Stage Lenght Density in Gated Queues with an Infinite Number of Servers

Dimitra Pinotsi and Michael Zazanis
Department of Statistics, Athens University of Economics, Athens, Greece, February 18, 2021

Stage length density in gated M/G/∞ systems

We are pursuing a number of research problems related to infinite server queues, extending the work in [4] and [5]. Here we present the problem of determining the stage length density in a gated $M/G/\infty$ system.

Customers arrive according to a Poisson process (rate λ) and service time are independent with distribution function G which we assume to have density g. Upon arrival the customers wait in a buffer until the gate to the service area opens. Then all customers present are admitted and served in parallel. The total length of this service stage is thus the maximum of the service requirements of the customers admitted. (See [1], [2]. Such models arise in manufacturing processes and communications systems.)

Here we study the process of service stage lengths $\{Y_n\}$ which constitutes a discrete time Markov chain with continuous state space $\mathbb{R}^+$ and transition density $f(y|x) := \mathbb{P}(Y_{n+1} \in dy \mid Y_n = x)$ given by

$$f(y|x) = \lambda x g(y) e^{-\lambda x \overline{G}(y)} + e^{-\lambda x} g(y). \tag{1}$$

Express $f(y|x)$ in series of λ: From (1),

$$f(y|x) = g(y)\left[1 + (\lambda x)^2\left(\frac{1}{2} - \overline{G}(y)\right) + \frac{(\lambda x)^3}{2!}\left(\frac{1}{3} - \overline{G}(y)^2\right) + \frac{(\lambda x)^4}{3!}\left(\frac{1}{4} - \overline{G}(y)^3\right) + \cdots\right].$$

The invariant density $f(x)$ of the stage lenght satisfies the relationship

$$f(y) = \int_0^\infty f(x) f(y|x) dx. \tag{2}$$

We plan to evaluate f as follows. If $\beta_k := \int_0^\infty x^k f(x) dx$, $k = 1, 2, \ldots$,

$$f(y) = g(y)\left[1 + \frac{\lambda^2 \beta_2}{2!}\left(1 - 2\overline{G}(y)\right) + \frac{\lambda^3 \beta_3}{3!}\left(1 - 3\overline{G}(y)^2\right) + \frac{\lambda^4 \beta_4}{4!}\left(1 - 4\overline{G}(y)^3\right) + \cdots\right].$$

Note that

$$\int_0^\infty y^m g(y)\left(1 - k\overline{G}(y)^{k-1}\right) dy = \gamma_{m,1} - \gamma_{m,k} \quad \text{where} \quad \gamma_{m,k} := \mathbb{E}[\min(\sigma_1, \sigma_2, \ldots, \sigma_k)^m]$$

and σ_i are i.i.d. random variables with density g. Thus

$$\int_0^\infty y^i f(y) dy = \int_0^\infty y^i g(y)\left[1 + \frac{\lambda^2\beta_2}{2!}\left(1 - 2\overline{G}(y)\right) + \frac{\lambda^3\beta_3}{3!}\left(1 - 3\overline{G}(y)^2\right) + \cdots\right] dy$$

or

$$\beta_i = \gamma_{i,1} + \frac{\lambda^2\beta_2}{2!}(\gamma_{i,1} - \gamma_{i,2}) + \frac{\lambda^3\beta_3}{3!}(\gamma_{i,1} - \gamma_{i,3}) + \cdots + \frac{\lambda^j\beta_j}{j!}(\gamma_{i,1} - \gamma_{i,j}) + \cdots \tag{3}$$

An infinite system for stage length density in gated M/G/∞ systems

The equations (3) for $i = 2, 3, \ldots$, describe a system with an infinite number of linear equations. We may truncate the systme (e.g. take $k = 3$) to obtain

$$\beta_2^{(3)} = \gamma_{2,1} + \frac{\lambda^2\beta_2^{(3)}}{2!}(\gamma_{2,1} - \gamma_{2,2}) + \frac{\lambda^3\beta_3^{(3)}}{3!}(\gamma_{2,1} - \gamma_{2,3})$$
$$\beta_3^{(3)} = \gamma_{3,1} + \frac{\lambda^2\beta_2^{(3)}}{2!}(\gamma_{3,1} - \gamma_{3,2}) + \frac{\lambda^3\beta_3^{(3)}}{3!}(\gamma_{3,1} - \gamma_{3,3})$$

Solving the system we obtain $\beta_2^{(3)}$, $\beta_3^{(3)}$, and hence

$$f^{(3)}(y) = g(y)\left[1 + \frac{\lambda^2\beta_2^{(3)}}{2!}\left(1 - 2\overline{G}(y)\right) + \frac{\lambda^3\beta_3^{(3)}}{3!}\left(1 - 3\overline{G}(y)^2\right)\right].$$

Infinite system for the moments β_i, $i = 2, 3, \ldots$

$$\beta_2 = \gamma_{2,1} + \frac{\lambda^2\beta_2}{2!}(\gamma_{2,1} - \gamma_{2,2}) + \frac{\lambda^3\beta_3}{3!}(\gamma_{2,1} - \gamma_{2,3}) + \frac{\lambda^4\beta_4}{4!}(\gamma_{2,1} - \gamma_{2,4}) + \cdots$$
$$\beta_3 = \gamma_{3,1} + \frac{\lambda^2\beta_2}{2!}(\gamma_{3,1} - \gamma_{3,2}) + \frac{\lambda^3\beta_3}{3!}(\gamma_{3,1} - \gamma_{3,3}) + \frac{\lambda^4\beta_4}{4!}(\gamma_{3,1} - \gamma_{3,4}) + \cdots$$
$$\beta_4 = \gamma_{4,1} + \frac{\lambda^2\beta_2}{2!}(\gamma_{4,1} - \gamma_{4,2}) + \frac{\lambda^3\beta_3}{3!}(\gamma_{4,1} - \gamma_{4,3}) + \frac{\lambda^4\beta_4}{4!}(\gamma_{4,1} - \gamma_{4,4}) + \cdots$$

We will investigate the conditions under which, this infinite system has a unique solution which is the limit of the solutions of the corresponding *truncated finite systems* using the techniques described in [3]. Thus we will be able to obtain arbitrarily accurate approximate solutions for f.

References

[1] Browne, S., E.G. Coffman, Jr., E.N. Gilbert, and P.E. Wright (1992). Gated, Exhaustive, Parallel Service, *Probab. Engrg. Inform. Sci.* **6**, 217-239.

[2] Browne, S., E.G. Coffman, Jr., E.N. Gilbert, and P.E. Wright (1992). The Gated Infinite-Server Queue: Uniform Service Times. *SIAM Journal on Applied Mathematics* **52,** 6, 1751-1762.

[3] L. V. Kantorovich, G. P. Akilov (1982). *Functional Analysis* Pergamon Press.

[4] Pinotsi, D. and M.A. Zazanis (2004). Stability Conditions for Gated $M/G/\infty$ Queues, *Probability in the Engineering and Informational Sciences*, 18(1) 103-110.