_s^H$ can be defined over stochastic processes F as the limit

$$\int_0^t F_s dW_s^H = \lim_{\Delta \to 0} \sum_{i=0}^{n-1} F_{t_i} \diamond (W_{t_{i+1}}^H - W_{t_i}^H)$$

where $\{t_i\}$ is some partition of the interval $(0,t)$, $\Delta = \sup_i \mid t_{i+1} - t_i \mid$. By $\diamond$ we denote the Wick product which is defined by

$$\varepsilon(f) \diamond \varepsilon(g) = \varepsilon(f+g)$$

where

$$\varepsilon(f) := exp\left\{ \int_0^\infty f_t dW_t^H - \frac{1}{2} \int_0^\infty \int_0^\infty f_s f_t \phi(s,t) ds dt \right\}$$

is the stochastic exponential of the deterministic function f which is such that

$$\mid \int_0^\infty \int_0^\infty f_s f_t \phi(s,t) ds dt \mid < \infty.$$

The Wick product was introduced in Wick (1950) as a tool to renormalize certain infinite quantities in quantum field theory. In stochastic analysis it was first introduced by Hida and Ikeda (1965). A systematic, general account of the traditions of both mathematical physics and probability theory regarding this subject was given in Dobrushin and Minlos (1977). Today the Wick product is also important in the study of stochastic, ordinary and partial, differential equations. For more on this subject one can see Holden, Oksendal, Uboe and Zhang (1996) and Janson (1997).

Duncan et al provide the following generalization of Itô 's lemma in the case of fractional Brownian motion. For a proof of this result and generalizations to more complicated integrands, we refer to Duncan et al.

Theorem 9 *(Duncan, Hu and Pasik-Duncan 2000)*

Let $f : R \to R$, and $f \in C^{1,2}$, then

$$f(W_T^H) - f(W_0^H) = \int_0^T f'(W_s^H) dW_s^H + H \int_0^T s^{2H-1} f''(W_s^H) ds, \quad a.s.$$

It is interesting to see that the above formula implies the usual Ito formula for Brownian motion when $H = \frac{1}{2}$.

Proposition 10 *(Duncan, Hu and Pasik-Duncan 2000)*

Let

$$\eta_t = \int_0^t a_s dW_s^H$$

where a_t is some deterministic function such that

$$\mid \int_0^\infty \int_0^\infty a_s a_t \phi(s,t) ds dt \mid < \infty.$$

Let $f \in C^{1,2}$ and assume that $\frac{\partial f}{\partial x}(s, \eta_s) a_s \in \mathcal{L}(0,T)$. Then,

$$\begin{aligned} f(t, \eta_t) &= f(0,0) + \int_0^t \frac{\partial f}{\partial s}(s, \eta_s) ds + \int_0^t \frac{\partial f}{\partial x}(s, \eta_s) a_s dW_s^H \\ &\quad + \int_0^t \frac{\partial^2 f}{\partial x^2}(s, \eta_s) a_s \int_0^s \phi(s, v) a_v dv ds, \quad a.s. \end{aligned}$$

1.2.4 Fractional Brownian Motion and Fractional White Noise Theory

The integration theory based on the Wick product that was introduced by Duncan, Hu and Pasik-Duncan (2000) for $\frac{1}{2} < H < 1$ was extended using a fractional white noise calculus setup and was applied to finance by Hu and Oksendal (2003), still for the case of $\frac{1}{2} < H < 1$. Then Elliot and Van der Hoek (2003) extended this theory and applied it to finance for $0 < H < 1$. We will review briefly the approach of Elliot and Van der Hoek (2003).

Fix H,$0 < H < 1$. The main idea is to relate fractional Brownian motion W_t^H to the classical Brownian motion through the following operator M :

Definition 11 *The operator $M = M^{(H)}$ is defined on functions $f \in S(R)$ by*

$$\hat{M}f(y) = |y|^{\frac{1}{2}-H}\hat{f}(y);\ y \in R$$

where

$$\hat{g}(y) := \int\limits_R e^{-ixy}g(x)dx$$

denotes the Fourier transform.

This can be restated as follows:

For $0 < H < \frac{1}{2}$ we have

$$Mf(x) = C_H \int\limits_R \frac{f(x-t) - f(x)}{|t|^{\frac{3}{2}-H}}dt,$$

where

$$C_H = \left[2\Gamma(H - \frac{1}{2})\cos(\frac{\pi}{2}(H - \frac{1}{2}))\right]^{-1} \left[\Gamma\left(2H+1\right)\sin(\pi H)\right]^{\frac{1}{2}}$$

with $\Gamma(.)$ denoting the $\Gamma-$function.

For $H = \frac{1}{2}$ we have

$$Mf(x) = f(x).$$

For $\frac{1}{2} < H < 1$ we have that

$$Mf(x) = C_H \int_R \frac{f(t)}{|t-x|^{\frac{3}{2}-H}} dt.$$

The operator M extends in a natural way from $S(R)$ to the space

$$\begin{aligned} L_H^2(R) \quad &: = \{f : R \to R \text{ (deterministic)}; |y|^{\frac{1}{2}-H} \hat{f}(y) \in L^2(R)\} \\ &= \{f : R \to R; Mf(x) \in L^2(R)\} \\ &= \{f : R \to R; ||y||_{L_H^2(R)} < \infty\}, \end{aligned}$$

where

$$||f||_{L_H^2(R)} = ||Mf||_{L_H^2(R)}.$$

The inner product on this space is

$$(f, g)_{L_H^2(R)} = (Mf, Mg)_{L^2(R)}.$$

In particular the indicator funtion $\chi_{[0,t]}(.)$ is easily seen to belong to this space, for fixed $t \in R$, and we write

$$M\chi_{[0,t]}(x) = M[0,t](x).$$

Note that if $f, g \in L^2(R) \cap L_H^2(R)$ then

$$\begin{aligned}(f, Mg)_{L^2(R)} &= \left(\hat{f}, \hat{M}g\right)_{L^2(R)} \\ &= \int_R |y|^{\frac{1}{2}-H} \hat{f}(y)\hat{g}(y)dy = \\ &= \left(\hat{M}f, \hat{g}\right)_{L^2(R)} \\ &= (Mf, g)_{L^2(R)}\,.\end{aligned}$$

We now define for $t \in R$

$$\tilde{B}^{(H)}(t) := \tilde{B}^{(H)}(t, \omega) :=< \omega, M[0, t]\,(.) >$$

Then $\tilde{B}^{(H)}(t)$ is Gaussian, $\tilde{B}^{(H)}(0) = E\left[\tilde{B}^{(H)}(t)\right] = 0$ for all $t \in R$ and

$$\begin{aligned}E\left[\tilde{B}^{(H)}(s)\tilde{B}^{(H)}(t)\right] &= \int_R M\,[0, s]\,(x) M\,[0, t]\,(x)dx \\ &= \int_R \hat{M}\,[0, s]\,(y)\hat{M}\,[0, t]\,(y)dy \\ &= \int_R |y|^{1-2H} \hat{\chi}\,[0, s]\,(y)\hat{\chi}\,[0, t]\,(y)dy \\ &= \frac{1}{2}\left[|t|^{2H} + |s|^{2H} - |s-t|^{2H}\right]\end{aligned}$$

Therefore the continuous version $B^{(H)}(t)$ of $\tilde{B}^{(H)}(t)$ is a fractional Brownian motion. Note that the underlying probability measure is the same as for $B(t)$. Then following Elliott and Van der Hoek (2003) we have that

$$\int_R f(t)dB^{(H)}(t) = \int_R Mf(t)dB(t), \;\; f \in L_H^2(R).$$

Denoting by $\{\xi_k\}_{k=1}^{\infty}$ the Hermite functions, it can be shown for the fractional white noise that it can then be defined using the expansion

$$W^{(H)}(t) = \sum_{k=1}^{\infty} M\xi_k(t)\mathcal{H}_{\varepsilon^{(k)}}(\omega).$$

Then it can be shown that

$$\frac{dB^{(H)}(t)}{dt} = W^{(H)}(t) \text{ in } (S)^*.$$

Definition 12 *(The fractional Wick/Ito Integral)*

Let $Y : R \to (S)^*$ be such that $Y(t) \diamond W^{(H)}(t)$ is dt-integrable in $(S)^*$. We say that Y is $dB^{(H)}$-integrable and we define the integral of $Y(t) = Y(t,\omega)$ with respect to $B^{(H)}(t)$ by

$$\int_R Y(t,\omega)dB^{(H)}(t) = \int_R Y(t) \diamond W^{(H)}(t)dt.$$

Theorem 13 *(A fractional Ito formula).*

Let $f(s,x) : R\times R \to R$ belong to $C^{1,2}(R\times R)$ and assume that the random variables

$$f(t, B^{(H)}(t)), \int_0^t \frac{\partial f}{\partial s}(s, B^{(H)}(s))ds \text{ and } \int_0^t \frac{\partial^2 f}{\partial x^2}(s, B^{(H)}(s))s^{2H-1}ds$$

all belong to $L^2(\mu)$. Then

$$\begin{aligned} f(t, B^{(H)}(t)) &= f(0,0) + \int_0^t \frac{\partial f}{\partial s}(s, B^{(H)}(s))ds + \int_0^t \frac{\partial f}{\partial x}(s, B^{(H)}(s))a_s dB_s^H \\ &+ \int_0^t \frac{\partial^2 f}{\partial x^2}(s, B^{(H)}(s))s^{2H-1}ds. \end{aligned}$$

For a proof of this Ito formula one can see Elliot and Van der Hoek (2003) and Biagini, Oksendal, Sulem, Wallner (2003). One can see also Lindstom (1993), for representations of fractional Brownian fields as integrals of white noise.

1.2.5 Pathwise Approach to Stochastic Itegration

A different approach to all above was taken by Dudley and Norvaisa (1999) and Zahle (1998) who used specific path properties of fractional Brownian motion, p-variation in Dudley and Norvaisa (1999) and Holder continuity in Zahle (1998).

We begin with some preliminaries for p-variation. Let $a < b$ be two real numbers. A real-valued function f on $[a, b]$ is called regulated if it has a left limit at each point of $(a, b]$ and a right limit at each point of $[a, b)$. We write then that $f \in \Re = \Re([a, b])$. Define also the following functions on $[a, b]$:

$$f_b^+(x) = f^+(x) = f(x+) = \lim_{y \downarrow x} f(y), \;\; a \leqslant x < b, \;\; f_b^+(b) = f(b)$$

and

$$f_a^-(x) = f^-(x) = f(x-) = \lim_{y \uparrow x} f(y), a < x \leqslant b, f_a^-(a) = f(a).$$

Let $\tau \subset [a, b]$, a non-degenerate interval, open or closed at either end. Let us define $\Delta_\tau^- f$ on τ by $\Delta_\tau^- f(x) = f(x) - f(x-)$ for each $x \in \tau$ which is not the left end-point on τ and $\Delta_\tau^- f(x) = 0$ at the left end point x whenever τ is left closed.

Similarly define $\Delta_\tau^+ f$ on τ by $\Delta_\tau^+ f(x) = f(x+) - f(x)$ for each $x \in \tau$ which is not the right end-point on τ and $\Delta_\tau^+ f(x) = 0$ at the right end point x whenever τ is right closed.

Definition 14 *The p-variation,* $0 < p < \infty$, *of a real valued function* f *on [a, b] is defined as*

$$\upsilon_p(f) = \upsilon_p(f; [a, b]) = \sup_{\kappa} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|^p,$$

where the supremum is taken over all subdivisions κ *of [a,b]:*

$$\kappa : \;\; a = x_0 < ... < x_n = b, \;\; n \geqslant 1.$$

If $\upsilon_p(f) < \infty$, f is said to have bounded variation on $[\alpha, b]$. All functions of bounded p-variation constitute the set

$$W_p = W_p([a,b]) = \{f : [a,b] \to R \; with \; v_p(f) < \infty\}.$$

Note that $W_q \subset W_p$ for $0 < p < q < \infty$ and every function of bounded p-variation is regulated, i.e. $W_p \subset \Re$.

Let us now review the classical Riemann-Stieltes integral and some of its extensions. If f, h are two real valued functions on $[a,b]$, a Riemann-Stieltjes sum is defined by

$$S(f,h,\kappa,\sigma) = \sum_{i=1}^{n} f(y_i)[h(x_i) - h(x_{i-1})].$$

Here κ is a subdivision of $[\alpha, b]$, and σ in an intermediate subdivision of κ, i.e. $x_{i-1} \leqslant y_i \leqslant x_i$, for $i = 1, ..., n$. The function f is Riemann-Stieltjes integrable with respect to h on $[a,b]$ if there exists a number I satisfying the following property: given $\varepsilon > 0$ one can find a $\delta > 0$ such that

$$|S(f,h,\kappa,\sigma) - I| < \varepsilon$$

for all subdivisions κ with mesh $\max(x_i - x_{i-1}) < \delta$ and for all intermediate subdivisions σ of κ. The number I, if it exists is unique and is denoted by

$$(RS)\int_a^b f dh.$$

If f is Riemann-Stieltjes integrable with respect to h then f and h cannot have a jump at the same point. The Moore - Pollard -Stieljtes integral requires less restrictive conditions at jump points. Its definition is the same as above with one exception: the convergence of the Riemman-Stieltjes sums as the mesh tends to zero is replaced by their convergence under refinements of subdivisions. More precisely we say that κ is a refinement of a subdivision λ if $k \supset \lambda$. Then the function h is Moore-Pollard-Stieltjes integrable, or MPS integrable with respect to h on [a,b], if there exists a number I satisfying the following property: given $\varepsilon > 0$ one canfind a subdivision λ such that $|S(f,h,\kappa,\sigma) - I| < \varepsilon$ holds for all refinements κ of λ and for all intermediate subdivisions σ of κ. The number I , if it exists is unique and will be denoted by

$$(MPS)\int_a^b fdh.$$

It is well known that the RS-integral and the MPS-integral above exist if h is of bounded variation and f is continuous. However both integrals may exist when none of the two functions have bounded variation. This was proved by L. C. Young (1936):

Theorem 15 *(L.C. Young, 1936). Assume $h \in W_p$ and $f \in W_q$ for some p,q >0 with $\frac{1}{p} + \frac{1}{q} > 1$. Then the following statements hold:*

i) $(RS)\int_a^b fdh$ if f and h do not have a common discontinuity at the same point.

ii) $(MPS)\int_a^b fdh$ if f and h do not have a common discontinuity on the same side at the same point.

Moreover, there exists a finite constant $K = K(p,q)$ such that, for any $y \in [a,b]$,the inequality

$$|\int_a^b fdh - f(y)[h(b) - h(a)]| \leq KV_p(h)V_q(f)$$

holds for both kinds of integral, provided it is defined.

If the sample paths of the stochastic process are only known to be regulated, the same results hold for if the MPS integral is replaced by another extension of the Riemann-Stieltjes integral. The following variants of the integral introduced by Young (1936) were proposed by Dudley and Norvaisa (1999).

Definition 16 *Assume $f, h \in \Re$. Define the left Young integral by*

$$(LY)\int_a^b f dh = (MPS)\int_a^b f_a^- dh_b^+ + [f(\Delta^+ h)](a) + \sum_{(a,b)} \Delta^- f \Delta^+ h$$

whenever the (MPS) exists and the sum converges absolutely. Define the right Young integral by

$$(RY)\int_a^b f dh = (MPS)\int_a^b f_b^+ dh_a^- + [f(\Delta^- h)](b) - \sum_{(a,b)} \Delta^+ f \Delta^- h$$

whenever the (MPS) exists and the sum converges absolutely. We say that f is LY integrable (or RY integrable) with respect to h on [a,b] provided the above integrals are defined.

The left and right Young integrals have the usual properties of integrals. For example they are bilinear and additive on adjacent intervals.

In stochastic analysis the Lebesgue-Stieltjes integral is used to integrate with respect to stochastic processes that have sample paths of bounded variation. In this case the values of the above extensions of the Riemann-Stieltjes integrals agree with the corresponding values of the Lebesgue-Stieltjes integral (or LS integral) as stated next:

Proposition 17 *If h is a right-continuous function of bounded variation and f is a regulated function on [a,b] then the following three integrals exist and are equal:*

$$(LY)\int_a^b f dh = (MPS)\int_a^b f_a^- dh = (LS)\int_a^b f_a^- dh$$

Theorem 18 *Assume h$\in W_p$ and f$\in W_q$ for some p,q>0 with $\frac{1}{p} + \frac{1}{q} > 1$. Then both, the right Young and the left Young integrals, exist.*

Lemma 19 *Let h∈ W_p and f∈ W_q for some p,q>0 with* $\frac{1}{p}+\frac{1}{q}>1$ *and assume h is continuous. Then the integrals* $(RS)\int_a^b f_a^- dh$ *and* $(RS)\int_a^b f dh$ *exist and are equal.*

It is clear from the above that the p-variation of the sample paths of a stochastic process is an indicator of its extended Riemman-Stieltjes integrability. We will state now a chain rule obtained by Mikosh and Norvaisa (2000).

Theorem 20 *Mikosh and Norvaisa (2000). Let* $h=(h_1,...,h_d):[a,b]\to R^d$*, where for every* $l=1,...,d$, $h_l\in W_p$ *, for some p∈* $(0,2)$*. Let* $g:R^d\to R$ *be a differentiable function with locally Lipschitz partial derivatives* $g_l^{'}, l=1,...,d$*. Then the integrals* $(LY)\int_a^b (g_l^{'}oh)dh_l$ *exist and satisfy the relation*

$$\begin{aligned}(goh)(b)-(goh)(a) &= \sum_{l=1}^{d}(LY)\int_a^b (g_l^{'}oh)dh_l+\sum_{(a,b)}[\Delta^+(goh)-\sum_{l=1}^{d}(g_l^{'}oh)\Delta^+h_l] \\ &+\sum_{(a,b)}[\Delta^-(goh)-\sum_{l=1}^{d}(g_l^{'}oh)^-\Delta^-h_l]\end{aligned}$$

where the two sums converge absolutely. Similarly, the integrals $(RY)\int_a^b (g_l^{'}oh)dh_l$ *exist and satisfy the relation*

$$\begin{aligned}(goh)(b)-(goh)(a) &= \sum_{l=1}^{d}(RY)\int_a^b (g_l^{'}oh)dh_l+\sum_{(a,b)}[\Delta^+(goh)-\sum_{l=1}^{d}(g_l^{'}oh)\Delta^+h_l] \\ &+\sum_{(a,b)}[\Delta^-(goh)-\sum_{l=1}^{d}(g_l^{'}oh)^-\Delta^-h_l]\end{aligned}$$

where the two sums converge absolutely.

The results of the paper of Mikosch and Norvaisa (2000) are applicable to sample paths of stochastic processes having bounded p-variation with $p\in(0,2)$. It is well known that standard Brownian motion does not satisfy this condition.The sample paths of Brownian motion have unbounded p-variation for $p=2$ and bounded p-variation for every $p>2$.

For the exact result, see Taylor (1972) . For fractional Brownian motion Mikosh and Norvaisa (2000) showed the folowing result:

Proposition 21 *(Mikosch and Norvaisa, 2000). Let W_H be fractional Brownian motion with index $H \in (0,1)$ and $p \in (H^{-1}, \infty)$. Then almost all sample paths of B_H are continuous and $\upsilon_p(B_H) < \infty$ with probability 1.*

The above claim is a combination of the results in Fernique (1964) and Kawada and Kono (1973).

1.2.6 Use of fractional Brownian motion as a modelling tool

Fractional Brownian motion is used to model a wide variety of stochastic data arising in enigineering and physics as well as in financial mathematics and telecommunications. For example fractional Brownian motion has been used to model the log returns of the stock prices see e.g. Shiryaev (1999), Hu and Oksendal (2003), Elliot and Van der Hoek (2003), Norvaisa (2000), the electricity price in a liberated electricity market, see e.g. Simonsen (2003), foreign exhange rates, see e.g. Los and Karuppiah (1997) and weather derivatives, see for example Brody, Syroka and Zervos (2002) and references therein. The use of fractional Brownian motion in finance can be divided in two categories. The first one is when the pathwise type of integral, or the integral defined by Lin (1995) or Shiryaev (1998) is used, which creates arbitrage opportunities when it is used for option pricing. The second one is the Wick-Ito type of integral or the integral based to the white noise calculus for fractional Brownian motion which, does not create arbitrage opportunities when it is used for option pricing. For the first category one can see for example Norvaisa (2000) who is using a real analysis approach to stock price modelling. It shows that classical calculus is applicable to market analysis whenever the local 2-variation of the return is zero, or is determined by jumps if the process is discontinuous. Fractional Brownian motion with $H \in (\frac{1}{2}, 1)$ can be treated in his setup using pathwise type of integration. Salopek (1998) constructs an arbitrage opportunity in a frictionless stock market when price process have continuous sample paths of bounded p-variation and Salopek (2002) shows the existence of arbitrage for stop-loss start-gain trading strategies when the stock price process is geometric fractional Brownian motion and the stochastic integral with respect to fractional Brownian motion is defined in the pathwise sense. Cheridito (2001) considers the construction of arbitrage for models based on fractional Brownian motion and shows how arbitrage can be ruled out by putting restrictions on the trading strategies. Cheridito is also showing how arbitrage can be excluded from fractional Brownian motion models regularizing the local path behaviour of fractional Brownian motion. He is introducing two different ways of regularizing fractional Brownian motion and he is considering the pricing of a European call option using the regularized fractional Brownian motion. Regularization in the first way is excluded by a change in the convolution kernel of the fractional Brownian motion. This yields a Gaussian semimartingale with a distribution similar to the one of fractional Brownian motion. Cheridito (2001) shows that the sum of a Brownian motion and a non-trivial multiple of an independent fractional Brownian motion with $H \in (\frac{3}{4}, 1)$ is equivalent to Brownian motion. As an application he obtains the price for a European call option on an asset driven by a linear combination of a Brownian motion and an independent fractional Brownian motion. Furthermore Dasgupta (1997), Dasgupta and Kallianpur (2000), Dasgupta and Kallianpur (1999), Rogers (1997) and Shiryaev (1998) show that when the pathwise type of integral is applied to option pricing leads to arbitrage. Aldabe, Barone-Adesi and Elliott (1998) consider also the problem of option pricing with regulated fractional Brownian motion.

For the second category one can see for example Hu and Oksendal (2003) and Elliot and Van der Hoek (2003) who are using a fractional white noise approach and they obtain a fractional Black-Scholes formula which does not create arbitrage opportunities. Furthermore Brody, Syroka and Zervos (2002) consider the case of pricing weather derivatives using the Wick - Ito type of integral developed by Duncan, Hu and Pasik-Duncan. One can see also Sottinnen and Valkeila (2003) who examine option pricing in a fractional Black-Scholes market using both cases of Wick-Ito and Riemann Stiltjes integrals. Besides Hu, Oksendal and Salopek (2001) derive a Meyer Tanaka formula involving weighted local time for fractional Brownian motion and geometric fractional Brownian motion. They use the results obtained to the study of the start-gain stop-loss portfolio and a fractional version of the Carr-Jarrow decomposition of the European call and put option into their intrinsic and time values. Benth (2003) also considers the valuation of pricing of weather derivatives both of European and of Asian type in an arbitrage free framework. Besides the integral approach developed by Bender in a series of papers allowed him to obtain option pricing formulas that exclude arbitrage opportunities, one can see Bender(2003a, 2003b).

Fractional Brownian motion has also been used as the driving stochastic process in highly-aggregated traffic in communication networks. Traditionally one assumed either the absence of any significant correlation between consecutive packet arrivals ('renewal input' for instance a Poisson process), or just a mild form of dependence (for instance, Markov modulated Poisson processes). The discovery of significant correlations on a broad range of time scales, as exhibited in many measurement studies during the 1990's, led to the examination of different classes of traffic models. Many models have been proposed to model this long - range dependence and one of them is fractional Brownian motion. One can see for more on these Debicki and Mandies (2004), Norros(1994), Norros (1995), Norros (1999), Taqqu, Teverovsky and Willinger (1997), Duncan, Yan and Yan (2001), Addie, Mannersalo and Norros (2002), Narayan (1998), Belly and Decreusefond (1997), Daley and Vesilo (1997) , O'Connell and Procissi (1998).

Furthermore, fractional Brownian motion (as a special case of self similar process) has been used to model the claims of an insurance business, see for example Michna (1998), Michna (1999) and there is also recent work on the extremes of fractional Brownian motion, probability of ruin and generalised Pickands constant, see for example Debicki (2002). Besides Husler and Piterbarg (2004), and Husler and Piterbarg (1999) consider respectively the probability of ruin for physical fractional Brownian motion and certain classes of Gaussian processes. The study of the ruin problem when the driving process is fractional Brownian motion is the subject of chapter 2 which follows.

Chapter 2

Fractional Brownian Motion and Probability of Ruin at a Given Date

The calculation of probability of ruin of an insurance company has been the subject of extended research. One can see for example Asmussen (2000), Rolski, Schmidli, Schmidt and Teugels (1998), Kalashnikov (1997) and Grandell (1991) and the references there in.

An insurer is exposed not only to the traditional liability risk related to the insurance portfolio, but also to asset risk related to the investment portfolio. The probability of ruin has been investigated in models with both kinds of risks by several authors. Schnieper (1983) combines the classical Poisson claims process with a force of interest that changes in accordance with a discrete Markov chain at Poisson times. Paulsen (1993) works with a quite general semimartingale setup with claims and interest driven by Poisson processes and Brownian motions. Dickson and Hipp (2003) studied a risk process in which claim inter-arrival times have an Erlang(2) distribution. Gerber (1971) and Harisson (1977) were among the first to incorporate in their studies of ruin the financial side of risk business in the form of non-stochastic interest. For more on ruin problem one can see for example Nyrhinen (2001), Nyrhinen (1999), Nyrhinen (1998), Paulsen (1993), Paulsen (1998a), Paulsen (1998b), Shimura (1983), Ramsay (1986), Ramsey and Usabel (1997), Kalashnikov and Norberg (2002), Norberg (1995), Kluppelberg and Stadtmuller (1998), Kluppelberg (1993), Embrechts and Villasenor (1988), Furrer (1998), Promislow (1991), Muller and Pflug (2001), Hoglund (1990), Dufresne and Gerber (1988), Gerber (1988), Garrido (1988), Delbaen and Haezendonck (1985), Delbaen and Haezendonck (1987), Moller (1995), Schmidli (1995), Furrer and Schmidli (1994), and Gjessing and Paulsen (1997).

In this chapter we model the liabilities of an insurance business assuming that they are driven by fractional Brownian motion and we study the ruin probability of the insurance company under the influence of interest force.This problem is interesting from the point of view of applications but presents also considerable theoretical interest. There is recent work on this problem by several authors as fractional Brownian motion has been used recently to model the claims an insurance business may face, one can see for example Michna (1998a), Michna (1998b), Michna (1999), and there is also recent work on the extremes of fractional Brownian motion, see for example Debicki (2002), Husler and Piterbarg (1999). Some of these works, deal with the asymptotic properties of ruin probability using probabilistic techniques and some provide upper and lower bounds for the ruin probability in certain limiting situations. Michna (1998,1999) investigates ruin probabilities and first passage times for self-similar prcesses. He proposes self-similar processes as a risk model with claims appearing in good and bad periods. Then, in particular, he gets the fractional Brownian motion with drift as a limit risk process. Some bounds and asymptotics for ruin probability on a finite interval for fractional Brownian motion are derived. A method of simulation of ruin probability over infinite horizon for fractional Brownian motion is presented. The moments of the first passage time of fractional Brownian motion are studied. As an application of his method he numerically computes the Pickands constant for fractional Brownian motion. An asymptotic behavior of the supremum of a Gaussian process X over infinite horizon is studied. In particular X can be a fractional Brownian motion, a nonlinearly scaled Brownian motion or an integrated stationary Gaussian processes.

