A new framework of semi-Markov processes for parameter estimation and Reliability Analysis

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All systems are designed to perform their intended tasks in a given environment. Some systems can perform their tasks with various distinctive levels of efficiency usually referred to as performance rates.

A system that can have a finite number of performance rates is called a multi-state system (MSS).

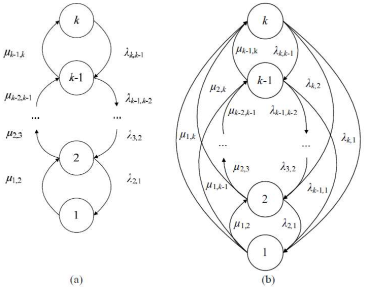


Figure 1: Multi state diagrams

- The simplest example of such a situation is the k-out-of-n system. It consists of n identical binary units and can have $n + 1$ states depending on the number of available units.
- The entire set of possible system states can be divided into two disjoint subsets corresponding to acceptable and unacceptable system functioning. The system entrance into the subset of unacceptable states constitutes a failure.
- MSS reliability: The system's ability to remain in acceptable states during the operation period.

Semi-Markov Processes and MSS

Semi-Markov (SM) processes are typical tools for the modeling of technical systems. Such classes of stochastic processes generalize typical Markov jump processes by allowing general distributions for sojourn times (Limnios, N. and Oprisan, G. (2001) and Barbu, V.S. and Limnios, N. (2008)).
If $(J, S) = (J_n, S_n)_{n\in\mathbb{N}}$ satisfies the relation

$$\mathbb{P}(J_{n+1} = j, S_{n+1} - S_n \leq t | J_0, \cdots, J_n; S_1, \cdots, S_n)$$
$$= \mathbb{P}(J_{n+1} = j, S_{n+1} - S_n \leq t | J_n), j \in E, t \in \mathbb{R}_+$$

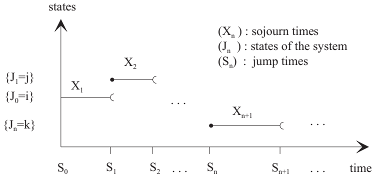


Figure 2: A trajectory of a SM process

- (J, S) Markov renewal chain
- $Z = (Z_t)_{t\in\mathbb{R}_+}$ – semi-Markov process associated to (J, S)

$$Z_t := J_{N(t)} \quad \Leftrightarrow \quad J_n = Z_{S_n}$$

$$N(t) := \max\{n \in \mathbb{N} \mid S_n \leq t\}, \; t \in \mathbb{R}_+.$$

Let also $X_n = S_n - S_{n-1}, \quad n \geq 1,$
be the sojourn times, with $S_0 := 0$.

The SM model is characterized by its initial distribution $\alpha = \{\alpha_1, \ldots, \alpha_N\}$

$$\alpha_j = \mathbb{P}(J_0 = j),\ j \in E,$$

and by the semi-Markov kernel

$$Q_{ij}(t) = \mathbb{P}(J_n = j, X_n \leq t | J_{n-1} = i).$$

Let us also consider the transition probabilities of the embedded Markov Chain $(J_n)_{n\in\mathbb{N}}$

$$p_{ij} = \mathbb{P}(J_n = j | J_{n-1} = i) = \lim_{t\to\infty} Q_{ij}(t),$$

and the conditional sojourn time distribution

$$\begin{aligned} W_{ij}(t) &= \mathbb{P}(S_n - S_{n-1} \leq t | J_{n-1} = i, J_n = j) \\ &= \mathbb{P}(X_n \leq t | J_{n-1} = i, J_n = j). \end{aligned}$$

Observe that

$$Q_{ij}(t) = p_{ij} W_{ij}(t). \tag{1}$$

- T_{ij} is the r.v. that can be seen as the "potential" times.

The dynamic of the system: the next state to be visited after state i is the one for which T_{il} is the minimum. Thus, in our framework, the next state to be visited, say j, is "chosen" through $j = argmin_{l \in E}(T_{il})$ and $\min_{j \in E} T_{ij}$ represents the sojourn time in state i before moving to the next state.

