A FLEXIBLE TRIVARIATE POISSON INTEGER-VALUED GARCH MODEL

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Objectives

Objectives for today:

- Introducing a Poisson INGARCH model.
- Stationarity and ergodicity of the model.
- Parameters' estimation.
- Forecasting.
- Implementation.

A new bivariate Poisson distribution

- Sarmanov distribution

$$f(y_1,y_2)=f(y_1)f(y_2)[1+\omega q_1(y_1)q_2(y_2)]$$

where

- $f_1(y_1)$ and $f_2(y_2)$ are two probability mass functions (p.m.f.) with supports defined on $A_1\subseteq\mathbb{R}$ and $A_2\subseteq\mathbb{R}$
- $q_i(y_i), i=1,2$ are bounded non constant functions

$$\sum_{y_i=-\infty}^{\infty}q_i(y_i)f_i(y_i)=0$$

- $[1+\omega q_1(y_1)q_2(y_2)]$ indicates the deviance of two variables Y_1,Y_2 of their dependence.
- ω is a real number that satisfies the condition

$$[1+\omega q_1(y_1)q_2(y_2)]\geq 0$$

FGM copula

$$C(u,v)=uv[1+\omega(1-u)(1-v)]$$
$$\Updownarrow$$
$$F(y_1,y_2)=F(y_1)F(y_2)[1+\omega(1-F(y_1)(1-F(y_2)))]$$

$$f(y_1,y_2)=\prod_{i=1}^{2}f(y_i)[1+\omega\prod_{i=1}^{2}((g(y_i)-\sum g(y_i)f(y_i)))]$$

$$f(y_1,y_2)=\prod_{i=1}^{2}f(y_i)[1+\omega\prod_{i=1}^{2}(\alpha_i^{-k_iy_i}-\sum\alpha_i^{-k_iy_i}f(y_i))]$$

where $k_1,k_2\in\mathbb{R}^+$

$$q_i(y_i)=\alpha_i^{-k_iy_i}-\sum_{y_i=0}^{\infty}\alpha_i^{-k_iy_i}f(y_i)$$

- Domain [0,1], $q_i(y_i)=1-2F(y_i)|1-2u$
- Domain $[0,\infty)$, $q_i(y_i)=exp(-y_i)-L_i(1)$

Considering Poisson distribution

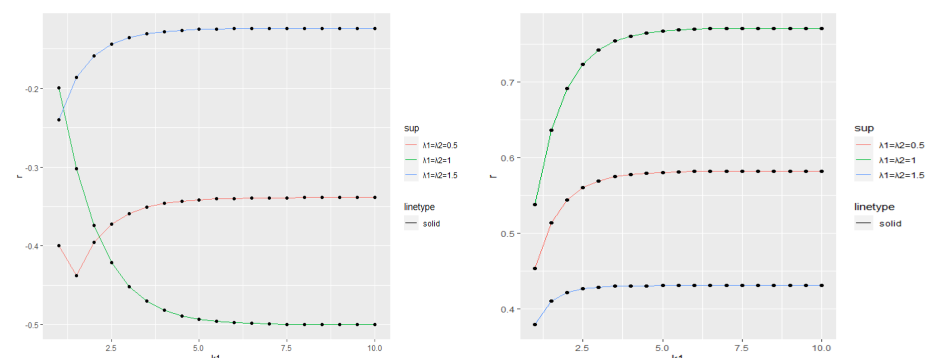
$$f(y_1,y_2)=\prod_{i=1}^{2}\frac{e^{-\lambda_i}\lambda_i^{y_i}}{y_i!}[1+\omega q_1(y_1)q(y_2)]$$

where

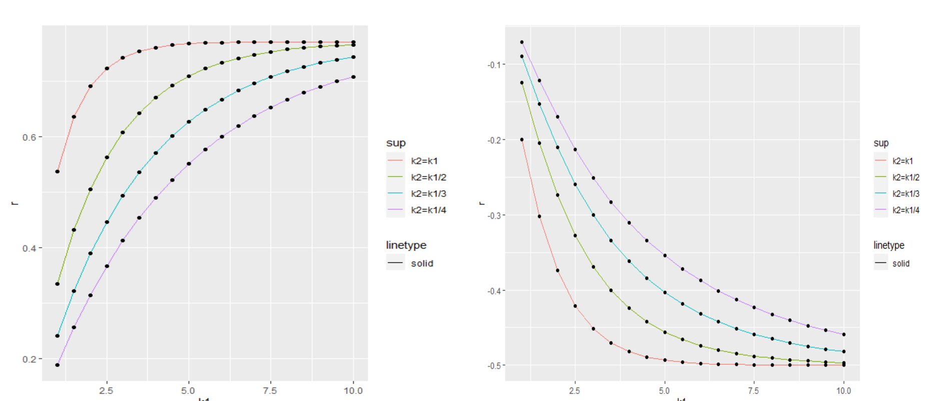
$$q(y_i)=\alpha_i^{-k_iy_i}-e^{\lambda_i(\alpha_i^{-k_iy_i}-1)}$$

$$\rho=\omega\prod_{i=1}^{2}\sqrt{\lambda_i}e^{\lambda_i(\alpha_i^{-k_i}-1)}\left(\alpha_i^{-k_i}-1\right),i=1,2$$