Husler and Piterbarg (1999) considered the extreme values of fractional Brownian motions, self-similar Gaussian processes and more general Gaussian processes which have a trend $-ct^{\beta}$ for some constants $c, \beta > 0$ and a variance t^{2H}. They derive the tail behaviour of these extremes and show that they occur in the neighborhood of the unique point t_0 where the related boundary function $(u + ct^{\beta})/t^H$ is minimal. They consider the case of $H < \beta$. Specifically they derive the following results:

Let $X(t), t \geq 0$, be a self-similar Gaussian process with index H, $0 < H < 1$, and c, β positive constants with $H < \beta$. Assume that the process $X(t)$ is locally stationary with $0 < a \leq 2$ and some positive constant D. Then for $u \to \infty$

i) if $a < 2$:

$$\begin{aligned} P\{X(t) &\geq u + c\beta^t \; for \; some \; t \geq 0\} \\ &\sim \frac{H_\alpha \sqrt{\pi} D^{\frac{1}{\alpha}}}{\sqrt{B} 2^{\frac{1}{a}-0.5}} A^{\frac{2}{a}-0.5} u^{(1-\frac{H}{\beta})(\frac{2}{a}-1)} \Psi(Au^{1-\frac{H}{\beta}}) \end{aligned}$$

ii) $a = 2$:

$$\begin{aligned} P\{X(t) &\geq u + c\beta^t \; for \; some \; t \geq 0\} \\ &\sim \sqrt{\frac{AD+B}{B}} \Psi(Au^{1-\frac{H}{\beta}}) \end{aligned}$$

where

$$A := \left(\frac{H}{c(\beta - H)} \right)^{\frac{-H}{\beta}} \frac{\beta}{b - H}$$

and

$$B := \left(\frac{H}{c(\beta - H)} \right)^{\frac{-(H+2)}{\beta}} H\beta,$$

and $H_a, a \leq 2$ is the constant defined by

$$H_a = \lim_{T \to \infty} \frac{1}{T} E(\exp[\max_{0 \leq t \leq T} \chi(t)]),$$

where $\chi(t)$ is a fractional Brownian motion with drift $E(\chi(t)) = -t^{-\alpha}$ and covariance function

$$Cov(\chi(s),\chi(t)) = t^\alpha + s^\alpha - |t-s|^\alpha.$$

The assumption of local stationarity is specified as follows: if we consider the standardized process

$$Y(t) = X(t)t^{-H}$$

in a neighborhood of a point s_0,for some $a > 0$ it is assumed that

$$\lim_{s\to s_0 . s'\to s_0} \frac{E(Y(s)-Y(s'))^2}{|s-s'|^\alpha} = D > 0.$$

Now let $X(t)$,be a fractional Brownian motion with index H, $0 < H < 1$. Then we get

$$\begin{aligned} P\{X(t) &\geq u + c\beta^t \ for\ some\ t \geq 0\} \\ &\sim \frac{H_{2H}\sqrt{\pi}D^{\frac{1}{2H}}A^{\frac{2-H}{2H}}}{\sqrt{B}2^{\frac{1-H}{2H}}} u^{(1-\frac{H}{\beta})(1-H)/H}\Psi(Au^{1-\frac{H}{\beta}}) \end{aligned}$$

Let $X(t)$ be a Gaussian process with mean zero and variance t^{2H} and $c, \beta > 0$ with $H < \beta$. Assume the conditions (i), (ii) written below and with $0 < \alpha < 2$, then for $u \to \infty$:

$$\begin{aligned} P\{\sup_{t>0} X(t) - c\beta^t &> u \text{ for some } t \geq 0\} \\ &\sim \frac{(\sqrt{D}v(s_0))^2 H_a 2^{-1/\alpha} e^{-(1/2)A^2 u^{2-2H/\beta}}}{\sqrt{AB}v(s_0)K^{-1}(u^{-1+H/\beta})u^{2-2H/\beta}} \end{aligned}$$

Condition (i), If G, γ, s, s' positive

$$\lim_u \sup E(X^{(u)}(s) - X^{(u)}(s'))^2 \leq G|s-s'|^\gamma$$

Condition (ii).

$$\lim_{u\to\infty} \frac{E[X^{(u)}(s)\upsilon(s) - X^{(u)}(s')\upsilon(s')]^2}{K^2(|s-s'|)} = D,$$

where $K^2(.)$ is a regular varying function (at zero).

Debicki (2002) considered the important role that Pickand constants play in the exact asymptotic of extreme values for Gaussian stochastic processes. The generalized Pickands constant $H_{n,}$ defined as

$$H_n = \lim_{T\to\infty} \frac{H_n(T)}{T},$$

where

$$H_n(T) = E[\exp(\max_{t\in[0,T]} \sqrt{2}\eta(t) - \sigma_\eta^2(t))]$$

and $\eta(t)$ is a centered Gaussian process with stationary increments and variance function $\sigma_\eta^2(t)$. Under some mild conditions on $\sigma_\eta^2(t)$ Debicki proves that H_n is well defined and he gives a comparison criterion for the generalised Pickand constants. Moreover he proves a theorem that extends the result of Pickands for certain stationary Gaussian processes. As an application he obtain the asymptotic behavior of $\psi(u) = P(\sup_{t\geq 0}\zeta(t) - ct > u)$ as $u \to \infty$, where $\zeta(x) = \int_0^x Z(s)ds$ and $Z(s)$ is a stationary Gaussian process with covariance function $R(t)$ fulfilling some integrability conditions. For some bounds and estimators of H_a one can see Shao (1996).

The approach we adopt here for the treatment of ruin probabilities in models where the claims may present long range dependence is very different from the approach adopted in the above works. In this paper we propose a model for an insurance business facing liabilities presenting long term correlations. The long term correlations are modelled with the use of a fractional brownian motion with Hurst exponent H. The insurance firm invests in an interest account which is assumed to be deterministic. It is shown that the cash balance process of the firm satisfies an Ornstein-Uhlenbeck stochastic differential equation driven by fractional Brownian motion. Using the recently developed tools of fractional stochastic calculus we show that the probability of ruin of the firm can be expressed as the solution of a linear parabolic partial differential equation. We have solved this partial differential equation analytically and we provide an exact expression for the ruin probability in terms of error functions, valid for all times. Using this exact expression one may derive asymptotic results using standard techniques. Finally, the partial differential equation allows an efficient numerical treatment of the problem which may be used as an alternative to Monte-Carlo type simulations. Our model and treatment is inspired by a very interesting model proposed by Norberg (1999) for the study of ruin probability in a model with diffusive type liabilities (Brownian motion type), with the use of partial differential equations. In some sense our treatment is an extension of Norberg's model to the case of fractional Brownian motion type liabilities. This extension is by no means trivial since the inclusion of fractional Brownian motion in the model presents difficulties which need different mathematical techniques in order to be overcome. More specifically Norberg (1999) studied ruin and related problems for a risk business with compounding assets when the cash flow and the cumulative interest rate are diffusion processes with coefficients depending on the time and on the current cash balance. Differential equations were obtained for the probabilities or ruin at a given date, in finite time, and in infinite time. Relationships between crossing probabilities and transition probabilities are considered and, in particular, existing results on the probability distribution of the running maximum of a Brownian motion and on the relationship between the probability of ruin and on the probability distribution of the discount total payments were generalized. The proofs of Norberg are based on a martingale technique. It is clear however that this technique cannot be applied in the case of fractional Brownian motion.

2.1 The model

Following the spirit of the original model proposed for liabilities of the Brownian motion type by Norberg (1999) let us consider the following model for an insurance firm: The firm is characterized by its value at time t, which is assumed to be a stochastic process X_t. The firm invests its value X_t to an interest account with logarithmic interest force δ_t. The interest is assumed to be a deterministic function of time. This assumption is not unreasonable for models valid for short times. The firm has to face liabilities B_t. Assuming an insurance portfolio that is made up of a large number of individual risks, none of which is large enough to affect the total result significantly we approximate the liabilities or the payment function B_t by a fractional Brownian motion with drift

$$dB_t = -b_t dt - \sigma_t dW_t^H \tag{2.1}$$

where b_t represents the expected gain per time unit due to a safety loading in the premium, and σ_t is the standard deviation of the liabilities per time unit and is thus a measure of the size of the liability risk. In the above W_t^H is a fractional Brownian motion with Hurst exponent H and b_t, σ_t are deterministic functions of time. Allowing b_t,σ_t to be given functions of time we allow for seasonality in the claims. This seasonality is relevant in a number of models, for instance road accidents are more likely to happen during holiday periods, fires which may lead to property damage are more likely to happen during the hot months of the summer etc. The introduction of the fractional Brownian motion allows for the modeling of correlations in the claims. We consider only the case where $H > 1/2$ which corresponds to positive correlation between the claims. Such models may be relevant in models of claims related to health, disability insurance, accident or whole life insurance. The case $H < 1/2$ will correspond to negative correlation between claims. An example of a risk process with long range dependence was developed by Michna (1998 a, b) , who constructed a risk model in which claims appear in good and bad periods (e.g. good weather and bad weather), and under the assumption that the claims in bad periods are bigger than the claims of the good periods.

Following Norberg (1999) the cash balance equation for the firm at time t has the following form

$$X_t = e^{\Delta_t}(X_0 - \int_0^t e^{-\Delta_s} dB_s) \tag{2.2}$$

where

$$\Delta_t = \int_0^t \delta_s ds$$

The cash-balance process is the solution of a stochastic differential equation, driven by fractional Brownian motion. We have the following proposition:

Proposition 22 *The cash-balance process X_t given by the book-keeping equation (2.2) is the solution of the fractional Ornstein-Uhlenbeck equation*

$$\begin{aligned} dX_t &= (\delta_t X_t + b_t)\, dt + \sigma dW_t^H \\ X_0 &= x. \end{aligned}$$

Proof: Define the process $K_t = exp(\int_0^t \delta_s ds)$. Then we may rewrite (2.2) as

$$X_t = xK_t + K_t \int_0^t K_s^{-1} b_s ds + K_t \int_0^t K_s^{-1} \sigma_s dW_s^H$$

Let us further define the stochastic process

$$\eta_s = \int_0^t \sigma_s K_s^{-1} dW_s^H$$

and the function

$$f(t, \eta) = xK_t + K_t \int_0^t K_s^{-1} b_s ds + K_t \eta$$

We see that $f(t, \eta_t) = X_t$. We now apply the fractional Itô lemma on the function $f(t, \eta)$. We have that

$$
\begin{aligned}
\frac{\partial f}{\partial t}(t,\eta) &= \delta_t f(t,\eta) + b_t \\
\frac{\partial f}{\partial \eta}(t,\eta) &= K_t \\
\frac{\partial^2 f}{\partial \eta^2}(t,\eta) &= 0
\end{aligned}
$$

A straightforward application of fractional Itô's lemma yields

$$
f(t,\eta_t) = f(0,0) + \int_0^t (\delta_s f(s,\eta_s) + b_s)ds + \int_0^t \sigma_s dW_s^H
$$

or equivalently

$$
X_t = x + \int_0^t (\delta_s X_s + b_s)ds + \int_0^t \sigma_s dW_s^H.
$$

This concludes the proof.

We may also state the following:

Proposition 23 *The cash-balance process is a Gaussian process with mean*

$$
m_t = xK_t + K_t \int_0^t b_s K_s^{-1} ds
$$

and variance

$$
V_t = K_t^2 \int_0^t \int_0^s \sigma_u \sigma_s K_u^{-1} K_s^{-1} \phi(u,s) du ds
$$

where

$$
K_t = \exp(\int_0^t \delta_s ds)
$$

and

$$
\phi(u,s) = H(2H-1) \mid u - s \mid^{2H-2} .
$$

Proof: The proof follows using the properties of the stochastic integral over fractional Brownian motion.

The mean and the variance can be computed using special functions for the particular case of constant parameters, $\delta_s = \delta$, $b_s = b$.

2.2 A partial differential equation for the ruin probability at a given date

We are interested in the derivation of the ruin probability at a given date $P(X_t \leq 0 \mid X_0 = x)$. We will show in this section that this ruin probability can be determined by the solution of a linear parabolic partial differential equation. The analysis follows the lines of Brody, Syroka and Zervos (2002) where the value of a weather derivative whose underlying (the temperature) is modelled by a fractional Brownian motion is expressed through the use of a partial differential equation. We have the following proposition:

Proposition 24 *Assume that $H > 0.5$ and σ_s has no singularities. The ruin probability at a given date*

$$u(t,x) := P(X_t \leq 0 \mid X_0 = x)$$

satisfies the following parabolic partial differential equation (Cauchy problem)

$$\begin{aligned} -\frac{\partial u}{\partial \tau} + (\delta_{t-\tau} x + b_{t-\tau})\frac{\partial u}{\partial x} + K_{t-\tau}\sigma_{t-\tau}\left(\int_0^{t-\tau} \phi(s, t-\tau)\sigma_s K_s^{-1} ds\right)\frac{\partial^2 u}{\partial x^2} &= 0 \qquad (2.3) \\ u(0,x) &= \mathbf{1}_{\{x \leq 0\}} \end{aligned}$$

Remark 1 *It is useful to make a comment on the meaning and use of the above equation. Since the equation depends on the parameter t, the solution of the equation is a function $u(\tau, x) = u(\tau, x; t)$. The ruin probability at time t, given that the initial capital is x is the solution of this equation calculated at $\tau = t$, i.e $u(t,x) = u(t,x;t)$. That means that fixing t we have to solve the equation for $u(\tau, x) = u(\tau, x; t)$ and then take the limit as $\tau \to t$.*

Proof: Consider the function

$$g(\tau_0, \eta; t) = w(t - \tau_0, f(\tau_0, \eta))$$

where

$$f(\tau_0, \eta) = xK_{\tau_0} + K_{\tau_0} \int_0^{\tau_0} K_s^{-1} \beta_s ds + K_{\tau_0} \eta.$$

We now apply the fractional Itô formula to $g(\tau_0, \eta_{\tau_0}; t)$ for τ_0 taking values between $\tau_0 = 0$ and $\tau = t$. Note that t is considered as a fixed parameter while τ_0, η_{τ_0} are considered as variables.

Since

$$\begin{aligned}
\frac{\partial g}{\partial \tau_0} &= -\frac{\partial w}{\partial \tau} + (\delta_{\tau_0} f(\tau_0, \eta) + b_{\tau_0}) \frac{\partial w}{\partial x} \\
\frac{\partial g}{\partial \eta} &= \frac{\partial w}{\partial x} K_{\tau_0} \\
\frac{\partial^2 g}{\partial \eta^2} &= \frac{\partial^2 w}{\partial x^2} K_{\tau_0}^2,
\end{aligned}$$

where we consider $w = w(\tau, x)$, with $\tau = t - \tau_0$, we see that

$$\begin{aligned}
g(t, \eta_t) = g(0, 0) &+ \int_0^t \frac{\partial g}{\partial \tau_0}(\tau_0, \eta_{\tau_0}) d\tau_0 + \int_0^t \frac{\partial g}{\partial \eta}(\tau_0, \eta_{\tau_0}) \sigma_{\tau_0} K_{\tau_0}^{-1} dW_{\tau_0}^H \\
&+ \int_0^t \frac{\partial^2 g}{\partial \eta^2}(\tau_0, \eta_{\tau_0}) \sigma_{\tau_0} K_{\tau_0}^{-1} \left(\int_0^{\tau_0} \phi(s, \tau_0) \sigma_s K_s^{-1} ds \right) d\tau_0
\end{aligned} \tag{2.4}$$

where

$$\phi(s, t) = H(2H - 1) \mid s - t \mid^{2H-2} .$$

We observe that

$$g(t, \eta_t) = w(0, X_t), \quad g(0, x) = w(t, x)$$

Thus, equation (2.4) assumes the form

$$
\begin{aligned}
w(0,X_t) &= w(t,x)+\int_0^t\left(-\frac{\partial w}{\partial \tau}+(\delta_{\tau_0}f+b_{\tau_0})\frac{\partial w}{\partial x}+K_{\tau_0}\sigma_{\tau_0}\left(\int_0^{\tau_0}\phi(s,\tau_0)\sigma_s K_s^{-1}ds\right)\frac{\partial^2 w}{\partial x^2}\right)d\tau_0\\
&+ \int_0^t \frac{\partial w}{\partial x}dW^H_{\tau_0}
\end{aligned}
$$

We now take expectations and use the properties of the stochastic integral to obtain

$$
\begin{aligned}
&E[w(0,X_t)] = w(t,x)\\
+E[&\int_0^t\left(-\frac{\partial w}{\partial \tau}+(\delta_{\tau_0}f+b_{\tau_0})\frac{\partial w}{\partial x}+K_{\tau_0}\sigma_{\tau_0}\left(\int_0^{\tau_0}\phi(s,\tau_0)\sigma_s K_s^{-1}ds\right)\frac{\partial^2 w}{\partial x^2}\right)d\tau_0] \quad (2.5)
\end{aligned}
$$

We rephrase the ruin probability as

$$
P(X_t \le 0 \mid x) = u(t,x) = E[\mathbf{1}_{\{X_t\le 0\}}]
$$

and add this to equation (2.5) to obtain

$$
\begin{aligned}
&E[w(0,X_t)] + u(t,x) = w(t,x) + E[\mathbf{1}_{\{X_t\le 0\}}] +\\
E[&\int_0^t\left(-\frac{\partial w}{\partial \tau}+(\delta_{\tau_0}f+b_{\tau_0})\frac{\partial w}{\partial x}+K_{\tau_0}\sigma_{\tau_0}\left(\int_0^{\tau_0}\phi(s,\tau_0)\sigma_s K_s^{-1}ds\right)\frac{\partial^2 w}{\partial x^2}\right)d\tau_0] \quad (2.6)
\end{aligned}
$$

where inside the integrals $w := w(t-\tau_0, X_{\tau_0}) = w(\tau, X_{\tau_0})$. If we choose w to be the solution of the PDE

$$
-\frac{\partial w}{\partial \tau}+(\delta_{\tau_0}f+b_{\tau_0})\frac{\partial w}{\partial x}+K_{\tau_0}\sigma_{\tau_0}\left(\int_0^{\tau_0}\phi(s,\tau_0)\sigma_s K_s^{-1}ds\right)\frac{\partial^2 w}{\partial x^2} = 0
$$

with $w(0,x) = \mathbf{1}_{\{x\le 0\}}$ we see that $u(t,x) = w(t,x)$. Observing that the coefficients of the equation are calculated in τ_0 whereas w is calculated in $\tau := t-\tau_0$ we may redefine time so as to express this equation in the equivalent form

$$-\frac{\partial w}{\partial \tau} + (\delta_{t-\tau}x + b_{t-\tau})\frac{\partial w}{\partial x} + K_{t-\tau}\sigma_{t-\tau}\left(\int_0^{t-\tau} \phi(s, t-\tau)\sigma_s K_s^{-1} ds\right)\frac{\partial^2 w}{\partial x^2} = 0$$
$$w(0, x) = \mathbf{1}_{\{x \le 0\}}$$

This concludes the proof of the proposition. □

We may further obtain a PDE for the computation of the ruin probability at a given date

$$u(t, y; s, x) := P(X_t \le y \mid X_s = x).$$

We have the following proposition:

Proposition 25 *The ruin probability at a given date*

$$u(t, x; s, y) := P(X_t \le y \mid X_s = x)$$

satisfies the following parabolic partial differential equation (Cauchy problem)

$$-\frac{\partial w}{\partial \tau} + (\delta_{t+s-\tau}x + b_{t+s-\tau})\frac{\partial w}{\partial x} + K_{t+s-\tau}\sigma_{t+s-\tau}\left(\int_s^{t+s-\tau} \phi(s', t+s-\tau)\sigma_{s'} K_{s'}^{-1} ds'\right)\frac{\partial^2 w}{\partial x^2} \tag{2.7}$$

$$= 0$$

$$w(s, x) = \mathbf{1}_{\{x \le y\}}$$

in the sense that

$$u(t, x; s, y) = w(t, x).$$

Proof: We may show that

$$X_t = \bar{K}_{s,t}x + K_t\left[\int_s^t K_{s'}^{-1} b_{s'} ds' + \int_s^t K_{s'}^{-1}\sigma_{s'} dW_{s'}^H\right]$$

where

$$\bar{K}_{s,t} = exp(\int_s^t \delta_{s'} ds').$$

We now apply the fractional Itô formula to the function

$$g(\tau, \eta_\tau) = w(t + s - \tau, f(\tau, \eta_\tau))$$

where

$$\begin{aligned} f(\tau, \eta) &= x\bar{K}_{s,\tau} + K_\tau \int_s^t K_{s'}^{-1} b_{s'} ds' + K_\tau \eta_\tau \\ \eta_\tau &= \int_s^t K_{s'}^{-1} \sigma_{s'} dW_{s'}^H \end{aligned}$$

The rest follows as in the proof of the previous proposition.

Remark 2 *The quantity* $P[\inf_{s\in[0,T]} X_s < 0 \mid X_0 = x]$ *is proposed in the literature as a measure for the ruin probability. The ruin probability at a given date we calculate in our model may serve as a lower bound for this quantity and may thus serve as an alert to the regulating authority of the company. In Norberg (1999) a partial differential equation was obtained for this quantity as well using the Markovian property of the Brownian motion driving the liabilities. However, in the model with fractional Brownian motion, the Markovian property is no longer valid and we do not expect similar results to hold. We content here to perform a numerical evaluation of*

$$P[\inf_{s\in[0,T]} X_s < 0 \mid X_0 = x]$$

using Monte-Carlo simulation.

2.3 Solution of the PDE

We now deal with the solution of the PDE for the ruin probability at a given date.

2.3.1 An analytical solution

The PDE for the ruin probability at a given date can be solved analytically in its most general form. This facilitates immensely the calculation of the ruin probability at a given date.

We start our presentation of the analytical solution of the PDE for the ruin probability at a given date in the case where the coefficients δ_t, σ_t and b_t are constants. This facilitates the arguments. Then we provide the solution for the general case of time dependent coefficients.

In the case of constant coefficients the time dependent factor multiplying the second derivative term becomes

$$\begin{aligned} f(t) &= e^{\delta t}\sigma^2 H(2H-1)\int_0^t \mid t-s \mid^{2H-2} e^{-\delta s} ds \\ &= \sigma^2 H(2H-1)\int_0^t s^{2H-2} e^{\delta s} ds \\ &= \sigma^2 H(2H-1)\sum_{n=0}^{\infty}\frac{\delta^n}{n!}\frac{t^{2H-1+n}}{2H-1+n} \\ &= \sigma^2 H(2H-1)(-1)^{2H-1}\delta^{1-2H}\gamma(2H-1,-\delta t) \end{aligned} \tag{2.8}$$

where $\gamma(z,a)$ is the incomplete gamma function, see for instance Lebedev (1972). In this case we may obtain an expression for the ruin probability in terms of the complementary error function. We have the following proposition.

Proposition 26 *In the case where δ_t, σ_t, b_t are constants the ruin probability may be expressed as*

$$P(X_t \leq 0 \mid X_0 = x) = \frac{1}{\sqrt{\pi}} erfc(k(t,x))$$

where

$$k(t,x) = \frac{e^{\delta t}(\delta x + b) - b}{\sqrt{2T(t)}}$$

with

$$T(t) = 2\delta^2 \int_0^t f(t - t_1^{'}) e^{2\delta t_1^{'}} dt_1^{'}.$$

Proof: We proceed to the solution of the PDE

$$-\frac{\partial w}{\partial t_1} + (\delta x + b)\frac{\partial w}{\partial x} + f(t - t_1)\frac{\partial^2 w}{\partial x^2} = 0$$

with initial condition $w(0, x) = \mathbf{1}_{\{x \leq 0\}}$. We will use the change of variables $(t, x) \to (T, X)$ where

$$\begin{cases} X = e^{\delta t_1}(\delta x + b) \\ T = t_1 \end{cases}$$

Since

$$\begin{aligned} \frac{\partial}{\partial x} &= \delta e^{\delta t}\frac{\partial}{\partial X} = \delta e^{\delta T}\frac{\partial}{\partial X}, \\ \frac{\partial^2}{\partial x^2} &= \delta^2 e^{2\delta t}\frac{\partial^2}{\partial X^2} = \delta^2 e^{2\delta T}\frac{\partial^2}{\partial X^2}, \\ \frac{\partial}{\partial t_1} &= \delta e^{\delta t}(\delta x + b)\frac{\partial}{\partial X} + \frac{\partial}{\partial T} = \delta X\frac{\partial}{\partial X} + \frac{\partial}{\partial T} \end{aligned}$$

the PDE becomes in the new variables

$$-\frac{\partial w}{\partial T} + \delta^2 f(T) e^{2\delta T}\frac{\partial^2 w}{\partial X^2} = 0$$

By further defining the new set of variables

$$\begin{cases} X' = X \\ T' = 2\delta^2 \int_0^T e^{2\delta t} f(t - t_1) dt_1 \end{cases}$$

we see that the PDE assumes the form of the heat equation

$$-\frac{\partial w}{\partial T'} + \frac{1}{2}\frac{\partial^2 w}{\partial X'^2} = 0$$

with initial condition $w(0, X') = \mathbf{1}_{\{X' \leq b\}}$. This can be solved using the Green's function (heat kernel) for the diffusion equation $G(X' - Y, T')$. The solution is given by the integral formula

$$w(T', X') = \int_{-\infty}^{\infty} G(X' - Y, T') w(0, Y) dY$$

where

$$G(X' - Y, T') = \frac{1}{\sqrt{2\pi T'}} \exp\left(-\frac{(X' - Y)^2}{2T'}\right)$$

and $w(0, Y) = \mathbf{1}_{\{Y \leq b\}}$. Using the integral formula

$$w(T', X') = \int_{-\infty}^{b} \frac{1}{\sqrt{2\pi T'}} \exp\left(-\frac{(X' - Y)^2}{2T'}\right) dY$$

The last integral may be expressed in terms of the complementary error function as follows

$$w(T', X') = \frac{1}{\sqrt{\pi}} erfc(\frac{X' - b}{\sqrt{2T'}})$$

and returning to the original variables we may find that

$$\begin{aligned} w(t_1, x) &= \frac{1}{\sqrt{\pi}} erfc(k(t_1, x), \\ k(t_1, x) &= \frac{e^{\delta t_1}(\delta x + b) - b}{\sqrt{2T'}} \\ T' &= 2\delta^2 \int_0^{t_1} f(t - t_1') e^{2\delta t_1'} dt_1' \end{aligned}$$

The ruin probability is obtained setting $t_1 = t$ in the above expression. This completes the proof. □

Proposition 27 ***Remark 3*** *The new variable T' may be expressed as a function of t_1 in the form of series using the expression*

$$T' = 2\delta^2 \sigma^2 H(2H - 1) \sum_{n,m=0}^{\infty} \frac{2^m \delta^{n+m}}{m! n! (n + 2H - 1)} t^{n+m+2H} B_{t_1/t}(m + 1, n + 2H)$$

where by $B_x(\alpha,\beta)$ *we denote the incomplete Beta function*

$$B_x(\alpha,\beta)=\int_0^x s^{\alpha-1}(1-s)^{\beta-1}ds$$

Setting $t=t_1$ *in the above series we may obtain a series expression for* $T(t)$ *of the form*

$$T(t)=2\delta^2\sigma^2H(2H-1)\sum_{n,m=0}^{\infty}\frac{2^m\delta^{n+m}}{m!n!(n+2H-1)}t^{n+m+2H}B(m+1,n+2H)$$

where by $B(\alpha,\beta)$ *we denote the complete Beta function*

$$B(\alpha,\beta)=\int_0^1 s^{\alpha-1}(1-s)^{\beta-1}ds.$$

We now give the solution of the ruin probability PDE in the general case of time dependent coefficients:

Proposition 28 *The solution of the ruin probability PDE in the general case is given in the form*

$$u(t,x)=\int_{-\infty}^0\frac{1}{\sqrt{2T'}}exp\left(-\frac{(X'-Y)^2}{2T'}\right)dY=\frac{1}{\sqrt{\pi}}erfc\left(\frac{X'}{\sqrt{2T'}}\right)$$

where

$$\begin{aligned}X' &= exp\left(\int_0^t\delta_{t-s}ds\right)x+\int_0^t b_{t-s}exp(\int_0^s\delta_{t-s'}ds')ds\\ T' &= \int_0^t exp\left(\int_0^{t_1}2\delta_{t-s}ds\right)f(t-t_1)dt_1\\ f(t-t_1) &= K_{t-t_1}\sigma_{t-t_1}\left(\int_0^{t-t_1}\phi(s,t-t_1)\sigma_sK_s^{-1}ds\right)\end{aligned}\qquad(2.9)$$

Proof: As before we seek for a new set of variables in which the PDE assumes the form of the heat equation. To this end we try the new set of variables

$$\begin{cases} X = f_1(t_1)x + f_2(t_1) \\ T = t_1 \end{cases}$$

where f_1 and f_2 are functions to be specified. Choosing f_1 and f_2 to be the solutions of the differential equations

$$\begin{aligned} -\frac{df_1}{dt_1}(t_1) + \delta_{t-t_1} f_1(t_1) &= 0 \\ -\frac{df_2}{dt_1} + b_{t-t_1} f_1(t_1) &= 0 \end{aligned}$$

we see that in the new coordinates the equation becomes

$$-\frac{\partial u}{\partial T} + f_1(t_1)^2 f(t-t_1) \frac{\partial^2 u}{\partial X^2} = 0 \tag{2.10}$$

where f_1 and f_2 can be readily found from the solution of the above ODEs as

$$\begin{aligned} f_1(t_1) &= exp(\int_0^{t_1} \delta_{t-s} ds) \\ f_2(t_1) &= \int_0^{t_1} b_{t-s} exp(\int_0^{s} \delta_{t-s'} ds') ds \end{aligned}$$

Equation (2.10) can be reduced to a diffusion equation of the form

$$-\frac{\partial u}{\partial T'} + \frac{1}{2} \frac{\partial^2 u}{\partial X'^2} = 0$$

through a further change of variables

$$\begin{cases} X' = X \\ T' = 2 \int_0^T f_1(t_1)^2 \, f(t-t_1) \, dt_1 \end{cases}$$

The solution of this equation can be given in terms of the Green's function (heat kernel) for the diffusion equation in a way analogous to the constant coefficient case. □

Remark: Note that we use Lebedev's (1972) convention for the complementary error function

$$erfc(x) = \int_x^\infty e^{-z^2} dz$$

To avoid confusion note that software packages such as e.g. Mathematica or Matlab use a slightly different definition. These two are related by a simple scaling factor of $\frac{2}{\sqrt{\pi}}$.