Thus, for our SM system, the SM kernel has the particular form

$$\begin{aligned} Q_{ij}(t) &= \mathbb{P}\left(\min_k T_{ik} \leq t, T_{ij} \leq T_{ik}, \forall k | J_{n-1} = i\right) \\ &= \mathbb{P}\left(\min_k T_{ik} \leq t | J_{n-1} = i, J_n = j\right) \times \mathbb{P}\left(T_{ij} \leq T_{ik}, \forall k | J_{n-1} = i\right) \\ &= p_{ij} W_i(t), \end{aligned}$$

- $p_{ij} = \mathbb{P}(J_n = j | J_{n-1} = i) = \mathbb{P}\left(T_{ij} \leq T_{ik}, \forall k | J_{n-1} = i\right)$
- $W_{ij}(t) = \mathbb{P}(S_n - S_{n-1} \leq t | J_{n-1} = i, J_n = j)$
  $= \mathbb{P}(\min_k T_{ik} \leq t | J_{n-1} = i, J_n = j) = W_i(t),$ independent of j.
- $\sum_j Q_{ij}(t) = W_i(t).$

I.N.I.D. random variables

It is natural to assume in MSS that sojourn times between different states are not necessarily identically distributed.

Indeed, if a system's component fails it is expected that the transition rate from this new state i to another state j **will not necessarily be equal** to the transition rate that resulted in the transition to state i.

Consider the general family of distribution functions with parameter a:

$$F(t;a) = 1 - (1 - F(t;1))^a . \tag{2}$$

$f(t;a)$: its associated density w.r.t. the Lebesgue measure (assumed to exist).

Theorem 1

Let $X_1, \ldots, X_N$ be inid r.v. such that $X_i \sim F(x;a_i)$ which belongs to class (2). Then the distribution function $F^{(1)}$ of the minimum order statistic $X_{(1)}$ belongs also to (2).

Examples: Geometric distribution, Exponential distribution, Weibull distribution, Rayleigh distribution, Pareto distribution, Kumaraswamy distribution

Based on Kumaraswamy distribution we create a generalized G-class of distributions which is entirely WITHIN the class (2) of distributions

$$F(t;a) := 1 - (1 - G(t)^c)^a \,. \tag{3}$$

- $F(t;1) = G(t)^c$
- $G(\cdot)$: a parent continuous distribution
- for any function G we identify a new member of the new G-class and all of them are inside the class (2)
- Kumaraswamy is the baseline distribution of (3) obtained for a special $G(t)$:

Remark 1

If $G(\cdot)$ is the identity function, the baseline (Kumarasawmy) distribution is obtained, i.e.

$$F(t;a) := 1 - (1 - t^c)^a \,, \quad t \in (0,1). \tag{4}$$

Idea of generating family of distributions

Based on Gompertz distribution

$$G(t) = \left(1 - \rho e^{-\lambda t}\right)^{\alpha}, \quad \text{for} \quad t > \lambda^{-1} \log \rho, \quad \text{where} \quad \rho, \alpha, \lambda \geq 0.$$

three distribution classes ("Exponentiated families") have been created (Lehmann 1953 and Nadarajah 2005):

- $F(t) = G(t)^{\alpha}$ (Exponentiated family of distributions)
- $F(t) = 1 - (1 - G(t))^{\alpha}$ (Lehmann alternative 2 family of distributions)
- $F(t) = 1 - (1 - G(e^{t}))^{\alpha}$ (Nadarajah family of distributions)

Reliability Parameter R

In order to evaluate the performance of the reliability system, let the strength X which is subject to a stress Y.
The system fails as soon as the stress exceeds its strength. The probability of exceedance is defined as

$$R = P(Y < X) = E[P(Y < X)|X]. \tag{5}$$

Theorem 2

Let X, Y be independent random variables from the G-class of distributions (3) with shape parameters α_1 and α_2 respectively and common shape parameter c. Then, the reliability parameter R given in (5), is a constant that depends only on the shape parameters α_1 and α_2, namely

$$R = \frac{a_2}{a_1 + a_2}.$$

More Reliability Indices

- Reliability or Survival Function: $R(t)$
- Availability (Instantaneous Reliability): $A(t)$
- Maintainability: $M(t)$
- Mean Time to Failure: $MTTF$

The characteristics of our SM system

- $Q_{ij}(t; a_{ik}; k = 1, \dots, N) := Q_{ij}(t)$

$$Q_{ij}(t) = \frac{a_{ij}}{\sum\limits_{k \in E} a_{ik}} \left[1 - (1 - G(t)^c)^{\sum\limits_{k \in E} a_{ik}} \right], \tag{6}$$

- By taking the limit, the transition probabilities are

$$p_{ij} = \lim_{t \to \infty} Q_{ij}(t) = \frac{a_{ij}}{\sum\limits_{k \in E} a_{ik}}.$$

