Control Charts for Two Versions of the Two-Parameter Lindley Distribution

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Two-parameter Lindley Distribution

- Lindley distribution first proposed by Lindley (1958)
- Asymmetric continuous probability distribution
- Many applications in medical and actuarial science, biology, genetics, ecology and environmental monitoring, sociology and demography, engineering, life testing, reliability and stress-strength investigation etc.
- Many generalizations and mixtures of the Lindley distribution in the literature
- First version of the two-parameter Lindley distribution proposed by Shanker et al. (2013)
- Second version of the two-parameter Lindley distribution proposed by Shanker and Mishra (2013)

First Version of the Two-parameter Lindley Distribution

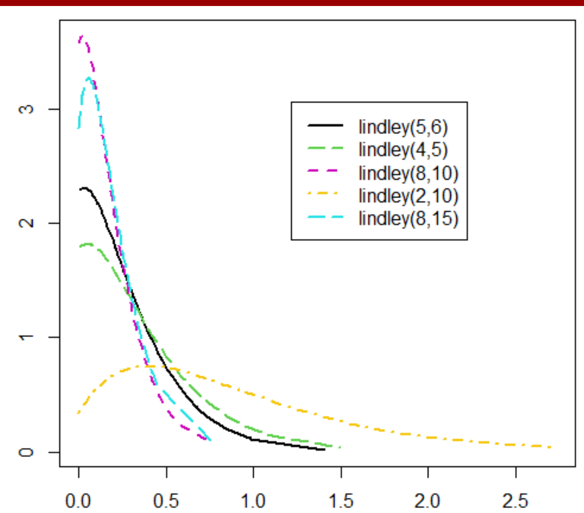

Fig. 1: Probability density function of the first two-parameter Lindley distribution for various values of the parameters

- p.d.f.: $f_X(x)=\frac{\theta^2}{\theta+r}(1+rx)e^{-\theta x}, x>0, \theta>0, r>-\theta$
- mean: $E(X)=\frac{\theta+2r}{\theta(\theta+r)}$,
- variance: $V(X)=\frac{\theta^2+4\theta r+2r^2}{\theta^2(\theta+r)^2}$.

Skewness Correction

- skewness correction for the mean:

$$c_4^*(\bar{x})=\frac{\frac{4}{3}[sk(\bar{x})]}{1+0.2[sk(\bar{x})]^2} \qquad sk(\bar{x})=sk\left(\frac{X}{\sqrt{n}}\right)=\frac{2(\theta^3+6\theta^2 r+6\theta r^2+2r^3)}{(\theta^2+4\theta r+2r^2)^{3/2}\sqrt{n}}$$

- skewness correction for the variability: $c_4^*(s^2)=\frac{\frac{4}{3}[sk(s^2)]}{1+0.2[sk(s^2)]^2}$

$$sk(s^2)=\frac{\left[\frac{\theta^2+4\theta r+2r^2}{\theta^2(\theta+r)^2}-n\left(\frac{\theta^2+4\theta r+2r^2}{\theta^2(\theta+r)^2}\right)^2\right]^3}{(n-1)\left[\frac{1}{n}\left\{E(X-\mu)^4-\frac{n-3}{n-1}\left(\frac{\theta^2+4\theta r+2r^2}{\theta^2(\theta+r)^2}\right)^2\right\}\right]^{3/2}}$$

Shewhart-type 1st Two-parameter Lindley Control Charts for Detecting Shifts in The Process Mean

$$UCL=\frac{\theta+2r}{\theta(\theta+r)}+[L+c_4^*(\bar{x})]\sqrt{\frac{\theta+4\theta r+2r^2}{n\theta^2(\theta+r)^2}}$$

$$CL=\frac{\theta+2r}{\theta(\theta+r)}$$

$$LCL=\frac{\theta+2r}{\theta(\theta+r)}+[-L+c_4^*(\bar{x})]\sqrt{\frac{\theta+4\theta r+2r^2}{n\theta^2(\theta+r)^2}}$$