2.3.2 Asymptotics

Using the well known asymptotic expansions for the error function (see e.g. Lebedev, 1972) we may obtain asymptotics for the probability of ruin for various limiting cases of interest.

One particularly interesting case is the limit of large initial capital $x \to \infty$. In the constant coefficients case for example we have

$$P(X_t \leq 0 \mid X_0 = x) \simeq \frac{1}{\sqrt{\pi}} exp(-k(t,x)^2) \left[\frac{1}{2k(t,x)} - \frac{1}{2^2 k(t,x)^3} + ... \right]$$

From that we see that the ruin probability decreases as $exp(-\lambda x^2)$ for large x, for some properly chosen constant λ. This is in accordance with the results obtained in Norberg (1999) for the Brownian motion case.

Of interest are also the large time asymptotics. The case of general H is complicated to handle (due to the complicated form of the integral defining T'), but some insight can be obtained by studying the special cases $H = 1/2$ and $H = 1$ (see next section).

Finally of interest is the asymptotic formulae for the ruin probability as the interest force tends to 0 ($\delta \to 0$). When $\frac{1}{2} < H < 1$ and $\delta \to 0$ we have from the asymptotics of the general solution that

$$\lim_{\delta \to 0} u(t,x) = \frac{1}{2} \left[1 - \Phi \left(\frac{tb + x}{t^H \sigma \sqrt{2}} \right) \right]$$

2.3.3 Two special cases

We now provide results for two special values for the Hurst parameter H. We will only consider the constant coefficient case.

The case of Brownian motion ($H = 1/2$)

In the case $H = 1/2$ the only term in the series (2.8) that survives is the term corresponding to $n = 0$. This gives $f(t) = \frac{\sigma^2}{2}$ which is a constant.

Then the PDE for the ruin probability becomes

$$\begin{aligned} -\frac{\partial w}{\partial t_1} + (\delta x + b)\frac{\partial w}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 w}{\partial x^2} &= 0 \\ w(0, x) &= \mathbf{1}_{\{x \leq 0\}} \end{aligned}$$

The ruin probability $u(t, x) = w(t, x)$. This is the same equation as the one derived by Norberg (1999) for the case of Brownian motion driven liabilities.

Using the consecutive transformations

$$\begin{cases} X = e^{\delta t_1}(\delta x + b) \\ T = t_1 \end{cases}$$

and

$$\begin{cases} X' = X \\ T' = \int_0^T \delta^2 \sigma^2 e^{2\delta t} dt = \frac{\delta \sigma^2}{2}(e^{2\delta T} - 1) \end{cases}$$

we see that the equation transforms to

$$-\frac{\partial w}{\partial T'} + \frac{1}{2}\frac{\partial^2 w}{\partial X'^2} = 0$$

The initial condition is

$$w(t_1 = 0, x) = \mathbf{1}_{\{x \leq 0\}}$$

which translates to

$$w(T^{'} = 0, \frac{X-b}{\delta}) = \mathbf{1}_{\{\frac{X-b}{\delta} \leq 0\}} = \mathbf{1}_{\{X \leq b\}}$$

The general solution to this equation is

$$\begin{aligned} w(X^{'}, T^{'}) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi T^{'}}} exp\left(-\frac{(X^{'} - Y)^2}{2T^{'}}\right) \mathbf{1}_{\{Y \leq b\}} dY \\ &= \frac{1}{\sqrt{\pi}} \int_{k}^{\infty} exp(-z^2) dz = \frac{1}{\sqrt{\pi}} erfc(k) \end{aligned}$$

where

$$k = k(T^{'}, X^{'}) = \frac{X-b}{\sqrt{2T^{'}}}$$

or in terms of the original coordinates

$$k = k(x,t) = \frac{1}{\sqrt{\delta}\sigma} \frac{e^{\delta t}(\delta x + b) - b}{\sqrt{e^{2\delta t} - 1}}$$

It is interesting to look at the limiting behaviour of the above formula.

$t \to 0$ In this case

$$k(x,t) \simeq \frac{x}{\sigma\sqrt{t}}$$

and

$$u(x,t) \simeq \frac{1}{\sqrt{\pi}} erfc\left(\frac{x}{\sigma\sqrt{t}}\right)$$

The limiting behaviour is different depending on whether x is posiitve or negative.

$t \to \infty$. In this case

$$k(x,t) \simeq \frac{\sqrt{2}}{\sigma\sqrt{\delta}}(\delta x + b - be^{-2\delta t})$$

so that

$$u(x,t) \simeq \frac{1}{\sqrt{\pi}} erfc\left(\frac{\sqrt{2}}{\sigma\sqrt{\delta}}(\delta x + b - be^{-2\delta t})\right)$$

The case $H = 1$

In the case $H = 1$ we have

$$f(t) = e^{\delta t}\sigma^2 \int_0^t e^{-\delta s} ds = \frac{\sigma^2}{\delta}(e^{\delta t} - 1)$$

The equation for the ruin probability becomes

$$-\frac{\partial w}{\partial t_1} + (\delta x + b)\frac{\partial w}{\partial x} + \frac{\sigma^2}{\delta}(e^{\delta(t-t_1)} - 1)\frac{\partial^2 w}{\partial x^2}$$

and the ruin probability is $u(t,x) = w(t_1 = t, x)$.

We perform the consecutive change of variables

$$X = e^{\delta t_1}(\delta x + b)$$
$$T = t_1$$

and

$$X' = X$$
$$T' = 2\delta\sigma^2 \int_0^T (e^{\delta(t-T)} - 1)e^{2\delta T} dT = 2\sigma^2 \left\{(e^{\delta T} - 1)e^{\delta t} - \frac{e^{2\delta T} - 1}{2}\right\}$$

In the new variables the equation becomes

$$-\frac{\partial w}{\partial T'} + \frac{1}{2}\frac{\partial^2 w}{\partial X'^2} = 0$$

with initial condition $w(t_1 = 0, x) = \mathbf{1}_{\{x \leq 0\}}$, or in the new variables

$$w(T^{'} = 0, X^{'}) = \mathbf{1}_{\{\frac{X^{'}-b}{\delta} \leq 0\}} = \mathbf{1}_{\{X^{'} \leq b\}}.$$

Using the integral formula for the solution of the diffusion equation we find that

$$w(T^{'}, X^{'}) = \frac{1}{\sqrt{2T^{'}}} \int_{-\infty}^{b} exp\left(-\frac{(X-Y)^2}{2T^{'}}\right) dY = \frac{1}{\sqrt{\pi}} \int_{\frac{X^{'}-b}{\sqrt{2T^{'}}}}^{\infty} exp(-z^2) dz$$

or in terms of the original variables

$$w(t_1, x) = \frac{1}{\sqrt{\pi}} \int_{k(t_1,x)}^{\infty} exp(-z^2) dz = \frac{1}{\sqrt{\pi}} erfc(k(t_1, x))$$

$$k(t_1, x) = \frac{e^{\delta t_1}(\delta x + b) - b}{2\sigma\sqrt{(e^{\delta t_1} - 1)e^{\delta t} - \frac{(e^{2\delta t_1} - 1)}{2}}}$$

The ruin probability $u(t, x) = w(t, x)$ is obtained by setting $t_1 = t$ in the above formula. We thus find

$$u(t, x) = \frac{1}{\sqrt{\pi}} erfc\left(\frac{e^{\delta t}(\delta x + b) - b}{2\sigma\sqrt{\frac{1}{2}e^{2\delta t} - e^{\delta t} + \frac{1}{2}}}\right) = \frac{1}{\sqrt{\pi}} erfc\left(\frac{e^{\delta t}(\delta x + b) - b}{\sigma\sqrt{2}(e^{\delta t} - 1)}\right)$$

Two limiting cases are interesting.

$t \to 0$. Then

$$k(t, x) \simeq \frac{x}{\sqrt{2}\sigma t}$$

and

$$u(t, x) \simeq \frac{1}{\sqrt{\pi}} erfc\left(\frac{x}{\sqrt{2}\sigma t}\right)$$

$t \to \infty$. Then

$$k(t, x) = \frac{1}{\sqrt{2}\sigma}(\delta x + b - be^{-\delta t})$$

and

$$u(t,x) \simeq \frac{1}{\sqrt{\pi}} erfc\left(\frac{1}{\sqrt{2}\sigma}(\delta x + b - be^{-\delta t})\right)$$

Finally we present the asymptotics for zero interest force in the $H = 1$ case. When $H = 1$ and $\delta \to 0$ we have from the asymptotics of the general solution that

$$\lim_{\delta \to 0} u(t,x) = \frac{1}{2}\left[1 - \Phi\left(\frac{tb + x}{t\sigma\sqrt{2}}\right)\right]$$

2.4 Numerical treatment of the problem

In this section we propose some possible approaches to the numerical study of ruin probabilities for our insurance business model.

2.4.1 Monte Carlo Method

As an alternative to the pde approach one can use the Monte Carlo method in order to find the probability of ruin $w(x,t)$. Furthermore since the Monte Carlo method attacks the problem from a quite different point of view it can be used as an independent test of the validity of the pde approach proposed here.

As we have seen the cash-balance process X_t is given by the solution of equations (2) and (3). In order to implement the Monte Carlo method we simulate a large number M of paths of X_t in the time interval $[0,T]$. We use for each path $N = 2^L$, number of points. Then the probability of ruin can be found as:

$$w(x_0, T) = \frac{\text{number of } X_T \leq 0}{M}.$$

In order to simulate the paths of fractional Brownian motion, we have used the method of Wood and Chan (1994). Some other methods that could be used are the methods described in Abry and Sellan (1996) and Norros, Mannersalo and Wang (1999), and the references therein.

2.4.2 Finite Difference Methods

As we have seen we have derived an analytical solution of the pde that governs the ruin probability. Instead of using the analytical solution we could alternatively use the finite difference method for the numerical solution of the pde. We have implemented the full implicit finite difference method and the Crank-Nicolson finite difference method. For the numerical solution of pdes using finite difference methods one can see Wilmott, Dewynne and Howison (1993), Wilmott (1998), Smith (1985), Tavella and Randal (2000) and Richtmyer and Morton (1967).

2.4.3 Numerical Results

In the following tables we present some numerical results using the analytic expressions, the Monte Carlo method and the finite difference method. We consider the following values for the parameters of the model. For the interest force we have assumed that $\delta = 0.05$, for the volatility in claims liabilities we assume that $\sigma = 0.20$, for the expected gain per time unit due to a safety loading in the premium we assume that $b = 0.10$. The parameters used for the implementation of the various numerical schemes are included in the Appendix.

As an indication of the results obtained for the ruin probability using the various methods proposed we present tables comparing the estimates for the ruin probability at time T for different values of the initial capital $X_0 = 0, 0.5, -0.5$.

Table 2.1. Probability of Ruin for Initial Capital $X_0 = 0$,at $T = 100$

H	$Exact$	$MonteCarlo$	$Implicit$	$Crank - Nicolson$	
0.5	0.00084174	0.000867	0.00084170	0.00084169	
0.6	0.0132523	0.0131667	0.01310549	0.01319379	
0.7	0.060585	0.0605333	0.06097709	0.06110057	
0.8	0.141854	0.145800	0.14438584	0.14459275	
0.9	0.231166	0.234100	0.23129410	0.23160310	
1	0.308538	0.309100	0.30000028	0.30017402	

Table 2.2. Probability of Ruin for Initial Capital $X_0 = 0.5$,at $T = 100$

H	$Exact$	$Monte\ Carlo$	$Implicit$	$Crank - Nicolson$	
0.5	0.000042186	0.00003333	0.00004218	0.000042178	
0.6	0.00274159	0.00316667	0.00270194	0.00272269	
0.7	0.026191	0.02706667	0.02636487	0.02639296	
0.8	0.0898221	0.09073300	0.09051296	0.09062079	
0.9	0.178783	0.17716600	0.17352670	0.17379322	
1	0.265707	0.26480000	0.24598212	0.24616236	

Table 2.3. Probability of Ruin for Initial Capital $X_0 = -0.5$,at $T = 100$

H	$Exact$	$Monte\ Carlo$	$Implicit$	$Crank - Nicolson$	
0.5	0.00937525	0.009133	0.00937540	0.00937526	
0.6	0.048428	0.04943333	0.04807283	0.04830152	
0.7	0.123069	0.12323333	0.12391473	0.12414554	
0.8	0.211218	0.20900000	0.21659835	0.21685958	
0.9	0.291155	0.29193330	0.29843905	0.29873668	
1	0.354146	0.35490000	0.35870696	0.35885493	

The following observation is of interest. For $H = 1$ and $x = 0$ the Monte-Carlo approach gives the following results for the ruin probability as a function of time:

Table 2.4. The probability of ruin for $H = 1$ at $x = 0$ for different times.

T	*Exact*	*Monte Carlo*
1	0.308538	0.3094333
10	0.308538	0.3106333
100	0.308538	0.309100

It is interesting to see that this behaviour, i.e. the fact that the ruin probability is independent with respect to variations in time is predicted by the exact analytical solution for $H = 1$. The case $H = 1$ is a limiting situation for which the results of this work are questionable since the theory of stochastic integration with respect to fractional Brownian motion used here is strictly valid for values of the Hurst index in the interval $(0, 1)$. However, the fact that this behaviour is reproduced by the Monte Carlo simulation poses questions on the validity of the theory in the limit $H = 1$. This is a point which probably deserves further attention.

For the Monte Carlo we have used M=30000 paths and L=14. For the Implicit method for H=0.5 we took initial capital steps = 5000, time steps = 50000, X_min=-10,X_max=10, and for the Crank - Nicolson, for $H = 0.5$, we took initial capital steps = 5000, time steps = 10000, $X_{min} = -10, X_{max} = 10$. For the Implicit method for $H \in (0.5, 1)$ we took initial capital steps = 1000, time steps = 1000, $X_{min} = -10, X_{max} = 10$. For the Crank - Nicolson method for H=0.5, we used initial capital steps=5000, time steps=10000, Xmin=-10, Xmax=10. For the Implicit method for $H \in (0.5, 1)$ we used initial capital steps=1000, time steps=1000, Xmin=-10, Xmax=10.

For the Crank - Nicolson method for $H \in (0.5, 1)$, we have some some singularities because of the discontinuity of the probability of ruin at x=0, and thus we used for the first five steps the implicit finite difference method and for the rest steps the Crank - Nicolson method. We have used initial capital steps=1000, time steps=1000, Xmin=-10, Xmax=10. For the Implicit method for $H = 1$ we used initial capital steps=2000, time steps=2000, Xmin=-10, Xmax=10. For the Crank - Nicolson method for $H = 1$,we used for the first five steps the implicit finite difference method and for the rest steps the Crank - Nicolson method. We have used initial capital steps=1000, time steps=1000, Xmin=-10, Xmax=10.

We now present graphically the dependence of probability of ruin with the Hurst index, the initial capital and time.

In figure 2-1 we present the variation of the ruin probability at $x = 0$ and time $T = 100$ with the Hurst exponent. We see that as H is taking bigger values the probability of ruin is also growing. This result indicates the effect of long time correlations in the probability of ruin for the insurance business.

In figure 2-2 we present the variation of the probability of ruin at time $T = 100$ as a function of the initial capital The probability of ruin decreases as the initial capital increases as is expected. The Hurst index is taken to be $H = 0.7$.

In figure 2-3 we present the variation of the probability of ruin with time for initial capital equal to $x = 0$ and $H = 0.7$. As time increases the probability of ruin decreases.

In all the above figures we have taken $\delta = 0.05$, $\sigma = 0.20$, $b = 0.10$.

As a final application of the simulation approach we present the calculation of a slightly different form of the ruin probability $P^* = P[\inf_{s\in[0,T]} X_s < 0 \mid X_0 = x]$. We have taken 10000 paths and 2^{14} points in each path, for initial capital $X_0 = 0, 0.25, 0.5$. The results are shown in Table 5. We observe that for initial capital zero as H increases the probability of ruin in a finite time decreases. For initial capital 0.25, 0.50 we see that as H increases the probability of ruin in a finite time is also increasing in general.Using Monte Carlo simulation we have calculated the $P[\inf_{s\in[0,T]} X_s < 0|X_0 = x]$. We have taken 10000 paths and 2^{14} points in each path, for initial capital $X_0 = 0, 0.25, 0.5$. We see that for initial capital zero as H increases the probability of ruin in a finite time decreases. For initial capital 0.25, 0.50 we see that as H increases the probability of ruin in a finite time is also increasing in general.

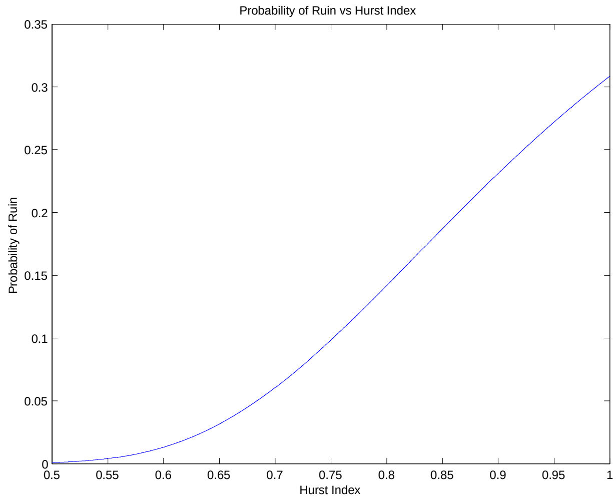

Figure 2-1:

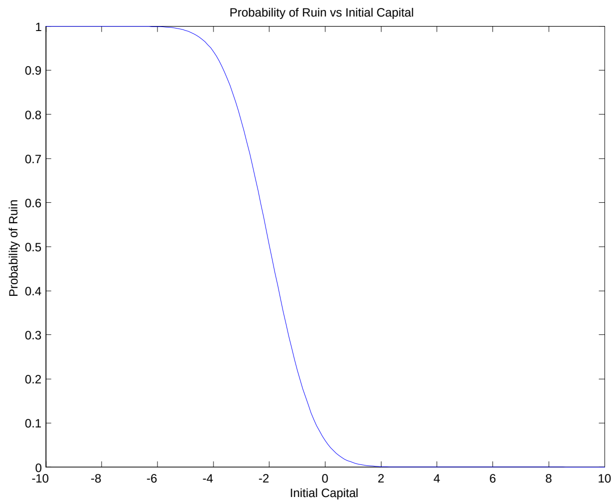

Figure 2-2:

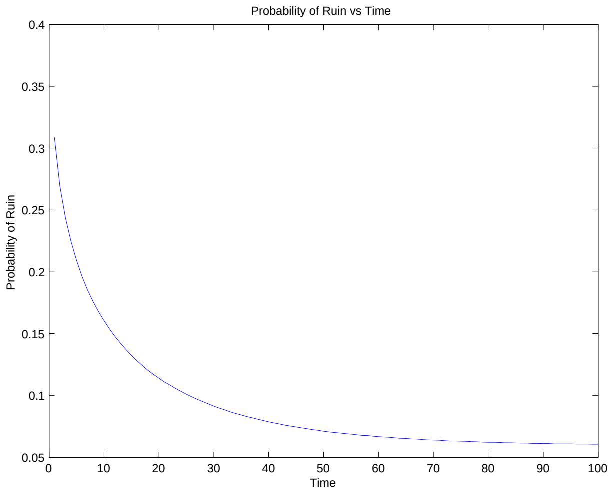

Figure 2-3:

Table 2.5. The probability of ruin $P_x^* = P[\inf_{s\in[0,T]} X_s < 0 \mid X_0 = x]$ for different values of H and x.

H	P_0^*	$P_{0.25}^*]$	$P_{0.5}^*$
0.5	0.941	0.228	0.047
0.6	0.915	0.232	0.078
0.7	0.871	0.263	0.126
0.8	0.798	0.292	0.189
0.9	0.671	0.314	0.244
1	0.310	0.285	0.287

2.4.4 Conclusions and extensions

In this chapter we have derived a linear parabolic partial differential equation for the ruin probability of an insurance firm with long correlated claims, modelled by a fractional Brownian motion with Hurst exponent $1/2 < H < 1$. The equation has been solved and an explicit expression for the ruin probability has been derived in terms of error functions. Alternatively, this viewpoint offers a convenient way of calculating the ruin probability using standard finite difference schemes for the solution of partial differential equations.

The derivation of a differential equation is interesting in its own right for the following reason. If in the model the liability was driven by a Brownian motion then using the martingale properties and the Markov properties of Brownan motion the celebrated Feynman-Kac representation of the solution of partial differential equations can be obtained (see e.g. Mao (1997)). Through the use of the Feynman-Kac formula one may derive a PDE for the ruin probability. This was the original approach of Norberg (1999). However, in the case where liabilities are driven by a fractional Brownian motion the validity of a Feynman-Kac representation is no longer straighforward, since the fractional Brownian motion is neither a semimartingale, nor a Markov process.

The model may be generalized along the following directions:

An obvious generalization would be to include stochasticity in the interest force as well. This would lead to a more complicated linear stochastic differential equation driven by fractional Brownian motions. In this case the derivation of an equation for the ruin probability is possible but its form will be different and most probably it will not be expressible in local form. Also, since the fractional Brownian motions driving the liabilities and the interest force will, in principle, have different Hurst exponents the fractional stochastic calculus set up proposed by Duncan et al may be insufficient and we may have to resort to the theory of stochastic integration on fractional Brownian motion proposed by Elliot and Van der Hoek (2003), where mixtures of fractional Brwonian motions of different Hurst exponents may be used. Another interesting direction towards the generalization of the model is the inclusion of Poisson jumps in the liability process. This will turn the partial differential equation for the ruin probability into a partial integrodifferential equation which may be treated using standard techniques for such problems.

Chapter 3

Stochastic Control and Insurance

In this chapter we formulate a simple problem of insurance control for liabilities of diffusion type, driven by fractional Brownian motion. More specifically we propose a model of insurance control for an insurance business facing liabilities presenting long term correlations. The long term correlations are modelled with the use of a fractional Brownian motion with Hurst exponent H. The insurance firm invests in an interest account which is assumed to be deterministic. The problem reduces to a version of the fractional linear quadratic regulator for which analytic solutions have recently been obtained.

Let us mention a few applications of stochastic control in insurance. Hipp and Taksar (1999) consider an insurance portfolio, where investment in new business is used to minimize the probability of ruin for the total position. This is a stochastic control problem for which solutions can be characterized and computed when the risk processes for old and new business are modelled by compound Poisson processes. Hipp and Plum (2000) also consider a risk process modelled as compound Poisson process. The ruin probability of the risk process is minimized by the choice of a suitable investment strategy for a capital market index. The optimal strategy is computed using the Bellman equation. They proved the existence of a smooth solution and a verification theorem, and give expicit solutions in some cases with exponential claim size distribution, as well as numerical results in a case with Pareto claim size, where the optimal amount invested is not bounded. Hipp and Vogt (2001) also consider a risk process modelled as a compound Poisson process. They found the optimal dynamic excess of loss reinsurance strategy to minimize infinite time ruin probability, and proved the existence of a smooth solution of the corresponding Hamilton-Jacobi-Bellman equation as well as a verification theorem. They also gave numerical examples with exponential, shifted exponential, and pareto claims. One can see also Schmidli (2004a), and Schmidli (2004b) for related work. Usually the functional that is chosen to be minimized in the above works is the probability of ruin of the insurance firm. We choose to minimize the distance of the (final) capital target of the insurance firm from a prespecified capital target. One can see for more on applications of optimal control in finance the classical papers of Merton (1969), Merton (1971), and Merton (1990). One can see also Karatzas, Lehoczky and Shreve (1987) and Korn and Korn (2000). For optimal trading under constraints and its relation to derivatives pricing one can see Cvitavic (1997). For more on forward backward stochastic differential equations and applications in finance and specifically on an alternative derivation of the Black-Scholes option pricing formula one can see also Ma and Yong (1999).

Before proceeding to the control problem let us introduce here the notion of the fundamental martingale from Norros, Valkeila and Virtamo (1999). The fundamental martingale associated with fractional Brownian motion, denoted with M_t^H, is the stochastic process defined by

$$M_t^H = \int_0^t k_H(t,s) dB_s^H \tag{3.1}$$

where

$$\begin{aligned} k_H(t, s0 &= k_H^{-1} s^{1/2-H} (t-s)^{1/2-H}; \\ k_H &= 2H\Gamma(3/2 - H)\Gamma(H + 1/2) \end{aligned}$$

and it is a Gaussian martingale. The quadratic variation process of this martingale is given by

$$
\begin{aligned}
< \ M^H >_t &= w_H(t) \\
w_H(t) = \lambda_H^{-1} t^{2-2H}; \quad \lambda_H &= \frac{2H\Gamma(3-2H)\Gamma(H+1/2)}{\Gamma(3/2-H)}
\end{aligned}
$$

Furthermore, for a suitably defined deterministic function $c(t)$ the following equation holds

$$
\int_0^t c(s) dB_s^H = \int_0^t \mathcal{K}_H^c(t,s) dM_s^H
$$

where

$$
\mathcal{K}_H^c(t,s) = H(2H-1) \int_s^t c(r) r^{H-1/2} (r-s)^{H-3/2} dr, \quad 0 \le s \le t
$$

The above formula can also be 'inverted', in the sense that for a suitably regular deterministic function $f(t)$

$$
\int_0^t f(s) dM_s^H = \int_0^t \mathcal{L}_H^f(t,s) dB_s^H
$$

where

$$
\mathcal{L}_H^f(t,s) = -k_H^{-1} s^{1/2-H} \frac{d}{ds} \int_s^t (r-s)^{1/2-H} f(r) dr
$$

3.1 The model

Let us consider the same model for an insurance firm as we did in the previous chapter: The firm is characterized by its value at time t, which is assumed to be a stochastic process X_t. The firm invests its value X_t to an interest account with logarithmic interest force δ_t. The interest is assumed to be a deterministic function of time. This assumption is not unreasonable for models valid for short times. The firm has to face liabilities B_t which are taken to be of the form

$$
dB_t = -b_t dt - \sigma_t \, dW_t^H
$$

where W_t^H is a fractional Brownian motion with Hurst exponent H and b_t, σ_t are deterministic functions of time. The cash balance equation for the firm at time t has the following form

$$X_t = e^{\Delta_t}(X_0 - \int_0^t e^{-\Delta_t} dB_t)$$

where

$$\Delta_t = \int_0^t \delta_s ds$$

The cash-balance process is the solution of a stochastic differential equation, driven by fractional Brownian motion. The following proposition has been proved in chapter 2. The cash-balance process X_t is the solution of the fractional Ornstein-Uhlenbeck equation

$$dX_t = (\delta_t X_t + b_t)dt + \sigma \, dW_t^H, \tag{3.2}$$

$$X_0 = x \tag{3.3}$$

We now assume that the firm may control its cash balance process by asking its customers for input $u_t$in the time interval $(t, t + dt)$. This input u_t may be considered as a control parameter which allows the firm to reach a desired target A at a specified time T. This input will modify the liabilities equation to

$$dB_t = -b_t dt - u_t dt + \sigma_t \, dW_t^H$$

Then the cash-balance equation is equivalent to the controlled fractional Ornstein-Uhlenbeck equation

$$dX_t = (\delta_t X_t + b_t + u_t)dt + \sigma \, dW_t^H, \tag{3.4}$$

$$X_0 = x \tag{3.5}$$

The control u_t will have to be chosen in such a manner as to minimize some functional.

A reasonable choice for such a functional will be

$$J(u) = \frac{1}{2} E \left[q_T (X_T - A)^2 + \int_0^T r(t) u_t^2 dt \right]$$

This choice for the functional ensures the minimization of the distance from the final target A but at the same time this has to be done in such a way as to keep the total cost over the interval $[0, T]$ as low as possible. The function $r(t)$ is some deterministic weight function assigning some time dependent weight on the costs of the input. One reasonable choice could be $r(t) = e^{\Delta t}$, i.e. we wish to minimize the discounted cost of the input.

Therefore, our model takes the following form

$$\begin{aligned} minJ(u) &:= \frac{1}{2} E \left[q_T (X_T - A)^2 + \int_0^T r(t) u_t^2 dt \right] \\ & \quad \text{subject to} \qquad\qquad\qquad\qquad \text{(Problem A)} \\ dX_t &= (\delta_t X_t + b_t + u_t) dt + \sigma \, dW_t^H, \\ X_0 &= x \end{aligned}$$

Other choices of the cost functional may be reasonable. For instance one may assume that the objective of the firm could be to keep as close as possible to some prescribed target path A_t. Such a situation would require the modification of the cost functional to

$$minJ(u) := \frac{1}{2} E \left[q_T (X_T - A_T)^2 + \int_0^T q_t (X_t - A_t)^2 dt + \int_0^T r(t) u_t^2 dt \right]$$

where q_t is some deterministic function modelling the cost of deviation from the desired target path at different times. The associated model would now take the form

$$\begin{aligned} minJ(u) :=& \quad \frac{1}{2} E \left[q_T (X_T - A)^2 + \int_0^T q_t (X_t - A_t)^2 dt + \int_0^T r(t) u_t^2 dt \right] \\ & \quad \text{subject to} \qquad\qquad\qquad\qquad \text{(Problem B)} \\ dX_t &= (\delta_t X_t + b_t + u_t) dt + \sigma \, dW_t^H, \\ X_0 &= x \end{aligned}$$

3.2 Solution of the control problem

We now present an analytic solution of the control problem. The problem is treated in two ways. One way is through a completion of squares method, inspired from the recent work of Kleptsyna, Le Breton and Viot (2002) on the linear-quadratic regulator problem under a fractional Brownian motion. The second method is through the application of a stochastic maximum principle for fractional brownian motion. This method was proposed for general control problems by Biagini, Hu, Oksendal and Sulem (2002). Both methods use the method of fractional forward backward stochastic differential equations for the determination of the control policy.