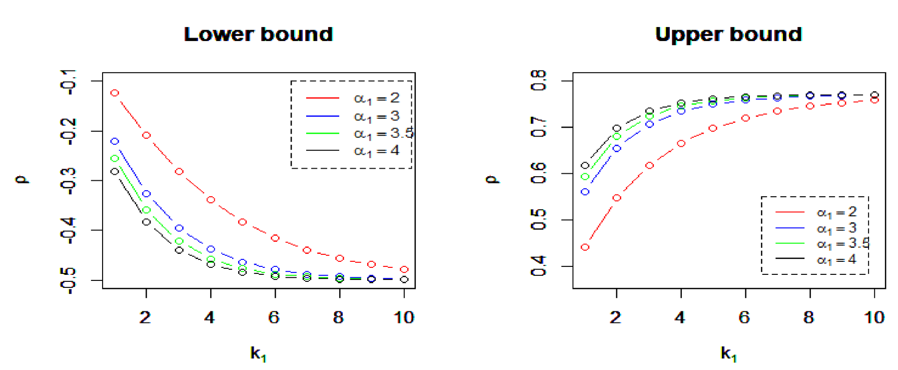
$$-0.5\leq\rho\leq 0.77$$



The optimal value for correlation ρ is taken when $\lambda_i=1,\ i=1,2$ and when $k_1=k_2$



At the cases where $k_1=k_2$ and $k_1=k_2/2$ bounds for ρ are closed at the same value. More specifically at each of two cases low bound is -0.49995 and -0.49660 accordingly. Moreover value in upper bound is 0.77072 and 0.76556 accordingly. In case of different values for parameters a_i we take best correlation coefficient values when $a_i=4$ and $\lambda_i=0.5$.



A trivariate INGARCH model

$$Y_t\mid\mathcal{F}_{t-1}^{Y}\sim TP(\lambda_t) \tag{1}$$

where

$$\lambda_t=\mathbf{d}+\mathbf{A}\lambda_{t-1}+\mathbf{B}\mathbf{Y}_{t-1} \tag{2}$$

- $\mathcal{F}_{t-1}^{Y}$ σ-algebra
- **d** is a 3-dimensional vector
- **A,B** are 3×3 unknown matrices.
- **d,A,B** are all positive considering that $\lambda_t>0$

Stationarity and ergodicity

Prpoposition 1. Consider model (1) and suppose that $||A||_1+||B||_1\leq 1$, the process $\{\lambda_t, t>0\}$ is an ergodic Markov chain.

We need to prove that

$$Cov(g('past'),g('future'))$$

become small as distance between 'past' and 'future' becomes large.

Equally we can prove that $Cov(g(\lambda_t,Y_t\mid\mathcal{F}_{t-1}^{Y}),g(\lambda_t,Y_t))$ tends to 0 based on definitions of θ-weak dependence and τ-weak dependence proposed by `Dedecker et al.(2007)`. The assumption that multivariate function $g:\mathbb{R}^n\rightarrow\mathbb{R}^m$ where $g(\lambda_{t-1},\ldots,\lambda_{t-p},\mathbf{y}_{t-1},\ldots,\mathbf{y}_{t-q})=\lambda_t$ satisfy the Lipschitz condition $|g(\lambda,\mathbf{y})-g(\lambda^*,\mathbf{y}^*)|\leq\sum_{i=1}^{p}||A_i||_1||\lambda_{i,t-i}-\tilde{\lambda}_{i,t-i}||+\sum_{j=1}^{q}||B_j||_1||Y_{j,t-j}-\tilde{Y}_{j,t-j}||_1$

Parameters' estimation

A generalization of the joint pmf based on the new construction of Sarmanov distribution in case of Poisson distribution.

