Optimal dynamic pricing of airline tickets and hotel rooms under random arrivals of customers

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1.1 The airplane ticket problem with cancellations

Introduction

We consider an airplane with N seats which departs after n days. The possible ticket prices are $a_1 < \cdots < a_k$. On day t, a customer arrives at the booking system and buy a ticket with probability which has distribution X_a(t)∈ $\{0,1,\dots\}$ if the ticket price is α with $E\left[X_{a_1}(\mathrm{t})\right] \geq \cdots \geq E\left[X_{a_k}(\mathrm{t})\right]$. Customer may cancel his/her ticket on every day t∈ $\{2,\dots,n\}$ and he/she receive a compensation c<a_1. The number of cancellations for day t if the number of empty seats is i,is a random variable $Y_i(t) \in \{0,1,\dots,N-i\}$. We would like to determine the pricing policy which maximizes the expected total revenue for n days.

Stochastic model equations

a) $V(N,1)= \max_{a\in A}\{aE[\min(X_a(1),N)] + E[N-\min(X_a(1),N),2)]$.

b) $V(i, t)= \max_{a\in A}\{aE[\min(X_a(t),i+Y_i(t))] + E[V(i+Y_i(t)-\min(X_a(t),i+Y_i(t)),t+1)]\} - cE[Y_i(t)], i=0,\dots,N, t=2,\dots,n-1$.

c) $V(i, n)= \max_{a\in A}\{aE[\min(X_a(t),i+Y_i(n))]\} - cE[Y_i(t)]$, $i=0,\dots,N$.

1.2 The airplane ticket problem with no cancellations

Stochastic model equations

If we set $Y_i(t)$=0 and c=0, then equations (b),(c) of section 1.1 take the following form:

b) $V(i,t) = \max_{a\in A}\{a\sum_{x=0}^{i-1}[1-F_{X_a}(x)] + \sum_{x=0}^{i-1} V(i-x,t+1)P(X_a=x)\ \}$

$i=0,\dots,N, t=1,\dots,n-1$.

c) $V(i,n) = \max_{a\in A}\{a\sum_{x=0}^{i-1}[1-F_{X_a}(x)]\}, i=0,\dots,N$

The following Proposition is proved by induction:

Proposition 1: For t=1,…,n , V(i,t) is non-decreasing in i. Many various cases of distributions for $X_{a_i}(t)$ have been examined in order to verify if the threshold-policy can be implemented.

Threshold-type policy means that the optimal prices satisfy the inequality: a_j(t)≥ a_{j+1}(t), t=1,…,n, j=0,…,N with a_j(t) the optimal price if on day t , the number of empty seats are j.

Although an analytical proof seems to be too difficult, various arithmetical results indicate that a threshold-type policy holds if $X_{\alpha_i}(t) \sim Poisson(\lambda_i)$.

Arithmetical example 1

We set $N=250$, $n=20$, $\alpha_1=80$, $\alpha_2=120$, $a_3=160$, $a_4=200$,$X_{\alpha_i}(t) \sim Poisson(\lambda_i)$,$\lambda_1=10$,$\lambda_2=5$,$\lambda_3=3$, $\lambda_4=2$.

The price α_j is optimal for $i : I_j \leq i < I_{j+1}$.

Table 1.2: The critical values for the example 2

t	$I_5(t)$	$I_4(t)$	$I_3(t)$	$I_2(t)$	$I_1(t)$
1	249	250	–	–	–
2	0	51	76	138	250
3	0	48	73	131	250
4	0	46	69	124	250
5	0	43	65	117	250
6	0	41	61	110	250
7	0	38	57	103	250
8	0	36	53	95	250
9	0	33	49	88	250
10	0	30	45	81	250
11	0	28	41	74	250
12	0	25	37	67	250
13	0	23	33	59	250
14	0	20	29	52	250
15	0	17	25	45	250
16	0	15	22	38	250
17	0	12	18	31	250
18	0	9	14	24	250
19	0	7	10	17	250
20	0	3	5	8	250

We study other cases for $X_{a_i}(t)$:

A)Modified discrete distribution demand: $X_{a_i}(t) \in$\{0,…,N\}and P$(X_{a_i}(t)=j) = f_a$, $j=1\dots,N$, $P(X_{a_i}(t)=j) = 1 - Nf_a$, *with* $f_a \leq \frac{1}{N}$ *and* $f_{a_1} \geq \cdots \geq f_{a_k}$.

Proposition 2: In the case of modified discrete uniform demand $X_{a_i}(t)$, the optimal policy is of the threshold-type.

B) Binomial distributed demand:

$X_{a_i}(t) \sim Binomial(N, p_{a_i}(t))$

with $p_{a_1}(t) \geq \cdots \geq p_{a_k}(t)$, $t=1,\dots,n$.

In the case of binomial distributed demand the threshold-type policy demand doesn't hold.

Counterexample:

For $N=250, n=20, \alpha_1=185, \alpha_2=220, \alpha_3=250$, $X_{\alpha_i}(t) \sim Binomial(250, p_{a_i})$ with $p_{a_1}=0.45$, $p_{a_2}=0.32$, $p_{a_3}=0.25$. We take for $t=19$ and $i=171$ that the optimal price is $a_{171}(19)= 220$ and for $t=19$ and $i=172$ the optimal price is $a_{172}(19) =250 > a_{171}(19)$, thus the threshold-type policy cannot not be implemented.