3.2.1 The completion of squares method

We first show that the optimal policy can be given by the solution of a forward backward stochastic differential equation driven by fractional Brownian motion.

Proposition 29 *The control that minimizes $J(u)$ over $\mathcal{U}_H$ is given by $u_t = -\frac{1}{r}P_t$ where P_t is given by the solution of the forward-backward stochastic differential equation*

$$\begin{aligned} dX_t &= (\delta_t X_t + b_t - \frac{1}{r}P_t)dt + \sigma\, dW_t^H, \\ dP_t &= \{-\delta_t P_t - q_t(X_t - A_t)\}dt + \beta_t dM_t^H \\ X_0 &= x \\ P_T &= q_T(X_T - A_T) \end{aligned}$$

where M_t^H is the fundamental martingale associated with fractional Brownian motion and β_t is some stochastic process.

Proof: Assume that we look for a solution of the form $u_t = -\frac{f_{1,t}}{r_t} P_t$ where P_t solves the backward stochastic differential equation

$$\begin{aligned} dP_t &= (-g_{1,t}P_t + g_{2,t}X_t + g_{3,t})dt + \beta_t dM_t^H \\ P_T &= \Lambda_T X_T + R_T \end{aligned} \tag{3.6}$$

where $f_{1,t}, g_{i,t},\ i = 1, 2, 3$ are to be specified and Λ_T, R_T are random variables to be specified. Let u^* be any control and u be the assumed control. Then

$$\begin{aligned} J(u^*) - J(u) &= \frac{1}{2}E\{q_T[(X_T^* - A_T)^2 - (X_T - A_T)^2] \\ &+ \int_0^T (q_t[(X_t^* - A_t)^2 - (X_t - A_t)^2] + r_t[u_t^{*2} - u_t^2])dt\} \end{aligned}$$

Following Kleptsyna, Le Breton and Viot (2002) we use the equality

$$y^{*2} - y^2 = (y^* - y)^2 + 2(y^* - y)y$$

to write

$$(X_T^* - A_T)^2 - (X_T - A_T)^2 = (X_T^* - X_T)^2 + 2(X_T^* - X_T)(X_T - A_T),$$

$$(X_t^* - A_t)^2 - (X_t - A_t)^2 = (X_t^* - X_t)^2 + 2(X_t^* - X_t)(X_t - A_t),$$

$${u_t^{*2}} - {u_t}^2 = (u_t^* - u_t)^2 + 2(u_t^* - u_t)u_t$$

and thus $J(u^*) - J(u) = I_1 + I_2$ where

$$\begin{aligned} I_1 &= \frac{1}{2}E\left\{q_T(X_T^* - X_T)^2 + \int_0^T [q_t(X_t^* - X_t)^2 + r_t(u_t^* - u_t)^2]dt\right\} > 0 \\ I_2 &= E\left\{q_T(X_T - A_T)(X_T^* - X_T) + \int_0^T [q_t(X_t - A_t)(X_t^* - X_t) - f_{1,t}P_t(u_t^* - u_t)]dt\right\} \end{aligned}$$

where we have used the ansatz $u_t = -\frac{f_{1,t}}{r_t} P_t$. Let us rewrite the integrand in I_2 as

$$I_2^{'} = (X_t^* - X_t)[q_t(X_t - A_t) + g_{1,t}P_t] - P_t[g_{1,t}(X_t^* - X_t) + f_{1,t}(u_t^* - u_t)]$$

Taking into account that P_t solves the BSDE (3.6) and X_t solves the forward equation

$$\begin{aligned} dX_t &= (\delta_t\, X_t + b_t + u_t)dt + \sigma\, dW_t^H, \\ X_0 &= x \end{aligned}$$

and that

$$\begin{aligned} dX_t^* &= (\delta_t\, X_t^* + b_t + u_t^*)dt + \sigma\, dW_t^H, \\ X_0^* &= X_0 = x \end{aligned}$$

we see that

$$d(X_t^* - X_t) = dX_t^* - dX_t = [\delta_t\,(X_t^* - X_t) + (u_t^* - u_t)]\, dt$$

and if we take $g_{1,t} = \delta_t$, $f_{1,t} = 1$ it is

$$I_2^{'} = (X_t^* - X_t)[q_t(X_t - A_t) + \delta_t P_t] - P_t d(X_t^* - X_t)\frac{1}{dt}$$

Our aim is to turn this expression into a perfect differential. To this end we have to choose

$$\begin{aligned} -dP_t &= \{q_t(X_t - A_t) + \delta_t P_t\}dt - \beta_t dM_t^H \\ P_T &= \Lambda_T X_T + R_T \end{aligned}$$

that is we have to take $g_{2,t} = -q_t, g_{3,t} = q_t A_t$. With this choice

$$\begin{aligned} I_2^{'} dt &= -(X_t^* - X_t) dP_t - P_t d(X_t^* - X_t) + \beta_t (X_t^* - X_t) dM_t^H \\ &= -d((X_t^* - X_t) P_t) + \beta_t (X_t^* - X_t) dM_t^H \end{aligned}$$

We now perform the integrations to get

$$\int_0^T I_2^{'} dt = \int_0^T -d((X_t^* - X_t) P_t) + \int_0^T \beta_t (X_t^* - X_t) dM_t^H$$

and

$$E\left[\int_0^T I_2^{'} dt\right] = E - \left[(X_T^* - X_T)\left(\Lambda_T X_T + R_T\right)\right]$$

where we have used the fact that the expectation of the stochastic integral over a martingale is zero. We now observe that for the choice

$$\Lambda_T = q_T, R_T = -A_T q_T$$

we get

$$I_2 = E\left[q_T (X_T^* - X_T)(X_T - A_T) - (X_T^* - X_T)\left(\Lambda_T X_T + R_T\right)\right]$$

$$I_2 = E\left[(X_T^* - X_T)(q_T X_T - q_T A_T - \Lambda_T X_T - R_T)\right] = 0$$

Thus if u is of the specified form where (X_t, P_t, β_t) solves the given FBSDE we have that $J(u^*) - J(u) > 0$ for any control u^*. This completes the proof. □

We now give the solution of the FBSDE which provides the optimal strategy.

Proposition 30 *The solution of the FBSDE can be given in the the form:*

$$
\begin{aligned}
P_t &= \pi(t)X_t - \pi(t)\int_0^t \sigma_s dW_s^H + \int_0^t \gamma(t,s)dM_s^H + v_t - q_T A_T \qquad (3.7)\\
\beta_t &= \bar{\gamma}(t) = \gamma(t,t)
\end{aligned}
$$

where the deterministic functions $\pi(t)$, $\gamma(t,s)$, $v_t$solve the ODEs

$$
\begin{aligned}
\dot{\pi} + 2\delta_t\pi - \frac{\pi^2}{r_t} + q_t &= 0 \qquad (3.8)\\
\dot{\gamma} + \left(\delta - \frac{\pi}{r_t}\right)\gamma + \left(\delta\pi - 2\frac{\pi^2}{r_t} + q_t\right)K_H^\sigma &= 0\\
\dot{v}_t + \left(\delta_t - \frac{\pi}{r_t}\right)v_t + \left(b_t + \frac{q_T A_T}{r_t}\right)\pi - q_t A_t - q_T A_T \delta_t &= 0
\end{aligned}
$$

with final conditions:

$$
\begin{aligned}
\pi(T) &= q_T\\
\gamma(T,s) &= \pi(T)K_H^\sigma(T,s)\\
\nu_T &= 0
\end{aligned}
$$

where

$$
K_H^\sigma(T,s) = H(2H-1)\int_s^T \sigma r^{H-\frac{1}{2}}(r-s)^{H-\frac{3}{2}}dr, 0 \leqslant s \leqslant T.
$$

Proof: Assume that q_T, A_T are deterministic. The associated BSDE is

$$\begin{aligned} dP_t &= \dot{\pi}(t)X_t dt + \pi(t)dX_t - \dot{\pi}(t)dt\int_0^t \sigma_s dW_s^H \\ &\quad -\pi(t)\sigma_t dW_t^H + \left(\int_0^t \dot{\gamma}(t,s)dM_s^H\right)dt + \bar{\gamma}(t)dM_t^H + \dot{\upsilon}_t dt \end{aligned}$$

We substitute the ansatz for P_t in the FBSDEs and we have that

$$dP_t = \{-\delta_t P_t - q_t(X_t - A_t)\}dt + \beta_t dM_t^H$$

and if we substitute $P_t$we have

$$\begin{aligned} dP_t &= \{-\delta_t \left[\pi(t)X_t - \pi(t)\int_0^t \sigma_s dW_s^H + \int_0^t \gamma(t,s)dM_s^H + \upsilon_t - q_T A_T\right] \\ &\quad -q_t(X_t - A_t)\}dt + \beta_t dM_t^H \end{aligned}$$

and that

$$\begin{aligned} dX_t &= (\delta_t X_t + b_t - \frac{1}{r_t} \\ &\quad \left[\pi(t)X_t - \pi(t)\int_0^t \sigma_s dW_s^H + \int_0^t \gamma(t,s)dM_s^H + \upsilon_t - q_T A_T\right])dt \\ &\quad +\sigma\, dW_t^H \end{aligned}$$

and then

$$
\begin{aligned}
dP_t &= \dot{\pi}(t)X_t dt + \pi(t)dX_t - \dot{\pi}(t)dt\int_0^t \sigma_s dW_s^H \\
&\quad -\pi(t)\sigma_t dW_t^H + \left(\int_0^t \dot{\gamma}(t,s)dM_s^H\right)dt + \bar{\gamma}(t)dM_t^H + \dot{\upsilon}_t dt \\
&= \dot{\pi}(t)X_t dt + \pi(t)\delta_t X_t dt + b_t\pi(t)dt \\
&\quad -\frac{1}{r_t}\pi(t)\left[\pi(t)X(t) - \pi(t)\int_0^t \sigma_s dW_s^H + \int_0^t \gamma(t,s)dM_s^H + \upsilon_t - q_T A_T\right]dt \\
&\quad +\sigma_t\pi(t)dW_t^H - \dot{\pi}(t)dt\int_0^t \sigma_s dW_s^H - \pi(t)\sigma_t dW_t^H \\
&\quad + \left(\int_0^t \dot{\gamma}(t,s)dM_s^H\right)dt + \bar{\gamma}(t)dM_t^H + \dot{\upsilon}_t dt
\end{aligned}
$$

From

$$
\begin{aligned}
dP_t &= \{-\delta_t\left[\pi(t)X_t - \pi(t)\int_0^t \sigma_s dW_s^H + \int_0^t \gamma(t,s)dM_s^H + \upsilon_t - q_T A_T\right] \\
&\quad -q_t(X_t - A_t)\}dt + \beta_t dM_t^H
\end{aligned}
$$

if we write it analytically we have that

$$
\begin{aligned}
dP_t &= -\delta_t\pi(t)X_t dt + \pi(t)\delta_t\int_0^t \sigma_s dW_s^H dt - \delta_t\int_0^t \gamma(t,s)dM_s^H dt - \delta_t\upsilon_t dt \\
&\quad +\delta_t q_T A_T dt - q_t X_t dt + q_t A_t dt + \beta_t dM_t^H
\end{aligned}
$$

and thus by taking the coefficients of $X_t dt$ we have

$$
\begin{aligned}
-\pi(t)\delta_t - q_t &= \dot{\pi}(t) + \delta_t \pi(t) - \frac{1}{r_t}\pi^2(t) \\
\dot{\pi}(t) + 2\delta_t\pi(t) - \frac{\pi^2(t)}{r_t} + q_t &= 0
\end{aligned}
$$

By taking the coefficients of dM_t^H we have

$$
\beta_t = \bar{\gamma}(t) = \gamma(t,t)
$$

From the terms with dt we have

$$
\begin{aligned}
-\delta_t \upsilon(t) + \delta_t q_T A_T + q_t A_t &= b_t \pi(t) - \frac{1}{r_t}\pi(t)\upsilon(t) + \frac{1}{r_t}\pi(t) q_T A_T + \dot{\upsilon}(t) \\
\dot{\upsilon}(t) + \upsilon(t)\left(\delta_t - \frac{\pi(t)}{r_t}\right) &= -\left(\frac{q_T A_T}{r_t} + b_t\right)\pi(t) + \delta_t q_T A_T + q_t A_t
\end{aligned}
$$

If we write the

$$
\int_0^t \sigma_s dW_s^H = \int_0^t K_H^\sigma(t,s) dM_s^H
$$

and from the terms with the integral with respect to the fundamental martingale we have

$$
\begin{aligned}
\delta_t\gamma(t,s) + \frac{1}{r_t}\pi^2(t)K_H^\sigma(t,s) - \frac{1}{r_t}\pi(t)\gamma(t,s) - \dot{\pi}(t)K_H^\sigma(t,s) + \dot{\gamma}(t,s) - \pi(t)\delta_t K_H^\sigma(t,s) &= 0 \\
\dot{\gamma}(t,s) + \left(\delta_t - \frac{\pi(t)}{r_t}\right)\gamma(t,s) + \left(\frac{\pi^2(t)}{r_t} - \dot{\pi}(t) - \pi(t)\delta_t\right) K_H^\sigma(t,s) &= 0
\end{aligned}
$$

Using the

$$2\delta_t\pi(t) - \frac{\pi^2(t)}{r_t} + q_t = -\dot{\pi}(t)$$

we have that

$$\begin{aligned}
\dot{\gamma}(t,s) + \left(\delta_t - \frac{\pi(t)}{r_t}\right)\gamma(t,s) + \left(\frac{\pi^2(t)}{r_t} + 2\delta_t\pi(t) - \frac{\pi^2(t)}{r_t} + q_t - \pi(t)\delta_t\right)K_H^\sigma(t,s) &= 0 \\
\dot{\gamma}(t,s) + \left(\delta_t - \frac{\pi(t)}{r_t}\right)\gamma(t,s) + \left(\delta_t\pi(t) + q_t\right)K_H^\sigma(t,s) &= 0
\end{aligned}$$

Thus we have found

$$\begin{aligned}
\dot{\pi}(t) + 2\delta_t\pi(t) - \frac{\pi^2(t)}{r_t} + q_t &= 0 \\
\dot{\upsilon}(t) + \upsilon(t)\left(\delta_t - \frac{\pi(t)}{r_t}\right) + \left(\frac{q_T A_T}{r_t} + b_t\right)\pi(t) - \delta_t q_T A_T + q_t A_t &= 0 \\
\dot{\gamma}(t,s) + \left(\delta_t - \frac{\pi(t)}{r_t}\right)\gamma(t,s) + \left(\delta_t\pi(t) + q_t\right)K_H^\sigma(t,s) &= 0
\end{aligned}$$

We assume that $q_t = 0$ for $t < T, \delta$ is constant and $r_t = e^{-\lambda t}$. In this case the equations become

$$\begin{aligned}
\dot{\pi(t)} + 2\delta_t\pi(t) - \frac{\pi^2(t)}{r_t} &= 0 \\
\dot{\gamma}(t,s) + (\delta_t - \frac{\pi(t)}{r_t})\gamma(t,s) + \delta\pi(t)K_H^\sigma(t,s) &= 0 \\
\dot{\upsilon}(t) + \upsilon(t)\left(\delta_t - \frac{\pi(t)}{r_t}\right) + \left(\frac{q_T A_T}{r_t} + b_t\right)\pi(t) - \delta_t q_T A_T &= 0
\end{aligned}$$

and for these equations we have the following final conditions that:

$$\begin{aligned}\pi(T) &= q_T \\ \gamma(T,s) &= \pi(T)K_H^{\sigma}(T,s) \\ \upsilon_T &= 0\end{aligned}$$

Derivation of Final Conditions

The final conditions for the last two equations are obtained in the following way. At time $t = T$, we have that

$$P_T = \pi(T)X(T) - \pi(T)\int_0^T \sigma_s dW_s^H + \int_0^T \gamma(T,s)dM_s^H + \upsilon_T - q_T A_T$$

and also that $P_T = \Lambda_T X_T + R_T, \pi(T) = q_T$, $\Lambda_T = q_T$ and $R_T = -A_T q_T$. Thus the following must hold:

$$-\pi(T)\int_0^T \sigma_s dW_s^H + \int_0^T \gamma(T,s)dM_s^H + \upsilon_T = 0,$$

and for this we must have that $\upsilon_T = 0$ and

$$-\pi(T)\int_0^T \sigma_s dW_s^H + \int_0^T \gamma(T,s)dM_s^H = 0$$

which becomes

$$\begin{aligned} -\pi(T)\int_0^T K_H^\sigma(T,s)dM_s^H + \int_0^T \gamma(T,s)dM_s^H &= 0 \\ \int_0^T \left[-\pi(T)K_H^\sigma(T,s) + \gamma(T,s)\right] dM_s^H &= 0 \\ -\pi(T)K_H^\sigma(T,s) + \gamma(T,s) &= 0 \end{aligned}$$

and thus the final condition for γ is that

$$\gamma(T,s) = \pi(T)K_H^\sigma(T,s).$$

3.2.2 The maximum principle

We now provide a treatment of the problem with the use of the maximum principle. According to Biagini, Hu, Oksendal and Sulem (2002) the key quantity in the study of the problem is the Hamiltonian

$$\begin{aligned} H(t,x,u,p,q) &= f(t,x,u) + b(t,x,u)p + \sigma\int_0^T Q(s)\phi_H(s,t)ds \\ &= -\frac{1}{2}q_t(x-A_t)^2 - \frac{1}{2}r_t u_t^2 + (\delta_t x + b_t + u)p + \sigma\int_0^T Q(s)\phi_H(s,t)ds \end{aligned}$$

The adjoint process p satisfies the fractional backward SDE

$$\begin{aligned} dp_t &= -H_x(t,x_t,u_t,p_t,q_t)dt + Q_t dB_t^H \\ p_T &= g_x(x_T) \end{aligned}$$

where $g(x_T)$ is the terminal time part of the functional which is to be minimized (in our case $g(x_T) = -\frac{1}{2}q_T(x_T - A)^2$). The maximum principle states that the optimal process

$(\hat{x}, \hat{p}, \hat{Q}, \hat{u})$is such that

$$H(t, \hat{x}, \hat{u}, \hat{p}, \hat{Q}) = \max_{v \in U} H(t, \hat{x}, v, \hat{p}, \hat{Q})$$

The maximization of the Hamiltonian is given by the solution of the equation $\frac{\partial H}{\partial v} = 0$ i.e. for $v = \frac{p}{r_t}$. For this choice we get

$$H(t, \hat{x}, \hat{u}, \hat{p}, \hat{Q}) = -\frac{1}{2} q_t (x - A_t)^2 + \frac{1}{2} \frac{p^2}{r_t} + (\delta_t x + b_t) p + \sigma \int_0^T Q_s \phi_H(s, t) ds$$

and the optimal process is the solution of the following fractional forward backward stochastic differential equation

$$\begin{aligned} dx_t &= \left(\delta_t x_t + \frac{p_t}{r_t} + b_t \right) dt + \sigma dB_t^H \\ dp_t &= (q_t (x_t - A_t) - \delta_t p_t) + Q_t dB_t^H \\ x_0 &= x \\ p_T &= -q_T (x_T - A) \end{aligned} \tag{3.9}$$

The optimal control is

$$\hat{u} = \frac{p_t}{r_t}$$

The above FBSDE is slightly different from the one derived in the previous subsection. The main difference comes from fact that the backward equation derived by the maximum principle is driven by the fractional Brownian motion itself rather than by the fundamental martingale. However, it can be shown quite easily that these two forms are equivalent. Indeed, let us first define $p = -P$. In terms of the new variables $(x, P, q)$the

FBSDE becomes

$$
\begin{aligned}
dx_t &= \left(\delta_t x_t - \frac{P_t}{r_t} + b_t\right) dt + \sigma dB_t^H \\
dp_t &= (-q_t(x_t - A_t) - \delta_t P_t) - Q_t dB_t^H \\
x_0 &= x \\
P_T &= q_T(x_T - A)
\end{aligned}
\tag{3.10}
$$

and the optimal control will be

$$\hat{u} = -\frac{P_t}{r_t}$$

The above FBSDE is of the same form as the FBSDE derived by the completion of squares technique with the slight difference that it is driven by the fractional Brownian motion rather than the fundamental martingale. We will now show the equivalence between these two forms.

Proposition 31 *The solution (x_t, p_t, β_t) of the FBSDE (3.9) can be used to derive a solution (X_t, P_t, Q_t) of FBSDE (3.10) by the relations*

$$
\begin{aligned}
X_t &= x_t, \\
P_t &= p_t, \\
Q_t &= k_H^{-1} t^{1/2-H} \frac{d}{dt} \int^T (r-t)^{1/2-H} \beta_r dr
\end{aligned}
$$

Alternatively, the solution (X_t, P_t, Q_t) of the FBSDE (3.10) can be used to derive a solution (x_t, p_t, β_t) of the FBSDE (3.9) by the relations

$$
\begin{aligned}
x_t &= X_t, \\
p_t &= P_t, \\
\beta_t &= -H(2H-1) \int_t^T Q_r r^{H-1/2} (r-t)^{H-3/2} dr
\end{aligned}
$$

Proof: *Using the integral form of the FBSDEs we see that in order to connect the solutions of the (3.9) and (3.10) we need to find relations between the integrands such that*

$$-\int_t^T Q_s dB_s^H = \int_t^T \beta_s dM_s^H$$

To make the connection we will use the property of the fundamental martingale rewritten over the whole interval $(0,T)$ *with the use of characteristic functions. That is*

$$\begin{aligned}\int_t^T c_s dB_s^H &= \int_0^T c_s \mathbf{1}_{[t,T]}(s) dB_s^H \\ &= H(2H-1)\int_0^T \int_0^T c_r \mathbf{1}_{[t,T]}(r) r^{H-1/2}(r-s)^{H-3/2}\mathbf{1}_{[s,T]}(r) dr dM_s^H \\ &= H(2H-1)\int_t^T \left[\int_s^T c_r r^{H-1/2}(r-s)^{H-3/2} dr\right] dM_s^H\end{aligned}$$

from which the claim follows. The other relations are proved using the property of the fundamental martingale. □

3.3 Properties of the solution

We now provide a discussion on the qualitative properties of the solution. Our first result concerns the solution of the Riccati equation

$$\begin{aligned}&\dot{\pi} + 2\delta_t \pi - \frac{1}{r_t}\pi^2 + q_t = 0 \\ &\pi(T) = q_T \geq 0\end{aligned} \tag{3.11}$$

Proposition 32 *The solution of the Ricatti equation (3.11) satisfies the property* $\pi(t) \geq 0, \ \forall t$.

Proof: We will use the following iterative scheme for the derivation of the solution:

$$\dot{\pi}_{i+1}(t) + \hat{A}_i \pi_{i+1}(t) + \hat{Q}_i = 0$$
$$\pi_{i+1}(T) = q_T \geq 0$$

where

$$\begin{aligned} \hat{A}_i &= 2\left(\delta_t - \frac{1}{r_t}\psi_i\right) \\ \hat{Q}_i &= \frac{\psi_i^2}{r_t} + q_t \end{aligned}$$

where we will take $\psi_i = \pi_i$. A fixed point of this iterative scheme is clearly a function $\pi(t)$ which is a solution of the Riccati equation. Observe also that $\hat{Q}_i > 0$. As the starting point of the iteration we will take $\pi_0(t) = q_T$.

By the positivity of $\hat{Q}_i$ we have that $\pi_{i+1} \geq 0$ for all i and all t. Therefore, all iterates of the scheme remain positive.

We will now show that the scheme is monotone: To this end define $\Delta_i = \pi_i - \pi_{i+1}$ and $\Lambda_i = \psi_i - \psi_{i-1}$. Using the scheme we see that

$$\begin{aligned} -\dot{\Delta}_i &= \dot{pi}_{i+1} - \dot{\pi}_i = \hat{A}_i \Delta_i + (\hat{A}_{i-1} - \hat{A}_i)\pi_i + \hat{Q}_{i-1} - \hat{Q}_i \\ &= \hat{A}_i \Delta_i + \frac{2}{r_t}\Lambda_i \pi_i + \frac{1}{r_t}(\psi_{i-1} - \psi_i)(\psi_{i-1} + \psi_i) \\ &= \hat{A}_i \Delta_i + \frac{\Lambda_i}{r_t}(2\pi_i - \psi_{i-1} - \psi_i) \\ &= \hat{A}_i \Delta_i + \frac{\Lambda_i}{r_t}(\pi_i - \pi_{i-1} = \hat{A}_i \Delta_i + \frac{\Lambda_i^2}{r_t} \end{aligned}$$

where we have used the fact that $\psi_i = \pi_i$. Rearranging we get

$$-(\dot{\Delta}_i + \hat{A}_i \Delta_i) = \frac{\Lambda_i^2}{r_t} \geq 0$$

with final condition $\Delta(T) = 0$. Integrating (backwards in time) this differential inequality we get that $\Delta_i(t) \geq 0$, $\forall t$ which leads to

$$\pi_i(t) \geq \pi_{i+1}(t), \quad \forall t$$

Since this is a monotone scheme with $\pi_i(t) \geq 0$, it converges to a limit $\pi \geq 0$. □

Theorem 33 *Let $\pi(t)$ be the solution of the linear equation*

$$\dot{\pi}(t) + \hat{A}(t)\pi(t) + \hat{Q}(t) = 0$$
$$\pi(T) = G \geq 0$$

where $Q(t) \geq 0$. Then $\pi(t) \geq 0$ for all t.

Proof: The proof follows by straightforward integration of the linear equation. □

Remark 4 *The proof of the positivity of the solution of the Riccati equation is based on the proof of similar results in Yong and Zhou (1999)*

The above result helps us characterize the type of optimal control needed. Since $u = -\frac{1}{r_t} P_t$ and

$$P_t = \pi(t)X_t - \pi(t)\int_0^t \sigma_s dW_s^H + \int_0^t \gamma(t,s)dM_s^H + v_t - q_T A_T$$

we see that for some $t \leq t_{cr}$ the sign of u will be negative. This means that the control may be treated as a divident control. This t_{cr} will depend on x and the stochastic terms. For $t \geq t_{cr}$ the control may no longer be treated as purely divident control and since u may take positive values, it may be thought as if the control corresponds to input from the customers to the firm.

3.4 Application

In this section we will give some numerical results for the insurance control problem. We have seen that the optimal control is given by the solution of the three ordinary differential equations (3.7) and (3.8). For the first equation with respect to $\pi(t)$ we have that it is a Ricatti ordinary differential equation and the other two equations for $\gamma(t,s)$ and $\upsilon(t)$ are non homogeneous first order linear differential equations. We assume that $q_T = 1$, $\delta = 0.05$, $\beta = 0.01, \lambda = 0.05$, $\sigma = 0.30$, $T = 1$, and that the initial capital is $X0 = 0, 0.1$ and the final target is $A_T = 0.1, 0.2, 0.3, 0.4, 0.5$. For the Hurst parameter we assume that $H = 0.6, 0.7, 0.8, 0.9$. As we can see from Table 1 and Table 2, the higher the Hurst exponent the higher the probability of ruin. Furthermore the higher the final capital target the smaller the probability of ruin, and the controlled process has a significantly smaller probability of ruin in comparison with the uncontrolled process, which for the parameters given is 0.4867 for $X_0 = 0$ and 0.3538 for $X_0 = 0.1$.

Table 3.1 Probability of Ruin, $X_0 = 0.0$

H	$A_T = 0.1$	$A_T = 0.2$	$A_T = 0.3$	$A_T = 0.4$	$A_T = 0.5$
0.6	0.301	0.160	0.065	0.023	0.003
0.7	0.319	0.187	0.081	0.035	0.012
0.8	0.325	0.217	0.087	0.040	0.016
0.9	0.330	0.220	0.103	0.043	0.016

Table 3.2 Probability of Ruin, $X_0 = 0.1$

H	$A_T = 0.2$	$A_T = 0.3$	$A_T = 0.4$	$A_T = 0.5$
0.6	0.099	0.046	0.017	0.004
0.7	0.116	0.047	0.020	0.005
0.8	0.130	0.063	0.021	0.008
0.9	0.131	0.066	0.029	0.011

Chapter 4

Reinsurance Pricing

Reinsurance is, broadly speaking, the insurance of insurance companies. If an individual risk is too big for an insurance company or the loss potential of its entire portfolio is too heavy - then the insurance company either decides, or is forced to by legal restrictions, to buy reinsurance protection. Often the reinsurance company does the same, i.e. it retrocedes part of risk or parts of its portfolio to a third company. By passing on parts of risks, large risks particularly are finally split up into a number of portions placed with many different carriers. Some of the most common types of reinsurance are proportional and excess of loss reinsurance. A proportional reinsurance treaty means that the ceding company cedes to the reinsurer a fixed percentage of each risk of the covered portfolio, the reinsurer in return pays the same percentage of each claim and receives the same percentage of the underlying gross premiums. In an excess of loss treaty, of each claim exceeding the retention of the ceding company the reinsurer pays the exceeding amount subject to a maximum. For more on reinsurance one can see for example Buhlmann (1970) and Straub (1997).