Shewhart-type 1st Two-parameter Lindley Control Charts for Detecting Shifts in The Process Variability

$$UCL=\frac{\theta^2+4\theta r+2r^2}{\theta^2(\theta+r)^2}+[L+c_4^*(s^2)]\sqrt{(n-1)\left[\frac{1}{n}\left\{E(X-\mu)^4-\frac{n-3}{n-1}\left(\frac{\theta^2+4\theta r+2r^2}{\theta^2(\theta+r)^2}\right)^2\right\}\right]^{3/2}}$$

$$CL=\frac{\theta^2+4\theta r+2r^2}{\theta^2(\theta+r)^2}$$

$$LCL=\frac{\theta^2+4\theta r+2r^2}{\theta^2(\theta+r)^2}+[-L+c_4^*(s^2)]\sqrt{(n-1)\left[\frac{1}{n}\left\{E(X-\mu)^4-\frac{n-3}{n-1}\left(\frac{\theta^2+4\theta r+2r^2}{\theta^2(\theta+r)^2}\right)^2\right\}\right]^{3/2}}$$

Example of Control Charts for the 1st Two-parameter Lindley Distribution

- simulated data (30 samples of 5 observations)
- data from a 1st two-parameter Lindley distribution with θ = 56 and r = 68 (15 samples of 5 observations)
- a change in the process mean of 1σ (15 samples of 5 observations) due to a shift either in θ or in r

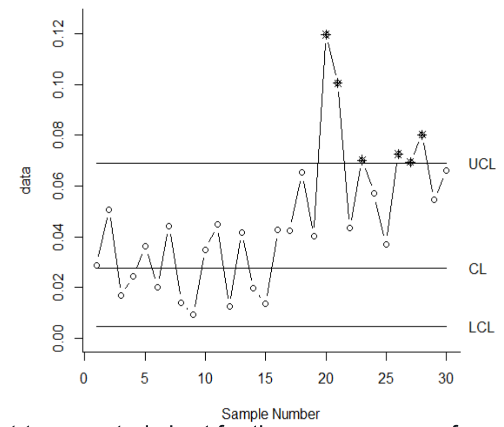

Fig. 2: Shewhart-type control chart for the process mean for a change in θ

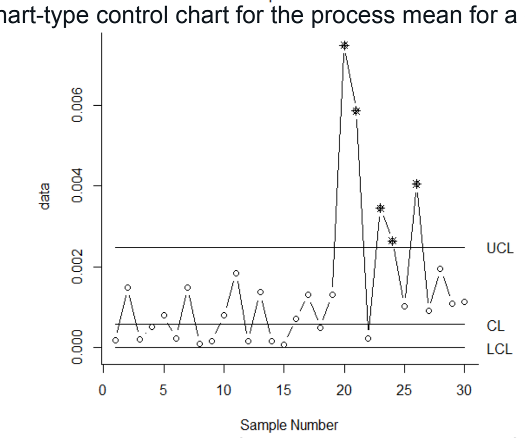

Fig. 3: Shewhart-type control chart for the process variability for a change in θ

Sensitivity of the 1st Two-parameter Lindley Control Charts

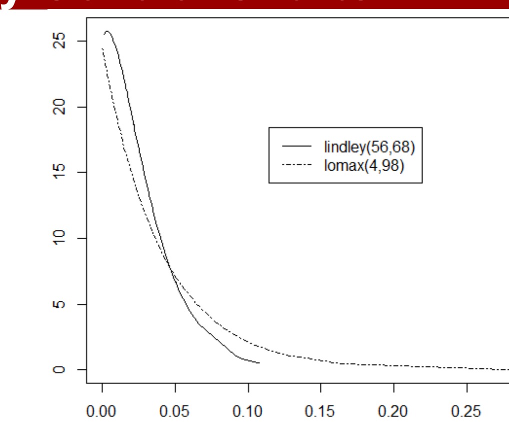

Fig. 4: Probability density function of the 1st two-parameter Lindley(56,68) and the Lomax(4,98) distribution