- The distribution of the minimum is

$$W_i(t) = 1 - [1 - G(t)^c]^{\sum\limits_{j=1}^{N} a_{ij}}$$

Inference with and without censoring

Several sample paths, no censoring

Given $\left\{j_0^{(l)}, x_1^{(l)}, j_1^{(l)}, x_2^{(l)}, \ldots, j_{N^l(M)}^{(l)}\right\}, l = 1, \ldots, L$, L sample paths of a semi-Markov process, then the associated likelihood for L trajectories is

$$
\begin{aligned}
\mathcal{L} &= \prod_{l=1}^{L} \alpha_{j_0}^{(l)} p_{j_0^{(l)} j_1^{(l)}} f_{j_0^{(l)}}(x_1^{(l)}) \ldots f_{j_{N^l(M)-1}^{(l)}}(x_{N^l(M)}^{(l)}) \\
&= \left(\prod_{i \in E} \alpha_i^{N_{i,0}(L)}\right) \left(\prod_{i,j \in E} p_{ij}^{\sum\limits_{l=1}^{L} N_{ij}^{(l)}(M)}\right) \\
&\quad \times \left(\prod_{l=1}^{L} \prod_{i \in E} \prod_{k=1}^{N_i^{(l)}(M)} f_i(x_i^{(l,k)})\right)
\end{aligned}
$$

where,

$N_{i,0}(L) = \sum_{l=1}^{L} \mathbf{1}_{\{J_0^{(l)}=i\}}$,

$N_i^{(l)}(M)$: the number of exits from state i up to time M of the l^{th} trajectory, $l = 1, \ldots, L$,

$N_{ij}^{(l)}(M)$: the number of transitions from state i to state j up to time M of the l^{th} trajectory, $l = 1, \ldots, L$,

$N_{ij}(L, M) = \sum_{l=1}^{L} N_{ij}^{(l)}(M)$,

$N^l(M)$: the number of jumps during the l^{th} trajectory, $l = 1, \ldots, L$.

$x_i^{(l,k)}$: the sojourn time in state i during the k^{th} visit, $k = 1, \ldots, N_i^{(l)}(M)$ of the l^{th} trajectory, $l = 1, \ldots, L$.

Several censored sample paths at the end

Given $\left\{j_0^{(l)}, x_1^{(l)}, j_1^{(l)}, x_2^{(l)}, \ldots, j_{N^l(M)}^{(l)}, u_M^{(l)}\right\}, l = 1, \ldots, L$, L sample paths of a semi-Markov process, then the associated likelihood with censoring at time M is

$$\begin{aligned} \mathcal{L} &= \left(\prod_{i \in E} \alpha_i^{N_{i,0}(L)}\right)\left(\prod_{i,j \in E} p_{ij}^{\sum_{l=1}^{L} N_{ij}^{(l)}(M)}\right) \\ &\quad \times \left(\prod_{l=1}^{L}\prod_{i \in E}\prod_{k=1}^{N_i^{(l)}(M)} f_i(x_i^{(l,k)})\right)\prod_{i \in E}\prod_{k=1}^{\overline{N}_{i\bullet}^{e}(L)}\left(1 - W_i(u_i^{(k)})\right). \end{aligned}$$

where,
$u_M^{(l)} := M - S_{N^l(M)}$ is the censored sojourn time in the last visited state of the l^{th} trajectory,
$\overline{N}^e_{i\bullet}(L) = \sum_k \sum_{l=1}^{L} \mathbb{1}_{\{J^{(l)}_{N^l(M)}=i, X^{(l)}_{j^{(l)}_{N^l(M)}} > k\}}$ is the number of trajectories ending in state i with censored last sojourn time in state i greater than k.

$u_i^{(k)}$ is the censored sojourn time in state i as the last visited state, during the k^{th} visit, $k = 1, \ldots, N_{i,M}(L)$.

Note that, if the censoring time M in a certain trajectory l is a jump time, then for the corresponding observed censored time we have $u_M^{(l)} = 0$.

Several censored sample paths at the beginning

Given $\left\{x_0^{(l)}, j_0^{(l)}, x_1^{(l)}, j_1^{(l)}, x_2^{(l)}, \dots, j_{N^l(M)}^{(l)}\right\}, l = 1, \dots, L$, L sample paths of a semi-Markov process, then the associated likelihood with censoring at the beginning is

$$\begin{aligned} \mathcal{L} &= \left(\prod_{i \in E} \alpha_i^{N_{i,0}(L)}\right) \prod_{i \in E} \prod_{k=1}^{\overline{N}^b_{i\bullet}(L)} \overline{W}_{i\bullet}(x_{i,0}^{(k)}) \left(\prod_{i,j \in E} p_{ij}^{\sum_{l=1}^{L} N_{ij}^{(l)}(M)}\right) \times \\ &\times \left(\prod_{l=1}^{L} \prod_{i \in E} \prod_{k=1}^{N_i^{(l)}(M)} f_i(x_i^{(l,k)})\right), \end{aligned} \tag{7}$$

where

- $\overline{N}^b_{i\bullet}(L) = \sum_t \sum_{l=1}^{L} \mathbb{1}_{\{J_0^{(l)} = i, S_1^{(l)} - S_0^{(l)} > t, S_1^{(l)} < M\}}$ is the number of trajectories starting in state i with censored first sojourn time in state i greater than t,
- $x_{i,0}^{(k)}$ is the censored sojourn time in state i as the first state, during the k^{th} visit, $k = 1, \dots, \overline{N}^b_{i\bullet}(L)$.