In this chapter we model the liabilities of an insurance business driven by a fractional Brownian motion and we study the valuation of a reinsurance policy both for excess of loss and for proportional reinsurance.

4.0.1 Excess of loss reinsurance

Let us assume that the claims process is in the form

$$
\begin{aligned}
dC_t &= b_t dt + \sigma_t dB_t^H \\
C_0 &= c
\end{aligned}
$$

where with C_t we denote the claims at time t and with b_t we denote the expectations of the claims which may model seasonalities. The term B_t^H is a fractional Brownian motion with Hurst exponent H which is used to model the long range dependence often present in insurance claims. Here we assume that $H \in \left(\frac{1}{2}, 1\right)$. We furthermore assume that $\sigma_t \in L_\phi^2, t\sigma_t \in L_\phi^2$.

Consider first the case of excess of loss reinsurance, i.e consider a reinsurance policy according to which if the total claim amount until time T is less than K the reinsurance company pays nothing whereas if the total claim amount is higher that K it pays the excess of K. This reinsurance scheme is an Asian type contigent claim and if we denote

$$
I_T := C_0 + \int_0^T C_t dt
$$

its payoff is given by

$$
\max\left(0, I_T - K\right) = \left(I_T - K\right)^+ = \left(\int_0^T C_t dt - K\right)^+ .
$$

Thus we have that:

$$
\begin{aligned}
dI_t &= C_t = \int_0^t b_s ds + \int_0^t \sigma_s dB_s^H \\
I_0 &= C_0 = c
\end{aligned}
$$

Integrating we have that

$$
\begin{aligned}
I_T &= I_0 + \int_0^T \int_0^t b_s ds dt + \int_0^T \int_0^t \sigma_s dB_s^H dt \\
I_0 &= C_0 = c
\end{aligned}
$$

Then using the stochastic Fubini theorem we have that

$$
\begin{aligned}
\int_0^T \int_0^t \sigma_s \psi(s,t) dB_s^H dt &= \int_0^T \int_0^t \sigma_s \psi(s,t) 1_{[0,t]}(s) \diamond dW_s^H dt = \\
&= \int_0^T \int_0^t 1_{[0,t]}(s) \psi(s,t) dt \sigma_s \diamond dW_s^H ds = \\
&= \int_0^T \left[\int_s^T \psi(s,t) dt \right] \sigma_s dB_s^H =
\end{aligned}
$$

and for $\psi(s,t) = 1$

$$
\int_0^T \int_0^t \sigma_s \psi(s,t) dB_s^H dt = \int_0^T (T-s) \, \sigma_s dB_s^H
$$

Thus if we denote as $\Theta_t = \int_0^t b_s ds$ and $\Sigma(T,s) = (T-s)\sigma_s$ and we integrate we take that

$$I_T = I_0 + \int_0^T \Theta_t dt + \int_0^T \Sigma(T,s) dB_s^H.$$

The price for the reinsurance policy will be equal to the expected value of the discounted payoff, i.e:

$$\upsilon(I_0, T) = E\left(e^{-\delta T}\left(I_T - K\right)^+\right).$$

Consider a function of three variables $f(t,x,y)$, $x = C_t$, $y = I_t$

$$f(T, C_T, I_T) = f(T, \int_0^T b_t dt + \int_0^T \sigma_t dB_t^H, I_0 + \int_0^T \Theta_t dt + T\int_0^T \sigma_t dB_t^H - \int_0^T t\sigma_t dB_t^H) =$$

$$= f(T, \int_0^T b_t dt + X_T, I_0 + \int_0^T \Theta_t dt + TX_T - Y_T)$$

where

$$X(T) \; : \; = \int_0^T \sigma_s dB_s^H,$$
$$Y(T) \; : \; = \int_0^T s\sigma_s dB_s^H.$$

We will use the following corollary.

Corollary 34 *(Benth, 2003). Assume* $X_t = \int_0^t a(s)dB_s^H$ *and* $Y_t = \int_0^t b(s)dB_s^H$ *where* $a, b \in L_\phi^2(R)$. *For a function* $f \in C^{1,2}(R_+ \times R^2)$ *with bounded derivatives we have*

$$\begin{aligned} f(T,X(T),Y(T)) &= f(0,0,0)+\int_0^T f_t(s,X(s),Y(s))ds+ \\ &+\int_0^T \left[f_x(s,X(s),Y(s))a(s)+f_y(s,X(s),Y(s))b(s)\right]dB_s^H+ \\ &+\int_0^T f_{xx}(s,X(s),Y(s))\left[a(s)\int_0^s \phi(s,u)a(u)du\right]ds+ \\ &+\int_0^T f_{yy}(s,X(s),Y(s))\left[b(s)\int_0^s \phi(s,u)b(u)du\right]ds+ \\ &+\int_0^T f_{xy}(s,X(s),Y(s))\left[a(s)\int_0^s \phi(s,u)b(u)du+b(s)\int_0^s \phi(s,u)a(u)ds\right] \end{aligned}$$

If we apply Itô 's lemma on

$$f(t,X_t,Y_t)=e^{-\delta t}w\left(T-t,\int_0^t b_s ds+X_t,I_0+\int_0^t \Theta_s ds+tX_t-Y_t\right)$$

we have that

$$f_t=e^{-\delta t}(-\delta w-w_1+b_t w_2+C_t w_3)$$

$$\begin{aligned} f_x &= e^{-\delta t}\left(w_2+tw_3\right) \\ f_y &= -e^{-\delta t}w_3 \\ f_{xx} &= e^{-\delta t}\left(w_{22}+2tw_{23}+t^2w_{33}\right) \\ f_{xy} &= -e^{-\delta t}w_{33} \\ f_{yy} &= e^{-\delta t}\left(-w_{23}-tw_{33}\right). \end{aligned}$$

Thus

$$\begin{aligned}
e^{-\delta T}w\left(0,C_T,I_T\right) &= w\left(T,C_0,I_0\right)+\int_0^T e^{-\delta s}\left(-\delta w-w_1+b_s w_2+C_s w_3\right)ds+\\
&+\int_0^T e^{-\delta s}\left[\left(w_2+sw_3\right)\sigma(s)-w_3 s\sigma(s)\right]dB_s^H\\
&+\int_0^T e^{-\delta s}\left[w_{22}+2sw_{23}+s^2w_{33}\right]\sigma_s\left(\int_0^s \phi(s,u)\sigma(u)du\right)ds+\\
&\int_0^T e^{-\delta s}w_{33}s\sigma(s)\left[\int_0^s \phi(s,u)\sigma(u)udu\right]ds+\\
&+\int_0^T e^{-\delta s}(-w_{23}-sw_{33})\left(\sigma_s\int_0^s \phi(s,u)\sigma(u)udu+s\sigma_s\int_0^s \phi(s,u)\sigma(u)du\right)ds
\end{aligned}$$

Let

$$\begin{aligned}
A_s &= \int_0^s \phi(s,u)\sigma(u)du\\
B_s &= \int_0^s \phi(s,u)u\sigma(u)du.
\end{aligned}$$

We have

$$
\begin{aligned}
e^{-\delta T}w\left(0,C_T,I_T\right) &= w\left(T,C_{0,}I_0\right)+\int_0^T e^{-\delta s}\left(-\delta w-w_1+b_s w_2+C_s w_3\right)ds+ \\
&+\int_0^T e^{-\delta s}\left[\left(w_2+s w_3\right)\sigma(s)-w_3 s\sigma(s)\right]dB_s^H \\
&+\int_0^T e^{-\delta s}\left(\sigma_s A_s w_{22}+\left(2sA_s\sigma_s-\sigma_s B_s-s\sigma_s A_s\right)w_{23}\right)ds
\end{aligned}
$$

Note: Term w_{33} cancels out. Taking expectations we have:

$$
\begin{aligned}
E\left[e^{-\delta t}w(0,C_T,I_T)\right] &= w\left(T,C_0,I_0\right)+E\left[\int_0^T e^{-\delta s}\left[-\delta w-w_1+b_s w_2+C_s w_3\right]ds\right]+ \\
&E\left[\int_0^T e^{-\delta s}\left[\sigma_s A_s w_{22}+\left(2sA_s\sigma_s-\sigma_s B_s-s\sigma_s A_s\right)w_{23}\right]ds\right]
\end{aligned}
$$

We add

$$
u(T,C_0,I_0)=E\left[e^{-\delta T}\left(I_T-k\right)^{+}\right]
$$

to the above and we have that

$$
u(T,C_0,I_0)+E\left[e^{-\delta t}w(0,C_T,I_T)\right]=w\left(T,C_0,I_0\right)+E\left[e^{-\delta T}\left(I_T-k\right)^{+}\right]+
E\left[\int_0^T e^{-\delta s}\left[-\delta w-w_1+b_s w_2+C_s w_3+\sigma_s A_s w_{22}+(sA_s\sigma_s-\sigma_s B_s)w_{23}\right]ds\right]
$$

We choose w to be the solution of the equation

$$
\begin{aligned}
-\delta w-w_1+b_s w_2+C_s w_3+\sigma_s A_s w_{22}+(sA_s\sigma_s-\sigma_s B_s)w_{23} &= 0 \\
w &= w(T-t,C,I)
\end{aligned}
$$

with initial condition $w(0,C_0,I)=(I-K)^{+}$.Then $w(T,C_0,I_0)$ will be the price of the reinsurance policy.

Let us look for a similarity solution. Assume without loss of generality that $b_t = b, \sigma_t = \sigma$.

We have

$$\begin{aligned} w &= w(t,x,y) \\ -\delta w - w_t + b w_x + x w_y + \sigma A w_{xx} + (sA\sigma - \sigma B) w_{xy} &= 0 \end{aligned}$$

Look for

$$w(t,x,y) = u(t, c_1 + c_2 x + c_3 y) = u(t,z)$$

$$z = c_1 + c_2 x + c_3 y$$

taking the derivatives we have

$$\begin{aligned} w_t &= u_t + u_z(c_1' + c_2' x + c_3' y) \\ w_x &= u_z c_2 \\ w_y &= u_z c_3 \\ w_{xx} &= u_{zz} c_2^2 \\ w_{xy} &= c_2 c_3 u_{zz} \end{aligned}$$

and the pde becomes

$$-\delta u - u_t - u_z(c_1' + c_2' x + c_3' y) + b c_2 u_z + x c_3 u_z + \sigma A c_2^2 u_{zz} + (s\sigma A - \sigma B) c_2 c_3 u_{zz} = 0$$

or

$$-\delta u - u_t + u_z(-c_1' - c_2'x - c_3'y + bc_2 + xc_3) + u_{zz}(\sigma A c_2^2 + s\sigma A c_2 c_3 - \sigma B c_2 c_3) = 0$$

Take

$$R_1 + R_2 z = R_1 + R_2 C_1 + R_2 C_2 x + R_2 C_3 y$$

$$\begin{aligned} -c_1' + bc_2 &= R_1 + R_2 c_1 \\ -c_2' + c_3 &= R_2 c_2 \\ c_3 &= -R_2 c_3 \end{aligned}$$

thus taking $R_1 = 0,\ R_2 = 0,\ c_3 = const = 1, c_2 = t, c_1' = bt, c_1 = b\frac{t^2}{2}$

we have

$$\begin{aligned} -\delta u - u_t + u_{zz}(2t^2\sigma A - \sigma B t) &= 0 \\ -\delta u - u_t + u_{zz}\Delta(t) &= 0 \\ \Delta(t) &= 2t^2\sigma A - \sigma B t \end{aligned}$$

Let

$$\begin{aligned} u &= e^{-\delta t}\tilde{u} \\ u_t &= -\delta e^{-\delta t}\tilde{u} + e^{-\delta t}\tilde{u}_t \\ u_z &= e^{-\delta t}\tilde{u}_z \\ u_{zz} &= e^{-\delta t}\tilde{u}_z \end{aligned}$$

and the pde becomes

$$\tilde{u}_t = \Delta(t)\tilde{u}_{zz}$$

let $\tilde{t}$ such that

$$\begin{aligned} \frac{\partial \tilde{t}}{\partial t} &= 2\Delta(t) \\ \tilde{t} &= 2\int_0^t \Delta(s)ds \end{aligned}$$

and

$$\frac{\partial \tilde{u}}{\partial t} = \frac{\partial \tilde{u}}{\partial \tilde{t}}\frac{\partial \tilde{t}}{\partial t}$$

and

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} = \frac{1}{2}\frac{\partial^2 \tilde{u}}{\partial z^2}$$

We had for $w(T-t, C, I)$ that the initial condition was given by

$$\begin{aligned} w(0, C, I) &= (I-k)^+ \\ w(t, x, y) &\rightarrow u(t, z), \\ z &= c_1 + c_2 x + c_3 y \\ c_1(0) &= 0 \\ c_2(0) &= 0 \\ c_3(0) &= 1 \\ &\rightarrow (z-k)^+ \end{aligned}$$

Thus the price of the reinsurance policy will be given by:

$$
\begin{aligned}
\tilde{u}\left(\tilde{t}, z\right) &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\tilde{t}}} \exp\left(-\frac{\left(z - z'\right)^2}{2\tilde{t}}\right)\left(z' - k\right)^+ dz' = \\
&= \\
&= \frac{\sqrt{\tilde{t}}}{\sqrt{2\pi}} \exp\left(-\frac{k_1^2}{2}\right) + \left[\frac{z}{\sqrt{2\pi}} - k\right]\left[1 - \Phi(k_1)\right], \\
k_1 &= \frac{k - z}{\sqrt{\tilde{t}}}
\end{aligned}
$$

4.0.2 Proportional Reinsurance

In this type of reinsurance the company pays a percentage equal to α of total claims up to time T. The payoff is given by

$$
\alpha I_T = \alpha \int_0^T C_t dt.
$$

The proportional reinsurance contract is an Asian type option with final payoff αI.

It solves the same equation with different initial condition

$$
\frac{\partial \tilde{u}}{\partial \tilde{t}} = \frac{1}{2}\frac{\partial^2 \tilde{u}}{\partial z^2}
$$

$$
u(0, I) = \alpha I
$$

Thus we have that the solution will be given by

$$
\tilde{u}\left(\tilde{t}, z\right) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\tilde{t}}} \exp\left(-\frac{\left(z - z'\right)^2}{2\tilde{t}}\right) az'dz' = \alpha z.
$$

Chapter 5

Reinsurance Control

Stochastic control in reinsurance has been addresssed by various researchers, such as Schmidli (2001), Schmidli (2002a), Schmidli (2002b), Schmidli (2001), Hald and Schmidli (2004), Hipp and Plum (2000) and Waters (1983), where among other problems they consider the problem of minimizing the probability of ruin by reinsurance and by investment, the maximization of the adjustment coefficient under proportional reinsurance and also the optimal distribution of dividends. The functional they use is usually the probability of ruin of the insurance company which they want to minimize. In this chapter we study the problem of reinsurance control in a model with liabilities exhibiting long range dependence and we choose to minimize the distance of the (final) capital target of the insurance firm from a prespecified capital target.

5.1 The model

Let us assume that an insurance firm uses a proportional reinsurance scheme according to which it is reinsured for a percentage $1 - p_t$ of claims where $p_t \in [0, 1]$. Let us also assume that the cost for reinsurance is some function of the expected claims and the percentage p_t which we will denote hereafter by $c(p_t)$. We will provide concrete forms for this quantity later on in the paper. It suffices to note here that in principle $c(p_t)$ can be calculated using some sort of premium calculation scheme.

5.1.1 The cash balance equation

Let us assume that the claims process is in the form

$$dC_t = b_t dt + \sigma_t dB_t^H$$

where in b_t the expectations of the claims which may model seasonalities. The term B_t^H is a fractional Brownian motion with Hurst exponent H which is used to model the long range dependence often present in insurance claims.

The cash balance equation for the firm, assuming deterministic interest force δ_t is of the form

$$\begin{aligned} dX_t &= (\delta_t X_t + r_t + p_t b_t - c(p_t))dt + p_t \sigma_t dB_t^H \\ X_0 &= x \end{aligned}$$

The p_t enters in the equation since this is the percentage of the claims that the firm is covering whereas the $(1 - p_t)$ of the claims is covered by the reinsurance firm. By r_t we denote the payments into the insurance firm in the form of premia for the contracts the firm issues.

5.1.2 The control problem

We will now assume that p_t is a control parameter which is chosen in such a way as to minimize some cost functional. One possible choice would be to minimize the ruin probability of the firm. In this paper we choose a different type of functional which may be of more relevance to practical applications. The functional we choose is a combination of the distance of the cash balance process from some predetermined target and the cost of reinsurance policy. A simple choice for such a functional is

$$J(p) = E\left[Q(X_T - A_T)^2 + \int_0^T q(t)(X_t - A_t)^2 dt + \int_0^T R(t)F(c(p_t))dt\right]$$

where $Q, q(t), R(t) > 0$ and play the rôle of weights for the various quantities in the cost functional, A_T and A_t is the predetermined target the firm wishes for the cash balance process and F is some utility function for the cost of reinsurance. As a measure for the distance from the predefined target we choose the L^2 distance.

The choice of reinsurance policy thus takes the form of the following optimal control problem:

$$\min_{p_t \in \mathcal{U}} J(p),$$

subject to

$$dX_t = (\delta_t X_t + r_t + p_t b_t - c(p_t))dt + p_t \sigma_t dB_t^H$$

where $\mathcal{U}$ is the set of admissible reinsurance policies.

5.1.3 The set of admissible reinsurance policies

A company will enter into a reinsurance policy which will be updated at specified (deterministic time instants). The policy can only change at the beginning of these time intervals. That means that a plausible form for the insurance policy will be a piecewise constant strategy of the form

$$p_t = \sum_{i=0}^{n-1} p_i \mathbf{1}_{[t_i, t_{i+1})}(t)$$

where $\{t_i\}$ is a partition of the time interval $[0, T]$ such that $t_0 = 0$ and $t_n = T$ and $p_i \in [0, 1]$ are constants. We will consider as the set of admissible reinsurance policies $\mathcal{U}$ the set of the functions p_t of the above form.

The fact that p_t is a deterministic function models the situation where the manager of the firm chooses the reinsurance policy at the beginning of the period $[0,T]$ by using her/his expectations of the customers claims.

5.2 Solution of the control problem

Due to the special class of reinsurance policies we are interested in we may provide a simple solution to the control problem in the form of algebraic equations. In this section we present the solution procedure.

5.2.1 Evaluation of the functional $J(b)$

Since p_t is a deterministic reinsurance policy, in the sense described above, we may obtain a solution for the cash balance equation in the form

$$\begin{aligned} X_t &= xK_t + K_t \int_0^t K_s^{-1}(r_s + p_s b_s - c(p_s))ds + K_t \int_0^t K_s^{-1} p_s \sigma_s dB_s^H \\ K_t &= exp\left(\int_0^t \delta_s ds \right) \end{aligned}$$

Using this exact solution we may calculate the value of the functional $J(p)$ for a given reinsurance plan p_t. We find that

$$\begin{aligned}
J(p_t) &= x^2K_T^2 + K_T^2\left[\int_0^T K_s^{-1}(r_s + p_sb_s - c(p_s))ds\right]^2 \\
&+ K_T^2\int_0^T\int_0^T K_s^{-1}p_s\sigma_sK_{s'}^{-1}p_{s'}\sigma_{s'}\phi_{s,s'}dsds' \\
&+ 2xK_T^2\int_0^T K_s^{-1}(r_s + p_sb_s - c(p_s))ds \\
&+ A_T^2 - 2A_TxK_T - 2A_TK_T\int_0^T K_s^{-1}(r_s + p_sb_s - c(p_s))ds \\
&+ \int_0^T q(t)x^2K_t^2dt + \int_0^T q(t)K_t^2\left[\int_0^t K_s^{-1}(r_s + p_sb_s - c(p_s))ds\right]^2 dt \\
&+ \int_0^T q(t)K_t^2\left[\int_0^t\int_0^t K_s^{-1}p_s\sigma_sK_{s'}^{-1}p_{s'}\sigma_{s'}\phi(s,s')dsds'\right]dt \\
&+ \int_0^T 2q(t)xK_t^2\left[\int_0^t K_s^{-1}(r_s + p_sb_s - c(p_s))ds\right]dt - \int_0^T 2q(t)A_txK_tdt \\
&- \int_0^T 2q(t)A_tK_t\int_0^t K_s^{-1}(r_s + p_sb_s - c(p_s))dsdt \\
&+ \int_0^T q(t)A_t^2dt + \int_0^T R(t)F(c(p_t))dt
\end{aligned}$$

where without loss of generality we have chosen $Q = 1$. In the above

$$\phi(s,s') = 2H(2H-1) \mid s - s' \mid^{2H-1} .$$

This is a functional of the deterministic function p_t which may be minimized using techniques from the calculus of variation. This problem is challenging since the functional to be minimized is non-local in time. The non locality arises because of the terms involving the kernel $\psi(s,s')$ which in turn arises because of the long range dependence in the claim process.

5.2.2 Restriction to the class $\mathcal{U}$

We now restrict to the class of controls $\mathcal{U}$ which is the class of piecewise constant functions. By restricting within this class we may explicitly evaluate all the integrals used in the definition of the functional $J(p)$. For instance:

$$\begin{aligned}\int_0^T K_s^{-1}(r_s + p_s b_s - c(p_s))ds &= \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} (r_s + p_i b_s - c(p_i))ds \\ &= G + \sum_{i=0}^{N-1} C_i p_i - \sum_{i=0}^{N-1} D_i c(p_i)\end{aligned}$$

where

$$C_i := \int_{t_i}^{t_{i+1}} K_s^{-1} b_s ds, \;\; D_i := \int_{t_i}^{t_{i+1}} K_s^{-1} ds, \;\; G := \int_0^T K_s^{-1} r_s ds$$

Similarly for the non-local terms

$$\begin{aligned}&\int_0^T \int_0^T K_s^{-1} p_s \sigma_s K_{s'}^{-1} p_{s'} \sigma_{s'} \phi(s, s') ds ds' \\ = &\sum_{i=0}^{N-1} \sum_{j=0}^{N-1} p_i p_j \int_0^t \int_0^T K_s^{-1} \sigma_s K_{s'}^{-1} \sigma_{s'} \phi(s, s') \mathbf{1}_{[t_i, t_{i+1})}(s) \mathbf{1}_{[t_j, t_{j+1})}(s') ds ds' \\ = &: \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} E_{ij} p_i p_j\end{aligned}$$

where

$$E_{ij} := \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} K_s^{-1} \sigma_s K_{s'}^{-1} \sigma_{s'} \phi(s, s') ds ds'$$

Thus the cost functional becomes (assuming without loss of generality that $q(t) = 0$, that is that the manager of the insurance firm is interested in the final target rather than intermediate targets)

$$\begin{aligned} J(p_t) &= J(p_0, \cdots, p_{N-1}) = x^2 K_T^2 + K_T^2 \left[G + \sum_{i=0}^{N-1} (C_i p_i - D_i c(p_i)) \right]^2 \\ &+ K_T^2 \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} E_{ij} p_i p_j + 2x K_T^2 \left[G + \sum_{i=0}^{N-1} (C_i p_i - D_i c(p_i)) \right] \\ &+ A_T^2 - 2A_T x K_T - 2A_T K_T \left[G + \sum_{i=0}^{N-1} (C_i p_i - D_i c(p_i)) \right] + \sum_{i=0}^{N-1} F_i(c(p_i)) \end{aligned}$$

where

$$F_i(c(p_i)) = \int_{t_i}^{t_{i+1}} R(t) F(c(p_i)) dt := R_i F(c(p_i))$$

with

$$R_i = \int_{t_i}^{t_{i+1}} R(t) dt.$$

Thus the reinsurance control problem is reduced to the minimization of a nonlinear algebraic function of N variables in the compact domain $[0,1]^N$.

5.3 Linear Reinsurance: Reduction to a Quadratic Programming Problem

We will show that in the case of linear reinsurance cost, i.e. $c(p_i) = a_i p_i + \beta_i$, that the functional $J(p_t)$ can be written in a quadratic form. Let us assume that

$$F_i(c(p_i)) = \gamma_i p_i^2 + \varepsilon_i p_i + \zeta_i,$$

then we have

$$
\begin{aligned}
J(p_t) &= J(p_0, \cdots, p_{N-1}) = x^2 K_T^2 + K_T^2 \left[G + \sum_{i=0}^{N-1} (C_i p_i - D_i c(p_i)) \right]^2 \\
&+ K_T^2 \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} E_{ij} p_i p_j + 2x K_T^2 \left[G + \sum_{i=0}^{N-1} (C_i p_i - D_i c(p_i)) \right] \\
&+ A_T^2 - 2A_T x K_T - 2A_T K_T \left[G + \sum_{i=0}^{N-1} (C_i p_i - D_i c(p_i)) \right] + \sum_{i=0}^{N-1} F_i(c(p_i)) =
\end{aligned}
$$

$$
\begin{aligned}
&= x^2 K_T^2 + K_T^2 \left[G + \sum_{i=0}^{N-1} (C_i p_i - D_i a_i p_i - D_i \beta_i) \right]^2 \\
&\quad + K_T^2 \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} E_{ij} p_i p_j + 2x K_T^2 \left[G + \sum_{i=0}^{N-1} (C_i p_i - D_i a_i p_i - D_i \beta_i) \right] \\
&\quad + A_T^2 - 2A_T x K_T - 2A_T K_T \left[G + \sum_{i=0}^{N-1} (C_i p_i - D_i a_i p_i - D_i \beta_i) \right] \\
&\quad + \sum_{i=0}^{N-1} (\gamma_i p_i^2 + \varepsilon_i p_i + \zeta_i)
\end{aligned}
$$

Consider

$$
\begin{aligned}
I \quad : &= \left[G + \sum_{i=0}^{N-1} (C_i p_i - D_i a_i p_i - D_i \beta_i) \right]^2 = \\
&= G^2 + 2G \sum_{i=0}^{N-1} (C_i p_i - D_i a_i p_i - D_i \beta_i) + \\
&\quad + \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} (C_i p_i - D_i a_i p_i - D_i \beta_i)(C_j p_j - D_j a_j p_j - D_j \beta_j)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & G^2 + 2G\sum_{i=0}^{N-1}(C_i p_i - D_i a_i p_i - D_i \beta_i) + \\
& + \sum_{i=0}^{N-1}\sum_{j=0}^{N-1}(C_i C_j p_i p_j - D_i C_j a_i p_i p_j - D_i C_j \beta_i p_j \\
& - C_i D_j a_j p_i p_j + D_i D_j a_i a_j p_i p_j + D_i D_j a_j \beta_i p_j + \\
& - C_i D_j \beta_j p_i + D_i D_j \beta_j \alpha_i p_i + D_i D_j \beta_i \beta_j)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & G^2 + 2G\sum_{i=0}^{N-1}(C_i - D_i a_i)p_i - 2G\sum_{i=0}^{N-1} D_i \beta_i + \\
& \sum_{i=0}^{N-1}\sum_{j=0}^{N-1}(C_i C_j - D_i C_j a_i - C_i D_j a_j + D_i D_j a_i a_j) p_i p_j \\
& + \sum_{i=0}^{N-1}\{-\left(\sum_{j=0}^{N-1} D_j \beta_j\right) C_i + \left(\sum_{j=0}^{N-1} D_j \beta_j\right) D_i a_i - \left(\sum_{j=0}^{N-1} D_j \beta_j\right) C_i \\
& + \left(\sum_{j=0}^{N-1} D_j \beta_j\right) D_i a_i\} p_i + \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} D_i D_j \beta_i \beta_j
\end{aligned}
$$

Thus we have that

$$
I := A_1 + \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \tilde{E}_{ij} p_i p_j + \sum_{i=0}^{N-1} A_{2i} p_i
$$

where

$$
A_1 = G^2 - 2G\sum_{i=0}^{N-1} D_i \beta_i + \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} D_i D_j \beta_i \beta_j,
$$

$$
\tilde{E}_{ij} = C_i C_j - D_i C_j a_i - C_i D_j a_j + D_i D_j a_i a_j,
$$

$$
\begin{aligned}
A_{2i} &= 2G(C_i - D_i a_i) + \\
&\{-\left(\sum_{j=0}^{N-1} D_j\beta_j\right) C_i + \left(\sum_{j=0}^{N-1} D_j\beta_j\right) D_i a_i - \left(\sum_{j=0}^{N-1} D_j\beta_j\right) C_i \\
&+ \left(\sum_{j=0}^{N-1} D_j\beta_j\right) D_i a_i\}
\end{aligned}
$$

$$
\begin{aligned}
&= 2G(C_i - D_i a_i) + 2\left(\sum_{j=0}^{N-1} D_j\beta_j\right)(D_i a_i - C_i) \\
&= (C_i - D_i a_i)[2G - 2\left(\sum_{j=0}^{N-1} D_j\beta_j\right)] \\
&= 2(C_i - D_i a_i)[G - \left(\sum_{j=0}^{N-1} D_j\beta_j\right)]
\end{aligned}
$$