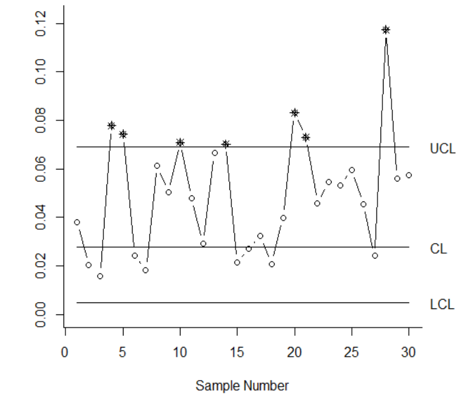

Fig. 5: Shewhart-type control chart for the mean of a process with control limits based on a 1st two-parameter Lindley(56,68) distribution and plotted data from a Lomax(4,98) distribution

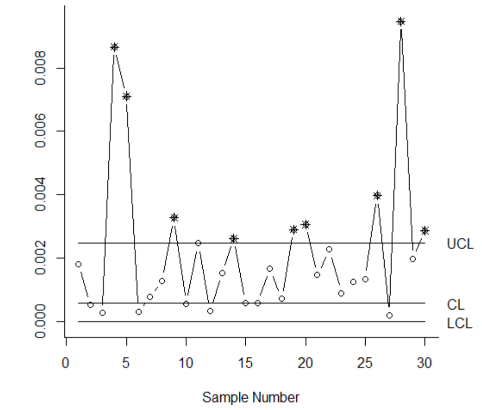

Fig. 6: Shewhart-type control chart for the mean of a process with control limits based on a 1st two-parameter Lindley(56,68) distribution and plotted data from a Lomax(4,98) distribution

Parameter Estimation

- solve equation $(2-k)b^2+4(2-k)b+2(3-2k)=0$
- k estimated using first and second sample moments about origin: $k=\frac{m'_2}{\bar{X}^2}$

$$\hat{r}=\frac{b+2}{b(b+1)\bar{X}} \qquad \hat{\theta}=\frac{b+2}{(b+1)\bar{X}}$$

- control limits constructed by replacing θ and r with their estimators

Real Data Example

Waiting times [Ghitany et al. (2008), Shanker et al. (2013)]

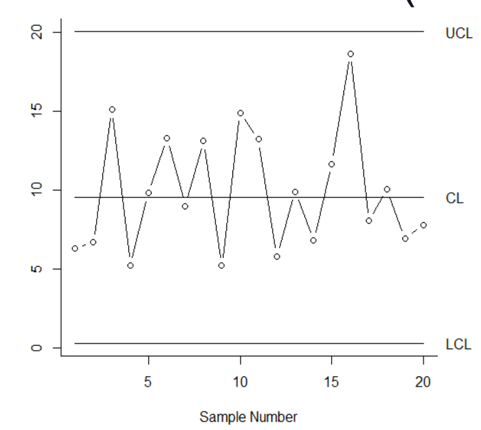

Fig. 7: Shewhart-type control chart for the process mean

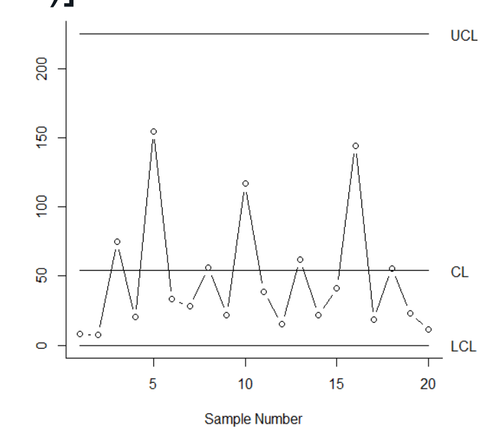

Fig. 8: Shewhart-type control chart for the process variability

Control Charts for Two Versions of the Two-Parameter Lindley Distribution

(continued)

