# Processes with Lévy Marginals and Ornstein-Uhlenbeck Correlation Structure

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## Processes with Lévy marginals and OU correlation structure (LMOU)

$\{X_t; t \geq 0\}$ is an Ornstein-Uhlenbeck process if it is the solution of the stochastic differential equation $dX_t = -\alpha X_t dt + \sigma dW_t$, $X_0$ given. $a > 0$ and $X_0$ is chosen so that $\{X_t\}$ is stationary. The *joint distribution* is

$$\mathbb{E} \exp \left( i \sum_{i=1}^{n} \theta_i X_{t_i} \right) = \exp \left( -\frac{1}{4a} \sum_{i=1}^{n} \theta_i^2 - \frac{1}{2a} \sum_{1 \leq i < j \leq n} \theta_i \theta_j e^{-a(t_j - t_i)} \right).$$

We introduce and study the following generalization of the OU process.

**Definition of LMOU process. (a)** *The process* $\{X_t\}$ *is stationary.*

***(b)*** *For all* $t \in \mathbb{R}$ $X_t$ *has* spectrally positive *Lévy distribution with*

$$\log \mathbb{E}\left[e^{i\theta X_t}\right] = \int_0^\infty (e^{i\theta x} - 1)\nu(dx)$$

*where* $\int_0^\infty x\nu(dx) < \infty$ *(but* $\int_0^\infty \nu(dx)$ *may or may not be finite)*

***(c)*** *The finite dimensional distributions are given by* $\log \mathbb{E}\left[e^{i(\theta_1 X_{t_1} + \theta_2 X_{t_2} + \cdots + \theta_n X_{t_n})}\right] =$

$$\sum_{1 \leq j \leq k \leq n} \int_0^\infty \nu(dx) \left[ \left( e^{i(\theta_j + \cdots + \theta_k)x} - 1 \right) \right] (1 - e^{-\mu(t_j - t_{j-1})}) e^{-\mu(t_k - t_j)} (1 - e^{-\mu(t_{k+1} - t_k)})$$

*The corresponding Gaussian case for which* $\log \mathbb{E}\left[e^{i\theta X_t}\right] = -\frac{1}{2}\theta^2$ *has finite dimensional distributions given by* $\log \mathbb{E}\left[e^{i(\theta_1 X_{t_1} + \theta_2 X_{t_2} + \cdots + \theta_n X_{t_n})}\right]$

$$= -\frac{1}{2} \sum_{1 \leq i \leq j \leq n} (\theta_i + \cdots + \theta_j)^2 \left(1 - e^{-\mu(t_i - t_{i-1})}\right) e^{-\mu(t_j - t_i)} \left(1 - e^{-\mu(t_{j+1} - t_j)}\right).$$

*The right hand side of the above equation reduces, after algebraic simplifications, to* $-\frac{1}{2}\sum_{i=1}^n \theta_i^2 - \sum_{1 \leq i < j \leq n} \theta_i \theta_j e^{-(t_j - t_i)}$.

## The $M/G/\infty$ process as an LMOU process

Let $X_t$ denote the number of customers at time $t$ in an $M/G/\infty$ queue with Poisson arrivals with rate $\lambda$ and service times with distribution $G$. Let $-\infty = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = +\infty$ and denote by $\phi_n(\theta_1, \ldots, \theta_n; t_1, \ldots, t_n) := \mathbb{E}[e^{i\theta_1 X_{t_1} + i\theta_2 X_{t_2} + \cdots + i\theta_n X_{t_n}}]$ the joint characteristic function of $(X_{t_1}, \ldots, X_{t_n})$. Also, for $1 \leq i \leq j \leq n$, denote by $Y_{ij}$ the number of customers who arrived in the system in the time interval $(t_{i-1}, t_i]$ and left in the time interval $(t_j, t_{j+1}]$. Also, let $A_{ij} := \{(x,t) : t_j < x + t \leq t_{j+1}; x \geq 0; t_{i-1} < t \leq t_i\}$ and $\mu(A_{ij}) = \iint_{A_{ij}} \lambda G(dx)dt$. Then the collection of random variables $\{Y_{ij}; 1 \leq i \leq j \leq n\}$ are independent, Poisson distributed, with parameters $\mu(A_{ij})$. Furthermore, $X_{t_i} = \sum_{j=i}^n \sum_{k=1}^i Y_{kj}$ and $\phi_n(\theta_1, \ldots, \theta_n; t_1, \ldots, t_n)$