And the equation becomes

$$
\begin{aligned}
J(p_t) &= x^2 K_T^2 + K_T^2 \left[A_1 + \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \tilde{E}_{ij} p_i p_j + \sum_{i=0}^{N-1} A_{2i} p_i\right] \\
&+ K_T^2 \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} E_{ij} p_i p_j + 2xK_T^2 G + 2xK_T^2 \sum_{i=0}^{N-1} (C_i - D_i a_i) p_i
\end{aligned}
$$

$$
\begin{aligned}
&-2xK_T^2 \sum_{i=0}^{N-1} D_i\beta_i + A_T^2 - 2A_T x K_T - 2A_T K_T G \\
&-2A_T K_T \sum_{i=0}^{N-1} (C_i - D_i a_i) p_i + 2A_T K_T \sum_{i=0}^{N-1} D_i\beta_i
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=0}^{N-1}\gamma_i p_i^2+\sum_{i=0}^{N-1}\varepsilon_i p_i+\sum_{i=0}^{N-1}\zeta_i \\
= \quad & M+\sum_{i=0}^{N-1}N_i p_i+\sum_{i=0}^{N-1}L_i p_i^2+\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\hat{L}_i p_i p_j
\end{aligned}
$$

where

$$
\begin{aligned}
M \quad : \quad & = x^2K_T^2+K_T^2A_1+2xK_T^2G \\
& -2xK_T^2\sum_{i=0}^{N-1}D_i\beta_i+A_T^2-2A_TxK_T-2A_TK_TG \\
& +2A_TK_T\sum_{i=0}^{N-1}D_i\beta_i+\sum_{i=0}^{N-1}\zeta_i
\end{aligned}
$$

and

$$N_i := K_T^2A_{2i}+2xK_T^2(C_i-D_ia_i)-2A_TK_T(C_i-D_ia_i)+\varepsilon_i$$

and

$$L_{ii} := K_T^2\tilde{E}_{ii}+K_T^2E_{ii}+\gamma_i, \text{ for } i=j$$

and

$$\hat{L}_{ij} := K_T^2\tilde{E}_{ij}+K_T^2E_{ij}, \text{ for } i\neq j$$

Thus the equation can be written as

$$J(p_t)=M+p^TL_1p+N^Tp,$$

where L_1 is a matrix with elements equal to $\hat{L}_{ij}$for $i\neq j$ and L_{ii} for $i=j$.We want to minimize $J(p_t)$ subject to $p_t\in[0,1]$, thus we have

$$\begin{aligned} &\min M + p^T L_1 p + N^T p \\ \textit{subject to } Ap &\leq b, p \geq 0 \end{aligned}$$

where $A = I$, and b has one everywhere. The above problem can be written as

$$\begin{aligned} &\min \frac{1}{2} p^T L_1^* p + N^T p \\ \textit{subject to } Ap &\leq b \end{aligned}$$

since M is constant and $L_1^* = 2L_1$. We then define

$$L_1^{**} = \frac{L_1^* + L_1^{*T}}{2}$$

and thus the above problem can be written as

$$\begin{aligned} &\min M + \frac{1}{2} p^T L_1^{**} p + N^T p \\ \textit{subject to } Ap &\leq b, p \geq 0 \end{aligned}$$

and thus we have written the initial problem in a quadratic form which can solved using standard numerical procedures. We have used *quadprog* from MatLab. One can see for more Coleman and Li (1996), Gill, Murray and Wright (1981) and Bazaraa, Sherali and Shetty (1993). The above problem can be written as

$$\begin{aligned} &\min \frac{1}{2} p^T L_1^{**} p + N^T p \\ \textit{subject to } Ap &\leq b, p \geq 0. \end{aligned}$$

Denoting the Langrangian multiplier vectors of the constraints by u and v and the vector of slack variables by y,the Karush - Kuhn - Tucker conditions can be written as

$$\begin{aligned} Ap + y &= b \\ -L_1^{**}p - A^t u + v &= N \\ p^t v &= 0 \\ u^t y &= 0 \\ p, y, u, v &\geq 0. \end{aligned}$$

Now letting

$$\begin{aligned} M &= \begin{bmatrix} 0 & -A \\ A^t & L_1^{**} \end{bmatrix}, \\ q &= \begin{bmatrix} b \\ N \end{bmatrix}, w = \begin{bmatrix} y \\ v \end{bmatrix}, z = \begin{bmatrix} u \\ p \end{bmatrix} \end{aligned}$$

we can rewrite the KKT conditions as the linear complementarity problem

$$\begin{aligned} w - Mz &= q, \\ w^t z &= 0 \\ w, z &\geq 0. \end{aligned}$$

Then by the theorem 11.2.4 of Bazaraa et al, the above problem can be solved using the complementary pivoting algorithm that stops in a finite number of iterations with a KKT point under any of the following assumptions:1. L_1^{**} is positive semidefinite and $N = 0$, 2. L_1^{**} is positive semidefinite, 3. L_1^{**} has nonegative elements with positive diagonal elements.

5.4 Non Linear Reinsurance Cost

Let us examine also the case of nonlinear reinsurance cost. We will assume that the reinsurance cost has the form $c(p_i) = \tanh(p_i)$. Then the functional $J(p_t)$

$$\begin{aligned} J(p_t) &= J(p_0, \cdots, p_{N-1}) = x^2 K_T^2 + K_T^2 \left[G + \sum_{i=0}^{N-1} (C_i p_i - D_i c(p_i)) \right]^2 \\ &+ K_T^2 \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} E_{ij} p_i p_j + 2x K_T^2 \left[G + \sum_{i=0}^{N-1} (C_i p_i - D_i c(p_i)) \right] \\ &+ A_T^2 - 2A_T x K_T - 2A_T K_T \left[G + \sum_{i=0}^{N-1} (C_i p_i - D_i c(p_i)) \right] + \sum_{i=0}^{N-1} F_i(c(p_i)) \end{aligned}$$

where

$$F_i(c(p_i)) = c(p_i)^2$$

and

$$F_i(c(p_i)) = \gamma_i p_i^2 + \varepsilon_i p_i + \zeta_i,$$

We want to

$$\begin{aligned} &\min\mathrm{imize}\ J(p_t) \\ \text{subject to } p &\in (0,1) \end{aligned}$$

A very useful transformation which can be applied is the Kruzkov transformation. We consider

$$
\begin{aligned}
p^* &= \log(\frac{p}{1-p}) \Rightarrow \\
p &= \frac{e^{p^*}}{1+e^{p^*}}
\end{aligned}
$$

and we have to minimize $J(p_t^*)$. Using this transformation we make sure that $p \in (0,1)$ and we can then use some of the standard numerical procedures in order to minimize $J(p_t^*)$. Please note that the Kruzkov transformation could have been also applied to the case of linear reinsurance cost.

5.5 Application

Let us give a few numerical examples in order to illustrate the model developed in the previous sections. We give some examples using the control problem of choosing p_t so as to be as close as possible to a final target A_T. We assume that we have $N = 5$ periods in which we want to calculate the optimal percent of proportional reinsurance. We consider that the initial capital of the company is equal to $X_0 = 20$, the volatility of the liabilities is assumed to be constant and equal to $\sigma = 0.30$, the operating period of the insurance company is $T = 5$, the final target is $A_T = 26, 28, 30$, the Hurst parameter $H \in (0.5, 1)$, here in the examples we let $H = 0.6, 0.7, 0.8, 0.9$, the parameter b_t take for the next five periods the values $b_t = -1.20, -1.21, -1.22, -1.21, -1.20$ in order to incorporate some sort of seasonality and time variability in b_t and in general $E[b] = -1.0867$. We use for the next five periods higher value for b_t than $E[b]$ in order to make sure that the insurance company will need the reinsurance. For the deterministic interest force we assume that it is constant and equal to $\delta = 0.05$.

Using Expected Value Principle

For the premia that the insurance company receives we assume in the first case that $r = -(1+\eta_I)E[b]$, where η_I is the safety loading imposed by the insurer. We also have that in the case of linear reinsurance cost with $c(p_t) = ap_t + \beta$,and the parameters a and β are constants and are chosen in such a way in order to have that $c(p_t) = (1-p)(1+\eta_R)r$, where the parameter η_R will denote the safety loading imposed by the reinsurer. In this case the parameters α, β are given by $\alpha = -r(1+\eta_R)$, $\beta = r(1+\eta_R)$. For the cost functional of the cost of reinsurance arising in the functional $J(p_t)$ we assume that $F[C(p_t)] = [C(p_t)]^2$. This functional can be written in the form of $F[C(p_t)] = \gamma p_t^2 + \varepsilon p_t + \zeta$, where $\gamma = \alpha^2$, $\varepsilon = 2\alpha\beta$ and $\zeta = \beta^2$. We would like to mention that the matrix L_1^{**} is, for the parameters we have chosen, positive definite, and thus the KKT conditions imply that a solution to this minimization problem will always exist.

We first examine the effect that the safety loading of the reinsurer has in the determination of the reinsurance strategy of the insurance company. We use $H = 0.6$, $X_0 = 20$, $A_T = 26$, $\eta_I = 0.05$ and $\sigma = 0.25$. As we see from Table 5.1 the higher the reinsurance safety loading, the higher the percent of the claims that the insurer pays himself and this is logical since higher values of reinsurance safety loading make the reinsurance cost higher driving the insurer to pay higher percent of the claims by himself .

Table 5.1. Reinsurance strategy for η_R=0.01,..0.05

p_0	p_1	p_2	p_3	p_4	η_R
0,300	0,197	0,105	0,274	0,427	0,01
0,467	0,358	0,260	0,419	0,563	0,02
0,629	0,514	0,411	0,560	0,696	0,03
0,787	0,667	0,558	0,698	0,825	0,04
0,940	0,815	0,701	0,832	0,950	0,05

We also examine the effect of the insurer safety loading η_I. We use $H = 0.6$, $X_0 = 20$, $A_T = 26$, $\sigma = 0.25$, $\eta_R = 0.01$. As we see from Table 5.2 the higher the insurer safety loading, the higher the percent of the claims that the insurer pays himself and this is logical since higher values of insurer safety loading make the insurance company stronger financially and thus the insurer has the ability to pay higher percent of the claims by himself .

Table 5.2. Reinsurance strategy for η_I=0.05,..0.10

p_0	p_1	p_2	p_3	p_4	η_I
0,300	0,197	0,105	0,274	0,427	0,05
0,458	0,348	0,250	0,410	0,556	0,06
0,613	0,498	0,394	0,545	0,683	0,07
0,766	0,645	0,535	0,678	0,808	0,08
0,916	0,789	0,674	0,809	0,930	0,09
1,000	0,931	0,811	0,937	1,000	0,1

We then examine the effect of the volatility of the claims σ. We use $H = 0.6$, $X_0 = 20$, $A_T = 26$, $\sigma = 0.25$, $\eta_I = 0.05$ and $\eta_R = 0.01$. As we see from Table 5.3 the higher the claims volatility, the lower the percent of the claims that the insurer pays himself and this is also logical since higher values of claims volatility make the insurance company exposed in higher risk and thus the insurer selects to be reinsured more.

Table 5.3. Reinsurance strategy for $\sigma = 0.25, 0.35, 0.45$

p_0	p_1	p_2	p_3	p_4	σ
0,300	0,197	0,105	0,274	0,427	0,25
0,290	0,188	0,097	0,266	0,420	0,35
0,278	0,177	0,086	0,256	0,410	0,45

We also examine the effect of the final capital target A_T. We use $H = 0.6$, $X_0 = 20$, $\sigma = 0.25$, $\eta_I = 0.05$ and $\eta_R = 0.01, 0.05$. As we see from Table 5.4, if the reinsurance is not expensive enough, $\eta_R = 0.01$, the higher the final target, the lower the percent of the claims that the insurer pays himself, but for higher reinsurance safety loading, $\eta_R = 0.05$ the percent of the claims that the insurer is paying himself is having a small decrease in comparison with $\eta_R = 0.01$.

Table 5.4. Reinsurance strategy for $A_T = 26, 28, 30$, $\eta_R = 0.01, 0.05$

p_0	p_1	p_2	p_3	p_4	A_T	η_R
0,300	0,197	0,105	0,274	0,427	26	0,01
0,216	0,099	0,000	0,185	0,358	28	0,01
0,130	0,000	0,000	0,095	0,288	30	0,01
0,940	0,815	0,701	0,832	0,950	26	0,05
0,937	0,796	0,667	0,815	0,948	28	0,05
0,934	0,776	0,633	0,797	0,945	30	0,05

Finally we examine the effect of the Hurst parameter H. We use $X_0 = 20$, $A_T = 26$, $\eta_I = 0.05$, $\eta_R = 0.05$. As we see from Table 5.5, the higher the value of H the lower the percent of the claims that the insurer pays himself. This means that for higher H the insurer has to deal with larger risk and thus he is reinsured more. The fact that for larger H the insurer is facing more risk has been studied extensively in the first chapter whre the probability of ruin at a given time for insurance claims driven by fractional Brownian motion has been studied and it was found that larger values of H give higher probabilities of ruin.

Table 5.5. Reinsurance strategy for $H = 0.6, ..., 0.9$

p_0	p_1	p_2	p_3	p_4	H
0,940	0,815	0,701	0,832	0,950	0,6
0,890	0,774	0,663	0,794	0,907	0,7
0,819	0,722	0,616	0,743	0,844	0,8
0,723	0,659	0,564	0,684	0,760	0,9

Using Zero Utility Principle

As a second case we assume that the insurer has an exponential utility function given by $U(W) = -e^{-a_1 W}$, $a_1 > 0$. In this case the premia that the insurance company receives will be given by $r = -b+\ 0.5\sigma^2 T^{2H-1} a_1$. We also have that $c(p_t) = ap_t + \beta$,and the parameters a and β are constants and are chosen in such a way in order to have that $c(p_t) = (1-p)(1+\eta_R)r$ and the parameters α, β are given by $\alpha = -r(1+\eta_R)$, $\beta = r(1+\eta_R)$.

We first examine the effect that the safety loading of the reinsurer has in the determination of the reinsurance strategy of the insurance company. We use $H = 0.6$, $X_0 = 20$, $A_T = 26$, $a_1 = 1.3$ and $\sigma = 0.25$. As we see from Table 5.6 the higher the reinsurance safety loading, the higher the percent of the claims that the insurer pays himself.

Table 5.6. Reinsurance strategy for η_R=0.01,..0.05

p_0	p_1	p_2	p_3	p_4	η_R
0.325	0.221	0.128	0.296	0.448	0.01
0.491	0.381	0.283	0.440	0.583	0.02
0.653	0.538	0.434	0.582	0.716	0.03
0.811	0.690	0.581	0.719	0.845	0.04
0.964	0.838	0.723	0.853	0.969	0.05

We also examine the effect of the parameter α_1 that appears in the exponential utility function and the premiums that are paid to the insurance company. We use $H = 0.6$, $X_0 = 20$, $A_T = 26$, $\sigma = 0.25$, $\eta_R = 0.01$. As we see from Table 5.7 the higher the parameter a_1 the higher the percent of the claims that the insurer pays himself and this is logical since higher values a_1 make the insurance company stronger financially, as it receive higher premiums, and thus the insurer has the ability to pay higher percent of the claims by himself .

Table 5.7. Reinsurance strategy for $a_1 = 1, ..., 1.5$

p_0	p_1	p_2	p_3	p_4	r	a_1
0.136	0.039	0.000	0.131	0.293	1.12978	1.0
0.200	0.100	0.013	0.186	0.345	1.13409	1.1
0.263	0.161	0.071	0.241	0.397	1.13841	1.2
0.325	0.221	0.128	0.296	0.448	1.14272	1.3
0.388	0.281	0.186	0.350	0.499	1.14703	1.4
0.450	0.341	0.243	0.404	0.550	1.15134	1.5

We then examine the effect of the volatility of the claims σ. We use $H = 0.6$, $X_0 = 20$, $A_T = 26$, $\sigma = 0.25$, $a_1 = 1.3$ and $\eta_R = 0.01$. As we see from Table 5.8 the higher the claims volatility, the higher the premium the insurer receives and thus he is able to pay a higher percent of the claims. We see also that as $r > b_t$ the insurer is able to pay the claims by himself without the need for insurance.

Table 5.8. Reinsurance strategy for $\sigma = 0.25, ..., 0.50$

p_0	p_1	p_2	p_3	p_4	r	H	X0	AT	sigma	au	etaR
0.325	0.221	0.128	0.296	0.448	1.14272	0.6	20	26	0.25	1.3	0.01
0.668	0.551	0.445	0.594	0.728	1.16738	0.6	20	26	0.30	1.3	0.01
1.000	0.916	0.796	0.923	1.000	1.19653	0.6	20	26	0.35	1.3	0.01
1.000	1.000	1.000	1.000	1.000	1.23016	0.6	20	26	0.40	1.3	0.01
1.000	1.000	1.000	1.000	1.000	1.26827	0.6	20	26	0.45	1.3	0.01
1.000	1.000	1.000	1.000	1.000	1.31087	0.6	20	26	0.50	1.3	0.01

We also examine the effect of the final capital target A_T. We use $H = 0.6$, $X_0 = 20$, $\sigma = 0.25$, $a_1 = 1.3$ and $\eta_R = 0.01, 0.05$. As we see from Table 5.9, we have analogous results with the expected value principle. If the reinsurance is not expensive enough, $\eta_R = 0.01$, the higher the final target, the lower the percent of the claims that the insurer pays himself, but for higher reinsurance safety loading, $\eta_R = 0.05$ the percent of the claims that the insurer is paying himself is having a small decrease.

Table 5.9. Reinsurance strategy for $A_T = 26, 28, 30$, $\eta_R = 0.01, 0.05$

p_0	p_1	p_2	p_3	p_4	A_T	η_R
0.325	0.221	0.128	0.296	0.448	26	0.01
0.244	0.126	0.021	0.209	0.381	28	0.01
0.162	0.030	0.000	0.123	0.314	30	0.01
0.964	0.838	0.723	0.853	0.969	26	0.05
0.964	0.822	0.692	0.838	0.970	28	0.05
0.964	0.805	0.661	0.823	0.970	30	0.05

Finally we examine the effect of the Hurst parameter H. We use $X_0 = 20$, $A_T = 26$, $\sigma = 0.25, a_1 = 1.3$, $\eta_R = 0.01$. As we see from Table 4.10, the higher the value of H the higher the percent of the claims that the insurer pays himself. This happens because the premium that the insurer receives is a function of H and the higher the H the higher the premium as is shown in table 5.10.

Table 5.10. Reinsurance strategy for $H = 0.6, ..., 0.9$

p_0	p_1	p_2	p_3	p_4	r	H
0.325	0.221	0.128	0.296	0.448	1.14272	0.6
0.596	0.486	0.384	0.535	0.669	1.164	0.7
0.905	0.804	0.693	0.817	0.913	1.19337	0.8
1.000	1.000	1.000	1.000	1.000	1.23389	0.9

Chapter 6

Asset Allocation with Derivatives

It is well known that options and other derivatives securities can be replicated by dynamic trading strategies involving simpler securities (i.e. stocks and bonds), such as the delta-hedging strategy. Haugh and Lo (2001) had considered the reverse implications of this correspondence by constructing an optimal buy and hold portfolio of stocks, bonds and options which can mimic the properties of a dynamic trading strategy over a period of time. Specifically given an investor's optimal dynamic investment policy for two assets, stocks and bonds, they constructed a buy and hold portfolio of stocks, bonds and options at the start of the investment horizon that will come closest to the optimal dynamic investment policy. Closest is defined in three distinct ways: expected utility, mean-squared error of terminal weath and utility weighted mean-squared error of terminal weatlh. Haugh and Lo (2001) considered three leading cases for the stock price process: geometric Brownian motion, the trending Ornstein-Uhlenbeck process and a bivariate linear difussion process with a stochastic mean reverting drift. In this chapter we generalize the above results for the case of geometric fractional Brownian motion. The stock price modelling under fractional Brownian motion has been considered by many reseachers recently such as Elliott and Van der Hoek (2003), Hu and Oksendal (2003), Brody, Syroka and Zervos (2002) and others. In the sequel we will take a look at Merton's asset allocation problem and the calculation of optimal terminal wealth and the derivation of Haugh and Lo of the optimal buy and hold portfolio that comes closest to the optimal terminal wealth. Then we will deal with the construction of such a portfolio when the stock price prococess presents long memory and we will show how to use such an allocation in asset - liability management. Useful comparisons will be made between the case of Brownian motion and fractional Brownian motion.

6.1 The optimal buy and hold portfolio

6.1.1 The Asset Allocation Problem

The asset allocation problem, see for more on the case of Brownian motion, Merton (1969), Samuelson (1969) and for the case of fractional Brownian motion, Hu, Oksendal and Sulem (1999) can be stated as follows: An investor wants to maximize the expected utility $E[u(Z_T)]$ of the end-of-period wealth Z_T by allocating his wealth between two assets, a stock and a bond over $[0, T]$.

The bond with price $A(t)$ yields a riskless instantaneous rate of return of rdt and with an initial market price of \$1, the bond price at any date t is simply $\exp(rt)$, and its dynamics are given by

$$dA_t = rA_t dt, \ A_0 = 1$$

where $r > 0$ is constant.

The stock has price S_t given by

$$dS_t = \mu S_t dt + \sigma S_t dB_t^{(H)},$$

where $S_0 = s > 0$, and μ, σ are constants.

A portfolio $\theta_t = (\xi_t, \eta_t)$ is an $F_t^{(H)}-$adapted 2-dimensional process giving the number of units ξ_t held at time t of the bond and the number of units η_t held at time t of the stock. The corresponding value process, or the wealth of the investor Z_t is given by

$$Z_t = \xi_t A_t + \eta_t S_t$$

and we assume that the portfolio is self-financing meaning that

$$dZ_t = \xi_t dA_t + \eta_t dS_t$$

and that in addition θ is admissible, meaning that Z_t^θ is nonnegative. We let Θ denote the set of admissible portfolios. With a given initial value z, the standard asset allocation problem or the optimal portfolio problem then is to find $V(z)$ and $\theta^* \in \Theta$ such that

$$V(z) = V_H(z) = \sup_{\theta \in \Upsilon} E^z[u(Z_T^\theta)],$$

where $T > 0$ is a given constant, E^z denotes expectation w.r.t μ_H when $Z_0^\theta = 0$ and $u : (0, \infty) \to R$ is a given utility function, assumed to be nondecreasing and concave. An example of such a utility function is

$$u(x) = \frac{x^\gamma}{\gamma},$$

where $\gamma \in (0, 1)$ is constant. The constant $1 - \gamma$ is interpreted as the risk aversion. Hu, Oksendal and Sulem (1999) have used an adaptation of the martingale approach, introduced by Cox and Huang (1989, 1991) to solve this problem. The following theorem was proved by Hu, Oksendal and Sulem (1999).

Theorem 35 *(Hu, Oksendal and Sulem, 1999). The value function $V(z)$ of the optimal portfolio problem is given by*

$$V(z) = V_H(z) = \frac{1}{\gamma} z^\gamma \exp\left(r\gamma T + \frac{\gamma}{2(1-\gamma)} (\frac{\mu - r}{\sigma})^2 \Lambda_H T^{2-2H} \right).$$

The corresponding optimal terminal value $Z_T^{\theta^*}$ is given by:

$$Z_T^{\theta^*} = z\exp\left(\frac{1}{1-\gamma}\int_0^T K(s)dB_s^H + rT + \frac{1-2\gamma}{2(1-\gamma)^2}(\frac{\mu-r}{\sigma})^2\Lambda_H T^{2-2H}\right)$$

where

$$\Lambda_H = \frac{\Gamma^2(\frac{3}{2}-H)}{2H(2-2H)\Gamma(2H)\Gamma(2-2H)\cos(\pi(H-\frac{1}{2}))}$$

and $K(s) =$

In order to find the optimal portfolio $\theta^* = (\xi^*, \eta^*)$ Hu, Oksendal an Sulem (1999) are using the Clark-Ocone formula developed in Hu and Oksendal (2003) and they find that

Theorem 36 *(Hu, Oksendal and Sulem, 1999).The optimal portfolio $\theta^* = (\xi^*, \eta^*)$ is given by*

$$\begin{aligned}\eta_t^* &= \exp(r(t-T))\sigma^{-1}S_t^{-1}\frac{K(t)}{1-\gamma}\cdot \\ &\exp\{\frac{1}{1-\gamma}\int_0^T K(s)d\hat{B}_s^H - \frac{(\mu-r)^2T^{2-2H}}{2(1-\gamma)^2\sigma^2}[\gamma\Lambda_H + \rho_H(\frac{t}{T}) + rT]\}\end{aligned}$$

and

$$\xi_t^* = A_t^{-1}(Z_t^{\theta^*} - \eta_t^* S_t)$$

where

$$\exp(-rt)Z_t^{\theta^*} = z + \int_0^T \exp(-rs)\sigma\eta_t^* S_t d\hat{B}_s^H.$$

Remark 5 *In order to apply the Clark-Ocone following Bender and Elliott (2002) we assume that the F_T^H-measurable $F(\omega) \geq 0$ that can be achieved as the terminal value of $Z^\theta(T,\omega)$ belongs to the space $|D_H^{1,2}|$. $|D_H^{1,2}|$ consists of the random variables $F \in (L_H^2)$ with fractional chaos decomposition such that*

$$\sum_{n=1}^{\infty} nn! \int_{R^n} (M^{H,n}(|f_n|)(t))^2 dt < \infty$$

Hu, Oksendal and Sulem (1999) provide also results for the case that the utility function is given by $u(x) = \log x$, $x > 0$.

6.1.2 Asset Allocation with Derivatives

How close can we come to the above optimal policy θ_t^* with a buy-and-hold portfolio of stocks, bonds and options? We measure closeness in three ways: maximizing expected utility, minimizing mean-square error, and minimizing weigted mean-square error. In all cases we describe it is assummed that a large number $N > n$ of options is specified, we solve the above problem for the $\binom{N}{n}$ possible combinations of options and we select the best combination.

Maximizing Expected Utility

Suppose we allow the investor to include up to n European call options in his portfolio at date 0 which expire at date T, and we do not allow him to trade after setting his initial portfolio of stocks, bonds and options. Specifically denote by D_i, the date-T payoff of a European call option with strike price k_i, hence

$$D_i = (S_T - k_i)^+.$$

Then the buy-and-hold asset allocation problem for the investor is given by

$$\max_{\{a,b,c,k_i\}} E[U(V_T)]$$

subject to

$$V_T \equiv \alpha \exp(rT) + bS_T + c_1 D_1 + c_2 D_2 + ... + c_n D_n,$$

$$W_0 = \exp(-rT)E^Q[V_T]$$

where α and b denote the investor's position in bonds and stock and $c_{1,}c_2, ..., c_n$ denotes the number of options with strike prices $k_{1,}..., k_n$ included in the portfolio. Note that the second constraint is under the equivalent martingale measure. Option pricing formulas are implicit in this second constraint. Option pricing theory for European call option when the stock price process is modelled using a geometric fractional Brownian motion has been developed, among others, from Eliott and Van der Hoek (2003) and Hu and Oksendal (2003), and we refer to them for the corresponding fornmulas. Under the CRRA utility function one needs addditional constraints to ensure non-negative wealth, thus the folloing constraints must be imposed along with the budget constraint.

$$\begin{aligned}
\alpha \exp(rT) &\geq 0 \\
\alpha \exp(rT) + bk_1 &\geq 0 \\
\alpha \exp(rT) + (b + c_1)k_2 - c_1 k_1 &\geq 0 \\
&. \\
&. \\
&. \\
\alpha \exp(rT) + (b + c_1 + ... + c_{n-1})k_n &\geq 0 \\
b + c_1 + c_2 + ... + c_n &\geq 0 \\
k_1 &\leq k_2 \leq ... \leq k_n
\end{aligned}$$

In order to solve the above problem, i.e. to maximize the concave objective function subject to the linear constraints the karush-Kuhn-Tucker conditions are sufficient for an optimal solution. One can see for more on this Bertsekas (1999).

Minimizing Mean-Square Error

In situations where the computational demands of the buy and hold asset allocation problem under the maximization of the expected utility are too great or when the dynamic investment policies are not derived by maximization of expected utility but by other treatments a mean squared error objective function may be more appropriate. In this case the buy and hold portfolio problem becomes:

$$\min_{\{a,b,c,k_i\}} E[(W_T^* - V_T)^2]$$

subject to

$$V_T \equiv \alpha \exp(rT) + bS_T + c_1 D_1 + c_2 D_2 + ... + c_n D_n,$$

$$W_0 = \exp(-rT) E^Q[V_T]$$

where α and b denote the investor's position in bonds and stock and $c_{1,} c_2, ..., c_n$ denotes the number of options with strike prices $k_{1,...,} k_n$ included in the portfolio. If we do not impose any additional constraints beyond the budget constraint then the corresponding subproblems can be solved using the first order conditions which are necesarry and sufficient and merely amount to the solution of a system of linear equations.

