Efficient Simulation of Ruin Probabilities When Claims are Mixtures of Heavy and Light Tails

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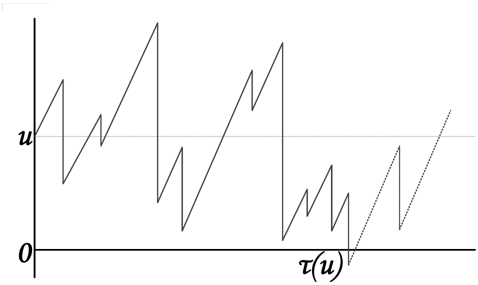
Insurance company with:

- 2 opposing cash flows
  - incoming cash premiums
  - outgoing claims, and
- an initial capital.

Question of interest: What is the probability of bankruptcy?



- λ: Poisson arrival rate ($N(t)$)
- U_k: i.i.d. claim sizes (G)
- $c = 1$: premium rate
- u: initial capital.

Risk reserve process

$$R(t) = u + t - \sum_{k=1}^{N(t)} U_k.$$

Ruin probability for the Cramér-Lundberg risk model

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Claim surplus: $M = \sup_{0 \leq t < \infty} \left(u - R(t)\right)$

Ruin probability

$$\psi(u) = \mathbb{P}(M > u).$$

Pollaczek-Khinchine (PK) formula

PK formula

If $\rho = \lambda \mathbb{E} U < 1$ (safety loading condition) then

$$\psi(u) = 1 - (1-\rho) \sum_{k=0}^{\infty} \rho^k (G^e)^{*k}(u),$$

where $G^e(u) = \int_0^u \big(1 - G(x)\big)\, dx / \mathbb{E} U$ (stationary excess claim size distribution).

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Ways to calculate the ruin probability:

- closed-form solutions (e.g. algebraic, analytic/Laplace transforms)
- approximations (e.g. PH, asymptotic/tail probabilities)
- simulations

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high accuracy computationally tractable error bounds	correct tail behavior

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Fitting distributions to data

1. higher order statistics for tail probabilities (r.v. C)
2. remaining data set phase-type distribution (r.v. B)

Claim size distribution

$$G(x) = (1-\epsilon)F_p(x) + \epsilon F_h(x), \quad x \geq 0.$$

New series expansion for the ruin probability

Theorem

We have

$$\psi(u) = \frac{1-\rho}{1-\rho^\bullet}\,\psi^\bullet(u) + \frac{1-\rho}{1-\rho^\bullet}\sum_{k=1}^{\infty}\left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^k \mathscr{A}_k(u),$$

where $\mathscr{A}_k(u) = \mathbb{P}(M_0^\bullet + M_1^\bullet + \cdots + M_k^\bullet + C_1^e + \cdots + C_k^e > u)$ *and* $M_k^\bullet \stackrel{\mathfrak{D}}{=} M^\bullet$*. This expansion converges for all values of* u*.*

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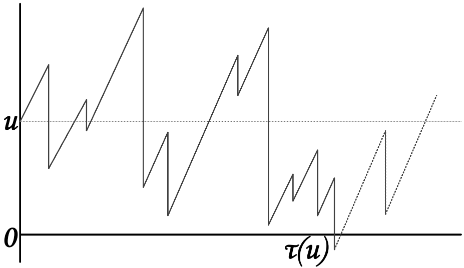

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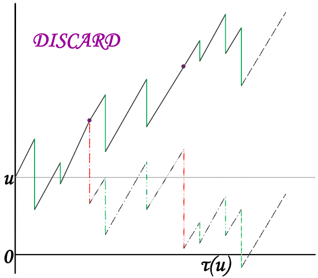

Question: can this series expansion improve simulations?

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Simulate:

$$V \stackrel{\mathfrak{D}}{=} M_0^\bullet + M_1^\bullet + C_1^e + \sum_{k=2}^{N+2}(M_k^\bullet + C_k^e), \quad N \sim Geom\left(\frac{1-\rho}{1-\rho^\bullet}\right).$$

Idea of the control variate techniques:

1. We must simulate a r.v. $Z(u)$ such that $\varphi(u) = \mathbb{E}Z(u)$.
2. We find another r.v. $W(u)$ that has a known expectation $\mathbb{E}W(u)$ and is strongly correlated with $Z(u)$.
3. We simulate them together, i.e. we take $\big(Z^{(i)}(u), W^{(i)}(u)\big)$, $i = 1, 2, \ldots, \kappa$, are independent copies of $\big(Z(u), W(u)\big)$.
4. We calculate

$$\hat{\varphi}_\kappa(u) := \hat{z}_\kappa(u) + \hat{\alpha}_\kappa\big(\hat{w}_\kappa(u) - \mathbb{E}W(u)\big),$$

where

$$\hat{z}_\kappa(u) = \frac{\sum_{i=1}^{\kappa} Z^{(i)}(u)}{\kappa}, \qquad \hat{w}_\kappa(u) = \frac{\sum_{i=1}^{\kappa} W^{(i)}(u)}{\kappa},$$

$$\hat{\alpha}_\kappa = -\frac{\sum_{i=1}^{\kappa}\big(Z^{(i)}(u) - \hat{z}_\kappa(u)\big)\big(W^{(i)}(u) - \hat{w}_\kappa(u)\big)}{\sum_{i=1}^{\kappa}\big(W^{(i)}(u) - \hat{w}_\kappa(u)\big)^2}.$$

Control variate: max of heavy tails

$$\varphi(u) = \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^2 \mathbb{P}(\underbrace{M_0^\bullet + M_1^\bullet + \cdots + M_{N+2}^\bullet + C_1^e + \cdots + C_{N+2}^e}_{\stackrel{\mathfrak{D}}{=} V} > u)$$

Obviously: $Z(u) = \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^2 \mathbb{1}_{\{V>u\}}$.