Several censored sample paths at the beginning and/or at the end

Given $\left\{ {x_0^{(l)}}^{\delta_b^{(l)}}, j_0^{(l)}, x_1^{(l)}, j_1^{(l)}, x_2^{(l)}, \ldots, j_{N^l(M)}^{(l)}, {u_M^{(l)}}^{\delta_e^{(l)}} \right\}, l = 1, \ldots, L,$

where $\delta_b^{(l)} = \begin{cases} 1, & \text{if the first sojourn time is considered to be censored,} \\ 0, & \text{if the first sojourn time is not considered to be censored.} \end{cases}$

and $\delta_e^{(l)} = \begin{cases} 1, & \text{if the last sojourn time is considered to be censored,} \\ 0, & \text{if the last sojourn time is not considered to be censored.} \end{cases}$

The associated likelihood where some of the sojourn times are censored either at the beginning and/or at the end is

$$\begin{aligned} \mathcal{L} &= \left(\prod_{i \in E} \alpha_i^{N_{i,0}(L)} \right) \left(\prod_{i,j \in E} p_{ij}^{\sum_{l=1}^{L} N_{ij}^{(l)}(M)} \right) \left(\prod_{l=1}^{L} \prod_{i \in E} \prod_{k=1}^{N_i^{(l)}(M)} f_i(x_i^{(l,k)}) \right) \times \\ &\times \left(\prod_{i \in E} \prod_{k=1}^{\overline{N}_{i\bullet}^{b}(L)} \overline{W}_{i\bullet}(x_{i,0}^{(k)}) \right) \left(\prod_{i \in E} \prod_{k=1}^{\overline{N}_{i\bullet}^{e}(L)} \overline{W}_{i\bullet}(u_i^{(k)}) \right). \end{aligned} \tag{8}$$

Estimation - General G-Class of Distributions

Theorem 3 (Estimators - no censoring)

$$\widehat{a}_{ij}(L, M) = -\frac{N_{ij}(L, M)}{\sum_{l=1}^{L} B_i^{(l)}(M)}, \tag{9}$$

where,

$$N_{ij}(L, M) = \sum_{l=1}^{L} N_{ij}^{(l)}(M),$$

$$B_i^{(l)}(M) = \sum_{k=1}^{N_i^{(l)}(M)} \log\left(1 - G\left(X_i^{(l,k)}\right)^c\right).$$

$$\widehat{\alpha}_i(L, M) = \frac{N_{i,0}(L)}{L}. \tag{10}$$

Theorem 4 (Estimators - censoring at the end)

$$\widehat{a}_{ij}(L,M) = -\frac{N_{ij}(L,M)}{\sum_{l=1}^{L} B_i^{(l)}(M) + \sum_{k=1}^{N_{i,M}(L)} \log\left(1 - G\left(U_i^{(k)}\right)^c\right)}. \tag{11}$$

$$\widehat{\alpha}_i(L,M) = \frac{N_{i,0}(L)}{L}. \tag{12}$$

Estimation - Kumaraswamy Distribution ($G(x) = x$)

Theorem 5 (Estimators - no censoring)

$$\widehat{a}_{ij}(L,M) = -\frac{N_{ij}(L,M)}{\sum_{l=1}^{L}\sum_{k=1}^{N_i^{(l)}(M)} \log\left(1-\left(X_i^{(l,k)}\right)^c\right)}. \tag{13}$$

Theorem 6 (Estimators - censoring at the end)

$$\widehat{a}_{ij}(L,M) = -\frac{N_{ij}(L,M)}{\sum_{l=1}^{L}\sum_{k=1}^{N_i^{(l)}(M)} \log\left(1-\left(X_i^{(l,k)}\right)^c\right) + \sum_{k=1}^{N_{i,M}(L)} \log\left(1-\left(U_i^{(k)}\right)^c\right)}. \tag{14}$$

For the MLE of the parameter c, one should solve the following equation:

$$\begin{aligned}\frac{\partial \log \mathcal{L}}{\partial c} \quad &= \quad \sum_{l,i,j,k}\left[-\frac{N_{ij}^{(l)}(M)}{\log\left(1-\left(X_i^{(l,k)}\right)^c\right)} - 1\right]\sum_{k=1}^{N_i^{(l)}(M)}\left[\frac{-\left(x_i^{(l,k)}\right)^c \log x_i^{(l,k)}}{1-\left(x_i^{(l,k)}\right)^c}\right] + \\ &\quad + \frac{\sum_{l,i,k}\left(1 + c\log x_i^{(l,k)}\right)}{c} = 0.\end{aligned}$$

Estimation of the SM process' quantities in our framework

- $\hat{p}_{ij}(L,M) = \frac{\hat{a}_{ij}(L,M)}{\sum\limits_{k\in E} \hat{a}_{ik}(L,M)}$.