Minimizing Weighted Mean-Square Error

The third method is to maximize expected utility, but using an approximation for the utility function. This yields a weighted mean-squared error objective function where the weighting function is the second derivative of the utility function evaluated at the optimal end-of-period wealth W_T^*.

$$\min_{\{a,b,c,k_i\}} E[-U^{''}(W_T^*)(W_T^* - V_T)^2]$$

subject to

$$V_T \equiv \alpha \exp(rT) + bS_T + c_1 D_1 + c_2 D_2 + ... + c_n D_n,$$

$$W_0 = \exp(-rT) E^Q [V_T]$$

where α and b denote the investor's position in bonds and stock and $c_{1,} c_2, ..., c_n$ denotes the number of options with strike prices $k_{1,...,} k_n$ included in the portfolio. For CRRA utility, we still need to impose certain non-negative weatlh conditions.

6.2 Use for Asset Allocation for Insurance Products

6.2.1 The Case of a Defined Benefit Pension Scheme

The objectives of the sponsors of such a scheme is to manage the pensions assets over time so as to be able to pay an amount equal to the liabilities at time T, denoted as L_T . Instead of creating a dynamic asset allocation strategy in bonds and stocks, that is instead of creating a portfolio of assets that is perfectly correlated with the liabilities, using the above method we create an optimal buy-and-hold portfolio of bonds, stocks and options which gives as an accumulated return at time T an amount as close as possible to L_T, using the measures of closeness described above. In this way we create a perfect or almost perfect hedge (99%, or 97.5%) in a single moment and and we do not have to worry about the big transaction costs of replicating portfolios. This is an alternative version of the standard liability immunising portfolio. Using more complex options, one can adjust for early exercise of the contract. Furthermore using such a portfolio a better measure of risk of the portfolio can be provided.

In the case of a defined benefit pension scheme, using the approach of minimizing mean-square error the objective of the sponsor is to minimize the following objective function:

$$\min_{\{a,b,c,k_i\}} E[(L - V_T)^2]$$

subject to

$$V_T \equiv \alpha \exp(rT) + bS_T + c_1 D_1 + c_2 D_2 + ... + c_n D_n,$$

$$W_0 = \exp(-rT) E^Q[V_T]$$

where α and b denote the investor's position in bonds and stock and $c_{1,}c_2, ..., c_n$ denotes the number of options with strike prices $k_{1,...,}k_n$ included in the portfolio. Additionally one can input some solvency constraints to the above optimization problem which guarantee that V_t will not be less from L_t for t in $[0, T]$,such as

$$V_t - L_t \geq 0$$

or

$$V_t \geq 0.99 L_t.$$

The liabilities L_t can be found from factors such as the final salary, the length of the pensionable service and the age of the member. For example a typical UK scheme provides a pension equal to 1.67% of final salary for each year of pensionable service, up to a maximum of 40 years, thus the maximum pension is the two-thirds of the final salary. One can see for more on such a scheme Blake (1998).

6.2.2 The Case of a Targeted Money Purchase Pension Scheme

In the case of a Targeted Money Purchase pension scheme the aim is to use a defined contribution pension scheme to target a particular pension at retirement (which may be the same as that resulting from a final salary scheme) but which also benefits from any upside potential in the value of the fund assets above that required to deliver this target level. In other words the TMP scheme aims to provide a minimum pension but not a maximum pension. The optimization problem in this case can be formulated using the maximization of expected utility measure of closeness. The buy-and-hold asset allocation problem for the sponsor is given by

$$\max_{\{a,b,c,k_i\}} E[U(V_T)]$$

subject to

$$V_T \equiv \alpha \exp(rT) + bS_T + c_1D_1 + c_2D_2 + ... + c_nD_n,$$

$$W_0 = \exp(-rT)E^Q[V_T]$$

$$V_T \geq L_T$$

where α and b denote the investor's position in bonds and stock and $c_{1,}c_2, ..., c_n$ denotes the number of options with strike prices $k_{1,}..., k_n$ included in the portfolio. Note that the second constraint is under the equivalent martingale measure. Option pricing formulas are implicit in this second constraint. Under some utility functions one needs addditional constraints to ensure non-negative wealth.

6.2.3 The Case of a Defined Contribution Pension Scheme

A defined contribution pension scheme uses the full value of the funds assets to determine the amount of pension which depending on the success of the fund manager might be high or low. In the case of a defined contribution pension scheme, the approach of maximizing the expected utility of terminal wealth is natural and the sponsor has the following buy-and-hold asset allocation problem

$$\max_{\{a,b,c,k_i\}} E[U(V_T)]$$

subject to

$$V_T \equiv \alpha \exp(rT) + bS_T + c_1 D_1 + c_2 D_2 + ... + c_n D_n,$$

$$W_0 = \exp(-rT) E^Q [V_T]$$

where α and b denote the investor's position in bonds and stock and $c_{1,} c_2, ..., c_n$ denotes the number of options with strike prices $k_{1,}..., k_n$ included in the portfolio. Note that the second constraint is under the equivalent martingale measure. Option pricing formulas are implicit in this second constraint. Under some utility functions one needs addditional constraints to ensure non-negative wealth.

6.2.4 Application

Let us consider here an application of the models presented above. Consider that the investor wants to construct an optimal buy and hold portfolio containing stocks, bonds and a maximum of two options, assuming that there are only three possible options to choose from with the following strikes: $k_1 = 368$, $k_2 = 673$, $k_3 = 1231$. We set $W_0 = \$100000$, $T = 20$ years, $S_0 = 50$, $r = 0.05$, $\mu = 0.15$ and $\sigma = 0.20$. We discretise the support of S_T using its probability mass and we use a grid of 50000 points.

We will consider first the case of maximizing the expected utility. We also set $\gamma = -4$ and $H = 0.5$. In the following table we see the optimal positions of the investor and the corresponding certainty equivalent of the optimal terminal wealth.

Table 6.1. Optimal Positions, $\gamma = -4$, $H = 0.5$

Options Used	% in Bonds	% in Stocks	% in Option	% in Option	$CE(V_T^*)$
1 and 2	45.744	58.077	-4.010	0.189	442166.56
1 and 3	45.977	57.670	-3.642	-0.005	442098.75
2 and 3	49.135	52.213	-1.473	0.124	438591.29

It is clear from the above that the optimal buy - and - hold strategy is to use the options with strikes $k_1 = 368$ and $k_2 = 673$. With only two options, the optimal buy-and-hold portfolio yields an estimated certainty equivalent which is 98.66% of the certainty equivalent of the optimal dynamic asset allocation strategy, a strategy that requires contiunuous trading over a 20-year period.

Let us examine now the case of $\gamma = -3, -4, 0.5$ and $H = 0.5$. In the following table we see the certainty equivalent of the optimal terminal wealth for the possible choice of strike prices.

Table 6.2. Certainty Equivalent, $\gamma = -3, -4, 0.5$, $H = 0.5$

Options Used	$CE(V_T^*)$, $\gamma = -3$	$CE(V_T^*), \gamma = -4$	$CE(V_T^*), \gamma = 0.5$
1 and 2	503304.18	442166.56	13344115
1 and 3	503268.01	442098.75	20419533
2 and 3	500184.43	438591.29	20320550

It is clear from the above that the optimal buy - and - hold strategy for $\gamma = -3, -4$ is to use the options with strikes $k_1 = 368$ and $k_2 = 673$, but for $\gamma = 0.5$ the optimal strategy is to use the options with strikes $k_1 = 368$ and $k_3 = 1231$. The optimal positions for each choice of γ are presented in the following table.

Table 6.3. Optimal Positions, $\gamma = -3, -4, 0.5$, $H = 0.5$

γ	% in Bonds	% in Stocks	% in Option	% in Option
-3 (options 1, 2)	32.541	71.306	-4.001	0.153
-4 (options 1, 2)	45.744	58.077	-4.010	0.189
0.5 (options 1, 3)	0.0000	0.2193	10.604	89.176

It is well known that the lower the value of γ the higher the risk aversion coefficient of the investor. For $\gamma = -4$ the investor has a risk aversion coefficient of $1 - \gamma = 5$. This fact is verified from table 3 where we see that for $\gamma = -3$ the investor invest a higher portion of his wealth in stocks in comparison with the investor with $\gamma = -4$ simply because the first investor is less risk averse than the second. The opposite is happening for the bonds. We see that in these two cases the investor is shorting the call with the lower strike price in order to create a hedged position. We would like to mention here that usually one sees in textbooks that $\gamma \in (0, 1)$. However as mentioned by Haugh and Lo (2001) and as is verified also by the present study, and other empirical and experimental accounts, these are very low levels of risk aversion, and examples of investors with such preferences are proprietary traders and hedge fund managers. We see that for $\gamma = 0.5$ the investor is using the options in order to increase its risk exposure and he is investing a small amount of his capital in bonds and stocks.

Let us examine now the case of geometric fractional Brownian motion. We assume $H = 0.6$. In the following table we see the certainty equivalent of the optimal terminal wealth for the three positions he may take.

Table 6.4. Certainty Equivalent, $\gamma = -3, -4, 0.5$, $H = 0.6$

Options Used	$CE(V_T^*)$, $\gamma = -3$	$CE(V_T^*), \gamma = -4$	$CE(V_T^*), \gamma = 0.5$
1 and 2	375504.89	350692.22	3190833.82
1 and 3	375383.34	350557.74	3578647.88
2 and 3	371460.86	347149.43	3487202.39

Table 6.5. Optimal Positions, $\gamma = -3, -4, 0.5$, $H = 0.6$

γ	% in Bonds	% in Stocks	% in Option	% in Option
-3 (options 1,2)	58.88	50.38	-10.08	0.815
-4 (options 1,2)	67.50	40.21	-8.45	0.745
0.5 (options 1,3)	0	4.819	26.09	69.08

If we compare the optimal strategies for $H = 0.5$ and for $H = 0.6$ we see that in both cases the options with the same strike prices are chosen. However in the case of a risk averse investor $\gamma = -4, -3$ and in the case of long range dependence $H = 0.6$ we see that smaller proportions of the wealth are invested in the stocks and higher in bonds. In the the case of a risk taker investor $\gamma = 0.5$ and in the case of long range dependence $H = 0.6$ we see that higher proportions of the wealth are invested in the stocks and the wealth invested in options is more balanced.

Chapter 7

Detection of Long-Range Dependence

Long - range dependence can provide an elegant explanation of the empirical law that is often reffered to as the Hurst effect or Hurst's law, discovered by a British climatologist Hurst who spent many years in Egypt as a participant of the Nile hydrology projects, studing the fluctuations of yearly run-offs of Nile and several other rivers, see Hurst (1951). Suppose we have a set of observations X_i, $i > 1$, the partial sum is

$$Y(n) = \sum_{1=1}^{n} X_i$$

and the sample variance is

$$S^2(n) = n^{-1} \sum_{1=1}^{n} (X_i - n^{-1} Y(n))^2, n \geq 1.$$

Then the rescaled adjusted range statistic or R/S statistic is defined by:

$$\frac{R}{S}(n) = \frac{1}{S(n)} [\max_{0 \leq t \leq n} (Y(t) - \frac{t}{n} Y(n)) - \min_{0 \leq t \leq n} (Y(t) - \frac{t}{n} Y(n))], n \geq 1.$$

Hurst found that many naturally occuring empirical records appear to be well represented by the relation

$$E[\frac{R}{S}(n)]\tilde{}c_1 n^H,$$

as $n \to \infty$, with typical values of the H in the interval (0.5,1). If observations X_i come from a short-range dependent model, then it is known from Feller (1951) and Ammis and Lloyd (1976) that

$$E[\frac{R}{S}(n)]\tilde{}c_2 n^{0.5},$$

as $n \to \infty$, where c_2 is independent of n, and finite and positive. The discrepancy between these two relations is generally referred to as the Hurst effect.

7.0.5 Long - Range Dependence and its Implications in Finance

Self-similar processes such as fractional Brownian motion are stochastic processes that are invariant in distribution under suitable scaling of time. These processes can typically be used to model random phenomena with long - range dependence. Naturally these processes are closely related to the notion of renormalization in statistical and high energy physics and they are of increasing importance in many fields of applications such as economics and finance.

Thus a way to incorporate dependence between returns is with the use of fractional Brownian motion, see for example Mandelbrot (1997), Shiryaev (1999) and Rogers(1997)

.

The detection and measurement of index H, is of major significance since it can be considered as a measure of the intensity of long-range dependence, when $H \in (\frac{1}{2}, 1)$. One can see for more Samordnitsky and Taqqu (1994).

The presence of long memory in stock price returns has important implications for many of the paradigms used in modern financial economics. It is inconsistent with the efficient market hypothesis, (a security market is efficient if every price reflects all available and relevant information, newly arrived information is promtly arbitraged away), capital asset pricing model and arbitrage pricing theory, while optimal consumption/savings and portfolio decisions may become extremely sensitive if stock returns are long-range dependent. Furthermore the pricing of derivative securities such as options and futures with martingale methods and the classical Black-Scholes model or its extensions, Black and Scholes (1973), is then problematic since the class of continuous time stochastic processes most commonly employed is inconsistent with long-range dependence. Specifically the Black-Scholes model assumes the dynamics of the stock prices is well described by geometric Brownian motion, which makes the assumption that stock returns have independent increments in disjoint time intervals. While it is commonly accepted that normality is a mathematical convenience that is not consistent with empirical stock price returns see for example Mandelbrot (1960), Fama (1963) and Mittnik and Rachev (1993), for the dependence structure of stock price returns there were various results. To mention some of them, Fama (1965) concluded that we can assume that succesive returns are independent, Lo and Makinlay (1988) found substantial short-range dependence in the data and strongly rejected the hypothesis of i.i.d. asset returns, and Mandelbrot (1967), Greene and Fielitz (1977) have found presence of long-range dependence in asset returns. For all the above reasons the detection and the measurement of the intensity of long-range dependence when it exists is of major importance.

Let us see what are the implications of H for financial series. For $\frac{1}{2} < H < 1$ the process is said to have long - range dependence, for $H = \frac{1}{2}$, the observations are uncorrelated and for $0 < H < \frac{1}{2}$, the process has short range dependence and the correlations sum up to zero. When $H > \frac{1}{2}$, a high value of H shows less noise, more persistence, and clearer trend than do lower values. As Peters (1991) mentions the Hurst exponent measures the impact of information on the series. $H = 0.50$ implies a random walk, confirming the Efficient Market Hypothesis. Yesterday's events do not impact today. Today's events do not impact tomorrow. The events are uncorrelated, old news have already been absorbed and discounted by the market. If H>0.50 the impact of information does not decay quicly. This means that today's events do impact tomorrow, information received today continues to be discounted by the market after it has been received, in contradiction with the EMH and the quantitative models derived from it.

As Beran (1994) is showing slowly decaying correlations make the estimation more difficult, the opposite is true for predictions of future observations. It is important to note that the more dependence there is between a future observation and past values, the better the future observation can be predicted provided that the existing dependence structure is exploited appropriately, and an appropriately model is used.

7.1 Estimation of the self - similarity parameter

The methods that can be used for this purpose are many. Some of them are the method of aggregated variance, of absolute moments, of the periodogram, of the modified periodogram, of the variance of residuals, the R/S method, Whittle's approximate MLE method , local Whittle, the ratio of variance of residuals and a method using wavelets. For more on these methods one can see for example Taqqu, Teverovsky and Willinger (1995), Taqqu and Teverovsky (1996), Lo (1991), or in the monograph of Beran (1994). We will discuss here the following methods: the R/S method, Lo's modified R/S method, the periodogram method, the modified periodogram method and the Whittle's method.

7.1.1 The R/S method and Lo's modified R/S Method

The R/S method is one of the oldest methods for estimating H. The method divides the time series of N observations into K blocks, each of size N/K. Then for each lag n, $n \leq N$, estimates of $R(k_m, n)/S(k_m, n)$ of the R/S statistic are computed by starting at the points, $k_m = (m-1)N/K + 1$, $m = 1, 2, ..., K$, and such that $k_m + n \leq N$. Thus for any given m all the data points before $k_m = mN/K + 1$ are ignored. For values of n smaller than N/K, there are K different estimates of $R(n)/S(n)$. The graphical R/S approach consists then of calculating the estimates of $R(n)/S(n)$ for logarithmically spaced values of n and plotting $\log R(k_{m,}n)/S(k_{m,}n)$ vs $\log(n)$, for all starting points k_m.

This results in the rescaled adjusted range plot, also known as the pox plot of R/S. One can see for more Mandelbrot and Wallis (1969), and also Mandelbrot and Taqqu (1979). In contrast Lo (1991), focuses only on the lag n=N, and instead of using the sample standard deviation S, to normalize R, he uses a weigted sum of autocovariances, defined as:

$$S_q(N) = (\frac{1}{N}\sum_{j=1}^{N}(X_j - \bar{X}_N)^2 + \frac{2}{N}\sum_{j=1}^{q}\omega_j(q)[\sum_{i=j+1}^{N}(X_i - \bar{X}_N)(X_{i-j} - \bar{X}_N)])^{\frac{1}{2}},$$

where $\bar{X}_N$ denotes the sample mean and the weights $\omega_j(q)$ are given by

$$\omega_j(q) = 1 - \frac{j}{q+1}, q < N.$$

Lo then defines the modified R/S statistic, $V_q(N)$, by setting

$$V_q(N) = N^{-\frac{1}{2}}R(N)/S_q(N),$$

where $R(N)$ is defined above. Since

$$\lim_{N\to\infty} P\{V_q(N) \in [0.809, 1.862]\} = 0.95,$$

Lo uses the interval as the 95% asymptotic acceptance region for testing the null hypothesis

$$H_0 = \{\text{no long - range dependence, i.e. } H = 0.5\}$$

vs the composite alternative

$$H_1 = \{\text{there is long - range dependence,}$$

$$\text{i.e. } 0.5 < H < 1\}.$$

Lo's method compensates for the presence of any 'extra' short - range dependence in the data, it indicates only if long - range dependence is present or not, but it does not give an estimate of H, the right values of q are not always obvious, and it has a strong preference in accepting the null hypothesis of no long range dependence, especially for large values of q. Thus as Willinger Taqqu and Teverovsky (1999) conclude it should not be used as the sole technique for testing for long memory in the data.

7.1.2 The Periodogram Method

Following Taqqu et al (1995), Taqqu et al (1996) , suppose that λ is the frequency, N is the number of terms in the series, and X_j are the data. The periodogram is defined by

$$I(\lambda) = \frac{1}{2\pi N} | \sum_{j=1}^{N} X_j e^{ij\lambda} |^2.$$

A series with long - range dependence has a spectral density proportional to $|\lambda|^{1-2H}$ close to the origin. Since $I(\lambda)$ is an estimator of the spectal density, a regression of the logarithm of the periodogram on the logarithm of the frequency λ should give a coefficient of 1-2H. This provides an approximation to the parameter H. In practice we use only the lowest 10% of the frequencies for the calculation, since this behavior holds only for frequencies close to zero. This method was first introduced by Geweke and Porter-Hudak (1983) in a slightly different version which is referred to as the GPH estimator. Their estimator regresses on $\log |2 \sin(\lambda/2)|$ instead of $\log|\lambda|$. One can see for more Taqqu and Teverovsky (1996) and the references therein.

7.1.3 The Modified Periodogram Method

One could use also the modified periodogram method. This method compensates for the fact that on a log-log plot most of the frequencies fall on the far right, thus exerting a very strong influence on the least-squared line fitted to the periodogram. The frequency axis is divided into logarithmically equally spaced boxes and the periodogram values corresponding to the frequencies inside the box are averaged. Several of the values at the very low frequencies are left untouched since there are so few of them to begin with. Another alternative would be also the cumulative periodogram method.

7.1.4 Whittle Estimator

The Whittle estimator assumes that the parametric form of the spectral density is known and is also based on the periodogram. It involves the function

$$Q(n) = \int_{-\pi}^{\pi} \frac{I(\lambda)}{f(\lambda;\eta)} d\lambda,$$

where $I(\lambda)$ is the periodogram and $f(\lambda;\eta)$ is the spectral density at frequency λ, and where η denotes the vector of unknown parameters. The Whittle estimator is the value of η which minimizes the function Q. When dealing with fractional Gaussian noise or fractional ARIMA, η is simply the parameter H. This estimator takes more time to obtain but one also confidence intervals. For details see Fox and Taqqu (1989) and Beran (1994).

7.2 Application - Estimation of H in ASE

The ASE is the only stock exchange in Greece and it has about 367 companies listed. Previous studies for ASE include among others Papaioannou (1982), Papaioannou (1984), which reports price dependencies in stock returns for a period of six days. Niarchos and Georgakopoulos (1986) test for market efficiency with respect to information contained in corporate profit reports and find that the market is not efficient with respect to this information set since investors react slowly and gradually to new information. Panas (1990), examining the stock returns of ten large Greek firms, provides evidence of the weak - form efficiency. Barkoulas and Travlos (1998) investigate the existence of deterministic nonlinear structure in Greek stock returns, and found that there is very weak, at best, evidence for such a structure. More recently Panas (1999), find evidence of the presence of long memory on daily returns based on the Hurst exponent, by using the estimated characteristic α of the Levy - stable probability density function ($\alpha = 1/H$), and also using the spectral regression method and specifically the GPH estimator. Barkoulas, Baum and Traulos (2000), test for the presence of long -memory on weekly data using the spectral regression method finds evidence of long memory. Using a number of stocks listed on the Athens Stock Exchange (ASE) and daily returns we apply the modified Lo's method, the periodogram method, the modified periodogram method and the Whittle's method and we examine if stock prices from ASE exhibit long memory.

The stocks we examine are listed below, in the second parenthesis there is the code of the stock: Alfa-Beta Vassilopoulos (CR), (ABK), AEGEK (CR), (AEGEK), Nexans Hellas S.A. (CR), (ALKAT), Bank of Attica (CR), (ATT), EEEK Coca-Cola (CB), (EEEK), Ethniki S.A. General Insurance (CR), (EEGA), Hellas Can (C), (ELASK), Bank of Greece (CR), (ELL), Eltrak (C), (ELTRAK), Commercial Bank of Greece (EMP), National Bank of Greece (CR) (ETE), Fourlis (CB) (FRLK), N.B.G. Real Estate Development Co. (GENAK), General Hellenic Bank, Heracles General Cement Co. (CR) (HRAK), Athens Medical C.S.A. (CR) (IATR), A. Kalpinis-N.Simos Steel Service Center (C) (KALSK), Katselis Sons S.A. Bread Ind. (C) (KATSK), Keranis Holding S.A. (CB) (KERK), Michaniki S.A. (CR) (MHXK), Michaniki S.A. (PR) (MHXK), D. Mouzakis (CB) (MOYZK), Papastratos Cigarette Co. (C) (PAPAK), Bank of Piraeus (CR) (PEIR), Petzetakis S.A. (CB) PETZK, Alpha Bank (CR) (PIST), The Greek Progress Fund S.A. (CB) (PROOD), Arcadia Metal Ind. C. Rokas S.A. (CR) (ROKKA), Shelman (C) (SELMK), Sportsman S.A. (CB) (SPKAN), Titan Cement Co. (CR) (TITK), Zampa S.A. (C) (ZAMPA).

The data we have are for each stock from the perods listed in Table 7.1. In Table 7.2, we see the estimates of the periodogram and the modified periodogram method. Using the periodogram method we see that out of 33 stocks, 24 of them have H higher than 0.5 and 12 of them have H higher than 0.55. These are AEGEK, Bank of Attica, Eltrak, Commercial Bank of Greece, Fourlis, N.B.G. Real Estate Development Co., Athens Medical, Michaniki, D. Mouzakis, Bank of Piraeus, Alpha Bank and Shelman.

Using the modified periodogram method we see that out of 33 stocks, 23 of them have H higher than 0.5 and 12 of them have H higher than 0.55. These are Bank of Attica, Ethniki S.A. General Insurance, Bank of Greece, Eltrak,Fourlis, General Hellenic Bank, Athens Medical, Keranis Holding, Michaniki (PR), Bank of Piraeus,Arcadia Metal Ind. C. Rokas and Shelman.

In table 7.3 we display the results of the Whittle's method which also provides a 95% confidence interval. It is assumed that the underlying model of the stock price returns is Fractional Gaussian noise. Almost all stocks seem to present long memory, and not only the estimate of H but the Whittle's 95% confidence interval is higher than 0.5. Only for six stocks we do not find to have (strong) evidence of long - range dependence and these are Alfa -Beta Vassilopoulos, Hellas Can, Athens Medical, Papastratos Cigarette, The Greek Progress Fund and Sportsman.

In table 7.4 we display the results of the Lo's modified R/S method for values of q, q=10, 20, 30, 40. Willinger et al for a series of N=6400 uses q= 90, 180, 270, 360 and considers them as large, thus since we have N to vary from 1505 to 3708 we use the above values of q. We are writing here the result of the test and not the values of $V_q(N)$. According to this method Alfa -Beta Vassilopoulos, Commercial Bank of Greece, Ergasias Bank, Fourlis, Katselis Sons Bread, Alpha Bank, Shelman and Zampas seem to present long memory, that is 9 out of 33. This can be explained due to the strong preference that Lo's method has to accept the null hypothesis.

A partial explanation of why we have long - range dependence could be given from the fact that as Niarchos and Georgakopoulos (1986) find, the market is not efficient with respect to information contained in corporate profit reports since investors react slowly and gradually to new information.

7.3 Conclusions

As we see using the Whittle' method the vast majority (87.9%) of the stocks present long-range dependence supporting the Fractional Gaussian noise model for the stock price returns. Using the periodogram and the modified periodogram we see that also many stocks present evidence of long-range dependence and using the modified R/S method of Lo we see that 9 out of 33 of the series present long memory, but this, as Willinger et al (1999) write, could be because this method shows a strong preference for the acceptance of the null hypothesis. We conclude that there is evidence of long -range dependence in some of the ASE stock returns but in order one to be more certain has to apply more methods and in a larger scale. Since the series realizations are not independent over time, realizations from the past can help predict future returns, giving rise to the probability of consistent speculative profits. Furthermore the more dependence there is between a future observation and past values, the better the future observation can be predicted provided that the existing dependence structure is exploited appropriately, and an appropriate model is used.