First Version of the Two-parameter Lindley Distribution

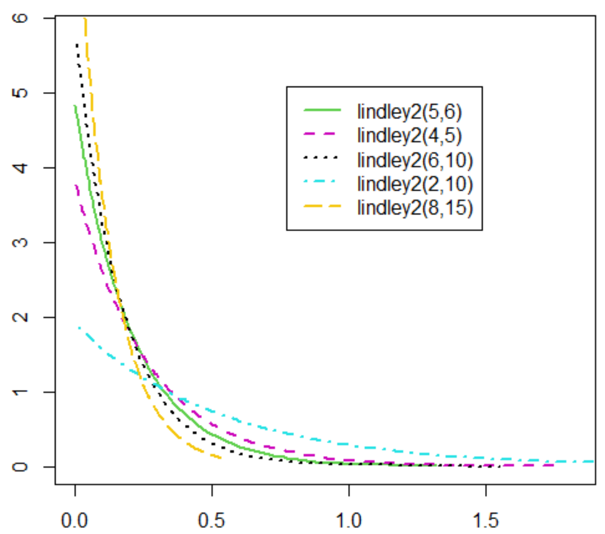

Fig. 9: Probability density function of the 2nd two-parameter Lindley distribution for various values of the parameters

- p.d.f.: $f_X(x)=\frac{\theta^2}{r\theta+1}(r+x)e^{-\theta x}, x>0, \theta>0, r\theta>-1$
- mean: $E(X)=\frac{r\theta+2}{\theta(r\theta+1)}$,
- variance: $V(X)=\frac{r^2\theta^2+4r\theta+2}{\theta(r\theta+1)^2}$

Skewness Correction

- skewness correction for the mean:

$$c_4^*(\bar{x})=\frac{\frac{4}{3}[sk(\bar{x})]}{1+0.2[sk(\bar{x})]^2} \qquad sk(\bar{x})=sk\left(\frac{X}{\sqrt{n}}\right)=\frac{2(r^3\theta^3+6r^2\theta^2+6r\theta+2)}{(r^2\theta^2+4r\theta+2)^{3/2}\sqrt{n}}$$

- skewness correction for the variability: $c_4^*(s^2)=\frac{\frac{4}{3}[sk(s^2)]}{1+0.2[sk(s^2)]^2}$

$$sk(s^2)=\frac{\left[\frac{r^2\theta^2+4r\theta+2}{\theta^2(r\theta+1)^2}-n\left(\frac{r^2\theta^2+4r\theta+2}{\theta^2(r\theta+1)^2}\right)^2\right]^3}{(n-1)\left[\frac{1}{n}\left\{E(X-\mu)^4-\frac{n-3}{n-1}\left(\frac{r^2\theta^2+42\theta+2}{\theta^2(r\theta+1)^2}\right)^2\right\}\right]^{3/2}}$$

Shewhart-type 2nd Two-parameter Lindley Control Charts for Detecting Shifts in The Process Mean

$$UCL=\frac{r\theta+2}{\theta(r\theta+1)}+[L+c_4^*(\bar{x})]\sqrt{\frac{r^2\theta+4r\theta+2}{n\theta^2(r\theta+1)^2}}$$

$$CL=\frac{r\theta+2}{\theta(r\theta+1)}$$

$$LCL=\frac{r\theta+2}{\theta(r\theta+1)}+[-L+c_4^*(\bar{x})]\sqrt{\frac{r^2\theta+4r\theta+2}{n\theta^2(r\theta+1)^2}}$$

