Large Deviations for the Time to Exit from a Stochastic Boundary

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Time to exit from a deterministic boundary

We examine linear SDE's in $\mathbb{R}$ and obtain large deviation results for the time to exit from a deterministic boundary which satisfies an ODE mimicking the behavior of its stochastic counterpart. The Wentzell-Freidlin approach is used which involves the solution of a variational problem in order to obtain the so-called rate function leading to the large deviation exponent. We use this approach to extend the results in [2].

Geometric Brownian Motion

Consider the Stochastic Differential Equation

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \quad X_0 = x_0 \text{ w.p. } 1$$

together with the deterministic function $x(t) = u_0 e^{\alpha t}$ where $u_0 > x_0$ and $\alpha > \mu$. We are interested in estimating the probability of the *rare event* $\mathbb{P}\left(\sup_{0\le t\le T}(X_t - x(t)) \ge 0\right)$. The Wentzell-Freidlin [3] approach involves a *family of SDE's* parametrized by ϵ: $dX_t^\epsilon = \mu X_t^\epsilon dt + \sqrt{\epsilon}\,\sigma X_t^\epsilon dW_t$. Then

$$\lim_{\epsilon\to 0} \epsilon^{-1} \log \mathbb{P}\left(\sup_{0\le t\le T}(X_t - x(t)) \ge 0\right) = -I(T)$$

where the *rate function* I is the solution to the following variational problem

$$I(T) = \inf_{x\in C[0,T]} \frac{1}{2}\int_0^t \left(\frac{\dot{x} - \mu x}{\sigma x}\right)^2 du$$

under the conditions $x(0) = x_0$, $x(t) = u_0 e^{\alpha t}$ where $u_0 > x_0$ and $\alpha \ge \mu$. Once the variational problem for a fixed time horizon is solved we may then find the value T of the "most likely meeting point" for the two curves by optimizing over T. Alternatively we may use (and have done so) the transversality conditions approach of the Calculus of Variations. The solution of the variational process that minimizes the action functional I and satisfies the boundary conditions the optimal path is $x_t = x_0 e^(2\alpha - \mu)t$ and the rate function

$$I = 2\frac{\alpha-\mu}{\sigma^2}\log\frac{u_0}{x_0} \text{ and } T = \frac{\log\frac{u_0}{x_0}}{\alpha-\mu}.$$

It is worth pointing out that, in this case, a closed form analytic expression can also be obtained. The solution of the SDE is $X_t^\epsilon = x_0 e^{(\mu - \frac{1}{2}\epsilon\sigma^2)t + \sqrt{\epsilon}\sigma W_t}$ and one may show that

$$\lim_{\epsilon\to 0}\epsilon \log P\left(\sup_{t\ge 0}(X_t^\epsilon - u_0 e^{at}) \ge 0\right) = -\frac{2}{\sigma^2}(a-\mu)\log\frac{u_0}{x_0}.$$

The exact solution agrees with the Wentzell-Freidlin asymptotic result. In the figure below the extreme path was selected by simulating a large number of paths and picking the largest among them.

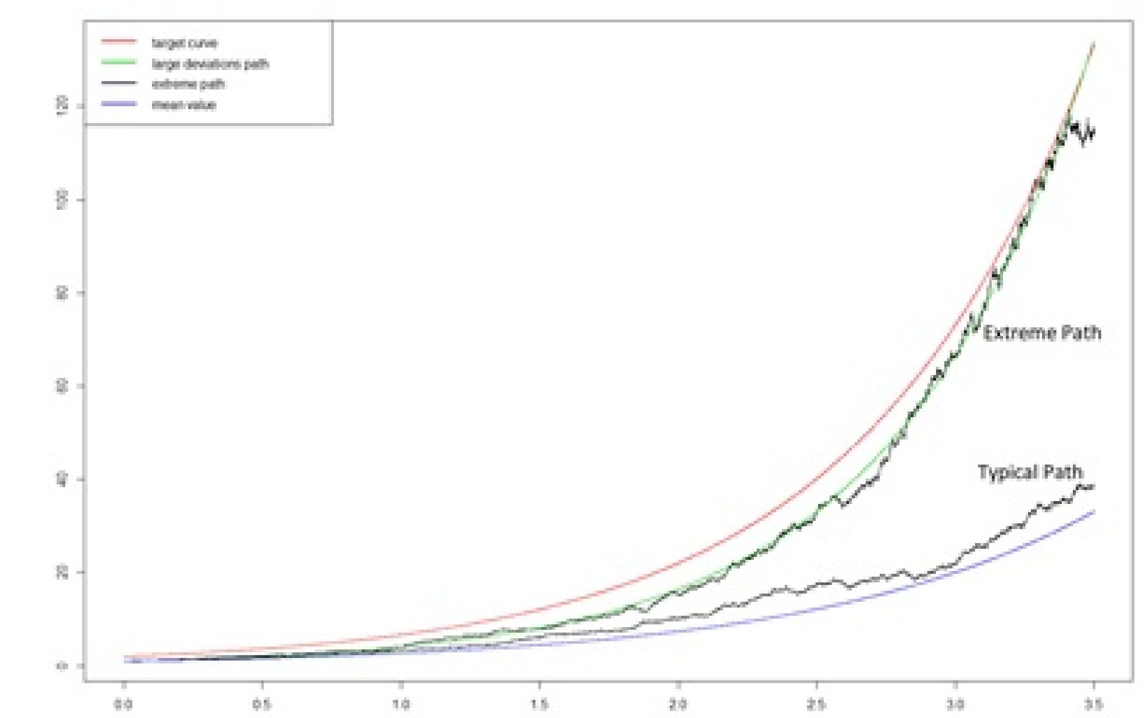


Fig. 1: Simulated sample path for $\alpha = 1, x_0 = 1, u_0 = 2$ and $\sigma = 0, 15$.