$$f_{12\ldots n}(y_1y_2\ldots y_n)=\prod_{i=1}^{n}f_i(y_i)[1+R_{q_1q_2\ldots q_n}(y_1y_2\ldots y_n)]$$

where

$$R_{q_1q_2\ldots q_n}(y_1y_2\ldots y_n)=\sum_{i<}^{n-1}\sum_{j=2}^{n}\omega_{ij}q_i(y_i)q_j(y_j)$$

where

$$q_i(y_i)=\left(\alpha_i^{-k_iy_i}-e^{\lambda_i(\alpha_i^{-k_i}-1)}\right).$$

Log-likelihood function for trivariate Poisson INGARCH model is given by:

$$l(\theta)=\sum_{t=1}^{n}\left(\sum_{i=1}^{3}[-\lambda_i(\theta)+Y_{i,t}\log(\lambda_i(\theta))-\log(Y_{i,t}!)]+\log\phi_t(\theta)\right)$$

where

$$\phi_t(\theta)=1+\sum_{i<}^{2}\sum_{j=2}^{3}\omega_{ij}q_i(y_i)q_j(y_j)$$

Theorem 1

For the INGARCH model as $n\rightarrow\infty$

$$\sqrt{n}(\hat{\theta}-\theta_0)\rightarrow N(0,I(\theta_0)^{-1})$$

where

$$I(\theta_0)=E\left(\frac{\partial l(\theta_0)}{\partial\theta}\frac{\partial l(\theta_0)}{\partial\theta^T}\right)=-E\left(\frac{\partial^2 l(\theta_0)}{\partial\theta\partial\theta^T}\right)$$

Considering boundaries for parameter ω

- $\phi_L\leq\phi_t(\theta)\leq\phi_U$
- $1+\sum_{i<}^{2}\sum_{j=2}^{3}\omega_{ij}q_i(x_{it})q_j(x_{jt})\leq c_1$
- $\lambda_L\leq\lambda_t\leq\lambda_U$

$$E\left(-\frac{\partial^2 l}{\partial\theta_{ij}^2}\mid F_{t-1}\right),\ E\left(-\frac{\partial^2 l}{\partial\theta_4^2}\right)$$

are measurable

One step Forecasting

After recursions we have:

$$\begin{aligned}\lambda_t &= \mathbf{d} + \mathbf{A}\lambda_{t-1} + \mathbf{B}Y_{t-1}\\ &= \dots\\ &= \sum_{i=0}^{k-1}\mathbf{A}^{\mathbf{i}}\mathbf{d} + \mathbf{A}^{\mathbf{k}}\lambda_{t-k} + \mathbf{B}\sum_{i=0}^{k-1}\mathbf{A}^{\mathbf{i}}Y_{t-i-1}\end{aligned}$$

Prediction based on method of conditional expectation. An one step ahead forecast is:

$$\hat{\lambda}_{t+1} = E(\lambda_{t+1} \mid \mathcal{F}) = \sum_{i=0}^{k-1}(\hat{\mathbf{A}}^{\mathbf{i}}\hat{\mathbf{d}} + \hat{\mathbf{A}}^{\mathbf{i}}\hat{\mathbf{B}}Y_{t-i}) + \hat{\mathbf{A}}^{\mathbf{t}}\lambda_1$$

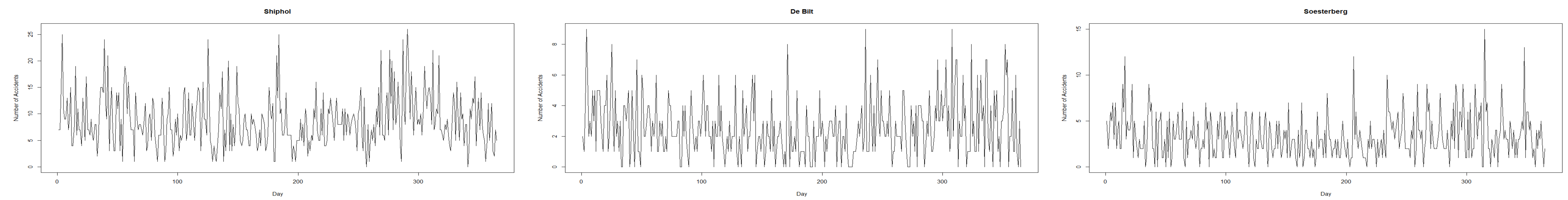
Application

We use daily crash counts in three areas of Netherlands that is, Shiphol, De Bilt and Soesterberg. The data share some similar environment, especially with respect to weather conditions, road characteristics and traffic exposure. For the trivariate Poisson model we calculate parameter vector $\theta = (d_1, d_2, d_3, a_{ij}, b_{ij}, \omega_{ij}, k_1, k_2, k_3),\ i, j = 1, 2, 3$