Shewhart-type 2nd Two-parameter Lindley Control Charts for Detecting Shifts in The Process Variability

$$UCL=\frac{r^2\theta^2+4r\theta+2}{\theta(r\theta+1)^2}+[L+c_4^*(s^2)]\sqrt{(n-1)\left[\frac{1}{n}\left\{E(X-\mu)^4-\frac{n-3}{n-1}\left(\frac{\theta^2+4\theta r+2r^2}{\theta^2(\theta+r)^2}\right)^2\right\}\right]^{3/2}}$$

$$CL=\frac{r^2\theta^2+4r\theta+2}{\theta(r\theta+1)^2}$$

$$LCL=\frac{r^2\theta^2+4r\theta+2}{\theta(r\theta+1)^2}+[-L+c_4^*(s^2)]\sqrt{(n-1)\left[\frac{1}{n}\left\{E(X-\mu)^4-\frac{n-3}{n-1}\left(\frac{\theta^2+4\theta r+2r^2}{\theta^2(\theta+r)^2}\right)^2\right\}\right]^{3/2}}$$

Example of Control Charts for the 2nd Two-parameter Lindley Distribution

- simulated data
- 30 samples of 5 observations
- data from a 2nd two-parameter Lindley distribution with θ = 5 and r = 6 (15 samples of 5 observations)
- a change in the process mean of 1σ (15 samples of 5 observations) due to a shift either in θ or in r

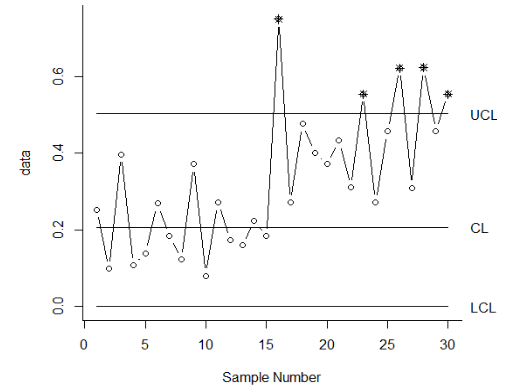

Fig. 10: Shewhart-type control chart for the process mean for a change in θ

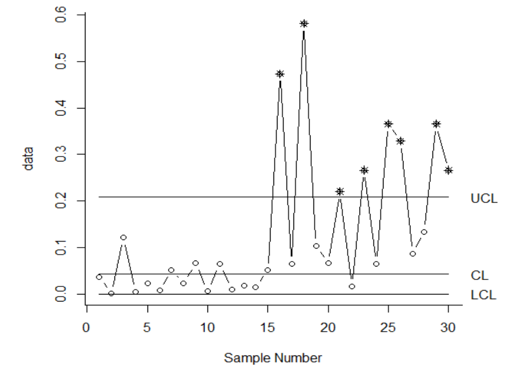

Fig. 11: Shewhart-type control chart for the process variability for a change in θ

Sensitivity of The 2nd Two-parameter Lindley Control Charts

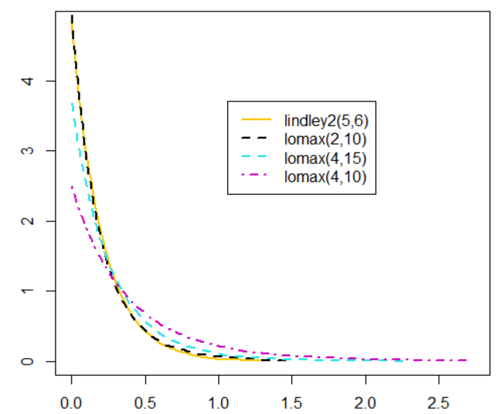

Fig. 12: Probability density function of the 2nd two-parameter Lindley(5,6) distribution and various Lomax distributions

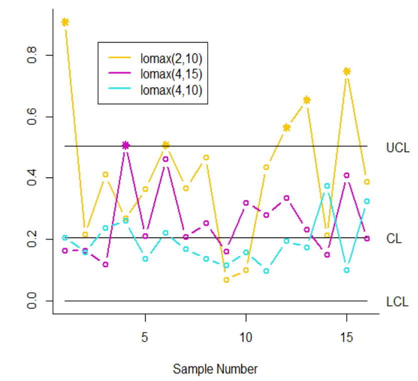

Fig. 13: Shewhart-type control chart for the mean of a process with control limits based on a 2nd two-parameter Lindley(5,6) distribution and plotted data from Lomax distributions