$$\mathbb{E} \prod_{1 \leq i \leq j \leq n} e^{i(\theta_i + \cdots + \theta_j)Y_{ij}} = e^{-\sum_{1 \leq i \leq j \leq n} \mu(A_{ij})\left(1 - e^{i(\theta_i + \cdots + \theta_j)}\right)}$$

where $\mu(A_{ij}) = \rho[G_I(t_j - t_{i-1}) - G_I(t_j - t_i) - G_I(t_{j+1} - t_{i-1}) + G_I(t_{j+1} - t_i)]$, $\rho = \lambda m$ and $G_I(x) := \frac{1}{m}\int_0^x \overline{G}(y)dy$ denotes the integrated tail distribution. Here $H(t) = G_I(t)$ and $\psi(\theta) = \rho\left(1 - e^{i\theta}\right)$. This process is then LMOU.

## Triangular arrays of on–off sources and limiting theorems

LMOU processes arise also as the limit of the superpositions of $n$ simple ON-OFF processes. For the $i$th source, the ON period last an exponential time with rate $\lambda_i$ and during it, the source generates fluid with rate $r_i$. These periods alternate with OFF periods during which no fluid is generated and whose duration is exponential with rate $\mu_i$. $\xi_i(t)$ designates the continuous time Markov process with state space $\{0, r_i\}$. We consider the effect of the superposition of a large number of such sources, each generating claims very infrequently.

Introduce a triangular array of independent on–off processes, $\{\xi_{in}(t); t \geq 0\}$ with off rates $\{\lambda_{in}\}$, on rates $\{\mu_{in}\}$, and fluid intensity rates $\{r_{in}\}$, $i = 1 \ldots n, n = 1, 2, \ldots$. Assume stationarity under the probability measure $P$. The second order behavior of these processes is then given by

$$\mathbb{E}\left[\xi_{in}(t)\right] = \frac{\lambda_{in}}{\lambda_{in} + \mu_{in}} r_{in}, \quad \text{Var}\left[\xi_{in}(t)\right] = \frac{\lambda_{in}\mu_{in}}{(\lambda_{in} + \mu_{in})^2} r_{in}^2,$$

and

$$\text{Cov}\left[\xi_{in}(0), \xi_{in}(t)\right] = \frac{\lambda_{in}\mu_{in}}{(\lambda_{in} + \mu_{in})^2} r_{in}^2 e^{-(\lambda_{in} + \mu_{in})t}.$$

Let

$$B_n^2 := \sum_{i=1}^n \text{Var}\left(\xi_{in}(t)\right) = \sum_{i=1}^n \frac{\lambda_{in}\mu_{in}}{(\lambda_{in} + \mu_{in})^2} r_{i,n}^2$$

and define the *scaled processes* via

$$\zeta_{jn}(s) = \frac{\xi_{jn}(s) - \mathbb{E}\left[\xi_{jn}(s)\right]}{B_n}.$$

We obtain the asymptotic distribution of the sum of scaled processes $Y_n(t) := \sum_{j=1}^n \zeta_{jn}(t)$. We then establish a Functional Central Limit Theorem which states that if

$$c(\tau) := \lim_{n \to \infty} \frac{1}{B_n^2} \sum_{j=1}^n \frac{\lambda_{jn}\mu_{jn} r_{jn}^2}{(\lambda_{jn} + \mu_{jn})^2} e^{-(\lambda_{jn} + \mu_{jn})\tau}$$

exists, then the sequence of processes $\{Y_n\}$ converges to a zero mean Gaussian process with covariance function $c(\tau)$. We obtain general conditions under which the limiting process is of the LMOU type.

### References

[1] G. Makatis and M.A. Zazanis. LMOU: Stationary Processes with Spectrally Positive Lévy Marginals and Ornstein-Uhlenbeck Covarince Structure, In preparation.