Stochastic Processes and Applications in Credit Risk

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Motivation

The **International Financial Reporting Standards (IFRS) 9** have amplified the need for rigorous mathematical methods, able to quantify, assess and optimize credit risk. Under IFRS 9 loans are classified into three Stages:

- Stage 1 contains all performing loans.
- Stage 2 contains all loans which have exhibited a "Significant increase in credit risk" (SICR event).
- Stage 3 contains all non-performing loans.

Main Aspects and Approaches

The research considered thus far consists of two main approaches:

- Markov chains and loan transition probabilities
- Continuous stochastic processes and credit risk

Markov chain lumpability

Markov chain lumpability is method of aggregating the states of a discrete-time Markov chain. We say that a Markov chain $M = (S, P)$ is **exactly lumpable** with respect to a partition S' if

$$\mathbb{P}(X_t \in A_j | X_{t-1} \in A_{i_1}, \dots, X_{t-k} \in A_{i_k}) = \mathbb{P}(X_t \in A_j | X_{t-1} \in A_{i_1})$$

for any n, k, j and any $A_{i_1}, \cdots, A_{i_k} \in S'$.

With U and V characterizing the partition, we formulate the exact lumpability condition using P:

We say that the transition matrix P is $U - V$ lumpable if

$$VUPV = PV$$

The reduced stochastic process on the aggregated state space S' retains the Markov property and the stochastic matrix $P^* = UPV$ is the transition matrix of the lumped system.

Approximate lumpability

In real life applications we may want to find a lumpable approximation. The problem can be reformulated as:

$$\Big((VU - I_{n\times n}) \otimes V^T\Big) p_L = \mathbf{0}_{mn},$$
$$(I_{n\times n} \otimes \mathbf{1}_n^T) p_L = \mathbf{1}_n.$$

Therefore, by defining the *lumpability condition matrix*:

$$A = \begin{pmatrix} (VU - I_n) \otimes V^T \\ I_{n\times n} \otimes \mathbf{1}_n^T \end{pmatrix}.$$

and $b = [\mathbf{0}_{mn}\ \mathbf{1}_n]^T$. we obtain the vector equation: $Ap_L = b$. From this we can get a closed form solution to any approximate lumpability problem:

$$p_L = A^\dagger (b - Ap) + p,$$

where $A^\dagger$ is the Moore-Penrose inverse of A. The result can be generalized to Markov chains of any dimension and their functions.

Application to IFRS 9

A quantity of interest is the fundamental matrix, defined by $N = I + Q + Q^2 + \cdots . = (I - Q)^{-1}$, where Q contains the transition probabilities of transient states. We can formulate the approximate lumpability problem as:

$$\min_{Q_L \in \mathbb{R}^{m\times m}} \|(I - Q_L)^{-1} - (I - Q)^{-1}\|_2$$
$$\text{subject to} \quad VUQ_LV = Q_LV,$$
$$Q_L\mathbf{1} = \mathbf{1} - R.$$

$$Q = \begin{pmatrix} 0.75 & 0.1 & 0.1 \\ 0.35 & 0.5 & 0.05 \\ 0.1 & 0.3 & 0.5 \end{pmatrix},\ R = \begin{pmatrix} 0.05 \\ 0.1 \\ 0.1 \end{pmatrix}.$$

$$Q_L^* = UQ_LV = \begin{pmatrix} 0.75 & 0.2 \\ 0.25 & 0.65 \end{pmatrix},$$

$$N_L^* = UN_LV = \begin{pmatrix} 9.31 & 5.34 \\ 6.63 & 6.68 \end{pmatrix}.$$

Therefore, $\mathbb{E}[\text{Lifetime}] = (N_L^*\mathbf{1}) = 14.65$ and $\mathbb{E}[\text{Time in Stage 2}] = 5.34$ years.

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Motivation

Of paramount importance in credit risk, and particularly under the IFRS 9 framework, is the Probability of Default (PD) modelling. To obtain more accuracte models, we wish to model the asset process using stochastic processes with jumps.

The Ornstein Uhlenbeck process

For the assets, we consider an Ornstein Uhlenbeck (OU) process with a Lévy jump. This is an example of a Lévy - Itô process. The OU process is a mean-reverting, Gaussian and Markov process, which is also temporally homogeneous. Adding the jump term:

$$dG_t = k(\theta - G_t)dt + \sigma dB_t + \int_{\mathbb{R}} zN(dz,dt), \; G_0 = x,$$

for some known x, where B_t represents the standard Brownian motion and k, θ and σ are real constants.

Probability of Default

We define the PD in a way analogous to the ruin probability of the stochastic process G. Therefore, we define:

$$\Psi(x,s,t) = \mathbb{P}(\inf_{s \leq u \leq t} G_u^x \leq 0) \equiv \mathbb{P}(\inf_{s \leq u \leq t} G_u \leq 0 | G_s = x)$$

and corresponding survival probability $\Phi(x,s,t)$, for any starting time s and maturity t.

Properties of the default (survival) probability:

1. $\Psi(x,s,t)$ a continuous function of x and s,t.

2. $\Psi(x,s,t)$ is weakly differentiable.

3. The PD $\Psi(x,s,t)$ can be equivalently considered as a function of two variables x,s or x,t. The two cases represent two different problems in credit risk; the case of a PD with a variable starting time and with a variable maturity. These are important in credit risk (e.g. scenario analysis and lifetime provision estimation).

Integro-differential equation for the PD

It can be proven that, in the case of a variable starting time, $\Phi(x,s)$ satifies:

$$\frac{\partial \Phi}{\partial r} + \frac{1}{2}\sigma^2 \frac{\partial^2 \Phi}{\partial x^2} + k(\theta - G_r)\frac{\partial \Phi}{\partial x} + \int_{\mathbb{R}} \Phi(r, G_r + z) - \Phi(r, G_{r\_})\nu(dz) = 0$$

for $s < r < t$. Due to time homogeneity, a change of variables $t - s = \omega$, will allow us to solve the above equation to estimate the survival probability in the case of a variable maturity t. Hence, in both cases we see that the survival probability satisfies:

$$\mathcal{A}\Phi(u,x) = 0,$$

where $\mathcal{A}$ represents the inifinitesimal generator of the process. The discretization of the equation allows us to obtain numerical solutions for the survival (and default) probabilities.

Application to IFRS 9

A main aspect of IFRS 9 modelling is classifying loans as Stage 2 (increased risk). The dynamics of the PD process allow us to model and predict future risky loans. Furthermore, for such exposures, lifetime expected losses must be calculated; using the evolution of the PD process we can estimate the provisions.

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