- In our study the parameters of the multivariate Poisson model are calculated by ML method.
- As constraints we consider conditions of stationarity for parameters $a_{ij}, b_{ij}\ i, j = 1, 2, 3$ and under the assumption $k_i \in [1, 10]$ we restrict parameters $\omega_{12}, \omega_{13}, \omega_{23}$ according to

$$\frac{-l_w}{max\left\{\left(1 - e^{\lambda_i(a_i^{-k_i}-1)}\right)\left(1 - e^{\lambda_j(a_2^{-k_j}-1)}\right), e^{\lambda_i(a_i^{-k_i-1})}e^{\lambda_j(a_j^{-k_j-1})}\right\}}$$

$$\frac{l_w}{max\left\{e^{\lambda_i(a_i^{-k_i-1})}\left(1 - e^{\lambda_j(a_j^{-k_j-1})}\right), e^{\lambda_j(a_j^{-k_j-1})}\left(1 - e^{\lambda_i(a_i^{-k_i-1})}\right)\right\}}$$



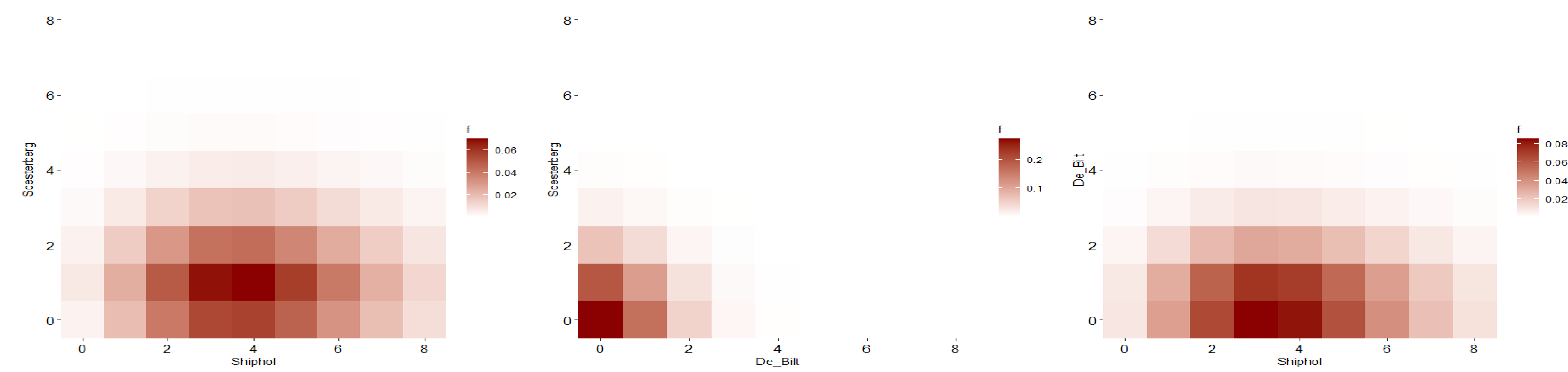
	Mean	Variance	Mean Variance
Shiphol	8.781	24.600	2.8
De Bilt	2.584	3.681	1.42
Soesterberg	3.370	5.673	1.68

○ For all three series the ratio of the variance to the mean is larger than 1 implying overdispersion relative to the simple Poisson distribution.

$\hat{d}_1(SE)$	$\hat{d}_2(SE)$	$\hat{d}_3(SE)$	$\hat{a}_{11}(SE)$	$\hat{a}_{12}(SE)$	$\hat{a}_{13}(SE)$	$\hat{a}_{21}(SE)$	$\hat{a}_{22}(SE)$	$\hat{a}_{23}(SE)$
1.002(0.002)	0.810(0.010)	0.814(0.019)	0.215(0.015)	0.192(0.007)	0.113(0.013)	0.128(0.028)	0.117(0.017)	0.188(0.012)
$\hat{a}_{31}(SE)$	$\hat{a}_{32}(SE)$	$\hat{a}_{33}(SE)$	$\hat{b}_{11}(SE)$	$\hat{b}_{12}(SE)$	$\hat{b}_{13}(SE)$	$\hat{b}_{21}(SE)$	$\hat{b}_{22}(SE)$	$\hat{b}_{23}(SE)$
0.299(0.001)	0.134(0.034)	0.039(0.061)	0.124(0.024)	0.123(0.023)	0.113(0.013)	0.084(0.016)	0.115(0.015)	0.105(0.014)
$\hat{b}_{31}(SE)$	$\hat{b}_{32}(SE)$	$\hat{b}_{33}(SE)$	$\hat{\omega}_{12}(SE)$	$\hat{\omega}_{13}(SE)$	$\hat{\omega}_{23}(SE)$	$\hat{k}_1(SE)$	$\hat{k}_2(SE)$	$\hat{k}_3(SE)$
0.243(0.043)	0.084(0.015)	0.122(0.022)	-2.465(0.035)	-1.448(0.051)	-1.961(0.039)	9.040(0.040)	9.011(0.011)	9.034(0.034)

Predictive bivariate marginals

- We predict λ at time $t = 366$ and then we calculate bivariate marginal predictive pmfs.



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