- $\widehat{W}_i(t,M) = \left[1 - (1 - G(t)^c)^{\sum\limits_{j\in E} \widehat{a}_{ij}(L,M)}\right]$.

- $\hat{Q}_{ij}(t,M) = \hat{p}_{ij}(t,M)\hat{W}_i(t,M)$.

- $\hat{R}_{ij} = \frac{\hat{a}_{ij_{(2)}}}{\hat{a}_{ij_{(1)}} + \hat{a}_{ij_{(2)}}}$

Reliability/survival analysis framework

Let the state space be partitioned as follows: $E = U \cup D$ and $E = U \cap D = \varnothing$, where we assume that $U = \{1, \ldots, n\}$ and $D = \{n+1, \ldots, N\}$. Each matrix/vector can be partitioned accordingly.

$$Q(t) = \begin{array}{c} \begin{array}{cc} \text{U} & \text{D} \end{array} \\ \begin{bmatrix} Q_{UU}(t) & Q_{UD}(t) \\ Q_{DU}(t) & Q_{DD}(t) \end{bmatrix} \end{array} \begin{array}{c} \text{U} \\ \text{D} \end{array}, \qquad W(t) = \begin{array}{c} \begin{array}{cc} \text{U} & \text{D} \end{array} \\ \begin{bmatrix} W_U(t) & 0 \\ 0 & W_D(t) \end{bmatrix} \end{array} \begin{array}{c} \text{U} \\ \text{D} \end{array}$$

For the matrices $\Psi(t)$ $\left(\text{where } \Psi_{ij}(t) = \mathbb{E}_i[N_j(t)]\right)$ and $P(t)$ $\left(\text{where } P_{ij}(t) = \mathbb{P}(Z_t = j | Z_0 = i)\right)$ we consider their partitions induced by the corresponding partitions of the semi-Markov kernel $Q(t)$:

$$P_{UU}(t) = \left(\Psi_{UU} \star (I_n - W_U)\right)(t)$$

and

$$\Psi_{UU}(t) = (I_n - Q_{UU})^{(-1)}(t).$$

Reliability

Consider the *reliability* or *survival function* of the system at time t :

$$R(t) = \mathbb{P}(T_D > t) = \mathbb{P}(Z_s \in U, s \leq t),$$

where $T_D := \inf\{t \mid Z_t \in D\}$ is the lifetime of the system. We know (Ouhbi, Limnios; 1996) $R(t) = \alpha_U P_{UU}(t)\mathbf{1_n}$.

Proposition 1

For a semi-Markov system, the estimator of the reliability at time $t > 0$ is

$$\widehat{R}(t, M) = \widehat{\alpha}_U(M)\widehat{P}_{UU}(t, M)\mathbf{1_n},$$

where $\widehat{\alpha}_U(M)$ is an estimator of α_U and $\widehat{P}_{UU}(t, M)$ is an estimator of $P_{UU}(t)$, with

$$(Q_{UU})_{ij}(t) = \frac{a_{ij}}{\sum\limits_{k \in E} a_{ik}} \left[1 - (1 - G(t)^c)^{\sum\limits_{k \in E} a_{ik}}\right], \quad i, j \in U.$$

Failure rate

Remark 2

Using the estimator of the reliability, we immediately have an estimator of the failure rate of the system, given by

$$\widehat{\lambda}(t, M) := -\frac{\widehat{R}'(t, M)}{\widehat{R}(t, M)}, t > 0.$$

Availability, maintainability

Proposition 2

For a semi-Markov system, the estimators of the availability and maintainability are given by:

$$\begin{aligned}\widehat{A}(t,M) &= \widehat{\alpha}(M)\widehat{P}(t,M)\mathbf{1}_{N;n},\\ \widehat{M}(t,M) &= 1-\widehat{\alpha}_D(M)\widehat{P}_{DD}(t,M)\mathbf{1_{N-n}},\end{aligned}$$

where $\mathbf{1}_{N;n} = (\underbrace{1,\cdots,1}_{n},\underbrace{0,\cdots,0}_{N-n})^\top$.