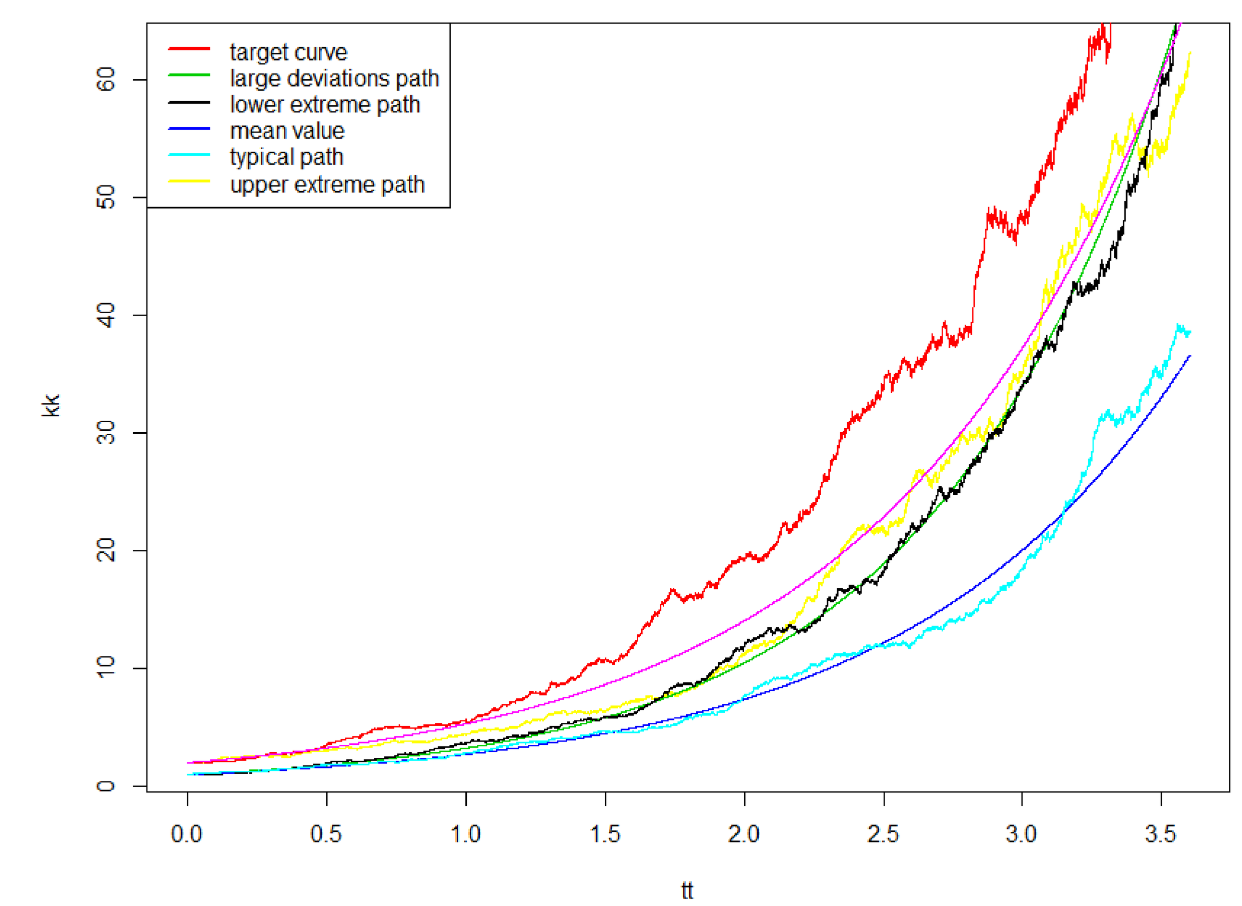
Two Correlated Geometric Brownian Motions

Consider the processes $dX_t = \alpha X_t dt + \sigma X_t dW_t$, $dY_t = \beta Y_t dt + bY_t dB_t$ where (W_t, B_t) are correlated standard Brownian motions. Assuming $\alpha > \beta$ and $x_0 > y_0$ we again obtain a large deviations estimate for $\mathbb{P}(\sup_{t\ge 0}(Y_t - X_t) \ge 0)$. Here the action functional is

$$I = \frac{1}{2}\int_0^T \left(\frac{x' - \alpha x}{x\sigma}\right)^2 + \frac{1}{1-\rho^2}\left(\frac{y' - \beta y}{yb} - \rho\frac{x' - \alpha x}{x\sigma}\right)^2 dt.$$

The variational problem consists in determining two unknown functions, x_t, y_t which minimize I over the (unknown) optimal time horizon T determined using again transversality conditions. The solution to the optimization problem gives

$$I = \frac{2(\alpha-\beta)\log\left(\frac{x_0}{y_0}\right)}{\sigma^2 + b^2 - 2\rho b\sigma} \text{ and } T = \frac{1}{\alpha-\beta}\log\left(\frac{x_0}{y_0}\right).$$



We have also considered two independent OU processes, two independent Geometric Brownian Motions, two correlated OU processes and two independent linear SDEs.

References

[1] H. U. Gerber and E.S.W. Shiu, "Optimal Dividends: Analysis with Brownian Motion", *North American Actuarial Journal*, 8(2), 1-20, 2004.

[2] H. U. Gerber and E.S.W. Shiu, "Geometric Brownian Motion Models for Assets and Liabilities: From Pension Funding to Optimal Dividends", *North American Actuarial Journal*, 7(3), 37-51, 2003.

[3] Mark I. Freidlin, Alexander D. Wentzell, *Random Perturbations of Dynamical Systems*, 3rd edition, Springer, 2012.