2.1 Optimal ticket price for a hotel room-finite time horizon

Introduction

We study the case of hotel with N rooms to be rent for a period of n days. Instead of days, another time unit can be used and apartments or warehouses can be used instead of hotel rooms. Customers book tickets with prices $a_1 < \cdots < a_k$ The demand of a room on day t is probabilistic and has distribution $X_a(\mathrm{t}) \in \{0,1,\dots\}$ with $E\left[X_{a_1}(\mathrm{t})\right] \geq \cdots \geq E\left[X_{a_k}(\mathrm{t})\right]$. At the beginning of day t a customer decides if he/she stays at the the hotel for another day with probability $p_a(t)$ independently of the other customers. Our goal is to determine the optimal price for the hotel rooms every day t, if the number of empty rooms is *i* in order to maximize the expected total revenue.

Stochastic model equations

i) $V(i,n) = \max_{a \in A}\{E\left(aY_i(n)\right) + E[a\min(X_a, N - Y_i(n))]\}, i = 0,\dots,N$

ii) $V(i,t) = \max_{a \in A}\{E\left(aY_i(t)\right) + E[a\min(X_a, N - Y_i(t))] + E[V(N - Y_i(t) - \min(X_a, N - Y_i(t)), t+1)]\}, i = 0,\dots,N, t = 2,\dots,n-1.$

iii) $V(N,1) = \max_{a \in A}\{aE[\min(N, X_a)] + E[V(N - \min(N, X_a), 2)]\}, i = 0,\dots,N$

where $Y_i(t) \sim Binomial(N-i, p_a(t))$:the number of customers who decided to stay at the hotel on day t.

Arithmetical example 2

We consider $\alpha_1=60, \alpha_2=75, \alpha_3= 80, \alpha_4=100, \alpha_5=120$, $X_{\alpha_i}(t) \sim Poisson(\lambda_i)$, $\lambda_1=14, \lambda_2=11, \lambda_3= 10, \lambda_4= 8, \lambda_5=7$, $N=20, n=7$, $(P_a(t))_{7x5}$=[0.45 0.4 0.4 0.38 0.3/0.45 0.4 0.38 0.35 0.35/ 0.5 0.5 0.5 0.5 0.5/ 0.55 0.5 0.48 0.45 0.4/ 0.6 0.54 0.5 0.5 0.5/0.5 0.5 0.4 0.4 0.35/0.6 0.31 0.28 0.24 0.2] and the optimal price for every (i,t), i=0,…,20,t=1,…,7 is seen below:

Table 2.1:The optimal prices for the example 3

0	120	120	100	120	120	120
0	120	120	100	120	120	120
0	75	120	100	120	120	120
0	60	120	75	120	120	120
0	60	120	75	120	120	120
0	60	120	75	120	120	120
0	60	120	75	120	120	120
0	60	120	60	120	75	120
0	60	120	60	120	75	120
0	60	120	60	120	75	120
0	60	120	60	120	75	120
0	60	120	60	60	75	120
0	60	120	60	60	75	120
0	60	60	60	60	75	120
0	60	60	60	60	60	120
0	60	60	60	60	60	120
0	60	60	60	60	60	120
0	60	60	60	60	60	120
0	60	60	60	60	60	120
0	60	60	60	60	60	120
60	60	60	60	60	60	120

2.2 Optimal ticket price for a hotel room-infinite horizon case

In this section we study the case of the infinite time horizon. Here, $p_t(a) = p_a$, thus the probability for a customer to remain at the hotel if the price of the room is a doesn't depend on t. Our goal is to determine the optimal pricing policy of the hotel room in order to maximize the total expected discounted revenue

V(i, a)=E[$\sum_{n=0}^{\infty} R(X_n, a_n)\lambda^n | X_0 = i$]

where X_n be the number of empty rooms of the hotel at the beginning of the n-th day, n=1,2,… and 0<λ<1 discount factor. Then, V(i,a) satisfies the optimality equation

V(i)= $\max_a\{R(i,a) + \lambda \cdot \sum_j P_{ij}(a) \cdot V(j)\}, i=0,\dots,N$

with $R(i,a) = aE(Y_i^a) + aE[\min(X_a, N - Y_i^a)]$ and

$p_{ij}(a) = P(X_{n+1} = j \,|X_n = i, a) = P(N - Y_i^a - \min(X_\alpha, N - Y_i^a) = j) = \sum_{k=0}^{N-i} P(\min(X_\alpha, N-k) = N - k - j) \cdot P(Y_i^a = k)$,

$p_{i0}(a) = \sum_{k=0}^{N-i}\left[1 - F_{X_a}(N-k-1)\right] P(Y_i^a = k)$,

$p_{ij}(a) = \sum_{k=0}^{N-i} P(X_a = N - k - j) \cdot P(Y_i^a = k)$.

Another optimality criterion is the maximization of the total expected revenue per unit time: g(a)= $\lim_{n\to\infty} \frac{V_n(i,a)}{n}$ where a one pricing policy. We implement the Policy Iteration Algorithm, the Value Iteration Algorithm, the Modified Policy Iteration Algorithm and the Value Iteration Algorithm with a relaxation factor. There is no statistical difference in computational times between the above algorithms for the infinite horizon case of the hotel room problem.

Arithmetical example 3

Suppose we rent $N=20$ rooms/apartments for an defined period of time. The possible prices for the apartments are $\alpha_1=600$, $\alpha_2=700$, $\alpha_3= 800$ and demand has Poisson distribution with $\lambda_1=9$, $\lambda_2=8.5$, $\lambda_3= 8$, where as $p_{a_1}=0.6$, $p_{a_2}=0.55$, $p_{a_3}=0.5$. The stochastic dynamic programming algorithms mentioned above were implemented and the optimal price of a room is described by the rule/policy a : a(i)=800 if the number of empty rooms is i=0,…,8, a(i)=700 if i=9,…,12 and a(i)=800 if i=13,…,20. The maximum expected revenue per unit time is g=12077 (for the Value Iteration Algorithms we set ε=0.0001).

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