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Obviously: $Z(u) = \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^2 \mathbb{1}_{\{V>u\}}$.

We define: $V_n := \max\{C_1^e, \ldots, C_{N+2}^e\}\mathbb{1}_{\{N+2\leq n\}}$, for fixed n.

Control variate: $W_n(u) = \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^2 \mathbb{1}_{\{V_n>u\}}$, for fixed n.

$$\varphi_n(u) = \mathbb{E}\,W_n(u) = \left(\frac{1-\rho}{1-\rho^\bullet}\right)\sum_{k=2}^{n}\left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^k \mathbb{P}\big(\max\{C_1^e, \ldots, C_k^e\} > u\big)$$

Alternative control variate: Conditional Monte Carlo

$$V \overset{\mathfrak{D}}{=} \underbrace{M_0^\bullet}_{=X_0^\star} + \underbrace{M_1^\bullet + C_1^e}_{=X_1} + \sum_{k=2}^{N+2} (\underbrace{M_k^\bullet + C_k^e}_{=X_k}) \overset{\mathfrak{D}}{=} X_0^\star + \sum_{k=1}^{N+2} X_k$$

If: $m_k := \max\{X_1, \ldots, X_k\}$, $\overline{F}_X$ is the c.c.d.f. of X_k's, and $S_\ell = \sum_{k=1}^{\ell} X_k$, $S_0 = 0$

Now: $Z^\star(u) = \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^2 (N+2)\overline{F}_X\big(m_{N+1} \vee (u - X_0^\star - S_{N+1})\big)$.

AK control variate: $W^\star(u) = \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^2 (N+2)\overline{F}_X(u)$

$$\varphi^\star(u) = \left(\frac{\epsilon\theta}{1-\rho^\bullet}\right)^2 \left(\frac{\epsilon\theta}{1-\rho} + 2\right)\overline{F}_X(u)$$

Numerical experiments

- Mixture claim size distribution
  - PH: $\overline{F_p}(u) = \overline{F_p^e}(u) = e^{-\mu u}$, and $\mu_B = 1/\mu$ $\underline{(\mu = 3)}$
  - HT: shifted Pareto with shape $a > 1$ and scale $b > 0$, i.e. $\overline{F_h}(u) = (1 + u/b)^{-a}$ and $\overline{F_h^e}(u) = (1 + u/b)^{-(a-1)}$, $u \geq 0$, with $\mu_C = b/(a-1)$ $\underline{(b = 1)}$
  - Perturbation parameter: $\epsilon \in \{0.1, 0.7\}$.
- Focus on $\rho \in \{0.9, 0.99, 0.999\}$.
- Order of $\varphi_n(u)$ equal to $n = 100$
- Number of simulations is $\kappa = 10,000$.

Figures for 1st control variate (max of heavy tails)

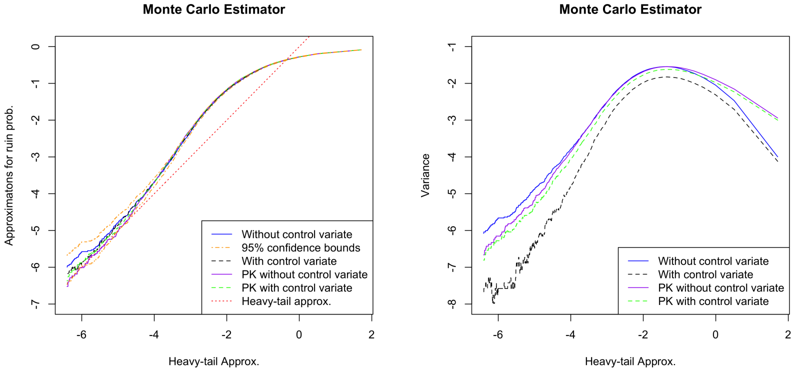

Figure: Plotted in a log-log scale. Model parameters: $a = 2$, $\epsilon = 0.1$, and $\rho = 0.99$.

Figures for 2nd control variate (AK estimator)

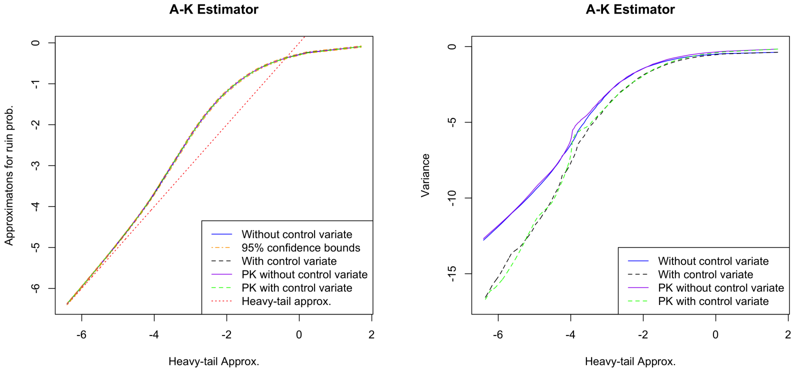

Figure: Plotted in a log-log scale. Model parameters: $a = 2$, $\epsilon = 0.1$, and $\rho = 0.99$.

Conclusions

- We introduced an alternative series expansion for the PK formula in the Cramér-Lundberg model for mixture claim sizes
- Significant improvement of simulation algorithms based on this series
- Proposed a control variate technique: fast and preferable in the heavy traffic regime
- Variance reduction is better with AK conditional Monte Carlo technique but the method is significantly slower
- For other mixtures that the 2nd term of the ruin probability cannot be evaluated, it can also be simulated
- Extension to the Sparre Andersen model,which also has a PK-type formula with respect to the ladder height distribution

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