Table 7.1. Stocks Used in Application

STOCK	Start	End	Length	STOCK	Start	End	Length
ABK	11/26/90	12/03/99	2256	IATR	08/29/91	11/26/99	2064
AEGEK	11/29/93	12/03/99	1505	KALSK	08/30/90	12/03/99	2307
ALKAT	03/12/90	12/03/99	2418	KATSK	11/19/90	12/03/99	2261
ATT	01/02/85	11/26/99	3704	KERK	01/02/90	12/03/99	2466
EEEK	07/15/91	11/26/99	2095	MHXK	07/19/90	12/03/99	2331
EEGA	01/02/85	11/26/99	3704	MHXP	04/13/92	12/03/99	1916
ELASK	01/07/92	12/03/99	1983	MOYZK	03/29/91	12/03/99	2172
ELL	01/02/85	11/26/99	3704	PAPAK	01/02/85	12/03/99	3170
ELTK	08/20/91	12/03/99	2076	PEIR	01/02/85	11/26/99	3704
EMP	01/02/85	11/26/99	3704	PETZK	01/02/85	12/03/99	3709
ERGA	01/02/85	11/26/99	3704	PIST	01/02/85	11/26/99	3704
ETE	01/02/85	11/26/99	3704	PROOD	08/23/90	12/03/99	2312
FRLK	04/21/88	12/03/99	2893	ROKKA	08/27/90	12/03/99	2311
GENAK	01/02/85	12/03/99	3709	SELMK	03/28/88	12/03/99	2909
GTE	01/02/85	11/26/99	3704	SPKAN	12/10/90	12/03/99	2246
HRAK	12/31/87	11/26/99	2692	TITK	01/02/85	11/26/99	3704
ZAMPA	01/02/85	12/03/99	3708				

Table 7.2. Estimation of H using the periodogram and modified periodogram method

Stock	Periodogram	Mod. Periodogram	Stock	Periodogram	Mod. Periodogram
ABK	0.39	0.48	KALSK	0.55	0.41
AEGEK	0.59	0.48	KATSK	0.49	0.49
ALKAT	0.52	0.54	KERK	0.50	0.66
ATT	0.58	0.60	MHXK	0.57	0.53
EEEK	0.53	0.53	MHXP	0.52	0.60
EEGA	0.49	0.58	MOYZK	0.57	0.53
ELASK	0.47	0.52	PAPAK	0.53	0.49
ELL	0.51	0.59	PEIR	0.71	0.60
ELTK	0.60	0.58	PETZK	0.40	0.48
EMP	0.58	0.53	PIST	0.59	0.51
ERGA	0.52	0.51	PROOD	0.51	0.54
ETE	0.54	0.50	ROKKA	0.49	0.61
FRLK	0.57	0.57	SELMK	0.66	0.59
GENAK	0.62	0.53	SPKAN	0.51	0.49
GTE	0.54	0.59	TITK	0.49	0.44
HRAK	0.53	0.52	ZAMPA	0.50	0.55
IATR	0.56	0.56			

Table 7.3 Estimation of H using Whittle's method

Stock	Whittle	Wh. LL	Wh.UL	Stock	Whittle	Wh. LL	Wh.UL
ABK	0.49	0.46	0.52	KALSK	0.56	0.53	0.59
AEGEK	0.63	0.60	0.67	KATSK	0.53	0.50	0.55
ALKAT	0.57	0.55	0.60	KERK	0.66	0.63	0.68
ATT	0.59	0.57	0.61	MHXK	0.63	0.60	0.66
EEEK	0.53	0.51	0.56	MHXP	0.67	0.64	0.70
EEGA	0.59	0.57	0.61	MOYZK	0.58	0.55	0.60
ELASK	0.51	0.49	0.54	PAPAK	0.47	0.45	0.49
ELL	0.61	0.59	0.63	PEIR	0.63	0.61	0.65
ELTK	0.54	0.51	0.56	PETZK	0.53	0.51	0.55
EMP	0.55	0.53	0.57	PIST	0.56	0.54	0.58
ERGA	0.57	0.55	0.59	PROOD	0.52	0.49	0.54
ETE	0.55	0.53	0.57	ROKKA	0.57	0.54	0.59
FRLK	0.55	0.53	0.58	SELMK	0.55	0.53	0.58
GENAK	0.57	0.55	0.59	SPKAN	0.52	0.49	0.54
GTE	0.61	0.58	0.63	TITK	0.60	0.58	0.62
HRAK	0.53	0.50	0.55	ZAMPA	0.62	0.60	0.64
IATR	0.51	0.48	0.53				

Table 7.4. Lo's Modified R/S Method

Stock	q=10	q=20	q=30	q=40	Stock	q=10	q=20	q=30	q=40
ABK	LRD	LRD	LRD	LRD	KALSK	-	-	-	-
AEGEK	LRD	-	-	-	KATSK	LRD	-	-	-
ALKAT	-	-	-	-	KERK	-	-	-	-
ATT	-	-	-	-	MHXK	-	-	-	-
EEEK	-	-	-	-	MHXP	-	-	-	-
EEGA	-	-	-	-	MOYZK	-	-	-	-
ELASK	-	-	-	-	PAPAK	-	-	-	-
ELL	-	-	-	-	PEIR	-	-	-	-
ELTK	-	-	-	-	PETZK	-	-	-	-
EMP	LRD	-	-	-	PIST	LRD	LRD	LRD	LRD
ERGA	LRD	LRD	LRD	LRD	PROOD	-	-	-	-
ETE	-	-	-	-	ROKKA	-	-	-	-
FRLK	LRD	LRD	LRD	LRD	SELMK	LRD	LRD	LRD	LRD
GENAK	-	-	-	-	SPKAN	-	-	-	-
GTE	-	-	-	-	TITK	-	-	-	-
HRAK	-	-	-	-	ZAMPA	LRD	LRD	LRD	LRD
IATR	-	-	-	-					

Bibliography

[1] Abry, P. and Sellan, D. (1996). The Wavelet - based Synthesis for Fractional Brownian motion Proposed by F. Sellan and Y. Meyer: Remarks and Fast Implementation, *Applied and Computational Harmonic Analysis*, 3, 377-383.

[2] Addie, R., Mannersalo, P. and Norros, I. (2002). Most probable paths and performance formulae for buffers with Gaussian input traffic. *European Transactions on Telecommunications*, 13, 3.

[3] Aldabe, F., Barone-Adesi, G. and Elliott, R. J. (1998). Option pricing with regulated fractional Brownian motions. *Applied Stochastic Models and Data Analysis*, 14, 285-294.

[4] Ammis, A. A. and Lloyd, E. H. (1976). The expected value of the adjusted rescaled Hurst range of independent normal summands. *Biometrika* 63, 111-116.

[5] Asmussen, S. (2000). *Ruin Probabilities.* World Scientific.

[6] Barkoulas, J. and Travlos, N. (1998). Chaos in an emerging capital market? The case of the Athens Stock Exchange. *Applied Financial Economics*, 8, 231-243.

[7] Barkoulas, J., Baum, C. and Travlos, N. (2000). Long Memory in the Greek Stock Market. *Applied Financial Economics*, 10, 2, 177-184.

[8] Bazaraa, M. S., Sherali, H. D. and Shetty, C.M. (1993). *Nonlinear Programming, Theory and Applications.* John Wiley & Sons.

[9] Belly, S. and Decreusefond, L. (1997). Multi-dimensional fractional Brownian Motion and some Applications to Queueing Theory. In *Fractals in Engineering*, Tricot, C., and Lévy–Véhel, (eds).

[10] Bender, C. (2003a). An Ito Formula for Generalized Functionals of a fractional Brownian motion with Arbitrary Hurst parameter. *Stochastic Processes and Their Applications,* 104, 81-106.

[11] Bender, C. (2003b). An S-transform approach to integration with respect to a fractional Brownian motion. *Bernoulli*, 9, 6, 955-983.

[12] Benth, F. E. (2003). On arbitrage-free pricing of weather derivatives based on fractional Brownian motion. Working paper.

[13] Beran, J. (1994). *Statistics for Long- Memory Processes.* New York: Chapman and Hall.

[14] Biagini, F., Hu, Y., Oksendal, B. and Sulem, A. (2002). A stochastic maximum principle for processes driven by fractional Brownian motion. *Stochastic Processes and their Applications*, 100, 233 - 253.

[15] Biagini, F., Oksendal, B., Sulem, A. and Wallner, N. (2003). An introduction to white noise theory and Malliavin calculus for fractional Brownian motion. *Working Paper.*

[16] Bichteler, K. (1981). Stochastic integration and L^p - theory of semimartingales. *Annals of Probability*, 9, 1, 49-89.

[17] Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 3, 637-654.

[18] Blake, D., (1998). Pension schemes as options on pension fund assets: implications for pension fund management, *Insurance: Mathematics and Economics*, 23, 263-286.

[19] Brody, D. C., Syroka, J. and Zervos, M. (2002). Dynamical pricing of weather derivatives. *Quantitative Finance*, 2, 189-198.

[20] Buhlmann, H. (1970). Mathematical Methods in Risk Theory. Springer - Verlag. Berlin.

[21] Cheridito, P. (2001a). Regularizing Fractional Brownian Motion With a View Towards Stock Price Modelling, PhD. Thesis.

[22] Cheridito, P. (2001b). Mixed fractional Brownian motion. *Bernoulli*, 7, 6, 913-934.

[23] Cheridito, P. (2001c). Arbitrage in fractional Brownian motion models. Working paper.

[24] Coleman, T. F. and Li, Y. (1996). A Reflective Newton Method for Minimizing a Quadratic Function Subject to Bounds on some of the Variables. *SIAM Journal on Optimization*, Vol. 6, 4, 1040-1058.

[25] Cutland, N. J., Kopp, P. E. and Willinger, W. (1995). Stock price returns and the Joseph effect: a fractional version of the Black-Scholes model. In E. Bolthausen, M. Dozzi, F. Russo (eds). *Seminar on Stochastic Analysis, Random Fields and Applications*. Boston , Birkhauser, pp 327- 351.Birkhauser, Basel.

[26] Cvitavic, J. (1997). Optimal Trading Under Constraints. In *Financial Mathematics*, Runggaldier, W. (eds), Lecture Notes in Mathematics, 1656, Springer - Verlag, Berlin.

[27] Daley, D. J. and Vesilo, R. (1997). Long range dependence of point processes with queueing examples. *Stochastic Processes and their Applications,* 70, 2, 265-282.

[28] Debicki, K. (2002). Ruin probability for Gaussian integrated processes. *Stochastic Processes and Their Appllications*, 98, 151-174.

[29] Debicki, K. and Mandies, M. (2004). Traffic with an FBM limit: convergence of the stationary workload process. *Queueing Systems,* 46, 113-127.

[30] Decreusefond, L. and Üstunel, A. S. (1999). Stochastic analysis of the fractional Brownian motion. *Potential ananlysis*, 10, 177-214.

[31] Delbaen, F. and Haezendonck, J. (1985). Inversed martingales in risk theory. *Insurance: Mathematics and Economics*, 4, 201-206.

[32] Delbaen, F. and Haezendonck, J. (1987). Classical risk theory in an economic environment. *Insurance: Mathematics and Economics*, 6, 85-116.

[33] Dellacherie, C. and Meyer, P. A. (1982). *Probabilities and Potential B, Theory of Martingales.* North Holland Publishing Company.

[34] Dickson, D. C. M. and Hipp, C. (2004). Ruin Probabilities for Erlang(2) risk processes, *Working Paper.*

[35] Dudley, R. M. and Norvaisa, R. (1999). *An introduction to p-variation and Young integrals.* MaPhySto Lecture Notes.

[36] Dufresne, F. and Gerber, H. U. (1998). The probability and severity of ruin for combinations of exponential claim amount distributions and their translations. *Insurance: Mathematics and Economics*, 7, 75-80.

[37] Duncan, T. E., Yan, Yi and Peng Yan. (2001). *J. App. Prob.* Exact Asymptotics for a Queue with Fractional Brownian input and applications in ATM networks , 38, 932-945.

[38] Duncan, T. E., Hu, Y. and Pasik-Duncan, B. (2000). Stochastic calculus for fractional Brownian motion I. Theory. *Siam Journal Control Optimization,* 38, 582-612.

[39] Elliott, R. and Van Der Hoek, J. (2003). A general fractional white noise theory and applications to finance. *Mathematical Finance*, 13, 2, 301-330.

[40] Embrechts, P. and Maejima, M. (2003). *Self-Similar Processes.* Priceton University Press.

[41] Embrechts, P. and Villasenor, A. (1998). Ruin estimates for large claims. *Insurance: Mathematics and Economics*, 7, 269-274.

[42] Fama, E. (1963). Mandelbrot and the stable Paretian hypothesis. *Journal of Business* 36, 420-429.

[43] Fama, E. (1965). The behavior of stock market prices. *Journal of Business* 38, 34-105.

[44] Feller, W. (1951). The asymptotic distributions of the range of sums of independen trandom variables. *Annals of Mathematical Statistics,* 22, 427-432.

[45] Fernique, X. (1964). Continuite des processus gaussiens. *C. R. Acad. Sci. Paris*, 258, 6058-6060.

[46] Fox, R. and Taqqu, M. S. (1989). Large - sample properties of parameter estimates for strongly dependent stationary Gaussian time series (with discussion). *The Annals of Statistics*, 38,1-50.

[47] Furrer, H. J. (1998). Risk processes perturbed by alpha-stable Levy motion. *Scandinavian Actuarial Journal*, 1, 59-74.

[48] Furrer, H. J. and Schmidli, H. (1994). Exponential inequalities for ruin probabilities of risk processes perturbed by diffusion. *Insurance: Mathematics and Economics*, 15, 23-36.

[49] Garrido, J. (1988). Diffusion premiums for claim severities subject to inflation. *Insurance: Mathematics and Economics*, 7, 123-129.

[50] Gerber, H. U. (1988). Mathematical fun with ruin theory. *Insurance: Mathematics and Economics*, 7, 15-23.

[51] Geweke, P. and Porter-Hudak, S. (1983). The estimation and application of long memory time series models. *Journal of Time Series Analysis,* 4, 221-238.

[52] Gill, P. E. and Murray, W. and Wright, M. H. (1981). *Practical Optimization*, Academic Press, London, UK.

[53] Gjessing, H. K. and Paulsen, J. (1997). Present value distributions with applications to ruin theory and stochastic equations. *Stochastic Processes and their Applications*, 71, 123-144.

[54] Grandel J. (1991). *Aspects of Risk Theory.* Heidelberg, Springer.

[55] Greene, M. T. and Fielitz, B. D. (1977). Long -term dependence in common stock returns. *Journal of Financial Economics*, 4, 339-349.

[56] Hald, M. and Schmidli H. (2004). On the maximisation of the adjustment coefficient under proportional reinsurance. ASTIN Bulletin, 34, 75-83.

[57] Hipp, C. and Schmidli, H. (2004). Asymptotics of ruin probabilities for controlled risk processes in the small claims case. *Scandinavian Actuarial Journal.*

[58] Harrison, J. M. (1977). Ruin problems with compounding assets. *Stochastic Processes and their Applications,* 5, 67-79.

[59] Hausmann J. A., Hall, B. H. and Griliches, Z. (1984). Econometric models for count data with an application to the patents-R&D relationship. *Econometrica*, 46, 1251-1271.

[60] Herzog, T. (1996). *Introduction to Credibility Theory.* Actex Publications, Winstead.

[61] Hoglund, T. (1990). An Asymptotic Expression for the Probability of Ruin within Finite Time. *Annals of Probability*, 18, 1, 378-389.

[62] Hida T. (1980). *Brownian Motion.* Springer, Berlin.

[63] Hille, E. and Phillips, R. (1957). *Functional Analysis and Semi-groups.* American Mathematical Society.

[64] Hipp, C. and Taksar, M. (1999). Stochastic control for optimal new business. *Insurance: Mathematics and Economics*, 26, 185-192.

[65] Hipp, C. and Plum, M. (2000). Optimal Investment for Insurers, *Insurance: Mathematics and Economics*, 27, 215-228.

[66] Hipp, C. and Vogt, M. (2001). Optimal Dynamic XL reinsurance. *Working Paper.*

[67] Hogg, R.V. and Klugman, S. A. (1984). *Loss Distributions.* John Wiley & Sons, New York.

[68] Hurst, H. E. (1951). Long-term storage capacity of reservoirs. *Trans. Amer. Soc. Civil Eng.* 116, 770-808.

[69] Husler, J. and Piterbarg, V. (1999). Extremes of a certain class of Gaussian processes. *Stochastic Processes and their Applications*, 83, 2, 257-271.

[70] Husler, J. and Piterbarg, V. (2004). On the ruin probability for physical fractional Brownian motion. *Stochastic Processes and their Applications*, 113, 315-332.

[71] Kalashnikov, V. and Norberg, R. (2002). Power tailed ruin probabilities in the presence of risky investments. *Stochastic Processes and their Applications*, 98, 2, 211-228.

[72] Karatzas, I., Lehoczky and Shreve, S. (1987). Optimal Portfolio and Consumption Decision for a 'Small Investor' on a Finite Horizon. *SIAM Journal of Control and Optimization*, 26, 1157-1186.

[73] Karatzas, I. and Shreve, S. E. (1991). *Brownian motion and Stochastic Calculus.* Springer, New York.

[74] Kawada, T. and Kono, N. (1973). On the variation of Gaussian Processes. *Lecture Notes in Mathematics*, 330, 176-192.

[75] Kleptsyna, M. L., Le Breton, A. and Viot, M. (2002). About the linear - quadratic regulator problem under a Fractional Brownian perturbation and complete observation. *Working Paper.*

[76] Kluppelberg, C. (1993). Asymptotic ordering of risks and ruin probabilities. *Insurance: Mathematics and Economics*, 12, 259-264.

[77] Kluppelberg, C. and Stadtmuller, U. (1998). Ruin probabilities in the presence of heavy-tails and interest rates. *Scandinavian Actuarial Journal,* 1, 49-58.

[78] Kolmogorov, A. N. (1940). Wienersche Spiralen und einige andere interessante Kurven im Hilbertschen Raum. *C.R. (Doklady) Acad. Sci. URSS (N.S.)* 26, 115-118.

[79] Korn, R. and Korn, E. (2000). Option Pricing and Portfolio Optimization. *Graduate Studies in Mathematics*, *American Mathematical Society*, 31.

[80] Kuo, H. H. (1996). *White Noise Distribution Theory.* CRC Press.

[81] Lebedev, N. N. (1972). *Special functions and their applications.* Dover.

[82] Lemaire, J. (1995). *Bonus - Malus Systems in Automobile Insurance.* Kluwer Academic Publishers, Massachusetts.

[83] Lindstom, T. (1993). Fractional Brownian Fields and Integrals of White Noise. *Bull. London Math. Soc.*, 25, 83-88.

[84] Lin, S. J. (1995). Stochastic Analysis of Fractional Brownian Motion. *Stochastics and Stochastics Reports*, 55, 121-140.

[85] Liptser, R. S. and Shiryayev, A.N. (1989). *Theory of Martingales.* Kluwer Academic Publishers, Dordercht.

[86] Lo, A. W. (1991). Long-term memory in stock market prices. *Econometrica,* 59, 1279-1313.

[87] Lo, A. W. and MacKinlay, A. C. (1988). Stock market prices do not follow random walks: Evidence from a simple specification test. *Rev. Financial Stud*, 1, 41-66.

[88] Los, C. and Karuppiah, J. (1997). Wavelet Multiresolution Analysis of High Frequency Asian Exhange Rates, *Working Paper*, Dept. of Finance, Kent State University.

[89] Ma, J. and Yong, J. (1999). *Forward-Bakward Stochastic Differential Equations and their Applications*, Springer - Verlag, New York.

[90] Mandelbrott, B. B. (1960). The Pareto-Levy law and the distribution of income. *International Econ. Review*, 1, 79-106.

[91] Mandelbrot, B. B. (1967). Forecast of future prices, unbiased markets and martingale models. *Journal of Business,* 39, 242-255.

[92] Mandelbrot, B. (1971). A fast fractional Gaussian noise generator. *Water Resources Research*, 7, 543-553.

[93] Mandelbrot, B. B. (1997). *Fractals and Scaling in Finance. Discontinuity, Concentration*, Risk. Springer.

[94] Mandelbrot, B. B. and Taqqu, M. (1979). Robust R/S analysis of long run serial correlation. *Proceedings of the 42nd Session of the International Statistical Institute*, Manilla 1979, Bulletin of the International Statistical Institute. Vol. 48, Book 2, pp. 69-104.

[95] Mandelbrot, B. and Van Ness, J. (1968). Fractional Brownian Motion, Fractional Noises and Applications. *SIAM Review*, 10, 422-437.

[96] Mandelbrott, B. B. and Wallis, J. R. (1968). Noah, Joseph and operational hydrology. *Water Resources Res*, 4, 909-918.

[97] Mandelbrott, B. B. and Wallis, J. R. (1969). Some long-run properties of geophysical records. *Water Resources Res,* 5, 321-340.

[98] Mao, X. (1997). *Stochastic Differential Equations and Applications.* Horwood Series in Mathematics and Applications, Chichester.

[99] Merton, R. (1969). Lifetime Portfolio Selection Under Uncertainty. The Continuous Time Case. *Review of Economics and Statistics*, 51, 247-257.

[100] Merton, R. (1971). Optimal Consumption and Portfolio Rules in a Continuous Time Model. *Journal of Economic Theory*, 3, 373-413.

[101] Merton, R. (1990). *Continous Time Finance.* Blackwell, Oxford.

[102] Moller, C. M. (1995). Stochastic Differential Equations for ruin problems. *Journal of Applied Probability*, 32, 74-89.

[103] Michna, Z. (1998a). Self-similar processes in collective risk theory. *Journal of Applied Mathematics and Stochastic Analysis*, 11, 429-248.

[104] Michna, Z. (1998b). Ruin probabilities and first passage times for self-similar processes. Ph. D. Thesis, Lund University.

[105] Michna, Z. (1999). On tail probabilties and first passage times for fractional Brownian motion. *Mathematical Methods for Operations Research*, 49, 335-354.

[106] Mikosch, T. and Norvaisa, R. (2000). Stochastic integral equations without probability. *Bernoulli*, 6, 3, 401-434.

[107] Mittnik, S. and Rachev, S.T. (1993). Modelling asset returns with alternative stable distributions. *Ecometr. Rev.*, 12, 261-330.

[108] Mulller, A. and Pflug, G. (2001), Asymptotic ruin probabilities for risk processes with dependent increments. *Insurance: Mathematics and Economics*, 28, 381-392.

[109] Narayan, O. (1998). Exact asymptotic queue length for fractional Brownian motion traffic. *Advances in Performance Analysis*, 1, 39-63.

[110] Neil O'Connell and Procissi, G. (1998). On the Build-Up of Large Queues in a Queueing model with Fractional Brownian motion input. Working paper.

[111] Niarchos, N. and Georgakopoulos, M. C. (1986). The effect of the annual corporate profit reports on the Athens Stock Exchange: an empirical investigation. *Management International Review,* 26, 64-72.

[112] Norberg, R. (1995). Differential equations for moments of present values in life insurance. *Insurance: Mathematics and Economics*, 17, 171-180.

[113] Norberg, R. (1999). Ruin problems with assets and liabilities of diffusion type. *Stochastic Processes and their Applications*, 81, 255-269.

[114] Norros, I. (1994). A storage model with self-similar input. *Queueing Systems*, 16, 387-396.

[115] Norros, I. (1995). On the use of fractional Brownian motion in the theory of connectionless networks. *IEEE Journal on Selected Areas in Communications*, 13, 6.

[116] Norros, I. (1999). Busy periods of fractional Brownian storage: a large deviations approach. *Adv. Perf. Anal.,* 2, 1,1-19.

[117] Norros, I., Mannersalo, P. and Wang, J. L. (1999). Simulation of fractional Brownian motion with conditionalized random midpoint displacement. *Adv. Perf. Anal*, 1, 77-101.

[118] Norros, I., Valkeila, E. and Virtamo, J. (1999). An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions. *Bernoulli,* 5(4), 571-587.

[119] Norvaisa, R. (2000). Modelling of stock price changes: A real analysis approach. *Finance and Stochastics*, 4, 343-369.

[120] Nualart, D. (2003). Stochastic integration with respect to fractional brownian motion and applications.

[121] Oksendal, B. and Hu, Y. (2003). Fractional white noise calculus and applications to finance. *Infin. Dimens. Anal. Quantum Prob. Related Top.*, 6, 1-32.

[122] Nyrhinen, H. (1998). Rough descriptions of ruin for a general class of stochastic processes. *Advances in Applied Probability*, 30, 1008-1026.

[123] Nyrhinen, H. (1999). On the ruin probabilities in a general economic environment. *Stochastic Processes and their Applications,* 83, 2, 319-330.

[124] Nyrhinen, H. (2001). Finite and infinite time ruin probabilities in a stochastic economic environment. *Stochastic Processes and their Applications*, 92, 2, 265-285.

[125] Panas, E. (1990). The behavior of the Athens Stock Prices, *Applied Economics*, 22, 1715-1727.

[126] Panas, E. (1999). Estimating fractal dimension using stable distributions and exploring long memory through ARFIMA models in the Athens Stock Exchange. *Technical Report*.

[127] Papaioannou, G. J. (1982). Thinness and short run price dependence in the Athens Stock Exchange, Greek *Economic Review*, 315-333.

[128] Papaioannou, G. J. (1984). Informational efficiency tests in the Athens Stock Market, in Hawawini and Michel, eds, European Equity Markets:Risk, *Return and Efficiency*, 367-381, Garland Publishing.

[129] Paulsen, J. (1993). Risk theory in a stochastic economic environment. *Stochastic Processes and their Applications,* 46, 327-361.

[130] Paulsen, J. (1998a). Sharp conditions for certain ruin in a risk process with stochastic return on investments. *Stochastic Processes and their Applications*, 75, 1, 135-148.

[131] Paulsen, J. (1998b). Ruin theory with compounding assets - a survey. *Insurance: Mathematics and Economics*, 22, 3-16.

[132] Peters, E. (1991). *Chaos and Order in the Capital Markets : A New View of Cycles, Prices, and Market Volatility*, Wiley Finance Editions.

[133] Pipiras, V. and Taqqu, M. (2000). Integration questions related to fractional Brownian motion. *Probability Theory and their Applications*, 118, 251-291.

[134] Promislow, S. D. (1991). The probability of ruin in a process with dependent increments. *Insurance: Mathematics and Economics*, 10, 99-107.

[135] Protter, P. (1990). *Stochastic Integration and Differential Equations.* Springer, Berlin.

[136] Ramsay, C. (1986). Ruin probabilities and the compound Poisson - Markov chain. *Scandinavian Actuarial Journal*, 3-12.

[137] Ramsey, C. and Usabel, M. (1997). Calculating Ruin Probabilities via Product Integration, *ASTIN Bulletin*, 27, 2, 263-271.

[138] Richtmyer, R. D. and Morton, K. W., (1967). *Difference Methods for Initial Value Problems*, Wiley.

[139] Rogers, L. C. G. (1997). Arbitrage with fractional Brownian motion, *Mathematical Finance*, 7, 95-105.

[140] Rolski, T., Schmidli, H., Schmidt, V. and Teugels J. (1998). *Stochastic Processes for Insurance and Finance*, John Wiley and Sons.

[141] Salopek, D. M. (1998). Tolerance to arbitrage. *Stochastic Processes and their Applications* 76, 2, 217-230.

[142] Salopek, D. M. (2002). A new class of nearly self-financing strategies. *Statistics and Probability Letters*, 56, 69-75.

[143] Samorodnitsky, G. and Taqqu, M. (1994). *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance.* Chapman and Hall, New York, London.

[144] Schmidli, H. (1995). Cramer-Lundberg approximations for ruin probabilities of risk processes perturbed by diffusion. *Insurance: Mathematics and Economics*, 16, 135-149.

[145] Schmidli, H. (2001). Optimal proportional reinsuranc policies in a dynamic setting. *Scandinavian Actuarial Journal*, 55-68.

[146] Schmidli, H. (2002a). On minimising the ruin probability by investment and reinsurance. *Ann. Appl. Prob.*, 12, 890-907.

[147] Schmidli, H. (2002b). Asymptotics of ruin probabilities for risk processes under optimal reisurance policies. *Working Paper*, Laboratory o Actuarial Mathematics, University of Copenhagen.

[148] Schmidli, H. (2004a). On optimal investment and subexponential claims, working paper, Laboratory of Actuarial Mathematics, University of Copenhagen.

[149] Schmidli, H. (2004b). *Optimisation in Life Insurance*, Lecture Notes from 3rd Conference in Actuarial Science and Finance, Samos.

[150] Shao, Q. M. (1996). Bounds and estimators of a basic constant in extreme value theory of Gaussian processes. *Statistica Sinica*, 6, 245-257.

[151] Shimura, M. (1983). A Class of Conditional Limit Theorems Related to Ruin Problem. *The Annals of Probability*, 11, 1, 40-45.

[152] Shiryaev, A. N. (1998). On arbitrage and replication for fractal models. *Research Report No 2*, MaPhySto, University of Aarhus.

[153] Shiryaev, A. N. (1999). *Essentials of Stochastic Finance: Facts, Models, Theories.* Singapore.

[154] Sottinnen, T. and Valkeila, E. (2003). On arbitrage and replication in the fractional Black-Scholes pricing model. *Statistics & Decisions*, 19, 331-336.

[155] Simonsen, I. (2003). Measuring anti-correlations in the Nordic electricity spot market by wavelets. *Physica A*, 322, 597-606.

[156] Smith, G. D. (1985), *Numerical Solution of Partial Differential Equations: Finite Difference Methods*, Oxford University Press.

[157] Tavella D. and Randal C., (2000), *Pricing Financial Instruments, The Finite Difference Method*, John Wiley and Sons.

[158] Straub, E., (1997). Non - Life Insurance Mathematics, Springer - Verlag.

[159] Taqqu, M. S. and Teverovsky, V. (1996). Semi-parametrical graphical estimation techniques for long - memory data. *Lecture Note in Statistics*, Vol. 115, 420-432, Springer - Verlag.

[160] Taqqu, M. S., Teverovsky, V. and Willinger, W. (1995). Estimators for long-range dependence: an empirical study. *Fractals*, 3 (4), 785-798.

[161] Taqqu, M. S., Teverovsky, V. and Willinger, W. (1997). Is network traffic self-similar or multifractal?. *Fractals,* 5, 63-73.

[162] Taqqu, M. S., Teverovsky, V. and Willinger, W. (1999). Stock price return indices and long-range dependence. *Finance and Stochastics*, 3, 1-13.

[163] Taylor, S. J. (1972). Exact Asymptotic Estimates of Brownian path variation. *Duke Mathematical Journal*, 39, 219-241.

[164] Waters, H. R. (1983). Some mathematical aspects of reinsurance. Insurance: Mathematics and Economics, 2, 17-26.

[165] Wilmott P., Dewynne J. and Howison S. (1993). *Option Pricing: Mathematical Models and Computation*, Oxford Financial Press.

[166] Wilmott, P. (1998). *Derivatives, The Theory and Practice of Financial Engineering*, John Wiley and Sons.

[167] Willinger, W., Taqqu, M. S. and Erramilli, A. (1996). A bibliographical guide to self-similar traffic and performance modeling for modern high speed networks. In F. P. Kelly, S. Zachary, I, Ziedins (eds). *Stochastic Networks: Theory and Applications.* Oxford, Clarenton Press, 339-366.

[168] Willinger, W., Taqqu, M. S. and Teverovsky, V. (1999). Stock market prices and long-range dependence, *Finance and Stochastics*, 3, 1-13.

[169] Wood, A. and Chan, G. (1994). Simulation of stationary Gaussian processes in $[0,1]^d$. *Journal of Computational and Graphical Statistics*, *3*, 4, 409-432.

[170] Young, L. C. (1936). An Inequality of the Holder Type Connected With Stieltjes integration. *Acta Mathematica*, 67, 251-282.