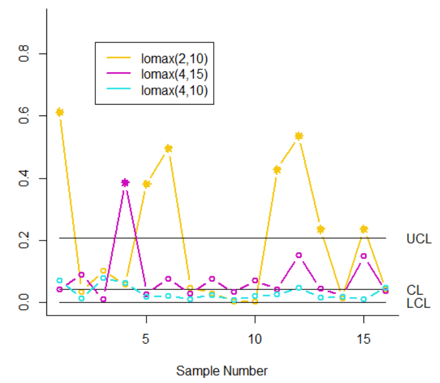

Fig. 14: Shewhart-type control chart for the variability of a process with control limits based on a 2nd two-parameter Lindley(5,6) distribution and plotted data from Lomax distributions

Parameter Estimation

- solve equation $(2-k)b^2+4(2-k)b+2(3-2k)=0$
- k estimated using first and second sample moments about origin: $k=\frac{m'_2}{\bar{X}^2}$

$$\hat{r}=\frac{b}{\hat{\theta}}=\frac{b(b+1)\bar{X}}{b+2} \qquad \hat{\theta}=\frac{b+2}{(b+1)\bar{X}}$$

- control limits constructed by replacing θ and r with their estimators

Real Data Example

Waiting times [Ghitany et al. (2008), Shanker et al. (2013)]

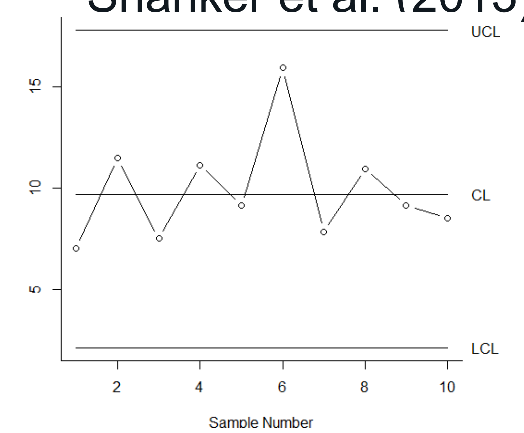

Fig. 15: Shewhart-type control chart for the process mean

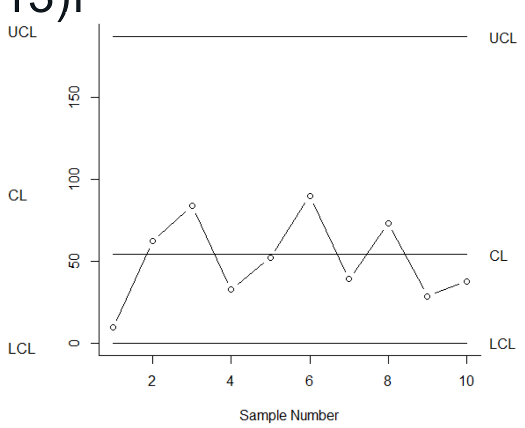

Fig. 16: Shewhart-type control chart for the process variability

References

[1] D.V. Lindley. Fiducial distributions and Bayes' theorem, *Journal of the Royal Statistical Society, Series B. (Methodological)*, 20:102-107,1958.

[2] R. Shanker, S. Sharma, and R. Shanker. A two-parameter Lindley distribution for modeling waiting and survival times data, *Applied Mathematics*, 4:363, 2013.

[3] R. Shanker, and A. Mishra. A two-parameter Lindley distribution, *STATISTICS IN TRANSITION-new series*, 14:45-56, 2013.

[4] M. E. Ghitany, B. Atieh, and S. Nadarajah. Lindley distribution and its application, *Mathematics and Computers in Simulation*, 78:493-506, 2008.