Mean time to failure

Let $m := (m_1, \ldots, m_N)^\top$, be the vector of the mean sojourn times, where the *mean sojourn time in state i* is

$$m_i := \mathbb{E}(S_1 \mid J_0 = i) = \int_0^\infty (1 - W_i(t))\, dt = \int_0^\infty \left((1 - G(t)^c)^{\sum_{j=1}^{N} a_{ij}} \right) dt.$$

We assume that $m_i < \infty, i \in E$ (it is the case for a regular and positive recurrent MRP).

We consider the *mean time to failure*:

$$MTTF := \mathbb{E}\left(T_D\right) = \alpha_U (I_n - p_{UU})^{-1} m_U.$$

We can estimate m_i by:

$$\widehat{m}_i^{(1)}(M) := \int_0^\infty \left(1 - \widehat{W}_i(t, M)\right) dt = \int_0^\infty \left((1 - G(t)^c)^{\sum_{j=1}^{N} \widehat{a}_{ij}(L,M)} \right) dt$$

and

$$\widehat{m}_i^{(2)}(M) := \frac{\sum_{k=1}^{N_i(M)} X_i^{(k)}}{N_i(M)}.$$

Simulations

Assume L sample paths where the sojourn times are considered to follow the Kumaraswamy distribution with parameter $c = 2$ and the observation time M is set to be 1000.

- Several sample paths, no censoring
- Several censored sample paths at the end
- Several censored sample paths at the beginning
- Several censored sample paths at the beginning and/or at the end

Real values of a_{ij} and p_{ij}

a_{ij}	1	2	3
1	0	0.9	2.1
2	1.5	0	0.3
3	1.2	1.8	0

p_{ij}	1	2	3
1	0	0.3	0.7
2	0.833	0	0.167
3	0.4	0.6	0

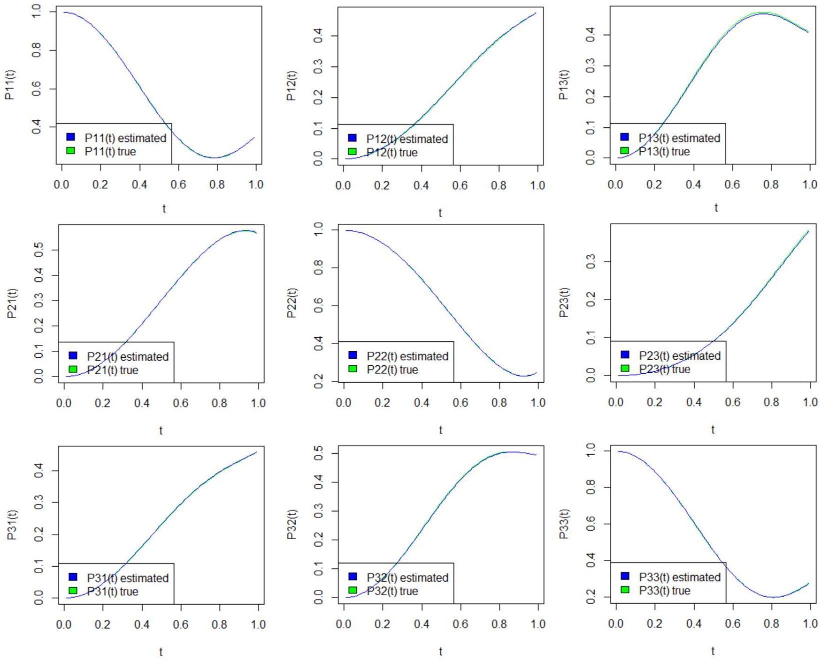
Several uncensored sample paths

i	1	2	3
$\hat{\alpha}_i(L, M)$	0.319	0.355	0.326

S.E. \ L	5	10	100	1000
$\hat{a}_{ij}(L, M)$	5.71×10^{-3}	1.05×10^{-3}	7×10^{-4}	4.35×10^{-5}
$\hat{p}_{ij}(L, M)$	2.6×10^{-4}	1.06×10^{-4}	1.47×10^{-5}	7.58×10^{-7}

S.E. \ t	0.1	0.2	0.5	0.9	0.99
$\hat{P}_{ij}(t; L, M)$	3.19×10^{-8}	4.58×10^{-5}	1.10×10^{-4}	1.85×10^{-4}	6.44×10^{-4}

- the bigger the L, the smaller the S.E. of $\hat{a}_{ij}(L, M)$ and $\hat{p}_{ij}(L, M)$
- the smaller the t the most accurate the $\hat{P}_{ij}(t; L, M)$


