Stage length density in gated M/G/∞ systems

We are pursuing a number of research problems related to infinite server queues, extending the work in [4] and [5]. Here we present the problem of determining the stage length density in a gated M/G/∞ system. Customers arrive according to a Poisson process (rate λ) and service time are independent with distribution function G which we assume to have density g. Upon arrival the customers wait in a buffer until the gate to the service area opens. Then all customers present are admitted and served in parallel. The total length of this service stage is thus the maximum of the service requirements of the customers admitted. (See [1], [2]. Such models arise in manufacturing processes and communications systems.) Here we study the process of service stage lengths \{Y_n\} which constitutes a discrete time Markov chain with continuous state space \(\mathbb{R}^+\) and transition density \(f(y|x) = P(Y_{n+1} \in dy \mid Y_n = x)\) given by

\[ f(y|x) = \lambda x g(y)e^{-\lambda x G(y)} + e^{-\lambda x} g(y). \]

Express \(f(y|x)\) in series of \(\lambda\): From (1),

\[ f(y|x) = g(y) \left[ 1 + (\lambda x)^2 \left( \frac{1}{2} - G(y) \right) + (\lambda x)^3 \left( \frac{1}{3} - G(y)^2 \right) + (\lambda x)^4 \left( \frac{1}{4} - G(y)^3 \right) + \cdots \right]. \]

The invariant density \(f(x)\) of the stage length satisfies the relationship

\[ f(y) = \int_0^\infty f(x) f(y|x) dx. \]

We plan to evaluate \(f\) as follows. If \(\beta_k = \int_0^\infty x^k f(x) dx, k = 1, 2, \ldots\),

\[ f(y) = g(y) \left[ 1 + \frac{\lambda^2 \beta_2}{2} (1 - 2G(y)) + \frac{\lambda^3 \beta_3}{3} (1 - 3G(y)^2) + \frac{\lambda^4 \beta_4}{4} (1 - 4G(y)^3) + \cdots \right]. \]

Note that

\[ \int_0^\infty y^m g(y) (1 - kG(y)^{k-1}) dy = \gamma_{m+1} - \gamma_m, \]

where \(\gamma_m := E[\min(\sigma_1, \sigma_2, \ldots, \sigma_k)^m]\) and \(\sigma_i\) are i.i.d. random variables with density \(g\). Thus

\[ \int_0^\infty f(y) dy = \int_0^\infty y^m f(y) dy = \sum \frac{\lambda^k \beta_k}{k!} \left( 1 - 2G(y) \right) + \frac{\lambda^k \beta_k}{k!} \left( 1 - 3G(y)^2 \right) + \cdots \] \[ \text{or} \]

\[ \beta_1 = \gamma_1 + \frac{\lambda^2 \beta_2}{2} (\gamma_1 - \gamma_2) + \frac{\lambda^3 \beta_3}{3} (\gamma_1 - \gamma_3) + \cdots + \frac{\lambda^4 \beta_4}{4} (\gamma_1 - \gamma_4) + \cdots \]

An infinite system for stage length density in gated M/G/∞ systems

The equations (3) for \(i = 2, 3, \ldots\) describe a system with an infinite number of linear equations. We may truncate the system (e.g. take \(k = 3\)) to obtain

\[ \beta_2^{(2)} = \gamma_2 + \frac{\lambda^2 \beta_2}{2} \left( \gamma_2 - \gamma_3 \right) + \frac{\lambda^3 \beta_3}{3} \left( \gamma_2 - \gamma_4 \right) \]

\[ \beta_3^{(2)} = \gamma_3 + \frac{\lambda^2 \beta_2}{2} \left( \gamma_3 - \gamma_4 \right) + \frac{\lambda^3 \beta_3}{3} \left( \gamma_3 - \gamma_5 \right) \]

Solving the system we obtain \(\beta_2^{(2)}, \beta_3^{(2)}\), and hence

\[ f^{(2)}(y) = g(y) \left[ 1 + \frac{\lambda^2 \beta_2^{(2)}}{2} (1 - 2G(y)) + \frac{\lambda^3 \beta_3^{(2)}}{3} (1 - 3G(y)^2) \right]. \]

Infinite system for the moments \(\beta_x, i = 2, 3, \ldots\)

\[ \beta_2 = \gamma_2 + \frac{\lambda^2 \beta_2}{2} (\gamma_2 - \gamma_3) + \frac{\lambda^3 \beta_3}{3} (\gamma_2 - \gamma_4) + \cdots \]

\[ \beta_3 = \gamma_3 + \frac{\lambda^2 \beta_2}{2} (\gamma_3 - \gamma_4) + \frac{\lambda^3 \beta_3}{3} (\gamma_3 - \gamma_5) + \cdots \]

We will investigate the conditions under which, this infinite system has a unique solution which is the limit of the solutions of the corresponding truncated finite systems using the techniques described in [3]. Thus we will be able to obtain arbitrarily accurate approximate solutions for \(f\).

References


