1. The airplane ticket problem with cancellations

Introduction

We consider an airplane with N seats which departs after n days. The possible ticket prices are \( a_1 < \cdots < a_k \). On day t, a customer arrives at the booking system and buy a ticket with probability which has distribution \( X_i(t) \in \{0,1,\ldots\} \) if the ticket price is \( a \) with
\[
E[X_i(t)] = \cdots \geq E[X_{a_k}(t)].
\]
Customer may cancel his/her ticket on every day \( t \in \{2, \ldots, n\} \) and he/she receive a compensation \( c < a_1 \). The number of cancellations for day \( t \) if the number of empty seats is \( i \), is a random variable \( Y_i(t) \in \{0,1,\ldots,N-i\} \). We would like to determine the pricing policy which maximizes the expected total revenue for \( n \) days.

Stochastic model equations

\( a \)
\[
a) V(N) = \max_{a \in A} \left\{ a \cdot \mathbb{E}[\min(X_a(1),N)] + \mathbb{E}[N - \min(X_a(1), N), 2] \right\}. \\
b) V(i, t) = \max_{a \in A} \left\{ a \cdot \mathbb{E}[\min(X_a(t), i + Y_i(t)) + \mathbb{E}[V(i + Y_i(t) - \min(X_a(t), i + Y_i(t), t + 1))] - c \mathbb{E}[Y_i(t)], i = 0, \ldots, N, t = 2, \ldots, n - 1 \right\}. \\
c) V(i, n) = \max_{a \in A} \left\{ a \cdot \mathbb{E}[\min(X_a(t), i + Y_i(n))] - c \mathbb{E}[Y_i(t)], i = 0, \ldots, N \right\}.
\]

Stochastic model equations

If we set \( Y_i(t) = 0 \) and \( c = 0 \), then equations (b), (c) of section 1.1 take the following form:
\[
b) V(i, t) = \max_{a \in A} \left\{ a \cdot \sum_{x=0}^{i-1} [1 - F_{X_a}(x)] + \sum_{x=0}^{i-1} V(i - x, t + 1) \cdot P(X_a = x) \right\} \\
i = 0, \ldots, N, t = 1, \ldots, n - 1. \\
c) V(i, n) = \max_{a \in A} \left\{ a \cdot \sum_{x=0}^{i-1} [1 - F_{X_a}(x)] \right\}, i = 0, \ldots, N \]
The following Proposition is proved by induction:

Proposition 1: For \( t = 1, \ldots, n \), \( V(i, t) \) is non-decreasing in \( i \). Many various cases of distributions for \( X_a(t) \) have been examined in order to verify if the threshold-policy can be implemented.

Threshold-type policy means that the optimal prices satisfy the inequality: \( a_i(t) \geq a_{i+1}(t), t = 1, \ldots, n \), with \( a_i(t) \) the optimal price if on day \( t \), the number of empty seats are \( j \).

Although an analytical proof seems to be too difficult, various arithmetical results indicate that a threshold-type policy holds if \( X_a(t) \sim \text{Poisson}(\lambda(t)) \).

Arithmetical example 1

We set \( N = 250, n = 20, a_1 = 80, a_2 = 120, a_3 = 160, a_4 = 200, X_a(t) \sim \text{Poisson}(\lambda(t)), \lambda_1 = 10, \lambda_2 = 5, \lambda_3 = 3, \lambda_4 = 2. \)

The price \( a_i \) is optimal for \( i : I_i \leq I_i+1 \).
Counterexample:
For \( N=250, n=20, a_1=185, a_2=220, a_3=250, \)
\( X_{a_1}(t) \sim \text{Binomial}(250, p_{a_1}) \) with \( p_{a_2}=0.45, \)
\( p_{a_2}=0.32, p_{a_3}=0.25. \) We take for \( t=19 \) and \( i=171 \) that the optimal price is \( a_{171}(19)=220 \) and for \( t=19 \) and \( i=172 \) the optimal price is \( a_{172}(19)=250 > a_{171}(19), \)
thus the threshold-type policy cannot be implemented.

2.1 Optimal ticket price for a hotel room-finite time horizon

Introduction

We study the case of hotel with \( N \) rooms to be rent for a period of \( n \) days. Instead of days, another time unit can be used and apartments or warehouses can be used instead of hotel rooms. Customers book tickets with prices \( a_1 < \cdots < a_k \)
The demand of a room on day \( t \) is probabilistic and has distribution \( X_{a_i}(t) \in \{0,1,\ldots\} \) with \( E[X_{a_i}(t)] \geq \cdots \geq E[X_{a_k}(t)]. \) At the beginning of day \( t \) a customer decides if he/she stays at the hotel for another day with probability \( p_{a}(t) \) independently of the other customers. Our goal is to determine the optimal price for the hotel rooms every day \( t \), if the number of empty rooms is \( i \) in order to maximize the expected total revenue.