P11(t)
P11(t) estimated
P11(t) true
P12(t)
P12(t) estimated
P12(t) true
P13(t)
P13(t) estimated
P13(t) true
P21(t)
P21(t) estimated
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P22(t)
P22(t) estimated
P22(t) true
P23(t)
P23(t) estimated
P23(t) true
P31(t)
P31(t) estimated
P31(t) true
P32(t)
P32(t) estimated
P32(t) true
P33(t)
P33(t) estimated
P33(t) true
t

Several censored sample paths at the beginning and/or at the end

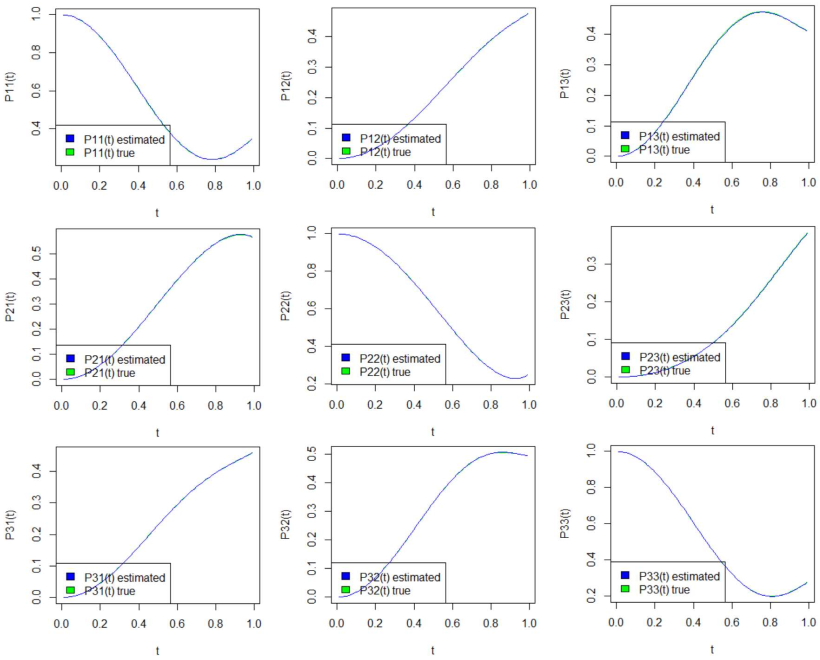
Censoring at the beginning: Using Uniform distribution choose the trajectories with censored sojourn time in the first visited state. Randomly cut the interval (computed as the first sojourn time) in two parts, the second part is the censored sojourn time in the first visited state.

i	1	2	3
$\hat{\alpha}_i(L, M)$	0.328	0.328	0.344

S.E. \ L	5	10	100	1000
$\hat{a}_{ij}(L, M)$	1.13×10^{-2}	6.18×10^{-3}	2.79×10^{-4}	3.27×10^{-5}
$\hat{p}_{ij}(L, M)$	2×10^{-4}	9.24×10^{-5}	1.13×10^{-5}	6.91×10^{-7}

S.E. \ t	0.1	0.2	0.5	0.9
$\hat{P}_{ij}(t; L, M)$	1.69×10^{-6}	2.39×10^{-5}	3.91×10^{-4}	5.37×10^{-4}

- $\hat{a}_{ij}(L, M)$ and $\hat{p}_{ij}(L, M)$ approach the true value as L increases
- $\hat{P}_{ij}(t; L, M)$ is better as t approaches the upper limit, and even better, in terms of the squared errors for small t


P11(t) estimated
P11(t) true
P12(t) estimated
P12(t) true
P13(t) estimated
P13(t) true
P21(t) estimated
P21(t) true
P22(t) estimated
P22(t) true
P23(t) estimated
P23(t) true
P31(t) estimated
P31(t) true
P32(t) estimated
P32(t) true
P33(t) estimated
P33(t) true

Influence of taking into account the censoring

Case(i) censoring at the end (target case)
Case(ii) not taking into account the censoring part

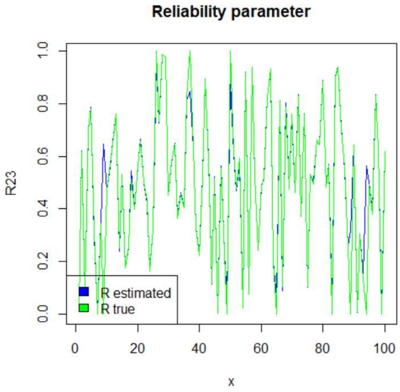
S.E. \ M	10	25	50	75	100	1000
$\hat{a}_{ij}(L,M))$ (i)	0.557899	0.110332	0.111098	0.073128	0.046231	0.007228
$\hat{a}_{ij}(L,M))$ (ii)	0.630267	0.119098	0.118435	0.079926	0.042860	0.006892

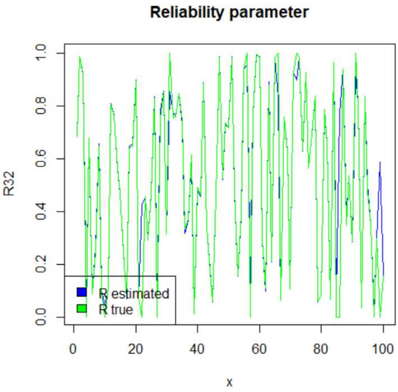
Average length of trajectories: 8.2, 18.6, 38.9, 56, 73.6, 740.2.
When the censored part represents approximately $5-12\%$ of average length of trajectories, target case gives better estimators.
When the censored part represents less than 2% of average length of trajectories, contribution of censored part not significant.

S.E. \ M	10	50	100	1000
$\hat{a}_{ij}(M)$ (case (i))	7.71	5.257648	0.775088	0.055301
$\hat{a}_{ij}(M)$ (case (ii))	18.48546	5.450699	0.757394	0.054022

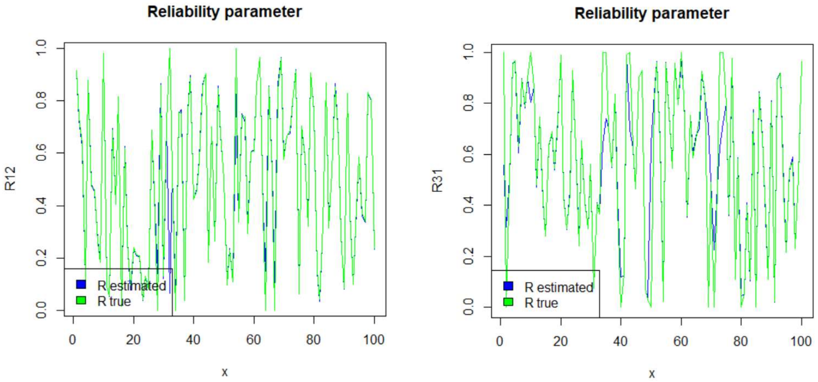
Length of trajectory: 8, 39, 70, 739.
When the censored part represents 10% of whole length of trajectory, target case gives better estimators.

Reliability Parameter for uncensored trajectories





Reliability Parameter for censored trajectories beginning and/or end



Awad, A., M., Azzam, M., M., Hamdan, M., A., (1981). Some inference results on $Pr(X < Y)$ in the bivariate exponential model. *Commun. Statist. Theory Methods* **10** (24), 2515-2524.

Balasubramanian, K., Beg, M. I. and Bapat, R. B. (1991). On families of distributions closed under extrema, *Sankhya: The Indian Journal of Statistics A*, 53, 375-388.

Barbu, V. S., Karagrigoriou, A., Makrides, A. (2017). Semi-Markov Modelling for Multi-State Systems, *Meth. & Comput. Appl. Prob.*, 19, 1011-1028.

Barlow, R. E. and Wu, A. S. (1978). Coherent systems with multi-state components, *Math. Oper. Res.*, **3**, 275-281.

Church, J., D., Harris, B. (1970). The estimation of reliability from stress strength relationships, *Technometrics*, **12**, 49-54.

Constantine, K., Karson, M., (1986). Estimation of P(Y¡X) in gamma case, *Commun. Statist. Comp. Simula.*, **15** (2), 365-388.

Cordeiro, G., M. and Castro, M. (2011) A new family of generalized distributions, *J. Stat. Comput. Simul.*, **81** (7), 883-898.

Eugene N., Lee C., and Famoye F. (2002). Beta-normal distribution and its applications, *Commun. Stat.-Theory Methods* **31**, 497-512.

Jones M.C. (2004). Families of distributions arising from distributions of order statistics (with discussion), Test 13 , 1-43.

Jones, M. C. (2008). Kumaraswamy's distribution: A beta-type distribution with some tractability advantages. *Stat. Methodol.* **6** (1), 70-81.

Kohansal, A. (2017). On estimation of reliability in a multicomponent stress-strength model for a Kumaraswamy distribution based on progressively censored sample. *Statistical Papers*, DOI:10.1007/s00362-017-0916-6

Kumaraswamy, P. (1980). A generalized probability density function for double-bounded random processes, Journal of Hydrology **46**, 79-88.

Slud, E.,V. and Vonta, F. (2005). Efficient semiparametric estimators via modified profile likelihood, *J. Stat. Plan. Inference*, **129** (1), 339-367.