Stochastic model equations

i) \( V(i,n) = \max\{E(aY_i(n)) + E(\min(X_{a_i}, N - Y_i(n)))] \}, \) \( i = 0, \ldots, N \)
ii) \( V(i,t) = \max\{E(aY_i(t)) + E(\min(X_{a_i}, N - Y_i(t))] + E[V(N - Y_i(t) - \min(X_{a_i}, N - Y_i(t))), t+1]\}], \) \( i = 0, \ldots, N, t = 2, \ldots, n-1. \)
iii) \( V(N,1) = \max\{aE[\min(Y_i(N), X_{a_i})] + E[V(N - \min(X_{a_i}, 2)))] \}, \) \( i = 0, \ldots, N \)

where \( Y_i(t) \sim \text{Binomial}(N - i, p_{a}(t)) \): the number of customers who decided to stay at the hotel on day \( t. \)

Arithmetical example 2

We consider \( a_1=60, a_2=75, a_3=80, a_4=100, a_5=120, \)
\( X_{a_i}(t) \sim \text{Poisson}(\lambda_i), \lambda_1=14, \lambda_2=11, \lambda_3=10, \lambda_4=8, \lambda_5=7, \)
\( N=20, n=7, (p_{a_1}(r),r)=0.45, 0.4, 0.38, 0.30, 0.45, \)
\( 0.38, 0.35, 0.35/0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.48, 0.45/0.4/0.6, 0.54, 0.5, 0.5, 0.5, 0.5, 0.4, 0.4, 0.4, 0.35/0.6, 0.31/0.28, 0.24, 0.2/0.4/0.2\) and the optimal price for every \( i(t), \)
i=0,\ldots,20, t=1,\ldots,7 \) is seen below:

Table 2.1: The optimal prices for the example 3

2.2 Optimal ticket price for a hotel room-infinite horizon case

In this section we study the case of the infinite time horizon. Here, \( p_{a}(t)=p_{a}, \) thus the probability for a customer to remain at the hotel if the price of the room is a doesn’t depend on \( t. \) Our goal is to determine the optimal pricing policy of the hotel room in order to maximize the total expected discounted revenue \( V(i,\gamma)=E[\sum_{n=0}^{\infty} R(X_i(n), a_n) \lambda^n | X_i= i] \)
where \( X_i \) be the number of empty rooms of the hotel at the beginning of the \( n \)-th day, \( n=1,2,\ldots, \) and \( 0<\gamma<1 \) discount factor. Then, \( V(i,a) \) satisfies the optimality equation \( V(i)=\max\{R(i,a)+\gamma \cdot \sum_{j} p_{ij}(a) \cdot V(j)\}, \) \( i=0,\ldots,N \)
with \( R(i,a) = aE(Y^a_i) + \gamma E[\min(X_i, a - Y^a_i)] \) and \( p_{ij}(a) = P(X_{n+1}= j | X_i = i,a) = P(N - Y^a_i - \min(X_i, a - Y^a_i) = j) = \sum_{N-k}^{\infty} P(\min(X_i, N - k) = \sum_{k=0}^{N-i} P(Y^a_i = k) \cdot P(Y^a_i = k), \)
\( p_{i0}(a) = \sum_{k=0}^{N-i} [1 - F_{X_i}(N - k - 1)] P(Y^a_i = k), \)
\( p_{ij}(a) = \sum_{k=0}^{N-i} P(X_i = N - k - j) \cdot P(Y^a_i = k). \)

Another optimality criterion is the maximization of the total expected revenue per unit time: \( g(a)=\lim_{n \to \infty} \frac{V_n(a)}{n} \) where a one pricing policy. We implement the Policy Iteration algorithm, the Value Iteration algorithm, the Modified Policy Iteration Algorithm and the Value Iteration Algorithm with a relaxation factor. There is no statistical difference in computational times between the above algorithms for the infinite horizon case of the hotel room problem.

Arithmetical example 3

Suppose we rent \( N=20 \) rooms/apartments for a defined period of time. The possible prices for the apartments are \( a_1=600, a_2=700, a_3=800 \) and demand has Poisson distribution with \( \lambda_1=9, \)
\( \lambda_2=8.5, \lambda_3=8, \) where as \( p_{a_1}=0.6, p_{a_2}=0.55, p_{a_3}=0.5. \) The stochastic dynamic programming algorithms mentioned above were implemented and the optimal price of a room is described by the rule/policy \( a(t)=800 \) if the number of empty rooms is \( i=0,\ldots,8, a(i)=700 \) if \( i=9,\ldots,12 \) and \( a(i)=800 \) if \( i=13,\ldots,20. \) The maximum expected revenue per unit time is \( g=12077 \) (for the Value Iteration Algorithms we set \( \varepsilon=0.0001). \)

Bibliography


